# RECTIFIABILITY OF METRIC FLAT CHAINS AND FRACTIONAL MASSES

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Abstract. It is proven that every real flat chain $T$ of finite mass in a complete separable metric space $E$ is rectifiable when $\mathbb{M}^\alpha(T) < +\infty$ for some $\alpha \in [0,1)$, where $\mathbb{M}^\alpha(T)$ stands for the $\alpha$-mass of $T$.

## 1. Introduction

The class of flat chains in $\mathbb{R}^n$ with real and integer coefficients has been introduced by Whitney in [12], used further by Federer and Fleming in [9], and later has been extended by Fleming in [10] to the flat chains with coefficients in general abelian group (a further extension to flat chains in a Banach space with coefficients in general abelian group has been made recently in [1]). One of the important achievements in the theory of flat chains in Euclidean spaces was the rectifiability theorem due to White proven in [11]. A consequence of the latter provides the rectifiability of every real flat chain $T$ in $\mathbb{R}^n$ with finite mass once one knows that $\mathbb{M}^\alpha(T) < +\infty$ for some $\alpha \in [0,1)$. Here $\mathbb{M}^\alpha(T)$ stands for the $\alpha$-mass of $T$ (which it is appropriate to call *fractional mass*, following [7]), that reduces to the usual notion of the mass in the case $\alpha = 1$, and to the notion of the size in the case $\alpha = 0$, while for a flat chain $T$ concentrated over a countably $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset \mathbb{R}^n$ and having multiplicity $\theta\colon \Sigma \to \mathbb{R}$ (such flat chains will be further denoted by $T = \theta[\![\Sigma]\!]$), one has

$$\mathbb{M}^\alpha(T) := \int_\Sigma |\theta|^\alpha \, d\mathcal{H}^k,$$

where $\mathcal{H}^k$ stands for the $k$-dimensional Hausdorff measure. Such masses, as well as more general concave functionals on flat chains, have been object of intensive study recently, see for instance [6, 7] and the recent book [3].

The present note is aimed at proving the analogous rectifiability result for flat chains defined over generic metric spaces rather than over Euclidean spaces. For this purpose we exploit the notion of metric currents introduced by Ambrosio and Kirchheim in [2], and define the notion of a flat chain in a metric space as it has been done in [1] (it is to be noted however that our definition differs from that given in [5]). We define the $\alpha$-mass of a flat chain $T$, so that the introduced notions extend the analogous ones in the Euclidean setting. We then prove that every real flat chain $T$ of finite mass in a separable complete metric space is rectifiable when $\mathbb{M}^\alpha(T) < +\infty$ for some $\alpha \in [0,1)$. Moreover, the same assertion is true, if $E$ is just assumed to contain a dense set, the cardinality of which is Ulam number (and it is consistent with the Zermelo-Fraenkel set theory to assume that every metric space is like that). The proof is based upon the rectifiable slices theorem for flat chains (Theorem 4.3) from [5], the Eilenberg-type inequality for $\alpha$-masses (Proposition 4.4) which is also of some independent interest, and the simple result on lower semicontinuity of functionals defined on measures (Lemma 5.1) in the style of [4]. Analogous rectifiability result can be proven also for more general masses over flat chains, e.g. for masses $\mathbb{M}_h$ with $h\colon [0,+\infty) \to [0,+\infty)$ being a given nondecreasing concave function satisfying $h(0) = 0$, $h'_+(0) = +\infty$. Such masses are defined over $k$-dimensional real flat chains $T = \theta[\![\Sigma]\!]$ in $\mathbb{R}^n$ by means of the formula

$\mathbb{M}_h(T) := \int_\Sigma h(\theta)\, d\mathcal{H}^k$, and extended by means of relaxation with respect to the flat norm topology to the class of all $k$-dimensional real flat chains (see [6, 7] for the Euclidean space setting). In particular one would have $\mathbb{M}_h = \mathbb{M}^\alpha$ for $h(t) := t^\alpha$, $\alpha \in [0,1]$ (with the convention $0^0 := 0$ in the case $\alpha = 0$). We prefer however to work with $\mathbb{M}^\alpha$ mainly for notational convenience.

## 2. Metric currents

Throughout the paper we will extensively use the notion of currents with finite mass due to Ambrosio and Kirchheim [2]. For metric spaces $X$ and $Y$ we denote

- $\mathrm{Lip}(X,Y)$ the set of Lipschitz maps $f\colon X \to Y$ with arbitrary Lipschitz constant,
- $\mathrm{Lip}_b(X,Y)$ the set of bounded Lipschitz maps $f\colon X \to Y$ with arbitrary Lipschitz constant.

If $Y = \mathbb{R}$, we will omit the reference to $Y$ and write simply $\mathrm{Lip}(X)$ and $\mathrm{Lip}_b(X)$ respectively. The Lipschitz constant of a map $f\colon X \to Y$ will be denoted by $\mathrm{Lip}(f)$. The metric spaces considered will further be tacitly assumed to be complete.

All the measures we will consider in the sequel are signed Borel measures with finite total variation over some metric space $E$. We will say that such a measure $\mu$ is *atomic*, if it is concentrated on an at most countable set, and *nonatomic*, if $\mu(\{x\}) = 0$ for all $x \in E$. The narrow topology on measures is defined by duality with the space $C_b(E)$ of continuous bounded functions.

For a metric space $E$ we denote

$$D^k(E) := \left\{(f, \pi_i, \ldots, \pi_k) \,:\, f \in \mathrm{Lip}_b(E), \pi \in \mathrm{Lip}(E;\mathbb{R}^k)\right\}.$$

The $k$-tuples $\omega = (f, \pi_i, \ldots, \pi_k) \in D^k(E)$ will be further called $k$-dimensional differential forms. For the form $\omega = (f, \pi_i, \ldots, \pi_k) \in D^k(E)$ we will adopt the notation

$$\omega = f\, d\pi_1 \wedge d\pi_2 \wedge \ldots \wedge d\pi_k = f\, d\pi.$$

The exterior derivative operator $d\colon D^k(E) \to D^{k+1}(E)$ is then defined by

$$d(f\, d\pi_1 \wedge d\pi_2 \wedge \ldots \wedge d\pi_k) := 1\, df \wedge d\pi_1 \wedge d\pi_2 \wedge \ldots \wedge d\pi_k.$$

Also, given an arbitrary Lipschitz map $\phi\colon F \to E$, with $F$ metric space, one defines the pull-back operator $\phi^\#\colon D^k(E) \to D^k(F)$ by setting

$$\phi^\#(f\, d\pi)(f\, d\pi_1 \wedge d\pi_2 \wedge \ldots \wedge d\pi_k) := (f\circ\phi\, d\pi_1\circ\phi \wedge d\pi_2\circ\phi \wedge \ldots \wedge d\pi_k\circ\phi).f\circ\phi\, d\pi\circ\phi$$

**Definition 2.1.** *A functional $T\colon D^k(E) \to \mathbb{R}$ is called real $k$-dimensional metric current (called further for simplicity current) over $E$, if the following conditions hold:*

(linearity) $(f, \pi_1, \ldots, \pi_k) \mapsto T(f, \pi_1, \ldots, \pi_k)$ *is multilinear, i.e. linear in $f$ and in each of $\pi_i$, $i = 1, \ldots, k$,*

(continuity) $T(f\, d\pi_\nu) \to T(f\, d\pi)$ *whenever $\pi_\nu \to \pi$ pointwise in* $\mathrm{Lip}(E;\mathbb{R}^k)$*, as $\nu \to \infty$,*

(locality) *if*

$$\{x \in E \,:\, f(x) \neq 0\} \subset \cup_{i=1}^k B_i,$$

*where each $B_i \subset E$ is a Borel set such that $\pi_i$ is constant over $B_i$, $i = 1, \ldots, k$, then $T(f\, d\pi) = 0$.*

The total mass of a current $T$ is defined by

$$\mathbb{M}(T) := \mu_T(E),$$

where $\mu_T$ is the minimum among finite Borel measures $\mu$ which satisfy

$$|T(f\, d\pi)| \le \prod_{i=1}^k \mathrm{Lip}(\pi_i) \int_E |f|\, d\mu$$

for every $f \in \mathrm{Lip}_b(E)$, $\pi \in \mathrm{Lip}(E, \mathbb{R}^k)$, with the convention $\prod_{i=1}^k \mathrm{Lip}(\pi_i) := 1$, if $k = 0$. The support $\operatorname{supp} T$ of a real $k$-dimensional metric current $T$ with finite mass is defined as the support of $\mu_T$. In what follows we will be always dealing with real $k$-dimensional metric currents with finite mass in $E$. The set of such currents will be denoted by $\mathcal{M}_k(E)$. The mass functional $\mathbb{M}$ is easily seen to define a norm in $\mathcal{M}_k(E)$.

We will say that a sequence of currents $\{T_\nu\} \subset \mathcal{M}_k(E)$ converges weakly to a current $T \in \mathcal{M}_k(E)$, and write $T_\nu \rightharpoonup T$, if $T_\nu(\omega) \to T(\omega)$ as $\nu \to \infty$, for every $\omega \in D^k(E)$. The mass is known to be lower semicontinuous with respect to weak convergence of currents [2].

Clearly, every metric current $T \in \mathcal{M}_k(E)$ may be extended by continuity from the space of forms $D^k(E)$ to the larger space of $(k+1)$-tuples $(f, \pi_i, \ldots, \pi_k)$, where $\pi \in \mathrm{Lip}(E; \mathbb{R}^k)$, while $f\colon E \to \mathbb{R}$ is a bounded Borel function on $E$. Thus, whenever $E$ is a complete metric space, then every $T \in \mathcal{M}_0(E)$ is represented by some signed Borel measure of finite total variation over $E$ (given by the set function $B \mapsto T(1_B)$ for every Borel set $B \subset E$, where $1_B$ stands for the characteristic function of $B$). Hence, when necessary, we will always identify a $T \in \mathcal{M}_0(E)$ with the respective signed measure. Note that the mass $\mathbb{M}$ over $\mathcal{M}_0(E)$ is nothing else than the total variation norm $\|\cdot\|$ over the space of such measures on $E$.

If $T \in \mathcal{M}_k(E)$ and $\omega = g\, d\tau \in D^m(E)$, $m \le k$, we define the restricted metric current $T \llcorner \omega \in \mathcal{M}_{k-m}(E)$ by the formula

$$T \llcorner \omega(f\, d\pi) := T(fg, \tau_1, \ldots, \tau_m, \pi_1, \ldots, \pi_{k-m}) \text{ for all } f\, d\pi \in D^{k-m}(E).$$

Since $T$ is assumed to have finite mass, then in the above formula one may admit in place of $f$ and $g$ any bounded Borel functions. In particular, whenever $\omega = 1_B \in D^0(E)$ for some Borel set $B \subset E$, we will simply write $T \llcorner B$ for $T \llcorner \omega$.

The boundary $\partial T$ of a $k$-dimensional current $T$ is a $(k-1)$-dimensional current defined by the formula

$$\partial T(\omega) := T(d\omega) \text{ for all } \omega \in D^{k-1}(E).$$

Further, for an arbitrary Lipschitz map $\phi\colon F \to E$, with $F$ metric space, we define the push-forward operator $\phi_\#\colon \mathcal{M}_k(F) \to \mathcal{M}_k(E)$ on currents by setting

$$(\phi_\# T)(\omega) := T(\phi^\# \omega) \text{ for all } \omega \in D^k(E).$$

We say that $T$ is a

- *normal current*, if $\mathbb{M}(T) < +\infty$ and $\mathbb{M}(\partial T) < +\infty$,
- *rectifiable current*, if $\mu_T \ll \mathcal{H}^k$ and $\mu$ is concentrated on some countably $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset E$.

It is worth remarking that in a finite-dimensional Euclidean space $E = \mathbb{R}^n$ every normal current (in the sense of metric currents) by theorem 11.1 from [2] may be indentified via a natural isomorphism with a Federer-Fleming normal current.

The sets of normal and rectifiable $k$-dimensional currents over a metric space $E$ will be further denoted $\mathbf{N}_k(E)$ and $\mathcal{R}_k(E)$ respectively. We will further need also some notions on the representation of rectifiable currents. If $E$ is a dual of a separable Banach space, then it has been shown in theorem 9.1 from [2] that for every $T \in \mathcal{R}_k(E)$ there exists a countably $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset E$, a nonnegative function $\theta \in L^1(\Sigma, \mathcal{H}^k \llcorner \Sigma)$ called *multiplicity* of $T$ (with negative multiplicities $\theta$ allowed only in the cases $k = 0$ or $k = n$, $E = \mathbb{R}^n$), and a $k$-vector $\tau(x)$ for $\mathcal{H}^k$-a.e. $x \in \Sigma$ giving the *orientation* of the approximate tangent space $S = S(x)$ to $\Sigma$ at point $x$, such that

$$T(f\, d\pi) = \int_\Sigma f(x)\theta(x)\langle \wedge_k d_x^S \pi, \tau\rangle\, d\mathcal{H}^k(x), \tag{2.1}$$

where $\wedge_k d_x^S \pi$ stands for the $k$-covector induced by the differential $d_x^S\pi$ of $\pi$ at $x$ along the tangent space $S(x)$, and $\langle\cdot,\cdot\rangle$ stands for the standard duality between $k$-vectors and $k$-covectors. In this case we will write $T = \theta[\![\Sigma]\!]$, whenever the orientation is not important. It is further proven (theorem 9.5 from [2]) that in this case

$$\mu_T = \theta\lambda_T \mathcal{H}^k \llcorner \Sigma, \tag{2.2}$$

where $\lambda_T \in L^\infty(\Sigma, \mathcal{H}^k\llcorner\Sigma)$ is a geometric factor satisfying

$$k^{-k/2} \leq \lambda_T(x) \leq 2^k/\omega_k$$

for $\mathcal{H}^k$-a.e. $x \in \Sigma$, $\omega_k$ standing for the volume of the unit $k$-dimensional ball in $\mathbb{R}^k$. Moreover, $\lambda_T = 1$ whenever the norm in $E$ is generated by an inner product (in particular, when $E$ is a finite-dimensional Euclidean space). Also $\lambda_T = 1$ when $k = 0$.

If $E$ is a general separable metric space, then there is an isometric embedding $j$ of $E$ in a dual $Y$ of a separable Banach space (for instance, one may take $Y = \ell^\infty$). It has been shown in [2] that whenever $T \in \mathcal{R}_k(E)$, then there still exists a countably $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset E$, a *multiplicity* function $\theta \in L^1(\Sigma, \mathcal{H}^k\llcorner\Sigma)$ (with $\theta > 0$ unless either $k = 0$ or $k = n$, $E = \mathbb{R}^n$), and a $k$-vector $\tau(x)$ for $\mathcal{H}^k$-a.e. $x \in j(\Sigma)$ giving the *orientation* of the approximate tangent space $S = S(x)$ to $j(\Sigma)$ at point $x$, such that the representation formula (2.1) holds for $j_\# T$, while the measure $\mu_T$ is represented by (2.2). Moreover, the triple $(j(\Sigma), \theta\circ j^{-1}, \tau)$ is independent on a particular choice of $j$ in the sense of a natural equivalence of such triples defined in theorem 9.2 from [2]. We still adopt the notation $T = \theta[\![\Sigma]\!]$ in this case, when the orientation is not important.

## 3. Flat chains and fractional masses

We define the *flat norm* $\mathcal{F}(T)$ of a current $T$ by setting

$$\mathcal{F}(T) := \inf\{\mathbb{M}(T - \partial B) + \mathbb{M}(B) \,:\, B \in \mathcal{M}_{k+1}(E)\}.$$

Clearly $\mathcal{F}(T) \leq \mathbb{M}(T)$ by definition, hence the topology on currents induced by the flat norm is weaker than that induced by mass. In what follows we will write $T_\nu \overset{\mathcal{F}}{\rightharpoonup} T$ when $\mathcal{F}(T_\nu - T) \to 0$ as $\nu \to \infty$.

The set $\mathcal{F}_k$ of real $k$-dimensional *flat chains* over $E$ is defined as the closure in the flat norm of the set of normal rectifiable currents. It is worth remarking that our definition of the class of flat chains is different from the one given in [5] (where it was defined as the norm closure of normal currents) and is closer to the original definition in the Euclidean space (where it is usually defined as flat closure of polyhedral chains). However, it is quite easy to observe that for every $T \in \mathcal{F}_k$ there is a sequence $\{T_\nu\}$ of normal currents converging to $T$ in the mass norm. In fact, let $T$ be a flat chain and $\{T_\nu\}$ be a sequence of normal rectifiable currents with $T_\nu \overset{\mathcal{F}}{\rightharpoonup} T$ as $\nu \to \infty$. This means the existence of a sequence $\{B_\nu\}$ of normal currents such that

$$\mathbb{M}(T_\nu - T - \partial B_\nu) + \mathbb{M}(B_\nu) \to 0,$$

hence in particular, $T_\nu - \partial B_\nu \to T$ in the mass norm as $\nu \to \infty$, while all the currents $T_\nu - \partial B_\nu$ are normal. The same argument shows that flat convergence implies weak convergence of currents, namely, since $T_\nu - \partial B_\nu \to T$ and $B_\nu \to 0$ in the mass norm (hence also weakly), then $\partial B_\nu \rightharpoonup 0$ and therefore $T_\nu \rightharpoonup T$ weakly as $\nu \to \infty$.

For a normal rectifiable current $T = \theta[\![\Sigma]\!]$, given a fixed $\alpha \in [0,1]$, one sets

$$\overline{\mathbb{M}}^\alpha(T) := \int_\Sigma |\theta\lambda_T|^\alpha \, d\mathcal{H}^k,$$

so that $\overline{\mathbb{M}}^1(T) = \mathbb{M}(T)$. For a generic flat chain $T$ one defines its $\alpha$-mass by means of the relaxation procedure, i.e.

$$\mathbb{M}^\alpha(T) := \inf\{\liminf_\nu \overline{\mathbb{M}}^\alpha(T_\nu) \,:\, T_\nu \overset{\mathcal{F}}{\rightharpoonup} T, \{T_\nu\} \subset \mathcal{R}_k(E) \cap \mathbf{N}_k(E)\}.$$

In the same way one may also define, as in [6], given a nondecreasing concave function $h\colon [0,+\infty) \to [0,+\infty)$ satisfying $h(0) = 0$, more general masses $\mathbb{M}_h$. Namely, for a $k$-dimensional normal rectifiable current $T = \theta[\![\Sigma]\!]$ we set

$$\overline{\mathbb{M}}_h(T) := \int_\Sigma h(|\theta\lambda_T|)\, d\mathcal{H}^k,$$

and then extend the latter by means of relaxation with respect to the flat norm topology to the class of all $k$-dimensional real flat chains, i.e. let

$$\mathbb{M}_h(T) := \inf\{\liminf_\nu \overline{\mathbb{M}}_h(T_\nu) \,:\, T_\nu \overset{\mathcal{F}}{\rightharpoonup} T, \{T_\nu\} \subset \mathcal{R}_k(E) \cap \mathbf{N}_k(E)\}.$$

In particular one has $\mathbb{M}_h = \mathbb{M}^\alpha$ for $h(t) := t^\alpha$, $\alpha \in [0,1]$ (with the convention $0^0 := 0$ in the case $\alpha = 0$).

## 4. Slicing

For the slices of currents we use the notation from [2]. Namely, if $u\colon E \to \mathbb{R}^m$, $m \le k$, is a Lipschitz-continuous function, we denote by $\langle T, u, r\rangle$ the slice of a $k$-dimensional normal current $T$ in the level set $u^{-1}(r) := \{x \in E\colon u(x) = r\}$, which are $(k-m)$-dimensional normal currents defined for $\mathcal{L}^m$-a.e. $r \in \mathbb{R}^m$, where $\mathcal{L}^m$ stands for the $m$-dimensional Lebesgue measure. Each $\langle T, u, r\rangle$ is concentrated over $u^{-1}(r)$ and possesses the following characteristic property

$$\int_{\mathbb{R}^m} \langle T, u, r\rangle \psi(r)\, d\mathcal{L}^m(r) = T \llcorner (\psi \circ u) du, \tag{4.1}$$

for every $\psi \in C_0(\mathbb{R}^m)$, and thus

$$\int_{\mathbb{R}^m} \mu_{\langle T,u,r\rangle}(B)\, d\mathcal{L}^m(r) = \mu_{T\llcorner du}(B) \tag{4.2}$$

for every Borel set $B \subset E$. Hence in particular we have

$$\int_{\mathbb{R}^m} \mathbb{M}(\langle T, u, r\rangle)\, d\mathcal{L}^m(r) \le (\operatorname{Lip} u)^m \mathbb{M}(T). \tag{4.3}$$

Further, the function $r \in \mathbb{R}^m \mapsto \langle T, u, r\rangle \in \mathcal{M}_{k-m}(E)$ is weakly measurable in the sense that the real-valued function $r \in \mathbb{R}^m \mapsto \langle T, u, r\rangle(f\, d\pi) \in \mathbb{R}$ is measurable for every $(k-m)$-form $f\, d\pi \in D^{k-m}(E)$.

The slices $\langle T, u, r\rangle$ are defined inductively as follows (see theorem 5.6 from [2]): for $m = 1$ one sets

$$\langle T, u, r\rangle := (\partial T)\llcorner\{u > r\} - \partial(T\llcorner\{u > r\}),$$

while for $m = k+1$, $k \in \mathbb{N}$,

$$\langle T, u, r\rangle := \langle\langle T, u_1, r_1\rangle, \tilde{u}, \tilde{r}\rangle\,,$$

where $\tilde{r} := (r_2, \ldots, r_{k+1}) \in \mathbb{R}^k$, $\tilde{u} := (u_2, \ldots, u_{k+1})\colon E \to \mathbb{R}^k$. It is an immediate consequence of the definition (proven by induction on $k \in \mathbb{N}$), that

$$\partial\langle T, u, r\rangle = (-1)^k \langle \partial T, u, r\rangle \tag{4.4}$$

for $\mathcal{L}^m$-a.e. $r \in \mathbb{R}^m$. Thus, from (4.3) and (4.4) we get

$$\int_{\mathbb{R}^m} \mathcal{F}(\langle T, u, r\rangle)\, d\mathcal{L}^m(r) \le (\operatorname{Lip} u)^m \mathcal{F}(T). \tag{4.5}$$

In the assertion below we collect some of the results of [5], namely, that the notion of slices can be extended to the flat norm closure of the set of normal currents. The only new observation here is that slices of flat chains are still flat chains.

**Lemma 4.1.** *Let $u\colon E \to \mathbb{R}^m$, $m \leq k$, be a Lipschitz-continuous function, and $T \in \mathcal{M}_k(E)$ belong to the flat norm closure of the set of normal currents. Then for $\mathcal{L}^m$-a.e. $r \in \mathbb{R}$ there is the current $\langle T, u, r\rangle \in \mathcal{M}_{k-m}(E)$ concentrated over $u^{-1}(r)$ such that the function $r \in \mathbb{R}^m \mapsto \langle T, u, r\rangle \in \mathcal{M}_{k-m}(E)$ is weakly measurable, while the relationships* (4.1), (4.2), (4.3), (4.4) *and* (4.5) *hold. Moreover, $\langle T, u, r\rangle$ is a flat chain when so is $T$.*

*Proof.* Consider a sequence of normal currents $\{T_i\}$ such that $T = \sum_{i=1}^\infty T_i$ in the sense of flat convergence. This implies in particular that $\sum_i \mathcal{F}(T_i) < +\infty$. Together with the estimate

$$\int_{\mathbb{R}^m} \mathcal{F}(\langle T_i, u, r\rangle)\, d\mathcal{L}^m(r) \leq (\operatorname{Lip} u)^m \mathcal{F}(T_i)$$

valid in view of (4.5), this implies the flat convergence of the series $\sum_i \langle T_i, u, r\rangle$ for $\mathcal{L}^m$-a.e. $r \in \mathbb{R}^m$. We call the resulting limit current $\langle T, u, r\rangle$ i.e.

$$\langle T, u, r\rangle := \sum_i \langle T_i, u, r\rangle.$$

This definition is clearly independent of the choice of the sequence $\{T_i\}$. In fact, if there is another sequence of normal currents $\{\bar T_i\}$ such that $T = \sum_{i=1}^\infty \bar T_i$ in the sense of flat convergence, then $\sum_{i=1}^\nu (\bar T_i - T_i) \stackrel{\mathcal{F}}{\rightharpoonup} 0$ as $\nu \to \infty$, and hence,

$$\int_{\mathbb{R}^m} \mathcal{F}\left(\sum_{i=1}^\nu \langle \bar T_i, u, r\rangle - \sum_{i=1}^\nu \langle T_i, u, r\rangle\right) d\mathcal{L}^m(r) = \\ \int_{\mathbb{R}^m} \mathcal{F}\left(\left\langle \sum_{i=1}^\nu (\bar T_i - T_i), u, r\right\rangle\right) d\mathcal{L}^m(r) \leq (\operatorname{Lip} u)^m \mathcal{F}\left(\sum_{i=1}^\nu (\bar T_i - T_i)\right) \to 0$$

as $\nu \to \infty$. Once one recalls that flat convergence implies weak convergence of currents, one gets the weak measurability of the function $r \in \mathbb{R} \mapsto \langle T, u, r\rangle \in \mathcal{M}_{k-m}(E)$ as well as the validity of (4.1) (and hence also of (4.2) and (4.3)) and (4.4) (hence also of (4.5)) immediately from the definition of $\langle T, u, r\rangle$, and finally, that $\langle T, u, r\rangle$ is concentrated over $u^{-1}(r)$ since so are all $\langle T_i, u, r\rangle$. In the case when $T \in \mathcal{F}_k(E)$ we choose $T_i \in \mathbf{N}_k(E) \cap \mathcal{R}_k(E)$ to see that $\langle T, u, r\rangle \in \mathcal{F}_{k-m}(E)$ as a limit of a sequence of normal rectifiable currents converging in the flat norm. □

As a consequence we get the following easy lemma.

**Lemma 4.2.** *Let $T \in \mathcal{M}_k(E)$ and $\{T_\nu\} \subset \mathbf{N}_k(E)$, where $E$ is a metric space, be such that $T_\nu \stackrel{\mathcal{F}}{\rightharpoonup} T$ (resp. $T_\nu \to T$ in the mass norm) as $\nu \to \infty$. Then for every $u \in \operatorname{Lip}(E, \mathbb{R}^m)$ one has, up to a subsequence of $\nu$ (not relabeled), $\langle T_\nu, u, r\rangle \stackrel{\mathcal{F}}{\rightharpoonup} \langle T, u, r\rangle$ (resp. $\langle T_\nu, u, r\rangle \to \langle T, u, r\rangle$ in the mass norm) as $\nu \to \infty$ for $\mathcal{L}^m$-a.e. $r \in \mathbb{R}^m$.*

*Proof.* Follows immediately from (4.5) (resp. (4.3)). □

In the sequel we will use the following rectifiable slices theorem proven in [5]. Note that though in [5] it is formulated formally only for the case of integer rectifiable currents, its proof is valid without changes also for real flat chains.

**Theorem 4.3.** *Let $E$ be a separable metric space. A $k$-dimensional flat chain of finite mass is rectifiable, if and only if for every Lipschitz-continuous function $u\colon E \to \mathbb{R}^k$ the current $\langle T, u, r\rangle \in \mathcal{M}_0(E)$ is rectifiable for $\mathcal{L}^k$-a.e. $r \in \mathbb{R}^k$.*

Finally, we will need the following result regarding the estimate of the $\alpha$-mass of a flat chain in terms of $\alpha$-masses of its slices.

**Proposition 4.4** (Eilenberg inequality)**.** *Let $E$ be a separable metric space. For every $T \in \mathcal{F}_k(E)$ and every Lipschitz-continuous function $u\colon E \to \mathbb{R}^m$ one has*

$$\int_{\mathbb{R}^m} \mathbb{M}^\alpha(\langle T, u, r\rangle)\, d\mathcal{L}^m(r) \le C(\mathrm{Lip}\, u)^m \mathbb{M}^\alpha(T).$$

*for some constant $C > 0$ depending only on $\alpha$ and on $k$.*

*Proof.* Without loss of generality we may assume that $E = \ell^\infty$ (since any separable metric space is isometrically imbedded in $\ell^\infty$). Consider first the case when $T$ is a $k$-dimensional rectifiable current of finite mass, i.e. $T = \theta[\![\Sigma]\!]$, where $\Sigma \subset E$ is some $(\mathcal{H}^k, k)$-rectifiable set, and $\theta \in L^1(\Sigma; \mathcal{H}^k)$. In this case in view of theorem 9.7 from [2] one has for $\mathcal{L}^m$-a.e. $r \in \mathbb{R}^m$ the relationship

$$\langle \theta^\alpha \lambda_T^{\alpha-1}[\![\Sigma]\!], u, r\rangle = \theta^\alpha \lambda_T^{\alpha-1}[\![\Sigma \cap u^{-1}(r)]\!]$$

with the appropriate orientation of the latter, and therefore,

$$\overline{\mathbb{M}}^\alpha(\langle \theta\lambda_T^{1-1/\alpha}[\![\Sigma]\!], u, r\rangle) = \mathbb{M}(\langle \theta^\alpha \lambda_T^{\alpha-1}[\![\Sigma]\!], u, r\rangle).$$

This implies

$$\begin{aligned}
\int_{\mathbb{R}^m} \overline{\mathbb{M}}^\alpha(\langle T, u, r\rangle)\, d\mathcal{L}^m(r) &\le C \int_{\mathbb{R}^m} \mathbb{M}(\langle \theta^\alpha \lambda_T^{\alpha-1}[\![\Sigma]\!], u, r\rangle)\, d\mathcal{L}^m(r) \\
&\le C(\mathrm{Lip}\, u)^m \mathbb{M}(\theta^\alpha \lambda_T^{\alpha-1}[\![\Sigma]\!]) = C(\mathrm{Lip}\, u)^m \overline{\mathbb{M}}^\alpha(T).
\end{aligned} \tag{4.6}$$

For a generic flat chain $T \in \mathcal{F}_k(E)$ we consider a sequence $\{T_\nu\} \in \mathcal{R}_k(E) \cap \mathbf{N}_k(E)$ such that $T_\nu \overset{\mathcal{F}}{\rightharpoonup} T$ and $\overline{\mathbb{M}}^\alpha(T_\nu) \to \mathbb{M}^\alpha(T)$ as $\nu \to \infty$. Since by Lemma 4.2 one has $\langle T_\nu, u, r\rangle \overset{\mathcal{F}}{\rightharpoonup} \langle T, u, r\rangle$ as $\nu \to \infty$ for $\mathcal{L}^m$-a.e. $r \in \mathbb{R}^m$, we get from (4.6) with the help of Fatou lemma

$$\begin{aligned}
\int_{\mathbb{R}^m} \mathbb{M}^\alpha(\langle T, u, r\rangle)\, d\mathcal{L}^m(r) &\le \liminf_\nu \int_{\mathbb{R}^m} \overline{\mathbb{M}}^\alpha(\langle T_\nu, u, r\rangle)\, d\mathcal{L}^m(r) \\
&\le C(\mathrm{Lip}\, u)^m \lim_\nu \overline{\mathbb{M}}^\alpha(T_\nu) = C(\mathrm{Lip}\, u)^m \mathbb{M}^\alpha(T),
\end{aligned}$$

concluding the proof. □

## 5. Rectifiability

We start with the proof of the following easy result on rectifiability of 0-dimensional flat chains.

**Lemma 5.1.** *Let $T \in \mathcal{F}_0(E)$, where $E$ is a separable metric space, and assume that $\mathbb{M}^\alpha(T) < +\infty$ for some $\alpha \in [0, 1)$. Then $T$ is rectifiable.*

*Proof.* We naturally identify the elements of $\mathcal{M}_0(E)$ with Borel measures over $E$ with finite total variation. But then $\bar{\mathbb{M}}^\alpha$, as the functional on measures, is given by

$$\bar{\mathbb{M}}^\alpha(\mu) = \begin{cases} \sum_{x\in E} |\mu|(\{x\})^\alpha, & \mu \text{ atomic},\\ +\infty, & \text{otherwise}, \end{cases}$$

where $|\mu|$ stands for the variation measure of the measure $\mu$. Thus $\mathbb{M}^\alpha$ is finite only on atomic measures. The right reference would be [4], where one shows that $\bar{\mathbb{M}}^\alpha$ (and even more general functionals) is l.s.c. with respect to the $*$-weak convergence on measures, which corresponds to weak convergence of 0-currents, hence $\mathbb{M}^\alpha = \bar{\mathbb{M}}^\alpha$, but since the result in this paper is proven for the case when $E$ is locally compact, we give the direct general argument. In fact, to show that $\mathbb{M}^\alpha$ is finite only on atomic measures, consider any Borel measure $\mu$ with nonzero nonatomic part $\mu'$ and let $\{\mu_k\}$ be an arbitrary sequence of atomic measures such that $\mu_k \to \mu$ in the narrow sense of measures as $k \to +\infty$. Now, for $\varepsilon \in (0, 1/2)$ and for every $n \in \mathbb{N}$, we may find disjoint open sets $\Omega_i \subset E$, $i = 1, \ldots, n$, such that

$$|\mu|(\Omega_i) \ge (1-\varepsilon)\frac{|\mu'|(E)}{2n}.$$

This construction immediately shows the statement since then

$$|\mu_k|(\Omega_i) \geq (1-\varepsilon)\frac{|\mu|(E)}{2n}.$$

for all sufficiently large $k \in \mathbb{N}$, and hence

$$\begin{aligned}\bar{\mathbb{M}}^\alpha(\mu_k) &= \sum_{i=1}^n \sum_{x\in\Omega_i} |\mu_k|(\{x\})^\alpha \geq \sum_{i=1}^n \left(\sum_{x\in\Omega_i} |\mu_k|(\{x\})\right)^\alpha \\ &= \sum_{i=1}^n |\mu_k|(\Omega_i)^\alpha \geq n\frac{C}{n^\alpha}\end{aligned}$$

where $C > 0$ is a constant independent of $n$. This shows $\mathbb{M}^\alpha(\mu) = +\infty$, and to conclude the proof its is enough to provide the construction of $\Omega_i$.

To construct $\Omega_i$, we proceed inductively as follows. Let $A_1 \subset E$ be such that $|\mu'|(A_1) = |\mu'|(E)/2n$, and let $C_1 \subset A_1$ be a closed set such that $(1-\varepsilon)|\mu'|(A_1) \leq |\mu'|(C_1) \leq |\mu'|(A_1)$. We choose now an $\varepsilon_1 > 0$ such that for

$$\Omega_1' := \{x \in E \,:\, \mathrm{dist}\,(x, C_1) < 2\varepsilon_1\}$$

one has $|\mu'|(\Omega_1') \leq (1+\varepsilon)|\mu'|(C_1)$ and set

$$\Omega_1 := \{x \in E \,:\, \mathrm{dist}\,(x, C_1) < \varepsilon_1\}.$$

Clearly, one has

$$(1-\varepsilon)\frac{|\mu'|(E)}{2n} \leq |\mu'|(\Omega_1) \leq |\mu'|(\Omega_1') \leq (1+\varepsilon)\frac{|\mu'|(E)}{2n}.$$

Then, once constructed disjoint open sets $\Omega_i$, $i = 1, \ldots, j$, the number $\varepsilon_j > 0$ and open $\Omega_j'$, where $j < n$, such that

$$\begin{gathered}\sqcup_{i=1}^j \Omega_i \subset \Omega_j', \qquad\qquad \mathrm{dist}\,(\Omega_i, E\setminus\Omega_j') \geq \varepsilon_j, \\ (1-\varepsilon)\frac{|\mu'|(E)}{2n} \leq |\mu'|(\Omega_i) \leq |\mu'|(\Omega_j') \leq (1+\varepsilon)\frac{j|\mu'|(E)}{2n},\end{gathered}$$

we let $A_{j+1} \subset E \setminus \Omega_j'$ be such that $|\mu'|(A_{j+1}) = |\mu'|(E)/2n$ (this is possible since by our choice of $\varepsilon$ the measure $E \setminus \Omega_j'$ is big enough), and let $C_{j+1} \subset A_{j+1}$ be a closed set such that $(1-\varepsilon)|\mu'|(A_{j+1}) \leq |\mu'|(C_{j+1}) \leq |\mu'|(A_{j+1})$. It is enough now to choose an $\varepsilon_{j+1} < \varepsilon_j$ such that, setting

$$\begin{aligned}\Omega_{j+1} &:= \{x \in E \,:\, \mathrm{dist}\,(x, C_{j+1}) < \varepsilon_{j+1}\}, \\ \Omega_{j+1}' &:= \{x \in E \,:\, \mathrm{dist}\,(x, \cup_{i=1}^{j+1}\Omega_i) < \varepsilon_{j+1}\},\end{aligned}$$

one has $|\mu'|(\Omega_{j+1}') \leq (1+\varepsilon)(j+1)|\mu'|(E)/2n$. Clearly then all $\Omega_i$ are disjoint and

$$|\mu|(\Omega_i) \geq |\mu'|(\Omega_i) \geq (1-\varepsilon)\frac{|\mu'|(E)}{2n},$$

completing the construction. □

We are now in a position to prove the main theorem of this paper.

**Theorem 5.2.** *Let $T \in \mathcal{F}_k(E)$, where $E$ is a separable metric space, and assume that $\mathbb{M}^\alpha(T) < +\infty$ for some $\alpha \in [0,1)$. Then $T$ is rectifiable.*

*Proof.* The case $k = 0$ is given by the above Lemma 5.1. For the generic case $k > 0$, considering an arbitrary $T \in \mathcal{F}_k(E)$ with $\mathbb{M}^\alpha(T) < +\infty$ and an arbitrary Lipschitz function $u\colon E \to \mathbb{R}^k$, we have by Proposition 4.4 that $\mathbb{M}^\alpha(\langle T, u, r\rangle) < +\infty$ for $\mathcal{L}^k$-a.e. $r \in \mathbb{R}^k$, and hence, the 0-dimensional flat chain $\langle T, u, r\rangle$ is rectifiable by Lemma 5.1. Then by rectifiable slices Theorem 4.3, we get that $T$ is rectifiable. □

The following corollary provides the result for generic metric spaces.

**Corollary 5.3.** *Let $T \in \mathcal{F}_k(E)$, where $E$ is a metric space having a dense subset, the cardinality of which is Ulam number. Then $\mathbb{M}^\alpha(T) < +\infty$ for some $\alpha \in [0, 1)$ implies that $T$ is rectifiable.*

*Proof.* We first observe that for every $S \in \mathcal{M}_k(E)$ and a fixed $X \subset E$ such that $\operatorname{supp} \mu_S \subset X$, denoting $\bar{S} \in \mathcal{M}_k(X)$ the current defined by

$$\bar{S}(\omega) := S(\omega) \text{ for all } \omega \in D^k(X),$$

we get from linearity and locality property of currents $S(\omega) = \bar{S}(\omega \llcorner X)$ for all $\omega \in D^k(E)$, since by theorem 2.2.16 from [8] every finite Borel measure is concentrated over its support. In particular one has $\mu_{\bar{S}} = \mu_S$ (as measures over $E$) and $\mathbb{M}(S) = \mathbb{M}(\bar{S})$. Further, one immediately deduces from the definition that $\partial\bar{S}(\omega) = \partial S(\omega)$ for all $\omega \in D^k(X)$, i.e. in other words $\overline{\partial S} = \partial\bar{S}$, and hence $\mathbb{M}(\partial\bar{S}) = \mathbb{M}(\partial S)$.

Let $\{T_\nu\} \subset \mathcal{R}_k(E)$ be such that $T_\nu \overset{\mathcal{F}}{\rightharpoonup} T$ and $\overline{\mathbb{M}}^\alpha(T_\nu) \to \mathbb{M}^\alpha(T)$ as $\nu \to \infty$. Then there exists a sequence $\{B_\nu\} \subset \mathbf{N}_{k+1}(E)$ such that $\mathbb{M}(T_\nu - T - \partial B_\nu) + \mathbb{M}(B_\nu) \to 0$ as $\nu \to \infty$. Letting

$$X := \operatorname{cl}\left(\operatorname{supp} \mu_T \cup \left(\bigcup_\nu (\operatorname{supp} \mu_{T_\nu} \cup \operatorname{supp} \mu_{B_\nu})\right)\right),$$

cl standing for the closure, we have

$$\mathbb{M}(\bar{T}_\nu - \bar{T} - \partial\bar{B}_\nu) = \mathbb{M}(T_\nu - T - \partial B_\nu),\ \mathbb{M}(\bar{B}_\nu) = \mathbb{M}(B_\nu),$$

hence $\bar{T}_\nu \overset{\mathcal{F}}{\rightharpoonup} \bar{T}$ as $\nu \to \infty$. Further, since $\overline{\mathbb{M}}^\alpha(\bar{T}_\nu) = \overline{\mathbb{M}}^\alpha(T_\nu)$, we get that $\mathbb{M}^\alpha(\bar{T}) < +\infty$. Since by theorem 2.2.16 from [8] the support of every finite Borel measure over $E$ satisfying the conditions of the corollary is separable, we observe that $X$ is a separable metric space, and hence by Theorem 5.2 that $\bar{T}$ is rectifiable, and therefore so is $T$. □

It is finally worth remarking that all the arguments used to prove the rectifiability results of this section remain valid also for general masses $\mathbb{M}_h$. Hence these results still hold true once one substitutes the condition $\mathbb{M}^\alpha(T) < +\infty$ for some $\alpha \in [0, 1)$ with the condition $\mathbb{M}_h(T) < +\infty$, where the concave function $h\colon [0, +\infty) \to [0, +\infty)$ is such that $h(0) = 0$, $h'_+(0) = +\infty$.

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