TRANSPORT DISTANCES AND IRRIGATION MODELS

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Abstract. Some variational models have been recently introduced to the aim of modeling ramified structures, such as trees, rivers and so on. We introduce a general scheme in which the notion of *transport distance* is introduced starting from a general transport cost functional, through relaxation arguments. Then we apply this general framework to the irrigation cost, which is a particular cost functional depending on a parameter $\alpha \in]0, 1[$. We discuss the equivalence between this abstract approach and the above models.

Contents

Introduction

Recently, starting from [21], [17], a variant of the Monge-Kantorovitch transport problem which leads to some variational models for ramified structures has been investigated. In [21] the functional is firstly defined on finite graphs and subsequently extended to real flat 1-chains. The first step of this approach goes back to Gilbert [14] who generalized the Steiner problem [15]. Therefore we shall refer to this formulation as to Gilbert-Steiner approach or Xia approach. In [17], on the contrary, the functional is proposed for families of curves parametrized on a set Ω equipped with a probability measure or, equivalently, as remarked in [3], for measures defined on a space of curves. The theory developed in these cases is based on the concavity properties of the model function $|x|^\alpha$, involved in all these functionals, with $0 < \alpha < 1$. Actually the value of α plays an important role with respect to some particular questions. More precisely, some problems are easy when $\alpha > \frac{1}{N'} = 1 - \frac{1}{N}$, if N is the dimension of the euclidean space in which the problem is studied, (large α) and much more subtle when $0 < \alpha \leq \frac{1}{N'}$ (small α), in which new questions, like the irrigability problem ([9]), arise. The assumption of α being large is not always explicitly remarked, since it is sometimes taken as implicitly assumed in this literature.

In this paper we moved from the idea of showing the equivalence of these various approaches, attaining to a more general point of view which sees these functionals as metrics induced by a *transport cost* defined on probability measures. By applying this general framework to a particular cost functional, depending on a parameter α, the *irrigation cost*, we consider besides the functional in [17] (in one of its equivalent variants), the functional obtained by a relaxation procedure which induces a weak lower semicontinuous metric (transport distance). Then we get the Gilbert-Steiner functional as the result of a two steps relaxation which firstly induces a metric and then its lower semicontinuous envelope. By using the appropriate version of the Pruning Theorem stated in [8], we show that for any α the three functionals are the same, getting in this way the equivalence of the two mentioned approaches and the more abstract formulation of the transport distances introduced here.

More recently, other variational models for branching structures have been proposed in literature (see [6]), these approaches exhibit similar structures to those studied here but keeping substantial differences. Our equivalence proof does not regard, for it could not, the model presented in [6].

The results in this paper do not answer completely the question of the equivalence of the approaches in [17], [3] and in [14], [21], since the functional used here is a variant of those in [17] and in [3], which are also different among themselves. A final part, with a few regularity properties which lead to establish the equivalence of all those variants of the functional in [17], had been planed as a conclusive part of this paper. However, since it requires several arguments and concepts in a different direction with respect to the theory developed here, we have preferred to leave it to a subsequent note [18], which will make the analysis complete.

In this paper we shall introduce a lot of definitions and, in order to avoid a too heavy notation, in some cases we will use the same symbol to denote different things. For instance, some microscopic objects induce their macroscopic counterparts and we will often keep the same notation for the two descriptions. The different use of the two objects and the different context in which they are employed should avoid any possible misunderstanding and let us simplify the exposition.

The paper begins with an introduction to the transport problems according to a kinematic interpretation, rather than the usual holes filling formulation. This allows us to introduce in a natural way the variables employed in [17] and [3], as well as any other concept which will be subsequently used. In the second section we shall introduce the notion of transport distance induced by a transport cost and in Section 3 we shall prove some general theorems obtained under abstract assumptions on a generic transport cost. In Section 4 we specialize these results by considering the particular cost functional which we shall call irrigation cost.

Throughout this paper X and Y will denote closed convex subsets of $\mathbb{R}^N$. However most of the arguments can be trivially extended to a more general setting which includes Polish spaces where no linear structure is needed or dual Banach spaces equipped with the weak* topology otherwise. We shall need to use some results at this higher level of generality and in such cases the assumption that X is a Polish space will be explicitly mentioned in the hypotheses, while in absence of any specification the assumption that X and Y are closed convex subsets of $\mathbb{R}^N$ will be always implicitly assumed.

We shall denote by Γ any space of X valued curves defined on an interval $I \subset \mathbb{R}$. Moreover, we shall denote by $\mathcal{P}(X)$, $\mathcal{P}(\Gamma)$, $\mathcal{P}(\mathbb{R}^N)$ the spaces of the probability measures defined respectively on X, Γ, $\mathbb{R}^N$. Though we shall refer to probability measures, we essentially need to work with positive Radon measures with finite total mass. Indeed, throughout the paper we will often use splitting operations or decompositions of the measures and nevertheless we will continue to refer to them as probability measures, by assuming that an underlying normalization operation has been made. This choice avoids a useless further notation and fits the standard setting adopted in the existing literature on the subject of this work.

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1. Mass transportation problems and kinematic interpretation

1.1. **The Monge-Kantorovitch problem and basic kinematic tools.** The classical setting of the *optimal mass transportation* deals with the problem, originally posed by Monge ([19]), of the minimization of the cost needed to transport a given mass of material from a starting placement to a final one (a pile of sand to some holes). Then, if the two prescribed initial and final distributions of mass are represented by two probability (by normalization) measures μ and ν, Monge problem relies in searching a transport map $T : \mathbb{R}^N \to \mathbb{R}^N$ which minimizes the transport cost

$$J_{\mathrm{M}}(T) = \int_{\mathbb{R}^N} c(x, T(x)) d\mu(x), \tag{1.1}$$

among all the admissible maps T, i.e. such that $T_\# \mu = \nu$, where we denote by $T_\# \mu$ the *push-forward measure* or *image measure* of μ through T, defined by

$$T_\# \mu(B) = \mu(T^{-1}(B)) \quad \text{for every} \quad B.$$

As it is well known, (see e.g. [1], [11], [12] [20], [22]), without suitable assumptions on the measures μ, ν and on the map $c : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, the problem may fail to admit any solution or this can be not unique or even the set of admissible maps can be empty. Then a natural generalization of the problem was introduced by Kantorovitch ([16]), whose main idea was to look for a measure on $\mathbb{R}^N \times \mathbb{R}^N$ instead of a map and thus the transport problem becomes a minimization problem for a cost functional defined on the space of admissible probability measures in two variables, i.e. probability measures with given marginals. Formally, let $\Pi(\mu, \nu) \subset \mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N)$ be the set of the probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals μ and ν, i.e. $p^0_\# \pi = \mu$, $p^1_\# \pi = \nu$, where $p^0 : (x_1, x_2) \mapsto x_1$ and $p^1 : (x_1, x_2) \mapsto x_2$ are the projections. A measure $\pi \in \Pi(\mu, \nu)$ is called a *transport plan* and $\Pi(\mu, \nu)$ is the set of transport plans between μ and ν. Let us notice that every transport map T induces a transport plan π_T through the formula $\pi_T = (Id, T)_\# \mu$, Id being the identity map of $\mathbb{R}^N$, and in this sense the transport plans can be seen as a generalization of the functions. This leads to the following weak formulation of the optimal mass transport, which is called Monge-Kantorovich problem, and asks for the minimization of the functional

$$J_{\mathrm{MK}}(\pi) = \int_{\mathbb{R}^N \times \mathbb{R}^N} c(x, y) d\pi(x, y), \tag{1.2}$$

among all the admissible measures $\pi \in \Pi(\mu, \nu)$ (see [7], [20], [22]).

Here we shall pursue a line of thoughts which can be better explained by a different physical interpretation of the problems modeled in the framework of mass transportation, as above introduced. More precisely, we can think to any probability (by normalization) measure as to an appropriate description of any material body in any of its configurations in the physical space. Thus the measures μ and ν, previously introduced, can be viewed as the initial and the final configurations of a material body undergoing to a transplacement, generally intended as a change of the density distribution. Obviously any $\mu \in \mathcal{P}(X)$ can be thought as the placement of a material body in X (but the physical interpretation is suited for $X = \mathbb{R}^N$ and, in particular, for $N = 3$). This point of view allows to deal with material points, represented by Dirac masses, as well as with continuous distribution of matter in their more general evolutions, regardless any topological restriction on the accessible spatial configurations. The sand piles considered by Monge are, of course, a particular case of physical bodies and the holes represent potential future positions (actually, in the Monge formulation, the mass density is replaced by a volume density on a two dimensional projection). Summarizing, we can say that the placement of a material body of normalized mass is represented by a probability measure, while a (macroscopic) change of position (transplacement) is represented by a pair of probability measures, which can be taken, in general, in two Polish spaces X and Y. In the Monge problem a macroscopic change of position (μ, ν) is assigned. The term *macroscopic* has been used in order to point out that we are just looking to the change of mass distributions and not to the displacements of the single particles. For instance, we consider null a macroscopic change of position if the mass density remains the same while the particles change place, as it happens, for instance, in the case of a rotation around the center of a spherical body of uniform density. On the other hand, if we want to assign a *microscopic* change of position of a body we must specify the change of position of every single particle, namely the density of mass which moves from any given point x to any other given point y. So this description will need a measure on the product space $X \times Y$ of the pairs (x, y) and this is precisely the concept of *transport plan*, which can be therefore intended as a *microscopic* change of position. Note that, roughly speaking, the passage from the *macro* description to the *micro* description requires something like a change of the order in the use of the concepts of measures and pairs, indeed a measure on pairs is used instead of a pair of measures. The same circumstance will be present in the other passages macro-micro which will be considered in the sequel. Clearly, every microscopic change of position induces a macroscopic change of position since the knowledge of the displacement of every particle allows to know the change of the global distribution of mass. This fact is reflected in the operation which to any transport plan π associates the pair of its marginals (μ, ν), which represent the macroscopic synthesis of the microscopic change of position π. From this point of view, the Monge-Kantorovich problem consists in searching the cheapest microscopic displacement, according to the cost (1.2), which induces the macro displacement (μ, ν).

We can coherently define a macroscopic motion as a continuous change of placements, namely as a narrowly continuous (see Section 1.6 below) curve $\mu : t \mapsto \mu(t) \in X$, for t varying in a given interval $I \subset \mathbb{R}$. In contrast, if we look at the micro description and we are interested in describing the motion of the elementary particles, we need a tool able to deal with the individual trajectories of the particles during the time interval I. This tool is just given by a measure on the space of trajectories. Therefore, let Γ be the space of continuous curves $\gamma : I \to X$ equipped with the topology of the local uniform convergence. Let us notice that although this notion of convergence could appear as a metric concept because of the uniformity requirement, it is a topological concept which only depends on the topology on X. Indeed, given any metric which induces the topology of X, we have that γ_n locally uniformly converges to γ if and only if

$$\text{(1.3)} \qquad for\ every\ t_n \to t \in I\ \ \gamma_n(t_n) \to \gamma(t).$$

Let $\mathcal{P}(\Gamma)$ be the space of probability measures on Γ, we define a *microscopic motion* or, equivalently, a *particle motion* as any $\sigma \in \mathcal{P}(\Gamma)$.

Let us observe that, as one can expect, *every microscopic motion induces a macroscopic motion*. Indeed, for every $t \in I$, let $p_t : \Gamma \to X$ be given by $p_t(\gamma) = \gamma(t)$ and let $\sigma \in \mathcal{P}(\Gamma)$ be a given microscopic motion. Then for every $t \in I$, by setting, with an evident but harmless abuse of notation,

$$\text{(1.4)} \qquad \sigma(t) = (p_t)_\# \sigma,$$

we get the macroscopic motion $\sigma : t \mapsto \sigma(t)$ induced by the particle motion σ. It is evident that, conversely, given a macroscopic motion σ we have, in general, many possible microscopic motions which induce it.

For instance, in the simple case of a body made of a finite number n of material points, each one of mass m_i $(i = 1, \ldots, n)$, which describe orbits $\gamma_i : t \mapsto \gamma_i(t) \in \mathbb{R}^N$, the particle motion is represented by the sum of n Dirac masses on Γ, placed on the orbits γ_i and of masses m_i; the position of the body at the time t is given by the measure $\mu(t)$ equal to the sum of n Dirac masses in $\mathbb{R}^N$, located at the points $\gamma_i(t)$, with masses m_i. The macroscopic description of such a motion is just given by $t \mapsto \mu(t) \in \mathcal{P}(\mathbb{R}^N)$.

In this kinematic framework, in the case $X = Y$, we can also look at a transport plan $\pi \in \Pi(\mu, \nu)$ as to a special microscopic motion. Indeed, by interpolating the pairs of points with the uniform rectilinear motions joining them, we can view transport plans as the particle motions concentrated on uniform rectilinear orbits. More precisely, let us take $I = [0, 1]$ (or any other bounded interval) and, for every $x, y \in X$, let $i : (x, y) \mapsto r_{xy}$, where r_{xy} is the uniform rectilinear motion $r_{xy} : t \mapsto ty + (1 - t)x$. The map i is a bijection between the space of pairs $X \times X$ and the space of orbits of uniform rectilinear motions Γ_r defined on $I = [0, 1]$, so if π is a transport plan, $i_\# \pi \in \mathcal{P}(\Gamma_r)$. Conversely, given a particle motion concentrated on the uniform rectilinear motions $\sigma \in \mathcal{P}(\Gamma_r)$, then $i^{-1}_\# \sigma \in \mathcal{P}(X \times X)$, namely it is a transport plan. Therefore, in such a case we can identify the transport plans with the particle motions in which every particle moves with a uniform rectilinear motion and their marginals with the endpoints of the corresponding macroscopic motion. Thus, every result established for particle motions can be referred, in particular, to transport plans. In general, if X and Y are general Polish spaces, any transport plan $\pi \in \mathcal{P}(X \times Y)$ can be only viewed as a discrete version of a particle motion.

If $\sigma \in \mathcal{P}(\Gamma)$ is a microscopic motion, we can consider the restriction σ_s of σ to the constant orbits $\{\gamma \in \Gamma \mid \gamma(t) = const.\ \forall t \in I\}$ and set $\sigma_m = \sigma - \sigma_s$ as the restriction of σ to the non constant orbits. With this notation we can split σ as

$$\text{(1.5)} \qquad \sigma = \sigma_s + \sigma_m.$$

By applying (1.4), the macroscopic motion induced by the microscopic motion σ can be decomposed as

$$\sigma(t) = \sigma_s(t) + \sigma_m(t), \ \ \forall t \in I$$

and, since σ_s is concentrated on the constant orbits, we have $\sigma_s(t) = \sigma_s$ for every t, then we can write

$$\sigma(t) = \sigma_s + \sigma_m(t), \ \ \forall t \in I. \tag{1.6}$$

The splitting (1.5) obviously applies to any transport plan $\pi \in \mathcal{P}(X, X)$ since it can be viewed as a particular microscopic motion. In this case we observe that π_s is the restriction of π to the diagonal set $\{(x, y) \in X \times X \mid x = y\}$.

Let $\pi \in \mathcal{P}(X \times Y)$ be any transport plan and let $\vartheta : X \times Y \to Y \times X$ defined by $\vartheta(x, y) = (y, x)$, we define the *symmetryc transport plan* $\pi^s \in P(X \times Y)$ as $\pi^s = \vartheta_\# \pi$.

1.2. **Restrictions and composition of microscopic motions.** Let $I \subset \mathbb{R}$ and let Γ be as above. If $J \subset I$ is any given subinterval and Γ_J is the space of the continuous curves defined on J, then the restriction map $R : \Gamma \to \Gamma_J$ is defined by $R(\gamma) = \gamma_{|J}$, where $\gamma_{|J}$ is the restriction of γ to J. Then, if $\sigma \in \mathcal{P}(\Gamma)$ is a particle motion on I, we can define the restriction of σ to J defined as $\sigma_J = R_\# \sigma$. We consider a partition of I in two subintervals $I = I_1 \cup I_2$ and we set $\bar{t} = \max I_1 = \min I_2$. Let Γ_1 and Γ_2 be the spaces of the continuous curves respectively defined on I_1 and I_2. Given two particle motions σ_1 and σ_2 on the time intervals I_1 and I_2 respectively, we say that a particle motion σ defined on the time interval I is a composition of σ_1 and σ_2 if these ones are the restrictions of σ to I_1 and I_2 respectively. We ask when two given σ_1 and σ_2 can be composed. Let us notice that two curves γ_1 and γ_2 can be composed if $\gamma_1(\bar{t}) = \gamma_2(\bar{t})$ and we shall say that two curves satisfying this condition are *compatible*. The set

$$\mathcal{C} = \big\{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid \gamma_1(\bar{t}) = \gamma_2(\bar{t})\big\}$$

is the set of the compatible pairs. We observe that a composition of microscopic motions induces the composition of the relative macroscopic motions, then the macroscopic compatibility condition $\sigma_1(\bar{t}) = \sigma_2(\bar{t})$, like the previous one stated for curves in $\mathbb{R}^N$, is necessary for such a composition exist.

We define the double restriction map $\mathbf{R} : \Gamma \to \Gamma_1 \times \Gamma_2$, $\mathbf{R} = (R^1, R^2)$, as follows:

$$\mathbf{R}(\gamma) = \left(\gamma_{|I_1}, \gamma_{|I_2}\right),$$

for every $\gamma \in \Gamma$. Through the map $\mathbf{R}$ we can see that the compatibility condition is also sufficient. Notice that $\mathbf{R}$ induces a bijection between Γ and $\mathcal{C}$. We have that for any $\sigma \in \mathcal{P}(\Gamma)$ $\pi = \mathbf{R}_\# \sigma \in \mathcal{P}(\Gamma_1 \times \Gamma_2)$ is a transport plan between Γ_1 and Γ_2 concentrated on $\mathcal{C}$. Conversely, if $\pi \in \mathcal{P}(\Gamma_1 \times \Gamma_2)$ is concentrated on $\mathcal{C}$, then $\sigma = R^{-1}_\# \pi \in \mathcal{P}(\Gamma)$. Then, through the restriction map $\mathbf{R}$, we have a canonical way to pass from a microscopic motion on I to a transport plan which has its restrictions (microscopic motions defined on the subintervals I_i) as marginals.

Therefore, if we want, given $\sigma_1 \in \mathcal{P}(\Gamma_1)$ and $\sigma_2 \in \mathcal{P}(\Gamma_2)$, to find a composition σ, we must look for a transport plan π between Γ_1 and Γ_2 with marginals σ_1 and σ_2, i.e $\pi \in \Pi(\sigma_1, \sigma_2)$, concentrated on $\mathcal{C}$. Notice that the simplest way to take a transport plan π having σ_1 and σ_2 as marginals is accomplished by taking $\pi = \sigma_1 \otimes \sigma_2$ but, to the aim of keeping the compatibility property, we must have such a π concentrated on $\mathcal{C}$ and so we cannot simply take the tensor product. Then we proceed as follows: firstly, let us use the compatibility condition $\sigma_1(\bar{t}) = \sigma_2(\bar{t})$ in order to set

$$\hat{\sigma} = \sigma_1(\bar{t}) = \sigma_2(\bar{t}). \tag{1.7}$$

Now, let us take the disintegration (see [2, Theorem 6.4.1]) of σ_1 and σ_2 with respect to $\hat{\sigma}$, namely

$$\sigma_1 = \int_X (\sigma_1)_x d\hat{\sigma}, \quad \sigma_2 = \int_X (\sigma_2)_x d\hat{\sigma}.$$

Then, for every x we have to find a transport plan in $\Pi((\sigma_1)_x, (\sigma_2)_x)$ and this is given, among the others, by $(\sigma_1)_x \otimes (\sigma_2)_x$. Therefore, with this choice, a transport plan in $\Pi(\sigma_1, \sigma_2)$ concentrated on $\mathcal{C}$ is given by

$$\pi = \int_X (\sigma_1)_x \otimes (\sigma_2)_x d\hat{\sigma}.$$

So,

$$\sigma = \mathbf{R}_{\#}^{-1} \pi \in \mathcal{P}(\Gamma)$$

gives a microscopic motion on Γ starting from two compatible microscopic motions on Γ_1 and Γ_2. This composition operation can be iterated to any finite or countable set of compatible particle motions. Indeed, let $I = \cup_i [t_{i-1}, t_i]$, if for every index i the compatibility condition

$$\sigma_i(t_i) = \sigma_{i+1}(t_i) \tag{1.8}$$

is satisfied by the macroscopic motions $t \mapsto \sigma_i(t)$ induced by the microscopic motions σ_i defined on $I_i = [t_{i-1}, t_i]$, then we can define a microscopic motion on I by taking, at each step k, the composition of the microscopic motions defined on $\cup_{i=1}^k [t_{i-1}, t_i]$.

1.3. **Multiple plans.** Whence a transport plan can be regarded as a microscopic motion, the above composition operation can be carried out on transport plans, provided the compatibility condition (1.8) is satisfied.

For every $k \in \mathbb{N}$, let us call *chain* any finite ordered sequence $(\mu_1, \mu_2, \dots, \mu_k) \in \mathcal{P}(X_1) \times \ldots \times \mathcal{P}(X_k)$. A chain of transport plans $\pi = (\pi_1, \pi_2, \dots, \pi_k) \in \mathcal{P}(X_1 \times Y_1) \times \ldots \mathcal{P}(X_k \times Y_k)$ will be said a *compatible chain* if, for $i = 2, \dots, k$, $X_i = Y_{i-1}$ and $p^1_{\#}(\pi_{i-1}) = p^0_{\#}(\pi_i)$. A compatible chain of transport plans represents a discrete microscopic motion. If $X_i = Y_i$ for every i, we can regard the transport plans π_i as microscopic motions on intervals $[t_{i-1}, t_i]$ and in such a case the chain turns out to be compatible if and only if the compatibility condition (1.8) is satisfied.

Moreover, given a compatible chain of transport plans $\pi = (\pi_1, \dots, \pi_k)$, we shall say that the chain $\xi = (\xi_0, \xi_1, \dots, \xi_k)$ is the chain of the marginals or of the vertices of π if for every $i = 1, \dots, k$, the component π_i has ξ_{i-1} and ξ_i respectively as marginals. Given ξ, we shall denote by $\Pi(\xi_0, \dots, \xi_k)$ the set of the admissible chains of transport plans corresponding to the chain of vertices ξ, i.e. the compatible chains of transport plans which have ξ as the chain of vertices.

Given a compatible chain of transport plans $\pi = (\pi_1, \dots, \pi_k)$, through the above defined composition, we can take the composition $\tilde{\pi}$ of π_i, $i = 1, \dots, k$ which turns out to be a microscopic motion concentrated on piecewise linear trajectories defined on I. Moreover, since any piecewise linear curve can be uniquely identified by its vertices, such a measure $\tilde{\pi}$ can be viewed as a measure on the set of ordered $(k+1)$-ples of points in X and so $\tilde{\pi} \in \mathcal{P}(X \times \cdots \times X)$. The $k+1$ marginals of such a measure $p^i_{\#}\tilde{\pi}, i = 1, \dots, k+1$ are the vertices ξ_i of the chain π and represent the macroscopic synthesis of π or $\tilde{\pi}$. We shall call (following the terminology of [2]) the measure $\tilde{\pi}$, so obtained, *multiple plan*. The above construction can be carried out on any sequence of Polish spaces $X_0, X_1, \dots, X_k$, even if, in such a case, we do not fall in the context of the previous section and we cannot speak of piecewise linear motion but only of discrete motions, which can only be interpolated in a piecewise linear way in presence of a vector structure. Indeed, in the kinematic picture, a sequence of k measures represents a discrete macroscopic motion, while a multiple plan, or equivalently, a measure on chains of points represents a discrete microscopic motion.

1.4. **Scaled transport plans.** When $X = Y$ we focus on another point of view on the transport plans, which will be involved in the sequel of this paper. As we have observed, any transport plan $\pi \in \mathcal{P}(X \times X)$ can be viewed as a particle motion in which each particle performs a rectilinear trajectory from the starting point $x \in X$ o the final one $y \in X$. In order to record the direction of this motion we introduce the mapping $\mathbf{d} : X \times X \to X \times \mathbb{R}^N$ defined by

$$\mathbf{d} : (x, y) \mapsto (x, \mathbf{v}), \quad \text{with } \mathbf{v} = (y - x),$$

which will be called *displacement* function. Let $\pi \in \mathcal{P}(X \times X)$ be any transport plan, the measure $\mathbf{d}_{\#}\pi$ is in $\mathcal{P}(X \times \mathbb{R}^N)$ and gives a description of π in terms of the variables $(x, \mathbf{v})$. Whenever any transport plan is expressed in such a way we shall refer to it as to a *displacement plan* to emphasize the displacement involved in it. Notice that the displacement plan $\mathbf{d}_{\#}\pi$ uses the direction in which the mass located at x moves for π. For every $\lambda \le 1$, we set $s_\lambda(x, \mathbf{v}) = (x, \lambda\mathbf{v})$ and we define the *scaled displacement plan* $\pi_\lambda = (s_\lambda)_{\#}\pi$. Since every transport plan can be viewed as a displacement plan and conversely, given a transport plan π, we can use the corresponding notion of *scaled transport plan* by taking $\pi_\lambda = (s'_\lambda)_{\#}\pi$, where $s'_\lambda : (x, y) \mapsto (x, x + \lambda(y - x))$, for $\lambda \in \mathbb{R}$.

1.5. **Lagrangian parameterizations.** The description of the position of a material body can be made by using a rest configuration of the body, so let (Ω, μ_Ω) be a given probability space which, in this kinematic interpretation, can be viewed as the reference configuration of a material body. If $\mu \in \mathcal{P}(X)$ is any placement of the material body, we can refer it to the rest configuration by using the so called *lagrangian description*.

Definition 1.1. *Let μ be a positive measure on X. We shall say that $f : \Omega \to X$ is a lagrangian parametrization of μ if $f_{\#}\mu_\Omega = \mu$.*

Thus the mapping f can be thought as a weak version of what in the classical setting of continuum mechanics is called a *deformation*. Note that a measure μ on X is a Borel measure if and only if f is a measurable map. If f is a lagrangian parametrization of μ we shall also say that f *induces* μ. By choosing a different configuration of the body, we can get a new lagrangian parametrization of the same placement μ. If two different lagrangian parameterizations f and g induce the same placement, then f and g will be called *equivalent* lagrangian parameterizations.

Remark 1.1. *If (Ω, μ_Ω) has no atom and $\mu_\Omega = 1$ we can find a lagrangian parametrization $f : \Omega \to X$ of any given probability measure μ. The proof of this assertion, or even more general versions, is an easy variant of* [17, Lemma 9.1]*, anyway it can be recovered by combining* [17, Lemma 9.1] *and* [10, Theorem 11.7.5].

In particular, a lagrangian parametrization of a transport plan $\pi \in \mathcal{P}(X \times Y)$ is given by a measurable function $g : \Omega \to X \times Y$, which amounts to assign two measurable functions $g^1 : \Omega \to X$ and $g^2 : \Omega \to Y$. More in general, for a multiple plan $\pi \in \mathcal{P}(X_1 \times \ldots \times X_k)$ a lagrangian parametrization is given by a measurable function $g : \Omega \to X_1 \times X_k$, which amounts to assign k measurable functions valued in X_i, for $i = 1 \ldots k$.

Analogously, in dealing with the microscopic motion σ of a material body we can obtain a lagrangian parametrization by using a map $\hat{\chi} : \Omega \to \Gamma$. Note that assigning $\hat{\chi}$ is equivalent to give a map $\chi : \Omega \times I \to \chi(p, t) \in X$ such that for a.e. material point $p \in \Omega$, $\chi_p : t \mapsto \chi(p, t)$ is continuous. Indeed χ is induced by $\hat{\chi}$ by setting $\chi(p, t) = [\hat{\chi}(p)](t)$ and, conversely, $\hat{\chi}$ is induced by χ as the map from Ω to Γ defined by $\hat{\chi} : p \to \chi_p$. Under this identification, we can consider the following definition as a particular case of Definition 1.1.

Definition 1.2. *Let σ be a positive measure on Γ. We shall say that $\chi : \Omega \times I \to X$ is a lagrangian parametrization of σ if $\hat{\chi}_{\#}\mu_\Omega = \sigma$.*

As just previously observed, checking the Borel regularity of a measure is equivalent to check the measurability of its lagrangian parameterizations $\hat{\chi}$. Furthermore, in terms of the lagrangian parametrization χ, we state the following assertion.

Proposition 1.1. *Let X be a Polish space, let σ be a positive measure on Γ and let $\chi : \Omega \times I \to X$ be any lagrangian parametrization of σ. Then the following statements are equivalent:*

i) *σ is a Borel measure (i.e. σ is a microscopic motion);*

ii) *χ is a measurable map ;*

iii) *for every* $t \in I$ $\chi(\cdot, t)$ *is a measurable map.*

Proof. *i)* $\Rightarrow$ *iii)* Fix $t \in I$. The map $p_t : \Gamma \to X$, defined by $p_t : \gamma \mapsto \gamma(t)$, is continuous and then $(p_t)_\# \sigma$ is a Borel measure. Moreover,

$$(p_t)_\# \sigma = (p_t)_\# (\hat{\chi}_\# \mu_\Omega) = (p_t \circ \hat{\chi})_\# \mu_\Omega = \chi(\cdot, t)_\# \mu_\Omega.$$

Therefore $\chi(\cdot, t)_\# \mu_\Omega$ is a Borel measure and then $\chi(\cdot, t)$ is a measurable map.

iii) $\Rightarrow$ *i)* Let $d : X \times X \to \mathbb{R}_+$ be a distance inducing the topology on X. Since $C(I)$ is separable, it is enough to show that, for K compact subset of I, $r > 0$ and $g \in C(I)$ fixed, the set

$$\Omega_{K,r} = \{p \in \Omega \mid d(g(t) - \chi(p, t)) \leq r, \ \forall t \in K\}$$

is measurable. To this aim, let $N \subset K$ be a countable set such that $K \subset \overline{N}$ and for every $t \in N$ let $\Omega_t = \{p \in \Omega \mid d(\chi(p, t) - g(t)) \leq r\}$. By continuity we have $\Omega_{K,r} = \bigcap_{t \in N} \Omega_t$ and, since for every $t \in N$ Ω_t is measurable, we get the claim.

ii) $\Rightarrow$ *iii)* Since χ is measurable, by Fubini Theorem ([13, Theorem 6.46]) we have that a.e. section of χ is measurable, namely for a.e. $t \in I$ $\chi(\cdot, t)$ is measurable. Since χ is continuous with respect to the variable t, we obtain the claim.

iii) $\Rightarrow$ *ii)* Fix $n \in \mathbb{N}$ and take a partition of I made of contiguous and disjoint subintervals I_i having endpoints a_i and $b_i = a_{i+1}$, with $|a_i - b_i| \leq \frac{1}{n}$ for every i. Let us define $\chi_n(p, t) = \chi(p, a_i)$ for $t \in I_i$. Now, χ_n is a measurable map and, since the fibers are continuous, we get $\chi_n \to \chi$ and so χ is measurable. □

Let us notice that, if Ω is a probability space with no atom and σ is a positive measure on Γ, by Remark 1.1 there exists a lagrangian parametrization of σ, namely $\hat{\chi} : \Omega \to \Gamma$ and so a lagrangian parametrization of σ given by the corresponding $\chi : \Omega \times I \to X$.

Remark 1.2. *We remark that in the following we shall use microscopic motions or lagrangian parameterizations of microscopic motions by considering them as two equivalent descriptions. That is, we can argue in terms of particle motions or lagrangian parameterizations in interchangeable way. Therefore any statement regarding one of these objects can be translated into a statement regarding the other one, as we shall often explicitly do. However, we shall consider any result established for particle motions automatically translated in terms of lagrangian parameterizations and conversely and we shall adopt one of the two descriptions instead of the other one without any further justification, even in the train of the same argument, according to the convenience of the exposition.*

Definition 1.3. *We shall say that a microscopic motion* $\sigma \in \mathcal{P}(\Gamma)$ *is regular if it is concentrated on the set of the absolutely continuous curves.*

Definition 1.4. *We shall say that a lagrangian parametrization* $\chi : \Omega \times I \to X$ *is regular if for a.e.* $p \in \Omega$, $\chi(p, \cdot)$ *is absolutely continuous with respect to* t.

Obviously, if a microscopic motion σ is regular then it only admits regular parameterizations.

Definition 1.5. *Let $\mu, \nu \in \mathcal{P}(X)$ be two given measures. We shall say that a regular $\sigma \in \mathcal{P}(\Gamma)$ is an admissible motion between μ and ν on the interval $I = [a, b]$ if $\sigma(a) = \mu, \sigma(b) = \nu$. Let $\Sigma_I(\mu, \nu)$ denote the set of the admissible σ and $\Sigma(\mu, \nu)$ be the union of $\Sigma_I(\mu, \nu)$ for all the closed bounded intervals $I \subset \mathbb{R}$.*

Proposition 1.2. *Let $\sigma \in \mathcal{P}(\Gamma)$ be regular and let $\chi : \Omega \times I \to X$ be a (regular) lagrangian parametrization of σ. Then for a.e. $t \in I$ χ is differentiable with respect to t for a.e. $p \in \Omega$.*

Proof. Let us point out that the set A of the pairs (p, t) where χ is differentiable with respect to t is a measurable set. Indeed, for every $(p, t) \in \Omega \times I$, let

$$\chi'_+(p, t) = \limsup_{h \to 0} \frac{\chi(p, t + h) - \chi(p, t)}{h},$$

$$\chi'_-(p, t) = \liminf_{h \to 0} \frac{\chi(p, t + h) - \chi(p, t)}{h},$$

with $h \in \mathbb{Q}$. Since $\chi : \Omega \times I \to X$ is measurable, it is easy to check that χ'_+ and χ'_- are measurable and so also the set where they coincide is measurable. By the continuity of χ in the t variable, this set is A. Then $A^c = (\Omega \times I) \setminus A$ is measurable and thus we can apply Fubini Theorem to compute $(\mu_\Omega \times \mathcal{H}^1)(A^c)$ in terms of the sections $S_p = \{t \in I \mid (p, t) \in A^c\}$ and $S^t = \{p \in \Omega \mid (p, t) \in A^c\}$, that is

$$(\mu_\Omega \times \mathcal{H}^1)(A^c) = \int_\Omega \mathcal{H}^1(S_p) d\mu_\Omega(p) = \int_I \mu_\Omega(S^t) d\mathcal{H}^1(t).$$

Since, for a.e. p, χ_p is a.e. differentiable with respect to t, we have $\mathcal{H}^1(S_p) = 0$ for a.e. p and so, by the previous equation, we get $\mu_\Omega(S^t) = 0$ for $\mathcal{H}^1$ a.e. t, which is just a restatement of the thesis. $\square$

If $\sigma \in \mathcal{P}(\Gamma)$ is a microscopic motion and $\chi : \Omega \times I \to X$ is any lagrangian parametrization of σ, we set

$$\Omega_s = \{p \in \Omega \mid \chi_p(t) = const. \ \forall t \in I\}, \quad \Omega_m = \Omega \setminus \Omega_s. \tag{1.9}$$

We denote by μ_s and μ_m the restrictions of μ_Ω to Ω_s and Ω_m respectively and finally we set

$$\chi_s = \chi_{|\Omega_s \times I}, \quad \chi_m = \chi_{|\Omega_m \times I}.$$

Thus, with the notation in (1.5), we have $\sigma_s = \chi_{s\#}\mu_s$ and $\sigma_m = \chi_{m\#}\mu_m$.

1.6. **Convergence of measures.** For the reader's convenience we recall some results, which are well known in Probability Theory literature, about convergence of measures. We state them, without proof, by using the terminology introduced here and we refer to [2], [4], [10] for the proofs and more details.

Let X be a Polish space and let $C_b(X)$ be the space of bounded and continuous real-valued functions defined on X.

Definition 1.6. *A sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(X)$ is narrowly convergent to $\nu \in \mathcal{P}(X)$, in symbols $\nu_n \rightharpoonup \nu$, if*

$$\lim_{n\to\infty} \int_X f d\nu_n = \int_X f d\nu \quad \forall f \in C_b(X).$$

Definition 1.7. *A sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(X)$ satisfies the tightness condition if*

$$\forall \varepsilon > 0 \ \exists\, K_\varepsilon \ \text{ compact in } X \text{ such that } \nu_n(X \setminus K_\varepsilon) < \varepsilon, \quad \forall n \in \mathbb{N}.$$

Theorem 1.1. (**Prokhorov**) *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}(X)$ satisfying the tightness condition. Then it has a narrowly convergent subsequence.*

Theorem 1.2. (**Skorohod**) *Let $(\mu_n)_{n\in\mathbb{N}}$ be any given sequence of probability measures on a Polish space X an let (Ω, μ_Ω) be a given probability space without atoms. Then $\mu_n \rightharpoonup \mu$ narrowly if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of lagrangian parameterizations of μ_n on Ω and there exists f lagrangian parametrization of μ on Ω such that f_n converges to f a.e.*

The previous result is essentially proved in [10, Theorem 11.7.2]. If $X = \Gamma$, Skorohod Theorem can be rephrased for microscopic motions, as stated in the next corollary. Then we say that $\hat{\chi}_n$ converges to $\hat{\chi}$, in symbols $\hat{\chi}_n \to \hat{\chi}$, if $\hat{\chi}_n$ converges to $\hat{\chi}$ a.e. in the topology of Γ or, more explicitly, in terms of the map χ, as in the next definition.

Definition 1.8. *Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence of lagrangian parameterizations defined on $\Omega \times I$ of microscopic motions. We say that χ_n converges to χ fiberwise, in symbol $\chi_n \to \chi$, if for a-e. $p \in \Omega$ $(\chi_n)_p \to \chi_p$ uniformly on compact subsets of I.*

Corollary 1.3. *Let $(\sigma_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\Gamma)$ be any given sequence of microscopic motions. Then $\sigma_n \rightharpoonup \sigma$ narrowly if and only if, for every n, there exists a lagrangian parametrization χ_n of σ_n such that $\chi_n \to \chi$, where χ is a lagrangian parametrization of σ.*

2. Transport costs and transport distances

2.1. **Macroscopic and microscopic transport costs.** Let us introduce the notion of *transport cost* as a positive functional defined on transplacements. More specifically, we shall define a *macroscopic transport cost* as a positive functional defined on macroscopic transplacements, namely on pairs of probability measures,

$$C : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

which satisfies the symmetry property $C(\mu,\nu) = C(\nu,\mu)$ for every μ, ν in $\mathcal{P}(X)$ and a *microscopic transport cost* as a positive functional defined on microscopic transplacements, namely on transport plans,

$$c : \mathcal{P}(X \times X) \to \mathbb{R}_+ \cup \{+\infty\}$$

which satisfies the symmetry property $c(\pi) = c(\pi^s)$ for every $\pi \in \mathcal{P}(X \times X)$. Then, the micro-macro relation, previously observed dually, reflects here the fact that *every macroscopic cost induces a microscopic cost.* On the other hand, we can consider the optimal macroscopic cost C induced by a given microscopic cost c, defined by

$$C(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} c(\pi). \tag{2.10}$$

Whenever the transport cost c will be defined on a restricted class of transport plans we shall take $c = +\infty$ out of this class.

Let C be a given macroscopic transport cost and let

$$\mathbb{A}_C = \{(\mu,\nu) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid C(\mu,\nu) < +\infty\},$$

we shall denote by D_C the domain of the relation $\mathbb{A}_C$.

2.2. **Transport distances.**

Definition 2.1. *A macroscopic transport cost* $d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ *is a* transport distance *if the following conditions hold true:*

- D1. *d is weakly lower semicontinuous (w.l.s.c.), that is l.s.c. with respect to the narrow convergence;*
- D2. *d satisfies the triangle inequality, i.e. $d(\mu,\nu) \le d(\mu,\zeta) + d(\zeta,\nu)$, for every μ, ν, ζ in $\mathcal{P}(X)$.*

Some remarkable examples of transport costs and transport distances are listed below.

Let $p \ge 1$, for every $\pi \in \mathcal{P}(X \times X)$, we define the microscopic cost

$$\|\pi\|_p = \left(\int_{X \times X} d(x_1,x_2)^p d\pi \right)^{\frac{1}{p}}.$$

Example 2.1. *The p-Wasserstein distance W_p between the measures μ and ν is defined as the optimal macroscopic cost induced by $\|\pi\|_p$, namely*

$$W_p(\mu,\nu) = \min_{\pi \in \Pi(\mu,\nu)} \|\pi\|_p. \tag{2.11}$$

It is worth to notice that the p-Wasserstein distance induces the L^p distance on the space of the parameterizations defined on a given Ω. Indeed, if $\pi = (f \times g)_{\#}\mu_\Omega$, then

$$\int_{X\times X} |x-y|^p d\pi = \int_\Omega |f(x)-g(x)|^p dx.$$

Following [10, Section 11.3], let us consider another example of transport distance. Let $A \subset X$ and $\varepsilon > 0$, we set

$$N_\varepsilon(A) = \{x \in X \mid d(x, A) < \varepsilon\}.$$

Example 2.2. *Let $\mu, \nu \in \mathcal{P}(X)$, the Prohorov distance ρ between the measures μ and ν is defined as*

$$\rho(\mu,\nu) = \inf\{\varepsilon > 0 \mid \mu(A) \le \nu(N_\varepsilon(A)) + \varepsilon \text{ for any Borel set } A \subset X\}.$$

Notice that if μ and ν have the same total mass then the symmetry property $\rho(\mu,\nu) = \rho(\nu,\mu)$ can be proved.

Let us remark that the Wasserstein distance metrizes the narrow topology on $\mathcal{P}(X)$ for X bounded, while the Prohorov distance does the same without any useless boundedness assumptions. In alternative we can employ the following distance which will be used several times in the sequel.

Example 2.3. *Let $\mu, \nu \in \mathcal{P}(X)$, we define the weak distance d_W as*

$$d_W(\mu,\nu) = \inf_{\alpha\in\mathbb{R}} \{\mu = \mu_1 + \mu_2,\ \nu = \nu_1 + \nu_2 | W_\infty(\mu_1,\nu_1) \le \alpha,\ \mu_2(X) \le \alpha, \nu_2(X) \le \alpha\}.$$

We leave to the reader to check that d_W satisfies the triangle inequality, this can be proved, for instance, by using a disintegration argument as [2, Theorem 6.4.1].

2.3. **Transport distance induced by a macroscopic cost and chain distance.**

Definition 2.2. *Let C be a given macroscopic transport cost and let $\mathcal{D}_C$ be the set of the transport distances d such that $d \le C$. We define the transport distance induced by C, d_C, as*

$$d_C(\mu,\nu) = \sup_{d\in\mathcal{D}_C} d(\mu,\nu). \tag{2.12}$$

Then d_C is defined as the relaxation of C in $\mathcal{D}_C$ and so it is the greatest weakly l.s.c. transport cost satisfying the triangle inequality and less or equal to C.

Let us point out that the relaxation in $\mathcal{D}_C$ is achieved by taking simultaneously two envelopes, indeed we take the greatest functional less or equal to C which, at the same time, satisfies the triangle inequality and is weakly l.s.c.. If we relax the function in two different steps by taking the envelope with respect to one of the two properties and then the envelope of this new function with respect to the other property, we may not get the same result. Indeed, in general the two properties are not preserved under a relaxation made with respect to the other one, as we are going to show in the following counterexamples.

Example 2.4. Let us take the function $d : \mathbb{R} \to \mathbb{R}_+$ defined by

$$d(x,y) = \begin{cases} |x-y| & \text{if } xy > 0 \text{ or } x = y = 0 \\ |x-y| + 1 & \text{otherwise.} \end{cases}$$

Let us note that d satisfies the triangle inequality but it is is not l.s.c., indeed let us fix $x < 0$ and consider $d(x, y_n)$ with $y_n \to 0^-$. We get $d(x, y_n) \to |x| < d(x, 0) = |x| + 1$. By relaxing, we get the function d^* defined as

$$d^*(x,y) = \begin{cases} |x-y| & \text{if } xy \geq 0 \\ |x-y| + 1 & \text{otherwise.} \end{cases}$$

Since $d^*(-1, 1) = 3 > 2 = d(-1, 0) + d(0, 1)$, d^* does not keep the triangle inequality, which is therefore not preserved under the relaxation operation.

Example 2.5. Let us consider $Q_2 = [0, 1]^2$ endowed with the natural topology and let us take the function $d : Q_2 \to \mathbb{R}_+$ given by

$$d(x,y) = \begin{cases} 1 & \text{if } x_2 \neq y_2, \\ |x_1 - y_1|^{1+x_2} & \text{if } x_2 = y_2. \end{cases}$$

The function d is l.s.c. but it does not satisfy the triangle inequality, if we take $\hat{d} = \sup_{g \in \mathcal{D}} g(x, y)$, where $\mathcal{D}$ denotes the set of the functions less or equal than d and satisfying the triangle inequality, we have

$$\hat{d}(x,y) = \begin{cases} 1 & \text{if } x_2 \neq y_2, \\ 0 & \text{if } x_2 = y_2 \neq 0, \\ |x_1 - y_1| & \text{if } x_2 = y_2 = 0. \end{cases}$$

It is easy to see that $\hat{d}$ is not l.s.c..

Let us remark that $\mathbb{R}$ and $\mathbb{R}^2$ can be embedded in $\mathcal{P}(\mathbb{R}^N)$ with each point identified with a Dirac mass and the natural topology sent in the narrow topology. Therefore, through this identification, the above counterexamples apply to our situation and so the two steps envelope which we are just going to introduce gives, in general, a different function.

Given a transport cost C, we denote by d_C^* the metric envelope of C, that is the greatest functional less or equal than C, satisfying the triangle inequality.

Definition 2.3. *Let C be a transport cost and let d_C^* be its metric envelope. We define the chain distance induced by C, $\overline{d}_C$, as the l.s.c. envelope of d_C^*.*

Remark 2.1. *In spite of its misleading name, the chain distance is not necessarily a distance. Indeed, it does not need to satisfy the triangle inequality, as shown by the above examples (Example 2.4, in particular). On the contrary, by definition, for every $\mu, \nu \in \mathcal{P}(X)$, we have*

$$d_C(\mu, \nu) \leq \overline{d}_C(\mu, \nu) \tag{2.13}$$

and the equality holds if and only if $\overline{d}_C$ satisfies the triangle inequality. So $\overline{d}_C$ is really a distance only when it is just equal to d_C.

Given any chain $\xi = (\xi_0, \xi_1, \dots, \xi_k)$, we introduce the projections p^+, p^- such that $p^+(\xi) = \xi_0$ and $p^-(\xi) = \xi_k$, moreover, given a macroscopic transport cost C, we define the cost of ξ as

$$C^*(\xi) = \sum_{i=1}^{k} C(\xi_{i-1}, \xi_i). \tag{2.14}$$

Let us recall that, given a microscopic cost c, for any i we get by (2.10) the induced optimal macroscopic cost $C(\xi_{i-1}, \xi_i) = \inf_{\pi_i \in \Pi(\xi_{i-1}, \xi_i)} c(\pi_i)$, from which a cost C^* is in turn induced by (2.14) on the chains. Alternatively, we define the microscopic cost of an admissible chain of transport plans $\tilde{\pi} \in \Pi(\xi_0, \dots \xi_k)$ as

$$c^*(\tilde{\pi}) = \sum_{i=1}^{k} c(\pi_i)$$

and we can characterize the cost of a chain of measures ξ as

$$C^*(\xi) = \inf_{\tilde{\pi} \in \Pi(\xi_0, \dots \xi_k)} c^*(\tilde{\pi}). \tag{2.15}$$

Let $\mathbb{P}(X)$ be the space of the chains on X.

Proposition 2.1. *Let C be a given macroscopic transport cost. Then for every $\mu, \nu \in \mathcal{P}(X)$ the following property holds true.*

$$\overline{d}_C(\mu, \nu) = \liminf_{\substack{\xi \in \mathbb{P}(X) \\ p^+(\xi) \rightharpoonup \mu, \\ p^-(\xi) \rightharpoonup \nu}} C^*(\xi). \tag{2.16}$$

Proof. Firstly we note that

$$d^*_C(\mu, \nu) = \inf_{\substack{\xi \in \mathbb{P}(X) \\ p^+(\xi) = \mu, \\ p^-(\xi) = \nu}} C^*(\xi).$$

Then by relaxing d^*_C we get (2.16). $\square$

Remark 2.2. *From (2.16) we can see clearly why $\overline{d}_C$ is not in general a distance. Indeed, we cannot join a chain used to evaluate $\overline{d}_C(\mu, \zeta)$ and a chain used to evaluate $\overline{d}_C(\zeta, \nu)$ in order to get a good chain for estimating $\overline{d}_C(\mu, \nu)$ because all what we know about the final point of the first chain and the initial point of the second chain is that they are both close to ζ in the narrow topology, so they are close among themselves. This does not allow, in general, to fill the gap between the two chains* *with a third chain with a small C^* cost. If this is the case, $\overline{d}_C$ turns out to be a distance. Nevertheless, we shall see that $\overline{d}_C$ can be a distance even if such a condition is not satisfied.*

2.4. **Integral of the infinitesimal costs.** Let c be any given microscopic cost on X, in the sequel we define the notion of *infinitesimal cost* of a given $\pi \in \mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N)$, which can be viewed as the derivative of c along the direction in which the transport plan π moves the mass. To this aim, we employ the notion of scaled transport plan π_λ introduced in Section 1.4.

Definition 2.4. *We define the infinitesimal cost c_0 as the l.s.c. relaxed of the cost $\overline{c}$ defined on $\mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N)$ as*

$$\overline{c}(\pi) = \limsup_{\lambda \to 0} \frac{c(\pi_\lambda)}{\lambda}.$$

Remark 2.3. *Note that we are defining c_0 on all of $\mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N)$ by assuming also c defined for all the transport plans on $\mathbb{R}^N$ by extending it by $+\infty$ out of $\mathcal{P}(X \times X)$, as assumed in general. Note that c_0 turns out to be valued $+\infty$ out of the set $\mathcal{T}$ of the transport plans π concentrated on the set*

$$T = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N \mid y \textit{ belongs to the tangent cone at } X \textit{ in } x\}. \tag{2.17}$$

So c_0 can be thought as defined on $\mathcal{T}$.

Given any regular microscopic motion $\sigma \in \mathcal{P}(\Gamma)$, for every t by taking the lagrangian parametrization $\mathbf{p}_t : \Gamma \to X \times X$ defined σ-a.e. on Γ

$$\mathbf{p}_t : \gamma \mapsto (\gamma(t), \gamma'(t)),$$

we can define the infinitesimal displacement plan at t as

$$\pi_t = (\mathbf{p}_t)_\# \sigma \in \mathcal{T}.$$

Analogously, if we take $\chi : \Omega \times I \to X$ as a lagrangian parametrization of σ, for a.e. t we set for a.e. p

$$\varphi_t : p \mapsto (\chi(p, t), \frac{\partial \chi}{\partial t}(p, t))$$

and then the infinitesimal displacement plan at t is also given by

$$\pi_t = (\varphi_t)_\# \mu_\Omega. \tag{2.18}$$

Note that π_t can be regarded as a transport plan on $\mathbb{R}^N$. Then, given any regular microscopic motion $\sigma \in \mathcal{P}(\Gamma)$, for a.e. t we define the infinitesimal cost as $c(t) = c_0(\pi_t)$.

Lemma 2.1. *The infinitesimal cost $t \mapsto c(t)$ is a measurable function.*

Proof. We can assume, without any restriction, $I = \mathbb{R}$ by extending every fiber by constant values on the two sides of I. Let

$$f^h : t \mapsto \pi_t^h = \left(\chi(\cdot, t), \frac{\chi(\cdot, t+h) - \chi(\cdot, t)}{h}\right)_{\#} \mu_\Omega.$$

Observe that the continuity of χ implies that $f^h : I \to \mathcal{P}(X \times \mathbb{R}^N)$, with $\mathcal{P}(X \times \mathbb{R}^N)$ endowed with the narrow topology, is continuous and so it is measurable. Since for a.e. $t \in I$ χ is differentiable with respect to t for a.e. $p \in \Omega$, we get that f^h a.e. converges in the narrow topology to π_t as $h \to 0$. Therefore the map $t \mapsto \pi_t$ is measurable, as a pointwise limit of a sequence of measurable mappings valued in a metrizable space. Moreover, the map $\pi \mapsto c_0(\pi)$ is l.s.c. by definition and so it is Borel measurable. So $t \mapsto c(t)$ turns out to be the composition of a measurable map with a Borel measurable map and thus it is measurable. □

For any regular $\sigma \in \mathcal{P}(\Gamma)$ we put

$$J(\sigma) = \int_I c(t)dt = \int_I c_0(\pi_t)dt. \tag{2.19}$$

We shall refer to $J(\sigma)$ as to the *integral of the infinitesimal costs* and, by recalling Remark 1.2, we shall use the notation $J(\chi)$ or $J(\pi)$ when we argue in terms of lagrangian parameterizations χ of microscopic motions or when, in particular, we are considering microscopic motions π concentrated in the space of uniform rectilinear orbits, which can be identified as transport plans.

Definition 2.5. *Let C be a given transport cost and let J be the corresponding integral of the infinitesimal costs. We define kinematic distance induced by C*

$$d_J(\mu, \nu) = \inf_{\sigma \in \Sigma(\mu,\nu)} J(\sigma). \tag{2.20}$$

It is easy to check that the kinematic distance d_J satisfies the triangle inequality. Moreover, let us remark that, by the homogeneity of c_0 with respect to the scaling operation of transport plans, we can replace $\Sigma(\mu, \nu)$ with $\Sigma_I(\mu, \nu)$ in the definition of d_J and the value of $d_J(\mu, \nu)$ does not depend on the choice of the interval I, since we are free to reparametrize t, obtaining the same total cost.

3. *A Priori* BOUNDS AND COMPACTNESS CRITERIA

3.1. **Cost assumptions.** Now we are going to introduce a set of general conditions, (**CA0**) – (**CA4**), involving a given microscopic transport cost functional $c : \mathcal{P}(X \times X) \to \mathbb{R}_+$ and the corresponding functions C, C^* and J constructed from c in Section 2.

Weak continuity

(**CA0**) $\quad \forall\, \varepsilon > 0\, \exists \delta > 0\; s.\,t.\, \forall \mu, \nu \in D_C\;\; s.\,t.\;\; d_W(\mu - \nu) < \delta : d^*_C(\mu, \nu) \leq \varepsilon.$

Coercivity

$$(\mathbf{CA1}) \qquad \exists\, p > 1 \;\; s.\,t.\, \forall \pi \in \mathcal{P}(X \times X) : c(\pi) \geq \|\pi\|_p\,.$$

Interpolation

$$(\mathbf{CA2}) \qquad \forall\, \pi \in \mathcal{P}(X \times X) : J(\pi) \leq c(\pi).$$

Lower semicontinuity

$$(\mathbf{CA3}) \qquad \forall\, \sigma_n \rightharpoonup \sigma \;\; on \;\; I : J(\sigma) \leq \liminf_n J(\sigma_n).$$

Discretization

$$(\mathbf{CA4}) \qquad \begin{gathered} \forall\, \sigma \in \Sigma(\mu, \nu),\ \forall \varepsilon > 0\, \exists\;\, \xi = (\xi_1, \dots, \xi_k)\ s.t. \\ d_W(p^+(\xi), \mu) < \varepsilon, \quad d_W(p^-(\xi), \nu) < \varepsilon, \quad C^*(\xi) < J(\sigma) + \varepsilon. \end{gathered}$$

By virtue of Corollary 1.3 the lower semicontinuity condition (**CA3**) can be rephrased in terms of lagrangian parameterizations of microscopic motions as follows.

$$(\mathbf{CA3'}) \qquad \forall\, \chi_n \rightarrow \chi \;\; on \;\; \Omega \times I : J(\chi) \leq \liminf_n J(\chi_n).$$

Let us remark that if C is a macroscopic cost satisfying (**CA0**), by Proposition 2.1 we infer that $\overline{d}_C$ is a metric. Indeed (**CA0**) is just the sufficient condition discussed in Remark 2.2. So we can state the following assertion.

Lemma 3.1. *If C is a macroscopic cost satisfying (**CA0**) then $\overline{d}_C = d_C$.*

We observe that (**CA0**) is in general not satisfied by the macroscopic costs in applications. Indeed, in Section 4 we will see that the irrigation cost, which depends on a parameter $\alpha \in]0, 1[$, satisfies (**CA0**) only if $\alpha > 1 - \frac{1}{N}$. Nevertheless, we will show the equality of the chain distance and the transport distance for the irrigation cost for every $\alpha \in]0, 1[$.

Let us point out that, if C is a transport cost satisfying (**CA1**), then $c_0(\pi) \geq \|\pi\|_p$ for every transport plan π, since $\|\pi\|_p$ is 1-homogeneous with respect to scalings and w.l.s.c..

The coercivity condition (**CA1**) allows to get a control on the time derivatives of a given microscopic motion σ in terms of the integral of the infinitesimal costs, as stated in the following proposition.

Proposition 3.1. *Let C be a transport cost satisfying (**CA1**). Then for any lagrangian parametrization of a microscopic motion χ and with p as in (**CA1**), the following estimate holds true.*

$$\int_I \int_\Omega \left| \frac{\partial \chi}{\partial t}(q,t)) \right|^p d\mu_\Omega \, dt \leq \int_I c(t)^p dt. \tag{3.21}$$

Proof. Let us consider a lagrangian parametrization χ defined on $\Omega \times I$. At any $t \in I$ let π_t be the infinitesimal transport plan, then we have by (**CA1**) that $c_0(\pi_t) \geq \|\pi_t\|_p$. Moreover, by writing (in the displacement plan version) π_t as in (2.18), we have

$$c(t) \geq \|\pi_t\|_p = \left(\int_\Omega \left| \frac{\partial \chi}{\partial t}(q,t)) \right|^p d\mu_\Omega \right)^{\frac{1}{p}}.$$

Therefore, after integrating with respect to t, the previous relation trivially leads to (3.21). □

As an immediate consequence of the previous result we have that, if c is in $L^p(I)$, then by (3.21) the Sobolev norm of the trajectories χ_q is finite, for a.e. q, that is $\chi_q \in H^p(I)$ and so we can immediately deduce the following statement.

Corollary 3.1. *Let C be a transport cost satisfying (**CA1**) and let $\chi : \Omega \times I \to X$ be any lagrangian parametrization of a given microscopic motion σ. If the infinitesimal cost function $c \in L^p(I)$ and p is as in (**CA1**), then $\chi_q \in C^{1-\frac{1}{p}}(I)$, for a.e. $q \in \Omega$.*

3.2. **Compactness Theorems.**

Theorem 3.2. (**Tightness**) *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a given sequence of particle motions. Let C be a transport cost satisfying (**CA1**) and for every $n \in \mathbb{N}$ let $c_n(t)$ be the infinitesimal cost at $t \in I$ of σ_n. If, for p as in (**CA1**), $(c_n)_{n\in\mathbb{N}}$ is bounded in $L^p(I)$, then, for any given $\bar{t} \in I$, $(\sigma_n)_{n\in\mathbb{N}}$ is tight if and only if $(\sigma_n(\bar{t}))_{n\in\mathbb{N}}$ is tight.*

Proof. If $(\sigma_n)_{n\in\mathbb{N}}$ is tight, then, since $\sigma_n(t) = (p_t)_\# \sigma_n$ for every $n \in \mathbb{N}$ and for every $t \in I$, we have the tightness of $(\sigma_n(\bar{t}))_{n\in\mathbb{N}}$. To prove the other implication, we fix $\varepsilon > 0$ and fix for every n a lagrangian parametrization χ_n of σ_n on $\Omega \times I$. Since $(\sigma_n(\bar{t}))_{n\in\mathbb{N}}$ is tight, for any given $\varepsilon > 0$ we have a compact subset $K_\varepsilon \subset X$ such that, for every n, $[\sigma_n(\bar{t})](X \setminus K_\varepsilon) < \frac{\varepsilon}{2}$, that is for every n we have $A_n \subset \Omega$ such that $\mu_\Omega(A_n) < \frac{\varepsilon}{2}$ and $\chi_n(q,\bar{t}) \in K_\varepsilon$ for every $q \in \Omega \setminus A_n$. Furthermore, by Proposition 3.1, if $\|c_n\|_{L^p(I)} \leq \bar{c}$, we can find another set $B_n \subset \Omega$ such that $\mu_\Omega(B_n) \leq \frac{\varepsilon}{2}$ and the following estimate holds

$$\forall q \in \Omega \setminus B_n : \quad \int_I \left| \frac{\partial \chi_n}{\partial t}(q,t)) \right|^p dt \leq \frac{2\bar{c}^p}{\varepsilon}. \tag{3.22}$$

Let $\Omega_n = \Omega \setminus (A_n \cup B_n)$. Let Q_n be the set of fibers corresponding to points in Ω_n Then $\sigma_n(\Gamma \setminus Q_n) = \mu_\Omega(A_n \cup B_n) < \varepsilon$. By (3.22) and Sobolev Embedding Theorem, we know that the fibers in Q_n are uniformly Hölder continuous and their value in $\bar{t}$

is contained in the compact set K_ε. So we can apply Ascoli-Arzelà Theorem and we get that Q_n is contained in a compact subset of Γ which does not depend on n. $\square$

Theorem 3.2 has two remarkable corollaries which we are going to state and which respectively follow by Prokhorov Theorem (Theorem 1.1) and by Skorohod Theorem (Corollary 1.3).

Corollary 3.3. (**Narrow compactness**) *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a given sequence of microscopic motions on $\mathbb{R}^N$. Let C be a transport cost satisfying (**CA1**) and for every $n \in \mathbb{N}$ let $c_n(t)$ be the infinitesimal cost function of σ_n. If, for p as in (**CA1**), $(c_n)_{n\in\mathbb{N}}$ is bounded in $L^p(I)$ and there exists $\overline{t} \in I$ such that $\sigma_n(\overline{t}) \rightharpoonup \overline{\sigma} \in \mathcal{P}(X)$, then $(\sigma_n)_{n\in\mathbb{N}}$ has a narrowly converging subsequence.*

Corollary 3.4. (**Fiberwise compactness modulo equivalence**) *Let $(\chi_n)_{n\in\mathbb{N}}$ be a given sequence of lagrangian parameterizations on $\Omega \times I$ of microscopic motions on $\mathbb{R}^N$. Let C be a transport cost satisfying (**CA1**) and for every $n \in \mathbb{N}$ let $c_n(t)$ be the infinitesimal cost function of χ_n. If, for p as in (**CA1**), $(c_n)_{n\in\mathbb{N}}$ is bounded in $L^p(I)$ and there exists $\overline{t} \in I$ such that $\chi_n(\cdot, \overline{t})$ converges a.e. to a measurable function χ, then for every $n \in \mathbb{N}$ there exists χ'_n equivalent to χ_n such that $(\chi'_n)_{n\in\mathbb{N}}$ has a fiberwise converging subsequence.*

3.3. **Main abstract results.** Now, in this abstract scheme of assumptions on a general transport cost, we are in a position to prove the following result stating the well posedness of the problem of minimizing the integral of the infinitesimal costs in a given class $\Sigma(\mu, \nu)$. This is achieved, thanks to the previous compactness properties, by applying the direct methods of the calculus of variations.

Theorem 3.5. (**Existence of minimizers**) *Let C be a transport cost satisfying (**CA1**), (**CA3**) and let $\mu, \nu \in \mathcal{P}(X)$ be given. Then there exists $\sigma \in \Sigma(\mu, \nu)$ such that*

$$d_J(\mu, \nu) = J(\sigma).$$

Proof. Fix I and let $(\sigma_n)_{n\in\mathbb{N}}$ be any minimizing sequence for J in $\Sigma_I(\mu, \nu)$. If $d_J(\mu, \nu) < +\infty$, otherwise we have nothing to prove, we know that the sequence of the infinitesimal costs $(c_n)_{n\in\mathbb{N}}$ turns out to be bounded in $L^1(I)$. Under a change of variable $t \mapsto \varphi(t)$, we can get it bounded in L^p for p as in (**CA1**). Then we can apply Corollary 3.3 and so we have a converging subsequence. By virtue of (**CA3**) J is lower semicontinuous and so we get the thesis. $\square$

Besides the previous existence result, we are going to prove the next theorem concerning the equivalence of the distances between measures, introduced here, in the abstract framework based on the cost assumptions before. This result will follow after some lemmas and it constitutes the main goal of this work.

Lemma 3.2. *Let c be a transport cost satisfying (**CA1**) – (**CA3**). Then*

$$d_J \leq \overline{d}_C.$$

Proof. By (2.16) there exists a sequence of chains $(\xi_n)_{n\in\mathbb{N}}$ such that $p^+(\xi_n) \rightharpoonup \mu$, $p^-(\xi_n) \rightharpoonup \nu$ and $C^*(\xi_n) \to \overline{d}_C(\mu,\nu)$. By (2.15) we know that for every n there exists a chain of compatible transport plans $\tilde{\pi}_n$ whose vertices are the components of ξ_n and such that

$$c^*(\tilde{\pi}_n) \leq C^*(\xi_n) + \frac{1}{n}.$$

We view every $\tilde{\pi}_n$ as a piecewise rectilinear microscopic motion σ_n on $I = [0,1]$ and by (**CA2**) we have

$$J(\sigma_n) \leq c^*(\tilde{\pi}_n),$$

so if $\overline{d}_C(\mu,\nu) < +\infty$, otherwise we have nothing to prove, then $J(\sigma_n)$ is bounded. We can make a change of variable $t \mapsto \varphi(t)$ in such a way the sequence of the infinitesimal costs $(c_n)_{n\in\mathbb{N}}$ turns out to be bounded in L^p for p as in (**CA1**). Then we can apply Corollary 3.3 and so we obtain a narrowly converging subsequence to a microscopic motion $\sigma \in \Sigma(\mu,\nu)$. Finally by (**CA3**)

$$d_J(\mu,\nu) \leq J(\sigma) \leq \lim_n J(\sigma_n) \leq \lim_n c^*(\tilde{\pi}_n) \leq \lim_n \left(C^*(\xi_n) + \frac{1}{n}\right) = \overline{d}_C(\mu,\nu).$$

□

The following statement trivially follows by Definition 2.3 and Definition 2.5.

Lemma 3.3. *Let c be a transport cost satisfying (**CA4**). Then*

$$\overline{d}_C \leq d_J.$$

Finally, we can prove the equivalence between the transport distance, the chain distance and the kinematic distance for a general transport cost satisfying the previous cost assumptions.

Theorem 3.6. *Let C be a transport cost satisfying (**CA1**) − (**CA4**). Then*

$$d_C = \overline{d}_C = d_J.$$

Proof. By the two previous lemmas we have $d_J = \overline{d}_C$. Thus $\overline{d}_C$ is a distance and so, as observed in Remark 2.1, we get $d_C = \overline{d}_C$. □

4. Irrigation cost and irrigation distance

4.1. **Irrigation cost and integral of the infinitesimal irrigation costs.** Let i be an index varying in a finite set, for every i let $x_i, y_i \in \mathbb{R}^N$, with $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$. Let $\pi = \sum_i m_i \delta_{x_i} \otimes \delta_{y_i}$ and $\alpha \in]0,1[$, we define the *irrigation cost* of π as

$$c_\alpha(\pi) = \sum_i m_i^\alpha |x_i - y_i|. \tag{4.23}$$

Then we define the microscopic transport cost functional $c_\alpha : \mathcal{P}(X \times X) \to \mathbb{R}_+$ defined as in (4.23) if the transport plan π is given by a finite sum of Dirac masses and, in accordance to Section 2, $c_\alpha(\pi) = +\infty$ for all the other transport plans $\pi \in \mathcal{P}(X \times X)$.

Let us compute the irrigation cost in terms of a lagrangian parametrization. Let $\pi = \sum_i m_i \delta_{x_i} \otimes \delta_{y_i}$, let (Ω, μ_Ω) be a measurable set and let $g = (g^1, g^2)$ be a lagrangian parametrization of π on Ω. For all i let $\Omega_i = g^{-1}(x_i, y_i)$, then $\mu_\Omega(\Omega_i) = m_i$. For $p \in \Omega$, we set

$$[p] = \{q \in \Omega \mid g(q) = g(p)\}$$

representing the equivalence class of material particles which have the same initial and final positions, so $p \in \Omega_i$ means $[p] = \Omega_i$. Moreover, we define the *solidarity function*

$$s_\alpha(p) = \mu_\Omega([p])^{\alpha-1},$$

so we have

$$(\mu_\Omega(\Omega_i))^\alpha = \int_{\Omega_i} s_\alpha(p)dp$$

and therefore

$$m_i^\alpha |x_i - y_i| = \int_{\Omega_i} s_\alpha(p)|g^1(p) - g^2(p)|dp,$$

from which

$$c_\alpha(\pi) = \int_\Omega s_\alpha(p)|g^1(p) - g^2(p)|\, dp \tag{4.24}$$

follows. To the aim of computing the infinitesimal irrigation cost let us observe that c_α being 1-homogeneous with respect to scalings, the infinitesimal cost coincides with the l.s.c. relaxed c_α^0 of c_α on $\mathcal{T}$ and so we argue as follows. It is easy to see by using (1.5) that if $\pi \in \mathcal{T}$ the l.s.c. relaxed is given by $c_\alpha^0(\pi) = \sum_i m_i^\alpha |x_i - y_i|$ when, with the notation introduced in (1.6) and in the subsequent comments, $\pi_m = \sum_i m_i \delta_{x_i} \otimes \delta_{y_i}$ is countably discrete and $c_\alpha^0(\pi) = +\infty$ otherwise. The finiteness of the integral in (4.24) implies that s_α is finite μ_Ω almost everywhere on Ω_m, where Ω_m is as defined in (1.9), and therefore μ_Ω a.e. $p \in \Omega_m$ satisfies $\mu_\Omega([p]) > 0$. Therefore Ω_m is decomposed in (countably many) equivalence classes $[p]$ of positive measure, excepted for a μ_Ω-negligible set. So $\int_\Omega s_\alpha(p)|g^1(p) - g^2(p)|\, dp$ gives the value of $c_\alpha^0(\pi)$ when it is finite and, obviously, when it is equal to $+\infty$. So the equality

$$c_\alpha^0(\pi) = \int_\Omega s_\alpha(p)|g^1(p) - g^2(p)|\, dp. \tag{4.25}$$

holds for $\pi = g_\# \mu_\Omega \in \mathcal{T}$ in every case.

Let a particle motion σ be given and fix a lagrangian parametrization χ on $\Omega \times I$. To compute the integral of the infinitesimal irrigation costs of σ, we just need to apply (4.25) to π_t for any $t \in I$, taking into account that $\pi_t \in \mathcal{T}$ and that

$$g = (\chi(\cdot, t), \chi(\cdot, t) + \frac{\partial \chi}{\partial t}(\cdot, t))$$

gives a lagrangian parametrization of π_t. So fix $t \in I$ and let

$$[p]_t^* = \left\{ q \in \Omega \mid \chi(q,t) = \chi(p,t), \;\; \frac{\partial \chi}{\partial t}(p,t) = \frac{\partial \chi}{\partial t}(q,t) \right\} \tag{4.26}$$

and

$$s_\alpha^*(p,t) = \mu_\Omega([p]_t^*)^{\alpha-1}. \tag{4.27}$$

By (4.25), we can write the infinitesimal irrigation cost at the time t as

$$c_\alpha^0(\pi_t) = \int_\Omega s_\alpha^*(p,t) \left| \frac{\partial \chi(p,t)}{\partial t} \right| \; dp \tag{4.28}$$

and then, provided we check the measurability of s_α^*, the integral of the infinitesimal irrigation costs is

$$J_\alpha(\chi) = \int_{\Omega \times I} s_\alpha^*(p,t) \left| \frac{\partial \chi(p,t)}{\partial t} \right| \; dp \; dt. \tag{4.29}$$

Let us show that (4.29) holds and also admits a simpler variant, to this aim we define

$$[p]_t = \{ q \in \Omega \mid \chi(q,t) = \chi(p,t) \} \tag{4.30}$$

and

$$s_\alpha(p,t) = \mu_\Omega([p]_t)^{\alpha-1}. \tag{4.31}$$

We shall prove that the integral of the infinitesimal irrigation costs takes the same value for the solidarity functions given by (4.27) and (4.31).

Proposition 4.1. *Let χ be any regular lagrangian parametrization of a microscopic motion. Then the functions s_α and s_α^* are measurable and, for a.e. $(p,t) \in \Omega \times I$, we have $s_\alpha(p,t) = s_\alpha^*(p,t)$.*

Proof. Let us consider the sets

$$E = \{ (p,q,t) \in \Omega^2 \times I \mid \chi(p,t) = \chi(q,t) \},$$

$$E^* = \{ (p,q,t) \in \Omega^2 \times I \mid \chi(p,t) = \chi(q,t), \frac{\partial \chi(p,t)}{\partial t} = \frac{\partial \chi(q,t)}{\partial t} \}$$

and their (p,t)-sections

$$E_{p,t} = \{ q \in \Omega \mid (p,q,t) \in E \},$$

$$E_{p,t}^* = \{ q \in \Omega \mid (p,q,t) \in E^* \}.$$

Since E and E^* are measurable sets for the measure $\mu_\Omega \times \mu_\Omega \times \mathcal{H}^1$, by Fubini Theorem [13, Theorem 6.46] we have that the section maps $(p,t) \mapsto \mu_\Omega(E_{p,t})$ and $(p,t) \mapsto \mu_\Omega(E_{p,t}^*)$ are both measurable and so also the maps s and s^* are measurable. Then we shall prove that $(\mu_\Omega \times \mu_\Omega \times \mathcal{H}^1)(E \setminus E^*) = 0$. To this aim we observe that, for a.e. $p, q \in \Omega$, since the maps χ_p and χ_q are a.e. differentiable, then for a.e. t where

$\chi_p(t) = \chi_q(t)$, we have $\frac{\partial}{\partial t}\chi_p(t) = \frac{\partial}{\partial t}\chi_q(t)$. Then for almost every (p,q)-section $S_{p,q}$ of $E \setminus E^*$ we have $\mathcal{H}^1(S_{p,q}) = 0$ and so, by Fubini Theorem again, we can conclude

$$(\mu_\Omega \times \mu_\Omega \times \mathcal{H}^1)(E_p \setminus E_p^*) = \int \mathcal{H}^1(S_{p,q})dp\, dq = 0.$$

So the section maps, and therefore the maps s and s^* agree for a.e. (p,t). $\square$

4.2. **Properties of the irrigation cost.** We shall check now the abstract conditions (**CA0-4**) for the irrigation cost c_α.

Lemma 4.1. *The irrigation cost c_α satisfies the cost assumption (**CA0**) if and only if α is large, (i.e. $\alpha > 1 - \frac{1}{N}$) and X is bounded.*

Proof. The fact that, if α is large and X is bounded, c_α satisfies (**CA0**) is essentially proved in [21, Theorem 4.1], so we shall prove the reverse implication. If X is not bounded, given any $\varepsilon, \delta > 0$, we can find $x, y \in X$ such that setting $\mu = \delta\delta_x$ and $\nu = \delta\delta_y$ one has $d_W(\mu,\nu) \le \delta$ and $\overline{d}_{c_\alpha}(\mu,\nu) \ge W_1(\mu,\nu) = \delta|x-y| > \varepsilon$, so (**CA0**) does not hold in such a case.

Let $\mu \in \mathcal{P}(X)$ be a given atomic measure, we fix $S \in X$ and denote by $e(\mu)$ the infimum of the energy $E(\chi)$ defined on the irrigation patterns $\chi \in \mathbf{P}_S(\Omega)$ with $\mu_\chi = \mu$ (see [17]), that is μ is the irrigated measure. Let $\nu \in \mathcal{P}(\mathbb{R}^N)$ be another atomic measure, and let $\pi \in \Pi(\mu,\nu)$. Then, by the definition of E in [17] and the Pruning Theorem [8, Theorem 7.1] we can easily check that $e(\nu) \le e(\mu) + c_\alpha(\pi)$. By iterating, we can pass to a chain of transport plans and so, for every μ, ν, we get

$$e(\nu) \le e(\mu) + d_C^*(\mu,\nu). \tag{4.32}$$

Let $\overline{\mu} \in \mathcal{P}(X)$ be any given measure, we take a sequence $(\mu_n)_{n\in\mathbb{N}}$, made of atomic measures, approximating $\overline{\mu}$, i.e. $\mu_n \rightharpoonup \overline{\mu}$. If (**CA0**) holds, since $(\mu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for d_W, then for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that for every $h, k \ge m_n$

$$d_C^*(\mu_k, \mu_h) \le \frac{1}{2^n}. \tag{4.33}$$

Set $\mu'_n = \mu_{m_n}$, for every $n \in \mathbb{N}$, we have

$$d_C^*(\mu'_n, \mu'_{n+1}) \le \frac{1}{2^n}.$$

By virtue of the l.s.c. of the functional E and by (4.32), (4.33), we get

$$e(\overline{\mu}) \le \lim_n e(\mu'_n) \le e(\mu'_1) + 1 < +\infty,$$

then $\overline{\mu}$ is an irrigable measure (see [9]). By [9, Corollary 1.2] and the arbitrariness of $\overline{\mu}$ this implies $\alpha > \frac{1}{N'}$. $\square$

Lemma 4.2. *For every $\alpha \in]0,1[$, the irrigation cost c_α satisfies the cost assumption (**CA1**) for $p = \frac{1}{\alpha}$.*

Proof. Let $\pi = \sum_i m_i \delta_{x_i} \otimes \delta_{y_i}$ (otherwise $c_\alpha(\pi) = +\infty$ and we have nothing to prove). For every i we put $d_i = |x_i - y_i|$, then $c_\alpha(\pi) = \sum_i m_i^\alpha d_i$. For every i

$$m_i^\alpha d_i \leq c_\alpha(\pi),$$

from which we deduce

$$m_i^{\alpha-1} \geq \frac{c_\alpha(\pi)^{\frac{\alpha-1}{\alpha}}}{d_i^{\frac{\alpha-1}{\alpha}}}.$$

Therefore

$$c_\alpha(\pi) = \sum_i m_i^{\alpha-1} m_i d_i \geq c_\alpha(\pi)^{\frac{\alpha-1}{\alpha}} \sum_i m_i d_i^{1-\frac{\alpha-1}{\alpha}}$$

and so

$$c_\alpha(\pi)^{\frac{1}{\alpha}} \geq \sum_i m_i d_i^{\frac{1}{\alpha}},$$

which gives

$$c_\alpha(\pi) \geq \left(\sum_i m_i d_i^{\frac{1}{\alpha}} \right)^\alpha = \|\pi\|_{\frac{1}{\alpha}}.$$

□

Lemma 4.3. *The irrigation cost c_α satisfies the cost assumption (**CA2**).*

Proof. Let $\pi = \sum_i m_i \delta_{x_i} \otimes \delta_{y_i}$ (otherwise $c_\alpha(\pi) = +\infty$ and we have nothing to prove), $I = [0, 1] \subset \mathbb{R}$ and χ be any lagrangian parametrization of π. We can compute by (4.28) the infinitesimal cost at any t as

$$c(t) = c_\alpha^0(\pi_t) = \sum_i m_i^\alpha |y_i - x_i|.$$

Therefore,

$$J_\alpha(\chi) = \int_I c(t)dt = \sum_i m_i^\alpha |y_i - x_i| = c_\alpha(\pi).$$

□

To the aim of proving that the irrigation cost c_α satisfies the cost assumption [**CA3**], we need the next result, for which we introduce the following notation. If $(\chi_n)_{n\in\mathbb{N}}$ is any sequence of lagrangian parameterizations of microscopic motions defined on $\Omega \times I$, for every $n \in \mathbb{N}$ we set

$$[p]_t^n = \{q \in \Omega \mid \chi_n(q,t) = \chi_n(p,t)\},$$
$$s_\alpha^n(p,t) = \mu_\Omega([p]_t^n)^{\alpha-1}.$$

Lemma 4.4. *If $\chi_n \to \chi$, then for a.e. p in Ω*

$$s_\alpha(p,\cdot) \leq \Gamma\text{--}\liminf_n s_\alpha^n(p,\cdot).$$

Proof. Firstly we recall that by definition, for every p,

$$\Gamma\text{-}\liminf_n s_\alpha^n(p,t) = \inf_{t_n \to t} (\liminf_n s_\alpha^n(p,t_n)).$$

We fix $p \in \Omega$, $t \in I$ and a sequence $t_n \to t$. We set $\overline{m} = \limsup_n \mu_\Omega([p]_{t_n}^n)$ and introduce the following sets:

$$A_n = \bigcup_{k \geq n} [p]_{t_k}^k, \quad A = \bigcap_n A_n.$$

Notice that $(A_n)_{n\in\mathbb{N}}$ is decreasing and, for every $n \in \mathbb{N}$, $\mu_\Omega(A_n) \geq \overline{m}$ and so $\mu_\Omega(A) \geq \overline{m}$. We claim that, for almost every choice of p, $A \subset [p]_t$ modulo a null set. To prove this inclusion we proceed as follows. By hypotheses we have that for a.e. $q \in \Omega$, $(\chi_q)_n$ locally uniformly converges to χ_q. We can assume that this property is enjoyed by p and by a given point $q \in A$. Then $q \in A_n$ for every n and so $q \in [p]_{t_n}^n$ for infinitely many n and for such values of n we have $\chi_n(p,t_n) = \chi_n(q,t_n)$. Since the convergence is locally uniform and $t_n \to t$, we can conclude by (1.3) that $\chi_p(t) = \chi_q(t)$, i.e. $q \in [p]_t$ and so we get the claim $A \subset [p]_t$. Therefore, we have

$$s_\alpha(p,t) = \mu_\Omega([p]_t)^{\alpha-1} \leq \mu_\Omega(A)^{\alpha-1} \leq \overline{m}^{\alpha-1} = \liminf_n \mu_\Omega([p]_{t_n}^n)^{\alpha-1} = \liminf_n s_\alpha^n(p,t_n)$$

and so, by taking the infimum in the last bound among the sequences $(t_n)_{n\in\mathbb{N}}$ converging to t, we get the thesis. □

Lemma 4.5. *The irrigation cost c_α satisfies the cost assumption [**CA3**].*

Proof. We shall prove [**CA3′**], so let $(\chi_n)_{n\in\mathbb{N}}$ be any fiberwise converging sequence to χ, we have to prove

$$J_\alpha(\chi) \leq \liminf_{n\to\infty} J_\alpha(\chi_n). \tag{4.34}$$

For every $n \in \mathbb{N}$ and any given p, let μ_n be the measure on I whose density is given by $\left|\frac{\partial \chi_n}{\partial t}(p,t)\right|$ and let μ be the measure on I whose density is given by $\left|\frac{\partial \chi}{\partial t}(p,t)\right|$. Since $\chi_n \to \chi$, then for a.e. p, $\frac{\partial \chi_n}{\partial t}(p,t) \rightharpoonup \frac{\partial \chi}{\partial t}(p,t)$ in the sense of distributions and thus we have

$$\mu(A) \leq \liminf_{n\to+\infty} \mu_n(A)$$

for every open subset $A \subset I$. Now, the above estimate and Lemma 4.4 allow to apply [5, Proposition 5.5] which states that, under these hypotheses, for a.e. p, the following relation holds true

$$\int_I s_\alpha(p,t)d\mu \leq \liminf_{n\to+\infty} \int_I s_\alpha^n(p,t)d\mu_n.$$

Finally, by integrating with respect to μ_Ω and by applying Fatou Lemma, we get (4.34). □

4.3. **Pruning Theorem and Discretization Property.** The proof that c_α satisfies the discretization property [**CA4**] is more complex and requires a result like the Pruning Theorem, stated in [8] in the case of the irrigation patterns. Then we proceed to state some preliminary lemmas which will allow us to prove the Pruning Theorem below, which is suited for the context studied in this paper. The first result we are going to prove says that a minimum of J_α in $\Sigma(\mu,\nu)$ enjoys a property called here *no-cycle property*, which is the variant of the analogous property established for the irrigation patterns in [8, Theorem 6.17] needed in this section. We limit ourselves to give a sketch of the proof and other related concepts and refer the reader to [18] for a more detailed study of this kind of properties.

Definition 4.1. *Let χ be a lagrangian parametrization defined on $\Omega \times I$ of a microscopic motion. We shall say that χ satisfies the* no-cycle property *if, fixed any $t_1, t_2 \in I$, $t_1 < t_2$, if A is the intersection of two equivalence classes of material points at t_1 and t_2 respectively, for a.e. $p, q \in A$ the equality $\chi_p(t) = \chi_q(t)$ holds for every $t \in [t_1, t_2]$.*

Let us call *flow curve* any measurable $\gamma : J \to \mathbb{R}^N$, where $J \subset I$ is an interval, such that there is a set of material points p with strictly positive measure such that $\chi(p,t) = \gamma(t)$ at every $t \in J$. Let D be the set of (p, t_1, t_2) such that χ_p coincides with a flow curve γ in t_1 and t_2 but it does not coincide with γ in $[t_1, t_2]$. Since the flow curves are continuous and on a fixed interval there are countably many flow curves, it is easy to check that D is a measurable set. The no-cycle property implies that every (t_1, t_2)-section of D is a μ_Ω-negligible set. Then, by applying Fubini Theorem as in Proposition 1.2 and Proposition 4.1, we deduce that for a.e. $p \in \Omega$ the p-section of D is a negligible set and so we can state the following property

$$\textit{for a.e. } t_1, t_2 \in I, t_1 < t_2 \textit{ and } \forall t \in [t_1, t_2] : [p]_{t_1} \cap [p]_{t_2} \subset [p]_t \tag{4.35}$$

modulo a negligible set.

Lemma 4.6. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$ be two given measures and let χ, defined on $\Omega \times I$, be a minimum of J_α in $\Sigma(\mu,\nu)$. Then χ has the no-cycle property.*

Proof. Let χ be a minimum of J_α as in the hypotheses. Fix $t_1 < t_2$ in I and let A denote the intersection of two equivalence classes of material points at t_1 and t_2 and assume that $\mu_\Omega(A) = a > 0$. Let us consider the modification of χ on $[t_1, t_2]$ given for every $p \in A$ by

$$\chi^p(q,t) = \begin{cases} \chi(p,t) & \text{if } q \in A,\ t \in [t_1,t_2] \\ \chi(q,t) & \text{otherwise.} \end{cases}$$

Let c, c^p be the infinitesimal cost functions of χ and χ^p respectively, let $\overline{c}$ be the average cost of χ^p, defined for $t \in I$ as

$$\overline{c}(t) = \frac{1}{a} \int_A c^p(t)\, d\mu_\Omega.$$

We claim that for almost every $t \in [t_1, t_2]$ $\overline{c}(t) \le c(t)$. Indeed, fix $t \in [t_1, t_2]$ such that $c(t)$ is finite and $\frac{\partial \chi}{\partial t}$ is constant a.e. in all the equivalence classes $[p]_t$. Proposition 4.1 ensures that this happens for a.e. $t \in I$. For such values of t we shall denote by A_i the (countably many) equivalence classes at t on which the constant value v_i of $|\frac{\partial \chi}{\partial t}(p,t)|$ is different from zero. Now we set $a_i = \mu_\Omega(A_i \cap A)$, $a = \mu_\Omega(A)$ and $b_i = \mu_\Omega(A_i \setminus A)$. With this notation, it is easy to check that

$$c(t) = \sum_i (b_i + a_i)^\alpha v_i,$$

$$\overline{c}(t) = \sum_i \left(\frac{a_i}{a}(b_i + a)^\alpha + \frac{a - a_i}{a} b_i^\alpha \right) v_i.$$

Note that $\frac{a_i}{a}(b_i + a) + \frac{a - a_i}{a} b_i = b_i + a_i$ and so the concavity of the mapping $x \mapsto x^\alpha$ yields the claim $\overline{c}(t) \le c(t)$, holding the equality only if for every i either $a_i = 0$ or $a_i = a$. Since χ minimizes J_α, then this must be the case for a.e. $t \in [t_1, t_2]$. Then, by applying standard arguments of measure theory and by virtue of the continuity of χ with respect to the time variable t, we get the thesis. □

Lemma 4.7. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$ be two atomic measures and let $\overline{\sigma}$ be a minimum of J in $\Sigma(\mu, \nu)$. Then $\overline{\sigma}$ has the structure of a finite graph and there exists a chain $\xi \in \mathbb{P}(\mathbb{R}^N)$, whose components are atomic measures, such that $p^+(\xi) = \mu$, $p^-(\xi) = \nu$ and $C^*_\alpha(\xi) = J_\alpha(\overline{\sigma})$.*

Proof. Let $\mu = \sum_i m_i \delta_{x_i}$, $\nu = \sum_j n_j \delta_{y_j}$, by denoting by $\overline{\chi}$ a lagrangian parametrization of $\overline{\sigma}$ on $\Omega \times [a, b]$, we introduce the sets

$$\Omega_{ij} = \{p \in \Omega \mid \overline{\chi}(p, a) = x_i,\ \ \overline{\chi}(p, b) = y_j\}$$

which constitute a finite partition of Ω. Now, by Lemma 4.6, almost all the points $p \in \Omega_{ij}$ have the same orbit $\chi(p, \cdot) = \gamma_{ij}$ and the boundary of the set of the points t in which two different curves γ_{ij} and γ_{hk} coincide contains at most two points. We denote by t_i such values of t, which are therefore in a finite number. Then we have

$$J_\alpha(\overline{\sigma}) = \sum_i J_\alpha(\overline{\sigma}_{|[t_i, t_{i+1}]}).$$

Let $\overline{\sigma}_i = \overline{\sigma}_{|[t_i, t_{i+1}]}$, we introduce the map $p_i : \Gamma \to \mathbb{R}^N \times \mathbb{R}^N$ given by $p_i(\gamma) = (\gamma(t_i), \gamma(t_{i+1}))$ and let

$$\pi_i = (p_i)_\# \overline{\sigma}_i. \tag{4.36}$$

We claim that, for every i,

$$J_\alpha(\overline{\sigma}_i) = c_\alpha(\pi_i).$$

Indeed, though $\overline{\sigma}_i$ is not necessarily a uniform rectilinear motion, we know by the definition of the points t_i that the motions of material particles either have the same trajectory or have always different positions and so we can reparametrize all the orbits with uniform velocity without changing the value of J_α. Now, let ξ be the chain of the vertices of the chain of transport plans $\tilde{\pi}$ whose components π_i are given by (4.36). We finally obtain

$$c^*_\alpha(\xi) = \sum_i c_\alpha(\pi_i) = \sum_i J_\alpha(\overline{\sigma}_i) = J_\alpha(\overline{\sigma}).$$

□

Let us introduce some notation and definitions which are essentially the two-sided version of those stated in [8]. Let σ be a particle motion and let χ be a lagrangian parametrization of σ on $\Omega \times I$, for every $p \in \Omega$ we set

$$\tau_-(p) = \sup\left\{t \in I : \mid \chi(p,s) = const.\ \forall s \le t\right\},$$
$$\tau_+(p) = \inf\left\{t \in I : \mid \chi(p,s) = const.\ \forall s \ge t\right\}.$$

Lemma 4.8. *τ_- and τ_+ are measurable functions.*

Proof. Let us consider the function τ_-. We show that for every $a \in \mathbb{R}$ the level set $L_a = \{p \in \Omega \mid \tau_-(p) \ge a\}$ is measurable. To this aim we observe that, for any real a, $\tau_-(p) \ge a$ if and only if for every $t_1, t_2 \le a$, $\chi(p,t_1) = \chi(p,t_2)$. Then, by introducing $Z_{t_1,t_2} = \{p \in \Omega \mid \chi(p,t_1) = \chi(p,t_2)\}$, we have that the set Z_{t_1,t_2} turns out to be measurable since the functions $\chi(\cdot,t_1)$ and $\chi(\cdot,t_2)$ are both measurable. By virtue of the continuity of χ with respect to t we get

$$L_a = \bigcap_{t_1,t_2 \le a} Z_{t_1,t_2} = \bigcap_{\substack{t_1,t_2 \le a \\ t_1,t_2 \in \mathbb{Q}}} Z_{t_1,t_2}$$

and so L_a is measurable since it is a countable intersection of measurable sets. Analogous proof can be carried out for τ_+. □

Let us notice that $\tau_-(p) < \tau_+(p)$ if and only if $p \in \Omega_m$. By (1.6), for every t, $\sigma(t)$ can be decomposed as $\sigma_s + \sigma_m(t)$. If $\sigma \in \Sigma_{[0,1]}(\mu,\nu)$,we put $\mu_1 = \nu_1 = \sigma_s$ $\mu_2 = \sigma_m(0)$, $\nu_2 = \sigma_m(1)$.

Lemma 4.9. *Let $\sigma \in \mathcal{P}(\Gamma)$ be a microscopic motion of minimal cost and let χ be a lagrangian parametrization of σ on (Ω,μ_Ω). Then, for a.e. $p \in \Omega_m$ and for every $\varepsilon > 0$ there exists $s \in \mathbb{Q} \cap [\tau_-(p), \tau_-(p)+\varepsilon]$ such that $\mu_\Omega([p]_s) > 0$.*

Proof. Fix $\varepsilon > 0$ and take $p \in \Omega_m$ such that $\int_I s(p,t)|\frac{\partial \chi_p}{\partial t}|\,dt < \infty$ and (4.35) holds. There exists $I^* \subset [\tau_-(p), \tau_-(p)+\varepsilon]$ with $\mathcal{H}^1(I^*) > 0$, such that for every $t \in I^*$ $\frac{\partial \chi_p}{\partial t}(t) \neq 0$. For a.e. $t \in I^*$ we have $s(p,t) < +\infty$, namely $\mu_\Omega([p]_t) > 0$. Therefore, since I^* is an uncountable set and $\mu_\Omega(\Omega) < +\infty$, there exist $t_1, t_2 \in I^*$ with $t_1 < t_2$ such that $\mu_\Omega([p]_{t_1} \cap [p]_{t_2}) > 0$. It is also easy to see that we can avoid the negligible set of the pairs (t_1,t_2) considered in (4.35). Thus, let us fix $s \in \mathbb{Q}$ such that $t_1 < s < t_2$, since σ is a minimizer then we can apply Lemma 4.6, so we know by (4.35) that $[p]_{t_1} \cap [p]_{t_2} \subset [p]_s$ and therefore $\mu_\Omega([p]_s) > 0$. □

We observe that for every $t \in \mathbb{Q} \cap I$, the set $\mathcal{V}_t$ of all the equivalence classes $[p]_t$ of positive measure is a countable set. Let us consider all the pairs (A, t) where t is a rational number and A is an equivalence class of positive measure at the time t. Such pairs are also countably many, so they can be represented as the terms of a sequence $(A_n, t_n)_{n\in\mathbb{N}}$. Then the previous lemma can be restated as follows.

Lemma 4.10. *For a.e. $p \in \Omega_m$ and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $[p]_{t_n} = A_n$, $t_n \in [\tau_-(p), \tau_-(p) + \varepsilon]$.*

By the continuity of χ in the t-variable, by keeping the same notation, we can deduce the following corollary.

Corollary 4.1. *For a.e. $p \in \Omega_m$ and for every $\varepsilon > 0$ there exists n such that $p \in A_n$ and $|\chi(p, 0) - \chi(p, t_n)| < \varepsilon$.*

Proof. Fix p and fix $\varepsilon > 0$, since χ is continuous with respect to t there exists $\delta(p)$ such that

$$|\chi(p, 0) - \chi(p, t)| = |\chi(p, \tau_-(p)) - \chi(p, t)| < \varepsilon, \quad \forall t \leq \tau_-(p) + \delta(p). \tag{4.37}$$

Then by applying Lemma 4.10 with ε replaced by $\delta(p)$, we get the thesis. □

Definition 4.2. *Let χ be a lagrangian parametrization of a microscopic motion and let $\overline{\tau}_-, \overline{\tau}_+ : \Omega \to I$, with $\overline{\tau}_- \leq \overline{\tau}_+$, be two measurable functions. Let us consider the mapping $\overline{\chi}$ defined by setting for a.e. p and for every $t \in I$*

$$\overline{\chi}(p, t) = \begin{cases} \chi(p, \overline{\tau}_-(p)) & \text{if } t < \overline{\tau}_-(p) \\ \chi(p, t) & \text{if } t \in [\overline{\tau}_-(p), \overline{\tau}_+(p)] \\ \chi(p, \overline{\tau}_+(p)) & \text{if } t > \overline{\tau}_+(p). \end{cases}$$

We shall say that $\overline{\chi}$ is the $(\overline{\tau}_-, \overline{\tau}_+)$-forced absorption of χ.

Theorem 4.2. *(**Pruning**) Let χ be a lagrangian parametrization of minimal cost on $\Omega \times [0, 1]$. For every $\varepsilon > 0$ there exists $\Omega_\varepsilon \subset \Omega_m$ and there exist two measurable functions $\overline{\tau}_-, \overline{\tau}_+ : \Omega_\varepsilon \to I$, with $\overline{\tau}_-(p) < \overline{\tau}_+(p)$ for a.e. p, such that the $(\overline{\tau}_-, \overline{\tau}_+)$-forced absorption $\overline{\chi}$ of $\chi_{|\Omega_\varepsilon \times I}$, inducing $\overline{\sigma}$, enjoys the following properties:*

(1) *$\overline{\sigma}(0)$ and $\overline{\sigma}(1)$ are measures of finite support.*

(2) *$d_W(\overline{\sigma}(0), \mu_2) \leq \varepsilon$, $d_W(\overline{\sigma}(1), \nu_2) \leq \varepsilon$.*

Proof. Fix $\varepsilon > 0$. With the notation in Corollary 4.1 we set

$$\Omega_n = \{p \in A_n \mid |\chi(p, t_n) - \chi(p, 0)| < \varepsilon\}.$$

For every n, Ω_n is a measurable set. Then, for a.e. $p \in \Omega_m$, we set by using the previous corollary

$$\overline{\tau}_-(p) = t_n \quad \text{with} \quad n = \min\{k \in \mathbb{N} \mid p \in \Omega_k\}.$$

By arguing in a symmetric way, we get the mapping $\overline{\tau}_+$ and so we obtain, for a fixed $\varepsilon > 0$, the $(\overline{\tau}_-, \overline{\tau}_+)$-forced absorption $\overline{\chi}$ of $\chi_{|\Omega_m}$. Then $\overline{\sigma}(0) = \chi(\cdot, \overline{\tau}_-(\cdot))_{\#}\mu_\Omega$ and $\overline{\sigma}(1) = \chi(\cdot, \overline{\tau}_+(\cdot))_{\#}\mu_\Omega$, by construction, are concentrated on two countable sets and so, by eliminating from Ω_m a set of arbitrarily small measure, we can reduce ourselves to the case that the supports are finite sets working on a subset $\Omega_\varepsilon \subset \Omega_m$ such that $\mu_\Omega(\Omega_m \setminus \Omega_\varepsilon) < \varepsilon$. Therefore we have just proved the statement (1) while (2) follows by the definition of Ω_n and the bound $\mu_\Omega(\Omega_m \setminus \Omega_\varepsilon) < \varepsilon$. □

Corollary 4.3. *Let $\sigma \in \Sigma(\mu, \nu)$ be any admissible microscopic motion. For every $\varepsilon > 0$ there exists a chain ξ such that $d_W(p^+(\xi), \mu) < \varepsilon$, $d_W(p^-(\xi), \nu) < \varepsilon$ and $C^*(\xi) \leq J_\alpha(\sigma)$.*

Proof. Fix $\varepsilon > 0$. By virtue of lemmas 4.2, 4.5 respectively, we know that the irrigation cost c_α satisfies the cost assumptions (**CA1**), (**CA3**) and so we are allowed to apply Theorem 3.5 and take a minimum σ^* of J_α in $\Sigma(\mu, \nu)$ and a corresponding lagrangian parametrization χ. Then we apply Theorem 4.2 to σ^* and so we get two atomic measures $\overline{\sigma}^*(0)$ and $\overline{\sigma}^*(1)$ such that $d_W(\overline{\sigma}^*(0), \mu_2) \leq \frac{\varepsilon}{2}$ and $d_W(\overline{\sigma}^*(1), \nu_2) \leq \frac{\varepsilon}{2}$.

Then we apply again Theorem 3.5 and find σ' minimizing J in $\Sigma(\overline{\sigma}^*(0), \overline{\sigma}^*(1))$. Now, since $\overline{\sigma}^*(0)$ and $\overline{\sigma}^{(}1)$ are atomic measures, by Lemma 4.7 we know that σ' has the structure of a finite graph and there is a chain $\xi \in \mathbb{P}(\mathbb{R}^N)$ whose components are atomic measures and such that $p^-(\xi) = \overline{\sigma}^*(0)$, $p^+(\xi) = \overline{\sigma}^*(1)$ and

$$C^*(\xi) = J_\alpha(\sigma') \leq J_\alpha(\overline{\sigma}^*) \leq J_\alpha(\sigma^*) \leq J_\alpha(\sigma).$$

Now let us focus the attention on Ω_s which is made of points p such that $\chi_p(t) = const$. The measure μ_1 $(= \nu_1)$ can be approximated in the weak distance by a finite supported measure $\overline{\mu}$ such that $d_W(\overline{\mu}, \mu_1) < \frac{\varepsilon}{2}$. After adding $\overline{\mu}$ to each term of ξ we get a new chain ξ' having the same cost of ξ, i.e. $C^*(\xi') = C^*(\xi)$ and connecting $\overline{\sigma}^*(0) + \overline{\mu}$ to $\overline{\sigma}^*(1) + \overline{\mu}$. Since

$$d_W(\overline{\sigma}^*(0) + \overline{\mu}, \mu) \leq d_W(\overline{\sigma}^*(0), \mu_2) + d_W(\overline{\mu}, \mu_1) < \varepsilon$$

and, analogously, $d_W(\overline{\sigma}^*(1) + \overline{\mu}, \nu) < \varepsilon$, the thesis follows. □

The following statement is an immediate consequence of the previous result.

Lemma 4.11. *For every $\alpha \in]0, 1[$ the irrigation cost c_α satisfies the cost assumption [**CA4**].*

4.4. **Equivalence of the irrigation models and existence of minimizers.** The following result establishes the existence of minimizers of the integral of the infinitesimal costs J_α in the class $\Sigma(\mu, \nu)$. Indeed, by Lemma 4.2 and Lemma 4.5, we are allowed to state the following particular case of Theorem 3.5.

Theorem 4.4. ***(Existence of minimizers)*** *Let c_α be the irrigation cost an let J_α be the corresponding integral of the infinitesimal costs. Then, for every $\alpha \in]0, 1[$*

there exists $\sigma \in \Sigma(\mu,\nu)$ such that

$$d_{J_\alpha}(\mu,\nu) = J_\alpha(\sigma).$$

Finally, we are in a position to state the following result which establishes the equivalence of the main irrigation models proposed in literature. More precisely, for the irrigation cost c_α the variational model proposed by Xia in [21], which is formalized in terms of flat chains on graphs, leads to a functional equal to the chain distance $\overline{d}_{c_\alpha}$. Indeed a finite chain of discrete measures induces a graph in an obvious way. The approach proposed in [3] is a variant of that in [17], the first one relies on a functional expressed in terms of microscopic motions, which are called *traffic plans* in [3], while the other one is formalized in terms of lagrangian parameterizations and regards the *irrigation patterns* which are characterized by having a single Dirac mass (the source) as initial measure. Both these approaches lead to a variational model which is equivalent to the irrigation model based on the kinematic distance d_{J_α}. Finally all of these approaches are equivalent to the formulation of the irrigation problem based on the more abstract notion of transport distance d_{c_α} presented here. We also remark that a close analysis of the full equivalence of the functionals involved in all these approaches will be pursued in [18].

In other terms, by Lemma 4.3, Lemma 4.11 and Theorem 4.4, we have the following statement.

Theorem 4.5. ***(Characterization of the irrigation distance)*** *Let c_α be the irrigation cost an let J_α be the corresponding integral of the infinitesimal costs. Then, for every $\alpha \in]0,1[$*

$$d_{c_\alpha} = \overline{d}_{c_\alpha} = d_{J_\alpha}.$$

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