The regularity of the inverses of Sobolev homeomorphisms with finite distortion

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ABSTRACT. Let Ω and Ω' be bounded open sets in $\mathbb{R}^n$, $n \geq 2$, and let $\mathrm{Hom}(\Omega;\Omega')$ the class of homeomorphisms $f\colon \Omega \to \Omega'$. If $f \in \mathrm{Hom}(\Omega;\Omega') \cap W^{1,n-1}(\Omega;\Omega')$ is a homeomorphism with finite inner distortion, we deduce regularity properties of the inverse f^{-1} from the regularity of the distortion function of f.

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1 Introduction

In the last few years, homeomorphisms with finite distortion have attracted a great interest thanks to their connection with relevant topics such as elliptic partial differential equations, differential geometry and calculus of variations (see [10] and the references therein).
Let Ω and Ω' be bounded open sets in $\mathbb{R}^n$, $n \geq 2$, and let $\mathrm{Hom}(\Omega;\Omega')$ the class of homeomorphisms $f\colon \Omega \to \Omega'$.

In [9], the class of bisobolev maps has been introduced as the class of homeomorphisms $f\colon \Omega \to \Omega'$ such that

$$f \in W^{1,1}_{\rm loc}(\Omega;\Omega') \qquad \text{and} \qquad f^{-1} \in W^{1,1}_{\rm loc}(\Omega';\Omega)$$

Bisobolev maps have a close connection with homeomorphisms with finite distortion.
Recall that a homeomorphism $f \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^n)$ has finite outer distortion if its Jacobian J_f is strictly positive a.e. on the set where $|Df| \neq 0$. In case $J_f(x) \geq 0$ a.e., we define its outer distortion function as

$$K_{O,f}(x) = \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{for } J_f(x) > 0 \\ 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

Similarly, we say that a homeomorphism $f \in W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^n)$ has finite inner distortion if its Jacobian is strictly positive a.e. on the set where the adjugate $\operatorname{adj} Df$ of the differential matrix does not vanish. In case $J_f(x) \geq 0$ a.e., we define its inner distortion function as

$$K_{I,f}(x) = \begin{cases} \frac{|\operatorname{adj} Df(x)|^n}{J_f(x)^{n-1}} & \text{for } J_f(x) > 0 \\ 1 & \text{otherwise.} \end{cases}$$

Obviously these two notions coincide in the planar case, while for $n > 2$ they are related by the inequality

$$K_{I,f}(x) \leq K_{O,f}^{n-1}(x).$$

The reverse estimate

$$K_{O,f}(x) \leq K_{I,f}^{n-1}(x)$$

holds when $J(x,f) > 0$.
In the planar case, i.e. for $n = 2$, it is known that each bisobolev map has finite outer distortion ([7, 2]). Such a conclusion is not valid in higher dimension. In fact, there exists a bisobolev map in $\mathbb{R}^n$, $n \geq 3$, such that its Jacobian determinant is zero a.e. and the modulus of its differential matrix is strictly positive on a set of positive measure ([9]). However in [9] it is proven that bisobolev maps have finite inner distortion (see Theorem below).
Obviously the question can be reversed, wondering what are the conditions on a homeomorphism f in the Sobolev class $W^{1,1}$ that guarantee that it is a bisobolev map.

Recall that the inverse of a homeomorphism $f \in W^{1,n-1}_{\text{loc}}(\Omega, \Omega')$ belongs to BV_{loc} only ([2]). On the other hand, in [2], it has been proved that if $f \in W^{1,n-1}(\Omega, \Omega')$ is a homeomorphism with finite outer distortion such that $|Df|$ belongs to the Lorentz space $L^{n-1,1}$ then the inverse map f^{-1} belongs to $W^{1,1}(\Omega', \Omega)$ and has finite distortion too.
Note that in case $n = 2$ the Lorentz space $L^{1,1}$ coincides with L^1, then for planar homeomorphisms the result is true without any additional assumption besides $f \in W^1(\Omega, \Omega')$ ([7]).
A stronger result has been established in [4], where it is shown that homeomorphisms in the Sobolev class $W^{1,n-1}_{loc}(\Omega, \Omega')$ with finite inner distortion are bisobolev maps, i.e. f^{-1} belongs to $W^{1,1}(\Omega', \Omega)$. The sharpness of previous Theorem has been shown in [8].
We can summarize the connection between homeomorphisms $f \in W^{1,n-1}$ with finite distortion and bisobolev maps as follows

- $n = 2$ $\qquad f$ bisobolev $\iff f$ has finite distortion
- $n > 2$ $\qquad f$ bisobolev $\iff f$ has finite inner distortion

In the planar case, homeomorphisms with finite distortion and their inverses enjoy the same regularity in the scale of Sobolev class while in the n-dimensional setting ($n > 2$) the inverses have a weaker degree of regularity. In fact, as we already mentioned, the inverse of a Sobolev homeomorphism $f \in W^{1,n-1}$ with finite inner distortion belongs to $W^{1,1}$ only.
Here we show that if we assume some regularity on the inner distortion function $K_{I,f}$ then the inverse of a Sobolev homeomorphism $f \in W^{1,n-1}$ with finite inner distortion belongs to $W^{1,n-1}$ too. More precisely we have the following

Theorem 1.1. *Let $f \in W^{1,n-1}(\Omega, \Omega')$ be a homeomorphism with finite inner distortion such that $|\operatorname{adj} Df|$ belongs to the space $L^{1,\frac{1}{n-1}}$ and that*

$$K_{I,f} \in L^{1,\infty}(\Omega).$$

Then

$$|Df^{-1}| \in L^{n-1}(\Omega').$$

We shall also prove by mean of a counterexample that Theorem 1.1 is sharp. More precisely, the assumption on K_I can not be removed nor weakened in the context of Lebesgue spaces. In fact, we shall construct a mapping $f \in W^{1,n-1}(\Omega,\Omega')$ with finite distortion , such that $|\operatorname{adj} Df| \in L^{1,n-1}$ and $K_{I,f} \notin L^{1,\infty}(\Omega)$ whose inverse $f^{-1} \notin W^{1,n-1}(\Omega,\Omega')$ (see Section 6).

Theorem 1.1 tells us that the regularity of the distortion function influences the regularity of the inverse mapping.

In the planar case, the L^1- integrability of the distortion K_f of a homeomorphism of the Sobolev class $W^{1,1}$ is a sufficient condition to the L^2-integrability of the differential matrix of the inverse mapping ([7]).
For $n > 2$ a similar result has been established in [17], under the stronger assumption $f \in W^{1,p}(\Omega,\Omega')$, for some $p > n-1$. In fact the analogous of the quoted result of [7] should yield the same conclusion under the assumption $f \in W^{1,n-1}(\Omega,\Omega')$. Here we are able to remove this stronger assumption, showing that

Theorem 1.2. *Let $f \in W^{1,n-1}(\Omega,\Omega')$ be a homeomorphism with finite inner distortion such that*

$$K_{I,f} \in L^1(\Omega).$$

Then

$$|Df^{-1}| \in L^n(\Omega')$$

and

$$\int_{\Omega'} |Df^{-1}(y)|^n \, dy = \int_\Omega K_{I,f}(x) \, dx. \tag{1.2}$$

Moreover we have that

$$\log\left(e + \frac{1}{J_f}\right) \in L^1_{\text{loc}}(\Omega). \tag{1.3}$$

It is worth pointing out that the regularity of the distortion influences the regularity of the inverse mapping also in the scale of Orlicz spaces. More precisely, we shall examine also the case in which the distortion is assumed to belong to a Orlicz class of functions not too far from L^1 and we shall obtain results in case the distortion belongs to a class smaller than L^1 and in case it has a degree of summability less than L^1. Previous results in this direction have been obtained in [7, 5] for planar homeomorphisms.
Our proofs strongly rely on the validity of the area formula for homeomorphisms ([3]) and on a chain rule formula proved in [4] for Sobolev functions. However, here we show that the area formula and the chain rule of [4] are valid in the more general context of approximate differentiable homeomorphisms (see Section 3).

2 Preliminaries

2.1 The area formula

Let Ω and Ω' be bounded domains in $\mathbb{R}^n$. We shall denote by $\mathrm{Hom}(\Omega;\Omega')$ the set of all homeomorphisms $f:\Omega\to\Omega'=f(\Omega)$, by $|Df|$ the operator norm of the differential matrix and by $\mathrm{adj}\,Df$ the adjugate of Df which is defined by the formula

$$Df\cdot\mathrm{adj}\,Df=\mathbf{I}\cdot J_f, \tag{2.1}$$

where, as usual, $J_f=\det Df$ and $\mathbf{I}$ is the identity matrix.
We will use the well known area formula for homeomorphisms in $W^{1,1}_{\mathrm{loc}}(\Omega)$, that is

$$\int_B \eta(f(x))\,|J_f(x)|\,dx\le\int_{f(B)}\eta(y)\,dy \tag{2.2}$$

where η is a nonnegative Borel measurable function on $\mathbb{R}^n$ and $B\subset\Omega$ is a Borel set (for more details we refer to [3]). The equality

$$\int_B \eta(f(x))\,|J_f(x)|\,dx=\int_{f(B)}\eta(y)\,dy \tag{2.3}$$

is verified if f is a homeomorphism that satisfies the *Lusin condition N*, i.e. the implication $|E|=0\implies|f(E)|=0$ holds for any measurable set $E\subset\Omega$.
Note that the function defined in (1.1) satisfies the so-called distortion inequality

$$|Df(x)|^n\le K_{O,f}(x)J_f(x)$$

Moreover, by virtue of (2.2), we have that $J_f\in L^1(B)$. Hence, the definition of homeomorphism with finite distortion coincides with the usual one, given for mappings which are not homeomorphisms (see [10]).
In [6] the authors proved that mappings $f\in W^{1,n}(\Omega,\mathbb{R}^n)$ of finite distortion satisfy the Lusin condition N. Here we are interested in mappings of finite distortion whose differential matrices belong to spaces slightly different from L^n. For this reason let us recall the definitions and some basic properties of these spaces.

2.2 Orlicz spaces

Let P be an increasing function from $P(0)=0$ to $\lim\limits_{t\to\infty}P(t)=\infty$ and continuously differentiable on $(0,\infty)$. The Orlicz space generated by the

function $P(t)$ will be denoted by $L^P(\Omega)$ and it consists of the functions h for which there exists a constant $\lambda = \lambda(h) > 0$ such that $\mathcal{P}\left(\frac{|h|}{\lambda}\right) \in L^1(\Omega)$. In particular we shall work with the Orlicz-Zygmund spaces $L^s \log^\alpha L$, $1 \leq s < \infty$, $\alpha \in \mathbb{R}$, which are Orlicz spaces generated by the function $P(t) = t^s \log^\alpha(e+t)$.
For $\alpha > 0$, the dual Orlicz space to $L \log^\alpha L(\Omega)$ is the space $Exp_{\frac{1}{\alpha}}(\Omega)$, generated by the function $Q(t) = \exp(t^{\frac{1}{\alpha}}) - 1$. The well known Young's inequality reads as

$$st \leq s \log^\alpha(e+s) + \exp(t^{\frac{1}{\alpha}}) - 1 \qquad \forall s, t \geq 0 \tag{2.4}$$

Observe that if $|g|^\beta \in L^1(\Omega)$ for some $\beta > 0$, then

$$\log^\alpha(e+|g|) \in Exp_{\frac{1}{\alpha}}(\Omega) \tag{2.5}$$

for all $\alpha > 0$. In fact, we have that

$$\int_\Omega \exp\left(\frac{\log^\alpha(e+|g|)}{\lambda}\right)^{\frac{1}{\alpha}} dx = \int_\Omega \exp\left(\frac{\log(e+|g|)}{\lambda^{\frac{1}{\alpha}}}\right) dx = \int_\Omega (e+|g|)^{\lambda^{-\frac{1}{\alpha}}} dx$$

Note that the last integral in previous equality is finite for every positive constant λ verifying the inequality $\lambda > \frac{1}{\beta^\alpha}$. Hence, by the definition, the function $\log^\alpha(e+|g|)$ belongs to the space $Exp_{\frac{1}{\alpha}}(\Omega)$. For more details on Orlicz spaces we refer to [12].

2.3 Lorentz Spaces

Let Ω be a bounded domain in $\mathbb{R}^n$ and $g : \Omega \to \mathbb{R}$ be a measurable function. For $t \geq 0$ we denote by

$$\Omega_t = \{x \in \Omega : |g(x)| > t\} \tag{2.6}$$

For $1 < p, q < +\infty$ the Lorentz space $L(p,q)$ consists of all measurable function g defined on Ω such that

$$||g||_{p,q}^q = p \int_0^{+\infty} |\Omega_t|^{\frac{q}{p}} t^{q-1}\, dt < +\infty$$

where $|\Omega_t|$ is the Lebesgue measure of Ω_t, then $||\cdot||_{p,q}$ is equivalent to a norm under which $L(p,q)$ is a Banach space ([16]). For $p = q$, the space $L(p,p)$ coincides with the usual L^p space and if $1 < q < r < +\infty$, we have

$$L(p,1) \subset L(p,q) \subset L(p,r)$$

For $q = \infty$, the class $L(p, \infty)$ consists of all functions g defined on Ω such that

$$||g||_{p,\infty}^p = \sup_{t>0} t^p |\Omega_t| < +\infty$$

and it is equivalent to the Marcinkiewicz class, weak-L^p.
For $1 < r < p$, $1 < q < \infty$

$$L(p, q) \subset L(p, \infty) \subset L^r$$

Whenever $1 < p < \infty$, $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $f \in L(p, q)$, $g \in L(p', q')$ we have the Hölder-type inequality

$$\int_\Omega |f(x)g(x)| \mathrm{d}x \leq ||f||_{p,q} ||g||_{p',q'} \tag{2.7}$$

2.4 Bisobolev maps

Let us recall that a homeomorphism $f : \Omega \xrightarrow{onto} \Omega'$ is said to be a bisobolev map if f belongs to the Sobolev space $W^{1,1}_{loc}(\Omega; \Omega')$ and its inverse f^{-1} belongs to $W^{1,1}_{loc}(\Omega'; \Omega)$. More specifically, if $f \in W^{1,p}_{loc}(\Omega; \Omega')$ and $f^{-1} \in W^{1,p}_{loc}(\Omega'; \Omega)$, $1 \leq p < \infty$, then we say that f is $W^{1,p}$- bisobolev.

The connection between bisobolev mappings and mappings with finite distortion is given by the following results.

Theorem 2.1. ([9]) *Let $f : \Omega \to \mathbb{R}^n$ be a bisobolev map. Suppose that for a measurable set $E \subset \Omega$ we have $J_f = 0$ a.e. on E. Then $|adjDf| = 0$ a.e. on E. If we moreover assume that $J_f \geq 0$ it follows that f has finite inner distortion.*

In the opposite direction, we have the following

Theorem 2.2. ([4]) *Let $f \in W^{1,n-1}(\Omega, \Omega')$ be a homeomorphism such that*

$$|\operatorname{adj} Df(x)|^n \leq K(x) J_f^{n-1}(x) \tag{2.8}$$

for some Borel function $K \colon \Omega \to [1, +\infty)$. Then f^{-1} is a $W^{1,1}(\Omega', \Omega)$ map of finite outer distortion. Moreover

$$|Df^{-1}(y)|^n \leq K(f^{-1}(y)) J_{f^{-1}}(y) \qquad \textit{a.e. in } \Omega$$

and

$$\int_{\Omega'} |Df^{-1}(y)|\, dy = \int_\Omega |\operatorname{adj} Df(x)|\, dx$$

3 The chain rule and the area formula

We need to recall the definition of approximate gradient of a Borel map. To this aim, for a measurable function $g : \Omega \to \mathbb{R}$, we will define its approximate limit as

$$\operatorname*{aplim}_{y\to x} g(y) = \inf\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{|\{g > t\} \cap B_r(x)|}{r^n} = 0\right\}$$

with the convention that

$$\operatorname*{aplim}_{y\to x} g(y) = +\infty \text{ if } \left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{|\{g > t\} \cap B_r(x)|}{r^n} = 0\right\} = \emptyset.$$

If $f \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, we will say that a point $x \in \Omega$ is a point of approximate continuity if there exists $z \in \mathbb{R}^N$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} |f(y) - z|\, dy = 0 \tag{3.1}$$

The precise representative of f is the function $f^* : \Omega \to \mathbb{R}^N$ defined by setting $f^*(x) = z$, where z is the vector appearing in (3.1), if x is a point of approximate continuity of f and $f^*(x) = 0$ otherwise.

Let x a point of approximate continuity of f. We will say that f is approximately differentiable at x if there exists a $N \times n$ matrix denoted by $Df(x)$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} \frac{|f(y) - f^*(x) - Df(x)(y - x)|}{r}\, dy = 0 \tag{3.2}$$

The approximate gradient $Df(x)$ is uniquely determined by (3.2), the set

$$\mathcal{D}_f = \{x \in \Omega : \ f \text{ is approximately differentiable at } x\}$$

is a Borel set and the map $Df : \mathcal{D}_f \to R^{Nn}$ is a Borel map. Recall that if $f, g \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ then

$$Df(x) = Dg(x) \quad \text{for a.e. } x \in \mathcal{D}_f \cap \mathcal{D}_g \cap \{f = g\} \tag{3.3}$$

If $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ then is approximately differentiable a.e. in Ω and its approximate gradient coincides a.e. with the distributional gradient ([1]).

Lemma 3.1. *Let $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and let $x \in \mathcal{D}_f$. Then we have*

$$\operatorname*{aplim\,sup}_{y\to x} \frac{|f^*(y)-f^*(x)|}{|y-x|} < +\infty$$

Proof. Without loss of generality we can suppose that $x = 0$. Let $t > |Df(0)|$ and set

$$A_t = \left\{ y \in \Omega : \frac{|f^*(y)-f^*(0)|}{|y|} > t \right\}.$$

It suffices to prove that

$$\lim_{r\to 0} \frac{|A_t \cap B_r|}{r^n} = 0 \tag{3.4}$$

To this aim, we argue by contradiction supposing that there exists a sequence $r_h \to 0$ such that

$$\lim_{h\to\infty} \frac{|A_t \cap B_{r_h}|}{r_h^n} = l > 0 \tag{3.5}$$

Since $Df^*(0) = Df(0)$, we have

$$\begin{aligned}
&\int_{B_{r_h}} \frac{|f^*(y)-f^*(0)-Df(0)\cdot y|}{r_h^{n+1}}\,dy \\
\geq\ & \int_{B_{r_h}\cap A_t} \frac{|f^*(y)-f^*(0)-Df(0)\cdot y|}{r_h^{n+1}}\,dy \\
\geq\ & \int_{B_{r_h}\cap A_t} \left[\frac{|f^*(y)-f^*(0)|-|Df(0)\cdot y|}{r_h^{n+1}}\right] dy \\
\geq\ & \frac{t-|Df(0)|}{r_h^{n+1}} \int_{B_{r_h}\cap A_t} |y|\,dy.
\end{aligned} \tag{3.6}$$

Let us consider ρ_h such that $|B_{\rho_h}| = |B_{r_h}\cap A_t|$ and so

$$\omega_n \rho_h^n = |B_{r_h}\cap A_t| \ \Rightarrow\ \rho_h = \left(\frac{|B_{r_h}\cap A_t|}{\omega_n}\right)^{\frac{1}{n}}. \tag{3.7}$$

Since the function $y \to |y|$ is radially symmetric, one easily gets

$$\int_{B_{r_h}\cap A_t} |y|\,dy \geq \int_{B_{\rho_h}} |y|\,dy = \int_0^{\rho_h} n\omega_n r^n\,dr = \frac{n}{n+1}\omega_n \rho_h^{n+1}. \tag{3.8}$$

Inserting (3.8) in (3.6) and using (3.7), we obtain

$$\int_{B_{r_h}} \frac{|f(y)-f^*(0)-Df(0)\cdot y|}{r_h^{n+1}}\,dy$$

$$
\begin{aligned}
&\geq \frac{t-|Df(0)|}{r_h^{n+1}}\frac{n}{n+1}\omega_n\rho_h^{n+1}\\
&= \frac{n[t-|Df(0)|]}{(n+1)\omega_n^{\frac{1}{n}}}\left(\frac{|B_{r_h}\cap A_t|}{r_h^n}\right)^{\frac{n+1}{n}}. \qquad (3.9)
\end{aligned}
$$

Taking the limit as $h \to \infty$ in estimate (3.9), using the assumption $0 \in \mathcal{D}_f$ and (3.5), we have

$$l = 0$$

which is clearly a contradiction. $\square$

Combining Lemma 3.1 with the area formula for Lipscitz maps and Theorem 3.1.8 in [3], as a particular case of Corollary 3.2.20 in [3], we have the following Area Formula

Theorem 3.2. (Area Formula) *For $f \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and $\psi : \mathbb{R}^N \to [0, +\infty]$ a Borel function, we have*

$$\int_{\mathcal{D}_f} \psi(x)|J_f(x)|\,dx = \int_{f^*(\mathcal{D}_f)} dy \int_{\{(f^*)^{-1}(y)\}} \psi(x)\,d\mathcal{H}^0(x).$$

From the Area Formula we can derive the following

Corollary 3.3. *Let $f : \Omega \to \mathbb{R}^N$ be a one to one, continuous map and let $\varphi : \mathbb{R}^N \to [0, +\infty]$ be a Borel function. Then*

$$\int_{\mathcal{D}_f} \varphi(f(x))|J_f(x)|\,dx = \int_{f(\mathcal{D}_f)} \varphi(y)\,dy.$$

Let us notice that, in general, if $f : \Omega \to \mathbb{R}^N$ is a one to one, continuous map and $\varphi : \mathbb{R}^N \to [0, +\infty]$ is a Borel function, one has

$$\int_{\Omega} \varphi(f(x))|J_f(x)|\,dx \leq \int_{f(\Omega)} \varphi(y)\,dy.$$

However, the equality is achieved when f satisfies the N property of Lusin. More precisely we have the following

Proposition 3.4. *Let $f : \Omega \to \mathbb{R}^N$ be a one to one, continuous map a.e. approximately differentiable. Then*

$$\int_\Omega \varphi(f(x))|J_f(x)|\,dx = \int_{f(\Omega)} \varphi(y)\,dy, \tag{3.10}$$

for every Borel function $\varphi : \mathbb{R}^N \to [0,+\infty]$, if and only if f satisfies the N property of Lusin.

Proof. Suppose that f satisfies the N property of Lusin. Since f is approximately differentiable a.e. in Ω, the N property of Lusin yields that $|f(\Omega \setminus \mathcal{D}_f)| = 0$. Hence, Corollary 3.3 implies

$$\int_\Omega \varphi(f(x))|J_f(x)|\,dx = \int_{\mathcal{D}_f} \varphi(f(x))|J_f(x)|\,dx = \int_{f(\mathcal{D}_f)} \varphi(y)\,dy = \int_{f(\Omega)} \varphi(y)\,dy.$$

Conversely, let us suppose that equality (3.10) holds, for every Borel function $\varphi : \mathbb{R}^N \to [0,+\infty]$. We need to show that, if $E \subset \Omega$ is a Borel set with $|E| = 0$ then we have $|f(E)| = 0$. Note that, since f is a one to one continuous map the image of a Borel set E is a Borel set and hence the characteristic function $\chi_{f(E)}$ is a Borel function. Since for $\chi_{f(E)}$, the equality (3.10) holds, we obtain

$$0 = \int_E |J_f(x)|\,dx = \int_\Omega \chi_{f(E)}(f(x))|J_f(x)|\,dx = \int_{f(\Omega)} \chi_{f(E)}(y)\,dy = |f(E)|$$

which concludes the proof. $\square$

A key tool for our aim is the following chain rule formula, which has been proved in [4] for bisobolev mappings. Actually the proof given in [4] works under slightly weaker assumptions. We give it here for the sake of completeness.

Lemma 3.5. *Let $f : \Omega \to \Omega'$ be a homeomorphism such that f and f^{-1} are approximately differentiable a.e.. Set*

$$F = \{y \in \mathcal{D}_{f^{-1}} : |J_{f^{-1}}(y)| > 0\}.$$

Then there exists a Borel set $A \subset F$ such that $|F \setminus A| = 0$, $f^{-1}(A) \subset \{x \in \mathcal{D}_f : |J_f(x)| > 0\}$, with the following property

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1} \quad \forall y \in A \tag{3.11}$$

Proof. The approximate differentiability of f^{-1} at any point of the Borel set F implies (see 3.1.8 of [3]) that we can cover F by an increasing family of Borel set F_i such that the restriction $f_{|F_i}$ is a Lipschitz map h. Hence, for every $\varepsilon > 0$ there exists a Lipschitz map h and a set $F_\varepsilon \subset F$ such that $|F \setminus F_\varepsilon| < \varepsilon$ and $f^{-1}(y) = h(y)$ for every $y \in F_\varepsilon$. Thanks to (3.3) we also have that $Df^{-1}(y) = Dh(y)$ and hence $|J_h(y)| > 0$ for all $y \in F_\varepsilon$. Thus, by the Lipschitz linearization Lemma, F_ε can be decomposed, up to a set of zero measure, into the union of countably many, pairwise disjoint, compact set H_i such that, for every integer i, the map $h_{|H_i}$ is invertible, $(h_{|H_i})^{-1}$ is Lipschitz, h is differentiable, $|J_h(y)| > 0$ and $Df^{-1}(y) = Dh(y)$ for all $y \in H_i$.

Now, let us denote by $g_i : \mathbb{R}^n \to \mathbb{R}^n$ a Lipschitz function such that $g_i(x) = (h_{|H_i})^{-1}(x)$ for all $x \in h(H_i)$. The equalities

$$h(g_i(x)) = x \quad \forall x \in h(H_i) \qquad\qquad g_i(h(y)) = y \quad \forall y \in H_i$$

together with the a.e. differentiability of Lipschitz functions yield

$$Dh(g_i(x)) = [Dg_i(x)]^{-1} \qquad \text{for a.e. } x \in h(H_i)$$

Since $g_i(x) = f(x)$ for every $x \in h(H_i)$, previous equality implies that there exists a null Borel set $M_i \subset h(H_i) = f^{-1}(H_i)$ such that f is approximately differentiable at every point $x \in f^{-1}(H_i) \setminus M_i$ and

$$Dh(f(x)) = [Df(x)]^{-1} \qquad \forall x \in f^{-1}(H_i) \setminus M_i,$$

i.e.

$$Dh(y) = [Df(f^{-1}(y))]^{-1} \qquad \forall y \in H_i \setminus f(M_i).$$

Since $f(M_i) = g_i(M_i)$ and g_i is a Lipschitz map, we deduce that $f(M_i)$ is a Borel set of zero Lebesgue measure. Recalling that

$$Df^{-1}(y) = Dh(y) \qquad \forall y \in \cup_i H_i$$

we have proven that the approximate gradient $Df(x)$ exists for every $x \in \bigcup_i \Big(f^{-1}(H_i) \setminus M_i\Big)$ and

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1} \qquad \forall y \in \bigcup_i \Big(H_i \setminus f(M_i)\Big).$$

Since $\bigcup_i \Big(H_i \setminus f(M_i)\Big)$ is a Borel subset of F_ε with full measure, the conclusion easily follows from previous equality. $\square$

4 Proof of Theorems 1.1 and 1.2

In this section we will give the proofs of the regularity results for the inverses of homeomorphisms with finite distortion in the scale of Sobolev classes.

Proof of Theorem 1.2 . Since $f \in W^{1,n-1}$ has finite inner distortion, from Theorem 2.2 it follows that $J_{f^{-1}}(y) \geq 0$ and $f^{-1} \in W^{1,1}$ is a homeomorphism with finite distortion. Hence f is approximately differentiable a.e. in Ω and f^{-1} is approximately differentiable a.e. in Ω'. Lemma 3.5 yields that there exists a subset A' of Ω', with full measure, such that chain rule holds at any point of A'. Then, we have

$$\begin{aligned} \int_{\Omega'} |Df^{-1}(y)|^n \, dy &= \int_{A'} |Df^{-1}(y)|^n \, dy \\ = & \int_{A'} \frac{|\operatorname{adj} Df(f^{-1}(y))|^n}{J_f^n(f^{-1}(y))} \, dy \\ \leq & \int_{A'} K_{I,f}(f^{-1}(y)) J_{f^{-1}}(y) \leq \int_{\Omega} K_{I,f}(x) \, dx \end{aligned} \tag{4.1}$$

where in the last line we applied area formula (2.2). Then the assumption $K_I \in L^1$ implies that $f^{-1} \in W^{1,n}(\Omega'; \Omega)$. It remains to prove the other inequality in (1.2). To this aim, note that as a consequence of the weak version of Sard Lemma we have

$$|f(\mathcal{D}_f \cap J_f^0)| = 0,$$

where we denoted by J_f^0 the zero set of the Jacobian determinant of f. Since $f^{-1} \in W^{1,n}(\Omega'; \Omega)$ we have that f^{-1} satisfies N property. Hence we have

$$|\mathcal{D}_f \cap J_f^0| = 0,$$

i.e. $J_f > 0$ a.e. in Ω. Thus, using area formula and chain rule again, we get

$$\begin{aligned} \int_{\Omega} K_{I,f}(x) \, dx &= \int_{\mathcal{D}_f} K_{I,f}(x) \, dx \\ = & \int_{\mathcal{D}_f \setminus J_f^0} \frac{|\operatorname{adj} Df(x)|^n}{J_f^n(x)} J_f(x) \, dx = \int_A \frac{|\operatorname{adj} Df(x)|^n}{J_f^n(x)} J_f(x) \, dx \\ = & \int_A |Df^{-1}(f(x))|^n J_f(x) \, dx \leq \int_{\Omega'} |Df^{-1}(y)|^n \, dy \end{aligned} \tag{4.2}$$

where A the Borel set determined by Lemma 3.5. This concludes the proof of the estimate (1.2).

As already observed, by Theorem 2.2 it follows that $J_{f^{-1}}(y) \geq 0$ and $f^{-1} \in W^{1,1}$ and we just proved that $f^{-1} \in W^{1,n}$. Therefore, by a well known result due to Müller ([15]), it is $J_{f^{-1}} \in L \log L_{\rm loc}(f(\Omega))$. By Lemma 3.5 there exists a Borel set $A \subset \Omega \setminus J_f^0$ such that $|A| = |\Omega|$ and such that the chain rule formula holds at every point of A. By the area formula and (3.11), for $x \in A$ and for every compact set $E \subset \Omega$, we have

$$\begin{aligned}
\int_E \log\left(e + \frac{1}{J_f(x)}\right) dx &= \int_{E\cap A} \frac{1}{J_f(x)} \log\left(e + \frac{1}{J_f(x)}\right) J_f(x)\, dx \\
&\leq \int_{f(E\cap A)} J_{f^{-1}}(y) \log(e + J_{f^{-1}}(y)) dy < \infty
\end{aligned}$$

and therefore the conclusion. $\square$

Proof of Theorem 1.1. Let A' the set determined by Lemma 3.5. From Theorem 2.2 it follows that $J_{f^{-1}}(y) \geq 0$ and $f^{-1} \in W^{1,1}$ is a homeomorphism with finite distortion. Using the chain rule of Lemma 3.5

$$\begin{aligned}
\int_{\Omega'} |Df^{-1}(y)|^{n-1}\, dy &= \int_{A'} |Df^{-1}(y)|^{n-1}\, dy \\
= & \int_{A'} \frac{|\operatorname{adj} Df(f^{-1}(y))|^{n-1}}{J_f^{n-1}(f^{-1}(y))}\, dy \\
\leq & \int_{\Omega} \frac{|\operatorname{adj} Df(x)|^{n-1}}{J_f^{n-2}(x)}\, dx
\end{aligned} \tag{4.3}$$

where in the last line we applied area formula (2.2). Hence, by the definition of the inner distortion function, it follows that

$$\begin{aligned}
\int_{\Omega'} |Df^{-1}(y)|^{n-1}\, dy &\leq \int_{\Omega} \frac{|\operatorname{adj} Df(x)|^{n-1}}{J_f^{n-2}(x)}\, dx \\
= & \int_{\Omega} \left(\frac{|\operatorname{adj} Df(x)|^{n}}{J_f^{n-1}(x)}\right)^{\frac{n-2}{n-1}} |\operatorname{adj} Df(x)|^{\frac{1}{n-1}}\, dx \\
= & \int_{\Omega} (K_{I,f}(x))^{\frac{n-2}{n-1}} |\operatorname{adj} Df(x)|^{\frac{1}{n-1}}\, dx
\end{aligned} \tag{4.4}$$

By a simple use of Holder's inequality in Lorentz spaces we get

$$\int_{\Omega'} |Df^{-1}(y)|^{n-1}\, dy$$

$$
\begin{aligned}
&\leq \left|\left|(K_{I,f}(x))^{\frac{n-2}{n-1}}\right|\right|_{L^{\frac{n-1}{n-2},\infty}} \left|\left|\,|\operatorname{adj} Df(x)|^{\frac{1}{n-1}}\right|\right|_{L^{n-1,1}} \\
&\leq ||K_{I,f}(x)||^{\frac{n-1}{n-2}}_{L^{1,\infty}} \, |||\operatorname{adj} Df(x)|||_{L^{1,\frac{1}{n-1}}} \qquad (4.5)
\end{aligned}
$$

i.e. the conclusion. □

Note that in the case $n = 2$, estimate (4.4) reduces to

$$
\int_{\Omega'} |Df^{-1}(y)|\, dy \leq \int_{\Omega} |Df(x)|\, dx
$$

and then no assumptions on the distortion function are needed to derive the analogous conclusion.
Next result shoes that assuming more regularity on the adjugate of the differential matrix allow to weaken the assumption on the regularity of the distortion and have the same conclusion of previous theorem.

Proposition 4.1. *Let $f \in W^{1,n-1}(\Omega, \Omega')$ be a homeomorphism with finite inner distortion such that $|\operatorname{adj} Df|$ belongs to the space L^p for some $p > 1$ and that*

$$
K^{\varepsilon}_{I,f} \in L^1(\Omega),
$$

for $\varepsilon = \frac{p(n-2)}{p(n-1)-1} < 1$. Then

$$
|Df^{-1}| \in L^{n-1}(\Omega').
$$

Proof. We argue as in the proof of previous Theorem until we arrive at the estimate

$$
\int_{\Omega'} |Df^{-1}(y)|^{n-1}\, dy \leq \int_{\Omega} (K_{I,f}(x))^{\frac{n-2}{n-1}} \, |\operatorname{adj} Df(x)|^{\frac{1}{n-1}}\, dx \qquad (4.6)
$$

Since by assumption $|\operatorname{adj} Df| \in L^p$, we can use Hölder's inequality of exponents $p(n-1)$ and $\frac{p(n-1)}{p(n-1)-1}$ thus having

$$
\int_{\Omega'} |Df^{-1}(y)|^{n-1}\, dy \leq \left(\int_{\Omega} (K_{I,f}(x))^{\varepsilon}\right)^{\frac{p(n-1)-1}{p(n-1)}} \left(\int_{\Omega} |\operatorname{adj} Df(x)|^p\, dx\right)^{\frac{1}{p(n-1)}}
$$

which concludes the proof. □

5 The regularity of the inverse in Orlicz spaces

In this section we shall examine how the regularity of the inner distortion function of a homeomorphism f reflects on the regularity of the inverse f^{-1} in the scale of Orlicz spaces. We shall confine ourselves to the case of spaces not too far from L^1 and we shall face both the case of spaces smaller than L^1 and of space slightly larger than L^1. Our first result is the following

Theorem 5.1. *Let $f \in W^{1,n-1}(\Omega, \Omega')$ be an homeomorphism with finite inner distortion such that*

$$K_{I,f} \in L \log^{\alpha} L(\Omega) \tag{5.1}$$

for some $\alpha \geq 0$. Then

$$|Df^{-1}| \in L^n \log^{\alpha} \log_{loc} L(\Omega') \tag{5.2}$$

Proof. Under our assumption $K_{I,f} \in L \log^{\alpha} L(\Omega)$, we can apply Theorem 1.2 to deduce that

$$\log \left(e + \frac{1}{J_f} \right) \in L^1_{\text{loc}}(\Omega). \tag{5.3}$$

Since f is a homeomorphism with finite inner distortion, we know, by Theorem 1.2, that f^{-1} is a homeomorphism of $W^{1,n}$ with finite inner distortion. Hence by the weak version of the Sard Lemma and by the Lusin property of f^{-1} we get that $J_f(x) > 0$ a.e. in Ω. Therefore, denoted by A' the set determined by Lemma 3.5, if E is a compact subset of Ω, by (2.2), we get

$$\begin{aligned}
& \int_{f(E)} |Df^{-1}(y)|^n \log^{\alpha}(e + \log(e + |Df^{-1}(y)|))\, dy \\
= & \int_{A'} |Df^{-1}(y)|^n \log^{\alpha}(e + \log(e + |Df^{-1}(y)|))\, dy \\
\leq & \int_E \frac{|\operatorname{adj} Df(x)|^n}{J_f^{n-1}(x)} \log^{\alpha} \left(e + \log \left(e + \frac{|\operatorname{adj} Df(x)|}{J_f(x)} \right) \right) dx \\
= & \int_{E \cap \{|\operatorname{adj} Df| \geq 1\}} K_{I,f}(x) \log^{\alpha} \left(e + \log \left(e + \frac{|\operatorname{adj} Df(x)|}{J_f} \right) \right) dx \\
+ & \int_{E \cap \{|\operatorname{adj} Df| < 1\}} K_{I,f}(x) \log^{\alpha} \left(e + \log \left(e + \frac{|\operatorname{adj} Df(x)|}{J_f} \right) \right) dx \\
= & \; I + II
\end{aligned} \tag{5.4}$$

Since $|\operatorname{adj} Df(x)| \le |\operatorname{adj} Df(x)|^{n/n-1}$ on the set $E \cap \{|\operatorname{adj} Df| \ge 1\}$, we have

$$
\begin{aligned}
I &\le \int_{E\cap\{|\operatorname{adj} Df|\ge 1\}} K_{I,f}(x)\log^{\alpha}\left(e+\log\left(e+\frac{|\operatorname{adj} Df(x)|^{n/n-1}}{J_f}\right)\right)dx \\
&\le c\int_E K_{I,f}\log^{\alpha}(e+\log(e+K_{I,f}))\,dx<+\infty \qquad (5.5)
\end{aligned}
$$

thanks to the assumption. In order to estimate II, we use Young's inequality at (2.4)

$$
\begin{aligned}
II &\le \int_{E\cap\{|\operatorname{adj} Df|<1\}} K_{I,f}(x)\log^{\alpha}\left(e+\log\left(e+\frac{1}{J_f}\right)\right)dx \\
&\le \int_{\Omega} K_{I,f}(x)\log^{\alpha}\left(e+K_{I,f}(x)\right)\,dx+\int_{\Omega}\exp\left(\log^{\alpha}\left(e+\log\left(e+\frac{1}{J_f}\right)\right)\right)^{\frac{1}{\alpha}}dx \\
&\le \int_{\Omega} K_{I,f}(x)\log^{\alpha}(e+K_{I,f}(x))\,dx+\int_{\Omega}\left(e+\log\left(e+\frac{1}{J_f}\right)\right)dx
\end{aligned}
$$

that is finite thanks to the assumption and (5.3), hence the conclusion. $\square$

Next result concerns homeomorphism whose inner distortion function has a degree of integrability less than L^1 and it is the analogous to the n-dimensional setting of a result contained in [7]. More precisely we have the following

Theorem 5.2. *Let $f \in W^{1,n-1}(\Omega,\Omega')$ be an homeomorphism with finite inner distortion such that $|\operatorname{adj} Df| \in L^p(\Omega)$ for some $p>1$. If*

$$\frac{K_{I,f}}{\log(e+|K_{I,f}|)} \in L^1(\Omega)$$

then

$$|Df^{-1}| \in \frac{L^n}{\log L}(\Omega').$$

Moreover we have that

$$\log\log(e+\frac{1}{J_f}) \in L^1_{loc}(\Omega).$$

Proof. Since $f \in W^{1,n-1}(\Omega, \Omega')$ is an homeomorphism with finite inner distortion by Theorem 2.2 we have that $f^{-1} \in W^{1,1}(\Omega'; \Omega)$. Let A' be the set determined by Lemma 3.5. Then we can use the chain rule and the area formula, thus getting

$$\begin{aligned}
&\int_{\Omega'} \frac{|Df^{-1}(y)|^n}{\log(e + |Df^{-1}(y)|)}\, dy = \int_{A'} \frac{|Df^{-1}(y)|^n}{\log(e + |Df^{-1}(y)|)}\, dy \\
&\leq \int_{\Omega} \frac{|\operatorname{adj} Df(x)|^n}{J_f(x)^{n-1} \log\left(e + \frac{|\operatorname{adj} Df(x)|}{J_f(x)}\right)}\, dx \leq c \int_{\Omega} \frac{K_{I,f}(x)}{\log\left(e + \frac{K_{I,f}(x)}{|\operatorname{adj} Df(x)|}\right)}\, dx \\
&= \int_{\{K_{I,f} \leq |\operatorname{adj} Df|^p\}} \frac{K_{I,f}(x)}{\log\left(e + \frac{K_{I,f}(x)}{|\operatorname{adj} Df(x)|}\right)}\, dx \\
&+ \int_{\{K_{I,f} > |\operatorname{adj} Df|^p\}} \frac{K_{I,f}(x)}{\log\left(e + \frac{K_{I,f}(x)}{|\operatorname{adj} Df(x)|}\right)}\, dx \\
&\leq \int_{\Omega} |\operatorname{adj} Df(x)|^p\, dx + \int_{\Omega} \frac{K_{I,f}(x)}{\log\left(e + K_{I,f}(x)\right)}\, dx \qquad (5.6)
\end{aligned}$$

Since, by our assumptions, the integrals in the right hand side of previous estimate are finite we obtain that $|Df^{-1}| \in \dfrac{L^n}{\log L}(\Omega')$.
At this point we can use a result of [14] to deduce that $J_{f^{-1}} \in L \log \log L_{loc}(\Omega')$. Theorem A in [11] implies that f^{-1} satisfies condition (N) and hence $|\mathcal{J}_f^0| = 0$. In fact, as we noticed before, Sard's Lemma yields that $|f(\mathcal{D}_f \cap \mathcal{J}_f^0)| = 0$ and hence the N- property of f^{-1} implies $|\mathcal{D}_f \cap \mathcal{J}_f^0| = 0$. So we can argue analogously to before, having

$$\begin{aligned}
&\int_E \log\left(e^2 + \log\left(e + \frac{1}{J_f(x)}\right)\right) dx \\
= &\int_{E \cap A} \frac{1}{J_f(x)} \log\left(e^2 + \log\left(e + \frac{1}{J_f(x)}\right)\right) J_f(x) dx \\
\leq &\int_{f(E \cap A)} J_{f^{-1}}(y) \log\left(e^2 + \log(e + J_{f^{-1}}(y))\right) dy < \infty
\end{aligned}$$

where A is the subset of Ω determined by Lemma 3.5. $\square$

We can weaken the regularity assumption on the adjugate matrix of the differential of the homeomorphism f and we arrive at the same conclusion of Theorem 5.2, slightly improving the regularity assumption on the inner distortion function. In fact we have

Theorem 5.3. *Let $f \in W^{1,n-1}(\Omega, \Omega')$ be a homeomorphism with finite inner distortion such that $|\operatorname{adj} Df|$ belongs to the space $L\log L(\Omega)$ and that*

$$\frac{K_{I,f}}{\log(e+\log(e+|K_{I,f}|))} \in L^1(\Omega).$$

Then

$$|Df^{-1}| \in \frac{L^n}{\log L}(\Omega').$$

Proof. Since $f \in W^{1,n-1}(\Omega, \Omega')$ is a homeomorphism with finite inner distortion by Theorem 2.2 we have that $f^{-1} \in W^{1,1}(\Omega'; \Omega)$. Then we can use the area formula, thus getting

$$\begin{aligned}
&\int_{\Omega'} \frac{|Df^{-1}(y)|^n}{\log(e+|Df^{-1}(y)|)}\, dy \\
\leq &\int_\Omega \frac{|\operatorname{adj} Df(x)|^n}{J_f(x)^{n-1}\log\left(e+\frac{|\operatorname{adj} Df(x)|}{J_f(x)}\right)}\, dx \leq c\int_\Omega \frac{K_{I,f}(x)}{\log\left(e+\frac{K_{I,f}(x)}{|\operatorname{adj} Df(x)|}\right)}\, dx \\
= &\int_{\{K_{I,f}\leq |\operatorname{adj} Df|\log(e+K_{I,f})\}} \frac{K_{I,f}(x)}{\log\left(e+\frac{K_{I,f}(x)}{|\operatorname{adj} Df(x)|}\right)}\, dx \\
+ &\int_{\{K_{I,f}> |\operatorname{adj} Df|\log(e+K_{I,f}(x))\}} \frac{K_{I,f}(x)}{\log\left(e+\frac{K_{I,f}(x)}{|\operatorname{adj} Df(x)|}\right)}\, dx \\
\leq &\int_\Omega |\operatorname{adj} Df(x)|\log(e+K_{I,f}(x))\, dx \\
+ &\int_\Omega \frac{K_{I,f}(x)}{\log\left(e+\log(e+K_{I,f}(x))\right)}\, dx
\end{aligned} \tag{5.7}$$

Since the second integral in the right hand side of (5.7) is finite thanks to the assumption, it remains to prove that the first integral is finite too. To this aim observe that

$$\begin{aligned}
&\int_\Omega |\operatorname{adj} Df(x)|\log(e+K_{I,f})\, dx = \int_\Omega |\operatorname{adj} Df(x)|\log\left(e+\frac{|\operatorname{adj} Df(x)|^n}{J_f^{n-1}}\right) dx \\
\leq \;\; & c\int_\Omega |\operatorname{adj} Df(x)|\log(e+|\operatorname{adj} Df(x)|)\, dx + \int_\Omega |\operatorname{adj} Df(x)|\log\left(e+\frac{1}{J_f}\right) dx \\
\leq \;\; & c\int_\Omega |\operatorname{adj} Df(x)|\log(e+|\operatorname{adj} Df(x)|)\, dx + c\int_\Omega \frac{|\operatorname{adj} Df(x)|}{(J_f(x))^{\frac{n-1}{n}}}\, dx
\end{aligned}$$

$$= \quad c\int_\Omega |\operatorname{adj} Df(x)| \log(e + |\operatorname{adj} Df(x)|)\,dx + \int_\Omega K_f(x)^{\frac{n-1}{n}}\,dx \tag{5.8}$$

where the first integral in the right hand side is finite by assumptions and the second one is finite thanks to the Young' inequality i.e.

$$\int_\Omega K_f(x)^{\frac{n-1}{n}}\,dx \le \int_\Omega \frac{K_f(x)}{\log\left(e + \log(e + K_f(x))\right)} + c(|\Omega|) \tag{5.9}$$

The proof is now complete. $\square$

This result has been established in case of planar homeomorphisms in [5].

6 Construction of Counterexamples

The following general construction of examples of mappings of finite distortion was introduced in [8] (see also [7]). Here we give only the brief overview of the construction, for details see [8, Section 5].

6.1 Canonical transformation

If $c \in \mathbb{R}^n$, $a, b > 0$, we use the notation

$$Q(c, a, b) := [c_1 - a, c_1 + a] \times \cdots \times [c_{n-1} - a, c_{n-1} + a] \times [c_n - b, c_n + b].$$

for the interval with center at c and halfedges a in the first $n-1$ coordinates and b in the last coordinate. For $Q = Q(c, a, b)$ we set

$$\varphi_Q(y) = (c_1 + ay_1, \ldots, c_{n-1} + ay_{n-1}, c_n + by_n).$$

Let P, P' be concentric intervals, $P = Q(c, a, b)$, $P' = Q(c, a', b')$, where $0 < a < a'$ and $0 < b < b'$. We set

$$\varphi_{P,P'}(t, y) = (1 - t)\varphi_P(y) + t\varphi_{P'}(y), \qquad t \in [0, 1],\ y \in \partial Q_0.$$

Now, we consider two rectangular annuli, $P' \setminus P^\circ$, and $\tilde{P}' \setminus \tilde{P}^\circ$, where $P = Q(c, a, b)$, $P' = Q(c, a', b')$, $\tilde{P} = Q(\tilde{c}, \tilde{a}, \tilde{b})$ and $\tilde{P}' = Q(\tilde{c}, \tilde{a}', \tilde{b}')$, The mapping

$$h = \varphi_{\tilde{P},\tilde{P}'} \circ (\varphi_{P,P'})^{-1}$$

is called the *canonical transformation* of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$.

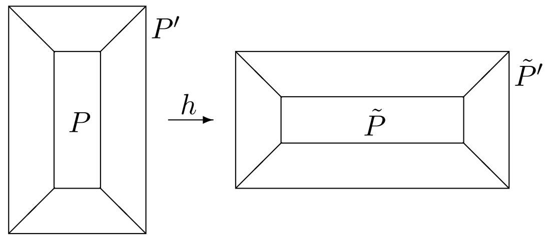

Fig. 1. The canonical transformation of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$ for $n = 2$

6.2 Construction of a mapping

By $\mathbb{V}$ we denote the set of 2^n vertices of the cube $[-1,1]^n =: Q_0$. The sets $\mathbb{V}^k = \mathbb{V} \times \ldots \times \mathbb{V}$, $k \in \mathbb{N}$, will serve as the sets of indices for our construction. If $\boldsymbol{w} \in \mathbb{V}^k$ and $v \in \mathbb{V}$, then the concatenation of $\boldsymbol{w}$ and v is denoted by $\boldsymbol{w}^\wedge v$. The following two results are proven in [8].

Lemma 6.1. *Let $n \geq 2$. Suppose that we are given two sequences of positive real numbers $\{a_k\}_{k\in\mathbb{N}_0}$, $\{b_k\}_{k\in\mathbb{N}_0}$,*

$$\begin{aligned}
&a_0 = b_0 = 1;\\
&a_k < a_{k-1},\ b_k < b_{k-1},\ \text{for } k \in \mathbb{N}.
\end{aligned}$$

Then there exist unique systems $\{Q_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$, $\{Q'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$ of intervals

$$Q_{\boldsymbol{v}} = Q(c_{\boldsymbol{v}}, 2^{-k}a_k, 2^{-k}b_k), \quad Q'_{\boldsymbol{v}} = Q(c_{\boldsymbol{v}}, 2^{-k}a_{k-1}, 2^{-k}b_{k-1})$$

such that

$$\begin{aligned}
&Q'_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathbb{V}^k, \quad \text{are nonoverlaping for fixed } k \in \mathbb{N},\\
&Q_{\boldsymbol{w}} = \bigcup_{v\in\mathbb{V}} Q'_{\boldsymbol{w}^\wedge v} \text{ for each } \boldsymbol{w} \in \mathbb{V}^k,\ k \in \mathbb{N},\\
&c_v = \frac{1}{2}v, \qquad v \in \mathbb{V},\\
&c_{\boldsymbol{w}^\wedge v} = c_{\boldsymbol{w}} + \sum_{i=1}^{n-1} 2^{-k}a_k v_i \varepsilon_i + 2^{-k}b_k v_n \varepsilon_n,\\
&\qquad \boldsymbol{w} \in \mathbb{V}^k,\ k \in \mathbb{N},\ v = (v_1, \ldots, v_n) \in \mathbb{V}.
\end{aligned}$$

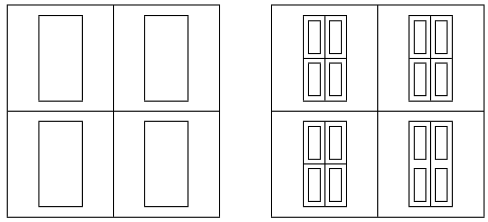

Fig. 2. Intervals $Q_{\boldsymbol{v}}$ and $Q'_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^1$ and $\boldsymbol{v} \in \mathbb{V}^2$ for $n = 2$.

Theorem 6.2. *Let $n \geq 2$. Suppose that we are given four sequences of positive real numbers $\{a_k\}_{k\in\mathbb{N}_0}$, $\{b_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{a}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{b}_k\}_{k\in\mathbb{N}_0}$,*

$$a_0 = b_0 = \tilde{a}_0 = \tilde{b}_0 = 1; \tag{6.1}$$

$$a_k < a_{k-1},\ b_k < b_{k-1},\ \tilde{a}_k < \tilde{a}_{k-1},\ \tilde{b}_k < \tilde{b}_{k-1},\ \text{for } k \in \mathbb{N}. \tag{6.2}$$

Let the systems $\{Q_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$, $\{Q'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$ of intervals be as in Lemma 6.1, and similarly systems $\{\tilde{Q}_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$, $\{\tilde{Q}'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$ of intervals be associated with the sequences $\{\tilde{a}_k\}$ and $\{\tilde{b}_k\}$. Then there exists a unique sequence $\{f^k\}$ of bilipschitz homeomorphisms of Q_0 onto itself such that

(a) *f^k maps each $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbb{V}^m$, $m = 1, \dots, k$, onto $\tilde{Q}'_{\boldsymbol{v}} \setminus \tilde{Q}_{\boldsymbol{v}}$ canonically,*

(b) *f^k maps each $Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbb{V}^k$, onto $\tilde{Q}_{\boldsymbol{v}}$ affinely.*

Moreover,

$$|f^k - f^{k+1}| \lesssim 2^{-k}, \qquad |(f^k)^{-1} - (f^{k+1})^{-1}| \lesssim 2^{-k}. \tag{6.3}$$

The sequence f^k converges uniformly to a homeomorphism f of Q_0 onto Q_0.

6.3 Counterexamples

Let us note that the assumption $K_I \in L^{1,\infty}$ in Theorem 1.1 can not be removed nor weakened in the context of Lebesgue spaces. In fact, we have

Example 6.3. *Let $n \geq 3$. There exists a homeomorphism $f : Q_0 \to Q_0$ of finite distortion such that $f \in W^{1,n-1}(Q_0, Q_0)$, $|\text{adj}Df| \in L^{1,\frac{1}{n-1}}$, $K_I \notin L^{1,\infty}$ and $f^{-1} \notin W^{1,n-1}(Q_0, Q_0)$.*

Proof. We start by setting

$$a_k = \frac{1}{k^\alpha} \qquad b_k = \frac{1}{k^\beta}$$

$$\tilde{a}_k = \frac{1}{k^\gamma} \qquad \tilde{b}_k = \frac{1}{k^\delta}$$

for

$$\alpha = \beta = 1 \qquad \delta = \gamma + \frac{n}{n-2}$$

for some $0 \le \gamma \le 1$. Note that this choice is possible only for $n \ge 3$. For $\alpha \ge \beta \ge 0$ and $\delta \ge \gamma \ge 0$ one can check that

$$|Df| \sim k^{\alpha-\gamma}$$

$$|Df^{-1}| \sim k^{\delta-\beta}$$

and

$$J_f \sim k^{(\alpha-\gamma)(n-1)+\beta-\delta}$$

(we refer to [2, 7] for more details). Therefore

$$\int_{Q_0} |Df|^{n-1} \sim C \sum_{k\in N} \frac{k^{(\alpha-\gamma)(n-1)}}{k^{\alpha(n-1)+\beta+1}}$$

$$\int_{Q_0} |\mathrm{adj} Df|^a \sim C \sum_{k\in N} \frac{k^{a(\alpha-\gamma)(n-1)}}{k^{\alpha(n-1)+\beta+1}}$$

$$\int_{Q_0} |Df^{-1}|^{n-1} \sim C \sum_{k\in N} \frac{k^{(\delta-\beta)(n-1)}}{k^{\gamma(n-1)+\delta+1}}$$

that yields

$$\sum_{k\in N} \frac{k^{(\alpha-\gamma)(n-1)}}{k^{\alpha(n-1)+\beta+1}} = \sum_{k\in N} \frac{1}{k^{\gamma(n-1)+2}} < \infty$$

for every $\gamma \ge 0$, which means that $f \in W^{1,n-1}(Q_0; Q_0)$. Moreover

$$\sum_{k\in N} \frac{k^{a(\alpha-\gamma)(n-1)}}{k^{\alpha(n-1)+\beta+1}} = \sum_{k\in N} \frac{1}{k^{n+1-a(1-\gamma)(n-1)}} < +\infty$$

for every $1 < a < \frac{n}{n-1}\frac{1}{1-\gamma}$, which implies that $|\mathrm{adj} Df| \in L^a$. Now, let us note that

$$\sum_{k\in N} \frac{k^{(\delta-\beta)(n-1)}}{k^{\gamma(n-1)+\delta+1}} = \sum_{k\in N} \frac{1}{k^{(\gamma-\delta)(n-1)+\delta+n}} = +\infty$$

since

$$(\gamma-\delta)(n-1) + \delta + n = \gamma + \frac{n}{n-2} + n - \frac{n(n-1)}{n-2} = \gamma \le 1$$

and hence $f^{-1} \notin W^{1,n-1}(Q_0; Q_0)$. Then observe that

$$K_I = \frac{|\mathrm{adj} Df|^n}{J_f^{n-1}} \sim \frac{k^{(\alpha-\gamma)n(n-1)}}{k^{(\alpha-\gamma)(n-1)^2+(\beta-\delta)(n-1)}} = k^{(\delta-\gamma)(n-1)}$$

In order to have that $K_I \notin L^{1,\infty}$ it suffices to show that $K_I^{\frac{n-2}{n-1}} \notin L^{\frac{n-1}{n-2},\infty}$. Since $L^{\frac{n-1}{n-2},\infty} \subset L^{\frac{n-1}{n-2}-\varepsilon}$ for every $0 < \varepsilon < \frac{1}{n-2}$ we will prove that there exists $0 < \varepsilon < \frac{1}{n-2}$ such that $K_I^{\frac{n-2}{n-1}} \notin L^{\frac{n-1}{n-2}-\varepsilon}$.
To this aim, we calculate

$$\int_{Q_0} \left(|K_I|^{\frac{n-2}{n-1}}\right)^{\left(\frac{n-1}{n-2}-\varepsilon\right)} = \int_{Q_0} |K_I|^{\left(1-\varepsilon\frac{n-2}{n-1}\right)}$$

$$\sim C \sum_{k\in N} \frac{k^{(\delta-\gamma)(n-1)-\varepsilon(\delta-\gamma)(n-2)}}{k^{n+1}}$$

Since

$$n - (\delta-\gamma)(n-1) + \varepsilon(\delta-\gamma)(n-2) = \varepsilon n - \frac{n}{n-2}$$

in order to have that $K_I^{\frac{n-2}{n-1}} \notin L^{\frac{n-1}{n-2}-\varepsilon}$ it suffices to choose

$$\varepsilon \leq \frac{1}{n-2}$$

□

On the other hand, arguing as in Example 6.3 and choosing in a suitable way δ, γ, we also have the following

Example 6.4. *Let $n \geq 3$ and $0 < \sigma < 1$. There exists a homeomorphism $f : Q_0 \to Q_0$ of finite distortion such that $f \in W^{1,n-1}(Q_0, Q_0)$, $K_I^\sigma \in L^1$ but $f^{-1} \notin W^{1,n-1}(Q_0, Q_0)$.*

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