Semicontinuity and relaxation of L^∞-functionals

Francesca Prinari
Dipartimento di Matematica "Ennio De Giorgi"
Universitá di Lecce
Via Prov.le Lecce-Arnesano, 73100 Lecce, Italy

Abstract

Fixed a bounded open set Ω of $\mathbf{R}^N$, we completely characterize the weak* lower semicontinuity of functionals of the form

$$F(u,A) = \operatorname*{ess\,sup}_{x\in A} f(x,u(x),Du(x))$$

defined for every $u \in W^{1,\infty}(\Omega)$ and for every open subset $A \subset \Omega$. Without a continuity assumption on $f(\cdot,u,\xi)$ we show that the *supremal* functional F is weakly* lower semicontinuous if and only if it can be represented through a *level convex* function. Then we study the properties of the lower semicontinuous envelope $\overline{F}$ of F. A complete relaxation theorem is shown in the case where f is a continuous function. In the case $f = f(x,\xi)$ is only a Carathéodory function, we show that $\overline{F}$ coincides with the level convex envelope of F.

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Contents

1 Introduction

In the last years the class of the L^∞- functionals has been studied with growing interest in the mathematical literature: these functionals are represented in the so called *supremal form*

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x,u(x),Du(x)) \tag{1.1}$$

where Ω is a bounded open set of $\mathbf{R}^N$ and $u \in W^{1,\infty}(\Omega)$. For this reason they are also referred as *supremal functionals* (see [1]) while we refer to the function f which represents F as an *admissible supremand.*

In the study of variational problems formulated through supremal functionals, it naturally arises the question of the necessary and sufficient conditions on the supremand f which imply the lower semicontinuity of F with respect to the weak* $W^{1,\infty}$ topology. The characterization of lower semicontinuity for a functional expressed by a supremum requires a new notion of convexity: the level convexity. A function $f = f(\xi)$ is said to be level convex (or quasi-convex in [2]) if it has convex sub levels. Namely f is level convex if the set $\{\xi : f(\xi) \le \lambda\}$ is convex for every $\lambda \in \mathbf{R}$; equivalently if

$$f(\theta\xi + (1-\theta)\eta) \le f(\xi) \vee f(\eta)$$

for every $\xi, \eta \in \mathbf{R}^N$ and $\theta \in [0,1]$. Under the assumption that $f(x,\cdot,\cdot)$ is lower semicontinuous, in [2] (see Theorem 3.4 therein) Barron, Jensen, Wang show that the level-convexity of f in the gradient variable is a sufficient condition for the weak* lower semicontinuity in $W^{1,\infty}(\Omega)$ of the functional (1.1). In Theorem 2.7 of the same paper, under a continuity assumption on $f(\cdot,u,\xi)$, they show that this condition is also necessary. Now, the proof of this result heavily relies on the continuity assumption on f: in fact, in the general case, a functional of the form (1.1) can be weak* l.s.c. without its supremand f being level convex, as put in evidence by Remark 3.1 of [12].

In spite of this fact, most of the L^∞-variational problems require that a weakly* l.s.c. functional F could be represented through a level convex function. Now, since a supremal functional does not admit a unique representation, the question whether a weakly* l.s.c. functional always admits a level convex supremand turns out to be interesting and useful for applications. In a previous paper (see [15]) we have considered the case $f = f(x,\xi)$ and we have given a positive answer to this question by showing that a a supremal functional of the form

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, Du(x)) \tag{1.2}$$

is weakly* lower semicontinuous on $W^{1,\infty}(\Omega)$ if and only if F is *a level convex functional* on $W^{1,\infty}(\Omega)$, i.e. the sub-level sets

$$E_\lambda := \{u \in W^{1,\infty}(\Omega) : \ F(u) \le \lambda\}$$

are convex. This result permits one to construct an admissible level convex supremand for a w* l.s.c. supremal functional of the form (1.2).

In this paper we extend our study to the class of functionals $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$ of the form

$$F(u, A) = \operatorname*{ess\,sup}_{A} f(x, u(x), Du(x)) \tag{1.3}$$

where $\mathcal{A}$ is the class of the open subsets of Ω and, without a continuity assumption on $f(\cdot,u,\xi)$, we show that $F(\cdot,A)$ is weakly* l.s.c. in $W^{1,\infty}(\Omega)$ for every $A \in \mathcal{A}$ if and only if it can be represented through a level convex supremand $\tilde{f}$ (see Theorem 2.4). This equivalence is not a trivial consequence of the result in [15]. In fact from the assumptions of Theorem 2.4 we cannot deduce that for every fixed $u_0 \in \mathbf{R}$ the functional $G(u,A) = \operatorname{ess\,sup}_A f(x,u_0,Du(x))$ is weakly* l.s.c. on $W^{1,\infty}(\Omega)$. The strategy we will follow is to consider the distances

$$d_\lambda^{u_0}(z,y) := \sup\left\{ |v(z) - v(y)| \ : \ v \in W^{1,\infty}(B_r(x)) : \operatorname*{ess\,sup}_{B_r(x)} f(y, u_0, Dv(y)) \le \lambda \right\} \tag{1.4}$$

and to apply the results established in [15] about the relationship between the functional F and the convex functionals

$$R(u) = \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{d_\lambda^{u_0}(x,y)}$$

referred to as *difference quotients*. This metric approach was introduced in [12] in the case of the 1-homogeneous supremal functionals and applied, in the general case, in [8] in order to characterize the absolute minimizers of a functional of the form (1.2) and in [15] in order to give equivalent conditions to the weak* lower semicontinuity.

The second part of this paper is devoted to study the w* l.s.c. envelope of a supremal functional. In the case $N = 1$ and $f = f(x,\xi)$, Barron and Liu prove, by using the L^p approximation, that

- the w* l.s.c. envelope $\overline{F}$ of a supremal functional of the form (1.1) coincides with the largest level convex functional minorant of F (see Corollary 2.7 and Theorem 5.4 in [4]);

- if F is represented by a continuous function f, then the level convex envelope f^{lc} of f represents $\overline{F}$ in a supremal form (see Theorem 5.6 in [4]).

With a completely different technique, we extend these results to the N-dimensional case for supremal functionals of the form

$$F(u,A) = \operatorname*{ess\,sup}_A f(x, Du(x)). \tag{1.5}$$

In the study of the relaxation problem, one encounters two open problems. First of all it is unknown if the functional $\overline{F}$ can be represented in some supremal form

$$\overline{F}(u,A) = \operatorname*{ess\,sup}_A \overline{f}(x, Du(x))$$

for every $u \in W^{1,\infty}(\Omega)$ and for every $A \in \mathcal{A}$. The difficult point is to check if $\overline{F}$ satisfies the property of countable supremality

$$\overline{F}\left(u, \bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} \overline{F}(u, A_i) \qquad \forall (A_i)_i \subset \mathcal{A}, \quad \forall u \in W^{1,\infty}(\Omega). \tag{1.6}$$

If $\overline{F}$ satisfies this property, thanks to Theorem 2.2 in [6], we can represent $\overline{F}$ as above. Moreover, by applying Theorem 2.7 in [15], we could deduce that $\overline{F}$ is a level convex functional and we could represent it through a level convex Carathéodory supremand $\overline{f}$.

The second question is that $\overline{f}$ may not coincide with f^{lc}: in fact it is possible to exhibit a supremal functional of the form (1.1) such that its weak* l.s.c. envelope is not represented by f^{lc} (see [12], Example 3.2). The question if $\overline{f}$ is the level convex envelope of some admissible supremand of F naturally arises.

This paper gives some new contributions to the study of the relaxation problem. First of all, we show that if $f = f(x, \xi)$ is a Carathéodory function satisfying a coercivity assumption, then the w* l.s.c. envelope $\overline{F}$ of the functional (1.2) is a level convex functional, i.e. for every $t \in \mathbf{R}$ the level set $\{u \in W^{1,\infty}(\Omega) : \overline{F}(u) \le t\}$ is convex. More precisely we show that $\overline{F}$ is the level convex envelope of F.

Then we discuss the problem of representing $\overline{F}$. In the case $f = f(x, u, \xi)$ globally continuous, we extend the result by Barron and Liu to the N-dimensional case by showing that $\overline{F}$ is itself a supremal functional represented by the level convex envelope f^{lc} of f (see Theorem 2.6). Recall that the same representation was shown in [14] in the 1-dimensional case under the hypothesis that f is only a Borel function. On the contrary, in the N-dimensional case, the analogous result is false without a continuity assumption on $f(\cdot, u, \xi)$, as shown by the already mentioned Example 3.2 in [12]. However, we conclude this paper by showing in Theorem 2.9 that if the w* l.s.c. envelope $\overline{F}$ of the functional (1.5) can be represented in a supremal form, then we can choose as supremand of $\overline{F}$ the level convex envelope of the admissible supremand $\tilde{f}$ of F defined by

$$\tilde{f}(x,\xi) := \inf \left\{ F(u, B_r(x)) \mid \ r > 0,\ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \ \right\}$$

where

$$\widehat{u} := \{x \in \Omega : x \text{ is a differentiability point of } u \text{ and a Lebesgue point of } Du\}.$$

This means, in particular, that the only possible supremal functional which represents $\overline{F}$ is the functional

$$\overline{F}(u,A) = \operatorname*{ess\,sup}_A \tilde{f}^{lc}(x, Du(x)).$$

Finally, we devote Section 3.4 to study the Γ-convergence, as $p \to \infty$, of the sequence of the integral functionals $\Phi_p : L^1(\Omega) \to \overline{\mathbf{R}}$

$$\Phi_p(u) = \begin{cases} \left(\displaystyle\int_\Omega f^p(x, u(x), Du(x)) dx \right)^{1/p} & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

already considered in [7]. In Theorem 3.4 therein, it is shown that if f is a level convex supremand satisfying a coercivity assumption in the gradient variable, then, as $p \to \infty$, the family $(\Phi_p)_p$ Γ-converges, with respect to the uniform convergence, to the functional Φ given by

$$\Phi(u) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), Du(x)) & \text{if } u \in W^{1,\infty}(\Omega), \\ \\ +\infty & \text{otherwise} \end{cases}$$

When f is not a level convex supremand, in [4] Barron and Liu consider in the particular case $N = 1$ the w* l.s.c. envelope $\overline{F}$ of the functional (1.2) defined on $W^{1,\infty}(\Omega)$ and the sequence of the lower semicontinuous envelopes $\overline{F}_p$ (w.r.t. the weak topology of $W^{1,p}(\Omega)$) of the functionals $F_p : W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$F_p(u) = \left(\int_\Omega f^p(x, u(x), Du(x))dx\right)^{1/p}.$$

Under the assumption that f is a globally continuous function, they show that $\overline{F}_p(u)$ converges, as $p \to \infty$, to $\overline{F}(u)$ for every $u \in W^{1,\infty}(\Omega)$ and they could deduce the relaxation formula

$$\overline{F}(u) = \operatorname*{ess\,sup}_{\Omega} f^{lc}(x, u(x), Du(x))$$

for every $u \in W^{1,\infty}(\Omega)$. We conclude this section by studying, in the general N-dimensional framework, the Γ-convergence of the sequence $(\Phi_p)_p$ when f is only a Carathéodory function and we obtain that the Γ-limit, with respect to the uniform convergence, is the functional $\Phi : L^1(\Omega) \to \bar{\mathbf{R}}$ given by

$$\Phi(u) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f^{lc}(x, u(x), Du(x)) & \text{if } u \in W^{1,\infty}(\Omega), \\ \\ +\infty & \text{otherwise.} \end{cases}$$

As a consequence, in the continuous case, the sequence $\overline{F}_p$ pointwise converges to $\overline{F}$ and we obtain a generalization to the case $N > 1$ of the result by Barron and Liu. On the contrary, since in general $\bar{F}(u) \neq \operatorname{ess\,sup}_{x\in\Omega} f^{lc}(x, u(x), Du(x))$, we obtain that it is not possible to approximate $\overline{F}$ through the sequence $\overline{F}_p$.

We shall fix some notations useful in the sequel.

Notations

We denote by Ω a generic open bounded domain of $\mathbf{R}^N$ and by $\mathcal{A}$ the family of open subsets of Ω.

For every $x \in \mathbf{R}^n$ and $r > 0$ we denote by $B_r(x)$ the open ball $\{y \in \mathbf{R}^N \ : \ |x - y| < r\}$ where $|\cdot|$ is the euclidean norm on $\mathbf{R}^N$.

For any set $B \subset \mathbf{R}^n$ we denote by $\mathcal{H}^1(B)$ its one dimensional Hausdorff measure. If moreover $B \subset \mathbf{R}^N$ is a measurable set then $|B|$ will denote its Lebesgue measure.

A modulus of continuity will be any continuous function $w : [0, +\infty) \to [0, +\infty)$ such that $w(0) = 0$.

For every $u \in W^{1,\infty}(\Omega)$ we denote by $\widehat{u}$ the set

$$\widehat{u} := \{x \in \Omega : x \text{ is a differentiability point of } u \text{ and a Lebesgue point of } Du\}.$$

2 The main results

Before stating the main results of this paper, we introduce the following definitions.

Definition 2.1 *A function $f : \Omega \times \mathbf{R}^N \to \overline{\mathbf{R}}$ is said to be*

(a) *a* Carathéodory supremand *if*

(i) for every $\xi \in \mathbf{R}^N$ the function $x \mapsto f(x, \xi)$ is measurable in Ω;

(ii) for a. a. $x \in \Omega$ the function $\xi \mapsto f(x, \xi)$ is continuous in $\mathbf{R}^N$;

(b) *a* level convex Carathéodory supremand *if f is a Carathéodory supremand and $f(x,\cdot)$ is level convex on $\mathbf{R}^N$ for almost every $x \in \Omega$ i.e. for every $t \in \mathbf{R}$ the level set $\{\xi \in \mathbf{R}^N \ : \ f(x,\xi) \le t\}$ is convex.*

Definition 2.2 *A function $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \overline{\mathbf{R}}$ is said to be*

(a) *a* Carathéodory supremand *if*

(i) for every $(u,\xi) \in \mathbf{R} \times \mathbf{R}^N$ the function $x \mapsto f(x,u,\xi)$ is measurable in Ω;

(ii) for a. a. $x \in \Omega$ the function $(u,\xi) \mapsto f(x,u,\xi)$ is continuous in $\mathbf{R} \times \mathbf{R}^N$;

(b) *a* level convex Carathéodory supremand *if f is a Carathéodory supremand and $f(x,u,\cdot)$ is level convex on $\mathbf{R}^N$ for almost every $x \in \Omega$ and for every $u \in \mathbf{R}$.*

Definition 2.3 *A functional $F : X \to \overline{\mathbf{R}}$ defined on a topological vector space X is said to be* level convex *if for every $t \in \mathbf{R}$ the level set $\{u \in X \ : \ F(u) \le t\}$ is convex.*

Now we are in a position to state the main theorems of this paper. First of all we show that a weakly* l.s.c. supremal functional $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$ of the form

$$F(u,A) = \operatorname*{ess\,sup}_A f(x,u(x),Du(x)) \tag{2.7}$$

can be represented through a level convex supremand $\tilde{f}$ and we give an explicit formula for $\tilde{f}$.

Theorem 2.4 *Let $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying the following assumptions:*

1) there exists an increasing continuous function $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\lim_{t\to+\infty} \alpha(t) = +\infty$ and

$$f(x,u,\xi) \ge \alpha(|\xi|) \qquad \text{for a.e } x \in \Omega, \text{ for every } u \in \mathbf{R} \text{ and } \xi \in \mathbf{R}^N; \tag{2.8}$$

2) for any $M > 0$ there exists a modulus of continuity ω_M such that

$$|f(x,u,\xi) - f(x,v,\eta)| \le \omega_M(|\xi - \eta| + |u - v|) \tag{2.9}$$

for a.e. $x \in \Omega$, for every $\xi, \eta \in B_M(0)$ and $|u|, |v| \le M$.

Let $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$ be the functional defined by (2.7). The following facts are equivalent:

1) $F(\cdot, A)$ is weakly l.s.c. on $W^{1,\infty}(\Omega)$ for every $A \in \mathcal{A}$;*

2) there exists a level convex Carathéodory supremand $\tilde{f} : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ such that

$$F(u,A) = \operatorname*{ess\,sup}_A \tilde{f}(x,u(x),Du(x)) \tag{2.10}$$

for all $u \in W^{1,\infty}(\Omega)$ and for all $A \in \mathcal{A}$. Moreover $\tilde{f}$ is given by

$$\tilde{f}(x,u,\xi) := \inf\left\{F(u,B_r(x))| \quad r>0,\ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } v(x) = u,\ Dv(x) = \xi \quad \right\}, \tag{2.11}$$

and $\tilde{f}$ satisfies (2.8), (2.9) (for a suitable family $(\omega'_M)_M$ of moduli of continuity) and for a.e. $x \in \Omega$ $\tilde{f}(x,\cdot,\cdot) \ge f(x,\cdot,\cdot)$.

Remark 2.5 **(i)** The supremand $\tilde{f}$ in (2.11) was introduced by P. Cardaliaguet and F. Prinari in [6] in order to represent a general weakly* lower semicontinuous functional F defined on $W^{1,\infty}(\Omega) \times \mathcal{A}$ in a supremal form (see Theorem 3.1 in the next section). But the authors cannot deduce if $\tilde{f}$ is a level convex function. The proof of this property is the main extension of Theorem 2.4 to the results in [6].

(ii) In the previous theorem, when $f(\cdot, u, \xi)$ is continuous for all $(u, \xi) \in \mathbf{R} \times \mathbf{R}^N$, it is easy to obtain the inequality $\tilde{f}(x, \cdot, \cdot) \leq f(x, \cdot, \cdot)$ for a.e. $x \in \Omega$ and therefore we have that if F is weakly* l.s.c. then f is a level convex supremand, i.e. the necessary condition shown in [2], Theorem 2.7. In the general case, under the assumptions of Theorem 2.4, it is not possible to deduce that $\tilde{f} = f^{lc}$: in fact, in the Example 8.1 of [15], the author exhibits a w^* l.s.c. functional F of the form (2.7) such that $\sup_\Omega f^{lc}(x, u(x), Du(x)) < F(u, \Omega)$.

(iii) The case $f = f(x, \xi)$ was completely studied in [15] where, among the results, it has been shown that a w* l.s.c. functional of the form

$$F(u) = \operatorname*{ess\,sup}_{\Omega} f(x, Du(x))$$

can be represented through a level convex supremand.

(iv) The proof of Theorem 2.4 uses in an essential way the fact the *localized* functional $F(\cdot, A)$ is w* l.s.c. *for every* $A \in \mathcal{A}$. For this reason it cannot be adapted to show that a w* l.s.c. functional of the form

$$F(u) = \operatorname*{ess\,sup}_{\Omega} f(x, u(x), Du(x)) \tag{2.12}$$

can be represented through a level convex supremand. Notice that Barron, Jensen and Wang need the same assumption in the case $f = f(\cdot, u, \xi)$ continuous.

The second part of this paper is devoted to the relaxation problem. First of all we give an explicit representation for the weak* lower semicontinuous envelope of a supremal functional on $W^{1,\infty}(\Omega)$ under a continuity assumption. We will prove that if the functional (2.12) is represented by a continuous and coercive function f, then

$$\overline{F}(u) := \sup\left\{G(u) \ : \ G : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}},\ G \text{ w}^*\text{l.s.c.},\ G \leq F \text{ on } W^{1,\infty}(\Omega)\right\} \tag{2.13}$$

is a supremal functional represented by the level convex envelope f^{lc} of f defined by

$$f^{lc}(x, u, \cdot) := \sup\{h : \mathbf{R}^N \to \overline{\mathbf{R}} :\ h \text{ lower semicontinuous and level convex},\ h(\cdot) \leq f(x, u, \cdot)\}$$

for every $x \in \Omega$ and $u \in \mathbf{R}$.

Theorem 2.6 *Let* $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ *be a continuous function satisfying (2.8) and let* $F : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}}$ *be the functional defined by (2.12). Then*

$$\overline{F}(u) = \operatorname*{ess\,sup}_{\Omega} f^{lc}(x, u(x), Du(x))$$

for every $u \in W^{1,\infty}(\Omega)$.

In particular this representation result holds when $f = f(\xi)$ is a continuous and coercive function. In the general case the previous result is false: it is possible to exhibit a supremal functional of the form (2.12) such that its w* lower semicontinuous envelope is not represented by f^{lc} (see [12], Example 3.2). However we can establish some properties of $\overline{F}$ when F is of the form

$$F(u) = \operatorname*{ess\,sup}_{x \in \Omega} f(x, Du(x)) \tag{2.14}$$

and f is only a Carathéodory supremand. First we show that $\overline{F}$ is a level convex functional.

Theorem 2.7 *Let* $\Omega \subset \mathbf{R}^N$ *be a bounded open set with a Lipschitz boundary. Let* $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ *be a Carathéodory supremand satisfying the following assumptions:*

1) there exists an increasing continuous function $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $\lim_{t \to +\infty} \alpha(t) = +\infty$ *and*

$$f(x, \xi) \geq \alpha(|\xi|) \qquad \text{for a.e } x \in \Omega, \text{ and } \xi \in \mathbf{R}^N; \tag{2.15}$$

2) for any $M > 0$ there exists a modulus of continuity ω_M such that

$$|f(x,\xi) - f(x,\eta)| \le \omega_M(|\xi - \eta|) \tag{2.16}$$

for a.e. $x \in \Omega$, for every $\xi, \eta \in B_M(0)$.

Let $F : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}}$ be the functional defined by (2.14). Then $\overline{F}$ is a level convex functional. In particular $\overline{F} = F^{lc}$ where

$$F^{lc}(u) := \sup \big\{ H(u) \ : \ H : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}}, \ H \ w^* \ \text{l.s.c. and level convex}, \ H \le F \ \text{on} \ W^{1,\infty}(\Omega) \big\}.$$

Remark 2.8 Consider now a supremal functional F on $W^{1,\infty}(\Omega) \times \mathcal{A}$ of the form

$$F(u, A) = \operatorname*{ess\,sup}_{x \in A} f(x, Du(x)). \tag{2.17}$$

According to the results in [6], if $\overline{F}$ satisfies the property of countable supremality (1.6), then $\overline{F}$ can be represented in a supremal form through the function $\overline{f}$ given by

$$\overline{f}(x,\xi) := \inf \big\{ \overline{F}(u, B_r(x)) \mid \ r > 0, \ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \ \ \big\}.$$

The previous theorem implies that $\overline{f}$ is a level convex function (see the proof of Theorem 2.9). Naturally the question arises if $\overline{f}$ is the level convex envelope of some supremand of F. In the Example 3.2 in [12], the authors exhibit a functional F of the form (2.14) such that $\sup_\Omega f^{lc}(x, Du) < \overline{F}(u, \Omega)$. This shows that in general $\overline{f} \ne f^{lc}$. However, according to the Theorem 3.4 in [15], we can write F also in the supremal form

$$F(u, A) = \operatorname*{ess\,sup}_{A} \tilde{f}(x, Du(x)) \qquad \forall u \in W^{1,\infty}(\Omega), \quad \forall A \in \mathcal{A},$$

where $\tilde{f}$ is given by

$$\tilde{f}(x,\xi) := \inf \big\{ F(u, B_r(x)) \mid \ r > 0, \ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \ \ \big\}. \tag{2.18}$$

The following result is based on the key remark that

$$\overline{f}(x, \cdot) = (\tilde{f}(x, \cdot))^{lc}$$

for a.e. $x \in \Omega$.

Theorem 2.9 *Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying (2.15). Then the following facts are equivalent:*

(1) *$\overline{F}$ satisfies the countable supremality (1.6);*

(2) *$\overline{F}(u, A) = \operatorname{ess\,sup}_A \tilde{f}^{lc}(x, Du(x))$ for every $u \in W^{1,\infty}(\Omega)$ and for every $A \in \mathcal{A}$.*

Moreover, under a stronger growth condition, we can show the following result:

Theorem 2.10 *Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying (2.15) and such that:*

$$c|\xi| \le f(x,\xi) \le B + C|\xi| \qquad \forall x \in \Omega, \forall \xi \in \mathbf{R}^N \tag{2.19}$$

where c, B, C are non-negative constants. Let $F(\cdot, A)$ be the functional defined by (2.17) and let $\tilde{f}$ be the function given by (2.18). Let $F_p : W^{1,p}(\Omega) \times \mathcal{A} \to \mathbf{R}^+$ be the functional defined by

$$F_p(u, A) = \left(\int_A \tilde{f}^p(x, Du(x)) dx \right)^{1/p}$$

and let $\overline{F}_p$ be the lower semicontinuous envelope (w.r.t. the weak topology of $W^{1,p}(\Omega)$) of F_p. Then

$$\lim_{p \to \infty} \overline{F}_p(u, A) = \operatorname*{ess\,sup}_{A} \tilde{f}^{lc}(x, Du(x))$$

for every $u \in W^{1,\infty}(\Omega)$ and for every $A \in \mathcal{A}$. In particular (1) and (2) in Theorem 2.10 are equivalent to:

(3) *for every* $A \in \mathcal{A}$ *the family* $(\overline{F}_p(\cdot, A))_p$ *converges to* $\overline{F}(\cdot, A)$ *as* $p \to +\infty$.

In particular, in the continuous case, we may put together the previous results and obtain the following generalization of Lemma 5.2 in [4] to the case $N > 1$:

Corollary 2.11 *Let* $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ *be a continuous function satisfying (2.19). Let* $F_p : W^{1,p}(\Omega) \to \mathbf{R}^+$ *be the functional defined by*

$$F_p(u) = \left(\int_\Omega f^p(x, Du(x))dx \right)^{1/p}$$

and let $\overline{F}_p$ *be the lower semicontinuous envelope (w.r.t. the weak topology of* $W^{1,p}(\Omega)$*) of* F_p*. Then, as* $p \to +\infty$*, the family* $\overline{F}_p(u)$ *converges to* $\overline{F}(u)$ *for every* $u \in W^{1,\infty}(\Omega)$.

Remark 2.12 Note that, without the continuity assumption, the previous result is false since in general

$$\overline{F}(u) > \operatorname*{ess\,sup}_\Omega f^{lc}(x, Du(x)).$$

In order to show all the results above, we introduce some tools, recall some known facts and prove further preliminary results. For these reasons, the proofs of the previous theorems are postponed until Section 4.

3 Preliminaries

3.1 Some representation theorems

In [6] the authors characterize the class of the weakly* lower semicontinuous functionals $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$ which can be written in the supremal form. Given an abstract functional F, they construct the function $\tilde{f}$ given by (2.11) and, under some suitable assumptions on F, they show that F can be represented as a supremal functional through the supremand $\tilde{f}$.

Theorem 3.1 *(Theorem 2.2 in [6]). Let* $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \overline{\mathbf{R}}$ *be a functional. Assume that* F *satisfies the following properties:*

(i) $F(u, A) = F(v, B)$ *for every* $u, v \in W^{1,\infty}(\Omega)$ *such that* $u(x) = v(x)$ *for any* $x \in A \cup B$ *and for every* $A, B \in \mathcal{A}$ *with* $\mathcal{L}(A \triangle B) = 0$*;*

(ii) $F\left(u, \bigcup_{i=1}^\infty A_i\right) = \bigvee_{i=1}^\infty F(u, A_i)$ *for every* $(A_i)_i \in \mathcal{A}$ *and for every* $u \in W^{1,\infty}(\Omega)$*;*

(iii) $\forall M > 0$ *there exists a modulus of continuity* ω_M *such that*

$$|F(u, A) - F(v, A)| \le \omega_M(||u - v||_{W^{1,\infty}(A)})$$

for every $A \in \mathcal{A}$ *and for every* $u, v \in W^{1,\infty}(\Omega)$ *s.t.* $||u||_{W^{1,\infty}(\Omega)}, ||v||_{W^{1,\infty}(\Omega)} \le M$;

(iv) $F(\cdot, A)$ *is weakly* l.s.c. for every* $A \in \mathcal{A}$*;*

(v) *there exists an increasing continuous function* $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $\lim_{t \to +\infty} \alpha(t) = +\infty$ *and for every* $A \in \mathcal{A}$, $F(\cdot, A) \ge \alpha(|| \cdot ||_{W^{1,\infty}(A)})$.

Then there exists a Carathéodory supremand $\tilde{f} : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ *defined by (2.11) such that*

$$F(u, A) = \operatorname*{ess\,sup}_A \tilde{f}(x, u(x), Du(x))$$

for any $u \in W^{1,\infty}(\Omega)$ *and any* $A \in \mathcal{A}$*. Moreover* $\tilde{f}$ *satisfies (2.9) (for a suitable family* $(\omega'_M)_M$ *of moduli of continuity) and (2.8).*

Remark 3.2 Note that, by applying Theorem 2.4, we can conclude that the function $\tilde{f}$ above is a level convex in the gradient variable.

Now we recall also some results shown in [15] and devoted to the construction of admissible supremands for supremal functionals F of the form (2.14). The following result is inspired by the proof of Lemma 3.3 in [6].

Proposition 3.3 *(Theorem 5.1 in [15]) Let Ω be an open subset of $\mathbf{R}^N$. Let $F : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}}$ be a functional such that for any $M > 0$ there exists a modulus of continuity ω_M such that*

$$|F(u) - F(v)| \leq \omega_M(||Du - Dv||_\infty) \tag{3.20}$$

for every $u, v \in W^{1,\infty}(\Omega)$ s.t. $||Du||_\infty, ||Dv||_\infty \leq M$. Let $\varphi : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be defined by

$$\varphi(x, \xi) := \inf \left\{ F(u) \mid \ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \ \right\}. \tag{3.21}$$

Then for every $\xi \in \mathbf{R}^N$ the function $x \mapsto \varphi(x, \xi)$ is measurable in Ω.
Moreover if there exists an increasing continuous function $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\lim_{t\to+\infty} \alpha(t) = +\infty$ and

$$F(u) \geq \alpha(||Du||_\infty) \tag{3.22}$$

then

(i) *φ is a Carathéodory supremand satisfying (2.15) and (2.16) for a suitable family $(\omega'_M)_M$ of moduli of continuity;*

(ii) *for any $u \in W^{1,\infty}(\Omega)$*

$$F(u) \geq \operatorname*{ess\,sup}_\Omega \varphi(x, Du(x)).$$

The following theorem shows that the function $\tilde{f}$ defined by (2.18) is an admissible supremand for a localized supremal functional $F(u, A) = \operatorname{ess\,sup}_A f(x, Du(x))$ without requiring that for every open set $A \subset \Omega$ $F(\cdot, A)$ is weakly* lower semicontinuous.

Theorem 3.4 *(Theorem 5.4 in [15]) Let Ω be an open subset of $\mathbf{R}^N$. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}^+$ be a Carathéodory supremand satisfying assumptions (2.16) and (2.15). Then*

(i) *the function $\tilde{f}$ given respectively by (2.18) is a Carathéodory supremand satisfying (2.15) and (2.16) for a suitable family $(\omega'_M)_M$ of moduli of continuity;*

(ii) *for every $u \in W^{1,\infty}(\Omega)$ and for every $A \in \mathcal{A}$*

$$\operatorname*{ess\,sup}_A f(x, Du(x)) = \operatorname*{ess\,sup}_A \tilde{f}(x, Du(x));$$

(iii) *there exists a negligible set $H \subset \Omega$ such that*

$$\tilde{f} \geq f \text{ on } (\Omega \setminus H) \times \mathbf{R}^N;$$

(iv) *if $f(\cdot, \xi)$ is continuous on Ω for every $\xi \in \mathbf{R}^N$ then there exists a negligible set $H \subset \Omega$ such that $f = \tilde{f}$ on $(\Omega \setminus H) \times \mathbf{R}^N$.*

3.2 Necessary and sufficient conditions for w* lower semicontinuity

In this subsection we recall some results shown in ([15]). For a large class of situations the level convexity of the functional F is a consequence of its weak* lower semicontinuity.

Theorem 3.5 *(Theorem 2.3 in [15]) Let $\Omega \subset \mathbf{R}^N$ be a connected open set with Lipschitz boundary. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying (2.16). If the functional F defined by (2.14) is weakly* l.s.c. on $W^{1,\infty}(\Omega)$ then F is a level convex functional.*

Unfortunately, in the general case, the above result does not give as a consequence the level convexity of $f(x,\cdot)$ (see Remark 3.1 of [12]). However, the next theorem states that there exists at least a level convex supremand $\tilde{f}$ for a level convex supremal functional F.

Theorem 3.6 *(Theorem 2.7 in [15]) Let Ω be an open subset of $\mathbf{R}^N$. Let $f:\Omega\times\mathbf{R}^N\to\mathbf{R}$ be a Carathéodory supremand satisfying (2.15) and (2.16). Let $F(\cdot,A)$ be the functional defined by (2.17). The following facts are equivalent:*

1) $F(\cdot,A)$ is weakly l.s.c. on $W^{1,\infty}(\Omega)$ for every $A\in\mathcal{A}$;*

2) $F(\cdot,A)$ is a level convex functional for every $A\in\mathcal{A}$;

3) there exists a level convex normal supremand $\tilde{f}:\Omega\times\mathbf{R}^N\to\mathbf{R}$ given by (2.18) such that

$$F(u,A)=\operatorname*{ess\,sup}_A \tilde{f}(x,Du(x))$$

for all $u\in W^{1,\infty}(\Omega)$ and for all $A\in\mathcal{A}$. Moreover $\tilde{f}$ satisfies (2.15), (2.16) (for a suitable family $(\omega'_M)_M$ of moduli of continuity) and for a.e. $x\in\Omega$ $\tilde{f}(x,\cdot)\geq f(x,\cdot)$.

3.3 The class of the difference quotients

Now we introduce the distances first introduced in [12] in the 1-homogeneous case and later considered in [15] in the general case (see Section 2 therein). Consider a supremal functional (2.14) represented through a Carathéodory supremand $f:\Omega\times\mathbf{R}^N\to\mathbf{R}^+$ satisfying (2.16) and (2.15). With every $\lambda\in\mathbf{R}$ such that the sub level set $E_\lambda:=\{u\in W^{1,\infty}(\Omega)\ :\ F(u)<\lambda\}$ is nonempty we can associate a distance d_λ in the following way:

$$d_\lambda(x,y):=\sup\Big\{|u(x)-u(y)|\ :\ u\in W^{1,\infty}(\Omega):\ F(u)\leq\lambda\Big\}. \tag{3.23}$$

The distance d_λ satisfies the following properties:

(i) if Ω is a connected open set, then for every $x,y\in\Omega$ it holds

$$d_\lambda(x,y)\leq\alpha^{-1}(\lambda)|x-y|_\Omega \tag{3.24}$$

where

$$|x-y|_\Omega=\inf\{\mathcal{L}(\gamma)\ :\ \gamma\in\Gamma_{x,y}(\Omega)\},$$

$\Gamma_{x,y}(\Omega)$ being the set of Lipschitz curves in Ω with end-points x and y, and $\mathcal{L}(\gamma)$ the Euclidean length of γ. In particular if $\partial\Omega$ is Lipschitz continuous then there exists a constant $C>0$ such that

$$d_\lambda(x,y)\leq|x-y|_\Omega\leq C|x-y|. \tag{3.25}$$

(ii) With every $\lambda\in\mathbf{R}$ there exists $\delta=\delta(\lambda)$ such that for every $x,y\in\Omega$

$$d_\lambda(x,y)\geq\delta|x-y|. \tag{3.26}$$

Now for every λ such that E_λ is nonempty, we consider the functional $R_\lambda:W^{1,\infty}(\Omega)\to\bar{\mathbf{R}}$ given by

$$R_\lambda(u):=\sup_{x,y\in\Omega,\,x\neq y}\frac{|u(x)-u(y)|}{d_\lambda(x,y)}. \tag{3.27}$$

The functional R_λ is referred to as the *difference quotient* associated with level set E_λ of F. The following proposition establishes the main properties of R_λ.

Proposition 3.7 *For every λ s.t. $E_\lambda\neq\emptyset$ the difference quotient R_λ is a convex lower semicontinuous functional with respect to the strong convergence in L^∞. Moreover $R_\lambda(u+v)\leq R_\lambda(u)+R_\lambda(v)$ for every $u,v\in W^{1,\infty}(\Omega)$.*

Proof. Let $u \in W^{1,\infty}(\Omega)$ and let $\{u_n\} \subset W^{1,\infty}(\Omega)$ be a sequence converging to u in $L^\infty(\Omega)$. We have that for every x, $y \in \Omega$ such that $0 < d_\lambda(x,y) < +\infty$

$$\frac{|u(x)-u(y)|}{d_\lambda(x,y)} = \lim_n \frac{|u_n(x)-u_n(y)|}{d_\lambda(x,y)} \le \liminf_n R_\lambda(u_n).$$

Taking the supremum for $x, y \in \Omega, x \neq y$ we get the thesis. The convexity and the sublinearity of R_λ are trivial. □

The key tool we will use in the sequel is the following lemma. It is an adaptation of Lemma 3.4 in [12]. For a proof see Lemma 3.2 in [15].

Lemma 3.8 *Let F be a supremal functional on $W^{1,\infty}(\Omega)$ represented by a Carathéodory supremand $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ satisfying (2.16) and (2.15). Let $v \in W^{1,\infty}(\Omega)$ be such that $R_\lambda(v) < 1$. Then there exists a sequence $\{v_n\} \subset W^{1,\infty}(\Omega)$ converging to v in $L^\infty(\Omega)$ with $F(v_n) \le \lambda$ for $n \in \mathbf{N}$.*

3.4 The L^p-approximation.

In this section we generalize the results shown in [2] and in [7]. In fact, we study the Γ-convergence of the sequence $\Phi_p : L^1(\Omega) \to \bar{\mathbf{R}}$ defined by

$$\Phi_p(u) := \begin{cases} \left(\int_\Omega f^p(x, u(x), Du(x))dx \right)^{1/p} & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{3.28}$$

without assuming that $f(x, u, \cdot)$ is level convex.

First of all, we recall that a given sequence of functionals G_p defined in a metric space X, Γ-*converges* to the functional G, as $p \to \infty$, if the following properties hold:

(a) for every $u \in X$ and for every sequence $\{u_p\}$ converging to u in X we have

$$G(u) \le \liminf_{p\to\infty} G_p(u_p).$$

(b) for every $u \in X$ there exists a sequence $\{u_p\}$ (recovery sequence) such that

$$G(u) \ge \limsup_{p\to\infty} G_p(u_p).$$

We will refer to (a) as the $\Gamma(X)$-liminf inequality and to (b) as the $\Gamma(X)$-limsup inequality.

The main result of this subsection is the following:

Theorem 3.9 *Let Ω be a bounded open set of $\mathbf{R}^N$. Let $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^+$ be a Carathéodory supremand satisfying (2.9) and such that*

$$c|\xi| \le f(x, u, \xi) \le B + C(|u| + |\xi|) \tag{3.29}$$

for every $x \in \Omega, u \in \mathbf{R}, \xi \in \mathbf{R}^N$ where c, B, C are non-negative constants. Let $\Phi_p : L^1(\Omega) \to \bar{\mathbf{R}}$ be the functional defined by (3.28). Then, as $p \to +\infty$, the family $(\Phi_p)_p$ Γ-converges with respect to the topology of the uniform convergence to the functional $\Phi : L^1(\Omega) \to \mathbf{R}^+$ given by

$$\Phi(u) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f^{lc}(x, u(x), Du(x)) & \textit{if } u \in W^{1,\infty}(\Omega), \\ +\infty & \textit{otherwise.} \end{cases} \tag{3.30}$$

Proof. Let $(f^p)^{**}$ be the convex envelope of the function f^p. Since $((f^p)^{**})^{1/p}$ is a level convex function such that $((f^p)^{**})^{1/p} \leq f$, then, by definition of f^{lc}, we have that

$$((f^p)^{**})^{1/p} \leq f^{lc}. \tag{3.31}$$

Now fix $p > N$ and let $\overline{F}_p$ the lower semicontinuous envelope (w.r.t. the weak topology of $W^{1,p}(\Omega)$) of the integral functional $F_p : W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$F_p(u) = \left(\int_\Omega f^p(x, u(x), Du(x))dx \right)^{1/p}.$$

It is well known (see, for example, Theorem 2 in [13]) that, for every $u \in W^{1,p}(\Omega)$,

$$\overline{F}_p(u) = \left(\int_\Omega (f^p)^{**}(x, u(x), Du(x))dx \right)^{1/p}.$$

Now let G_p be the weak* lower semicontinuous envelope of F_p w.r.t. the uniform convergence. We show that

$$\overline{F}_p(u) = G_p(u) \quad \forall u \in W^{1,p}(\Omega).$$

In fact if $(u_n)_n \subset W^{1,p}(\Omega)$ converges uniformly to u and $\liminf_{n\to\infty} \overline{F}_p(u_n) < +\infty$, since $(f^p)^{**}(x, u, \xi) \geq c\|\xi\|^p$ we have that the sequence $(\|Du_n\|_{L^p(\Omega)})_n$ is bounded. In particular $u_n \to u$ weakly in $W^{1,p}(\Omega)$. Then $\overline{F}_p(u) \leq \liminf_{n\to\infty} \overline{F}_p(u_n)$ which means that $\overline{F}_p$ is lower semicontinuous w.r.t. the uniform convergence. Then $\overline{F}_p \leq G_p$. On the other hand, thanks to the compact embedding of $W^{1,p}(\Omega)$ in $C(\overline{\Omega})$, G_p is lower semicontinuous w.r.t. the weak topology of $W^{1,p}(\Omega)$. Then $G_p = \overline{F}_p$. In particular it is easy to show that for every $p > N$ the weak* lower semicontinuous envelope of the functional Φ_p w.r.t. the uniform convergence is the functional

$$\overline{\Phi}_p(u) = \begin{cases} \overline{F}_p(u) & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

By applying Proposition 6.1 in [10], this implies that

$$\Gamma(L^\infty)\text{-}\limsup_{p\to\infty} \Phi_p = \Gamma(L^\infty)\text{-}\limsup_{p\to\infty} \overline{F}_p. \tag{3.32}$$

Now, thanks to (3.31), we have that

$$\Gamma(L^\infty)\text{-}\limsup_{p\to\infty} \overline{F}_p \leq \Phi.$$

Therefore it holds

$$\Gamma(L^\infty)\text{-}\limsup_{p\to\infty} \Phi_p \leq \Phi,$$

i.e. the Γ-limsup inequality. Now let $H_p : L^1(\Omega) \to \mathbf{R}^+$ be the functional defined by

$$H_p(u) = \begin{cases} \left(\int_\Omega (f^{lc}(x, u(x), Du(x))^p dx \right)^{1/p} & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

By Theorem 4.1 in [7], we know that

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty} H_p = \Phi.$$

This implies that for every $u \in W^{1,\infty}(\Omega)$

$$\Phi \leq \Gamma(L^\infty)\text{-}\liminf_{p\to\infty} H_p \leq \Gamma(L^\infty)\text{-}\liminf_{p\to\infty} \Phi_p, \tag{3.33}$$

i.e. the Γ-liminf inequality. $\square$

Remark 3.10 We notice that, in Theorem 3.9, if $f = f(x, \xi)$ then the assumption (2.9) can be dropped.

The previous result will be applied in the proof of Theorem 2.9. However here we conclude this section by discussing some easy but not trivial consequences of the result above. A first consequence is the following:

Corollary 3.11 *Let $f : \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying*

$$c|\xi| \leq f(\xi) \leq B + C|\xi|$$

for every $\xi \in \mathbf{R}^N$ where B, C are non-negative constants. Then for every $\xi \in \mathbf{R}^N$

$$\lim_{p\to\infty} ((f^p)^{**})^{1/p}(\xi) = f^{lc}(\xi).$$

Proof. Thanks to (3.31), we have that $\limsup_{p\to\infty}((f^p)^{**})^{1/p}(\xi) \leq f^{lc}(\xi)$. Moreover, by (3.33), it follows that for every $\xi \in \mathbf{R}^N$, if $u_\xi := x \cdot \xi$, we have that

$$\begin{aligned} f^{lc}(\xi) &= \operatorname*{ess\,sup}_{x\in\Omega} f^{lc}(Du_\xi(x)) \leq \liminf_{p\to\infty} \left(\int_\Omega (f^p)^{**}(Du_\xi(x))dx \right)^{1/p} \\ &= \liminf_{p\to\infty} ((f^p)^{**})^{1/p}(\xi)|\Omega|^{1/p} = \liminf_{p\to\infty} ((f^p)^{**})^{1/p}(\xi). \end{aligned}$$

□

Remark 3.12 Notice that we can provide a direct proof of the previous corollary using a characterization of f^{lc} given in [4]. In this paper the authors study the problem of giving an explicit representation of the greatest l.s.c. level convex minorant of a general function f defined on a Banach space: the level convex conjugates of f introduced therein are useful, for example, in expressing Hopf-Lax formulas for HJB equations. Among the results, in Theorem 5.1 in [4] it is shown that for every coercive function g and for every $\xi \in \mathbf{R}^N$

$$g^{lc}(\xi) = \min \left\{ \max_{1\leq i\leq N+1} \lambda_i g(\xi_i) : \sum_{i=1}^{N+1} \lambda_i \xi_i = \xi,\ \lambda_i \geq 0\ \forall i, \sum_{i=1}^{N+1} \lambda_i = 1 \right\}.$$

Now, since for every function g and for every $\xi \in \mathbf{R}^N$ we know that

$$g^{**}(\xi) = \min \left\{ \sum_{i=1}^{N+1} \lambda_i g(\xi_i) : \sum_{i=1}^{N+1} \lambda_i \xi_i = \xi,\ \lambda_i \geq 0\ \forall i, \sum_{i=1}^{N+1} \lambda_i = 1 \right\},$$

for every $p \geq 1$ and for every $\xi \in \mathbf{R}^N$ we can write $[(f^p)^{**}]^{1/p}$ as

$$[(f^p)^{**}]^{1/p}(\xi) = \min \left\{ \big(\sum_{i=1}^{N+1} \lambda_i f^p(\xi_i) \big)^{1/p} : \sum_{i=1}^{N+1} \lambda_i \xi_i = \xi,\ \lambda_i \geq 0\ \forall i, \sum_{i=1}^{N+1} \lambda_i = 1 \right\}.$$

Then it is easy to see that

$$\lim_{p\to\infty} [(f^p)^{**}]^{1/p}(\xi) = \min \left\{ \max_{1\leq i\leq N+1} f(\xi_i) : \sum_{i=1}^{N+1} \lambda_i \xi_i = \xi,\ \lambda_i \geq 0\ \forall i, \sum_{i=1}^{N+1} \lambda_i = 1 \right\} = f^{lc}(\xi).$$

Under the assumption that f is a level convex Carathéodory function, in [7] the authors show that if u_p is a sequence of functions such that

$$F_p^p(u_p, \Omega) \leq \inf F_p^p(., \Omega) + \varepsilon_p$$

and if $u_p \to u_0$ uniformly, then u_0 is a AML for the supremal functional

$$F(u, V) = \operatorname*{ess\,sup}_V f(x, u(x), Du(x))$$

i.e. for all open subset $V \subset\subset \Omega$

$$F(u_0, V) \le F(v, V) \quad \forall v \in W^{1,\infty}(V) \quad \text{ s.t. } v = u_0 \text{ on } \partial V.$$

By applying Theorem 3.9 and just repeating the proof of Proposition 4.3 in [7], we can determine the asymptotic behavior of the sequence $(u_p)_p$ also when f is not level convex.

Corollary 3.13 *Under the assumptions of Theorem 3.9 or of Theorem 3.10 if $(u_p)_p$ is a sequence of functions such that*

$$F_p^p(u_p, \Omega) \le \inf F_p^p(., \Omega) + \varepsilon_p$$

where $(\varepsilon_p)_p$ is such that $\lim_{p\to\infty} \sqrt[p]{\varepsilon_p} = 0$ then, every cluster points of a sequence $(u_p)_p$ is an AML for the functional

$$F(u, V) = \operatorname*{ess\,sup}_V f^{lc}(x, u(x), Du(x)).$$

4 The proofs

Now we are in a position to show the main theorems of this paper. In the following proof it is fundamental the application of Lemma 3.8 in order to deduce the convexity of the sub-level sets of a weakly* lower semicontinuous functional.

Proof of Theorem 2.4. 1)$\Longrightarrow$ 2). It is easy to show that the functional (2.7) satisfies all the assumptions of Theorem 3.1. Then the function $\tilde{f}$ given by (2.11) is a Carathéodory supremand satisfying (2.9) (for a suitable family $(\omega'_M)_M$ of moduli of continuity) and (2.8) and such that

$$F(u, A) = \operatorname*{ess\,sup}_A \tilde{f}(x, u(x), Du(x))$$

for any $u \in W^{1,\infty}(\Omega)$ and any $A \in \mathcal{A}$. Now we show that $\tilde{f}$ is level convex w.r.t. ξ. Fix $x \in \Omega$, $u \in \mathbf{R}$, $\xi, \eta \in \mathbf{R}^N$ and $\lambda \in (0, 1)$ such that $\tilde{f}(x, u, \xi) \le \lambda$ and $\tilde{f}(x, u, \eta) \le \lambda$. By definition of $\tilde{f}$ there exist $B_r(x) \subset \Omega$, $u_\varepsilon, v_\varepsilon \in W^{1,\infty}(\Omega)$, differentiable at x, such that $u_\xi(x) = u_\eta(x) = u$, $Du_\xi(x) = \xi$, $Du_\eta(x) = \eta$ and $\tilde{f}(x, u, \xi) \ge F(u_\xi, B_r(x)) - \varepsilon$ and $\tilde{f}(x, u, \eta) \ge F(u_\eta, B_r(x)) - \varepsilon$. In particular we can choose $0 < r' \le r$ such that $||u_\xi - u||_{L^\infty(B_{r'}(x))} \le \varepsilon$ and $||u_\eta - u||_{L^\infty(B_{r'}(x))} \le \varepsilon$. Define $v := \theta u_\eta + (1-\theta)u_\xi$. Thus $v(x) = u$ and $Dv(x) = \theta\eta + (1-\theta)\xi$. Notice that $||v - u||_{L^\infty(B_{r'}(x))} \le 2\varepsilon$. Let $M = max\{||u_\eta||_{L^\infty(B_{r'}(x))}, ||u_\xi||_{L^\infty(B_{r'}(x))}, |u|\}$. Then, for every $y \in B_{r'}(x)$, we have that

$$\tilde{f}(y, u, Du_\xi(y)) \le \tilde{f}(y, u_\xi(y), Du_\xi(y)) + w_M(||u_\xi - u||_{L^\infty(B_{r'}(x))})$$
$$\le F(u_\xi, B_r(x)) + w_M(\varepsilon) \le \lambda + 2\varepsilon + w_M(\varepsilon).$$

Therefore, if we set $\lambda_\varepsilon = \lambda + 2\varepsilon + w_M(\varepsilon)$, we have that

$$\operatorname*{ess\,sup}_{y \in B_{r'}(x)} \tilde{f}(y, u, Du_\xi(y)) \le \lambda_\varepsilon.$$

Anaguously it holds

$$\operatorname*{ess\,sup}_{y \in B_{r'}(x)} \tilde{f}(y, u, Du_\eta(y)) \le \lambda_\varepsilon.$$

For every $z, y \in B_{r'}(x)$ and for every $v \in W^{1,\infty}(\Omega)$ we define

$$d^u_{\lambda_\varepsilon}(z, y) := \sup\left\{|v(z) - v(y)| \ : \ v \in W^{1,\infty}(B_{r'}(x)) : \operatorname*{ess\,sup}_{B_{r'}(x)} \tilde{f}(y, u, Dv(y)) \le \lambda_\varepsilon\right\} \tag{4.34}$$

and

$$R(v) := \sup_{z,y\in B_{r'}(x)} \frac{|v(z)-v(y)|}{d^u_{\lambda_\varepsilon}(z,y)}.$$

Since $R(u_\eta) \leq 1$ and $R(u_\xi) \leq 1$, thanks to Proposition 3.7 we have that

$$R(\theta u_\eta + (1-\theta)u_\xi) \leq 1.$$

Moreover, thanks to Lemma 3.8, there exists $(v_n)_n \subset W^{1,\infty}(B_{r'}(x))$ such that $v_n \to v$ weakly* and

$$\operatorname*{ess\,sup}_{y\in B_{r'}(x)} \tilde{f}(y,u,Dv_n(y)) \leq \lambda_\varepsilon.$$

Let $n_0 \in \mathbf{N}$ be such that $||v_n - v||_{L^\infty(B_{r'}(x))} \leq \varepsilon$ for every $n \geq n_0$. Then we have that

$$||v_n - u||_{L^\infty(B_{r'}(x))} \leq ||v_n - v||_{L^\infty(B_{r'}(x))} + ||v-u||_{L^\infty(B_{r'}(x))} \leq 3\varepsilon.$$

This implies that, for every $n \geq n_0$,

$$\operatorname*{ess\,sup}_{y\in B_{r'}(x)} \tilde{f}(y,v_n(y),Dv_n(y)) \leq \lambda_\varepsilon + w_{M'}(||v_n-u||_{L^\infty(B_{r'}(x))}) \leq \lambda_\varepsilon + w_{M'}(3\varepsilon)$$

where $M' = 1 + M$.

Since $F(\cdot, B_{r'}(x))$ is a weakly* l.s.c. functional, we have that

$$\operatorname*{ess\,sup}_{y\in B_{r'}(x)} \tilde{f}(y,v(v),Dv(y)) \leq \lambda_\varepsilon + w_{M'}(3\varepsilon).$$

This yields to

$$\tilde{f}(x,u,\theta\eta + (1-\theta)\xi) \leq \operatorname*{ess\,sup}_{y\in B_{r'}(x)} \tilde{f}(y,v(v),Dv(y)) \leq \lambda_\varepsilon + w_{M'}(3\varepsilon)$$

for every $\varepsilon > 0$. Letting $\varepsilon \to 0$ we obtain

$$\tilde{f}(x,u,\theta\eta + (1-\theta)\xi) \leq \lambda.$$

It remains to show that for a.e. $x \in \Omega$ $\tilde{f}(x,\cdot,\cdot) \geq f(x,\cdot,\cdot)$. Fix $q \in \mathbf{Q}$ and for every $x \in \Omega$ define

$$g_q(x,\xi) := \inf\left\{ \operatorname*{ess\,sup}_{B_r(x)} f(y,q,Du(y)) \mid \ r>0,\ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \right\}.$$

Thanks to Theorem 3.4, there exists a negligible set $H_q \subset \Omega$ be such that $\mathcal{L}(N_q) = 0$ and

$$g_q(x,\xi) \geq f(x,q,\xi) \qquad \forall x \in \Omega \setminus H_q, \quad \forall \xi \in \mathbf{R}^N.$$

Now, fix $(x,\xi) \in (\Omega \setminus H_q) \times \mathbf{R}^N$. Then, by definition of $\tilde{f}$, for every $\varepsilon > 0$ there exist $B_r(x) \subset \Omega$, $u \in W^{1,\infty}(\Omega)$, differentiable at x, such that $u(x) = q$, $Du(x) = \xi$ and

$$\tilde{f}(x,q,\xi) \geq \operatorname*{ess\,sup}_{B_r(x)} f(y,u(y),Du(y)) - \varepsilon.$$

In particular, after choosing $0 < r' \leq r$ such that $||u-q||_{L^\infty(B_{r'}(x))} \leq \varepsilon$, we have that

$$\tilde{f}(x,q,\xi) \geq \operatorname*{ess\,sup}_{B'_r(x)} f(y,u(y),Du(y)) - \varepsilon \geq \operatorname*{ess\,sup}_{B'_r(x)} f(y,q,Du(y)) - w_M(||u-q||_\infty) - \varepsilon$$

$$\geq g_q(x,\xi) - w_M(\varepsilon) - \varepsilon \geq f(x,q,\xi) - w_M(\varepsilon) - \varepsilon$$

whre $M = ||u||_\infty + q$. Since ε is arbitrary, this yields

$$\tilde{f}(x,q,\xi) \geq f(x,q,\xi) \qquad \forall x \in \Omega \setminus H_q, \quad \forall \xi \in \mathbf{R}^N.$$

Now, define $H = \bigcup_{q\in\mathbf{Q}} H_q$. Then H is a negligible subset of Ω and, since $\tilde{f}(x,\cdot,\xi)$ and $f(x,\cdot,\xi)$ are continuous functions for every $(x,\xi) \in \Omega \times \mathbf{R}^N$, we obtain that

$$\tilde{f}(x,u,\xi) \geq f(x,u,\xi) \qquad \forall x \in \Omega \setminus H, \quad \forall (u,\xi) \in \mathbf{R} \times \mathbf{R}^N.$$

2)$\Longrightarrow$ 1). It is sufficient to apply Theorem 3.4 in [2].

□

In order to show the relaxation Theorem 2.6 we will apply the following theorem (see Theorem 2.10 in [9]) to the sub levels of f^{lc}.

Theorem 4.1 *Let $\Omega \subset \mathbf{R}^N$ be an open set and $E \subset \mathbf{R}^N$. Let $\varphi \in W^{1,\infty}(\Omega)$ satisfy $D\varphi(x) \in E \cup$ int co$E$, for a.e.$x \in \Omega$ (where int co$E$ stands for the interior of the convex hull of E); then for every $\varepsilon > 0$ there exists $u \in W^{1,\infty}(\Omega)$ such that*

$$\begin{cases} \|u - \varphi\|_{L^\infty(\Omega)} \le \varepsilon \\ Du(x) \in E \qquad \text{for a.e. } x \in \Omega, \\ u(x) = \varphi(x) \qquad x \in \partial\Omega. \end{cases} \tag{4.35}$$

Remark 4.2 It is easy to show that if f is a continuous function satisfying (2.8), then f^{lc} is a continuous function. Moreover if f satisfies (2.15), then f^{lc} satisfies (2.15). In fact, since α is an increasing real function, then α is level convex and the inequality (2.15) implies

$$f^{lc}(x,u,\xi) \ge \alpha(|\xi|) \qquad \text{for a.e } x \in \Omega, \text{ for every } u \in \mathbf{R} \text{ and } \xi \in \mathbf{R}^N.$$

Proof of Theorem 2.6. Without loss of generality, we can assume that $\alpha(t) = t$. Let $G : W^{1,\infty}(\Omega) \times \mathcal{A} \to \overline{\mathbf{R}}$ be the functional defined by

$$G(u,B) := \operatorname*{ess\,sup}_B f^{lc}(x,u(x),Du(x)).$$

By Remark 4.4 in [1] and by using the fact that $f^{lc}(x,\cdot,\xi)$ is continuous, it is easy to show that G is weakly* lower semicontinuous in $W^{1,\infty}(\Omega)$. This implies that $G(u,\Omega) \le \overline{F}(u)$ for every $u \in W^{1,\infty}(\Omega)$. For the converse inequality, fix $u \in W^{1,\infty}(\Omega)$ and $\lambda \in \mathbf{R}$ s.t. $G(u,\Omega) < \lambda$. If we find a sequence $w_n \in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} ||w_n - u||_{L^\infty(\Omega)} \le \frac{1}{n}, \\ F(w_n) \le \lambda + \varepsilon_n \end{cases}$$

where $\varepsilon_n \to 0$, then

$$\overline{F}(u) \le \liminf_n F(w_n) \le \lambda$$

which implies $\overline{F}(u) \le G(u,\Omega)$. Now, fix $n \in \mathbf{N}$. Thanks to Remark 4.2, for every $x_0 \in \Omega$ and for every $M > 0$ there exists $\delta = \delta(x_0,n,M)$ such that

$$|f^{lc}(x_0,u(x_0),\xi) - f^{lc}(y,u(y),\xi)| \le \frac{1}{n} \tag{4.36}$$

and

$$|f(x_0,u(x_0),\xi) - f(y,u(y),\xi)| \le \frac{1}{n} \tag{4.37}$$

for every $|\xi| \le M$ and for every $y \in \Omega$ such that $|x_0 - y| \le \delta$. Now fix $x_0 \in \Omega$, let $M := ||Du||_\infty$ and let $\delta > 0$ be such that (4.36) and (4.37) hold. Then, choose $0 < r_0 < \delta$ such that $B_{r_0}(x_0) \subset \Omega$, let $G_{x_0} : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}}$ be the functional given by

$$G_{x_0}(v) = \operatorname*{ess\,sup}_{B_{r_0}(x_0)} f^{lc}(x_0,u(x_0),Dv(x)).$$

From (4.36) we have that

$$G_{x_0}(u) \le \operatorname*{ess\,sup}_{B_{r_0}(x_0)} f^{lc}(x,u(x),Du(x)) + \frac{1}{n} \le G(u,\Omega) + \frac{1}{n}. \tag{4.38}$$

Let $n_0 \in \mathbf{N}$ be such that $G(u,\Omega)+\frac{1}{n} < \lambda$ for every $n \geq n_0$, for all $x_0 \in \Omega$ and for every $r_0 < \delta_n$. Since f^{lc} is a continuous function, we have that $E(x_0) = \{\xi \in \mathbf{R}^N \ : \ f(x_0, u(x_0), \xi) < \lambda\}$ is an open set. By the properties of the convex hull, we have that $co\ E(x_0)$ is an open set. In particular, by (4.38), for a.e. $x \in B_{r_0}(x_0)$ we have that

$$\begin{aligned} Du(x) &\in \{\xi \in \mathbf{R}^N \ : \ f^{lc}(x_0, u(x_0), \xi) \leq G(u,\Omega) + \frac{1}{n}\} \\ &= \overline{int\ co}\big(\{\xi \in \mathbf{R}^N \ : \ f(x_0, u(x_0), \xi) \leq G(u,\Omega) + \frac{1}{n}\} \\ &\subset co\ E(x_0) \subset int\ co\ E(x_0). \end{aligned}$$

Then, by Theorem 4.1, we can construct a function $u_n^{x_0,r_0} \in W^{1,\infty}(B_{r_0}(x_0))$ such that

$$\begin{cases} \|u - u_n^{x_0,r_0}\|_{L^\infty(B_{r_0}(x_0))} \leq \frac{1}{n} \\ Du_n^{x_0,r_0}(x) \in E(x_0) \qquad a.e.\ x \in B_{r_0}(x_0), \\ u(x) = u_n^{x_0,r_0}(x) \qquad x \in \partial B_{r_0}(x_0). \end{cases}$$

According to Vitali covering Theorem (see for instance Corollary 10.6 of [9]), we can find a countable family of pairwise disjoint balls $B_k = B_{r_k}(x_k)$ such that $\mathcal{L}\,(\Omega \setminus \bigcup_k B_k) = 0$. For simplicity, denote by u_n^k the function $u_n^{x_k,r_k}$. By definition, we have that

$$\begin{cases} \|u - u_n^k\|_{L^\infty(B_k)} \leq \frac{1}{n} \\ Du_n^k(x) \in E(x_k) \qquad a.e.\ x \in B_k, \\ u(x) = u_n^k(x) \qquad x \in \partial B_k. \end{cases}$$

Note that, since $Du_n^k(x) \in E(x_k)$ for a.e. $x \in B_k$, by making use of the coercivity assumption, we have that

$$\|Du_n^k\|_{L^\infty(B_k)} \leq \sup_{B_k} f(x_k, u(x_k), Du_n^k(x))$$

and thus,

$$\|Du_n^k\|_{L^\infty(B_k)} \leq \lambda. \tag{4.39}$$

Let us now set

$$w_n(x) = u_n^k(x) \quad \text{if } x \text{ belongs to some } B_k \text{ and} \quad w_n(x) = u(x) \quad \text{otherwise.}$$

Then w_n belongs to $W^{1,\infty}(\Omega)$, since it is the pointwise limit of the Lipschitz maps v_k defined inductively by $v_0 = u$, and

$$v_{k+1}(x) = u_n^{k+1}(x) \text{ if } x \text{ belongs to } B_{k+1} \text{ and } v_{k+1}(x) = v_k(x) \text{ otherwise.}$$

Thanks to (4.39), the maps v_k are equilipschitz continuous with a Lipschitz constant independent of n. Hence the function w_n belongs to $W^{1,\infty}(\Omega)$ and, since $w_n = u$ on $\Omega \setminus \bigcup_k B_k$, we have that

$$\|w_n - u\|_{L^\infty(\Omega)} = \sup_k \|u - u_n^k\|_{L^\infty(B_k)} \leq \frac{1}{n},$$

i.e. the sequence $(w_n)_n$ w* converges to u in $L^\infty(\Omega)$. Now, from (4.36) and Proposition 3.7 it follows that for every $n > n_0$

$$\begin{aligned} \operatorname*{ess\,sup}_{\Omega} f(x, w_n(x), Dw_n(x)) &= \bigvee_k \operatorname*{ess\,sup}_{B_k} f(x, w_n(x), Dw_n(x)) \\ &= \bigvee_k \operatorname*{ess\,sup}_{B_k} f(x, u_n^k(x), Du_n^k(x)) \\ &\leq \bigvee_k \big\{ \operatorname*{ess\,sup}_{B_k} f(x_k, u_n^k(x_k), Du_n^k(x)) + \frac{1}{n} \big\} \end{aligned}$$

and since $Du_n^k(x) \in E(x_k)$ for a.e. $x \in B_k$ we get that

$$\operatorname*{ess\,sup}_{\Omega} f(x, w_n(x), Dw_n(x)) \leq \lambda + \frac{1}{n}.$$

This means that $(w_n)_n$ is the sequence that we are looking for. $\square$

Proof of Theorem 2.7. Fix an open set $A \subset \Omega$ with Lipschitz continuous boundary and let $F(u) = F(u, A)$. Let $\lambda \in \mathbf{R}$ be such that the sub level set $F_\lambda := \{u \in W^{1,\infty}(A) \ : \ \overline{F}(u) \leq \lambda\}$ is nonempty. If $E_\lambda := \{u \in W^{1,\infty}(A) \ : \ F(u) < \lambda\}$ is nonempty too, then let R_λ be the corresponding difference quotient defined by (3.27). Now we show that

$$F_\lambda = \bigcap_{\varepsilon>0} \{u \in W^{1,\infty}(A) \ : \ R_{\lambda+\varepsilon}(u) \leq 1\}. \tag{4.40}$$

In fact, assume that $R_{\lambda+\varepsilon}(u) \leq 1$ for every $\varepsilon > 0$. Then, by applying Lemma 3.8, for every $\varepsilon > 0$ there exists a sequence $\{u_n^\varepsilon\} \subset W^{1,\infty}(A)$ converging to u in $L^\infty(A)$ s.t. $F(u_n^\varepsilon) \leq \lambda + \varepsilon$. Then $\overline{F}(u) \leq \lambda + \varepsilon$ for every $\varepsilon > 0$ which implies $\overline{F}(u) \leq \lambda$. Vice versa, assume that $\overline{F}(u) \leq \lambda$. Then there exists a sequence $(u_n)_n \subset W^{1,\infty}(\Omega)$ such that $u_n \to u$ weakly* and $\lim_n F(u_n) = \overline{F}(u)$. Fix $\varepsilon > 0$. Then there exists $n_0 \in \mathbf{N}$ such that $F(u_n) \leq \lambda + \varepsilon$ for every $n \geq n_0$. By definition, this implies that $R_{\lambda+\varepsilon}(u_n) \leq 1$ for every $n \geq n_0$ and, since $R_{\lambda+\varepsilon}$ is w* l.s.c., we obtain that $R_{\lambda+\varepsilon}(u) \leq 1$. From (4.40) it follows that F_λ is a convex set. Finally, if E_λ is empty, note that $E_{\lambda+\varepsilon} \neq \emptyset$ for every $\varepsilon > 0$ and from the first part of this proof it follows that $F_{\lambda+\varepsilon}$ is a convex set. Since

$$F_\lambda = \bigcap_{\varepsilon>0} F_{\lambda+\varepsilon}$$

then F_λ is a convex set too. $\square$

Proof of Theorem 2.9. 1)$\Longrightarrow$ 2). Without loss of generality, we can assume that $c = 1$. Let $\overline{f} : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be the function given by

$$\overline{f}(x, \xi) := \inf\left\{\overline{F}(u, B_r(x)) \mid \ \ r > 0, \ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \ \ \right\}. \tag{4.41}$$

We show that $\overline{f}$ is a level convex Carathéodory supremand such that

$$(\tilde{f}(x, \cdot))^{lc} = \overline{f}(x, \cdot) \tag{4.42}$$

for a.e. $x \in \Omega$. With this aim, first of all we show that for every $A \in \mathcal{A}$ $\overline{F}(\cdot, A)$ satisfies all the assumptions of Proposition 3.3. In order to show that $\overline{F}$ satisfies (3.20), we first show that $\overline{F}$ is locally bounded. Namely,

$$\forall M > 0 \ , \ \exists k_M > 0 \ \text{ s.t. } \ \ \overline{F}(u, A) \leq k_M \quad \forall A \in \mathcal{A} \ , \forall u \in W^{1,\infty}(A) \ \text{ s.t. } \ \|Du\|_{L^\infty(A)} \leq M \ . \tag{4.43}$$

In fact, let $(u_n)_n \subset W^{1,\infty}(\Omega)$ be such that $u_n \to 0$ w.r.t. the weak* convergence and $\lim_n F(u_n, \Omega) = \overline{F}(0, \Omega)$. From the coercivity assumption and the fact that $\|u_n\|_\infty \to 0$,

$$\limsup_n \|Du_n\|_\infty \leq \ \limsup_n F(u_n, \Omega) = \overline{F}(0, \Omega) \ .$$

Therefore, for n large enough, we have $\|Du_n\|_\infty \leq \overline{F}(0, \Omega) + 1$. Let now $M > 0$ be fixed. Let $A \in \mathcal{A}$ and $u \in W^{1,\infty}(\Omega)$ be such that $\|Du\|_{L^\infty(A)} \leq M$. Define $M' = \overline{F}(0, \Omega) + 1 + M$. Then

$$\overline{F}(u, A) \leq \limsup_n \overline{F}(u + u_n, A) \leq \limsup_n \left(F(u_n, A) + \omega_{M'}(\|Du\|_{L^\infty(A)})\right) \leq k + \omega_{M'}(M) \ .$$

So we have proved that (4.43) holds with $k_M = \overline{F}(0,\Omega) + 1 + \omega_{M'}(M)$.
Now we show that for every $M > 0$ there exists a modulus of continuity ω'_M such that for every $A \in \mathcal{A}$

$$|F(u,A) - F(v,A)| \leq \omega_M(||Du - Dv||_{L^\infty(A)}) \tag{4.44}$$

for every $u, v \in W^{1,\infty}(A)$ s.t. $||Du||_{L^\infty(A)}, ||Dv||_{L^\infty(A)} \leq M$. Let $M > 0$ be fixed, $A \in \mathcal{A}$, u and v be such that $\|Du\|_{L^\infty(A)}, \|Dv\|_{L^\infty(A)} \leq M$. Let $(u_n)_n \in W^{1,\infty}(A)$ be some recovery sequence for u: $\limsup_n F(u_n, A) = \overline{F}(u, A)$. Arguing as above we can prove that $\|Du_n\|_{L^\infty(A)} \leq (F''(u,A) + 1)$ for n large enough. Using (4.43), we deduce that $\|Du_n\|_{L^\infty(A)} \leq \max\{(k_M + 1), \|u\|_\infty + 1\} \leq \max\{(k_M + 1), M + 1\}$ for n large enough. Then, fixed $M' = M'(M) = \max\{(k_M + 1), M + 1\} + M$, we have that

$$\begin{aligned}
\overline{F}(v, A) &\leq \limsup_n F(v + u_n - u, A) \\
&\leq \limsup_n \left(F(u_n, A) + \omega_{M'}(\|v - u\|_{L^\infty(A')}\right) \\
&\leq F(u, A) + \omega_{M'}(\|v - u\|_{L^\infty(A)}) \ .
\end{aligned}$$

Now we prove that $\overline{F}(u,A) \geq ||Du||_{L^\infty(A)}$ for every $A \in \mathcal{A}$ and for every $u \in W^{1,\infty}(A)$. Let $A \in \mathcal{A}$ and $u \in W^{1,\infty}(A)$. Let (u_n) be a recovery sequence for u in A: $\limsup_n F_n(u_n, A) = \overline{F}(u, A)$. Then, from the coercivity assumption on F and the lower semicontinuity of the L^∞ norm of the gradient with respect to the uniform convergence, we have

$$\|Du\|_{L^\infty(A)} \leq \liminf_n \|Du_n\|_{L^\infty(A)} \leq \limsup_n F_n(u_n, A) = \overline{F}(u, A) \ .$$

Now let us choose a countable base $(A_n)_{n\in\mathbf{N}}$ of open subsets of Ω. By applying Proposition 3.3 to the functional $\overline{F}(\cdot, A_n)$ we obtain that for every $n \in \mathbf{N}$ the function $\varphi_n : A_n \times \Omega \to \mathbf{R}^N$ defined by

$$\varphi_n(x,\xi) := \inf\left\{\overline{F}(\cdot, A_n) \mid \quad u \in W^{1,\infty}(A_n) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \quad \right\}$$

is a Carathéodory supremand such that

$$\operatorname*{ess\,sup}_{A_n} \varphi_n(x, Du) \geq \overline{F}(\cdot, A_n) \tag{4.45}$$

for any $u \in W^{1,\infty}(A_n)$.

Since

$$\overline{f}(x,\xi) = \inf\{\varphi_n(x,\xi) \ : \ x \in A_n\}$$

we have that $\overline{f}$ is a Borel function.

Moreover, by applying Proposition 3.3, for any $M > 0$ there exists some modulus of continuity ω'_M (independent on n thanks to (4.44)) such that

$$|\varphi_n(x,\xi) - \varphi_n(x,\eta)| \leq \omega'_M(|\xi - \eta|)$$

for a.e. $x \in A_n$ and for every $\xi, \eta \in B_M(0)$ and for every $n \in \mathbf{N}$. This implies that

$$|\overline{f}(x,\xi) - \overline{f}(x,\eta)| \leq \omega'_M(|\xi - \eta|)$$

for a.e. $x \in \Omega$ and for every $\xi, \eta \in B_M(0)$. Therefore $\bar{f}$ is a Carathéodory supremand satisfying (2.16) and (2.15). Now it is easy to show that there exists a negligible set N such that

$$\overline{f} = (\tilde{f})^{lc} \text{ on } (\Omega \setminus N) \times \mathbf{R}^N$$

and therefore that

$$\operatorname*{ess\,sup}_A \overline{f}(x, Du) = \operatorname*{ess\,sup}_A (\tilde{f})^{lc}(x, Du).$$

In fact, by the definition of $\overline{f}$ and $\tilde{f}$, we have that $\overline{f} \leq \tilde{f}$. If we show that $\overline{f}$ is a level convex function then $\overline{f} \leq (\tilde{f})^{lc}$. Fix $x \in \Omega$, $\xi, \eta \in \mathbf{R}^N$ and $\lambda \in (0,1)$. By the definition of $\tilde{f}$ there exists $B_r(x)$,

$u_\varepsilon, v_\varepsilon \in W^{1,\infty}(\Omega)$, differentiable at x such that $Du_\varepsilon(x) = \xi,\ Dv_\varepsilon(x) = \eta$ and $\overline{f}(x,\xi) \geq \overline{F}(u_\varepsilon, B_r(x)) - \varepsilon$ and $\overline{f}(x,\eta) \geq \overline{F}(v_\varepsilon, B_r(x)) - \varepsilon$. Thanks to Theorem 2.7 $\overline{F}(\cdot, B_r(x))$ is a level convex functional. Then

$$\overline{f}(x, \lambda\xi + (1-\lambda)\eta) \leq \overline{F}(\lambda u_\varepsilon + (1-\lambda)v_\varepsilon, B_r(x)) \leq \overline{F}(u_\varepsilon, B_r(x)) \vee \overline{F}(v_\varepsilon, B_r(x)) \leq (\overline{f}(x,\xi) - \varepsilon) \vee (\overline{f}(x,\eta) - \varepsilon).$$

Letting $\varepsilon \to 0$ we obtain the thesis. Vice versa, thanks to Theorem 3.4, we have that $F(u, A) = \operatorname{ess\,sup}_A \tilde{f}(x, Du)$. Since the functional $H(u,A) = \operatorname{ess\,sup}_A (\tilde{f})^{lc}(x, Du)$ is w* lower semicontinuous and

$$\operatorname*{ess\,sup}_A (\tilde{f})^{lc}(x, Du) \leq \operatorname*{ess\,sup}_A \tilde{f}(x, Du) = F(u, A)$$

we have that

$$\operatorname*{ess\,sup}_A (\tilde{f})^{lc}(x, Du) \leq \overline{F}(u, A). \tag{4.46}$$

Therefore if we define

$$h(x,\xi) := \inf \left\{ \operatorname*{ess\,sup}_{B_r(x)} (\tilde{f})^{lc}(x, Du) \mid \ r > 0,\ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \right\} \tag{4.47}$$

we deduce that $h \leq \overline{F}$. Moreover by Theorem 3.4 there exists a negligible set N such that

$$h \geq (\tilde{f})^{lc} \text{ on } (\Omega \setminus N) \times \mathbf{R}^N.$$

Therefore

$$\overline{f} \geq (\tilde{f})^{lc} \text{ on } (\Omega \setminus N) \times \mathbf{R}^N.$$

In particular if $\overline{F}$ satisfies also the property of countable supremality, by applying Theorem 3.1 we have that

$$\overline{F}(u, A) = \operatorname*{ess\,sup}_A \overline{f}(x, Du)$$

for every $A \in \mathcal{A}$ and for every $u \in W^{1,\infty}(A)$ and therefore

$$\overline{F}(u, A) = \operatorname*{ess\,sup}_A \tilde{f}^{lc}(x, Du(x))$$

for every $A \in \mathcal{A}$ and for every $u \in W^{1,\infty}(A)$.
2)$\Longrightarrow$ 1)It is trivial. □

Proof of Theorem 2.10. By Theorem 4.4.1 and Remark 4.4.5 in [5], we have that

$$\overline{F}_p(u, A) = \left(\int_A (\tilde{f}^p)^{**}(x, Du(x))dx \right)^{1/p}$$

for every $A \in \mathcal{A}$ and for every $u \in W^{1,\infty}(A)$. Since $(\tilde{f}^p)^{**} \leq (\tilde{f}^{lc})^p$ (thanks to (3.31)), it holds that

$$\begin{aligned}\limsup_{p\to\infty} \overline{F}_p(u, A) &= \limsup_{p\to\infty} \left(\int_A (\tilde{f}^p)^{**}(x, Du(x)) \right)^{1/p} \\ &\leq \limsup_{p\to\infty} \left(\int_A (\tilde{f}^{lc})^p(x, Du(x)) \right)^{1/p} = \operatorname*{ess\,sup}_A \tilde{f}^{lc}(x, Du(x))\end{aligned}$$

for every $A \in \mathcal{A}$ and for every $u \in W^{1,\infty}(A)$.
On the other hand, thanks to Theorem 3.10, we have that

$$\operatorname*{ess\,sup}_A \tilde{f}^{lc}(x, Du(x)) \leq \liminf_{p\to\infty} \left(\int_A (\tilde{f}^p)^{**}(x, Du(x)) \right)^{1/p} = \liminf_{p\to\infty} \overline{F}_p(u, A)$$

for every $A \in \mathcal{A}$ and for every $u \in W^{1,\infty}(A)$. □

Proof of Corollary 2.11. Thanks to part (iv) of Theorem 3.4, we have that for every $u \in W^{1,\infty}(\Omega)$

$$F_p(u) = \left(\int_\Omega f^p(x, Du(x))dx \right)^{1/p} = \left(\int_\Omega \tilde{f}^p(x, Du(x))dx \right)^{1/p}$$

Then the thesis follows by applying Theorems 2.6 and 2.10. $\square$

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