DIFFERENTIAL FORMS IN CARNOT GROUPS: A Γ-CONVERGENCE APPROACH

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ABSTRACT. Carnot groups (connected simply connected nilpotent stratified Lie groups) can be endowed with a complex (E_0^*, d_c) of "intrinsic" differential forms. In this paper we prove that, in a free Carnot group of step κ, intrinsic 1-forms as well as their intrinsic differentials d_c appear naturally as limits of usual "Riemannian" differentials d_ε, $\varepsilon > 0$. More precisely, we show that L^2-energies associated with $\varepsilon^{-\kappa} d_\varepsilon$ on 1-forms Γ-converge, as $\varepsilon \to 0$, to the energy associated with d_c.

1. INTRODUCTION

In the last few years, sub-Riemannian structures have been largely studied in several respects, such as differential geometry, geometric measure theory, subelliptic differential equations, complex variables, optimal control theory, mathematical models in neurosciences, non-holonomic mechanics, robotics. Roughly speaking, a sub-Riemannian structure on a manifold M is defined by a subbundle H of the tangent bundle TM, that defines the "admissible" directions at any point of M (typically, think of a mechanical system with non-holonomic constraints). Usually, H is called the *horizontal* bundle. If we endow each fiber H_x of H with a scalar product, there is a naturally associated Carnot-Carathéodory (CC) distance d on M, defined as the Riemannian length of the horizontal curves on M, i.e. of the curves γ such that $\gamma'(t) \in H_{\gamma(t)}$. In the spirit of the present paper, it is worth recalling that CC-distances can be seen as limits of "Riemannian" distances (see e.g. [17] and [23]). Basically, this is obtained by penalizing the directions of the tangent bundle that are orthogonal to the horizontal bundle H.

Among sub-Riemannian spaces, a privileged role is played by Carnot groups (see below for precise definition and [5] for a general survey), a role akin to that of Euclidean spaces versus Riemannian manifolds, acting in some sense as rigid "tangent" spaces to general sub-Riemannian spaces (rigid because they are invariant under left translations and group dilations). Roughly speaking, we can always think of a Carnot group $\mathbb{G}$ as of the Lie group $(\mathbb{R}^n, \cdot)$, where $\cdot$ is a (non-commutative) multiplication such that its Lie algebra $\mathfrak{g}$ is nilpotent and admits a *step κ stratification.* This means

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that there exist linear subspaces $V_1, ..., V_\kappa$ (the layers of the stratification) such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. We refer to the first layer V_1 as to the *horizontal layer*, which plays a key role in our theory, since it generates the all of $\mathfrak{g}$ by commutation.

The stratification of the Lie algebra induces a family of anisotropic dilations δ_λ $(\lambda > 0)$ on $\mathfrak{g}$ and therefore, through exponential map, on $\mathbb{G}$.

It is well known that the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ can be identified with the tangent space at the origin e of $\mathbb{G}$, and hence the horizontal layer of $\mathfrak{g}$ can be identified with a subspace $H\mathbb{G}_e$ of $T\mathbb{G}_e$. By left translation, $H\mathbb{G}_e$ generates a subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ and eventually a sub-Riemannian structure on $\mathbb{R}^n$. We stress that Carnot-Carathéodory geometry is not Riemannian at any scale (see [26]).

The first Heisenberg groups $\mathbb{H}^1$ provides the simplest example of noncommutative Carnot groups (of step $\kappa = 2$). It can be identified with $\mathbb{R}^3$ with variables (x, y, t). Set $X := \partial_x - \frac{1}{2} y \partial_t$, $Y := \partial_y + \frac{1}{2} x \partial_t$, $T := \partial_t$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V_1 \oplus V_2$, where $V_1 = \text{span } \{X, Y\}$ and $V_2 = \text{span } \{T\}$.

From now on, we use the word "intrinsic" when we want to stress a privileged role played by the horizontal layer and by group translations and dilations.

Starting from de Rham complex (Ω^*, d) of differential forms in $\mathbb{R}^n$, we look for a complex of differential forms that has to be "intrinsic" for $\mathbb{G}$ in our sense. On one side, since the "intrinsic" vector fields are naturally sections of the horizontal bundle, and hence are vector fields of the first layer of $\mathfrak{g}$, "intrinsic" 1-forms should be their dual forms (for instance, if $\mathbb{G} = \mathbb{H}^1$, dx and dy are dual of X and Y, respectively). On the other side, it is not so evident how to choose a class of "intrinsic" forms of degree 2 or higher, but, even more, the complex we are looking for can not be merely a subcomplex of de Rham complex. Indeed, already in $\mathbb{H}^1$, consider a smooth function $f : \mathbb{H}^1 \to \mathbb{R}$; as we have seen, a "natural" differential would be $d_H f := (Xf) dx + (Yf) dy$. Clearly, this is no more de Rham differential $df = (Xf) dx + (Yf) dy + (Tf)\theta$ (here $\theta = dt + \frac{1}{2}(y dx - x dy)$ is the so-called contact form of $\mathbb{H}^1$). In addition, if we iterate this "differential", we get $d_H^2 f = [X, Y] f \, dx \wedge dy$, that does not vanish precisely because of the lack of commutativity of the group or, equivalently, of its Lie algebra. In other words, we do not have anymore the structure of a complex. In fact, we need a more sophisticated notion of "intrinsic" exterior differential to obtain a complex of differential forms that reflects the lack of commutativity of the group. It turns out that such a complex (E_0^*, d_c) , with $E_0^* \subset \Omega^*$, has been defined and studied by M. Rumin in [24] and [22] ([21] for contact structures). Rumin's theory needs a quite technical introduction that is sketched in Section 3 to make the paper self-consistent. For a more exhaustive presentation, we refer to original Rumin's papers, as well as to the presentation in [2]. The main properties of (E_0^*, d_c) can be summarized in the following points:

- Intrinsic 1-forms are horizontal 1–forms, i.e. forms that are dual of horizontal vector fields, where by duality we mean that, if v is a vector field in $\mathbb{R}^n$, then its dual form $v^\natural$ acts as $v^\natural(w) = \langle v, w\rangle$, for all $w \in \mathbb{R}^n$.
- The "intrinsic" exterior differential d_c on a smooth function is its horizontal differential (that is dual operator of the gradient along a basis of the horizontal bundle).
- The complex (E_0^*, d_c) is exact and self-dual under Hodge $*$-duality.

The first two properties above clearly fit our request for an "intrinsic" theory. Another evidence is provided by Theorem 3.16 in [15], that proves what we can call the "weak naturality" of the complex under homogeneous homomorphisms of the group $\mathbb{G}$. (notice homogeneous homomorphisms between Carnot groups appear naturally as Pansu differentials of maps between Carnot groups, see [19]). In fact, let T be a homogeneous homomorphism of $\mathbb{G}$ (where homogeneous means that $T(\delta_\lambda x) = \delta_\lambda(Tx)$). In exponential coordinates, T can be identified with linear map $T : \mathbb{R}^n \to \mathbb{R}^n$. Suppose now that also ${}^tT$ is a homogeneous homomorphism. Then the pull-back $T^\#$ maps E_0^* into E_0^* and the following diagram is commutative:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots \\
 & & {\scriptstyle T^\#}\big\downarrow & & {\scriptstyle T^\#}\big\downarrow & & \\
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots
\end{array}$$

Since the class of homogeneous homomorphisms well reflects both the group structure and the stratification, the naturality of d_c under homogeneous homomorphisms shows the intimate connection between the complex and the Carnot group. On the other hand, the "artificial assumption" on ${}^tT$ is extensively discussed in Remarks 3.13 and 3.17 of [15], and is basically motivated by the fact that we are working with classes of "true differential forms" and not with quotient classes.

Recently, Rumin's theory has been fruitfully used for several questions in differential geometry, as well as in pde's theory in Carnot groups.

We stress now that a crucial property of d_c relies on the fact that it is generally a *non-homogeneous higher order differential operator*. In this perspective, let us give a gist of how non-homogeneous higher order horizontal derivatives appear in d_c. We need now the notion of *weight* of vectors in $\mathfrak{g}$ and, by duality, of covectors. Elements of the j-th layer of $\mathfrak{g}$ are said to have (pure) weight $w = j$; by duality, a 1-covector that is dual of a vector of (pure) weight $w = j$ will be said to have (pure) weight $w = j$.

This procedure can be extended to h-forms. Clearly, there are forms that have no pure weight, but we can decompose E_0^h in the direct sum of orthogonal spaces of pure weight forms, and therefore we can find a basis of E_0^h given by orthonormal forms of increasing pure weights. We refer to such a basis as to a basis adapted to the filtration of E_0^h induced by the weight.

Then, once suitable adapted bases of h-forms and $(h+1)$-forms are chosen, d_c can be seen as a matrix-valued operator such that, if α has weight p, then the component of weight q of $d_c\alpha$ is given by an homogeneous differential operator in the horizontal derivatives of order $q-p \geq 1$, acting on the components of α.

In order to provide a concrete example of these phenomena, let us consider again the case $\mathbb{G}=\mathbb{H}^1$. We remind that $X^\natural = dx$, $Y^\natural = dy$, $T^\natural = \theta$. In this case

$$\begin{aligned} E_0^1 &= \text{span } \{dx, dy\}; \\ E_0^2 &= \text{span } \{dx\wedge\theta, dy\wedge\theta\}; \\ E_0^3 &= \text{span } \{dx\wedge dy\wedge\theta\}. \end{aligned}$$

The action of d_c on E_0^1 is the following ([21], [13], [3]): let $\alpha = \alpha_1 dx + \alpha_2 dy \in E_0^1$ be given. Then

$$\begin{aligned} d_c\alpha = &(X^2\alpha_2 - 2XY\alpha_1 + YX\alpha_1)dx\wedge\theta \\ &+ (2YX\alpha_2 - Y^2\alpha_1 - XY\alpha_2)dy\wedge\theta. \end{aligned}$$

We see that d_c is a homogeneous operator of order 2 in the horizontal derivatives, since 2-forms have weight 3 and 1-forms have weight 1.

In this paper we want to provide another evidence of the intrinsic character of Rumin's complex, in the spirit of the Riemannian approximation, like in [17] and [23]. More precisely, we want to show that the intrinsic differential d_c is a limit of suitably weighted usual first order de Rham differentials d_ε. For this purpose, we notice preliminarily that the usual exterior differential d acting on a form α of pure weight splits as

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_\kappa\alpha,$$

where $d_0\alpha$ does not increase the weight, $d_1\alpha$ increases the weight by 1, and, more generally, $d_i\alpha$ increases the weight by i when $i = 0, 1, \ldots, \kappa$. Then, we define the ε-differential weighting the different terms of d according to their different actions with respect to the stratification of the Lie algebra $\mathfrak{g}$. Therefore we set

$$d_\varepsilon = d_0 + \varepsilon d_1 + \cdot + \varepsilon^\kappa d_\kappa.$$

The issue now is to specify in what sense the d_ε (that is a first order operator) converges to d, that is, in general, a higher order differential operator, as it has already been pointed out. Keep in mind somehow similar phenomena in elasticity theory, where, roughly speaking, the equations for vibrating plates (that are 4th-order differential equations) can be seen as limits of usual lower order equations for elastic materials. In these cases, the natural approach relies in the use of De Giorgi's Γ-convergence ([9], [8], and see also Section 4 below for precise definitions in our setting) for variational functionals (see, for instance [7] and the references therein). Indeed, we are able to prove that the L^2-energies associated with $\varepsilon^{-\kappa}d_\varepsilon$ on 1-forms Γ-converge, as $\varepsilon\to 0$, to the energy associated with d_c. We stress that intrinsic 1-forms in groups appear in several applications, like H-convergence of elliptic operators on groups ([3], [2]) and Maxwell's equations in Carnot groups ([4], [14], [15]).

More precisely, the main theorem of the present paper reads as follows. If we denote by $W^{\kappa,2}(\mathbb{G}, \bigwedge^1\mathfrak{g})$ the space of differential 1-forms on $\mathbb{G}$ with coefficients belonging to the Folland-Stein space $W^{\kappa,2}(\mathbb{G})$ (see Definition 2.2), we have:

Theorem 1.1. *Let $\mathbb{G}$ be a free Carnot group of step κ. If $\omega \in W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$, we set*

$$F_\varepsilon(\omega) = \frac{1}{\varepsilon^{2\kappa}} \int_{\mathbb{G}} |d_\varepsilon \omega|^2 \, dV,$$

where

$$d_\varepsilon = d_0 + \varepsilon d_1 + \cdot + \varepsilon^\kappa d_\kappa.$$

Then F_ε sequentially Γ-coverges to F in the weak topology $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$, as $\varepsilon \to 0$, where

$$F(\omega) = \begin{cases} \displaystyle\int_{\mathbb{G}} |d_c \omega|^2 \, dV & \text{if } \omega \in W^{\kappa,2}(\mathbb{G}, E_0^1) \\ +\infty & \text{otherwise.} \end{cases}$$

We remind that the group $\mathbb{G}$ is said to be free if its Lie algebra is free, i.e. the commutators satisfy no linear relationships other than antisymmetry and the Jacobi identity. This is a large and relevant class of Carnot groups. We remind also that Carnot groups can always be "lifted" to free groups (see [20] and [5], Chapter 17). For our purposes, the main property of free Carnot groups relies on the fact that intrinsic 1-forms and 2-forms on free groups have all the same weight (see Theorem 3.10). This helps at several steps of the proofs. Unfortunately, the same assertion fails to hold for higher order forms (see Remark 3.11).

Finally, another point has to be put in evidence, i.e. the choice of the topology. Indeed, we prove a Γ-limit result with respect to the *sequential weak convergence*, and it is natural to ask whether we could get rid off the restriction "sequential". This would be possible if we had some kind of coercitivity of the functionals (see [8], Chapter 8). However, this is not the case, since our functionals contain only the L^2-norm of the differential and not of the codifferential, where it is well known that, already in the classical Euclidean setting, the differential alone does not control the $W^{1,2}$-norms (think of Gaffney's inequality: see e.g. [25], Corollary 2.1.6). For the same reasons our convergence result is not meant to derive a convergence of minima, but only to show in what sense Rumin's differential can be seen as the limit of "Riemannian" differentials.

2. Carnot groups

Let $(\mathbb{G}, \cdot)$ be a *Carnot group of step κ* identified to $\mathbb{R}^n$ through exponential coordinates (see [5] for details). By definition, the Lie algebra $\mathfrak{g}$ has dimension n, and admits a *step κ stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ (the layers of the stratification) such that

$$(1) \qquad \mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Set $m_i = \dim(V_i)$, for $i = 1, \dots, \kappa$ and $h_i = m_1 + \cdots + m_i$ with $h_0 = 0$. Clearly, $h_\kappa = n$. Choose now a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ adapted to the stratification, i.e. such that

$$e_{h_{j-1}+1}, \dots, e_{h_j} \text{ is a basis of } V_j \text{ for each } j = 1, \dots, \kappa.$$

We refer to the first layer V_1 as to the *horizontal layer*. It plays a key role in our theory, since it generates the all of $\mathfrak{g}$ by commutation.

Let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$. Given (1), the subset $X_1, \dots, X_{m_1}$ generates by commutations all the other vector fields; we will refer to $X_1, \dots, X_{m_1}$ as to the *generating vector fields* of the algebra, or as to the *horizontal derivatives* of the group.

The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle\cdot,\cdot\rangle$, making $\{X_1, \dots, X_n\}$ be an orthonormal basis.

We can write the elements of $\mathbb{G}$ in *exponential coordinates*, identifying p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and we identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{d_1} x_1, ..., \lambda^{d_n} x_n), \tag{2}$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [12] Chapter 1) and is defined as

$$d_j = i \quad \text{whenever } h_{i-1} + 1 \le j \le h_i, \tag{3}$$

hence $1 = d_1 = ... = d_{m_1} < d_{m_1+1} = 2 \le ... \le d_n = \kappa$.

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$.

We denote also by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q := \sum_{i=1}^{\kappa} i \dim(V_i).$$

The Euclidean space $\mathbb{R}^n$ endowed with the usual (commutative) sum of vectors provides the simplest example of Carnot group. It is a trivial example, since in this case the stratification of the algebra consists of only one layer, i.e. the Lie algebra reduces to the horizontal layer.

Definition 2.1. Let $m \ge 2$ and $\kappa \ge 1$ be fixed integers. We say that $\mathfrak{f}_{m,\kappa}$ is the free Lie algebra with m generators $x_1, \dots, x_m$ and nilpotent of step κ if:

i) $\mathfrak{f}_{m,\kappa}$ is a Lie algebra generated by its elements $x_1, \dots, x_m$, i.e. $\mathfrak{f}_{m,\kappa} = \mathrm{Lie}(x_1, \dots, x_m)$;
ii) $\mathfrak{f}_{m,\kappa}$ is nilpotent of step κ;
iii) for every Lie algebra $\mathfrak{n}$ nilpotent of step κ and for every map ϕ from the set $\{x_1, \dots, x_m\}$ to $\mathfrak{n}$, there exists a (unique) homomorphism of Lie algebras Φ from $\mathfrak{f}_{m,\kappa}$ to $\mathfrak{n}$ which extends ϕ.

The Carnot group $\mathbb{G}$ is said free if its Lie algebra $\mathfrak{g}$ is isomorphic to a free Lie algebra.

When $\mathbb{G}$ is a free group, we can assume $\{X_1, \dots, X_n\}$ a Grayson-Grossman-Hall basis of $\mathfrak{g}$ (see [16], [5], Theorem 14.1.10). This makes several computations much simpler. In particular, $\{[X_i, X_j], X_i, X_j \in V_1, i < j\}$ provides an orthonormal basis of V_2.

From now on, following [12], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi–index, we set $X^I = X_1^{i_1} \cdots X_n^{i_n}$. By the Poincaré–Birkhoff–Witt theorem (see, e.g. [6], I.2.7), the differential operators X^I form a basis for the algebra of left invariant differential operators in $\mathbb{G}$. Furthermore, we set $|I| := i_1 + \cdots + i_n$ the order of the differential operator X^I, and $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of homogeneity with respect to group dilations. From the Poincaré–Birkhoff–Witt theorem, it follows, in particular, that any homogeneous linear differential operator in the horizontal derivatives can be expressed as a linear combination of the operators X^I of the special form above.

Since here we are dealing only with integer order Folland-Stein function spaces, we can this simpler definition (for a general presentation, see e.g. [11]).

Definition 2.2. If $1 < s < \infty$ and $m \in \mathbb{N}$, then the space $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ is the space of all $u \in L^s(\mathbb{G})$ such that

$$X^I u \in L^s(\mathbb{G}) \quad \text{for all multi-index } I \text{ with } d(I) = m,$$

endowed with the natural norm.

We remind that

Proposition 2.3 ([11], Corollary 4.14)**.** *If $1 < s < \infty$ and $m \geq 0$, then the space $W^{m,s}_{\mathbb{G}}(\mathbb{G})$ is independent of the choice of $X_1, \dots, X_{m_1}$.*

Proposition 2.4. *If $1 < s < \infty$ and $m \geq 0$, then $\mathcal{S}(\mathbb{G})$ and $\mathcal{D}(\mathbb{G})$ are dense subspaces of $W^{m,s}_{\mathbb{G}}(\mathbb{G})$.*

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $X_1, \cdots, X_n$, is the family of covectors $\{\theta_1, \cdots, \theta_n\}$. We indicate by $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{g}$ that makes $\theta_1, \cdots, \theta_n$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1, \cdots, \theta_n$ may have polynomial (hence variable) coefficients.

Following Federer (see [10] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as $\bigwedge_* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge_h \mathfrak{g}$ and $\bigwedge^* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge^h \mathfrak{g}$ where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \leq h \leq n$,

$$\bigwedge_h \mathfrak{g} := \operatorname{span}\{X_{i_1} \wedge \cdots \wedge X_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\},$$

$$\bigwedge^h \mathfrak{g} := \operatorname{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}.$$

The elements of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ are called *h-vectors* and *h-covectors*.

We denote by Θ^h the basis $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}$ of $\bigwedge^h \mathfrak{g}$. We remind that $\dim \bigwedge^h \mathfrak{g} = \dim \bigwedge_h \mathfrak{g} = \binom{n}{h}$.

The dual space $\bigwedge^1(\bigwedge_h \mathfrak{g})$ of $\bigwedge_h \mathfrak{g}$ can be naturally identified with $\bigwedge^h \mathfrak{g}$. The action of a h-covector φ on a h-vector v is denoted as $\langle \varphi | v \rangle$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_h \mathfrak{g}$ and to $\bigwedge^h \mathfrak{g}$ making the bases $X_{i_1} \wedge \cdots \wedge X_{i_h}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ orthonormal.

Starting from $\bigwedge_* \mathfrak{g}$ and $\bigwedge^* \mathfrak{g}$, by left translation, we can define now two families of vector bundles (still denoted by $\bigwedge_* \mathfrak{g}$ and $\bigwedge^* \mathfrak{g}$) over $\mathbb{G}$ (see [2] for details). Sections of these vector bundles are said respectively vector fields and differential forms.

Definition 2.5. If $0 \le h \le n$, $1 \le s \le \infty$ and $m \ge 0$, we denote by $W^{m,s}_{\mathbb{G}}(\mathbb{G}, \bigwedge^h \mathfrak{g})$ the space of all sections of $\bigwedge^h \mathfrak{g}$ such that their components with respect to the basis Θ^h belong to $W^{m,s}_{\mathbb{G}}(\mathbb{G})$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself.

Sobolev spaces of vector fields are defined in the same way.

We conclude this section recalling the classical definition of Hodge duality: see [10] 1.7.8.

Definition 2.6. We define linear isomorphisms

$$* : \bigwedge_h \mathfrak{g} \longleftrightarrow \bigwedge_{n-h} \mathfrak{g} \quad \text{and} \quad * : \bigwedge^h \mathfrak{g} \longleftrightarrow \bigwedge^{n-h} \mathfrak{g},$$

for $1 \le h \le n$, putting, for $v = \sum_I v_I X_I$ and $\varphi = \sum_I \varphi_I \theta_I$,

$$*v := \sum\nolimits_I v_I (*X_I) \quad \text{and} \quad *\varphi := \sum\nolimits_I \varphi_I (*\theta_I)$$

where

$$*X_I := (-1)^{\sigma(I)} X_{I^*} \quad \text{and} \quad *\theta_I := (-1)^{\sigma(I)} \theta_{I^*}$$

with $I = \{i_1, \cdots, i_h\}$, $1 \le i_1 < \cdots < i_h \le n$, $X_I = X_{i_1} \wedge \cdots \wedge X_{i_h}$, $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$, $I^* = \{i^*_1 < \cdots < i^*_{n-h}\} = \{1, \cdots, n\} \setminus I$ and $\sigma(I)$ is the number of couples (i_h, i^*_ℓ) with $i_h > i^*_\ell$.

The following properties of the $*$ operator follow readily from the definition: $\forall v, w \in \bigwedge_h \mathfrak{g}$ and $\forall \varphi, \psi \in \bigwedge^h \mathfrak{g}$

$$\begin{aligned} &**v = (-1)^{h(n-h)} v, \qquad **\varphi = (-1)^{h(n-h)} \varphi, \\ &v \wedge *w = \langle v, w \rangle X_{\{1,\cdots,n\}}, \qquad \varphi \wedge *\psi = \langle \varphi, \psi \rangle \theta_{\{1,\cdots,n\}}, \\ &\langle *\varphi | *v \rangle = \langle \varphi | v \rangle. \end{aligned} \tag{4}$$

We refer to $dV := \theta_{\{1,\cdots,n\}}$ as to the canonical volume form of $\mathbb{G}$.

If $v \in \bigwedge_h \mathfrak{g}$ we define $v^\natural \in \bigwedge^h \mathfrak{g}$ by the identity $\langle v^\natural | w \rangle := \langle v, w \rangle$, for all $w \in \bigwedge_h \mathfrak{g}$, and analogously we define $\varphi^\natural \in \bigwedge_h \mathfrak{g}$ for $\varphi \in \bigwedge^h \mathfrak{g}$.

3. Differential forms in Carnot groups

The notion of intrinsic form in Carnot groups is due to M. Rumin ([24], [22]). A more extended presentation of the results of this section can be found in [2], [15].

The notion of weight of a differential form plays a key role.

Definition 3.1. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has *pure weight* p, and we write $w(\alpha) = p$, if $\alpha^\natural \in V_p$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight p if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = p$.

In particular, the canonical volume form dV has weight Q (the homogeneous dimension of the group).

Remark 3.2. If $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$. Indeed, it is enough to notice that, if $w(\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}) \neq w(\theta_{j_1} \wedge \cdots \wedge \theta_{j_h})$, with $i_1 < i_2 < \cdots < i_h$ and $j_1 < j_2 < \cdots < j_h$, then for at least one of the indices $\ell = 1, \ldots, h$, $i_\ell \neq j_\ell$, and hence $\langle \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}, \theta_{j_1} \wedge \cdots \wedge \theta_{j_h} \rangle = 0$.

We have ([2], formula (16))

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g}, \tag{5}$$

where $\bigwedge^{h,p}\mathfrak{g}$ is the linear span of the h–covectors of weight p and $M_h^{\min}$, $M_h^{\max}$ are respectively the smallest and the largest weight of left-invariant h-covectors.

Since the elements of the basis Θ^h have pure weights, a basis of $\bigwedge^{h,p}\mathfrak{g}$ is given by $\Theta^{h,p} := \Theta^h \cap \bigwedge^{h,p}\mathfrak{g}$. In other words, the basis $\Theta^h = \cup_p \Theta^{h,p}$ is a basis adapted to the filtration of $\bigwedge^h \mathfrak{g}$ associated with (5).

We denote by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p}\mathfrak{g}$. We have

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{6}$$

The following crucial property of the weight follows from Cartan identintity: see [24], Section 2.1:

Lemma 3.3. *We have $d(\bigwedge^{h,p}\mathfrak{g}) \subset \bigwedge^{h+1,p}\mathfrak{g}$, i.e., if $\alpha \in \bigwedge^{h,p}\mathfrak{g}$ is a left invariant h-form of weight p with $d\alpha \neq 0$, then $w(d\alpha) = w(\alpha)$.*

Definition 3.4. Let now $\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, \theta_i^h \in \Omega^{h,p}$ be a (say) smooth form of pure weight p. Then we can write

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_\kappa\alpha,$$

where

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$$

does not increase the weight,

$$d_1\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{m_1} (X_j\alpha_i)\theta_j \wedge \theta_i^h$$

increases the weight of 1, and, more generally,

$$d_i\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{X_j \in V_i} (X_j\alpha_i)\theta_j \wedge \theta_i^h,$$

when $i = 0, 1, \ldots, \kappa$. In particular, d_0 is an algebraic operator.

Definition 3.5 (M. Rumin)**.** If $0 \le h \le n$ we set

$$E_0^h := \ker d_0 \cap \ker \delta_0 = \ker d_0 \cap (\mathrm{Im}\, d_0)^\perp \subset \Omega^h$$

In the sequel, we refer to the elements of E_0^h as to *intrinsic h-forms on* $\mathbb{G}$. Since the construction of E_0^h is left invariant, this space of forms can be seen as the space of sections of a fiber subbundle of $\bigwedge^h \mathfrak{g}$, generated by left translation and still denoted by E_0^h. In particular E_0^h inherits from $\bigwedge^h \mathfrak{g}$ the scalar product on the fibers.

Moreover, there exists a left invariant orthonormal basis $\Xi_0^h = \{\xi_j\}$ of E_0^h that is adapted to the filtration (5).

Since it is easy to see that $E_0^1 = \text{span}\,\{\theta_1, \dots, \theta_m\}$, where the θ_i's are dual of the elements of the basis of V_1, without loss of generality, we can take $\xi_j = \theta_j$ for $j = 1, \dots, m$.

Finally, we denote by $N_h^{\min}$ and $N_h^{\max}$ respectively the lowest and highest weight of forms in E_0^h.

Definition 3.6. If $0 \le h \le n$, $1 \le s \le \infty$ and $m \ge 0$, we denote by $W_{\mathbb{G}}^{m,s}(\mathbb{G}, E_0^h)$ the space of all sections of E_0^h such that their components with respect to the basis Ξ_0^h belong to $W_{\mathbb{G}}^{m,s}(\mathbb{G})$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself.

Moreover, as in Proposition 2.4, $\mathcal{D}(\mathbb{G}, E_0^h)$ and $\mathcal{S}(\mathbb{G}, E_0^h)$ are dense in $W_{\mathbb{G}}^{m,s}(\mathbb{G})$.

Lemma 3.7 ([2], Lemma 2.11)**.** *If $\beta \in \bigwedge^{h+1} \mathfrak{g}$, then there exists a unique $\alpha \in \bigwedge^h \mathfrak{g} \cap (\ker d_0)^\perp$ such that*

$$d_0^* d_0 \alpha = d_0^* \beta. \quad \textit{We set} \quad \alpha := d_0^{-1}\beta.$$

Here $d_0^* : \bigwedge^{h+1} \mathfrak{g} \to \bigwedge^h \mathfrak{g}$ is the adjoint of d_0 with respect to our fixed scalar product. In particular

$$\alpha = d_0^{-1}\beta \quad \textit{if and only if} \quad d_0\alpha - \beta \in \mathcal{R}(d_0)^\perp.$$

In particular

i) $(\ker d_0)^\perp = \mathcal{R}(d_0^{-1})$*;*
ii) $d_0^{-1} d_0 = Id$ *on* $(\ker d_0)^\perp$*;*
iii) $d_0 d_0^{-1} - Id : \bigwedge^{h+1} \mathfrak{g} \to \mathcal{R}(d_0)^\perp$.

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [24] and [2], Section 2) .

Theorem 3.8. *The de Rham complex (Ω^*, d) splits in the direct sum of two sub-complexes (E^*, d) and (F^*, d), with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1} d) \quad \textit{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(d d_0^{-1}).$$

We have

i) *Let Π_E be the projection on E along F (that is not an orthogonal projection). Then for any $\alpha \in E_0^{h,p}$, if we denote by $(\Pi_E\alpha)_j$ the component of $\Pi_E\alpha$ of weight j, then*

$$\begin{aligned} &(\Pi_E\alpha)_p = \alpha \\ &(\Pi_E\alpha)_{p+k+1} = -d_0^{-1}\Big(\sum_{1\le \ell \le k+1} d_\ell (\Pi_E\alpha)_{p+k+1-\ell}\Big). \end{aligned} \tag{7}$$

Notice that $\alpha \to (\Pi_E\alpha)_{p+k+1}$ is an homogeneous differential operator of order $k+1$ in the horizontal derivatives.

ii) *Π_E is a chain map, i.e.*

$$d\Pi_E = \Pi_E d.$$

iii) *Let* Π_{E_0} *be the orthogonal projection from* Ω^* *on* E_0^**, then*

$$\Pi_{E_0} = Id - d_0^{-1} d_0 - d_0 d_0^{-1}, \quad \Pi_{E_0^\perp} = d_0^{-1} d_0 + d_0 d_0^{-1}. \tag{8}$$

Notice that, since d_0 *and* d_0^{-1} *are algebraic, then formulas* (8) *hold also for covectors.*

iv) $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ *and* $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$.

Set now

$$d_c = \Pi_{E_0}\, d\, \Pi_E : E_0^h \to E_0^{h+1}, \quad h = 0, \dots, n-1.$$

We have:

v) $d_c^2 = 0$*;*
vi) *the complex* $E_0 := (E_0^*, d_c)$ *is exact;*
vii) *with respect to the bases* Ξ_0^* *the intrinsic differential* d_c *can be seen as a matrix-valued operator such that, if* α *has weight* p*, then the component of weight* q *of* $d_c\alpha$ *is given by an homogeneous differential operator in the horizontal derivatives of order* $q - p \geq 1$*, acting on the components of* α*.*

Remark 3.9. Let us give a gist of the construction of E. The map $d_0^{-1}d$ induces an isomorphism from $\mathcal{R}(d_0^{-1})$ to itself. Thus, since $d_0^{-1}d_0 = Id$ on $\mathcal{R}(d_0^{-1})$, we can write $d_0^{-1}d = Id + D$, where D is a differential operator that increases the weight. Clearly, $D : \mathcal{R}(d_0^{-1}) \to \mathcal{R}(d_0^{-1})$. As a consequence of the nilpotency of $\mathbb{G}$, $D^k = 0$ for k large enough, and therefore the Neumann series of $d_0^{-1}d$ reduces to a finite sum on $\mathcal{R}(d_0^{-1})$. Hence there exist a differential operator

$$P = \sum_{k=1}^{N} (-1)^k D^k, \quad N \in \mathbb{N} \text{ suitable},$$

such that

$$P d_0^{-1} d = d_0^{-1} d P = \mathrm{Id}_{\mathcal{R}(d_0^{-1})}.$$

We set $Q := P d_0^{-1}$. Then Π_E is given by

$$\Pi_E = Id - Qd - dQ.$$

From now on, we restrict ourselves to assume $\mathbb{G}$ is a free group of step κ (see Definition 2.1 above). The technical reason for this choice relies in the following property.

Theorem 3.10 ([15], Theorem 5.9)**.** *Let* $\mathbb{G}$ *be a free group of step* κ*. Then all forms in* E_0^1 *have weight* 1 *and all forms in* E_0^2 *have weight* $\kappa + 1$*.*

In particular, the differential $d_c : E_0^1 \to E_0^2$ *can be identified, with respect to the adapted bases* Ξ_0^1 *and* Ξ_0^2*, with a homogeneous matrix-valued differential operator of degree* κ *in the horizontal derivatives.*

Moreover, if $\xi \in \bigwedge^{2,p}\mathfrak{g}$ *with* $p \neq \kappa + 1$*, then* $\Pi_{E_0}\xi = 0$*. Indeed,* $\Pi_{E_0}\xi$ *has weight* p*, and therefore has to be zero, since* $\Pi_{E_0}\xi \in \bigwedge^{2,\kappa+1}\mathfrak{g}$*.*

Remark 3.11. Theorem 3.10 might suggest that in free groups all forms in E_0^* have pure weight. Unfortunately, this assertion fails to hold, at least in this naïf form. Indeed, A. Ottazzi showed us a counterexample for E_0^3 in the free group of step 2 with 3 generators. Actually, this is a general phenomenon, due to the fact that, denoting as usual by Q the homogeneous dimension of the group, in this case $n = 6$ (even), so that $E_0^3 = *E_0^3$, but $Q = 9$ (odd), yielding a contradiction with $w(*\alpha) = w(\alpha)$ when $\alpha \in E_0^3$, since $w(*\alpha) = Q - w(\alpha)$. Clearly, this situation occurs whenever n is even and Q is odd.

Lemma 3.12. *If $\mathbb{G}$ is a free group of step $\kappa \geq 2$, then*

$$\bigwedge^{2,2}\mathfrak{g} \subset d_0(\bigwedge^{1,2}\mathfrak{g}) \subset \mathcal{R}(d_0),$$

or, equivalently,

$$\Omega^{2,2} \subset d_0(\Omega^{1,2}) \subset d_0(\Omega^1).$$

Proof. Since a basis of $\bigwedge^{2,2}\mathfrak{g}$ is given by covectors θ of the form $\theta = \theta_i \wedge \theta_j$, with $\theta_i = X_i^\natural$, $\theta_j = X_j^\natural$, $X_i, X_j \in V_1$, $i < j$, we need only to prove that

$$d_0([X_i, X_j]^\natural) = -\theta_i \wedge \theta_j.$$

Thus, if $X, Y \in \mathfrak{g}$, we are left to show that

$$\langle d_0([X_i, X_j]^\natural) | X \wedge Y\rangle = -\langle \theta_i \wedge \theta_j | X \wedge Y\rangle.$$

Since d_0 preserves the weights, we may assume that $X \wedge Y$ has weight 2. Therefore, without loss of generality, we can take $X = X_k, Y = X_h$, with $X_k, X_h \in V_1$. Therefore

$$\begin{aligned}&\langle d_0([X_i, X_j]^\natural) | X_k \wedge X_h\rangle = \langle d([X_i, X_j]^\natural) | X_k \wedge X_h\rangle \\ &= -\langle [X_i, X_j]^\natural | [X_k, X_h]\rangle = -\langle [X_i, X_j], [X_k, X_h]\rangle.\end{aligned}$$

On the other hand, as pointed out in Definition 2.1,

- $\langle [X_i, X_j], [X_k, X_h]\rangle = 0$ if $\{i, j\} \neq \{k, h\}$,
- $\langle [X_i, X_j], [X_k, X_h]\rangle = 1$ if $(i, j) = (k, h)$, and
- $\langle [X_i, X_j], [X_k, X_h]\rangle = -1$ if $(i, j) = (h, k)$,

whereas

$$\langle \theta_i \wedge \theta_j | X_k \wedge X_h\rangle = \det \begin{pmatrix} \langle \theta_i | X_k\rangle & \langle \theta_i | X_h\rangle \\ \langle \theta_j | X_k\rangle & \langle \theta_j | X_h\rangle \end{pmatrix}.$$

This achieves the proof of the lemma. □

4. Γ-convergence

Definition 4.1. Let X be a separated topological space, and let

$$F_\varepsilon,\ F:\ X \longrightarrow [-\infty, +\infty]$$

with $\varepsilon > 0$ be functionals on X. We say that $\{F_\varepsilon\}_{\varepsilon>0}$ sequentially Γ-converges to F on X as ε goes to zero if the following two conditions hold:

1) for every $u \in X$ and for every sequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$, which converges to u in X, there holds

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) \geq F(u); \tag{9}$$

2) for every $u \in X$ and for every sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$ there exists a subsequence (still denoted by $\{\varepsilon_k\}_{k\in\mathbb{N}}$) such that $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ converges to u in X and

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k}) \leq F(u) \tag{10}$$

For a deep and detailed survey on Γ-convergence, we refer to the monograph [8].

We recall the following reduction Lemma. The proof is only a minor variant of the one given in [18], Lemma IV (see also [1]), hence we shall omit such a proof.

Lemma 4.2. *Let X be a separated topological space, let F_h, $F : M \longrightarrow [-\infty, +\infty]$ with $h \in \mathbb{N}$; consider $D \subset M$ and $x \in M$. Let us suppose that*

1) *for every $y \in D$ there exists a sequence $(y_h)_{h\in\mathbb{N}} \subset M$ such that $y_h \to y$ in M and $\limsup_{h\to\infty} F_h(y_h) \leq F(y)$;*
2) *there exists a sequence $(x_h)_{h\in\mathbb{N}} \subset D$ such that $x_h \to x$ and $\limsup_{h\to\infty} F(x_h) \leq F(x)$;*

then there exists a sequence $(\overline{x}_h)_{h\in\mathbb{N}} \subset M$ such that $\limsup_{h\to\infty} F_h(\overline{x}_h) \leq F(x)$.

To avoid cumbersome notations, from now on we write systematically $\lim_{\varepsilon\to} 0$ to mean a limit with $\varepsilon = \varepsilon_k$, where $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is any sequence with $\varepsilon_k \to 0$ as $k \to \infty$.

5. Intrinsic differential as a Γ-limit

Let $\varepsilon > 0$ be given. If $\omega \in W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$, we set

$$F_\varepsilon(\omega) = \frac{1}{\varepsilon^{2\kappa}} \int_{\mathbb{G}} |d_\varepsilon \omega|^2 \, dV,$$

where

$$d_\varepsilon = d_0 + \varepsilon d_1 + \cdot + \varepsilon^\kappa d_\kappa.$$

We stress that $F_\varepsilon(\omega)$ is always finite, since the coefficients of $d_i\omega$ contain horizontal derivatives of order $i \leq \kappa$ of the coefficients of ω.

Theorem 5.1. *Let $\mathbb{G}$ be a free Carnot group of step κ. Then*

F_ε sequentially Γ-coverges to F in the weak topology $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$,

as $\varepsilon \to 0$, where

$$F(\omega) = \begin{cases} \displaystyle\int_{\mathbb{G}} |d_c\omega|^2 \, dV & \text{if } \omega \in W^{\kappa,2}(\mathbb{G}, E_0^1) \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Let $\omega^\varepsilon \to \omega$ as $\varepsilon \to 0$ weakly in $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$. We want to show that

$$F(\omega) \le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega^\varepsilon). \tag{11}$$

In particular, it follows that $\omega \in W^{\kappa,2}(\mathbb{G}, E_0^1)$ provided $\liminf_{\varepsilon\to 0} F_\varepsilon(\omega^\varepsilon) < \infty$.

Keeping in mind (6), we write

$$\omega^\varepsilon = \omega_1^\varepsilon + \cdots + \omega_\kappa^\varepsilon,$$

with $\omega_i^\varepsilon \in \Omega^{1,i}$, $i = 1, \dots, \kappa$. Reordering the terms of $d_\varepsilon \omega_\varepsilon$ according to their weights, as in (6), we have the following orthogonal decomposition:

$$\begin{aligned} d_\varepsilon \omega^\varepsilon = & \big(d_0\omega_2^\varepsilon + \varepsilon d_1 \omega_1^\varepsilon\big) \\ & + \big(d_0\omega_3^\varepsilon + \varepsilon d_1 \omega_2^\varepsilon + \varepsilon^2 d_2\omega_1^\varepsilon\big) \\ & + \cdots \\ & + \big(d_0\omega_\kappa^\varepsilon + \varepsilon d_1 \omega_{\kappa-1}^\varepsilon + \cdots \varepsilon^{\kappa-1} d_{\kappa-1}\omega_1^\varepsilon\big) \\ & + \big(\varepsilon d_1\omega_\kappa^\varepsilon + \cdots + \varepsilon^\kappa d_\kappa \omega_1^\varepsilon\big) \\ & + \cdots \\ & + \big(\varepsilon^{\kappa-1} d_{\kappa-1}\omega_\kappa^\varepsilon + \varepsilon^\kappa d_\kappa \omega_{\kappa-1}^\varepsilon\big) \\ & + \varepsilon^\kappa d_\kappa \omega_\kappa^\varepsilon \\ = & \sum_{2\le p\le\kappa} \sum_{i=0}^{p-1} \varepsilon^i d_i \omega_{p-i}^\varepsilon \\ & + \big(\varepsilon d_1\omega_\kappa^\varepsilon + \cdots + \varepsilon^\kappa d_\kappa \omega_1^\varepsilon\big) \\ & + \sum_{\kappa+2\le p\le 2\kappa} \sum_{i=p-\kappa}^{\kappa} \varepsilon^i d_i \omega_{p-i}^\varepsilon. \end{aligned} \tag{12}$$

Therefore we can write

$$\begin{aligned} F_\varepsilon(\omega_\varepsilon) = & \ \varepsilon^{-2\kappa} \sum_{2\le p\le\kappa} \int_{\mathbb{G}} \Big\| \sum_{i=0}^{p-1} \varepsilon^i d_i \omega_{p-i}^\varepsilon \Big\|^2 dV \\ & + \varepsilon^{2(1-\kappa)} \int_{\mathbb{G}} \big\| d_1\omega_\kappa^\varepsilon + \cdots + \varepsilon^{\kappa-1} d_\kappa \omega_1^\varepsilon \big\|^2 dV \\ & + \sum_{\kappa+2\le p\le 2\kappa} \varepsilon^{2(p-2\kappa)} \int_{\mathbb{G}} \Big\| \sum_{i=p-\kappa}^{\kappa} \varepsilon^{i-p+\kappa} d_i \omega_{p-i}^\varepsilon \Big\|^2 dV. \end{aligned} \tag{13}$$

Without loss of generality, we may assume $\liminf_{\varepsilon\to 0} F_\varepsilon(\omega_\varepsilon) < \infty$. This implies that, if $2 \le p \le \kappa$, then, if $\varepsilon \in (0,1)$,

$$\varepsilon^{-\kappa} \sum_{i=0}^{p-1} \varepsilon^i d_i \omega_{p-i}^\varepsilon \quad \text{is uniformly bounded in } L^2(\mathbb{G}, \textstyle\bigwedge^2 \mathfrak{g})). \tag{14}$$

Moreover, again if $\varepsilon \in (0,1)$,

$$\varepsilon^{1-\kappa}\big(d_1\omega_\kappa^\varepsilon + \cdots + \varepsilon^{\kappa-1} d_\kappa \omega_1^\varepsilon\big) \quad \text{is uniformly bounded in } L^2(\mathbb{G}, \textstyle\bigwedge^2 \mathfrak{g})). \tag{15}$$

In particular,

$$\sum_{i=0}^{p-1} \varepsilon^i d_i \omega^\varepsilon_{p-i} \longrightarrow 0 \quad \text{in } L^2(\mathbb{G}, \bigwedge^2 \mathfrak{g})) \tag{16}$$

as $\varepsilon \to 0$, since we can write (16) as

$$d_0\omega^\varepsilon_p + \varepsilon \sum_{i=1}^{p-1} \varepsilon^{i-1} d_i \omega^\varepsilon_{p-i} \longrightarrow 0 \tag{17}$$

as $\varepsilon \to 0$. By assumption, we know that $\omega^\varepsilon_p \to \omega_p$ weakly in $L^2(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ for $p \geq 1$, and therefore

$$d_0\omega^\varepsilon_p \to d_0\omega_p \quad \text{in } L^2(\mathbb{G}, \bigwedge^1 \mathfrak{g}), \tag{18}$$

since d_0 is algebraic.

Combining (17) with the boundedness of $\{\omega^\varepsilon\}$ in $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ and with (18), it follows that

$$d_0\omega_p = 0 \quad \text{for } p = 2, \dots, \kappa \tag{19}$$

(obviously, (19) holds also for $p = 1$ since $d_0(\bigwedge^{1,1} \mathfrak{g}) = \{0\}$). Hence $\omega \in \ker d_0 = E^1_0$, and therefore $\omega = \omega_1$.

Recall now that, by definition, $d_c\omega = \Pi_{E_0} d \Pi_E \omega$. But, by Theorem 3.10, Π_{E_0} vanishes on all 2-forms of weight $p \neq \kappa+1$. Therefore, the full expression of $d_c\omega$ reduces to

$$d_c(\omega) = \Pi_{E_0}\Big(\sum_{\ell=1}^{\kappa} d_\ell (\Pi_E\omega)_{\kappa+1-\ell}\Big). \tag{20}$$

Let us show now that

$$d_j(\Pi_E\omega)_{\kappa+1-\ell} = \lim_{\varepsilon\to 0} \varepsilon^{\ell-\kappa} d_j \omega^\varepsilon_{\kappa+1-\ell}, \tag{21}$$

in the sense of distributions for $\ell = 1, \dots, \kappa$, $j = 0, \dots, \ell$, i.e.

$$\int \langle \varepsilon^{\ell-\kappa} d_j \omega^\varepsilon_{\kappa+1-\ell}, \varphi\rangle \, dV \to \int \langle d_j(\Pi_E\omega)_{\kappa+1-\ell}, \varphi\rangle \, dV \tag{22}$$

for $\ell = 1, \dots, \kappa$, $j = 0, \dots, \ell$, and for any $\varphi \in \mathcal{D}(\mathbb{G}, \bigwedge^2 \mathfrak{g})$. Notice $(\Pi_E\omega)_{\kappa+1-\ell} \in W^{\ell,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$, by Theorem 3.8 i), so that (21) follows straightforwardly if we prove that

$$\varepsilon^{\ell-\kappa} \omega^\varepsilon_{\kappa+1-\ell} \overset{\varepsilon\to 0}{\longrightarrow} (\Pi_E\omega)_{\kappa+1-\ell} \tag{23}$$

in the sense of distributions for $\ell = 1, \dots, \kappa$, $j = 0, \dots, \ell$. Indeed, denoting by δ_j the formal adjoint of d_j in $L^2(\mathbb{G}, \bigwedge^* \mathfrak{g})$,

$$\begin{aligned} &\int \langle \varepsilon^{\ell-\kappa} d_j \omega^\varepsilon_{\kappa+1-\ell}, \varphi\rangle \, dV = \int \langle \varepsilon^{\ell-\kappa} \omega^\varepsilon_{\kappa+1-\ell}, \delta_j\varphi\rangle \, dV \\ &\overset{\varepsilon\to 0}{\longrightarrow} -\int \langle (\Pi_E\omega)_{\kappa+1-\ell}, \delta_j\varphi\rangle \, dV = -\int \langle d_j(\Pi_E\omega)_{\kappa+1-\ell}, \varphi\rangle \, dV. \end{aligned} \tag{24}$$

In order to prove (23), we set $p := \kappa+1-\ell$, $p = 1, \dots, \kappa$. Thus, formulae (23) and (21) become

$$\varepsilon^{1-p} \omega^\varepsilon_p \overset{\varepsilon\to 0}{\longrightarrow} (\Pi_E\omega)_p \tag{25}$$

in the sense of distributions, and

$$\varepsilon^{1-p} d_j \omega_p^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} d_j(\Pi_E\omega)_p \tag{26}$$

in the sense of distributions, respectively.

To prove (25) and (26), we argue by iteration in p.

Step 1 ($\mathbf{p=1}$): the proof is trivial, since, by assumption,

$$\omega_1^\varepsilon \to \omega_1 := (\Pi_E\omega)_1 \tag{27}$$

weakly in $L^2(\mathbb{G}, \bigwedge^2 \mathfrak{g})$ as $\varepsilon \to 0$.

Step 2 ($\mathbf{p=2}$): by (14), we have

$$d_0\omega_2^\varepsilon + \varepsilon d_1\omega_1^\varepsilon = O(\varepsilon^\kappa) \quad \text{in } L^2(\mathbb{G}, \textstyle\bigwedge^2 \mathfrak{g}). \tag{28}$$

Remember now that, by Lemma 3.7, ii), $d_0^{-1}d_0\omega_2^\varepsilon = \omega_2^\varepsilon$, since ω_2^ε has weight 2 and hence is orthogonal to $\Omega^{1,1} = \ker d_0$. Thus, keeping in mind that d_0^{-1} is algebraic, it follows from (28) that

$$\frac{1}{\varepsilon}\omega_2^\varepsilon + d_0^{-1}d_1\omega_1^\varepsilon = O(\varepsilon^{\kappa-1}) \quad \text{in } L^2(\mathbb{G}, \textstyle\bigwedge^2 \mathfrak{g}) \tag{29}$$

Moreover, since d_1 is an homogeneous differential operator in the horizontal derivatives of order $1 \le \kappa$ and, again, d_0^{-1} is algebraic, then

$$d_0^{-1}d_1\omega_1^\varepsilon \to d_0^{-1}d_1\omega_1 \quad \text{weakly in } L^2(\mathbb{G}, \textstyle\bigwedge^2 \mathfrak{g}). \tag{30}$$

Combining (29) and (30), we obtain

$$\varepsilon^{-1}\omega_2^\varepsilon \to -d_0^{-1}d_1\omega_1 = (\Pi_E\omega)_2, \tag{31}$$

weakly in $L^2(\mathbb{G}, \bigwedge^2 \mathfrak{g})$, as $\varepsilon \to 0$. This proves (25) for $p = 2$.

Step 3: suppose (25) and (26) have been proved for $p = 1, \dots, q-1 < \kappa$, with $q > 1$. Let us prove the assertion for $p = q$. We argue as follows: using (14) with $p = q$, we get

$$\begin{aligned}\frac{1}{\varepsilon^{q-1}}d_0\omega_q^\varepsilon + &\frac{1}{\varepsilon^{q-2}}d_1\omega_{q-1}^\varepsilon + \dots + \frac{1}{\varepsilon}d_{q-2}\omega_2^\varepsilon + d_{q-1}\omega_1^\varepsilon \\ &= O(\varepsilon^{\kappa-q+1}) = o(1)\end{aligned} \tag{32}$$

as $\varepsilon \to 0$. Remember now that, by Lemma 3.7, ii), $d_0^{-1}d_0\omega_q^\varepsilon = \omega_q^\varepsilon$, since ω_q^ε has weight q and hence is orthogonal to $\Omega^{1,1} = \ker d_0$. Thus, keeping in mind that d_0^{-1} is algebraic, it follows from (32) that

$$\begin{aligned}\frac{1}{\varepsilon^{q-1}}\omega_q^\varepsilon + &\frac{1}{\varepsilon^{q-2}}d_0^{-1}d_1\omega_{q-1}^\varepsilon + \dots + \frac{1}{\varepsilon}d_0^{-1}d_{q-2}\omega_2^\varepsilon + d_0^{-1}d_{q-1}\omega_1^\varepsilon \\ &= o(1)\end{aligned} \tag{33}$$

as $\varepsilon \to 0$. Hence, by the inductive hypothesis and keeping in mind (7),

$$\begin{aligned}\frac{1}{\varepsilon^{q-1}}\omega_q^\varepsilon \longrightarrow &-d_0^{-1}\Big(d_1(\Pi_E\omega)_{q-1} + \dots + d_{q-2}(\Pi_E\omega)_2 + d_{q-1}(\Pi_E\omega)_1\Big) \\ &= (\Pi_E\omega)_q\end{aligned} \tag{34}$$

in the sense of distribution, as $\varepsilon \to 0$.

This achieves the proof of (23) and hence of (21).

So far, we have used the equiboundedness of the first sum in (13) for ε close to zero. We proceed now to estimate the $\liminf$ of the second term in (13).

To this end, we take $j = \ell$ in (21)and we sum up for $\ell = 1, \dots, \kappa$. We obtain

$$\frac{1}{\varepsilon^{\kappa-1}}\Big(d_1\omega_\kappa^\varepsilon + \cdots + \varepsilon^{\kappa-1} d_\kappa \omega_1^\varepsilon\Big) \longrightarrow \sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell} \tag{35}$$

as $\varepsilon \to 0$ in the sense of distributions. On the other hand, the limit $\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell}$ belongs to $L^2(\mathbb{G}, \bigwedge^2 \mathfrak{g})$ (since $d_\ell(\Pi_E\omega)_{\kappa+1-\ell}$ is an homogeneous differential operator in the horizontal derivatives of order κ, by Theorem 3.8, i) and Definition 3.4), and

$$\Big\{\frac{1}{\varepsilon^{\kappa-1}}\big(d_1\omega_\kappa^\varepsilon + \cdots + \varepsilon^{\kappa-1} d_\kappa \omega_1^\varepsilon\big)\Big\}_{\varepsilon>0} \quad \text{is equibounded in } L^2(\mathbb{G}, \textstyle\bigwedge^2 \mathfrak{g}), \tag{36}$$

as $\varepsilon \to 0$, by (15). Combining (36) and (35) we obtain that the limit in (35) is in fact a weak limit in $L^2(\mathbb{G}, \bigwedge^2 \mathfrak{g})$. Thus, by (20), (13) and taking into account that Π_{E_0} is an othogonal projection, we obtain eventually

$$\begin{aligned} F(\omega) &= \int_{\mathbb{G}} \|\Pi_{E_0}\big(\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell}\big)\|^2\, dV \\ &\le \int_{\mathbb{G}} \|\sum_{\ell=1}^{\kappa} d_\ell(\Pi_E\omega)_{\kappa+1-\ell}\|^2\, dV \\ &\le \liminf_{\varepsilon\to 0} \varepsilon^{2(1-\kappa)} \int_{\mathbb{G}} \big\|d_1\omega_\kappa^\varepsilon + \cdots + \varepsilon^{\kappa-1} d_\kappa\omega_1^\varepsilon\big\|^2\, dV \le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega_\varepsilon). \end{aligned}$$

This proves (11).

We prove now that, if $\omega \in W^{\kappa,2}(\mathbb{G}, E_0^1)$, then there exists a sequence $(\omega_\varepsilon)_{\varepsilon>0}$ in $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$ such that

i) $\omega_\varepsilon \to \omega$ weakly in $W^{\kappa,2}(\mathbb{G}, \bigwedge^1 \mathfrak{g})$;
ii) $F_\varepsilon(\omega_\varepsilon) \to F(\omega)$ as $\varepsilon \to 0$.

By Lemma 4.2, without loss of generality we may assume $\omega \in \mathcal{D}(\mathbb{G}, E_0^1)$.

We choose

$$\omega_\varepsilon = \omega + \varepsilon(\Pi_E\omega)_2 + \cdots + \varepsilon^{\kappa-1}(\Pi_E\omega)_\kappa \tag{37}$$

I we write the identity $d^2 = 0$ gathering all terms of the same weight, we get

$$0 = \sum_{p=0}^{\kappa}\sum_{j=0}^{p} d_{p-j} d_j.$$

and therefore

$$\sum_{j=0}^{p} d_{p-j} d_j = 0 \quad \text{for } p = 0, \dots, \kappa, \tag{38}$$

since these terms are mutually orthogonal when applied to a form of pure weight. In particular,

$$d_0^2 = 0, \quad d_0 d_1 = -d_1 d_0, \quad d_0 d_2 = -d_2 d_0 - d_1^2, \quad \cdots \tag{39}$$

Thus,

$$
\begin{aligned}
F_\varepsilon(\omega_\varepsilon) &= \frac{1}{\varepsilon^{2\kappa}} \int_{\mathbb{G}} \|d_\varepsilon\big(\sum_{i=1}^{\kappa} \varepsilon^{i-1}(\Pi_E\omega)_i\big)\|^2 \, dV \\
&= \frac{1}{\varepsilon^{2\kappa}} \Bigg(\int_{\mathbb{G}} \|\Pi_{E_0}\big(d_\varepsilon\big(\sum_{i=1}^{\kappa} \varepsilon^{i-1}(\Pi_E\omega)_i\big)\big)\|^2 \, dV \\
&\qquad + \int_{\mathbb{G}} \|\Pi_{E_0^\perp}\big(d_\varepsilon\big(\sum_{i=1}^{\kappa} \varepsilon^{i-1}(\Pi_E\omega)_i\big)\big)\|^2 \, dV \Bigg).
\end{aligned}
\tag{40}
$$

Arguing as in (12), we can write

$$
\begin{aligned}
&d_\varepsilon\big(\sum_{i=1}^{\kappa} \varepsilon^{i-1}(\Pi_E\omega)_i\big) \\
&\quad = \sum_{2\le p\le\kappa} \varepsilon^{p-1} \sum_{i=0}^{p-1} d_i(\Pi_E\omega)_{p-i} \\
&\qquad + \varepsilon^{\kappa}\big(d_1(\Pi_E\omega)_\kappa + \cdots + d_\kappa(\Pi_E\omega)_1\big) \\
&\qquad + \sum_{\kappa+2\le p\le 2\kappa} \varepsilon^{p-1} \sum_{i=p-\kappa}^{\kappa} d_i(\Pi_E\omega)_{p-i} \\
&\quad := I_1 + I_2 + I_3.
\end{aligned}
$$

Now, by Theorem 3.10,

$$\Pi_{E_0} I_1 = 0.$$

Let us prove that

$$\Pi_{E_0}^\perp I_1 = \Pi_{E_0}^\perp I_2 = 0. \tag{41}$$

To this end, we have but to prove that

$$\sum_{i=0}^{p-1} d_i(\Pi_E\omega)_{p-i} \in E_0^2 \quad \text{for } 2 \le p \le \kappa+1. \tag{42}$$

Recalling that $E_0^2 = \ker d_0 \cap \mathcal{R}(d_0)^\perp$, we prove by induction that

i) $d_0\big(d_0(\Pi_E\omega)_p + d_1(\Pi_E\omega)_{p-1} + \cdots + d_{p-1}(\Pi_E\omega)_1\big) = 0$

ii) $d_0(\Pi_E\omega)_p + d_1(\Pi_E\omega)_{p-1} + \cdots + d_{p-1}(\Pi_E\omega)_1 \in \mathcal{R}(d_0)^\perp$,

for $2 \le p \le \kappa+1$. Obviously, by our choice of $\omega \in E_0^1$, (42) holds also for $p = 1$.

Step 1 ($\mathbf{p = 2}$): by Lemma 3.12, $d_0 d_0^{-1} d_1(\Pi_E\omega)_1 = d_1(\Pi_E\omega)_1$, since $d_1(\Pi_E\omega)_1 \in \Omega^{2,2}$. Hence we can write

$$d_0(\Pi_E\omega)_2 + d_1(\Pi_E\omega)_1 = d_0\big((\Pi_E\omega)_2 + d_0^{-1}d_1(\Pi_E\omega)_1\big) = 0,$$

by (7). Trivially, i) and ii) hold.

Step 2 (if $\mathbf{p}$ then $\mathbf{p+1}$): first of all, we notice that $p < \kappa+1$ (since $p+1 \le \kappa+1$). Hence, by (42) and by Theorem 3.10, we get in particular

$$d_0(\Pi_E\omega)_j + d_1(\Pi_E\omega)_{j-1} + \cdots + d_{j-1}(\Pi_E\omega)_1 = 0 \quad \text{for } j \le p < \kappa+1. \tag{43}$$

Now, using (39), we get

$$
\begin{aligned}
&d_0\big(d_0(\Pi_E\omega)_{p+1} + d_1(\Pi_E\omega)_p + \cdots + d_p(\Pi_E\omega)_1\big)\\
&\quad = -d_1 d_0(\Pi_E\omega)_p - \big(d_2 d_0(\Pi_E\omega)_{p-1} + d_1^2(\Pi_E\omega)_{p-1}\big)\\
&\quad - \cdots - (d_p d_0(\Pi_E\omega)_1 + d_{p-1} d_1(\Pi_E\omega)_1 + \cdots + d_1 d_{p-1}(\Pi_E\omega)_1)\\
&\quad = -\sum_{i=1}^{p-1} d_i \sum_{j=0}^{p-i} d_j(\Pi_E\omega)_{p+1-i-j} = 0
\end{aligned}
$$

by (43), and the assertion i) holds for $p+1$.

In order to prove ii), we remind that, by (7),

$$
(\Pi_E\omega)_p = -d_0^{-1}\big(d_1(\Pi_E\omega)_{p-1} + \cdots + d_{p-1}(\Pi_E\omega)_1\big),
$$

so that, by Lemma 3.7, iii),

$$
d_0(\Pi_E\omega)_p = -\big(d_1(\Pi_E\omega)_{p-1} + \cdots + d_{p-1}(\Pi_E\omega)_1\big) + \xi,
$$

with $\xi \in \mathcal{R}(d_0)^\perp$. This proves ii) and eventually (42).

Coming back to (40) we get,

$$
\begin{aligned}
F_\varepsilon(\omega_\varepsilon) &= \frac{1}{\varepsilon^{2\kappa}} \int_{\mathbb{G}} \|\Pi_{E_0} I_2\|^2\, dV + \frac{1}{\varepsilon^{2\kappa}} \int_{\mathbb{G}} \|\ I_3\|^2\, dV\\
&= \int_{\mathbb{G}} \|\Pi_{E_0}\big(d_1(\Pi_E\omega)_\kappa + \cdots + d_\kappa(\Pi_E\omega)_1\big)\|^2\, dV\\
&+ \frac{1}{\varepsilon^{2\kappa}} \int_{\mathbb{G}} \| \sum_{\kappa+2\le p\le 2\kappa} \varepsilon^{p-1} \sum_{i=p-\kappa}^{\kappa} d_i(\Pi_E\omega)_{p-i}\|^2\, dV;
\end{aligned}
$$

observing that the second term in previous expression goes to zero as $\varepsilon \to 0$, we get $\lim_{\varepsilon\to 0} F_\varepsilon(\omega_\varepsilon) = F(\omega)$ in $\mathcal{D}(\mathbb{G}, E_0^1)$. This achieves the proof of the theorem.

$\square$

References

[1] G. Alberti. Variational models for phase transitions, an approach via Γ-convergence. In *Calculus of variations and partial differential equations (Pisa, 1996)*, pages 95–114. Springer, Berlin, 2000.

[2] Annalisa Baldi, Bruno Franchi, Nicoletta Tchou, and Maria Carla Tesi. Compensated compactness for differential forms in Carnot groups and applications. *Adv. Math.*, 223(5):1555–1607, 2010.

[3] Annalisa Baldi, Bruno Franchi, and Maria Carla Tesi. Compensated compactness in the contact complex of Heisenberg groups. *Indiana Univ. Math. J.*, 57:133–186, 2008.

[4] Annalisa Baldi, Bruno Franchi, and Maria Carla Tesi. Differential forms, Maxwell equations and compensated compactness in Carnot groups. *Lecture Notes of Seminario Interdisciplinare di Matematica*, 7:21–40, 2008.

[5] Andrea Bonfiglioli, Ermanno Lanconelli, and Francesco Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[6] Nicolas Bourbaki. *Éléments de mathématique. XXVI. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie.* Actualités Sci. Ind. No. 1285. Hermann, Paris, 1960.

[7] M. Camar-Eddine and P. Seppecher. Determination of the closure of the set of elasticity functionals. *Arch. Ration. Mech. Anal.*, 170(3):211–245, 2003.

[8] Gianni Dal Maso. *An introduction to* Γ*-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.

[9] Ennio De Giorgi and Tullio Franzoni. Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 58(6):842–850, 1975.

[10] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[11] Gerald B. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, 13(2):161–207, 1975.

[12] Gerald B. Folland and Elias M. Stein. *Hardy spaces on homogeneous groups*, volume 28 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1982.

[13] Bruno Franchi, Nicoletta Tchou, and Maria Carla Tesi. Div-curl type theorem, H-convergence and Stokes formula in the Heisenberg group. *Commun. Contemp. Math.*, 8(1):67–99, 2006.

[14] Bruno Franchi and Maria Carla Tesi. Faraday's form and Maxwell's equations in the Heisenberg group. *Milan J. Math.*, 77:245–270, 2009.

[15] Bruno Franchi and Maria Carla Tesi. Wave and Maxwell's Equations in Carnot Groups. preprint, 2010.

[16] Matthew Grayson and Robert Grossman. Models for free nilpotent Lie algebras. *J. Algebra*, 135(1):177–191, 1990.

[17] Mikhael Gromov. *Structures métriques pour les variétés riemanniennes*, volume 1 of *Textes Mathématiques [Mathematical Texts]*. CEDIC, Paris, 1981. Edited by J. Lafontaine and P. Pansu.

[18] Luciano Modica and Stefano Mortola. Un esempio di Γ^--convergenza. *Boll. Un. Mat. Ital. B (5)*, 14(1):285–299, 1977.

[19] Pierre Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.

[20] Linda Preiss Rothschild and Elias M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137(3-4):247–320, 1976.

[21] Michel Rumin. Formes différentielles sur les variétés de contact. *J. Differential Geom.*, 39(2):281–330, 1994.

[22] Michel Rumin. Differential geometry on C-C spaces and application to the Novikov-Shubin numbers of nilpotent Lie groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(11):985–990, 1999.

[23] Michel Rumin. Sub-Riemannian limit of the differential form spectrum of contact manifolds. *Geom. Funct. Anal.*, 10(2):407–452, 2000.

[24] Michel Rumin. Around heat decay on forms and relations of nilpotent Lie groups. In *Séminaire de Théorie Spectrale et Géométrie, Vol. 19, Année 2000–2001*, volume 19 of *Sémin. Théor. Spectr. Géom.*, pages 123–164. Univ. Grenoble I, Saint, 2001.

[25] Günter Schwarz. *Hodge decomposition—a method for solving boundary value problems*, volume 1607 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[26] Stephen Semmes. On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights. *Rev. Mat. Iberoamericana*, 12(2):337–410, 1996.

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