Uniqueness of solutions to Hamilton-Jacobi equations arising in the Calculus of Variations

Gianni Dal Maso* and Hélène Frankowska†

Abstract.
We prove the uniqueness of the viscosity solution to the Hamilton-Jacobi equation associated with a Bolza problem of the Calculus of Variations, assuming that the Lagrangian is autonomous, continuous, superlinear, and satisfies the usual convexity hypothesis. Under the same assumptions we prove also the uniqueness, in a class of lower semicontinuous functions, of a slightly different notion of solution, where classical derivatives are replaced only by subdifferentials. These results follow from a new comparison theorem for lower semicontinuous viscosity supersolutions of the Hamilton-Jacobi equation, that is proved in the general case of lower semicontinuous Lagrangians.

Key words. Discontinuous Lagrangians, Hamilton-Jacobi equations, viability theory, viscosity solutions.

AMS-MOS subject classification: 49L20 (primary), 49L25 (secondary).

1 Introduction

Let us consider a *Bolza problem* of the Calculus of Variations

$$\min\left\{\int_0^t L(y(s),y'(s))ds \;+\; \varphi(y(t)) : y\in W^{1,1}(0,t;\mathbb{R}^n),\ y(0)=x\right\}, \tag{1.1}$$

where the final cost $\varphi\colon \mathbb{R}^n \mapsto \mathbb{R}_+ \cup \{+\infty\}$ is *lower semicontinuous* and the Lagrangian $L\colon \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}_+$ is *locally bounded* and *lower semicontinuous*. We assume also that $L(x,\cdot)$ is *convex* for every $x \in \mathbb{R}^n$ and that the following Tonelli

*SISSA, via Beirut 2-4, 34014 Trieste, Italy

†CNRS, ERS2064, Centre de Recherche Viabilité, Jeux, Contrôle, Université de Paris-Dauphine, 75775 Paris Cx 16, France

type *coercivity assumption* is satisfied: there exists a function $\Theta\colon \mathbb{R}^n \mapsto \mathbb{R}_+$ such that

$$\text{(1.2)} \qquad \lim_{|u|\to\infty} \frac{\Theta(u)}{|u|} = +\infty\,, \qquad \forall\,(x,u) \in \mathbb{R}^n \times \mathbb{R}^n,\;\; L(x,u) \geq \Theta(u)\,.$$

These assumptions guarantee the existence of absolutely continuous minimizers (see, e.g., [7]). The classical Lagrange problem (with the fixed final condition $y(t) = z$) may be reduced to the form (1.1) by simply setting $\varphi(z) := 0$ and $\varphi := +\infty$ elsewhere.

The *value function* $V\colon \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+ \cup \{+\infty\}$ for the Bolza problem (1.1) is defined by

$$V(t,x) := \min\left\{\int_0^t L(y(s),y'(s))ds \;+\; \varphi(y(t)) : y \in W^{1,1}(0,t;\mathbb{R}^n),\; y(0) = x\right\}.$$

Under our assumptions on L and φ the value function is *lower semicontinuous* on $\mathbb{R}_+ \times \mathbb{R}^n$ and *locally Lipschitz* on $\mathbb{R}_+^\star \times \mathbb{R}^n$, where $\mathbb{R}_+^\star := \mathbb{R}_+ \setminus \{0\}$. Moreover it satisfies the *initial condition*

$$\forall\, x \in \mathbb{R}^n, \quad \liminf_{h\to 0+,\; y\to x} V(h,y) = \varphi(x)\,.$$

If V is smooth and L is continuous, then V is a classical solution to the *Hamilton-Jacobi equation*

$$\text{(1.3)} \qquad \begin{cases} V_t + H(x,-V_x) = 0 & \text{in } \mathbb{R}_+^\star \times \mathbb{R}^n, \\[2ex] V(0,\cdot) = \varphi & \text{in } \mathbb{R}^n, \end{cases}$$

with Hamiltonian $H\colon \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ defined by

$$\text{(1.4)} \qquad H(x,p) := \sup_{u\in\mathbb{R}^n} \left(\langle p,u\rangle - L(x,u)\right),$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$. In other words, $H(x,\cdot)$ is the Legendre-Fenchel conjugate of $L(x,\cdot)$.

It is well known that (1.3) may not have smooth solutions, even if H and φ are smooth. To overcome this difficulty, different notions of generalized solutions have been proposed. The notion of viscosity solution can be introduced by means of subdifferentials and superdifferentials.

We recall that the *subdifferential* $\partial_- W(x)$ of a function $W\colon \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ at a point $x \in \mathrm{dom}(W)$ is defined by

$$\partial_- W(x) := \left\{p \in \mathbb{R}^n \;:\; \liminf_{y\to x} \frac{W(y) - W(x) - \langle p, y-x\rangle}{|y-x|} \geq 0\right\}.$$

while the *superdifferential* $\partial_+ W(x)$ is defined by $\partial_+ W(x) := -\partial_-(-W)(x)$.

A *continuous function* $W: \mathbb{R}_+^\star \times \mathbb{R}^n \mapsto \mathbb{R}$ is said to be a *viscosity solution to the Hamilton-Jacobi equation (1.3)* if the following conditions are satisfied:

$$\forall\, (t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n,\ \forall\, (p_t,p_x) \in \partial_- W(t,x),\ \ p_t + H(x,-p_x) \geq 0\,, \tag{1.5}$$

$$\forall\, (t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n,\ \forall\, (p_t,p_x) \in \partial_+ W(t,x),\ \ p_t + H(x,-p_x) \leq 0\,. \tag{1.6}$$

In [8] and [9] the uniqueness of a bounded uniformly continuous viscosity solution of (1.3) is proved under some assumptions on H, which imply in particular that H is continuous and $H(\cdot,p)$ is uniformly continuous for every $p \in \mathbb{R}^n$. In [10, Theorem 4.5] we proved the following result, that can be applied also to unbounded solutions.

Theorem 1.1 *Assume that L is continuous. Let $W: \mathbb{R}_+^\star \times \mathbb{R}^n \mapsto \mathbb{R}_+$ be a locally Lipschitz viscosity solution of (1.3) which satisfies the initial condition*

$$\forall\, x \in \mathbb{R}^n,\ \ \liminf_{h\to 0+, y\to x} W(h,y) = \varphi(x)\,. \tag{1.7}$$

Then $W = V$ on $\mathbb{R}_+^\star \times \mathbb{R}^n$.

To describe the new uniqueness results, we introduce the notion of *contingent directional derivatives* of a function $W: \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$. These are defined, for every $x \in \mathrm{dom}(W)$ and for every $u \in \mathbb{R}^n$, by

$$D_\uparrow W(x)(u) := \liminf_{\substack{h\to 0+\\ v\to u}} \frac{W(x+hv) - W(x)}{h}\quad . \tag{1.8}$$

The main result of this paper is the following theorem, which shows that the value function V is the unique viscosity solution in the larger class of continuous functions whose contingent derivatives satisfy the following very weak assumptions:

$$\forall\, (t,x) \in \mathrm{dom}(W),\ \ t > 0,\ \ D_\uparrow W(t,x)(0,0) = 0\,, \tag{1.9}$$

$$\forall\, (t,x) \in \mathrm{dom}(W),\ \ t > 0,\ \ \exists\, u \in \mathbb{R}^n,\ \ D_\uparrow W(t,x)(-1,u) < +\infty\,. \tag{1.10}$$

Theorem 1.2 *Assume that L is continuous. Let $W: \mathbb{R}_+^\star \times \mathbb{R}^n \mapsto \mathbb{R}_+$ be a continuous viscosity solution of (1.3) which satisfies (1.9), (1.10), and the initial condition (1.7). Then $W = V$ on $\mathbb{R}_+^\star \times \mathbb{R}^n$.*

In [10, Theorem 4.4] we considered also a different notion of generalized solution and we proved the following uniqueness result in the class of locally Lipschitz functions.

Theorem 1.3 *Assume that L is continuous. Let $W\colon \mathbb{R}_+^\star \times \mathbb{R}^n \mapsto \mathbb{R}_+$ be a locally Lipschitz function which satisfies the initial condition (1.7) and solves the Hamilton-Jacobi equation (1.3) in the following sense :*

$$(1.11) \qquad \forall\, (t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n,\ \forall\, (p_t,p_x) \in \partial_- W(t,x),\ \ p_t + H(x,-p_x) = 0\,.$$

Then $W = V$ on $\mathbb{R}_+^\star \times \mathbb{R}^n$.

In this paper we shall prove the following result, which provides uniqueness in the larger class of lower semicontinuous functions W satisfying (1.9) and (1.10).

Theorem 1.4 *Assume that L is continuous. Let $W\colon \mathbb{R}_+^\star \times \mathbb{R}^n \mapsto \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function which satisfies the initial condition (1.7), the technical conditions (1.9) and (1.10), and solves the Hamilton-Jacobi equation (1.3) in the sense of (1.11). Then $W = V$ on $\mathbb{R}_+^\star \times \mathbb{R}^n$.*

In the proof of Theorem 1.4 we use the following comparison result, which follows immediately from [10, Theorem 4.1, Remark 4.2 and Proposition 4.3].

Theorem 1.5 *Assume that L is continuous. Let $W\colon \mathbb{R}_+^\star \times \mathbb{R}^n \mapsto \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function which satisfies the initial condition (1.7). Suppose that W is a subsolution of the Hamilton-Jacobi equation (1.3) in the following sense:*

$$(1.12) \qquad \forall\, (t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n,\ \forall\, (p_t,p_x) \in \partial_- W(t,x),\ \ p_t + H(x,-p_x) \le 0$$

Then $W \le V$ on $\mathbb{R}_+^\star \times \mathbb{R}^n$.

In the proof of Theorems 1.2 and 1.5 we use also a very general comparison result (Theorem 1.7) for lower semicontinuous viscosity supersolutions of the Hamilton-Jacobi equation (1.3). To our knowledge the strongest result in this direction, dealing with possibly discontinuous Lagrangians, is the following theorem proved in [10, Theorem 5.1], where the notion of supersolution is given by using contingent inequalities.

Theorem 1.6 *Let $W\colon \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function which satisfies the initial condition $W(0,\cdot) = \varphi$. Suppose that W is a supersolution of (1.3) in the following sense :*

$$\forall\, (t,x) \in \mathrm{dom}(W),\ t > 0,\ \exists\, u \in \mathbb{R}^n,\ \ D_\uparrow W(t,x)(-1,u) \le -L(x,u)\,.$$

Then $W \ge V$ on $\mathbb{R}_+ \times \mathbb{R}^n$.

The comparison result for viscosity supersolutions we are going to prove is the following theorem, where we need the additional assumptions (1.9) and (1.10).

Theorem 1.7 *Let $W: \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function which satisfies (1.9) and (1.10). Suppose that $W(0,\cdot) = \varphi$ and that W is a viscosity supersolution of the Hamilton-Jacobi equation (1.3), i.e., W satisfies (1.5). Then $W \geq V$ on $\mathbb{R}_+ \times \mathbb{R}^n$.*

2 Preliminaries

Let $K \subset \mathbb{R}^n$ be a nonempty subset and $x \in K$. The contingent cone $T_K(x)$ to K at x is defined by

$$v \in T_K(x) \iff \liminf_{h \to 0+} \frac{\mathrm{dist}(x + hv, K)}{h} = 0\,.$$

The negative polar cone T^- to a subset $T \subset \mathbb{R}^n$ is given by

$$T^- := \{v \in \mathbb{R}^n \ : \ \forall\, w \in T\,,\ \langle v, w\rangle \leq 0\}\,.$$

When K is convex , then $[T_K(x)]^-$ coincides with the the usual normal cone $N_K(x)$ of convex analysis.

The epigraph $\mathcal{E}pi(W)$ of a function $W: \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{E}pi(W) := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : r \geq W(x)\}\,.$$

We shall need the following version of Rockafellar's result (see [12]).

Lemma 2.1 *Let $x \in \mathrm{dom}(W)$ and let $(p, 0) \in \left[T_{\mathcal{E}pi(W)}(x, W(x))\right]^-$ be such that $p \neq 0$. Then there exist x_ε converging to x (as $\varepsilon \to 0+$) and*

$$(p_\varepsilon, q_\varepsilon) \in \left[T_{\mathcal{E}pi(W)}(x_\varepsilon, W(x_\varepsilon))\right]^-$$

converging to $(p, 0)$ as $\varepsilon \to 0+$ such that $q_\varepsilon < 0$ for every $\varepsilon > 0$.

A closed subset K of $\mathbb{R}^n$ is called a viability domain of a set-valued map $G: K \rightsquigarrow \mathbb{R}^n$ if for every $x \in K$

$$G(x) \cap T_K(x) \neq \emptyset\,.$$

The following statement summarizes several versions of the viability theorem (see [2]).

Theorem 2.2 (Viability) *Let $K \subset \mathbb{R}^n$ be a closed set and let $G: K \rightsquigarrow \mathbb{R}^n$ be an upper semicontinuous set-valued map with compact convex values. The following conditions are equivalent:*

(a) *K is a viability domain of G;*

(b) $G(x) \cap \overline{co}\, T_K(x) \neq \emptyset$ *for every* $x \in K$;

(c) *for every* $x \in K$ *there exist* $\varepsilon > 0$ *and a solution* $y: [0, \varepsilon[\mapsto K$ *to the Cauchy problem*

$$\begin{cases} y'(t) \in G(y(t)), \\ y(0) = x. \end{cases} \tag{2.1}$$

The equivalence (a) $\Longleftrightarrow$ (b) was proved in [11]. This proof was simplified in [2, page 85]. The fact that (a) $\Longleftrightarrow$ (c) was first proved by Bebernes and Schuur in [6]. A proof can be found in [3] or [2].

The next theorem allows to deal with some unbounded set-valued maps with closed convex values. As usual, B_R denotes the closed ball with centre 0 and radius R.

Theorem 2.3 *Let* $K \subset \mathbb{R}^n$ *be a closed set and let* $G: K \rightsquigarrow \mathbb{R}^n$ *be an upper semicontinuous set-valued map with closed convex values. We assume that for every* $x \in K$ *there exists* $R > 0$ *such that for all small* $h > 0$

$$\text{dist}(x + hG(x), K) = \text{dist}(x + h(G(x) \cap B_R), K).$$

Then the following statements are equivalent:

(a) K *is a viability domain of* G;

(b) *for every* $x \in K$ *and for every* $p \in [T_K(x)]^-$ *we have* $\inf_{u \in G(x)} \langle p, u \rangle \leq 0$.

Theorem 2.3 is a direct consequence of the following more technical result, which will be crucial in the proof of Theorem 1.7.

Theorem 2.4 *Let* $K \subset \mathbb{R}^n$ *be a closed set, let* $x \in K$, *and let* $G: K \rightsquigarrow \mathbb{R}^n$ *be a set-valued map with non-empty closed convex values. Assume that there exists* $R > 0$ *such that for all small* $h > 0$

$$\text{dist}(x + hG(x), K) = \text{dist}(x + h(G(x) \cap B_R), K). \tag{2.2}$$

Assume also that the support function, defined by

$$\sigma(y, p) := \sup_{u \in G(y)} \langle p, u \rangle,$$

satisfies the following upper semicontinuity condition at x: for every $\varepsilon > 0$ there exists a neighbourhood U of x such that

$$\sigma(x,p) + \varepsilon|p| > \sigma(y,p) \tag{2.3}$$

for every $y \in U \cap K$ and for every p in the set

$$N_{G(x)}(B_R) := \{p \in \mathbb{R}^n : \exists\, u \in G(x) \cap B_R,\ p \in N_{G(x)}(u)\}\,.$$

Finally, assume that for every $y \in K$ in a suitable neighbourhood of x we have

$$\sigma(y,-p) \geq 0 \tag{2.4}$$

for every $p \in [T_K(y)]^-$ such that $-p \in N_{G(x)}(B_R)$. Then $G(x) \cap T_K(x) \cap B_R \neq \emptyset$.

From the proof given below it follows that the same result holds if $[T_K(y)]^-$ is replaced by the set of proximal normals to K at y.

Proof — Let us define the function $g\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ by

$$g(h) := \frac{1}{2}\mathrm{dist}(x + hG(x), K)^2.$$

Observe that g is locally Lipschitz around zero and $g(0) = 0$. For all small $h > 0$ let us consider $u_h \in G(x) \cap B_R$ and $x_h \in K$ such that $\mathrm{dist}(x+hG(x), K) = |x+hu_h-x_h|$. Then $x_h \to x$ when $h \to 0+$.

Consider $h > 0$ such that $g'(h)$ does exist and fix any $u \in G(x)$. Since $G(x)$ is convex, for all nonnegative $h,\ k$ we have $(h+k)G(x) = hG(x) + kG(x)$. Therefore

$$g'(h) \leq \frac{1}{2}\lim_{k\to 0+} \frac{|x + hu_h + ku - x_h|^2 - |x + hu_h - x_h|^2}{k} = \langle p_h, u\rangle\,,$$

where $p_h := x + hu_h - x_h$. Consequently

$$g'(h) \leq \inf_{u\in G(x)} \langle p_h, u\rangle = -\sup_{u\in G(x)} \langle -p_h, u\rangle = -\sigma(x,-p_h)\,. \tag{2.5}$$

As x_h is a point of K with minimum distance from $x + hu_h$, the vector p_h is a proximal normal to K, therefore it belongs to $[T_K(x_h)]^-$. On the other hand, u_h is the point of $G(x)$ with minimum distance from $(x_h - x)/h$. Thus $-p_h \in N_{G(x)}(u_h) \subset N_{G(x)}(B_R)$. By (2.4) we have

$$-\sigma(x_h, -p_h) \leq 0\,. \tag{2.6}$$

By the uniform upper semicontinuity (2.3) of σ for every $\varepsilon > 0$ there exists $h_\varepsilon > 0$ such that for $0 < h < h_\varepsilon$

$$-\sigma(x, -p_h) \leq -\sigma(x_h, -p_h) + |p_h|\varepsilon \leq -\sigma(x_h, -p_h) + hR\varepsilon\,, \tag{2.7}$$

where the last inequality follows from the fact that $|p_h| \leq hR$. From (2.5), (2.6), and (2.7) we obtain that $g'(h) \leq hR\varepsilon$ for every $h \in (0, h_\varepsilon)$ at which the derivative $g'(h)$ exists.

Integrating g' we deduce that for $0 < h < h_\varepsilon$

$$g(h) \leq \frac{Rh^2}{2}\varepsilon\,.$$

This implies that for $0 < h < h_\varepsilon$

$$\frac{\operatorname{dist}(x + hu_h, K)}{h} = \frac{\operatorname{dist}(x + hG(x), K)}{h} \leq \sqrt{R\varepsilon}\,.$$

Let $h_i \to 0+$ be a sequence such that u_{h_i} converges to some $u \in G(x)$. From the very definition of contingent cone we deduce that $u \in T_K(x)$. $\square$

3 Proof of the comparison theorem

This section is devoted to the proof of the new comparison theorem for viscosity supersolutions.

Proof of Theorem 1.7 — We first claim that H is locally bounded. Indeed for all $x \in I\!\!R^n$ we have $H(x,p) \geq -L(x,0)$, thus H is locally bounded from below. On the other hand, $H(x,p) \leq \Theta^*(p)$, where Θ^* denotes the Legendre-Fenchel conjugate of Θ. Since the function Θ has a superlinear growth, the convex function Θ^* takes only finite values, so it is locally bounded. This shows that H is also locally bounded from above.

Consequently, the function $H(x,\cdot)$ is locally Lipschitz with respect to p, locally uniformly with respect to x. By (1.2) $H(\cdot,p)$ is upper semicontinuous with respect to x. These two properties together imply that for every $x \in I\!\!R^n$, for every $M > 0$, and for every $\varepsilon > 0$ there exists a neighbourhood U of x such that

$$\forall\, y \in U\,,\ \forall\, p \in B_M\,, \quad H(x,p) + \varepsilon > H(y,p)\,. \tag{3.1}$$

Let us define the set-valued map $G\colon I\!\!R^n \rightsquigarrow I\!\!R \times I\!\!R^n \times I\!\!R$ with closed convex values by

$$G(x) := \{(-1, u, -L(x,u) - \rho) : \rho \geq 0,\ u \in I\!\!R^n\}\,.$$

Let $K := \mathcal{E}pi(W)$, let $(t,x) \in \mathrm{dom}(W)$ with $t > 0$, and let $z := (t, x, W(t,x))$. We want to show that all assumptions of Theorem 2.4 are satisfied (here z plays the role of x). To prove (2.2) we show that there exists $R > 0$ such that for all small $h > 0$

$$\exists\, u \in I\!\!R^n, \quad |(-1, u, -L(x,u))| \leq R\,,$$
$$\mathrm{dist}\,((t,x,W(t,x)) + h(-1,u,-L(x,u)), K) = \mathrm{dist}\,((t,x,W(t,x)) + hG(x), K)\ .$$

As $W \geq 0$, it is easy to see that for every $h > 0$ there exist $u_h \in I\!\!R^n$ and $z_h = (t_h, x_h, r_h) \in I\!\!R \times I\!\!R^n \times I\!\!R$, with $r_h \geq W(t_h, x_h)$, such that

$$\begin{aligned} &|(t,x,W(t,x)) + h(-1,u_h,-L(x,u_h)) - (t_h,x_h,r_h)| \\ &= \mathrm{dist}((t,x,W(t,x)) + hG(x), K) \leq h(L(x,0)+1)\,. \end{aligned}$$

If $W(t,x) - hL(x,u_h) - r_h > 0$, by increasing r_h we could make $W(t,x) - hL(x,u_h) - r_h = 0$, which would contradict the definition of distance. Therefore $W(t,x) - hL(x,u_h) - r_h \leq 0$, hence

$$hL(x,u_h) + W(t_h,x_h) - W(t,x) \leq hL(x,u_h) + r_h - W(t,x) \leq h(L(x,0)+1)\,.$$

We claim that u_h is bounded for all $h > 0$ small enough. Assume by contradiction that there exist $h_i \to 0+$ such that $|u_{h_i}| \to \infty$.

Case 1. Assume first that for a subsequence, still denoted by h_i, we have $h_i |u_{h_i}| \geq c$ for some $c > 0$. Since $W(t_h, x_h) \geq 0$, we have

$$h_i L(x, u_{h_i}) - W(t,x) \leq h_i(L(x,0)+1)\,;$$

dividing by $h_i|u_{h_i}|$ and taking the limit we get

$$\limsup_{i\to\infty} \frac{L(x,u_{h_i})}{|u_{h_i}|} < +\infty\,,$$

which contradicts (1.2).

Case 2. It remains to consider the case $h_i|u_{h_i}| \to 0$. Since

$$\max\{|t_h + h - t|, \ \ |x_h - x - hu_h|\} \leq h(L(x,0)+1),$$

we deduce that for some $v \in I\!\!R^n$ and for a subsequence, still denoted by h_i, we have

$$\lim_{i\to\infty} \frac{t_{h_i} - t}{h_i|u_{h_i}|} = 0 \qquad \text{and} \qquad \lim_{i\to\infty} \frac{x_{h_i} - x}{h_i|u_{h_i}|} = v\,.$$

Furthermore,

$$h_i L(x, u_{h_i}) + W(t_{h_i}, x_{h_i}) - W(t, x) \leq h_i(L(x, 0) + 1)\,;$$

dividing by $h_i |u_{h_i}|$ and taking the limit yields

$$\lim_{i\to\infty} \frac{W(t_{h_i}, x_{h_i}) - W(t, x)}{h_i |u_{h_i}|} = -\infty\,.$$

Hence $D_{\uparrow} W(t, x)(0, v) = -\infty$, which implies $D_{\uparrow} W(t, x)(0, 0) = -\infty$ (see [5]). This contradicts (1.9) and completes the proof of our claim.

Thus $|u_h|$ is uniformly bounded when $h > 0$ is small. As L is locally bounded, there exists $R > 0$ such that $|(-1, u_h, -L(x, u_h))| \leq R$ for all small $h > 0$.

Observe next that, if $(p_t, p_x, q) \in N_{G(x)}(B_R)$, then $q \geq 0$. Moreover, if $q = 0$, then $p_x = 0$; if $q > 0$, then there exists $u \in B_R$ such that $p_x/q \in \partial_u L(x, u)$, the subdifferential of $L(x, \cdot)$ at u. As L is locally bounded, this implies that there exists a constant M such that $p_x/q \in B_M$ for every $(p_t, p_x, q) \in N_{G(x)}(B_R)$ with $q > 0$.

To prove (2.3) it is enough to show that for every $\varepsilon > 0$ there exists a neighbourhood U of x such that

$$\sup_{u\in I\!\!R^n} (-p_t + \langle p_x, u\rangle - qL(x, u)) + \varepsilon|(p_t, p_x, q)| > \sup_{u\in I\!\!R^n} (-p_t + \langle p_x, u\rangle - qL(y, u)) \tag{3.2}$$

for every $y \in U$ and for every $(p_t, p_x, q) \in N_{G(x)}(B_R)$. If $q = 0$, then $p_x = 0$ and (3.2) is trivial. If $q > 0$, then (3.2) can be written as

$$-p_t + qH(x, \frac{p_x}{q}) + \varepsilon|(p_t, p_x, q)| > -p_t + qH(y, \frac{p_x}{q})\,,$$

which follows easily from (3.1).

Let us check that (2.4) holds true. Fix $(s, y, r) \in \mathcal{E}pi(W)$, with $s > 0$, and $(p_t, p_x, q) \in \left[T_{\mathcal{E}pi(W)}(s, y, r)\right]^-$. Since $(0, 0, 1) \in T_{\mathcal{E}pi(W)}(s, y, r)$, we have $q \leq 0$. Therefore (2.4) is equivalent to

$$\sup_{u\in I\!\!R^n} (p_t + \langle -p_x, u\rangle + qL(y, u)) \geq 0\,. \tag{3.3}$$

If $q < 0$, then $(p_t/|q|, p_x/|q|, -1) \in \left[T_{\mathcal{E}pi(W)}(s, y, r)\right]^-$, hence $(p_t/|q|, p_x/|q|) \in \partial_- W(s, y)$ (see [5, page 249]) and we deduce (3.3) from (1.5). If $q = 0$ and

$p_x \neq 0$, then the supremum in (3.3) is $+\infty$. If $q = 0$ and $p_x = 0$, then $(p_t, 0, 0) \in \left[T_{\mathcal{E}pi(W)}(s, y, r)\right]^-$. By (1.10) there exists $u \in \mathbb{R}^n$ such that $z := D_\uparrow W(s, y)(-1, u) < +\infty$. As $\mathcal{E}pi(D_\uparrow W(s, y)(\cdot, \cdot)) = T_{\mathcal{E}pi(W)}(s, y, W(s, y))$, the vector $(-1, u, z)$ belongs to $T_{\mathcal{E}pi(W)}(s, y, W(s, y))$, which is contained in $T_{\mathcal{E}pi(W)}(s, y, r)$. By the definition of $\left[T_{\mathcal{E}pi(W)}(s, y, r)\right]^-$ we obtain $-p_t \leq 0$, which yields (3.3) when $q = 0$ and $p_x = 0$.

From Theorem 2.4 we deduce that

$$G(x) \cap \left(T_{\mathcal{E}pi(W)}(t, x, W(t, x))\right) \neq \emptyset\,.$$

As $\mathcal{E}pi(D_\uparrow W(t, x)(\cdot, \cdot)) = T_{\mathcal{E}pi(W)}(t, x, W(t, x))$, we obtain that

$$\exists\, u \in \mathbb{R}^n,\ D_\uparrow W(t, x)(-1, u) \leq -L(x, u).$$

This and Theorem 1.6 imply that $W \geq V$ on $\mathbb{R}_+ \times \mathbb{R}^n$. □

Remark 3.1 From the proof of Theorem 1.7 we see that condition (1.10) yields, for $t > 0$,

$$(p_t, 0, 0) \in \left[T_{\mathcal{E}pi(W)}(t, x, W(t, x))\right]^- \implies p_t \geq 0\,.$$

If, in addition, the subsolution inequality (1.12) is satisfied, and

$$(p_t, 0, 0) \in \left[T_{\mathcal{E}pi(W)}(t, x, W(t, x))\right]^-\,,$$

then by Rockafellar's Lemma 2.1 there exist $(t_i, x_i) \to (t, x)$ and $(p_t^i, p_x^i, q_i) \in \left[T_{\mathcal{E}pi(W)}(t_i, x_i, W(t_i, x_i))\right]^-$, with $q_i < 0$, such that $(p_t^i, p_x^i, q_i) \to (p_t, 0, 0)$. Then $(-p_t^i/q_i, -p_x^i/q_i) \in \partial_- W(t_i, x_i)$ and from (1.12) we obtain $-p_t^i/q_i - L(x_i, 0) \leq 0$, hence $p_t^i + q_i L(x_i, 0) \leq 0$. Taking the limit we get $p_t \leq 0$. This shows that (1.10) and (1.12) together imply the following geometric condition for $t > 0$:

$$(p_t, 0, 0) \in \left[T_{\mathcal{E}pi(W)}(t, x, W(t, x))\right]^- \implies p_t = 0\,.$$

4 Proofs of the uniqueness results

We begin with the proof of the uniqueness theorem for viscsity solutions.

Proof of Theorem 1.2 — By Theorem 1.7 we have $W \geq V$. Recall that the hypograph of W is defined by

$$\mathcal{H}yp(W) := \{(t, x, r) \in \mathbb{R}_+^\star \times \mathbb{R}^n \times \mathbb{R} : r \leq W(t, x)\}\,.$$

Define the closed set

$$K := \mathcal{H}yp(W) \cup (\mathbb{R}_- \times \mathbb{R}^n \times \mathbb{R})\,.$$

We claim that for all $(t,x,r) \in K$

$$\forall\, u \in \mathbb{R}^n,\ \ (-1, u, -L(x,u)) \in \overline{co}\, T_K(t,x,r)\,. \tag{4.1}$$

It is enough to prove it in the case $t > 0$ and $r = W(t,x)$, since the other cases are evident. Fix $u \in \mathbb{R}^n$. Then by (1.6)

$$\forall\, (p_t, p_x) \in \partial_+ W(t,x),\ p_t + \langle -p_x, u\rangle - L(x,u) \le 0\,. \tag{4.2}$$

We want to prove that

$$\forall\, (-p_t, -p_x, q) \in \Big[T_{\mathcal{H}yp(W)}(t,x,W(t,x))\Big]^- \quad p_t + \langle -p_x, u\rangle - qL(x,u) \le 0\,. \tag{4.3}$$

When $q > 0$ we have $(p_t/q, p_x/q) \in \partial_+ W(t,x)$, thus (4.3) follows from (4.2). By Lemma 2.1, applied to $-W$, if $(0,0,0) \ne (-p_t, -p_x, 0) \in \Big[T_{\mathcal{H}yp(W)}(t,x,W(t,x))\Big]^-$, then for some $(t_i, x_i) \to (t,x)$ and $(-p_t^i, -p_x^i, q_i) \in \Big[T_{\mathcal{H}yp(W)}(t_i,x_i,W(t_i,x_i))\Big]^-$, with $q_i > 0$, we have $(p_t^i, p_x^i, q_i) \to (p_t, p_x, 0)$. So

$$p_t^i + \langle -p_x^i, u\rangle - q_i L(x_i, u) \le 0.$$

Taking the limit we get $p_t + \langle -p_x, u\rangle \le 0$, which concludes the proof of (4.3).

By the separation theorem, (4.1) follows from (4.3). Since the lower set-valued limit of contingent cones is equal to Clarke's tangent cone (see for instance [5]), from the continuity of L we deduce that, for all $(t,x) \in \mathbb{R}_+^\star \times \mathbb{R}^n$,

$$(-1, u, -L(x,u)) \in C_{\mathcal{H}yp(W)}(t,x,W(t,x)).$$

Fix $\varepsilon > 0$. Then it is not difficult to check that

$$\forall\, u \in \mathbb{R}^n,\ \ (-1, u, -L(x,u) - \varepsilon) \in \mathrm{Int}\Big(C_{\mathcal{H}yp(W)}(t,x,W(t,x))\Big)\,.$$

By [4, Proposition 13, p. 425] this yields

$$\liminf_{h\to 0+,\, v\to u} \frac{W(t-h, x+hv) - W(t,x)}{h} \ge -L(x,u) - \varepsilon. \tag{4.4}$$

We have to show that for all $t > 0$, $x \in \mathbb{R}^n$, $V(t,x) \ge W(t,x)$. Let y be a minimizer of the Bolza problem (1.1). Since L is continuous, $y' \in L^\infty(0,t;\mathbb{R}^n)$ by

[1]. Consider a sequence of continuous functions $u_i\colon [0,t] \mapsto I\!R^n$ which is bounded in $L^\infty(0,t;I\!R^n)$ and converges to y' almost everywhere in $[0,t]$, and let $t_i \to 0+$ and $x_i \to y(t)$ be such that $W(t_i,x_i) \to \varphi(y(t))$. Define

$$y_i(s) := x_i - \int_s^{t-t_i} u_i(\tau)d\tau, \quad \forall\, s \in [0,t-t_i].$$

and $y_i(s) := x_i$ for $s > t-t_i$. Then y_i converges to y uniformly in $[0,t]$. Fix i and set $\psi(s) := W(t-s, y_i(s))$ for $s \in [0,t-t_i]$. By (4.4) for every $s \in [0,t-t_i[$ we have

$$\limsup_{h\to 0+} \frac{\psi(s+h)-\psi(s)}{h} \geq -L(y_i(s),u_i(s)) - \varepsilon. \tag{4.5}$$

Consider the system

$$\begin{cases} (\alpha'(s), z'(s)) = (1, -L(y_i(s),u_i(s)) - \varepsilon), \quad s \geq 0 \\ (\alpha(0), z(0)) = (0, W(t, y_i(0)))\,. \end{cases}$$

where we have set $u_i(s) := u_i(t)$ for all $s \geq t-t_i$. It has the unique solution

$$(\alpha(s), z(s)) := \left(s, W(t,y_i(0)) - \int_0^s L(y_i(\tau),u_i(\tau))d\tau - \varepsilon s\right).$$

According to Theorem 2.2 and (4.5), this solution is viable in $\mathcal{H}yp(\psi) \cup ([t-t_i,+\infty[\times I\!R)$, i.e., for all $s \in [0,t-t_i]$ we have $(\alpha(s),z(s)) \in \mathcal{H}yp(\psi)$. Thus for all $s \in [0,t-t_i]$

$$W(t-s, y_i(s)) \geq W(t, y_i(0)) - \int_0^s L(y_i(\tau),u_i(\tau))d\tau - \varepsilon s\,.$$

In particular

$$W(t, y_i(0)) \leq W(t_i, x_i) + \int_0^{t-t_i} L(y_i(\tau),u_i(\tau))d\tau + \varepsilon(t-t_i)\,.$$

Since the functions L and W are continuous, taking the limit we get

$$W(t,x) - \varepsilon t \leq \varphi(y(t)) + \int_0^t L(y(\tau),y'(\tau))d\tau V(t,x)\,.$$

Finally, as $\varepsilon \to 0+$ we obtain $W(t,x) \leq V(t,x)$. $\Box$

Proof of Theorem 1.4 — The inequality $W \geq V$ is proved in Theorem 1.7, while the inequality $W \leq V$ follows from Theorem 1.5. $\Box$

References

[1] AMBROSIO L., ASCENZI O. & BUTTAZZO G. (1989) *Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands*, J. Math. Anal. Appl. **142**, 301-316.

[2] AUBIN J.-P. (1991) VIABILITY THEORY, Birkhäuser, Boston.

[3] AUBIN J.-P. & CELLINA A. (1984) DIFFERENTIAL INCLUSIONS, Grundlehren Math. Wiss. 264, Springer-Verlag, Berlin.

[4] AUBIN J.-P. & EKELAND I. (1984) APPLIED NONLINEAR ANALYSIS, Wiley & Sons, New York.

[5] AUBIN J.-P. & FRANKOWSKA H. (1990) SET-VALUED ANALYSIS, Birkhäuser, Boston.

[6] BEBERNES J.W. & SCHUUR J.D. (1970) *The Ważewski topological method for contingent equations*, Ann. Mat. Pura Appl. **87**, 271-280.

[7] CESARI L. (1983) OPTIMIZATION THEORY AND APPLICATIONS. PROBLEMS WITH ORDINARY DIFFERENTIAL EQUATIONS, Appl. Math. 17, Springer-Verlag, Berlin.

[8] CRANDALL M.G. & LIONS P.-L. (1983) *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277**, 1-42.

[9] CRANDALL M.G. & LIONS P.-L. (1986) *On existence and uniqueness of solutions of Hamilton-Jacobi equations*, Nonlinear Anal. **10**, 342-370.

[10] DAL MASO G. & FRANKOWSKA (to appear) *Value functions for Bolza problems with discontinuous Lagrangians and Hamilton-Jacobi inequalities*, ESAIM Control Optim. Calc. Var..

[11] GUSEINOV H.G., SUBBOTIN A.I. & USHAKOV V.N. (1985) *Derivatives for multivalued mappings with application to game-theoretical problems of control*, Problems Control and Inform. **14**, 155-168.

[12] ROCKAFELLAR T. (1981) *Proximal subgradients, marginal values and augmented Lagrangians in nonconvex optimization*, Math. Oper. Res. **6**, 424-436.