THE DISINTEGRATION OF THE LEBESGUE MEASURE ON THE FACES OF A CONVEX FUNCTION

L. CARAVENNA AND S. DANERI

Abstract. We consider the disintegration of the Lebesgue measure on the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ w.r.t. the partition into its faces, which are convex sets and therefore have a well defined linear dimension, and we prove that each conditional measure is equivalent to the k-dimensional Hausdorff measure of the k-dimensional face on which it is concentrated. The remarkable fact is that a priori the directions of the faces are just Borel and no Lipschitz regularity is known. Notwithstanding that, we also prove that a Green-Gauss formula for these directions holds on special sets.

Keywords: disintegration of measures, faces of a convex function, conditional measures, Hausdorff dimension, absolute continuity, Divergence Formula

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1. Introduction

In this paper we deal with the explicit disintegration of the Lebesgue measure on the graph of a convex function w.r.t. the partition given by its faces. As the graph of a convex function naturally supports the Lebesgue measure, its faces, being convex, have a well defined linear dimension, and then they naturally support a proper dimensional Hausdorff measure.
Our main result is that the conditional measures induced by the disintegration are equivalent to the Hausdorff measure on the faces on which they are concentrated.

Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\mathscr{H}^n$ be the Hausdorff measure on its graph. Define a face of f as the convex set obtained by the intersection of its graph with a supporting hyperplane and consider the partition of the graph of f into the relative interiors of the faces $\{\mathsf{F}_\alpha\}_{\alpha\in\mathsf{A}}$.*
Then, the Lebesgue measure on the graph of the convex function admits a unique disintegration

$$\mathscr{H}^n = \int_{\mathsf{A}} \lambda_\alpha \, dm(\alpha)$$

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w.r.t. this partition and the conditional measure λ_α which is concentrated on the relative interior of the face F_α is equivalent to $\mathscr{H}^k \llcorner \mathsf{F}_\alpha$, where k is the linear dimension of F_α.

This apparently intuitive fact does not always hold. Indeed, on one hand existence and uniqueness of a disintegration are obtained by classical theorems, thanks to measurability conditions satisfied by the faces. Nevertheless, even for a partition given by a Borel measurable collection of segments in $\mathbb{R}^3$ (1-dimensional convex sets), if any Lipschitz regularity of the directions of these segments is not known, it may happen that the conditional measures induced by the disintegration of the Lebesgue measure are Dirac deltas (see the couterexamples in [Lar71a], [AKP]).
Also in our case, up to our knowledge, the directions of the faces of a convex function are just Borel measurable. Therefore, our result, other than answering a quite natural question, enriches the regularity properties of the faces of a convex function, which have been intensively studied for example in [ELR70], [KM71], [Lar71a], [LR71], [AKP04], [PZ07]. As a byproduct, we recover the Lebesgue negligibility of the set of relative boundary points of the faces, which was first obtained in [Lar71b].

Our result is also interesting for possible applications. Indeed, the disintegration theorem is an effective tool in dimensional reduction arguments, where it may be essential to have an explicit expression for the conditional measures. In particular, the problem of the absolute continuity of the conditional measures w.r.t. a partition given by affine sets arose naturally in a work by Sudakov ([Sud79]).

In absence of any Lipschitz regularity for the directions of the faces, the proof of our theorem does not rely on Area or Coarea Formula, which in several situations allow to obtain in one step both the existence and the absolute continuity of the disintegration (in applications to optimal mass transport problem, see for example [TW01], [FM02], [AP03]). The basis of the technique we use was first presented in order to solve a variational problem in [BG07] and it has been successfully applied to the existence of optimal transport maps for strictly convex norms in [Car08].

Just to give an idea of how this technique works, focus on a collection of 1-dimensional faces $\mathscr{C}$ which are transversal to a fixed hyperplane $H_0 = \{x \in \mathbb{R}^n : x \cdot \mathrm{e} = 0\}$ and such that the projection of each face on the line spanned by the fixed vector e contains the interval $[h^-, h^+]$, with $h^- < 0 < h^+$. Indeed, we will obtain the disintegration of the Lebesgue measure on the k-dimensional faces, with $k > 1$, from a reduction argument to this case.
First, we slice $\mathscr{C}$ with the family of affine hyperplanes $H_t = \{x \cdot \mathrm{e} = t\}$, where $t \in [h^-, h^+]$, which are parallel to H_0. In this way, by Fubini-Tonelli Theorem, the Lebesgue measure $\mathscr{L}^n$ of $\mathscr{C}$ can be recovered by integrating the $(n-1)$-dimensional Hausdorff measures of the sections of $\mathscr{C} \cap H_t$ over the segment $[h^-, h^+]$ which parametrizes the parallel hyperplanes. Then, as the faces in $\mathscr{C}$ are transversal to H_0, one can see each point in $\mathscr{C} \cap H_t$ as the image of a map σ^t defined on $\mathscr{C} \cap H_0$ which couples the points lying on the same face.
Suppose that the $(n-1)$-dimensional Hausdorff measure $\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_t)$ is absolutely continuous w.r.t. the pushforward measure $\sigma^t_\#(\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_0))$ with Radon-Nikodym derivative α^t. Then we can reduce each integral over the section $\mathscr{C} \cap H_t$ to an integral over the section $\mathscr{C} \cap H_0$:

$$\int_{\mathscr{C}} d\mathscr{L}^n = \int_{[h^-,h^+]} \mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_t)\, dt = \int_{[h^-,h^+]} \int_{\mathscr{C}\cap H_0} \alpha^t(\sigma^t(z))\, d\mathscr{H}^{n-1}(z)\, dt.$$

Exchanging the order of the last iterated integrals, we obtain the following:

$$\int_{\mathscr{C}} d\mathscr{L}^n = \int_{\mathscr{C}\cap H_0} \int_{[h^-,h^+]} \alpha^t(\sigma^t(z))\, dt\, d\mathscr{H}^{n-1}(z).$$

Since the sets $\{\sigma^{[h^-,h^+]}(z)\}_{z\in\mathscr{C}\cap H_0}$ are exactly the elements of our partition, the last equality provides the explicit disintegration we are looking for: in particular, the conditional measure concentrated on $\sigma^{[h^-,h^+]}(z)$ is absolutely continuous w.r.t. $\mathscr{H}^{n-1} \llcorner \sigma^{[h^-,h^+]}(z)$.

The core of the proof is then to show that

$$\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_t) \ll \sigma^t_\sharp(\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_0)).$$

We prove this fact as a consequence of the following quantitative estimate: for all $0 \le t \le h^+$ and $S \subset \mathscr{C} \cap H_0$

$$\mathscr{H}^{n-1}(\sigma^t(S)) \le \left(\frac{t-h^-}{-h^-}\right)^{n-1} \mathscr{H}^{n-1}(S). \tag{1.1}$$

This fundamental estimate, as in [BG07], [Car08], is proved approximating the 1-dimensional faces with

a sequence of finitely many cones with vertex in $\mathscr{C} \cap H_{h^-}$ and basis in $\mathscr{C} \cap H_t$.
At this step of the technique, the construction of such approximating sequence heavily depends on the nature of the partition one has to deal with. In this case, our main task is to find the suitable cones relying on the fact that we are approximating the faces of a convex function.
One can also derive an en estimate symmetric to the above one, showing that $\sigma^t_\#(\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_0))$ is absolutely continuous w.r.t. $\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_t)$: as a consequence, α^t is strictly positive and therefore the conditional measures are not only absolutely continuous w.r.t. the proper Hausdorff measure, but equivalent to it.

The fundamental estimate (1.1) implies moreover a Lipschitz continuity and BV regularity of $\alpha^t(z)$ w.r.t t: this yields an improvement of the regularity of the partition that now we are going to describe.
Consider a vector field v which at each point x is parallel to the face through that point x. If we restrict the vector field to an open Lipschitz set Ω which does not contain points in the relative boundaries of the faces, then we prove that its distributional divergence is the sum of two terms: an absolutely continuous measure, and a $(n-1)$-rectifiable measure representing the flux of v through the boundary of Ω. The density $(\mathrm{div}\,\mathrm{v})_{\mathrm{a.c.}}$ of the absolutely continuous part is related to the density of the conditional measures defined by the disintegration above.
In the case of the set $\mathscr{C}$ previously considered, if the vector field is such that $\mathrm{v} \cdot \mathrm{e} = 1$, the expression of the density of the absolutely continuous part of the divergence is

$$\partial_t \alpha^t = (\mathrm{div}\,\mathrm{v})_{\mathrm{a.c.}} \alpha^t.$$

Up to our knowledge no piecewise BV regularity of the vector field v of faces directions is known. Therefore, it is a remarkable fact that a divergence formula holds.

The divergence of the whole vector field v is the limit, in the sense of distributions, of the sequence of measures which are the divergence of truncations of v on the elements $\{\mathscr{K}_\ell\}_{\ell \in \mathbb{N}}$ of a suitable partition of $\mathbb{R}^n$. However, in general, it fails to be a measure.

In the last part, we change point of view: instead of looking at vector fields constrained to the faces of the convex function, we describe the faces as an $(n+1)$-uple of currents, the k-th one corresponding to the family of k-dimensional faces, for $k = 0, \dots, n$. The regularity results obtained for the vector fields can be rewritten as regularity results for these currents. More precisely, we prove that they are locally flat chains. When truncated on a set Ω as above, they are locally normal, and we give an explicit formula for their border; the $(n+1)$-uple of currents is the limit, in the flat norm, of the truncations on the elements of a partition.

An application of this kind of further regularity is presented in Section 8 of [BG07]. Given a vector field v constrained to live on the faces of f, the divergence formula we obtain allows to reduce the transport equation

$$\mathrm{div}\,\rho\mathrm{v} = g$$

to a PDE on the faces of the convex function. We do not pursue this issue in the paper.

1.1. **Outline of the article.** In the following we describe the structure of the paper.

In Section 2 we first give the definition of disintegration of a measure consistent with a fixed partition of the ambient space. Then, we report an abstract disintegration theorem which guarantees the existence and uniqueness of the disintegration under quite general assumptions on the σ-algebra of the ambient space and on the partition; under these hypothesis, the conditional probabilities given by the disintegration are concentrated on the sets of the partition.

In Section 3, after giving the basic definitions and notations we will be working with, we apply the disintegration theorem recalled in Section 2 and get the existence and uniqueness of the disintegration of the Lebesgue measure on the faces of a convex function. For notational convenience, we work with the projections of the faces on $\mathbb{R}^n$ and we neglect the set where the convex function is not differentiable.
In Subsection 3.2 we state our main theorem on the equivalence between the conditional probabilities and the k-Hausdorff measure on the k-dimensional faces where they are concentrated. As the conditional measures on the 0-dimesional and n-dimesional faces are already determined (they must be respectively given by Dirac deltas and by the $\mathscr{H}^n$-measure on the corresponding faces), we focus on the disintegration of the Lebesgue measure on the k-dimensional faces for $k = 1, ..., n-1$.

In Section 4 we prove the explicit disintegration theorem.
In Subsection 4.1 we explain the first idea of our disintegration technique, which consists in the reduction to countably many model sets like $\mathscr{C}$ and in the application of the Fubini-Tonelli technique on these sets

which has been briefly sketched in the introduction.
In Subsection 4.2 we address the Borel measurability of the multivalued function $\mathcal{D}$ which assigns to each point $x \in \mathbb{R}^n$ the directions of the face passing through x. This is needed in the following in order to reduce the ambient space into countably many model sets.
In Subsection 4.3 we define the partition of $\mathbb{R}^n$ into the model sets (called $\mathcal{D}$-cylinders) on which we will first prove our disintegration theorem. When $k = 1$, the k-dimensional faces are partitioned into sets like $\mathscr{C}$. When $k > 1$, each model set $\mathscr{C}^k$ is defined taking a collection of k-dimensional faces which are transversal to a fixed $(n-k)$-dimensional affine plane (as, e.g., $H = \{x \in \mathbb{R}^n : x \cdot \mathrm{e}_1 = \ldots = x \cdot \mathrm{e}_k = 0\}$) and considering the points of these faces whose projection on the perpendicular k-plane ($H^\perp = \{x \in \mathbb{R}^n : x \cdot \mathrm{e}_{k+1} = \ldots = x \cdot \mathrm{e}_n = 0\}$) is contained in a fixed rectangle (as, e.g., $\{x \in \mathbb{R}^n : x \cdot \mathrm{e}_{k+1} = \ldots = x \cdot \mathrm{e}_n = 0,\ x \cdot \mathrm{e}_i \in [h_i^-, h_i^+]$ for $i = 1, \ldots, k$ and $h_i^\pm \in \mathbb{R}\}$).
Subsection 4.4 is devoted to the proof of the quantitative estimate (1.1). Actually, in Lemma 4.7 we prove that an estimate like (1.1) holds for the pushforward of the $\mathscr{H}^{n-k}$-dimensional measure on the sections of a model set $\mathscr{C}^k$ which are obtained cutting it with transversal $(n-k)$-dimensional affine planes. The core of the proof, which is the construction of a suitable sequence of approximating cones for the 1-dimensional faces of f, is contained in Lemma 4.14.
In Subsection 4.5 we study some regularity properties of the Radon-Nikodym derivative α^t; they will be used in Section 5 to study the regularity of the divergence of vector fields parallel to the faces.
In Subsection 4.6 we prove the explicit disintegration theorem on the model sets $\mathscr{C}^k$ of the partition.
In Subsection 4.7 we collect the disintegrations obtained on the model set and obtain the global result regarding the disintegration of the Lebesgue measure on all $\mathbb{R}^n$.

Section 5 deals with the divergence of the faces directions, with two equivalent approaches.
In Subsection 5.1 we consider the divergence of any vector field which at each point x is parallel to the face of f through x. In Subsection 5.1.1 we truncate this vector field to k-dimensional $\mathcal{D}$-cylinders. The divergence of these truncated vector fields turns out to be a Radon measure. The density of the absolutely continuous part of this distributional divergence involves the density α defined in (4.67). The precise statement is given in Corollary 5.4.
In Subsection 5.1.2 we consider the vector field v in the whole $\mathbb{R}^n$. Its divergence is the limit, in the sense of distributions, of finite sums of the Radon measures corresponding to the divergence of the truncations of v on a family of $\mathcal{D}$-cylinders as above constituting a partition of $\mathbb{R}^n$. In general, the divergence of v fails to be a measure.
In Section 5.2 we consider the border of k-dimensional currents associated to k-faces, where each k-face is thought as a k-covector field, for $k = 0, \ldots, n$. We rephrase the results of Section 5.1 in this formalism. Subsection 5.2.1 is devoted to recalls on tensors and currents in order to fix the notation.
In Subsection 5.2.2 we fix the attention on the current associated to a k-vector field that gives the direction of the k-faces on $\mathscr{C}^k$ and vanishes elsewhere. The border of this current is the sum of two currents, both representable by integration. One is the integral on $\mathscr{C}^k$ of the divergence of the k-vector field truncated to $\mathscr{C}^k$ and it is again related to α. The other one is concentrated on $\partial\mathscr{C}^k$ and arises from the truncation of the faces to the $\mathcal{D}$-cylinder. The statement is given in Lemma 5.9.
In Subsection 5.2.3, we consider the $(n+1)$-uple of currents associated to the faces of f, the k-th one acting on k-forms on $\mathbb{R}^n$. By means of a partition of $\mathbb{R}^n$ into $\mathcal{D}$-cylinders as above, we recover each of them as the limit, in the flat norm, of the normal currents defined as truncations of this $(n+1)$-uple to the elements of the partition.

The last section contains a long Table of Notations, for the reader's convenience.

2. An Abstract Disintegration Theorem

A disintegration of a measure over a partition of the space on which it is defined is a way to write that measure as a "weighted sum" of probability measures which are possibly concentrated on the elements of the partition.

Let (X, Σ, μ) be a measure space (which will be called the *ambient space* of the disintegration), i.e. Σ is a σ-algebra of subsets of X and μ is a measure with finite total variation on Σ and let $\{X_\alpha\}_{\alpha \in \mathsf{A}} \subset X$ be a partition of X. After defining the following equivalence relation on X

$$x \sim y \quad \Leftrightarrow \quad \exists\, \alpha \in \mathsf{A} :\ x, y \in X_\alpha,$$

we make the identification $\mathsf{A} = X/_\sim$ and we denote by p the quotient map $p : x \in X \mapsto [x] \in \mathsf{A}$.

Moreover, we endow the quotient space A with the measure space structure given by the largest σ-algebra that makes p measurable, i.e.

$$\mathscr{A} = \{F \subset \mathsf{A} : p^{-1}(F) \in \Sigma\},$$

and by the measure $\nu = p_\# \mu$.

Definition 2.1 (Disintegration). A *disintegration* of μ consistent with the partition $\{X_\alpha\}_{\alpha \in \mathsf{A}}$ is a family $\{\mu_\alpha\}_{\alpha \in \mathsf{A}}$ of probability measures on X such that

1. $\forall\, E \in \Sigma, \quad \alpha \mapsto \mu_\alpha(E)$ is ν-measurable;

2. $\mu = \int \mu_\alpha \, d\nu$, i.e.

$$\mu(E \cap p^{-1}(F)) = \int_F \mu_\alpha(E)\, d\nu(\alpha), \quad \forall\, E \in \Sigma,\, F \in \mathscr{A}. \tag{2.1}$$

The disintegration is *unique* if the measures μ_α are uniquely determined for ν-a.e. $\alpha \in \mathsf{A}$.
The disintegration is *strongly consistent with* p if $\mu_\alpha(X \setminus X_\alpha) = 0$ for ν-a.e. $\alpha \in \mathsf{A}$.
The measures μ_α are also called conditional probabilities of μ w.r.t. ν.

Remark 2.2. When a disintegration exists, formula (2.1) can be extended by Beppo Levi theorem to measurable functions $f : X \to \mathbb{R}$ as

$$\int f\, d\mu = \int \left(\int f\, d\mu_\alpha \right) d\nu(\alpha).$$

The existence and uniqueness of a disintegration can be obtained under very weak assumptions which concern only the ambient space. Nevertheless, in order to have the strong consistency of the conditional probabilities w.r.t. the quotient map we have to make structural assumptions also on the quotient measure algebra, otherwise in general $\mu_\alpha(X_\alpha) \neq 1$ (i.e. the disintegration is consistent but not strongly consistent). The more general result of existence of a disintegration which is consistent with a given partition is contained in [Pac79], while a weak sufficient condition in order that a consistent disintegration is also strongly consistent is given in [HJ71].

In the following we recall an abstract disintegration theorem, in the form presented in [BC]. It guarantees, under suitable assumptions on the ambient and on the quotient measure spaces, the existence, uniqueness and strong consistency of a disintegration. Before stating it, we recall that a measure space (X, Σ) is countably-generated if Σ coincides with the σ-algebra generated by a sequence of measurable sets $\{B_n\}_{n \in \mathbb{N}} \subset \Sigma$.

Theorem 2.3. *Let (X, Σ) be a countably-generated measure space and let μ be a measure on X with finite total variation. Then, given a partition $\{X_\alpha\}_{\alpha \in \mathsf{A}}$ of X, there exists a unique consistent disintegration $\{\mu_\alpha\}_{\alpha \in \mathsf{A}}$. Moreover, if there exists an injective measurable map from $(\mathsf{A}, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, where $\mathscr{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$, the disintegration is strongly consistent with p.*

Remark 2.4. If the total variation of μ is not finite, a disintegration of μ consistent with a given partition as defined in (2.1) in general does not exists, even under the assumptions on the ambient and on the quotient space made in Theorem 2.3 (take for example $X = \mathbb{R}^n$, $\Sigma = \mathscr{B}(\mathbb{R}^n)$, $\mu = \mathscr{L}^n$ and $X_\alpha = \{x : x \cdot z = \alpha\}$, where z is a fxed vector in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$).

Nevertheless, if μ is σ-finite and (X, Σ), $(\mathsf{A}, \mathscr{A})$ satisfy the hypothesis of Theorem 2.3, as soon as we replace the possibly infinite-valued measure $\nu = p_\# \mu$ with an equivalent σ-finite measure m on $(\mathsf{A}, \mathscr{A})$, we can find a family of σ-finite measures $\{\tilde{\mu}_\alpha\}_{\alpha \in \mathsf{A}}$ on X such that

$$\mu = \int \tilde{\mu}_\alpha \, dm(\alpha) \tag{2.2}$$

and

$$\tilde{\mu}_\alpha(X \setminus X_\alpha) = 0 \quad \text{for } m\text{-a.e. } \alpha \in \mathsf{A}. \tag{2.3}$$

For example, we can take $m = p_\# \theta$, where θ is a finite measure equivalent to μ.

We recall that two measures μ_1 and μ_2 are *equivalent* if and only if

$$\mu_1 \ll \mu_2 \quad \text{and} \quad \mu_2 \ll \mu_1. \tag{2.4}$$

Moreover, if λ and $\{\tilde{\lambda}_\alpha\}_{\alpha\in\mathsf{A}}$ satisfy (2.2) and (2.3) as well as m and $\{\tilde{\mu}_\alpha\}_{\alpha\in\mathsf{A}}$, then λ is equivalent to m and

$$\tilde{\lambda}_\alpha = \frac{dm}{d\lambda}(\alpha)\tilde{\mu}_\alpha,$$

where $\frac{dm}{d\lambda}$ is the Radon-Nikodym derivative of m w.r.t. λ.

In the following, whenever μ is a σ-finite measure with infinite total variation, by disintegration of μ strongly consistent with a given partition we will mean any family of σ-finite measures $\{\tilde{\mu}_\alpha\}_{\alpha\in\mathsf{A}}$ which satisfy the above properties; in fact, whenever μ has finite total variation we will keep the definition of disintegration given in (2.1).

Finally, we recall that any disintegration of a σ-finite measure μ can be recovered by the disintegrations of the finite measures $\{\mu \llcorner K_n\}_{n\in\mathbb{N}}$, where $\{K_n\}_{n\in\mathbb{N}} \subset X$ is a partition of X into sets of finite μ-measure.

3. Statement of the Main Theorem

In this section, after setting the notation and some basic definitions, we apply Theorem 2.3 to get the existence, uniqueness and strong consistency of the disintegration of the Lebesgue measure on the faces of a convex function. Then, we give a rigorous formulation of the problem we are going to deal with and state our main theorem.

3.1. Setting.

Let us consider the **ambient space**

$$(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mathscr{L}^n \llcorner K),$$

where $\mathscr{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$, $\mathscr{B}(\mathbb{R}^n)$ is the Borel σ-algebra, K is any set of finite Lebesgue measure and $\mathscr{L}^n \llcorner K$ is the restriction of the Lebesgue measure to the set K. Indeed, by Remark 2.4, the disintegration of the Lebesgue measure w.r.t. a given partition is determined by the disintegrations of the Lebesgue measure restricted to finite measure sets.

Then, let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function.
We recall that the subdifferential of f at a point $x \in \mathbb{R}^n$ is the set $\partial^- f(x)$ of all $r \in \mathbb{R}^n$ such that

$$f(w) - f(x) \geq r \cdot (w - x), \quad \forall\, w \in \mathbb{R}^n.$$

From the basic theory of convex functions, as f is real-valued and is defined on all $\mathbb{R}^n$, $\partial^- f(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$ and it consists of a single point if and only if f is differentiable at x. Moreover, in that case, $\partial^- f(x) = \{\nabla f(x)\}$, where $\nabla f(x)$ is the differential of f at the point x.
We denote by $\operatorname{dom} \nabla f$ a σ-compact set where f is differentiable and such that $\mathbb{R}^n \setminus \operatorname{dom} \nabla f$ is Lebesgue negligible. $\nabla f : \operatorname{dom} \nabla f \to \mathbb{R}$ denotes the differential map and $\operatorname{Im} \nabla f$ the image of $\operatorname{dom} \nabla f$ with the differential map.

The **partition** of $\mathbb{R}^n$ on which we want to decompose the Lebesgue measure is given by the sets

$$\nabla f^{-1}(y) = \{x \in \mathbb{R}^n : \nabla f(x) = y\}, \quad y \in \operatorname{Im} \nabla f,$$

along with the set $\Sigma^1(f) = \mathbb{R}^n \setminus \operatorname{dom} \nabla f$.
By the convexity of f, we can moreover assume w.l.o.g. that the intersection of $\nabla f^{-1}(y)$ with $\operatorname{dom} \nabla f$ is convex

Since ∇f is a Borel map and $\Sigma^1(f)$ is a $\mathscr{L}^n$-negligible Borel set (see e.g. [AAC92], [AA99]), we can assume that the quotient map p of Definition 2.1 is given by ∇f and that the quotient space is given by $(\operatorname{Im} \nabla f, \mathscr{B}(\operatorname{Im} \nabla f))$, which is measurably included in $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$.

Then, this partition satisfies the hypothesis of **Theorem 2.3** and there exists a family

$$\{\mu_y\}_{y\in\operatorname{Im}\nabla f}$$

of probability measures on $\mathbb{R}^n$ such that

$$\mathscr{L}^n \llcorner K(B \cap \nabla f^{-1}(A)) = \int_A \mu_y(B)\, d\nabla f_\#(\mathscr{L}^n \llcorner K)(y), \quad \forall\, A, B \in \mathscr{B}(\mathbb{R}^n).$$

In the following we give the formal definition of **face of a convex function** and relate this object to the sets $\nabla f^{-1}(y)$ of our partition.

Definition 3.1. A *tangent hyperplane* to the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of $\mathbb{R}^{n+1}$ of the form

$$H_y = \{(z, h_y(z)) : \ z \in \mathbb{R}^n, \text{ and } \ h_y(z) = f(x) + y \cdot (z - x)\}, \tag{3.1}$$

where $x \in \nabla f^{-1}(y)$.

We note that, by convexity, the above definition is independent of $x \in \nabla f^{-1}(y)$.

Definition 3.2. A *face of a convex function* $f : \mathbb{R}^n \to \mathbb{R}$ is a set of the form

$$H_y \cap \operatorname{graph} f_{|\operatorname{dom} \nabla f}. \tag{3.2}$$

It is easy to check that, $\forall\, y \in \operatorname{Im} \nabla f$ and $\forall\, z$ such that $(z, f(z)) \in H_y \cap \operatorname{graph} f_{|\operatorname{dom} \nabla f}$, we have that $y = \nabla^{-1} f(z)$.

If we denote by $\pi_{\mathbb{R}^n} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection map on the first n coordinates, one can see that, for all $y \in \operatorname{Im} \nabla f$,

$$\nabla f^{-1}(y) = \pi_{\mathbb{R}^n}(H_y \cap \operatorname{graph} f_{|\operatorname{dom} \nabla f}).$$

For notational convenience, the set $\nabla f^{-1}(y)$ will be denoted as F_y.
We also write F_y^k instead of F_y whenever we want to emphasize the fact that the latter has dimension k, for $k = 0, \ldots, n$ (where the dimension of a convex set C is the dimension of its affine hull $\operatorname{aff}(C)$) and we set

$$F^k = \bigcup_{\{y:\, \dim(F_y) = k\}} F_y. \tag{3.3}$$

3.2. **Absolute continuity of the conditional probabilities.** Since the measure we are disintegrating ($\mathscr{L}^n$) has the same Hausdorff dimension of the space on which it is concentrated ($\mathbb{R}^n$) and since the sets of the partition on which the conditional probabilities are concentrated have a well defined linear dimension, we address the problem of whether this absolute continuity property of the initial measure is still satisfied by the conditional probabilities produced by the disintegration: we want to see if

$$\dim(F_y) = k \quad \Rightarrow \quad \mu_y \ll \mathscr{H}^k \llcorner F_y. \tag{3.4}$$

The answer to this question is not trivial. Indeed, when $n \geq 3$ one can construct sets of full Lebesgue measure in $\mathbb{R}^n$ and Borel partitions of those sets into convex sets such that the conditional probabilities of the corresponding disintegration do not satisfy property (3.4) for $k = 1$ (see e.g. [AKP04]).

However, for the partition given by the faces of a convex function, we show that the absolute continuity property is preserved by the disintegration. Our main result is the following:

Theorem 3.3. *Let $\{\mu_y\}_{y \in \operatorname{Im} \nabla f}$ be the family of probability measures on $\mathbb{R}^n$ such that*

$$\mathscr{L}^n \llcorner K(B \cap \nabla f^{-1}(A)) = \int_A \mu_y(B)\, d\nabla f_\#(\mathscr{L}^n \llcorner K)(y), \quad \forall\, A, B \in \mathscr{B}(\mathbb{R}^n). \tag{3.5}$$

Then, for $\nabla f_\#(\mathscr{L}^n \llcorner K)$-a.e. $y \in \operatorname{Im} \nabla f$, the conditional probability μ_y is equivalent to the k-dimensional Hausdorff measure $\mathscr{H}^k$ restricted to $F_y^k \cap K$, i.e.

$$\mu_y \ll \mathscr{H}^k \llcorner (F_y^k \cap K) \quad \textit{and} \quad \mathscr{H}^k \llcorner (F_y^k \cap K) \ll \mu_y. \tag{3.6}$$

Remark 3.4. The result for $k = 0, n$ is trivial. Indeed, for all y such that $F_y \cap K \neq \emptyset$ and $\dim(F_y) = 0$ we must put $\mu_y = \delta_{\{F_y\}}$, where δ_{x_0} is the Dirac mass supported in x_0, whereas if $\dim(F_y \cap K) = n$ we have that $\mu_y = \frac{\mathscr{L}^n \llcorner F_y}{|\mathscr{L}^n \llcorner F_y|}$.

Remark 3.5. Since the map

$$\begin{aligned} \operatorname{id} \times f : &\mathbb{R}^n \to \mathbb{R}^{n+1} \\ &x \mapsto (x, f(x)) \end{aligned}$$

is locally Lipschitz and preserves the Hausdorff dimension of sets, Theorem 3.3 holds also for the disintegration of the $(n+1)$-dimensional Lebesgue measure over the partition of the graph of f given by the faces defined in (3.2). We have chosen to deal with the disintegration of the Lebesgue measure over the projections of the faces on $\mathbb{R}^n$ only for notational convenience.

Theorem 3.3 will be proved in Section 4.7, where we provide also an explicit expression for the conditional probabilities.

If we knew some Lipschitz regularity for the field of directions of the faces of a convex function, we could try to apply the Area or Coarea Formula in order to obtain within a single step the disintegration of the Lebesgue measure and the absolute continuity property (3.6).

However, such regularity is presently not known and for this reason we have to follow a different approach.

4. The Explicit Disintegration

4.1. **A disintegration technique.** In this paragraph we give an outline of the technique we use in order to prove Theorem 3.3.
This kind of strategy was first used in order to disintegrate the Lebesgue measure on a collection of disjoint segments in [BG07], and then in [Car08].
For simplicity, we focus on the disintegration of the Lebesgue measure on the 1-dimensional faces and, in the end, we give an idea of how we will extend this technique in order to prove the absolute continuity of the conditional probabilities on the faces of higher dimension.

The disintegration on model sets: Fubini-Tonelli theorem and absolute continuity estimates on affine planes which are transversal to the faces. First of all, let us suppose that the projected 1-dimensional faces of f are given by a collection of disjoint segments $\mathscr{C}$ whose projection on a fixed direction $\mathrm{e} \in \mathbf{S}^{n-1}$ is equal to a segment $[h^- \mathrm{e}, h^+ \mathrm{e}]$ with $h^- < 0 < h^+$, more precisely

$$\mathscr{C} = \bigcup_{z \in Z_t} [a(z), b(z)], \tag{4.1}$$

where Z_t is a compact subset of an affine hyperplane of the form $\{x \cdot \mathrm{e} = t\}$ for some $t \in \mathbb{R}$ and $a(z) \cdot \mathrm{e} = h^-$, $b(z) \cdot \mathrm{e} = h^+$. Any set of the form (4.1) will be called a *model set* (see also Figure 1).

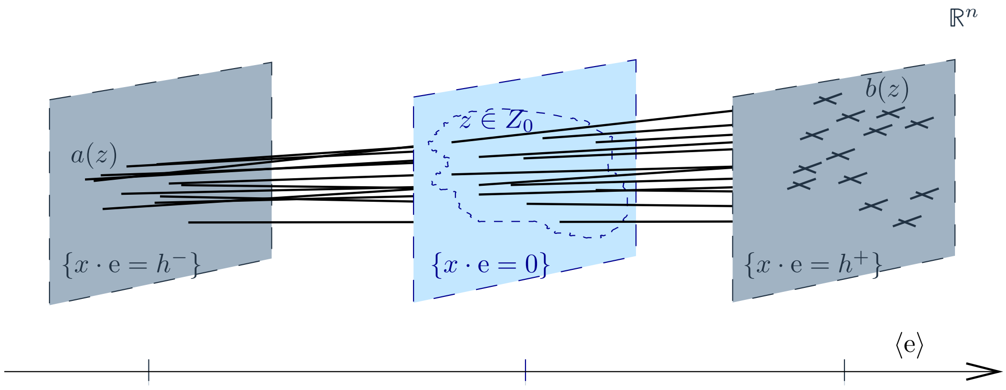

Figure 1: A model set of one dimensional projected faces. Given a subset Z_0 of the hyperplane $\{x \cdot \mathrm{e} = 0\}$, the above model set is made of the one dimensional faces of f passing through some $z \in Z_0$, truncated between $\{x \cdot \mathrm{e} = h^-\}$, $\{x \cdot \mathrm{e} = h^+\}$ and projected on $\mathbb{R}^n$.

We want to find the conditional probabilities of the disintegration of the Lebesgue measure on the segments which are contained in the model set $\mathscr{C}$ and see if they are absolutely continuous w.r.t. the $\mathscr{H}^1$ measure.

The idea of the proof is to obtain the required disintegration by a Fubini-Tonelli argument, that reverts the problem of absolute continuity w.r.t. $\mathscr{H}^1$ of the conditional probabilities on the projected 1-dimensional faces to the absolute continuity w.r.t. $\mathscr{H}^{n-1}$ of the push forward by the flow induced by the directions of the faces of the $\mathscr{H}^{n-1}$-measure on transversal hyperplanes.

First of all, we cut the set $\mathscr{C}$ with the affine hyperplanes which are perpendicular to the segment $[h^- \mathrm{e}, h^+ \mathrm{e}]$, we apply Fubini-Tonelli theorem and we get

$$\int_{\mathscr{C}} \varphi(x) \, d\mathscr{L}^n(x) = \int_{h^-}^{h^+} \int_{\{x \cdot \mathrm{e} = t\} \cap \mathscr{C}} \varphi \, d\mathscr{H}^{n-1} \, dt, \quad \forall \varphi \in C_c^0(\mathbb{R}^n). \tag{4.2}$$

Then we observe the following: for every $s, t \in [h^-, h^+]$, the points of $\{x \cdot \mathrm{e} = t\} \cap \mathscr{C}$ are in bijective correspondence with the points of the section $\{x \cdot \mathrm{e} = s\} \cap \mathscr{C}$ and a bijection is obtained by pairing the points that belong to the same segment $[a(z), b(z)]$, for some $z \in Z_t$.
For example, a map which sends the transversal section $Z = \{x \cdot \mathrm{e} = 0\} \cap \mathscr{C}$ into the section $Z_t = \{x \cdot \mathrm{e} = t\} \cap \mathscr{C}$ (for any $t \in [h^-, h^+]$) is given by

$$\begin{aligned} \sigma^t : &Z \to \sigma^t(Z) = \{x \cdot \mathrm{e} = t\} \cap \mathscr{C} \\ &z \mapsto z + t \frac{v_{\mathrm{e}}(z)}{|v_{\mathrm{e}}(z) \cdot \mathrm{e}|} = \{x \cdot \mathrm{e} = t\} \cap [a(z), b(z)], \end{aligned}$$

where $[a(z), b(z)]$ is the segment of $\mathscr{C}$ passing through the point z and $v_{\mathrm{e}}(z) = \frac{b(z)-a(z)}{|b(z)-a(z)|}$.

Therefore, as soon as we fix a transversal section of $\mathscr{C}$, say for e.g. $Z = \{x \cdot \mathrm{e} = 0\} \cap \mathscr{C}$, we can try to rewrite the inner integral in the r.h.s. of (4.2) as an integral of the function $\varphi \circ \sigma^t$ w.r.t. to the $\mathscr{H}^{n-1}$ measure of the fixed section Z.
This can be done if

$$(\sigma^t)^{-1}_{\#}(\mathscr{H}^{n-1} \llcorner \sigma^t(Z)) \ll \mathscr{H}^{n-1} \llcorner Z. \tag{4.3}$$

Indeed,

$$\int_{\sigma^t(Z)} \varphi(y)\, d\mathscr{H}^{n-1}(y) = \int_Z \varphi(\sigma^t(z))\, d(\sigma^t)^{-1}_{\#}(\mathscr{H}^{n-1} \llcorner \sigma^t(Z))(z) \tag{4.4}$$

and if (4.3) is satisfied for all $t \in [h^-, h^+]$, then

$$(4.2) = \int_{h^-}^{h^+} \int_Z \varphi(\sigma^t(z))\, \alpha(t, z)\, d\mathscr{H}^{n-1}(z)\, dt,$$

where $\alpha(t, z)$ is the Radon-Nikodym derivative of $(\sigma^t)^{-1}_{\#}(\mathscr{H}^{n-1} \llcorner \sigma^t(Z))$ w.r.t. $\mathscr{H}^{n-1} \llcorner Z$.

Having turned the r.h.s. of (4.2) into an iterated integral over a product space isomorphic to $Z + [h^-\mathrm{e}, h^+\mathrm{e}]$, the final step consists in applying Fubini-Tonelli theorem again so as to exchange the order of the integrals and get

$$\int_{\mathscr{C}} \varphi(x)\, d\mathscr{L}^n(x) = \int_Z \int_{h^-}^{h^+} \varphi(\sigma^t(z))\, \alpha(t, z)\, dt\, d\mathscr{H}^{n-1}(z). \tag{4.5}$$

This final step can be done if α is Borel-measurable and locally integrable in (t, z).

By the uniqueness of the disintegration stated in Theorem 2.3 we have that

$$d\mu_z(t) = \frac{\alpha(t, z) \cdot d\mathscr{H}^1 \llcorner [a(z), b(z)](t)}{\int_{h^-}^{h^+} \alpha(s, z)\, ds}, \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } z \in Z. \tag{4.6}$$

The same reasoning can be applied to the case $k > 1$. Indeed, let us consider a collection $\mathscr{C}^k$ of k-dimensional faces whose projection on a certain k-plane $\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$ is given by a rectangle $\prod_{i=1}^k [h_i^- \mathrm{e}_i, h_i^+ \mathrm{e}_i]$, with $h_i^- < 0 < h_i^+$ for all $i = 1, \dots, k$ (see Figure 2).

Then, as soon as we fix an affine $(n-k)$-dimensional plane which is perpendicular to the k-plane $\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$, as for example $H^k = \bigcap_{i=1}^k \{x \cdot \mathrm{e}_i = 0\}$, and we denote by $\pi_{\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle} : \mathbb{R}^n \to \langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$ the projection map on the k-plane $\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$, the k-dimensional faces in $\mathscr{C}^k$ can be parametrized with the map

$$\sigma^{t\mathrm{e}}(z) = z + t \frac{v_{\mathrm{e}}(z)}{|\pi_{\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle}(v_{\mathrm{e}}(z))|}, \tag{4.7}$$

where $z \in Z^k = H^k \cap \mathscr{C}^k$, e is a unit vector in the k-plane $\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$, $t \in \mathbb{R}$ satisfies $t\mathrm{e} \cdot \mathrm{e}_i \in [h_i^-, h_i^+]$ for all $i = 1, \dots, k$ and $v_{\mathrm{e}}(z)$ is the unit direction contained in the face passing through z which is such that $\frac{\pi_{\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle}(v_{\mathrm{e}}(z))}{|\pi_{\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle}(v_{\mathrm{e}}(z))|} = \mathrm{e}$.

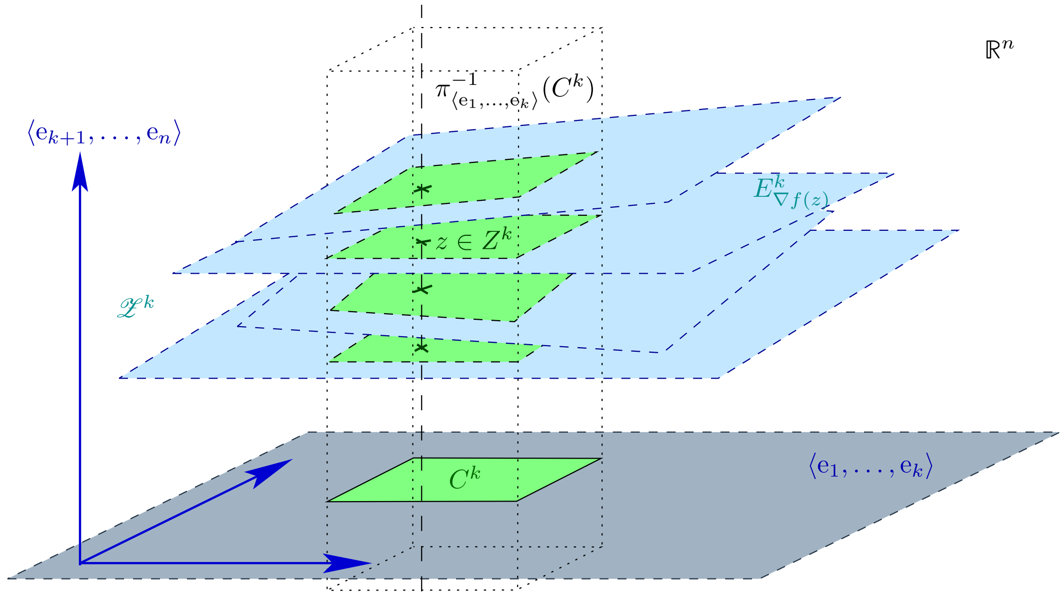

Figure 2: Sheaf sets and $\mathcal{D}$-cylinders (Definitions 4.3, 4.5). Roughly, a sheaf set $\mathscr{Z}^k$ is a collection of k-faces of f, projected on $\mathbb{R}^n$, which intersect exactly at one point some set Z^k contained in a $(n-k)$-dimensional plane. A $\mathcal{D}$-cylinder $\mathscr{C}^k$ is the intersection of a sheaf set with $\pi^{-1}_{\langle e_1,\dots,e_k\rangle}(C^k)$, for some rectangle $C^k = \mathrm{conv}(\{t_i^- e_i, t_i^+ e_i\}_{i=1,\dots,k})$, where $\{e_1,\dots,e_n\}$ are an orthonormal basis of $\mathbb{R}^n$. Such sections Z^k are called basis, while the k-plane $\langle e_1,\dots,e_k\rangle$ is an axis.

If we cut the set $\mathscr{C}^k$ with affine hyperplanes which are perpendicular to e_i for $i = 1,\dots,k$ and apply k-times the Fubini-Tonelli theorem, the main point is again to show that, for every e and t as above,

$$(\sigma^{te})^{-1}_{\#}(\mathscr{H}^{n-k} \llcorner Z^k) \ll \mathscr{H}^{n-k} \llcorner Z^k \tag{4.8}$$

and, after this, that the Radon-Nikodym derivative between the above measures satisfies proper measurability and integrability conditions.

Then, to prove Theorem 3.3 on model sets that are, up to translations and rotations, like the set $\mathscr{C}^k$, it is sufficient to prove (4.8) and some weak properties of the related density function, such as Borel-measurability and local integrability.

Actually, the properties of this function will follow immediately from our proof of (4.8), which is given in a stronger form in Lemma 4.31.

Partition of $\mathbb{R}^n$ into model sets and the global disintegration theorem. In the next section we show that the set F^k defined in (3.3), for $k = 1,\dots,n-1$, can be partitioned, up to a negligible set, into a countable collection of Borel-measurable model sets like $\mathscr{C}^k$. After proving the disintegration theorem on the model sets we will see how to glue the "local" results in order to obtain a global disintegration theorem for the Lebesgue measure over the whole faces of the convex function (restricted to a set of $\mathscr{L}^n$-finite measure).

4.2. **Measurability of the directions of the k-dimensional faces.** The aim of this subsection is to show that the set of the projected k-dimensional faces of a convex function f can be parametrized by a $\mathscr{L}^n$-measurable (and multivalued) map. This will allow us to decompose $\mathbb{R}^n$ into a countable family of Borel model sets on which to prove Theorem 3.3.

First of all we give the following definition, which generalizes Definition 3.1.

Definition 4.1. A *supporting hyperplane* to the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is an affine hyperplane in $\mathbb{R}^{n+1}$ of the form

$$H = \{w \in \mathbb{R}^{n+1} : w \cdot b = \beta\},$$

where $b \neq 0$, $w \cdot b \leq \beta$ for all $w \in \operatorname{epi} f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t \geq f(x)\}$ and $w \cdot b = \beta$ for at least one $w \in \operatorname{epi} f$. As f is defined and real-valued on all $\mathbb{R}^n$, every supporting hyperplane is of the form

$$H_y = \{(z, h_y(z)) : z \in \mathbb{R}^n, h_y(z) = f(x) + y \cdot (z - x)\}, \tag{4.9}$$

for some $y \in \partial^- f(x)$. Whenever $y \in \operatorname{Im} \nabla f$, H_y is a tangent hyperplane to the graph of f according to Definition 3.1.

Then we define the map

$$x \mapsto \mathcal{P}(x) = \left\{ z \in \mathbb{R}^n : \exists\, y \in \partial^- f(x) \text{ such that } f(z) - f(x) = y \cdot (z - x) \right\}. \tag{4.10}$$

By definition, $\mathcal{P}(x) = \bigcup_{y \in \partial^- f(x)} \pi_{\mathbb{R}^n} (H_y \cap \operatorname{graph}(f))$.

Moreover, the map

$$\operatorname{dom} \nabla f \ni x \mapsto \mathcal{R}(x) := \mathcal{P}(x) \cap \operatorname{dom} \nabla f,$$

gives precisely the set F_y of our partition that passes through the point x.

As the disintegration over the 0-dimensional faces is trivial, we will restrict our attention to the set

$$\mathcal{T} = \{x \in \operatorname{dom} \nabla f : \mathcal{R}(x) \neq \{x\}\}.$$

For all such points there is at least one maximal segment $[w, z] \subset \mathcal{R}(x)$ such that $w \neq z$.

We can also define the multivalued map giving the unit directions contained in the faces passing through the set $\mathcal{T}$, that is

$$\mathcal{T} \ni x \mapsto \mathcal{D}(x) = \left\{ \frac{z - x}{|z - x|} : z \in \mathcal{R}(x),\ z \neq x \right\}. \tag{4.11}$$

We recall that a multivalued map is defined to be Borel measurable if the counterimage of any open set is Borel.

The measurability of the above maps is proved in the following lemma:

Lemma 4.2. *The graph of the multivalued function $\mathcal{P}$ is a closed set in $\mathbb{R}^n \times \mathbb{R}^n$. As a consequence, $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{D}$ are Borel measurable multivalued maps and $\mathcal{T}$ is a Borel set.*

Proof. The closedness of the graph of $\mathcal{P}$ follows immediately from the continuity of f and from the upper-semicontinuity of its subdifferential. Then, the graph of $\mathcal{P}$ is σ-compact in $\mathbb{R}^n \times \mathbb{R}^n$ and, due to the continuity of the projections from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$. It follows then that the map is Borel.

Moreover, since we chose $\operatorname{dom} \nabla f$ to be σ-compact, also the graph of $\mathcal{R}$ is σ-compact, thus $\mathcal{R}$ is a Borel map.

The same reasoning that is made for the map $\mathcal{P}$ can be applied to the multifunction $\mathcal{P} \backslash \mathcal{I}$ (where $\mathcal{I}$ denotes the identity map), thus giving the mesurability of the set $\mathcal{T}$, since

$$\mathcal{T} = \pi\big(\operatorname{graph}(\mathcal{P} \backslash \mathcal{I})\big) \cap \operatorname{dom} \nabla f,$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the projection on the first n coordinates.

The measurability of $\mathcal{D}$ follows by the continuity of the map $\mathbb{R}^n \times \mathbb{R}^n \ni (x, z) \mapsto \frac{z-x}{|z-x|}$ out of the diagonal. □

4.3. **Partition into model sets.** First of all, we introduce some preliminary notation.
If $K \subset \mathbb{R}^d$ is a convex set and $\operatorname{aff}(K)$ is its affine hull, we denote by $\operatorname{ri}(K)$ the relative interior of K, which is the interior of K in the topology of $\operatorname{aff}(K)$, and by $\operatorname{rb}(K)$ its relative boundary, which is the boundary of K in $\operatorname{aff}(K)$.

In order to find a countable partition of F^k into model sets like the set $\mathscr{C}^k$ which was defined in Section 4.1, we have to neglect the points that lie on the relative boundary of the k-dimensional faces.
More precisely, from now onwards we look for the disintegration of the Lebesgue measure over the sets

$$E_y = \operatorname{ri}(F_y), \quad y \in \operatorname{Im} \nabla f. \tag{4.12}$$

As we did for the sets F_y, we set

$$E_y^k = E_y, \quad \text{if } \dim(E_y) = k$$

and

$$E^k = \bigcup_{\{y \in \operatorname{Im} \nabla f :\, \dim(E_y) = k\}} E_y^k. \tag{4.13}$$

This restriction will not affect the characterization of the conditional probabilities because, as we will prove in Lemma 4.19, the set

$$\mathcal{T} \setminus \bigcup_{k=1}^{n} E^k$$

is Lebesgue negligible.

Now we can start to build the partition of E^k into model sets.

Definition 4.3. For all $k = 1, \dots, n$, we call *sheaf set* a σ-compact subset of E^k of the form

$$\mathscr{Z}^k = \underset{z \in Z^k}{\cup} \mathrm{ri}(\mathcal{R}(z)), \tag{4.14}$$

where Z^k is a σ-compact subset of E^k which is contained in an affine $(n-k)$-plane in $\mathbb{R}^n$ and is such that

$$\mathrm{ri}(\mathcal{R}(z)) \cap Z^k = \{z\}, \quad \forall\, z \in Z^k. \tag{4.15}$$

We call *sections* of $\mathscr{Z}^k$ all the sets Y^k that satisfy the same properties of Z^k in the definition.
A *subsheaf* of a sheaf set $\mathscr{Z}^k$ is a sheaf set $\mathscr{W}^k$ of the form

$$\mathscr{W}^k = \underset{w \in W^k}{\cup} \mathrm{ri}(\mathcal{R}(w)),$$

where W^k is a σ-compact subset of a section of the sheaf set $\mathscr{Z}^k$.

Similarly to Lemma 2.6 in [Car08], we prove that the set E^k can be covered with countably many disjoint sets of the form (4.14).
First of all, let us take a dense sequence $\{V_i\}_{i\in\mathbb{N}} \subset \mathbf{G}(k,n)$, where $\mathbf{G}(k,n)$ is the compact set of all the k-planes in $\mathbb{R}^n$ passing through the origin, and fix, $\forall\, i \in \mathbb{N}$, an orthonormal set $\{\mathrm{e}_{i_1}, \dots, \mathrm{e}_{i_k}\}$ in $\mathbb{R}^n$ such that

$$V_i = \langle \mathrm{e}_{i_1}, \dots, \mathrm{e}_{i_k} \rangle. \tag{4.16}$$

Denoting by $\mathbf{S}^{n-1} \cap V$ the k-dimensional unit sphere of a k-plane $V \subset \mathbb{R}^n$ w.r.t. the Euclidean norm and by $\pi_i = \pi_{V_i} : \mathbb{R}^n \to V_i$ the projection map on the k-plane V_i, for every fixed $0 < \varepsilon < 1$ the following sets form a disjoint covering of the k-dimensional unit spheres in $\mathbb{R}^n$:

$$\mathbf{S}_i^{k-1} = \Big\{ \mathbf{S}^{n-1} \cap V : V \in \mathbf{G}(k,n), \inf_{x \in \mathbf{S}^{n-1} \cap V} \|\pi_i(x)\| \geq 1 - \varepsilon \Big\} \setminus \bigcup_{j=1}^{i-1} \mathbf{S}_j^{k-1}, \qquad i = 1, \dots, I, \tag{4.17}$$

where $I \in \mathbb{N}$ depends on the ε we have chosen.

In order to determine a countable partition of E^k into sheaf sets we consider the k-dimensional rectangles in the k-planes (4.16) whose boundary points have dyadic coordinates. For all

$$\mathrm{l} = (l_1, \dots, l_k),\ \mathrm{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k \quad \text{with } l_j < m_j \ \ \forall\, j = 1, \dots, k \tag{4.18}$$

and for all $i = 1, \dots, I,\ p \in \mathbb{N}$, let $C^k_{ip\mathrm{lm}}$ be the rectangle

$$C^k_{ip\mathrm{lm}} = 2^{-p} \prod_{j=1}^{k} [l_j\, \mathrm{e}_{i_j}, m_j\, \mathrm{e}_{i_j}]. \tag{4.19}$$

Lemma 4.4. *The following sets are sheaf sets covering E^k: for $i = 1, \dots, I$, $p \in \mathbb{N}$, and $S \subset \mathbb{Z}^k$ take*

$$\mathscr{Z}^k_{ipS} = \Big\{ x \in E^k : \mathcal{D}(x) \subset \mathbf{S}_i^{k-1} \text{ and } S \subset \mathbb{Z}^k \text{ is the maximal set such that}$$
$$\underset{\mathrm{l} \in S}{\cup} C^k_{ip\mathrm{l}(\mathrm{l}+1)} \subset \pi_i[\mathrm{ri}(\mathcal{R}(x))] \Big\}. \tag{4.20}$$

Moreover, a disjoint family of sheaf sets that cover E^k is obtained in the following way: in case $p = 1$ we consider all the sets $\mathscr{Z}^k_{ipS}$ as above, whereas for all $p > 1$ we take a set $\mathscr{Z}^k_{ipS}$ if and only if the set $\underset{\mathrm{l}\in S}{\cup} C^k_{ip\mathrm{l}(\mathrm{l}+1)}$ does not contain any rectangle of the form $C^k_{ip'\mathrm{l}(\mathrm{l}+1)}$ for every $p' < p$.

As soon as a nonempty sheaf set $\mathscr{Z}^k_{ipS}$ belongs to this partition, it will be denoted by $\bar{\mathscr{Z}}^k_{ipS}$.

For the proof of this lemma we refer to the analogous Lemma 2.6 in [Car08].

Then, we can refine the partition into sheaf sets by cutting them with sections which are perpendicular to fixed k-planes.

Definition 4.5. (See Figure 2) A k-dimensional $\mathcal{D}$-cylinder is a σ-compact set of the form

$$\mathscr{C}^k = \mathscr{Z}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(C^k), \tag{4.21}$$

where $\mathscr{Z}^k$ is a k-dimensional sheaf set, $\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle$ is any fixed k-dimensional subspace which is perpendicular to a section of $\mathscr{Z}^k$ and C^k is a rectangle in $\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle$ of the form

$$C^k = \prod_{i=1}^k [t_i^- \mathrm{e}_i, t_i^+ \mathrm{e}_i],$$

with $-\infty < t_i^- < t_i^+ < +\infty$ for all $i = 1,\dots,k$, such that

$$C^k \subset \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}[\mathrm{ri}(\mathcal{R}(z))] \quad \forall\, z \in \mathscr{Z}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(C^k). \tag{4.22}$$

We set $\mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k, C^k)$ when we want to refer explicitily to a sheaf set $\mathscr{Z}^k$ and to a rectangle C^k that can be taken in the definition of $\mathscr{C}^k$.
The k-plane $\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle$ is called the *axis* of the $\mathcal{D}$-cylinder and every set Z^k of the form

$$\mathscr{C}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{w}), \quad \text{for some } \mathrm{w} \in \mathrm{ri}(C^k)$$

is called a *section* of the $\mathcal{D}$-cylinder.

We also define the border of $\mathscr{C}^k$ transversal to $\mathcal{D}$ and its outer unit normal as

$$\begin{aligned} \mathfrak{d}\mathscr{C}^k &= \mathscr{C}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{rb}(C^k)), \\ \hat{n}_{|_{\mathfrak{d}\mathscr{C}^k}}(x) &= \text{ outer unit normal to } \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(C^k) \text{ at } x, \quad \text{for all } x \in \mathfrak{d}\mathscr{C}^k. \end{aligned} \tag{4.23}$$

Lemma 4.6. *The set E^k can be covered by the $\mathcal{D}$-cylinders*

$$\mathscr{C}^k(\mathscr{Z}^k_{ipS}, C^k_{ip\mathrm{l}(\mathrm{l}+1)}), \tag{4.24}$$

where $S \subset \mathbb{Z}^k$, $\mathrm{l} \in S$ and $\mathscr{Z}^k_{ipS}$, $C^k_{ip\mathrm{l}(\mathrm{l}+1)}$ are the sets defined in (4.20),(4.19).

Moreover, there exists a countable covering of E^k with $\mathcal{D}$-cylinders of the form (4.24) such that

$$\pi_i\Big[\mathscr{C}^k(\mathscr{Z}^k_{ipS}, C^k_{ip\mathrm{l}(\mathrm{l}+1)}) \cap \mathscr{C}^k(\mathscr{Z}^k_{ip'S'}, C^k_{ip'\mathrm{l}'(\mathrm{l}'+1)})\Big] \subset \mathrm{rb}[C^k_{ip\mathrm{l}(\mathrm{l}+1)}] \cap \mathrm{rb}[C^k_{ip'\mathrm{l}'(\mathrm{l}'+1)}] \tag{4.25}$$

for any couple of $\mathcal{D}$-cylinders which belong to this countable family (if $i \neq i'$, it follows from the definition of sheaf set that $\mathscr{C}^k(\mathscr{Z}^k_{ipS}, C^k_{ip\mathrm{l}(\mathrm{l}+1)}) \cap \mathscr{C}^k(\mathscr{Z}^k_{i'p'S'}, C^k_{i'p'\mathrm{l}'(\mathrm{l}'+1)})$ must be empty).

Proof. The fact that the $\mathcal{D}$-cylinders defined in (4.24) cover E^k follows directly from Definitions 4.3 and 4.5 as in [Car08].

Our aim is then to construct a countable covering of E^k with $\mathcal{D}$-cylinders wich satisfy property (4.25). First of all, let us fix a nonempty sheaf set $\bar{\mathscr{Z}}^k_{ipS}$ which belongs to the countable partition of E^k given in Lemma 4.4.
In the following we will determine the $\mathcal{D}$-cylinders of the countable covering which are contained in $\bar{\mathscr{Z}}^k_{ipS}$; the others can be selected in the same way starting from a different sheaf set of the partition given in Lemma 4.4.
Then, the $\mathcal{D}$-cylinders that we are going to choose are of the form

$$\mathscr{C}^k\big(\mathscr{Z}^k_{i\hat{p}\hat{S}}, C^k_{i\hat{p}\hat{\mathrm{l}}(\hat{\mathrm{l}}+1)}\big),$$

where $\mathscr{Z}^k_{i\hat{p}\hat{S}}$ is a subsheaf of the sheaf set $\bar{\mathscr{Z}}^k_{ipS}$.

The construction is done by induction on the natural number $\hat{p}$ which determines the diameter of the squares $C^k_{i\hat{p}\hat{\mathrm{l}}(\hat{\mathrm{l}}+1)}$ obtained projecting the $\mathcal{D}$-cylinders contained in $\bar{\mathscr{Z}}^k_{ipS}$ on the axis $\langle \mathrm{e}_{i_1},\dots,\mathrm{e}_{i_k}\rangle$. Then, as the induction step increases, the diameter of the k-dimensional rectangles associated to the $\mathcal{D}$-cylinders that we are going to add to our countable partition will be smaller and smaller (see Figure 3).

By definition (4.20) and by the fact that $\bar{\mathscr{Z}}^k_{ipS}$ is a nonempty element of the partition defined in Lemma 4.4, the smallest natural number $\hat{p}$ such that there exists a k-dimensional rectangle of the form $C^k_{i\hat{p}\hat{\mathrm{l}}(\hat{\mathrm{l}}+1)}$ which is contained in $\pi_i(\bar{\mathscr{Z}}^k_{ipS})$ is exactly p; then, w.l.o.g., we can assume in our induction argument that $p = 1$.

For all $\hat{p} \in \mathbb{N}$, we call $Cyl_{\hat{p}}$ the collection of the $\mathcal{D}$-cylinders which have been chosen up to step $\hat{p}$.

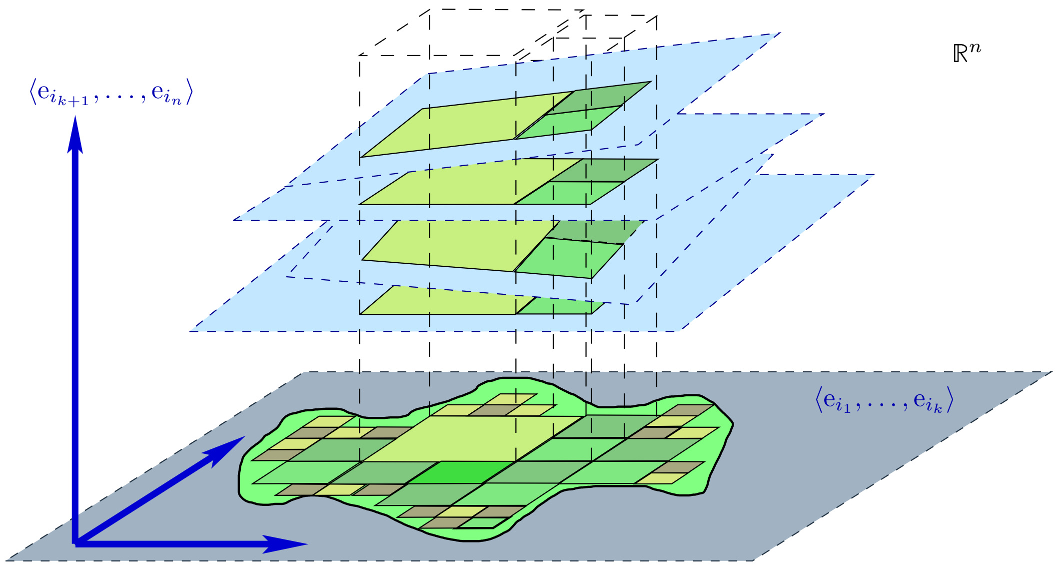

Figure 3: Partition of E^k into $\mathcal{D}$-cylinders (Lemma 4.6).

When $\hat{p} = 1$ we set

$$Cyl_1 = \{\mathscr{C}^k(\bar{\mathscr{Z}}^k_{i1S}, C^k_{i1\mathrm{l}(\mathrm{l}+1)}) : \mathrm{l} \in S\}.$$

Now, let us suppose to have determined the collection of $\mathcal{D}$-cylinders $Cyl_{\hat{p}}$ for some $\hat{p} \in \mathbb{N}$.

Then, we define

$$Cyl_{\hat{p}+1} = Cyl_{\hat{p}} \bigcup \Big\{ \mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k_{i(\hat{p}+1)\tilde{S}}, C^k_{i(\hat{p}+1)\tilde{\mathrm{l}}(\tilde{\mathrm{l}}+1)}) : \ \mathscr{Z}^k_{i(\hat{p}+1)\tilde{S}} \text{ is a subsheaf of } \bar{\mathscr{Z}}^k_{i\hat{p}S} \text{ and}$$

$$\mathscr{C}^k \not\subseteq \mathscr{C}^k(\mathscr{Z}^k_{ip'S'}, C^k_{ip'\mathrm{l}'(\mathrm{l}'+1)}) \text{ for all } \mathscr{C}^k(\mathscr{Z}^k_{ip'S'}, C^k_{i'p'\mathrm{l}'(\mathrm{l}'+1)}) \in Cyl_{\hat{p}} \Big\}. \tag{4.26}$$

□

As we did in (4.7), any k-dimensional $\mathcal{D}$-cylinder $\mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k, C^k)$ can be parametrized in the following way: if we fix $\mathrm{w} \in \mathrm{ri}(C^k)$, then

$$\mathscr{C}^k = \Big\{ \sigma^{\mathrm{w}+t\mathrm{e}}(z) : \ z \in Z^k = \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{w}) \cap \mathscr{C}^k, \ \mathrm{e} \in \mathbf{S}^{n-1} \cap \langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$$

$$\text{and } t \in \mathbb{R} \text{ is such that } (\mathrm{w} + t\mathrm{e}) \cdot \mathrm{e}_j \in [t^-_j, t^+_j] \quad \forall\, j = 1, \dots, k \Big\}, \tag{4.27}$$

where

$$\sigma^{\mathrm{w}+t\mathrm{e}}(z) = z + t \frac{v_{\mathrm{e}}(z)}{|\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(v_{\mathrm{e}}(z))|}, \tag{4.28}$$

and $v_{\mathrm{e}}(z) \in \mathcal{D}(z)$ is the unit vector such that $\frac{\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(v_{\mathrm{e}}(z))}{|\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(v_{\mathrm{e}}(z))|} = \mathrm{e}$.

We observe that, according to our notation,

$$(\sigma^{\mathrm{w}+t\mathrm{e}})^{-1} = \sigma^{(\mathrm{w}+t\mathrm{e})-t\mathrm{e}}. \tag{4.29}$$

4.4. **An absolute continuity estimate.** According to the strategy outlined in Section 4.1, in order to prove Theorem 3.3 for the disintegration of the Lebesgue measure on the $\mathcal{D}$-cylinders we have to show that, for every $\mathcal{D}$-cylinder $\mathscr{C}^k$ parametrized as in (4.27)

$$(\sigma^{\mathrm{w}+t\mathrm{e}})^{-1}_{\#}(\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{w}+t\mathrm{e}}(Z^k)) \ll \mathscr{H}^{n-k} \llcorner Z^k. \tag{4.30}$$

This will allow us to make a change of variables from the measure space $(\sigma^{\mathrm{w}+t\mathrm{e}}(Z^k), \mathscr{H}^{n-k} \llcorner (\sigma^{\mathrm{w}+t\mathrm{e}}(Z^k))$ to $(Z^k, \alpha \cdot \mathscr{H}^{n-k} \llcorner Z^k)$, where α is an integrable function w.r.t. $\mathscr{H}^{n-k} \llcorner Z^k$ (see Section 4.1).

It is clear that the domain of the parameter t, which can be interpreted as a time parameter for a flow $\sigma^{\mathrm{w}+t\mathrm{e}}$ that moves points along the k-dimensional projected faces of a convex function, depends on the section Z^k which has been chosen for the parametrization of $\mathscr{C}^k$ and on the direction e.
Then, if $\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$ is the axis of a $\mathcal{D}$-cylinder $\mathscr{C}^k$, for every $\mathrm{w} \in \mathrm{ri}(C^k)$ and for every $\mathrm{e} \in \mathbf{S}^{n-1} \cap \langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$, we define the numbers

$$h^-(\mathrm{w}, \mathrm{e}) = \inf\{t \in \mathbb{R} : \ \mathrm{w} + t\mathrm{e} \in C^k\}, \quad h^+(\mathrm{w}, \mathrm{e}) = \sup\{t \in \mathbb{R} : \ \mathrm{w} + t\mathrm{e} \in C^k\}.$$

We observe that, as $\mathrm{w} \in \mathrm{ri}(C^k)$, $h^-(\mathrm{w}, \mathrm{e}) < 0 < h^+(\mathrm{w}, \mathrm{e})$.

We obtain (4.30) in Corollary 4.15 as a consequence of the following fundamental lemma.

Lemma 4.7 (Absolutely continuous push forward)**.** *Let $\mathscr{C}^k$ be a k-dimensional $\mathcal{D}$-cylinder parametrized as in (4.27). Then, for all $S \subset Z^k$ the following estimate holds:*

$$\begin{aligned} \left(\frac{h^+(\mathrm{w}, \mathrm{e}) - t}{h^+(\mathrm{w}, \mathrm{e}) - s}\right)^{n-k} \mathscr{H}^{n-k}(\sigma^{\mathrm{w}+s\mathrm{e}}(S)) &\leq \mathscr{H}^{n-k}(\sigma^{\mathrm{w}+t\mathrm{e}}(S)) \\ &\leq \left(\frac{t - h^-(\mathrm{w}, \mathrm{e})}{s - h^-(\mathrm{w}, \mathrm{e})}\right)^{n-k} \mathscr{H}^{n-k}(\sigma^{\mathrm{w}+s\mathrm{e}}(S)), \end{aligned} \tag{4.31}$$

where $h^-(\mathrm{w}, \mathrm{e}) < \ s \leq \ t < \ h^+(\mathrm{w}, \mathrm{e})$.

Moreover, if $s = h^-(\mathrm{w}, \mathrm{e})$ the left inequality in (4.31) still holds and if $t = h^+(\mathrm{w}, \mathrm{e})$ the right one.

Lemma 4.7 will be proven at page 22.

The idea to prove this lemma, as in [BG07] and [Car08], is to get the estimate (4.31) for the flow $\sigma_j^{\mathrm{w}+t\mathrm{e}}$ induced by simpler vector fields $\{v_j\}_{j \in \mathbb{N}}$ and then to show that they approximate the initial vector field v_e in such a way that the inequalities in (4.31) pass to the limit.

The main problem in our proof is then to find a suitable sequence of vector fields $\{v_j\}_{j \in \mathbb{N}}$ that approximate, in a certain region, the geometry of the projected k-dimensional faces of a convex function in the direction e, which is described by the vector field v_e.
For the construction of this family of vector fields we strongly rely on the fact that the sets on which we want to disintegrate the Lebesgue measure are, other than disjoint, the projections of the k-dimensional faces of a convex function.

For simplicity, we first prove the estimate (4.31) for 1-dimensional $\mathcal{D}$-cylinders.
In this case, if $\langle \mathrm{e} \rangle$ is the axis of a 1-dimensional $\mathcal{D}$-cylinder $\mathscr{C}$, there are only two possible directions $\pm\mathrm{e}$ that can be chosen to parametrize it. Up to translations by a multiple of the same vector, we can assume that $\mathrm{w} = 0$. Moreover, since choosing $-\mathrm{e}$ instead of e in the definition of the parametrization map (4.28) simply reverses the order of s and t in (4.31), in order to prove (4.31) it is sufficient to show that, for all $0 \leq t \leq h^+$ and for all $S \subset \sigma^t(Z)$

$$\mathscr{H}^{n-1}(S) \leq \left(\frac{t - h^-}{-h^-}\right)^{n-1} \mathscr{H}^{n-1}((\sigma^t)^{-1}(S)), \tag{4.32}$$

where $\sigma^t = \sigma^{0+t\mathrm{e}}$ and $h^\pm = h^\pm(0, \mathrm{e})$.

In our construction we first approximate the 1-dimensional faces that lie on the graph of f restricted to the given $\mathcal{D}$-cylinder and then we get the approximating vector fields $\{v_j\}_{j \in \mathbb{N}}$ simply projecting the directions of those approximations on the first n coordinates.

Before giving the details we recall and introduce some useful notation:

$\mathbf{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$;

$\mathrm{e} \in \mathbf{S}^{n-1}$ a fixed vector;

$H_t := \{x \in \mathbb{R}^n : x \cdot \mathrm{e} = t\}$, where $t \in [h^-, h^+]$ and $h^-, h^+ \in \mathbb{R} : h^- < 0 < h^+$;

$\mathbf{B}^{n-1}_R(x) = \{z \in H_{\{x\cdot \mathrm{e}\}} : \|z - x\| \leq R\}$;

Z is the σ-compact section of the 1-dimensional $\mathcal{D}$-cylinder $\mathscr{C}$ which is contained in H_0;

$v_{\mathrm{e}}(x) \in \mathcal{D}(x)$ is the unit vector such that $\pi_{\mathrm{e}}(v_{\mathrm{e}}(x)) = |\pi_{\mathrm{e}}(v_{\mathrm{e}}(x))|\, \mathrm{e}$, $\forall\, x \in \mathscr{C}$;

$\mathscr{C} = \{\sigma^t(z) : z \in Z,\, t \in [h^-, h^+]\}$, $\sigma^t(z) = z + t\dfrac{v_{\mathrm{e}}(z)}{|\pi_{\mathrm{e}}(v_{\mathrm{e}}(z))|}$;

$\mathscr{C}_t = \bigcup_{s\in[h^-,t]} H_s \cap \mathscr{C}$;

$l_t(x) = \mathcal{R}(x) \cap \mathscr{C}_t$, $\forall\, x \in \mathscr{C}_t$;

$\forall\, x \in \mathbb{R}^n$, $\tilde{x} := (x, f(x)) \in \mathbb{R}^{n+1}$ and $\forall\, A \subset \mathbb{R}^n$, $\tilde{A} := \mathrm{graph}\, f_{|A}$.

Moreover, we recall the following definitions:

Definition 4.8. The *convex envelope* of a set of points $X \subset \mathbb{R}^n$ is the smaller convex set $\mathrm{conv}(X)$ that contains X. The following characterization holds:

$$\mathrm{conv}(X) = \left\{ \sum_{j=1}^{J} \lambda_j\, x_j : x_j \in X,\, 0 \leq \lambda_j \leq 1,\, \sum_{j=1}^{J} \lambda_j = 1,\, J \in \mathbb{N} \right\}. \tag{4.33}$$

Definition 4.9. The *graph of a compact convex set* $C \subset \mathbb{R}^{n+1}$, that we denote by $\mathrm{graph}(C)$, is the graph of the function $g : \pi_{\mathbb{R}^n}(C) \to \mathbb{R}$ which is defined by

$$g(x) = \min\{t \in \mathbb{R} : (x, t) \in C\}. \tag{4.34}$$

Definition 4.10. A *supporting k-plane* to the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is an affine k-dimensional subspace of a supporting hyperplane to the graph of f (see Definition 4.1) whose intersection with $\mathrm{graph}\, f$ is nonempty.

Definition 4.11. An R-face of a convex set $C \subset \mathbb{R}^d$ is a convex subset C' of C such that every closed segment in C with a relative interior point in C' has both endpoints in C'. The zero-dimensional R-faces of a convex set are also called *extreme points* and the set of all extreme points in a convex set C will be denoted by $\mathrm{ext}(C)$.

The definition of R-face corresponds to the definition of face of a convex set in [Roc70].

We also recall the following propositions, for which we refer to Section 18 of [Roc70].

Proposition 4.12. *Let $C = \mathrm{conv}(D)$, where D is a set of points in $\mathbb{R}^d$, and let C' be a nonempty R-face of C. Then $C' = \mathrm{conv}(D')$, where D' consists of the points in D which belong to C'.*

Proposition 4.13. *Let C be a bounded closed convex set. Then $C = \mathrm{conv}(\mathrm{ext}(C))$.*

The key to get fundamental estimate (4.32) is contained in the following lemma:

Lemma 4.14 (Construction of regular approximating vector fields)**.** *For all $0 \leq t \leq h^+$, there exists a sequence of $\mathscr{H}^{n-1}$-measurable vector fields*

$$\{v^t_j\}_{j\in\mathbb{N}}, \quad v^t_j : \sigma^t(Z) \to \mathbf{S}^{n-1}$$

such that

$$\textbf{1. } v^t_j \text{ converges } \mathscr{H}^{n-1}\text{-a.e. to } v_{\mathrm{e}} \text{ on } \sigma^t(Z); \tag{4.35}$$

$$\textbf{2. } \mathscr{H}^{n-1}(S) \leq \left(\frac{t-h^-}{-h^-}\right)^{n-1} \mathscr{H}^{n-1}((\sigma^t_{v^t_j})^{-1}(S)), \quad \forall\, S \subset \sigma^t(Z),$$
$$\text{where } \sigma^t_{v^t_j} \text{ is the flow map associated to the vector field } v^t_j. \,. \tag{4.36}$$

Indeed, if we have such a sequence of vector fields, the proof of the estimate (4.32) follows as in [Car08].

Proof. **Step 1** *Preliminary considerations*

First of all, let us fix $t \in [0, h^+]$.

Eventually partitioning $\mathscr{C}$ into a countable collection of sets, we can assume that $\sigma^t(Z)$ and $\sigma^{h^-}(Z)$ are bounded, with $\sigma^t(Z) \subset \mathbf{B}^{n-1}_{R_1}(x_1) \subset H_t$ and $\sigma^{h^-}(Z) \subset \mathbf{B}^{n-1}_{R_2}(x_2) \subset H_{h^-}$. Then, if we call K_t the convex envelope of $\mathbf{B}^{n-1}_{R_1}(x_1) \cup \mathbf{B}^{n-1}_{R_2}(x_2)$, the function $f_{|K_t}$ is uniformly Lipschitz with a certain Lipschitz constant L_f.

Step 2 *Construction of approximating functions* (see Figure 4)

Now we define a sequence of functions $\{f_j\}_{j\in\mathbb{N}}$ whose 1-dimensional faces approximate, in a certain

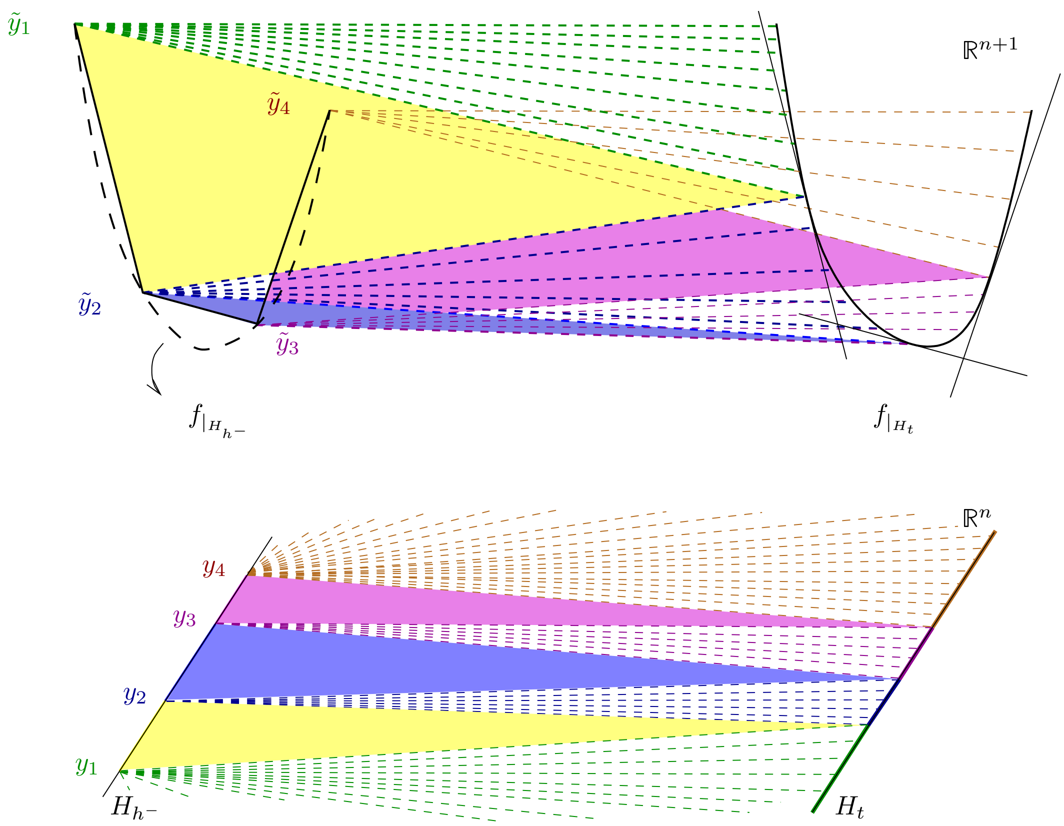

Figure 4: Illustration of a vector field approximating the one dimensional faces of f (Lemma 4.14). One can see in the picture the graph of f_4, which is the convex envelope of $\{\tilde{y}_i\}_{i=1,\dots,4}$ and $f_{|H_t}$. The faces of f_j connect $\mathscr{H}^{n-1}$-a.e. point of H_t to a single point among the $\{\tilde{y}_i\}_i$, while the remaining points of H_t correspond to some convex envelope $\mathrm{conv}(\{\tilde{y}_{i_\ell}\}_\ell)$ — here represented by the segments $[\tilde{y}_i, \tilde{y}_{i+1}]$. The region where the vector field v_4^t, giving the directions of the faces of f_j, is multivalued corresponds to the 'planar' faces of f_4. The affine span of these planar faces, restricted to suitable planes contained in H_t, provides a supporting hyperplane for the restriction of f to these latter planes — in the picture they are depicted as tangent lines. The intersection of $\sigma^t(Z) \subset H_t$ with any supporting plane to the graph of $f_{|H_t}$ must contain just one point, otherwise $\mathcal{D}$ would be multivalued at some point of $\sigma^t(Z)$.

sense, the pieces of the 1-dimensional faces of f which are contained in $\mathscr{C}_t$. The directions of a properly chosen subcollection of the 1-dimensional faces of f_j will give, when projected on the first n coordinates, the approximate vector field v_j^t.

First of all, take a sequence $\{\tilde{y}_i\}_{i\in\mathbb{N}} \subset \tilde{\sigma}^{h^-}(Z)$ such that the collection of segments $\{\tilde{l}_t(y_i)\}_{i\in\mathbb{N}}$ is dense in $\bigcup_{y\in\sigma^{h^-}(Z)} \tilde{l}_t(y)$.

For all $j \in \mathbb{N}$, let C_j be the convex envelope of the set

$$\{\tilde{y}_i\}_{i=1}^j \quad \cup \quad \operatorname{graph} f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}} \tag{4.37}$$

and call $f_j : \pi_{\mathbb{R}^n}(C_j) \to \mathbb{R}$ the function whose graph is the graph of the convex set C_j.
We note that $\pi_{\mathbb{R}^n}(C_j) \cap H_{h^-} = \operatorname{conv}(\{y_i\}_{i=1}^j)$ and $\operatorname{graph} f_{j|_{\operatorname{conv}(\{y_i\}_{i=1}^j)}} = \operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^j))$.

We claim that the graph of f_j is made of segments that connect the points of $\operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^j))$ to the graph of $f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ (indeed, by convexity and by the fact that $\tilde{y}_i = (y_i, f(y_i))$, $f_j = f$ on $\mathbf{B}^{n-1}_{R_1}(x_1)$). In order to prove this, we first observe that, by definition, all segments of this kind are contained in the set C_j. On the other hand, by (4.33), all the points in C_j are of the form

$$w = \sum_{i=1}^{J} \lambda_i \, w_i, \tag{4.38}$$

where $\sum_{i=1}^{J} \lambda_i = 1$, $0 \le \lambda_i \le 1$ and $w_i \in \{\tilde{y}_i\}_{i=1}^j \cup \operatorname{graph} f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$. In particular, we can write

$$w = \alpha z + (1-\alpha) r, \quad \text{where } 0 \le \alpha \le 1, \quad z \in \operatorname{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big) \quad \text{and } r \in \operatorname{epi} f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}. \tag{4.39}$$

Moreover, if we take two points $z' \in \operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^j))$, $r' \in \operatorname{graph} f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ such that $\pi_{\mathbb{R}^n}(z') = \pi_{\mathbb{R}^n}(z)$ and $\pi_{\mathbb{R}^n}(r') = \pi_{\mathbb{R}^n}(r)$, we have that the point

$$w' = \alpha z' + (1-\alpha) r' \tag{4.40}$$

belongs to C_j, lies on a segment which connects $\operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^j))$ to $\operatorname{graph} f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ and its $(n+1)$ coordinate is less than the $(n+1)$ coordinate of w.

The graph of f_j contains also all the pieces of 1-dimensional faces $\{\tilde{l}_t(y_i)\}_{i=1}^j$, since by construction it contains their endpoints and it lies over the graph of $f_{|_{\pi_{\mathbb{R}^n}(C_j)}}$.

Step 3 *Construction of approximating vector fields* (see Figure 4)

Among all the segments in the graph of f_j that connect the points of $\operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^j))$ to the graph of $f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$, we select those of the form $[\tilde{x}, \tilde{y_k}]$, where $x \in \sigma^t(Z)$, $y_k \in \{y_i\}_{i=1}^j$, and we show that for $\mathscr{H}^{n-1}$-a.e. $x \in \sigma^t(Z)$ there exists only one segment within this class which passes through $\tilde{x}$. The approximating vector field will be given by the projection on the first n coordinates of the directions of these segments.

First of all, we claim that for all $x \in \mathbf{B}^{n-1}_{R_1}(x_1)$ the graph of f_j contains at least a segment of the form $[\tilde{x}, \tilde{y}_i]$ for some $i \in \{1, \dots, j\}$.
Indeed, we show that if $\tilde{x}$ is the endpoint of a segment of the form $[\tilde{x}, (y, f_j(y))]$ where $y \in \operatorname{conv}\big(\{y_i\}_{i=1}^j\big)$ but $(y, f_j(y)) \notin \operatorname{ext}\big(\operatorname{conv}\big(\{y_i\}_{i=1}^j\big)\big)$, then there are at least two segments of the form $[\tilde{x}, \tilde{y}_k]$ with $\tilde{y_k} \in \operatorname{ext}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^j)) \subset \{\tilde{y}_i\}_{i=1}^j$ (here we assume that $j \ge 2$).
In order to prove this, take a point $(z, f_j(z))$ in the open segment $(\tilde{x}, (y, f_j(y)))$ and a supporting hyperplane $H(z)$ to the graph of f_j that contains that point. By definition, $H(z)$ contains the whole segment $[\tilde{x}, (y, f_j(y))]$ and the set $H(z) \cap \big(H_{h^-} \times \mathbb{R}\big)$ is a supporting hyperplane to the set $\operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^j))$ that contains the point $(y, f_j(y))$.
Now, take the smallest R-face C of $\operatorname{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big)$ which is contained in $\operatorname{graph}(\operatorname{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big))$ and contains the point $(y, f_j(y))$, that is given by the intersection of all R-faces which contain $(y, f_j(y))$.
By Propositions 4.12 and 4.13, $C = \operatorname{conv}\big[\operatorname{ext}\big(\operatorname{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big)\big) \cap C\big]$ and as $(y, f_j(y)) \notin \operatorname{ext}\big(\operatorname{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big)\big)$, $\dim(C) \ge 1$ and the set $\operatorname{ext}\big(\operatorname{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big)\big) \cap C$ contains at least two points $\tilde{y}_k$, $\tilde{y}_l$.
In particular, since both C and $\tilde{x}$ belong to $H(z) \cap \operatorname{graph}(f_j)$, by definition of supporting hyperplane we have that the graph of f_j contains the segments $[\tilde{x}, \tilde{y}_k]$, $[\tilde{x}, \tilde{y}_l]$ and our claim is proved.

Now, for each $j \in \mathbb{N}$, we define the (possibly multivalued) map $\mathcal{D}^t_j : \mathbf{B}^{n-1}_{R_1}(x_1) \to \mathbb{R}^n$ as follows:

$$\mathcal{D}^t_j : \, x \mapsto \left\{ \frac{y_i - x}{|y_i - x|} : \, [\tilde{x}, \tilde{y}_i] \subset \operatorname{graph}(f_j) \right\} \tag{4.41}$$

and we prove that the set

$$B_j := \sigma^t(Z) \cap \{x \in \mathbf{B}^{n-1}_{R_1}(x_1) : \mathcal{D}^t_j(x) \text{ is multivalued } \} \tag{4.42}$$

is $\mathscr{H}^{n-1}-$negligible, $\forall\, j \in \mathbb{N}$.
Thus, if we neglect the set $B = \bigcup_{j\in\mathbb{N}} B_j$, we can define our approximating vector field as

$$v^t_j(x) = \{\mathcal{D}^t_j(x)\}, \quad \forall\, x \in \sigma^t(Z)\backslash B, \quad \forall\, j \in \mathbb{N}. \tag{4.43}$$

In order to show that $\mathscr{H}^{n-1}(B_j) = 0$ we first prove that, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathbf{B}^{n-1}_{R_1}(x_1)$, whenever $\mathcal{D}^t_j(x)$ contains the directions of two segments, f_j must be linear on their convex envelope.
Indeed, suppose that the graph of f_j contains two segments $[\tilde{x}, \tilde{y}_{i_k}]$, where $i_k \in \{1, \ldots, j\}$ and $k = 1, 2$, and consider two points $(z_k, f_j(z_k)) \subset [\tilde{x}, \tilde{y}_{i_k}]$ such that

$$\begin{aligned} z_1 &= x + s\mathrm{e} + a_1 v_1, \quad s \in [h^- - t, 0),\ v_1 \in H_0; \\ z_2 &= x + s\mathrm{e} + a_2 v_2, \quad s \in [h^- - t, 0),\ v_2 \in H_0. \end{aligned} \tag{4.44}$$

As f_j is linear on $[x, y_{i_k}]$, we have that

$$f_j(z_k) = f_j(x) + r_k \cdot (s\mathrm{e} + a_k v_k), \tag{4.45}$$

where $r_k \in \partial^- f_j(x), \quad k = 1, 2$.
Moreover, since

$$\pi_{H_0}(\partial^- f_j(x)) = \partial^- f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}(x) \tag{4.46}$$

and the set where $\partial^- f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ is multivalued is $\mathscr{H}^{n-2}$-rectifiable (see for e.g. [Zaj78, AA99]), we have that, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathbf{B}^{n-1}_{R_1}(x_1)$

$$r \cdot v = \nabla(f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}})(x) \cdot v, \quad \forall\, r \in \partial^- f_j(x),\ \forall\, v \in H_0. \tag{4.47}$$

Then, if we put $w = \nabla(f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}})(x)$, (4.45) becomes

$$f_j(z_k) = f_j(x) + r_k \cdot s\mathrm{e} + w \cdot a_k v_k. \tag{4.48}$$

If $z_\lambda = (1-\lambda)z_1 + \lambda z_2$, we have that

$$\begin{aligned} f_j(z_\lambda) &\le (1-\lambda) f_j(z_1) + \lambda f_j(z_2) \\ &\overset{(4.48)}{=} f_j(x) + s((1-\lambda)r_1 + \lambda r_2) \cdot \mathrm{e} + w \cdot ((1-\lambda)a_1 v_1 + \lambda a_2 v_2). \end{aligned} \tag{4.49}$$

As $((1-\lambda)r_1 + \lambda r_2) \in \partial^- f_j(x)$, we also obtain that

$$\begin{aligned} f_j(z_\lambda) &\ge f_j(x) + s((1-\lambda)r_1 + \lambda r_2) \cdot \mathrm{e} \\ &\quad + ((1-\lambda)r_1 + \lambda r_2) \cdot ((1-\lambda)a_1 v_1 + \lambda a_2 v_2) = \\ &= f_j(x) + s((1-\lambda)r_1 + \lambda r_2) \cdot \mathrm{e} + w \cdot ((1-\lambda)a_1 v_1 + \lambda a_2 v_2) = \\ &\overset{(4.48)}{=} (1-\lambda) f_j(z_1) + \lambda f_j(z_2). \end{aligned} \tag{4.50}$$

Thus, we have that $f_j((1-\lambda)z_1 + \lambda z_2) = (1-\lambda) f_j(z_1) + \lambda f_j(z_2)$ and our claim is proved.

In particular, there exists a supporting hyperplane to the graph of f_j which contains the affine hull of the convex envelope of $\{[\tilde{x}, \tilde{y}_{i_k}]\}_{k=1,2}$ and then this affine hull must intersect $H_t \times \mathbb{R}$ into a supporting line to the graph of $f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ which is parallel to the segment $[\tilde{y}_{i_1}, \tilde{y}_{i_2}]$.

Thus, if all the supporting lines to the graph of $f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ which are parallel to a segment $[\tilde{y}_k, \tilde{y}_m]$ (with $k, m \in \{1, \ldots, j\}$, $k \neq m$) are parametrized as

$$l_{k,m} + w, \tag{4.51}$$

where $l_{k,m}$ is the linear subspace of $\mathbb{R}^{n+1}$ which is parallel to $[\tilde{y}_k, \tilde{y}_m]$ and $w \in W_{k,m} \subset H_t \times \mathbb{R}$ is perpendicular to $l_{k,m}$, we have that

$$B_j = \sigma^t(Z) \cap \Bigg[\bigcup_{\substack{k,m \in \{1,\ldots,j\} \\ k<m}} \ \bigcup_{w \in W_{k,m}} \pi_{\mathbb{R}^n}(l_{k,m} + w) \Bigg]. \tag{4.52}$$

By this characterization of the set B_j and by Fubini theorem on H_t w.r.t. the partition given by the lines which are parallel to $\pi_{\mathbb{R}^n}(l_{k,m})$ for every k and m, in order to show that $\mathscr{H}^{n-1}(B_j)=0$ it is sufficient to prove that, $\forall\, w\in W_{k,m}$,

$$\mathscr{H}^{n-1}(\sigma^t(Z)\cap\pi_{\mathbb{R}^n}(l_{k,m}+w))=0. \tag{4.53}$$

Finally, (4.53) follows from the fact that a supporting line to the graph of $f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ cannot contain two distinct points of $\tilde\sigma^t(Z)$, because otherwise they would be contained in a higher dimensional face of the grapf of f contraddicting the definition of $\tilde\sigma^t(Z)$.

Then, the vector field defined in (4.43) is defined $\mathscr{H}^{n-1}$-a.e..

Step 4 *Convergence of the approximating vector fields*

Here we prove the convergence property of the vector field defined in (4.43) as stated in (4.35). This result is obtained as a consequence of the uniform convergence of the approximating functions f_j to the function $\hat f$ which is the graph of the set

$$\hat C=\mathrm{conv}\big(\{\tilde l_t(y_i)\}_{i\in\mathbb{N}}\big). \tag{4.54}$$

First of all we observe that, since $C_j\nearrow\hat C$,

$$\mathrm{dom}\, f_j=\pi_{\mathbb{R}^n}(C_j)\nearrow\mathrm{dom}\,\hat f=\pi_{\mathbb{R}^n}(\hat C)\quad\text{and}\quad f_j(x)\searrow\hat f(x)\quad\forall\, x\in\mathrm{ri}(\pi_{\mathbb{R}^n}(\hat C)), \tag{4.55}$$

where $f_j(x)$ is defined $\forall\, j\geq j_0$ such that $x\in\pi_{\mathbb{R}^n}(C_{j_0})$.

In order to prove that $f_j(x)\searrow\hat f(x)$ uniformly, we show that the functions f_j are uniformly Lipschitz on their domain, with uniformly bounded Lipschitz constants.

We recall that the graph of f_j is made of segments that connect the points of graph $f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ to the points of graph$\big(\mathrm{conv}\big(\{\tilde y_i\}_{i=1}^j\big)\big)$.

In order to find and upper bound for the incremental ratios between points $z,w\in\mathrm{dom}\, f_j$, we distinguish two cases.

Case 1: $[z,w]\subset[x,y_k]$, where $x\in\mathbf{B}^{n-1}_{R_1}(x_1)$, $y_k\in\{y_i\}_{i=1}^j$ and $[\tilde x,\tilde y_k]\subset\mathrm{graph}(f_j)$.

In this case we have that

$$\frac{|f_j(z)-f_j(w)|}{|z-w|}=\frac{|f_j(x)-f_j(y_k)|}{|x-y_k|}=\frac{|f(x)-f(y_k)|}{|x-y_k|}\leq L_f, \tag{4.56}$$

where L_f is tha Lipschitz constant of f on K_t.

Case 2: Otherwise we observe that, since f_j is convex,

$$|f_j(z)-f_j(w)|\leq\sup_{r\in\partial^-f_j(z)\cup\partial^-f_j(w)}|r\cdot(z-w)|. \tag{4.57}$$

Let then $r\in\partial^-f_j(z)\cup\partial^-f_j(w)$ be a maximizer of the r.h.s. of (4.57) and let us suppose, without loss of generality, that $r\in\partial^-f_j(z)$. If $x\in\mathbf{B}^{n-1}_{R_1}(x_1)$ is such that $(z,f_j(z))\subset[(y,f_j(y)),\tilde x]\subset\mathrm{graph}(f_j)$ for some $y\in\mathrm{conv}(\{y_i\}_{i=1}^j)$, we have the following unique decomposition

$$w-z=\beta_j(z,w)\left(\frac{x-z}{|x-z|}\right)+\gamma_j(z,w)q, \tag{4.58}$$

where $q\in\mathbf{S}^{n-1}\cap H_0$ and $\beta_j(z,w)$, $\gamma_j(z,w)\in\mathbb{R}$.

Then,

$$r\cdot(w-z)=\beta_j(z,w)\left(r\cdot\frac{x-z}{|x-z|}\right)+\gamma_j(z,w)(r\cdot q). \tag{4.59}$$

The first scalar product in (4.59) can be estimated as in Case 1.

As for the second term, we note that the supporting hyperplane to the graph of f_j given by the graph of the affine function $h(p)=f_j(z)+r\cdot(p-z)$ contains the segment $[(z,f_j(z)),\tilde x]$ and its intersection with the hyperplane $H_t\times\mathbb{R}$ is given by a supporting hyperplane to the graph of $f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$ which contains the point $\tilde x$.

Moreover, as $q\in H_0$, we have that

$$r\cdot q=\pi_{H_0}(r)\cdot q, \tag{4.60}$$

and we know that $\pi_{H_0}(r)\in\partial^-f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}(x)$.

By definition of subdifferential, for all $s \in \partial^- f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}(x)$ and for all $\lambda > 0$ such that $x + \lambda q,\ x - \lambda q \in \mathbf{B}^{n-1}_{R_1}(x_1)$,

$$\frac{f(x) - f(x - \lambda q)}{\lambda} \leq s \cdot q \leq \frac{f(x + \lambda q) - f(x)}{\lambda} \tag{4.61}$$

and so the term $|r \cdot q|$ is bounded from above by the Lipschitz constant of f.
As the scalar products $\beta_j(z, w)$, $\gamma_j(z, w)$ are uniforlmly bounded w.r.t. j on $\operatorname{dom} f_j \subset \operatorname{dom} \hat{f}$, we conclude that the functions $\{f_j\}_{j\in\mathbb{N}}$ are uniformly Lipschitz on the sets $\{\operatorname{dom} f_j\}_{j\in\mathbb{N}}$ and their Lipschitz constants are uniformly bounded by some positive constant $\hat{L}$.
If we call $\hat{f}_j$ a Lipschitz extension of f_j to the set $\operatorname{dom} \hat{f}$ which has the same Lipschitz constant (Mac Shane lemma), by Ascoli-Arzelá theorem we have that

$$\hat{f}_j \to \hat{f} \quad \text{uniformly on } \operatorname{dom} \hat{f}.$$

Now we prove that, for $\mathscr{H}^{n-1}$-a.e. $x \in \sigma^t(Z)\backslash B$, $v^t_j(x) \to v_{\mathrm{e}}(x)$.
Given a point $x \in \sigma^t(Z)\backslash B$, we call $\tilde{y}_{j(x)}$, where $j \in \mathbb{N}$, the unique point $\tilde{y}_k \in \{\tilde{y}_i\}_{i=1}^j$ such that

$$v^t_j(x) = \frac{y_k - x}{|y_k - x|}.$$

By compactness of graph(conv($\{\tilde{y}_i\}_{i\in\mathbb{N}}$)) , there is a subsequence $\{j_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\tilde{y}_{j_n(x)} \to \hat{y} \in \operatorname{graph} f,$$

hence

$$v^t_{j_n}(x) \to \hat{v} = \frac{\hat{y} - x}{|\hat{y} - x|}.$$

As the functions f_j converge to $\hat{f}$ uniformly, the point $\hat{y}$ and the whole segment $[\tilde{x}, \hat{y}]$ belong to the graph of $\hat{f}$.
So, there are two segments $\tilde{l}_t(x)$ and $[\tilde{x}, \hat{y}]$ which belong to the graph of $\hat{f}$ and pass through the point $\tilde{x}$. Since $\hat{f}_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}} = f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}$, we can apply the same reasoning we made in order to prove that the set (4.42) was $\mathscr{H}^{n-1}$-negligible to conclude that the set

$$\sigma^t(Z) \cap \Big\{ x \in \mathbf{B}^{n-1}_{R_1}(x_1) : \exists \text{ more than two segments in the graph of } \hat{f}$$
$$\text{that connect } \tilde{x} \text{ to a point of graph(conv}(\{\tilde{y}_i\}_{i\in\mathbb{N}})) \Big\}$$

has zero $\mathscr{H}^{n-1}$-measure.
Then, $[\tilde{x}, \hat{y}] = \tilde{l}_t(x)$ and $\hat{v} = v_{\mathrm{e}}(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \sigma^t(Z)$, so that property (4.35) is proved.

Step 4 *Proof of the estimate* (4.36) (see Figure 5)
The estimate for the map $\sigma^t_{v^t_j}$ induced by the approximating vector fields v^t_j follows as in [BG07] and [Car08] from the fact that the collection of segments with directions given by v^t_j and endpoints in $\operatorname{dom} v^t_j$, $\sigma^{h^-}(Z)$ form a finite union of cones with bases in $\operatorname{dom} v^t_j$ and vertex in $\{y_i\}_{i=1}^j$.
Indeed, if we define the sets

$$\Omega_{ij} = \left\{ x \in \sigma^t(Z) : \mathcal{D}^t_j(x) = \{v^t_j(x)\} \text{ and } v^t_j(x) = \frac{y_i - x}{|y_i - x|} \right\}, \quad j \in \mathbb{N}, \quad i = 1, \ldots, j, \tag{4.62}$$

for all $S \subset \sigma^t(Z)\backslash B$ we have that

$$\mathscr{H}^{n-1}(S) = \sum_{i=1}^j \mathscr{H}^{n-1}(S \cap \Omega_{ij})$$

and

$$\mathscr{H}^{n-1}((\sigma^t_{v^t_j})^{-1}(S)) = \sum_{i=1}^j \mathscr{H}^{n-1}((\sigma^t_{v^t_j})^{-1}(S \cap \Omega_{ij})).$$

Then it is sufficient to prove (4.36) when the vector field v^t_j is defined as

$$v^t_j(x) = \frac{y_i - x}{|y_i - x|}.$$

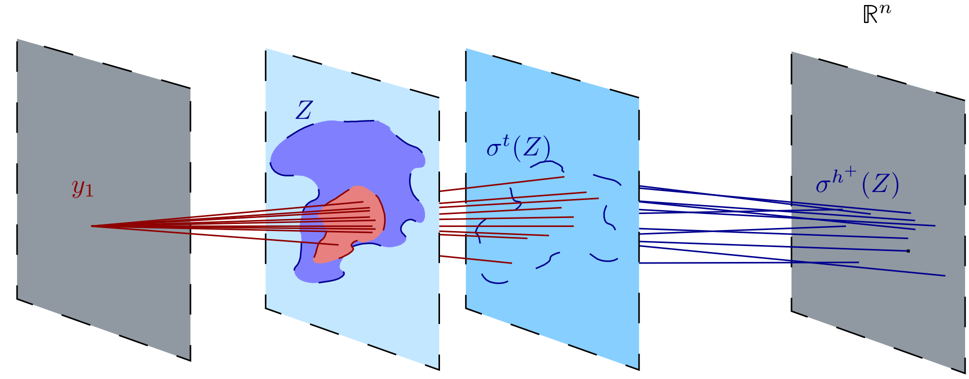

Figure 5: The vector field v_{e} is approximated by directions of approximating cones, in the picture one can see the first one. At the same time, Z is approximated by the push forward of $\sigma^t(Z)$ with the approximating vector field: compare the blue area with the red one.

After these preliminary considerations, (4.36) follows from the fact that the set

$$\bigcup_{s\in[0,t-h^-]} \sigma^{-s}_{v^t_j}(S) \tag{4.63}$$

is a cone with with base $S \subset H_t$ and vertex $y_i \in H_{h^-}$ and $\sigma^{-t}_{v^t_j}(S)$ is the intersection of this cone with the hyperplane H_0. □

Proof of Lemma 4.7. Given a k-dimensional $\mathcal{D}$-cylinder $\mathscr{C}^k$ parametrized as in (4.27), the collection of segments

$$\cup_{z\in Z^k} \{\sigma^{\mathrm{w}+t\mathrm{e}}(z) : t \in [h^-(\mathrm{w},\mathrm{e}), h^+(\mathrm{w},\mathrm{e})]\} \tag{4.64}$$

is a 1-dimensional $\mathcal{D}$-cylinder of the convex function f restricted to the $(n-k+1)$-dimensional set

$$\pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\{\mathrm{w}+t\mathrm{e} : \, t \in [h^-(\mathrm{w},\mathrm{e}), h^+(\mathrm{w},\mathrm{e})]\}). \tag{4.65}$$

Then, as in Lemma 4.14, we can construct a sequence of approximating vector fields also for the directions of the segments (4.64). The only difference with respect to the approximation of the 1-dimensional faces of f is that the domain of the approximating vector fields will be a subset of an $(n-k)$-dimensional affine plane of the form $\pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{w})$ and so the measure involved in the estimate (4.32) will be $\mathscr{H}^{n-k}$ instead of $\mathscr{H}^{n-1}$. Finally, we pass to the limit as with the approximating vector fields given in Lemma 4.14 and we obtain the fundamental estimate (4.31) for the k-dimensional $\mathcal{D}$-cylinders. □

4.5. **Properties of the density function.** In this subsection, we show that the quantitative estimates of Lemma 4.14 allow not only to derive the absolute continuity of the push forward with $\sigma^{\mathrm{w}+t\mathrm{e}}$, but also to find regularity estimates on the density function. This regularity properties will be used in Section 5.

Corollary 4.15. *Let $\mathscr{C}^k$ be a k-dimensional $\mathcal{D}$-cylinder parametrized as in (4.27) and let $\sigma^{\mathrm{w}+s\mathrm{e}}(Z^k)$, $\sigma^{\mathrm{w}+t\mathrm{e}}(Z^k)$ be two sections of $\mathscr{C}^k$ with s and t as in (4.31). Then, if we put $\mathrm{s} = \mathrm{w}+s\mathrm{e}$ and $\mathrm{t} = \mathrm{w}+t\mathrm{e}$, we have that*

$$\sigma^{\mathrm{t}-|\mathrm{s}-\mathrm{t}|\mathrm{e}}_{\#}\big(\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{t}}(Z^k)\big) \ll \mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{s}}(Z^k) \tag{4.66}$$

and by the Radon-Nikodym theorem there exists a function $\alpha(\mathrm{t},\mathrm{s},\cdot)$ which is $\mathscr{H}^{n-k}$-a.e. defined on $\sigma^{\mathrm{s}}(Z^k)$ and is such that

$$\sigma^{\mathrm{t}-|\mathrm{s}-\mathrm{t}|\mathrm{e}}_{\#}\big(\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{t}}(Z^k)\big) = \alpha(\mathrm{t},\mathrm{s},\cdot) \cdot \mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{s}}(Z^k). \tag{4.67}$$

Proof. Without loss of generality we can assume that $s = 0$. If $\mathscr{H}^{n-k}(A) = 0$ for some $A \subset Z^k$, by definition of push forward of a measure we have that

$$(\sigma^{\mathrm{w}+t\mathrm{e}})^{-1}_{\#}\big(\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{w}+t\mathrm{e}}(Z^k)\big)(A) = \mathscr{H}^{n-k}(\sigma^{\mathrm{w}+t\mathrm{e}}(A)) \tag{4.68}$$

and taking $s = 0$ in (4.31) we find that $\mathscr{H}^{n-k}(A) = 0$ implies that $\mathscr{H}^{n-k}(\sigma^{\mathrm{w}+t\mathrm{e}}(A)) = 0$. □

Remark 4.16. The fuction $\alpha = \alpha(\mathrm{t}, \mathrm{s}, y)$ defined in (4.67) is measurable w.r.t. y and, for $\mathscr{H}^{n-k}$-a.e. $y' \in \sigma^{\mathrm{w}+t\mathrm{e}}(Z^k)$, we have that

$$\alpha(\mathrm{s}, \mathrm{t}, y') = \alpha(\mathrm{t}, \mathrm{s}, \sigma^{\mathrm{t}-|\mathrm{s}-\mathrm{t}|\mathrm{e}}(y'))^{-1}. \tag{4.69}$$

Moreover, from Lemma 4.7 we immediately get the uniform bounds:

$$\begin{aligned} &\left(\frac{h^+(\mathrm{t},\mathrm{e}) - u}{h^+(\mathrm{t},\mathrm{e})}\right)^{n-k} \le \alpha(\mathrm{t}+u\mathrm{e}, \mathrm{t}, \cdot) \le \left(\frac{u - h^-(\mathrm{t},\mathrm{e})}{-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k} \quad \text{if } u \in [0, h^+(\mathrm{t},\mathrm{e})], \\ &\left(\frac{u - h^-(\mathrm{t},\mathrm{e})}{-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k} \le \alpha(\mathrm{t}+u\mathrm{e}, \mathrm{t}, \cdot) \le \left(\frac{h^+(\mathrm{t},\mathrm{e}) - u}{h^+(\mathrm{t},\mathrm{e})}\right)^{n-k} \quad \text{if } u \in [h^-(\mathrm{t},\mathrm{e}), 0]. \end{aligned} \tag{4.70}$$

We conclude this section with the following proposition:

Proposition 4.17. *Let $\mathscr{C}^k(\mathscr{Z}^k, C^k)$ be a k-dimensional $\mathfrak{D}$-cylinder parametrized as in (4.27) and assume without loss of generality that $\mathrm{w} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(Z^k) = 0$. Then, the function $\alpha(\mathrm{t}, 0, z)$ defined in (4.67) is locally Lipschitz in $\mathrm{t} \in \mathrm{ri}(C^k)$ (and so jointly measurable in (t, z)). Moreover, for $\mathscr{H}^{n-k}$-a.e. $y \in \sigma^{\mathrm{t}}(Z)$ the following estimates hold:*

***1.** Derivative estimate*

$$-\left(\frac{n-k}{h^+(\mathrm{t},\mathrm{e}) - u}\right)\alpha(\mathrm{t}+u\mathrm{e}, \mathrm{t}, y) \le \frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e}, \mathrm{t}, y) \le \left(\frac{n-k}{u - h^-(\mathrm{t},\mathrm{e})}\right)\alpha(\mathrm{t}+u\mathrm{e}, \mathrm{t}, y); \tag{4.71}$$

***2.** Integral estimate*

$$\left(\frac{|h^+(\mathrm{t},\mathrm{e}) - u|}{|h^+(\mathrm{t},\mathrm{e})|}\right)^{n-k} (-1)^{\mathbb{1}_{\{u<0\}}} \le \alpha(\mathrm{t}+u\mathrm{e}, \mathrm{t}, y)\, (-1)^{\mathbb{1}_{\{u<0\}}} \le \left(\frac{|h^-(\mathrm{t},\mathrm{e}) - u|}{|h^-(\mathrm{t},\mathrm{e})|}\right)^{n-k} (-1)^{\mathbb{1}_{\{u<0\}}}; \tag{4.72}$$

***3.** Total variation estimate*

$$\int_{h^-(\mathrm{t},\mathrm{e})}^{h^+(\mathrm{t},\mathrm{e})} \left|\frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e}, 0, z)\right| du \le 2\alpha(\mathrm{t}, 0, z)\left[\frac{|h^+ - h^-|^{n-k}}{|h^+|^{n-k}} + \frac{|h^+ - h^-|^{n-k}}{|h^-|^{n-k}} - 1\right], \tag{4.73}$$

where h^+, h^- stand for $h^+(\mathrm{t},\mathrm{e}), h^-(\mathrm{t},\mathrm{e})$.

Proof. Lipschitz regularity estimate First we prove the local Lipschitz regularity of $\alpha(\mathrm{t}, 0, z)$ w.r.t. $\mathrm{t} \in \mathrm{ri}(C^k)$.
Given $\mathrm{s}, \mathrm{t} \in C^k$, we set $\mathrm{e} = \frac{\mathrm{s}-\mathrm{t}}{|\mathrm{s}-\mathrm{t}|}$.
As

$$\sigma^{\mathrm{s}-|\mathrm{s}|\frac{\mathrm{s}}{|\mathrm{s}|}} = \sigma^{\mathrm{t}-|\mathrm{t}|\frac{\mathrm{t}}{|\mathrm{t}|}} \circ \sigma^{\mathrm{s}-|\mathrm{s}-\mathrm{t}|\mathrm{e}},$$

then

$$\begin{aligned} \sigma_{\#}^{\mathrm{s}-|\mathrm{s}|\frac{\mathrm{s}}{|\mathrm{s}|}}(\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{s}}(Z)) &= \sigma_{\#}^{\mathrm{t}-|\mathrm{t}|\frac{\mathrm{t}}{|\mathrm{t}|}}\big(\sigma_{\#}^{\mathrm{s}-|\mathrm{s}-\mathrm{t}|\mathrm{e}}\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{s}}(Z)\big) \\ &= \sigma_{\#}^{\mathrm{t}-|\mathrm{t}|\frac{\mathrm{t}}{|\mathrm{t}|}}\Big(\alpha(\mathrm{s}, \mathrm{t}, y) \cdot \mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{t}}(Z)\Big) \\ &= \alpha(\mathrm{t}, 0, z) \cdot \alpha(\mathrm{s}, \mathrm{t}, \sigma^{\mathrm{t}}(z)) \cdot \mathscr{H}^{n-k}_{|z}. \end{aligned} \tag{4.74}$$

By definition of α it follows that

$$\alpha(\mathrm{s}, 0, z) - \alpha(\mathrm{t}, 0, z) = \alpha(\mathrm{t}, 0, z)[\alpha(\mathrm{s}, \mathrm{t}, \sigma^{\mathrm{t}}(z)) - 1]. \tag{4.75}$$

Now we want to estimate the term $[\alpha(\mathrm{s},\mathrm{t},\sigma^{\mathrm{t}}(z))-1]$ with the lenght $|\mathrm{s}-\mathrm{t}|$ times a constant which is locally bounded w.r.t. t. In order to do this, we proceed as in the Corollary 2.19 of [Car08] using the estimate

$$\begin{aligned}\left(\frac{h^+(\mathrm{t},\mathrm{e})-u_2}{h^+(\mathrm{t},\mathrm{e})-u_1}\right)^{n-k}\mathscr{H}^{n-k}(\sigma^{\mathrm{t}+u_1\mathrm{e}}(S)) &\leq \mathscr{H}^{n-k}(\sigma^{\mathrm{t}+u_2\mathrm{e}}(S))\\ &\leq \left(\frac{u_2-h^-(\mathrm{t},\mathrm{e})}{u_1-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k}\mathscr{H}^{n-k}(\sigma^{\mathrm{t}+u_1\mathrm{e}}(S)),\end{aligned} \tag{4.76}$$

which holds $\forall\, h^-(\mathrm{t},\mathrm{e})<u_1\leq u_2<h^+(\mathrm{t},\mathrm{e})$ and $\forall\, S\subset\sigma^{\mathrm{t}}(Z)$.

Indeed, (4.76) can be rewritten in the following way:

$$\begin{aligned}\left(\frac{h^+(\mathrm{t},\mathrm{e})-u_2}{h^+(\mathrm{t},\mathrm{e})-u_1}\right)^{n-k}\int_S\alpha(\mathrm{t}+u_1\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y) &\leq \int_S\alpha(\mathrm{t}+u_2\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y)\\ &\leq \left(\frac{u_2-h^-(\mathrm{t},\mathrm{e})}{u_1-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k}\int_S\alpha(\mathrm{t}+u_1\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y).\end{aligned} \tag{4.77}$$

Therefore, there is a dense sequence $\{u_i\}_{i\in\mathbb{N}}$ in $(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ such that for $\mathscr{H}^{n-k}$-a.e. $y\in S$ and for all $u_i\leq u_j$, $i,j,\in\mathbb{N}$ the following inequalities hold

$$\begin{aligned}\left[\left(\frac{h^+(\mathrm{t},\mathrm{e})-u_j}{h^+(\mathrm{t},\mathrm{e})-u_i}\right)^{n-k}-1\right]\alpha(\mathrm{t}+u_i\mathrm{e},\mathrm{t},y) &\leq \alpha(\mathrm{t}+u_j\mathrm{e},\mathrm{t},y)-\alpha(\mathrm{t}+u_i\mathrm{e},\mathrm{t},y)\\ &\leq \left[\left(\frac{u_j-h^-(\mathrm{t},\mathrm{e})}{u_i-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k}-1\right]\alpha(\mathrm{t}+u_i\mathrm{e},\mathrm{t},y).\end{aligned} \tag{4.78}$$

Thanks to the uniform bounds (4.70), for all $y\in\sigma^{\mathrm{t}}(Z)$ such that (4.78) holds, the function $\alpha(\mathrm{t}+\cdot\mathrm{e},\mathrm{t},y)$ is locally Lipschitz on $\{u_i\}_{i\in\mathbb{N}}$ and for every $[a,b]\subset(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ the Lipschitz constants of α on $\{u_i\}_{i\in\mathbb{N}}\cap[a,b]$ are uniformly bounded w.r.t. y.
Then, on every compact interval $[a,b]\subset(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ there exists a Lipschitz extension $\tilde\alpha(\mathrm{t}+\cdot\mathrm{e},\mathrm{t},y)$ of $\alpha(\mathrm{t}+\cdot\mathrm{e},\mathrm{t},y)$ which has the same Lipschitz constant.
By the dominated convergence theorem, whenever $\{u_{j_n}\}_{n\in\mathbb{N}}\subset\{u_j\}_{j\in\mathbb{N}}$ converges to some $u\in[a,b]$ we have

$$\int_S\alpha(\mathrm{t}+u_{j_n}\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y)\quad\longrightarrow\quad\int_S\tilde\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y),\quad\forall\, S\subset\sigma^{\mathrm{t}}(Z).$$

However, the integral estimate (4.77) implies that

$$\int_S\alpha(\mathrm{t}+u_{j_n}\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y)\quad\longrightarrow\quad\int_S\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y),$$

so that the Lipschitz extension $\tilde\alpha$ is an $L^1(\mathscr{H}^{n-k})$ representative of the original density α for all $u\in[a,b]$.
Repeating the same reasoning for an increasing sequence of compact intervals $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ that converge to $(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$, we can assume that the density function $\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)$ is locally Lipschitz in u with a Lipschitz constant that depends continuously on t and on e.
Then, by (4.75), the local Lipschitz regularity in t of the function $\alpha(\mathrm{t},0,z)$ is proved.

Derivative estimate If we derive w.r.t. u the pointwise estimate (4.78) (which holds for all $u\in(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ by the first part of the proof) we obtain the derivative estimate (4.71).

Integral estimate (4.71) implies the monotonicity of the following quantities:

$$\frac{d}{du}\left(\frac{\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)}{(h^+(\mathrm{t},\mathrm{e})-u)^{n-k}}\right)\geq 0,\qquad\frac{d}{du}\left(\frac{\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)}{(u-h^-(\mathrm{t},\mathrm{e}))^{n-k}}\right)\leq 0.$$

Integrating the above inequalities from $u\in(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ to 0 we obtain (4.72).

Total variation estimate In order to prove (4.73) we proceed as in Corollary 2.19 of [Car08].

$$\begin{aligned}\int_{h^-(\mathrm{t},\mathrm{e})}^0 \left|\frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)\right| du &\le \int_{\{\frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)>0\}\cap\{u\in(h^-(\mathrm{t},\mathrm{e}),0)\}} \frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)\,du \\ &+ \int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{(n-k)\alpha(\mathrm{t}+u\mathrm{e},0,z)}{|h^+(\mathrm{t},\mathrm{e})-u|}\,du \\ &\le \int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)\,du + \\ &+ 2\int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{(n-k)\alpha(\mathrm{t}+u\mathrm{e},0,z)}{|h^+(\mathrm{t},\mathrm{e})-u|}\,du \\ &\le \alpha(\mathrm{t},0,z) + 2\int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{(n-k)\alpha(\mathrm{t}+u\mathrm{e},0,z)}{|h^+(\mathrm{t},\mathrm{e})-u|}\,du. \end{aligned} \tag{4.79}$$

From (4.75) we know that $\alpha(\mathrm{t}+u\mathrm{e},0,z)=\alpha(\mathrm{t},0,z)\,\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},\sigma^{\mathrm{t}}(z))$.
Moreover, since $u<0$

$$\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},\sigma^{\mathrm{t}}(z)) \underset{(4.72)}{\le} \left(\frac{|h^+(\mathrm{t},\mathrm{e})-u|}{|h^+(\mathrm{t},\mathrm{e})|}\right)^{n-k}.$$

If we substitute this inequality in (4.79) we find that

$$\begin{aligned}(4.79) &\le \alpha(\mathrm{t},0,z) + 2\alpha(\mathrm{t},0,z)\int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{(n-k)|h^+(\mathrm{t},\mathrm{e})-u|^{n-k-1}}{|h^+(\mathrm{t},\mathrm{e})|^{n-k}}\,du \\ &= -\alpha(\mathrm{t},0,z) + 2\alpha(\mathrm{t},0,z)\frac{|h^+(\mathrm{t},\mathrm{e})-h^-(\mathrm{t},\mathrm{e})|^{n-k}}{|h^+(\mathrm{t},\mathrm{e})|^{n-k}}. \end{aligned} \tag{4.80}$$

Adding the symmetric estimate on $(0,h^+(\mathrm{t},\mathrm{e}))$ we obtain (4.73). □

4.6. **The disintegration on model sets.** Now we conclude the proof of Theorem 3.3 on the model sets, giving also an explicit formula for the conditional probabilities.

We consider a k-dimensional $\mathcal{D}$-cylinder $\mathscr{C}^k=\mathscr{C}^k(\mathscr{Z}^k,C^k)$ parametrized as in (4.27) and we assume, without loss of generality, that $\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(Z^k)=0\in\mathbb{R}^n$. We also set $h_j^\pm=h^\pm(0,\mathrm{e}_j)$, $\forall\, j=1,\dots,k$, and we omit the point $w=0$ in the notation for the map (4.28).

Theorem 4.18. *Let $\mathscr{C}^k$ be a k-dimensional $\mathcal{D}$-cylinder parametrized as in (4.27). Then, $\forall\,\varphi\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\int_{\mathscr{C}^k}\varphi\,d\mathscr{L}^n=\int_{Z^k}\int_{h_k^-}^{h_k^+}\!\!\dots\int_{h_1^-}^{h_1^+}\alpha(t_1\mathrm{e}_1+\dots+t_k\mathrm{e}_k,0,z)\,\varphi(\sigma^{(t_1\mathrm{e}_1+\dots+t_k\mathrm{e}_k)}(z))\,dt_1\dots dt_k\,d\mathscr{H}^{n-k}(z). \tag{4.81}$$

Then, as $(Z^k,\mathscr{B}(Z^k))$ is isomorphic to the quotient space determined by the map ∇f on $\mathscr{C}^k$, by the uniqueness of the disintegration the conditional probabilities of the disintegration of the Lebesgue measure on the pieces of k-dimensional faces of f which are contained in $\mathscr{C}^k$ are given by

$$\mu_z(dt_1\dots dt_k)=\frac{\alpha(t_1\mathrm{e}_1+\dots+t_k\mathrm{e}_k,0,z)\,\mathscr{H}^k\llcorner[\mathrm{ri}(\mathcal{R}(z))\cap\mathscr{C}^k](dt_1\dots dt_k)}{\int_{h_k^-}^{h_k^+}\dots\int_{h_1^-}^{h_1^+}\alpha(s_1\mathrm{e}_1+\dots+s_k\mathrm{e}_k,0,z)\,ds_1\dots ds_k}, \tag{4.82}$$

for $\mathscr{H}^{n-k}$-a.e. $z\in Z^k$.

Proof. We proceed using the disintegration technique which was presented in Section 4.1.

$$\begin{aligned}
\int_{\mathscr{C}^k} \varphi(x)\, d\mathscr{L}^n(x) &= \int_{h_k^-}^{h_k^+} \cdots \int_{h_1^-}^{h_1^+} \int_{\mathscr{C}^k \cap \{x\cdot e_k = t_k\}\cap\cdots\cap\{x\cdot e_1=t_1\}} \varphi\, d\mathscr{H}^{n-k} \\
&\underset{(4.66)}{=} \int_{h_k^-}^{h_k^+} \cdots \int_{h_1^-}^{h_1^+} \int_{Z^k} \alpha(t_k e_k, 0, z) \ldots \alpha(t_1 e_1 + \cdots + t_k e_k, t_2 e_2 + \cdots + t_k e_k, \sigma^{(t_2 e_2+\cdots+t_k e_k)}(z)) \\
&\qquad\qquad \cdot \varphi(\sigma^{(t_1 e_1+\cdots+t_k e_k)}(z))\, d\mathscr{H}^{n-k}(z)\, dt_1 \ldots dt_k \\
&\underset{(4.74)}{=} \int_{h_k^-}^{h_k^+} \cdots \int_{h_1^-}^{h_1^+} \int_{Z^k} \alpha(t_1 e_1 + \cdots + t_k e_k, 0, z)\, \varphi(\sigma^{(t_1 e_1+\cdots+t_k e_k)}(z))\, d\mathscr{H}^{n-1}(z)\, dt_1 \ldots dt_k \\
&\underset{\substack{(4.70)\\ \text{Prop } 4.17}}{=} \int_{Z^k} \int_{h_k^-}^{h_k^+} \cdots \int_{h_1^-}^{h_1^+} \alpha(t_1 e_1 + \cdots + t_k e_k, 0, z)\, \varphi(\sigma^{(t_1 e_1+\cdots+t_k e_k)}(z))\, dt_1 \ldots dt_k\, d\mathscr{H}^{n-1}(z). \qquad (4.83)
\end{aligned}$$

□

4.7. **The global disintegration.** In this section we prove Theorem 3.3, concerning the disintegration of the Lebesgue measure (restricted to a set of finite Lebesgue measure $K \subset \mathbb{R}^n$) on the whole k-dimensional faces of a convex function.

The idea is to put side by side the disintegrations on the model $\mathcal{D}$-cylinders which belong to the countable family defined in Lemma 4.6, so as to obtain a global disintegration.

What will remain apart will be set $\mathcal{T} \setminus \cup_{k=1}^n E^k$, projection of those points which do not belong to the relative interior of any face. Nevertheless, the following lemma ensures that this set is $\mathscr{L}^n$-negligible. Indeed, the union of the borders of the n-dimensional faces has zero Lebesgue measure by convexity and by the fact that the n-dimensional faces of f are at most countable.

For faces of dimension k, with $1 < k < n$, the proof is by contradiction: one considers a Lebesgue point of suitable subsets of $\cup_y F_y^k$ and applies the fundamental estimate (4.31) in order to show that the complementary is too big.

Equation (4.84) below was first proved using a different technique in [Lar71b] — where it was shown that the union of the relative boundaries of the R-faces (see Definition 4.11) of an n-dimensional convex body C which have dimension at least 1 has zero $\mathscr{H}^{n-1}$-measure.

Lemma 4.19. *The set of points which do not belong to the relative interior of any face is $\mathscr{L}^n$-negligible:*

$$\mathscr{L}^n\Big(\mathcal{T} \setminus \bigcup_{k=1}^n E^k\Big) = 0, \qquad \text{where } E^k = \bigcup_y \mathrm{ri}\,(F_y^k). \tag{4.84}$$

Proof. Consider any n-dimensional face F_y^n. Being convex, it has nonempty interior. As a consequence, since two different faces cannot intersect, there are at most countably many n-dimensional faces $\{F_{y_i}^n\}_{i\in\mathbb{N}}$; moreover, by convexity, each $F_{y_i}^n$ has an $\mathscr{L}^n$-negligible boundary. Thus

$$\mathscr{L}^n\Big(\bigcup_i \mathrm{rb}\,(F_{y_i}^n)\Big) = 0.$$

Since $\mathcal{T} \subset \bigcup_{k=1}^n F^k$, the thesis is reduced to showing that, for $0 < k < n$,

$$\mathscr{L}^n\Big(F^k \setminus E^k\Big) = 0. \tag{4.85}$$

Given a k dimensional subspace $V \in \mathbf{G}(k,n)$, a unit direction $e \in \mathbf{S}^{n-1} \cap V$, and $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, define the set $\mathcal{A}^{p,e,V}$ of those $x \in \mathcal{T} \setminus \mathrm{ri}(F_{\nabla f(x)}^k)$ which satisfy the two relations

$$\inf_{d\in\mathcal{D}(x)} \|\pi_V(d)\| \geq 1/\sqrt{2} \tag{4.86}$$

$$\pi_V\big(F_{\nabla f(x)}^k\big) \supset \mathrm{conv}\Big(\big\{\pi_V(x)\big\} \cup \pi_V(x) + 2^{-p+1}e + 2^{-p}\big(\mathbf{S}^{n-1} \cap V\big)\Big). \tag{4.87}$$

Choosing (p, e, V) in a sequence $\{(p_i, e_i, V_i)\}_{i\in\mathbb{N}}$ which is dense in $\mathbb{N}_0 \times \big(\mathbf{S}^{n-1} \cap V\big) \times \mathbf{G}(k,n)$, the family $\big\{\mathcal{A}^{p_i,e_i,V_i}\big\}_{i\in\mathbb{N}}$ provides a countable covering of $F^k \setminus E^k$ with measurable sets. The measurability of each

$\mathcal{A}^{p,\mathrm{e},V}$ can be deduced as follows. The set defined by (4.86) is exactly

$$\mathcal{D}^{-1}\circ\pi_V^{-1}\left(V\setminus\mathrm{ri}\left(\frac{1}{\sqrt{2}}\mathbf{B}^n\right)\right).$$

Moreover, (4.87) is equivalent to

$$\pi_V\big(\mathcal{R}(x)-x\big)\supset\mathrm{conv}(2^{-p+1}\mathrm{e}+2^{-p}\big(\mathbf{S}^{n-1}\cap V)\big).$$

Since $\mathcal{R}$ and $\mathcal{D}$ are measurable (Lemma 4.2), then the measurability of $\mathcal{A}^{p,\mathrm{e},V}$ follows.

In particular, if by absurd (4.85) does not hold, then there exists a subset $\mathcal{A}^{p,\mathrm{e},V}$ of $F^k\backslash E^k$ with positive Lebesgue measure. Up to rescaling, one can assume w.l.o.g. that $p=0$, $V=\langle\mathrm{e}_1,\dots,\mathrm{e}_k\rangle$, where $\{\mathrm{e}_1,\dots,\mathrm{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$, and $\mathrm{e}=\mathrm{e}_1$. Moreover, we will denote $\mathcal{A}^{p,\mathrm{e},V}$ simply with $\mathcal{A}$.

Before reaching the contradiction $\mathscr{L}^n(\mathcal{A})=0$, we need the following remarks.
First of all we notice that, for $0\leq h\leq 3$ and $\mathrm{t}\in\pi_V(\mathcal{A})$, one can prove the fundamental estimate

$$\mathscr{H}^{n-k}\big(\sigma^{\mathrm{t}+h\mathrm{e}}(S)\big)\geq\left(\frac{3-h}{3}\right)^{n-k}\mathscr{H}^{n-k}(S)\qquad\forall S\subset\mathcal{A}\cap\pi_V^{-1}(\mathrm{t})\tag{4.88}$$

exactly as in Lemma 4.7, with the approximating vector field given in Step 3, Page 18. Indeed, the $(n-k+1)$-plane $\pi_V^{-1}(\mathbb{R}\mathrm{e})$ cuts the face of each $z\in\mathcal{A}\cap\pi_V^{-1}(\mathrm{t})$ into exactly one line l; this line has projection on V containing at least $[\mathrm{t},\mathrm{t}+3\mathrm{e}]$.
Notice moreover that, by (4.87), each point $x\in\mathrm{l}$, with $\pi_V(x)\in\mathrm{ri}\big([\mathrm{t},\mathrm{t}+3\mathrm{e}]\big)$, is a point in the relative interior of the face. In particular, it does not belong to $\mathcal{A}$.

Let us now prove the claim, assuming by contradiction that $\mathscr{L}^n(\mathcal{A})>0$ (see also Figure 6). Fix any $\varepsilon>0$ small enough. w.l.o.g. one can suppose the origin to be a Lebesgue point of $\mathcal{A}$. Therefore, for every $0<r<\bar{r}(\varepsilon)<1$, there exists $T\subset\prod_{i=1}^k[0,r\mathrm{e}_i]$, with $\mathscr{H}^k(T)>(1-\varepsilon)r^k$, such that

$$\mathscr{H}^{n-k}\big(\mathcal{A}\cap\pi_V^{-1}(\mathrm{t})\cap[0,r]^n\big)\geq(1-\varepsilon)r^{n-k}\qquad\text{for all }\mathrm{t}\in T.\tag{4.89}$$

Moreover, there is a set $Q\subset[0,r\mathrm{e}]$, with $\mathscr{H}^1(Q)>(1-2\varepsilon)r$, such that

$$\mathscr{H}^{k-1}\big(T\cap\pi_{\langle\mathrm{e}\rangle}^{-1}(\mathrm{q})\big)>(1-\varepsilon)r^{k-1}\qquad\text{for }\mathrm{q}\in Q.\tag{4.90}$$

Consider two points $\mathrm{q},\mathrm{s}:=\mathrm{q}+2\varepsilon r\mathrm{e}\in Q$, and take $\mathrm{t}\in T\cap\pi_{\langle\mathrm{e}\rangle}^{-1}(\mathrm{q})$. By the fundamental estimate (4.88), one has

$$\mathscr{H}^{n-k}\big(\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}(S_{\mathrm{t},r})\big)\geq(1-\varepsilon)^{n-k}\mathscr{H}^{n-k}(S_{\mathrm{t},r})\qquad\text{where }S_{\mathrm{t},r}:=\mathcal{A}\cap\pi_V^{-1}(\mathrm{t})\cap[0,r]^n.$$

Furthermore, condition (4.85) implies that $\|x+2\varepsilon r\mathrm{e}-\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}(x)\|\leq 2\varepsilon r$ for each $x\in A\cap\pi_V^{-1}(\mathrm{t})$. Moving points within $\pi_V^{-1}(\mathrm{t})\cap[0,r]^n$ by means of the map $\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}$, they can therefore reach only the square $\pi_V^{-1}(\mathrm{s})\cap[-2\varepsilon r,(1+2\varepsilon)r]^n$. Notice that for ε small, since our proof is needed for $n\geq 3$ and $k\geq 1$,

$$\mathscr{H}^{n-k}([-2\varepsilon r,(1+2\varepsilon)r]^n\setminus[0,r]^n)=(1+4\varepsilon)^{n-k}r^{n-k}-r^{n-k}\leq 4(n-k)\varepsilon r^{n-k}+o(\varepsilon)<n2^n\varepsilon r^{n-k}.$$

As a consequence, the portion which exceeds $\pi_V^{-1}(\mathrm{s})\cap[0,r]^n$ can be estimated as follows:

$$\mathscr{H}^{n-k}\big(\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}(S_{\mathrm{t},r})\cap[0,r]^n\big)\geq\mathscr{H}^{n-k}\big(\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}(S_{\mathrm{t},r})\big)-n2^n\varepsilon r^{n-k}.$$

As notice before, condition (4.87) implies that the points $\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}(S_{\mathrm{t},r})\cap[0,r]^n$ belong to the complementary of $\mathcal{A}$. By the above inequalities we obtain then

$$\begin{aligned}\mathscr{H}^{n-k}\big(\mathcal{A}^{\mathrm{c}}\cap\pi_V^{-1}(\mathrm{t}+2\varepsilon r\mathrm{e})\cap[0,r]^n\big)&\geq\mathscr{H}^{n-k}\big(\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}(S_{\mathrm{t},r})\cap[0,r]^n\big)\\&\geq(1-\varepsilon)^{n-k}\mathscr{H}^{n-k}(S_{\mathrm{t},r})-n2^n\varepsilon r^{n-k}\\&\overset{(4.89)}{\geq}(1-\varepsilon)^{n-k+1}r^{n-k}-n2^n\varepsilon r^{n-k}\\&\geq\frac{1}{2}r^{n-k}.\end{aligned}$$

The last estimate shows that, for each $\mathrm{t}\in T\cap\pi_{\langle\mathrm{e}\rangle}^{-1}(\mathrm{q})$, the point $\mathrm{s}=\mathrm{t}+2\varepsilon r\mathrm{e}$ does not satisfy the inequality in (4.89): thus $\big(T\cap\pi_{\langle\mathrm{e}\rangle}^{-1}(\mathrm{q})\big)+2\varepsilon r\mathrm{e}$ lies in the complementary of T. In particular

$$\mathscr{H}^{k-1}\big(T\cap\pi_{\langle\mathrm{e}\rangle}^{-1}(\mathrm{s})\big)<r^{k-1}-\mathscr{H}^{k-1}\big(T\cap\pi_{\langle\mathrm{e}\rangle}^{-1}(\mathrm{q})\big).$$

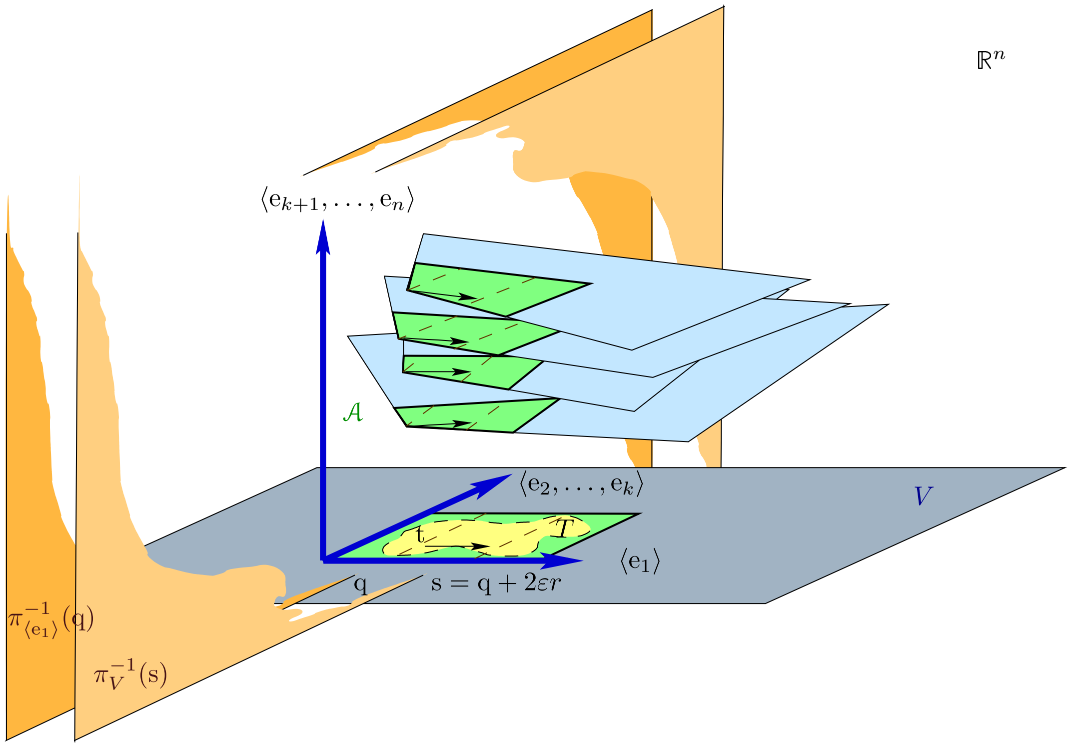

Figure 6: Illustration of the construction in the proof of Lemma 4.19. $\mathcal{A}$ is the set of points on the border of k-faces of f, projected on $\mathbb{R}^n$, having directions close to $V = \langle e_1, \dots, e_k \rangle$ and such that, for each point $x \in \mathcal{A}$, $\pi_V(F^k_{\nabla f(x)})$ contains a fixed half k-cone centered at x with direction e_1. T is a subset of the square $\prod_{i=1}^k [0, re_i]$ such that, for every $t \in T$, $\pi_V^{-1}(t) \cap \mathcal{A}$ is 'big'. Finally, q, $s = q + 2\varepsilon r e_1$ are points on $[0, re_1]$ such that the intersection of T with the affine hyperplanes $\pi^{-1}_{\langle e_1 \rangle}(q)$, $\pi^{-1}_{\langle e_1 \rangle}(s)$ is 'big'. The absurd arises from the following. Due to the fundamental estimate, translating by $2\varepsilon r e_1$ the points $T \cap \pi^{-1}_{\langle e_1 \rangle}(q)$, one finds points in the complementary of T. Since $T \cap \pi^{-1}_{\langle e_1 \rangle}(q)$ was 'big', then $T \setminus \pi^{-1}_{\langle e_1 \rangle}(s)$ should be big, contradicting the fact that $T \cap \pi^{-1}_{\langle e_1 \rangle}(s)$ is 'big'.

However, by construction both t and s belong to Q. This yields the contradiction, by definition of Q:

$$\frac{1}{2} r^{k-1} \overset{(4.90)}{<} \mathscr{H}^{k-1}\big(T \cap \pi^{-1}_{\langle e \rangle}(s)\big) < r^{k-1} - \mathscr{H}^{k-1}\big(T \cap \pi^{-1}_{\langle e \rangle}(t)\big) \overset{(4.90)}{<} \frac{1}{2} r^{k-1}. \qquad \square$$

Proof of Theorem 3.3. As we observed in Remark 3.4, it is sufficient to prove the theorem for the disintegration of the Lebegue measure on the set F^k when $k \in \{1, \dots, n-1\}$.

Thanks to Lemma 4.19, we can further restrict the disintegration to the set E^k defined in (4.13); moreover, by (5.4), for all $k = 1, \dots, n-1$ there exists a $\mathscr{L}^n$-negligible set N^k such that

$$E^k \setminus N^k = \bigcup_{j \in \mathbb{N}} \mathscr{C}^k_j \setminus \mathfrak{d}\mathscr{C}^k_j,$$

where $\{\mathscr{C}^k_j\}_{j \in \mathbb{N}}$ is the countable collection of k-dimensional $\mathcal{D}$-cylinders covering E^k which was constructed in Lemma 4.6, so that the sets $\hat{\mathscr{C}}^k_j = \mathscr{C}^k_j \setminus \mathfrak{d}\mathscr{C}^k_j$ are disjoint.

The fundamental observation is the following:

$$\bigcup_{j \in \mathbb{N}} \hat{\mathscr{C}}^k_j = \bigcup_{j \in \mathbb{N}} \bigcup_{y \in \mathrm{Im}\, \nabla f_{|_{E^k}}} E^k_{y,j} = \bigcup_{y \in \mathrm{Im}\, \nabla f_{|_{E^k}}} \bigcup_{j \in \mathbb{N}} E^k_{y,j} = \bigcup_{y \in \mathrm{Im}\, \nabla f_{|_{E^k}}} E^k_y \setminus N^k, \tag{4.91}$$

where $E^k_{y,j} = E^k_y \cap \hat{\mathscr{C}}^k_j$.

For all $j \in \mathbb{N}$, we set

$$Y_j = \{y \in \mathrm{Im}\, \nabla f_{|_{E^k}} : E^k_{y,j} \neq \emptyset\}, \tag{4.92}$$

we denote by $p_j : \hat{\mathscr{C}}_j^k \to Y_j$ the quotient map corresponding to the partition

$$\hat{\mathscr{C}}_j^k = \bigcup_{y \in \operatorname{Im} \nabla f_{|_{E^k}}} E_{y,j}^k$$

and we set $\nu_j = p_{j\#}\mathscr{L}^n \llcorner \hat{\mathscr{C}}_j^k$.

Since the quotient space $(Y_j, \mathscr{B}(Y_j))$ is isomorphic to $(Z_j^k, \mathscr{B}(Z_j^k))$, where Z_j^k is a section of $\mathscr{C}_j^k$, by Theorem 4.18 we have that

$$\mathscr{L}^n \llcorner \mathscr{C}_j^k(E_j \cap p_j^{-1}(F_j)) = \int_{F_j} \mu_y^j(E_j)\, d\nu_j(y), \quad \forall E_j \in \mathscr{B}(\mathscr{C}_j^k),\ F_j \in \mathscr{B}(Y_j), \tag{4.93}$$

where μ_y^j is equivalent to $\mathscr{H}^k \llcorner E_{y,j}^k$ for ν_j-a.e. $y \in Y_j$.

Moreover, for every $E \in \mathscr{B}(\mathbb{R}^n) \cap E^k$ there exist sets $E_j \in \mathscr{B}(\mathscr{C}_j^k)$ such that

$$E = \bigcup_{j \in \mathbb{N}} E_j$$

and for all $F \in \mathscr{B}(Y)$, where $Y = \bigcup_{j\in\mathbb{N}} Y_j = \operatorname{Im} \nabla f_{|_{E^k}}$, there exist sets $F_j \in \mathscr{B}(Y_j)$ such that

$$F = \bigcup_{j \in \mathbb{N}} F_j \quad \text{and} \quad \nabla f^{-1}(F) = \bigcup_{j\in\mathbb{N}} p_j^{-1}(F_j).$$

Then,

$$\begin{aligned}
\mathscr{L}^n \llcorner K(E \cap \nabla f^{-1}(F)) &= \sum_{j=1}^{+\infty} \mathscr{L}^n \llcorner \mathscr{C}_j^k(E_j \cap p_j^{-1}(F_j)) \\
&\underset{(4.93)}{=} \sum_{j=1}^{+\infty} \int_{F_j} \mu_y^j(E_j)\, d\nu_j(y) \\
&= \sum_{j=1}^{+\infty} \int_{Y_j} \mathbb{1}_{F_j}(y) \mu_y^j(E_j)\, d\nu_j(y) \\
&= \sum_{j=1}^{+\infty} \int_{Y} \mathbb{1}_{F_j}(y) \mu_y^j(E_j) f_j(y)\, d\nu(y),
\end{aligned} \tag{4.94}$$

where f_j is the Radon-Nikodym derivative of ν_j w.r.t. the measure ν on Y given by $\nabla f_\# \mathscr{L}^n \llcorner K$.

Since, as we proved in Section 3.1, there exists a unique disintegration $\{\mu_y\}_{y \in \operatorname{Im} \nabla f_{|_{E^k}}}$ such that

$$\mathscr{L}^n \llcorner K(E \cap \nabla f^{-1}(F)) = \int_F \mu_y(E)\, d\nu(y) \quad \text{for all} \quad E \in \mathscr{B}(\mathbb{R}^n),\ F \in \mathscr{B}(Y),$$

we conclude that the last term in (4.94) converges and

$$\mu_y = \sum_{j=1}^{+\infty} f_j(y)\, \mu_y^j \quad \text{for } \nu\text{-a.e. } y \in Y, \tag{4.95}$$

so that the Theorem is proved. □

5. A Divergence Formula

The previous section led to a definition of a function α, on any $\mathcal{D}$-cylinder $\mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k, C^k)$, as the Radon-Nikodym derivative in (4.67).

In the present section we find that on $\mathscr{C}^k$ the function α satisfies the system of ODEs

$$\partial_{t_\ell}\alpha\Big(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x \cdot \mathrm{e}_i \mathrm{v}_i(x)\Big) = (\operatorname{div} \mathrm{v}_\ell)_{\mathrm{a.c.}}(x)\alpha\Big(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x \cdot \mathrm{e}_i \mathrm{v}_i(x)\Big)$$

for $\ell = 1, \dots, k$, where we assume w.l.o.g. that $0 \in C^k$, $\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$ is an axis of $\mathscr{C}$, $\mathrm{v}_i(x)$ is the vector field

$$x \mapsto \mathbb{1}_{\mathscr{C}^k}(x)(\langle \mathcal{D}(x) \rangle \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{e}_i))$$

and $(\operatorname{div} \mathrm{v}_i)_{\mathrm{a.c.}}(x)$ is the density of the absolutely continuous part of the divergence of v_i, that we prove to be a measure.

This is a consequence of the Disintegration Theorem 4.18 and of the regularity estimates on α in Proposition 4.17.
Notice that even the fact that the divergence of v_i is a measure is not trivial, since the vector field is just Borel.

Heuristically, the ODEs above can be formally derived as follows.
In Section 4 we saw that $\mathscr{C}^k$ is the image of the product space $C^k + Z^k$, where $Z^k = \mathscr{C}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(0)$ is a section of $\mathscr{C}^k$, with the change of variable

$$\Phi(\mathrm{t}+\mathrm{z}) = \mathrm{z} + \sum_{i=1}^k t_i \mathrm{v}_i(\mathrm{z}) = \sigma^{\mathrm{t}}(\mathrm{z}) \qquad \text{for all } \mathrm{t} = \sum_{i=1}^k t_i \mathrm{e}_i \in C^k,\ \mathrm{z} \in Z^k. \tag{5.1}$$

In Theorem 4.18 we found that the weak Jacobian of this change of variable is defined, and given by

$$|\mathfrak{J}(\mathrm{t}+\mathrm{z})| = \alpha(\mathrm{t},0,\mathrm{z}).$$

From (5.1) one finds that, if v_i was smooth instead of only Borel, this Jacobian would be

$$\mathfrak{J}(\mathrm{t}+\mathrm{z}) = \det\left(\left[[\mathrm{v}_j\cdot \mathrm{e}_i]_{\substack{i=1,\dots,n\\ j=1,\dots,k}} \middle| \left[\sum_{\ell=1}^k t_\ell \partial_{z_j}\langle \mathrm{v}_\ell(\mathrm{z})\cdot \mathrm{e}_i\rangle + \delta_{i,j}\right]_{\substack{i=1,\dots,n\\ j=k+1,\dots,n}}\right]\right);$$

by direct computations with Cramer rule and the multilinearity of the determinant, moreover, from the last two equations above one would prove the relation

$$\partial_{t_\ell}\mathfrak{J}(\mathrm{t}+\mathrm{z}) = \mathrm{trace}\left(J\mathrm{v}_\ell(\mathrm{z})\,(J\Phi(\mathrm{t}+\mathrm{z}))^{-1}\right)\mathfrak{J}(\mathrm{t}+\mathrm{z}),$$

where Jg denotes the Jacobian matrix of a function g.
By the Lipschitz regularity of α w.r.t. the $\{t_i\}_{i=1}^k$ variables given in Proposition 4.17, one could then expect that

$$\partial_{t_\ell}\alpha(\mathrm{t},0,\mathrm{z}) = \left(\sum_{j=1}^n \partial_{x_j}(\mathrm{v}_i(\Phi^{-1}(x))\cdot \mathrm{e}_j)|_{x=\Phi(\mathrm{t}+\mathrm{z})}\right)\alpha(\mathrm{t},0,\mathrm{z}). \tag{5.2}$$

Notice that $\sum_j \partial_{x_j}(\mathrm{v}_i(\Phi^{-1}(x))\cdot \mathrm{e}_j)|_{x=\Phi(\mathrm{t}+\mathrm{z})}$ is the pointwise divergence of the vector field $\mathrm{v}_i(\Phi^{-1}(x))$ evaluated at $x=\Phi(\mathrm{t}+\mathrm{z})$. In this article, we denote it with $(\mathrm{div}(\mathrm{v}_i\circ\Phi^{-1}))_{\mathrm{a.c.}}$.
Finally, given a regular domain $\Omega\subset\mathbb{R}^n$, by the Green-Gauss-Stokes formula one should have

$$\int_\Omega (\mathrm{div}(\mathrm{v}_i\circ\Phi^{-1}))_{\mathrm{a.c.}}\,d\mathscr{L}^n(x) = \int_{\partial\Omega}\mathrm{v}_i(\Phi^{-1}(x))\cdot\hat{n}\,d\mathscr{H}^{n-1}(x), \tag{5.3}$$

where $\hat{n}$ is the outer normal to the boundary of Ω.

The analogue of Formulas (5.2) and (5.3) is the additional regularity we prove in this section, in a weak context, for vector fields parallel to the faces and for the current of k-faces. Actually, for simplicity of notations we will continue working with the projection of the faces on $\mathbb{R}^n$ instead of with the faces themselves. We give now the idea of the proof, in the case of one dimensional faces.

Fix the attention on a 1-dimensional $\mathfrak{D}$-cylinder $\mathscr{C}$ with axis e and basis $Z=\mathscr{C}\cap\pi^{-1}_{\langle\mathrm{e}\rangle}(0)$. Consider the distributional divergence of the vector field v giving pointwise on $\mathscr{C}$ the direction of projected faces, normalized with $\mathrm{v}\cdot\mathrm{e}=1$, and vanishing elsewhere. The Disintegration Theorem 4.18 decomposes integrals on $\mathscr{C}$ to integrals first on the projected faces, with the additional density factor α, then on Z. By means of it, one then reduces the integral $\int_{\mathscr{C}}\nabla\varphi\cdot\mathrm{v}$, defining the distributional divergence, to the following integrals on the projected faces:

$$-\int_{[h^-\mathrm{e},h^+\mathrm{e}]}\nabla\varphi(x)|_{x=z+t_1\mathrm{v}(z)}\cdot\mathrm{v}(z)\alpha(\mathrm{t},0,z)\,d\mathscr{H}^1(\mathrm{t}) \qquad \text{where } z \text{ varies in } Z.$$

Since α is Lipschitz in t and $\nabla\varphi|_{x=\sigma^{\mathrm{w}+t_1\mathrm{e}}(z)}\cdot\mathrm{v}=\partial_{t_1}(\varphi\circ\sigma^{\mathrm{w}+\mathrm{t}}(z))$, by integrating by parts one arrives to

$$\int_{[h^-\mathrm{e},h^+\mathrm{e}]}\varphi\circ\sigma^{\mathrm{w}+\mathrm{t}}(z)\partial_{t_1}\alpha(\mathrm{t},0,\mathrm{z})\,d\mathscr{H}^1(\mathrm{t}) - [\varphi\circ\sigma^{\mathrm{w}+\mathrm{t}}(z)\alpha(\mathrm{t},0,z)]\Big|_{\mathrm{t}=h^-\mathrm{e}}^{\mathrm{t}=h^+\mathrm{e}}.$$

Applying again the disintegration theorem in the other direction, by the invertibility of α, one comes back to integrals on the $\mathcal{D}$-cylinder, where in the first addend φ is now integrated with the factor $\partial_{t_1}\alpha/\alpha$.

An argument of this kind yields an explicit representation of the distributional divergence of the truncation of a vector field v, parallel at each point x to the projected face through x, to $\mathscr{C}^k$. This divergence is a Radon measure, the absolutely continuous part is basically given by (5.2) and, as in (5.3), there is moreover a singular term representing the flux through the border of $\mathscr{C}^k$ transversal to $\mathcal{D}$, already defined as

$$\mathfrak{d}\mathscr{C}^k = \mathscr{C}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{rb}(C^k)), \qquad \hat{n}|_{\mathfrak{d}\mathscr{C}^k} \text{ outer unit normal to } \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(C^k). \tag{5.4}$$

As $\mathscr{C}^k$ are not regular sets, but just σ-compact, there is a loss of regularity for the divergence of v in the whole $\mathbb{R}^n$. In general, the distributional divergence will just be a series of measures.

5.1. **Vector fields parallel to the faces.** In the present subsection, we study the regularity of a vector field parallel, at each point, to the corresponding face through that point.

5.1.1. *Study on $\mathcal{D}$-cylinders.* As a preliminary step, fix the attention on the $\mathcal{D}$-cylinder

$$\mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k, C^k).$$

One can assume w.l.o.g. that the axis of $\mathscr{C}^k$ is identified by vectors $\{\mathrm{e}_1,\dots,\mathrm{e}_k\}$ which are the first k coordinate vectors of $\mathbb{R}^n$ and that C^k is the square

$$C^k = \prod_{i=1}^k [-\mathrm{e}_i, \mathrm{e}_i].$$

Denote with Z^k the section $\mathscr{Z}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(0)$.

Definition 5.1 (Coordinate vector fields). We define on $\mathbb{R}^n$ *k-coordinate vector fields* for $\mathscr{C}^k$ as follows:

$$\mathrm{v}_i(x) = \begin{cases} 0 & \text{if } x \notin \mathscr{C}^k \\ \mathrm{v} \in \langle \mathcal{D}(x)\rangle \qquad \text{such that } \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}\mathrm{v} = \mathrm{e}_i & \text{if } x \in \mathscr{C}^k. \end{cases}$$

The k-coordinate vector fields are a basis for the module on the algebra of measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ constituted by the vector fields with values in $\langle \mathcal{D}(x)\rangle$ at each point $x \in \mathscr{C}^k$, and vanishing elsewhere.

Consider the distributional divergence of v_i, denoted by $\operatorname{div}\mathrm{v}_i$. As a consequence of the absolute continuity of the push forward with σ, and by the regularity of the density α, one gains more regularity of the divergence.

Let us fix a notation. Given any vector field $\mathrm{v}:\mathbb{R}^n \to \mathbb{R}^n$ whose distributional divergence is a Radon measure, we will denote with $(\operatorname{div}\mathrm{v})_{\mathrm{a.c.}}$ the density of the absolutely continuous part of the measure $\operatorname{div}\mathrm{v}$.

Lemma 5.2. *The distribution* $\operatorname{div}\mathrm{v}_i$ *is a Radon measure. Its absolutely continuous part has density*

$$(\operatorname{div}\mathrm{v}_i)_{\mathrm{a.c.}}(x) = \frac{\partial_{t_i}\alpha\big(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\big)}{\alpha\big(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\big)} \mathbb{1}_{\mathscr{C}^k}(x). \tag{5.5}$$

Its singular part is $\mathscr{H}^{n-1} \llcorner \big(\mathscr{C}^k \cap \{x\cdot \mathrm{e}_i = -1\}\big) - \mathscr{H}^{n-1} \llcorner \big(\mathscr{C}^k \cap \{x\cdot \mathrm{e}_i = 1\}\big)$.

Proof. Consider any test function $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and apply the Disintegration Theorem 4.18:

$$\langle \operatorname{div}\mathrm{v}_i, \varphi\rangle := -\int_{\mathscr{C}^k} \nabla\varphi(x)\cdot \mathrm{v}_i(x)\, d\mathscr{L}^n(x) = -\int_{Z^k}\int_{C^k} \alpha(\mathrm{t},0,z)\,\nabla\varphi(\sigma^{\mathrm{t}}(z))\cdot \mathrm{v}_i(z)\, d\mathscr{H}^k(\mathrm{t})\, d\mathscr{H}^{n-k}(z),$$

where we used that v_i is constant on the faces, i.e. $\mathrm{v}_i(z) = \mathrm{v}_i(\sigma^{\mathrm{t}}(z))$. Being $\sigma^{\mathrm{t}}(z) = z + \sum_{i=1}^k t_i \mathrm{v}_i(z)$, one has

$$\nabla_x\varphi(x = \sigma^{\mathrm{t}}(z))\cdot \mathrm{v}_i(z) = \nabla_x\varphi(x = \sigma^{\mathrm{t}}(z))\cdot \partial_{t_i}(\sigma^{\mathrm{t}}(z)) = \partial_{t_i}(\varphi(\sigma^{\mathrm{t}}(z))).$$

The inner integral is thus

$$\int_{C^k} \nabla\varphi(\sigma^{\mathrm{t}}(z))\cdot \mathrm{v}_i(z)\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t}) = \int_{C^k} \partial_{t_i}(\varphi(\sigma^{\mathrm{t}}z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t}).$$

Since Proposition 4.17 ensures that α is Lipschitz in t, for $\mathrm{t} \in C^k$, one can integrate by parts:

$$\begin{aligned}\int_{C^k} \partial_{t_i}(\varphi(\sigma^{\mathrm{t}}(z)))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t}) = &-\int_{C^k} \varphi(\sigma^{\mathrm{t}}(z))\partial_{t_i}\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t})\\ &+\int_{C^k\cap\{t_i=1\}} \varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^{k-1}(\mathrm{t})\\ &-\int_{C^k\cap\{t_i=-1\}} \varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^{k-1}(\mathrm{t}).\end{aligned}$$

Substitute in the first expression. Recall moreover the definition of α in (4.67), as a Radon-Nikodym derivative of a push-forward measure, and its invertibility and Lipschitz estimates (Remark 4.16, Proposition 4.17), among with in particular the L^1 estimate on the function $\partial_{t_i}\alpha/\alpha$. Then, pushing the measure from $\mathrm{t} = 0$ to a generic t, one comes back to the integral on the $\mathcal{D}$-cylinder

$$\begin{aligned}\langle \operatorname{div} \mathrm{v}_i, \varphi\rangle = &\int_{Z^k}\int_{C^k} \varphi(\sigma^{\mathrm{t}}(z))\partial_{t_i}\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t})\,d\mathscr{H}^{n-k}(z)\\ &-\int_{Z^k}\int_{C^k\cap\{t_i=1\}} \varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^{k-1}(\mathrm{t})\,d\mathscr{H}^{n-k}(z)\\ &+\int_{Z^k}\int_{C^k\cap\{t_i=-1\}} \varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^{k-1}(\mathrm{t})\,d\mathscr{H}^{n-k}(z)\\ = &\int_{\mathscr{C}^k} \varphi(x)(\operatorname{div} \mathrm{v}_i)_{\mathrm{a.c.}}(x)\,d\mathscr{L}^n(x) - \int_{\mathscr{C}^k\cap\{x\cdot \mathrm{e}_i=1\}} \varphi(x)\,d\mathscr{H}^{n-1}(x) + \int_{\mathscr{C}^k\cap\{x\cdot \mathrm{e}_i=-1\}} \varphi(x)\,d\mathscr{H}^{n-1}(x).\end{aligned}$$

where $(\operatorname{div} \mathrm{v}_i)_{\mathrm{a.c.}}$ is the function $\frac{\partial_{t_i}\alpha}{\alpha}$ precisely written in the statement. Thus we have just proved the thesis, consisting in the last formula. ☐

Remark 5.3. Consider a function $\lambda \in L^1(\mathscr{C}^k;\mathbb{R})$ constant on each face, meaning that $\lambda(\sigma^{\mathrm{t}}(z)) = \lambda(z)$ for $\mathrm{t} \in C^k$ and $z \in Z^k$. One can regard this λ as a function of $\nabla f(x)$. Then the same statement of Lemma 5.2 applies to the vector field $\lambda \mathrm{v}_i$, but the divergence is clearly $\operatorname{div}(\lambda \mathrm{v}_i) = \lambda \operatorname{div} \mathrm{v}_i$. The proof is the same, observing that

$$\begin{aligned}\langle \operatorname{div}(\lambda \mathrm{v}_i), \varphi\rangle &:= -\int_{\mathscr{C}^k} \nabla\varphi(x)\cdot\lambda(x)\mathrm{v}_i(x)\,d\mathscr{L}^n(x)\\ &\overset{4.18}{=} -\int_{Z^k}\int_{C^k} \lambda(z)\nabla\varphi(\sigma^{\mathrm{t}}(z))\cdot \mathrm{v}_i(z)\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t})\,d\mathscr{H}^{n-k}(z)\\ &= -\int_{Z^k}\int_{C^k} \lambda(z)\partial_{t_i}(\varphi(\sigma^{\mathrm{t}}(z)))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t})\,d\mathscr{H}^{n-k}(z)\\ &= \int_{Z^k}\int_{C^k} \lambda(z)\varphi(\sigma^{\mathrm{t}}(z))\partial_{t_i}\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t})\,d\mathscr{H}^{n-k}(z)\\ &\qquad -\int_{Z^k}\int_{C^k\cap\{t_i=1\}} \lambda(z)\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^{k-1}(\mathrm{t})\,d\mathscr{H}^{n-k}(z)\\ &\qquad +\int_{Z^k}\int_{C^k\cap\{t_i=-1\}} \lambda(z)\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^{k-1}(\mathrm{t})\,d\mathscr{H}^{n-k}(z)\\ &\overset{4.18}{=} \int_{\mathscr{C}^k} \varphi(x)\lambda(x)(\operatorname{div} \mathrm{v}_i)_{\mathrm{a.c.}}(x)\,d\mathscr{L}^n(x) - \int_{\mathscr{C}^k\cap\{x\cdot \mathrm{e}_i=1\}} \varphi(x)\lambda(x)\,d\mathscr{H}^{n-1}(x)\\ &\qquad + \int_{\mathscr{C}^k\cap\{x\cdot \mathrm{e}_i=-1\}} \varphi(x)\lambda(x)\,d\mathscr{H}^{n-1}(x).\end{aligned} \tag{5.6}$$

Suitably adapting the integration by parts in the above equality (5.6) with

$$\begin{aligned}&\int_{C^k} \lambda(\sigma^{\mathrm{t}}(z))\partial_{t_i}(\varphi(\sigma^{\mathrm{t}}(z)))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t}) =\\ &-\int_{C^k} \lambda(\sigma^{\mathrm{t}}(z))\varphi(\sigma^{\mathrm{t}}(z))\partial_{t_i}\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t}) - \int_{C^k} \partial_{t_i}\lambda(\sigma^{\mathrm{t}}(z))\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^k(\mathrm{t})\\ &+\int_{C^k\cap\{t_i=1\}} \lambda(\sigma^{\mathrm{t}}(z))\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^{k-1}(\mathrm{t}) - \int_{C^k\cap\{t_i=-1\}} \lambda(\sigma^{\mathrm{t}}(z))\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\,d\mathscr{H}^{k-1}(\mathrm{t})\end{aligned}$$

one finds moreover that for all $\lambda \in L^1(\mathbb{R}^n;\mathbb{R})$ continuously differentiable along v_i with integrable directional derivative $\partial_{\mathrm{v}_i}\lambda$, the following relation holds:

$$\operatorname{div}(\lambda \mathrm{v}_i) = \lambda \operatorname{div} \mathrm{v}_i + \partial_{\mathrm{v}_i}\lambda \, d\mathscr{L}^n \tag{5.7}$$

Notice that in (5.7) there is the addend $\lambda\mathscr{H}^{n-1} \llcorner (\mathscr{C}^k \cap \{x\cdot \mathrm{e}_i = 1\})$, which would make no sense for a general $\lambda \in L^1(\mathbb{R}^n;\mathbb{R})$. Now we prove that the restriction to $\mathscr{C}^k \cap \{x \cdot \mathrm{e}_i = 1\}$ of each representative of λ which is $\mathcal{C}^1(F^k_{\nabla f(z)} \cap \mathscr{C}^k)$, for $\mathscr{H}^{n-k}$-a.e. $z \in Z^k$, identifies the same function in $L^1(\mathscr{C}^k \cap \{x\cdot \mathrm{e}_i = 1\})$. Indeed, any two representatives $\tilde\lambda$, $\hat\lambda$ of the L^1-class of λ can differ only on a $\mathscr{L}^n$-negligible set N. By the Disintegration Theorem 4.18, and using moreover Fubini theorem for reducing the integral on C^k to integrals on lines parallel to e_i, one has that the intersection of N with each of the lines on the projected faces with projection on $\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle$ parallel to e_i is almost always negligible:

$$\mathscr{H}^1\big(N \cap \{q + \langle \mathrm{v}_i(q)\rangle\}\big) = 0 \qquad \text{for } q \in \mathscr{C}^k \cap \{x\cdot \mathrm{e}_i = 0\}\setminus M, \text{ with } \mathscr{H}^{n-1}(\mathrm{M}){=}0.$$

Being continuously differentiable along v_i, one can redefine $\tilde\lambda$, $\hat\lambda$ in such a way that $N \cap \{q + \langle \mathrm{v}_i(q)\rangle\} = \emptyset$ for all $q \in \mathscr{C}^k \cap \{x \cdot \mathrm{e}_i = 0\} \setminus M$. As a consequence $N \cap \{x\cdot \mathrm{e}_i = t\}$ is a subset of $\tau^{t\mathrm{e}_i}(M)$, where $\tau^{t\mathrm{e}_i}$ is the map moving along each projected face with $t\mathrm{v}_i$:

$$\mathscr{C}^k \cap \{x\cdot \mathrm{e}_i = 0\} \ni q \mapsto \tau^{t\mathrm{e}_i}(q) := q + t\mathrm{v}_i = \sigma^{(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(q))+t\mathrm{e}_1}(q).$$

By the push forward formula (4.67), denoting $\mathrm{w}_q := \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(q)$ and $z_q := \pi_{\langle \mathrm{e}_{k+1},\dots,\mathrm{e}_n\rangle}(q)$

$$\mathscr{H}^{n-1}\llcorner(\tau^{t\mathrm{e}_i}(S)) = \alpha(\mathrm{w}_q, \mathrm{w}_q + t\mathrm{e}_i, z_q)\tau^{t\mathrm{e}_i}_\sharp(\mathscr{H}^{n-1}(q)\llcorner S)) \qquad \text{for } S \subset \mathscr{C}^k \cap \{x\cdot \mathrm{e}_1 = 0\}.$$

Therefore, as $\mathscr{H}^{n-1}(M) = 0$, one has that $\tilde\lambda$ and $\hat\lambda$ identify the same integrable function on each section of $\mathscr{C}^k$ perpendicular to e_i, showing that the measure $\lambda\mathscr{H}^{n-1}\llcorner(\{x\cdot \mathrm{e}_i = 1\})$ is well defined.

Actually, the same argument as above should be used in (5.6) in order to show that $\lambda(z)$ is integrable on Z^k, so that one can separate the three integrals as we did. Indeed, being constant on each face by assumption, the restriction of λ to a section is trivially well defined as associating to a point the value of λ corresponding to the face of that point, but the integrability w.r.t. $\mathscr{H}^{n-1}$ on each slice is a consequence of the push forward estimate.

As a direct consequence of (5.7), by linearity, one gets a divergence formula for any sufficiently regular vector field which, at each point of $\mathscr{C}^k$, is parallel to the corresponding projected face of f.

Corollary 5.4. *Consider any vector field $\mathrm{v} = \sum_{i=1}^k \lambda_i \mathrm{v}_i$ with $\lambda_i \in L^1(\mathscr{C}^k;\mathbb{R})$ continuously differentiable along v_i, with directional derivative $\partial_{\mathrm{v}_i}\lambda_i$ integrable on $\mathscr{C}^k$. Then the divergence of v is a Radon measure and for every $\varphi \in \mathcal{C}^1_c(\mathbb{R}^n)$*

$$\langle \operatorname{div} \mathrm{v}, \varphi\rangle = \int_{\mathscr{C}^k} \varphi(x)(\operatorname{div}\mathrm{v})_{\mathrm{a.c.}}(x)\, d\mathscr{L}^n(x) - \int_{\mathfrak{d}\mathscr{C}^k} \varphi(x)\,\mathrm{v}(x)\cdot \hat n(x)\, d\mathscr{H}^{n-1}(x),$$

where $\mathfrak{d}\mathscr{C}^k$, the border of $\mathscr{C}^k$ transversal to $\mathcal{D}$, and $\hat n$, the outer unit normal, are define in Formula (5.4). Moreover, for $x \in \mathscr{C}^k$

$$(\operatorname{div}\mathrm{v})_{\mathrm{a.c.}}(x) = \sum_{i=1}^k \lambda_i(x) \frac{\partial_{t_i}\alpha(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x))}{\alpha(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i\mathrm{v}_i(x))} + \sum_{i=1}^k \partial_{\mathrm{v}_i}\lambda_i(x). \tag{5.8}$$

Remark 5.5. The result is essentially based on the application of the integration by parts formula when the integral on $\mathscr{C}^k$ is reduced, by the Disintegration Theorem, to integrals on C^k: this is why we assume the C^1 regularity of the λ_i, w.r.t. the directions of the k-face passing through each point of $\mathscr{C}^k$. Such regularity could be further weakened, however we do not pursue this issue here. As a consequence, one can easily extend the statement of the previous corollary to sets of the form $\mathscr{C}^k_\Omega = F^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_n\rangle}(\overline{\Omega})$, for an open set $\Omega \subset \langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle$ with piecewise Lipschitz boundary, defining $\mathfrak{d}\mathscr{C}^k_\Omega := F^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_n\rangle}(\mathrm{rb}(\Omega))$.

5.1.2. *Global Version.* We study now the distributional divergence of an integrable vector field v on $\mathcal{T}$, as we did in Subsection 5.1.1 for such a vector field truncated on $\mathcal{D}$-cylinders.

Corollary 5.6. *Consider a vector field $\mathrm{v} \in L^1(\mathcal{T};\mathbb{R}^n)$ such that $\mathrm{v}(x) \in \langle \mathcal{D}(x)\rangle$ for $x \in \mathbb{R}^n$, where we define $\mathcal{D}(x) = 0$ for $x \notin \mathcal{T}$. Suppose moreover that the restriction to every face E_y, for $y \in \operatorname{Im}\nabla f$, is*

continuously differentiable with integrable derivatives.
Then, for every $\varphi \in \mathcal{C}^1_c(\mathbb{R}^n)$ one can write

$$\langle \operatorname{div} \mathrm{v}, \varphi \rangle = \lim_{\ell\to\infty} \sum_{i=1}^{\ell} \left\{ \int_{\mathscr{C}_i} \varphi(x)(\operatorname{div}(\mathbb{1}_{\mathscr{C}_i}\mathrm{v}))_{\mathrm{a.c.}}(x)\, d\mathscr{L}^n(x) - \int_{\mathfrak{d}\mathscr{C}_i} \varphi(x)\,\mathrm{v}(x)\cdot \hat{n}_i(x)\, d\mathscr{H}^{n-1}(x) \right\}. \tag{5.9}$$

where $\{\mathscr{C}_\ell\}_{\ell\in\mathbb{N}}$ is the countable partition of $\mathcal{T}$ in $\mathcal{D}$-cylinders given in Lemma 4.6, while $(\operatorname{div}(\mathbb{1}_{\mathscr{C}_i}\mathrm{v}))_{\mathrm{a.c.}}$ is the one of Corollary 5.4 and $\mathfrak{d}\mathscr{C}_i$, $\hat{n}_i$ are defined in Formula (5.4).

Remark 5.7. By construction of the partition, each of the second integrals in the r.h.s. of (5.9) appears two times in the series, with opposite sign. Intuitively, the finite sum of these border terms is the integral on a perimeter which tends to the singular set.

Remark 5.8. Suppose that $\operatorname{div}\mathrm{v}$ is a Radon measure. Then Corollary 5.6 implies that

$$\mathbb{1}_{\mathscr{C}^k}(\operatorname{div}\mathrm{v})_{\mathrm{a.c.}} \equiv (\operatorname{div}(\mathbb{1}_{\mathscr{C}^k}\mathrm{v}))_{\mathrm{a.c.}}.$$

Proof of Corollary 5.6. The partition of $\cup_{k=1}^n E^k$ into such sets $\{\mathscr{C}_\ell\}_{\ell\in\mathbb{N}}$ is given exactly by Lemma 4.6. Moreover, Lemma 4.19 shows that the set $\mathcal{T}\setminus\cup_{k=1}^n E_k$ is Lebesgue negligible. Therefore, by dominated convergence theorem one finds that

$$\langle \operatorname{div}\mathrm{v}, \varphi\rangle = -\int_{\mathcal{T}} \mathrm{v}(x)\cdot\nabla\varphi(x)\, d\mathscr{L}^n(x) = -\lim_{\ell\to\infty}\sum_{i=1}^{\ell}\int_{\mathscr{C}_i} \mathrm{v}(x)\cdot\nabla\varphi(x)\, d\mathscr{L}^n(x).$$

The addends in the r.h.s. are, by definition, the distributional divergence of the vector fields $\mathrm{v}\mathbb{1}_{\mathscr{C}_i}$ applied to φ. In particular, by Corollary 5.4, they are equal to

$$-\int_{\mathscr{C}_i} \mathrm{v}(x)\cdot\nabla\varphi(x)\, d\mathscr{L}^n(x) = \int_{\mathscr{C}_i} \varphi(x)\big(\operatorname{div}\mathrm{v}\big)_{a.c.}(x)\, d\mathscr{L}^n(x) + \int_{\mathfrak{d}\mathscr{C}_i^k} \varphi(x)\,\mathrm{v}(x)\cdot\hat{n}_i(x)\, d\mathscr{H}^{n-1}(x),$$

proving the thesis. □

5.2. **The currents of k-faces.** In the present subsection, we change point of view. Instead of looking at vector fields constrained to the faces of f, we regard the k-dimensional faces of f as a k-dimensional current. We establish that this current is a locally flat chain, providing a sequence of normal currents converging to it in the mass norm. The border of these normal currents has the same representation one would have in a smooth setting.

Before proving it, we devote Subsection 5.2.1 to recalls on this argument, in order to fix the notations. They are taken mainly from Chapter 4 of [Mor00] and Sections 1.5.1, 4.1 of [Fed69].

5.2.1. *Recalls.* Let $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ be a basis of $\mathbb{R}^n$. The **wedge product** between vectors is multilinear and alternating, i.e.:

$$\Big(\sum_{i=1}^n \lambda_i \mathrm{e}_i\Big)\wedge u_1\wedge\cdots\wedge u_m = \sum_{i=1}^n \lambda_i(\mathrm{e}_i\wedge u_1\wedge\cdots\wedge u_m) \qquad m\in\mathbb{N},\ \lambda_1,\dots,\lambda_n\in\mathbb{R}$$
$$u_0\wedge\cdots\wedge u_i\wedge\cdots\wedge u_m = (-1)^i u_i\wedge u_0\wedge\cdots\wedge\widehat{u_i}\wedge\cdots\wedge u_m \qquad 0<i\le m,\ u_0,\dots,u_m\in\mathbb{R}^n,$$

where the element under the hat is missing. The space of all linear combinations of

$$\big\{\mathrm{e}_{i_1\dots i_m} := \mathrm{e}_{i_1}\wedge\cdots\wedge\mathrm{e}_{i_m} :\ i_1<\cdots<i_m \text{ in } \{1,\dots,n\}\big\}$$

is the space of **m-vectors**, denoted by $\Lambda_m\mathbb{R}^n$. The space $\Lambda_0\mathbb{R}$ is just $\mathbb{R}$. $\Lambda_m\mathbb{R}^n$ has the **inner product** given by

$$\mathrm{e}_{i_1\dots i_m}\cdot\mathrm{e}_{j_1\dots j_m} = \prod_{k=1}^m \delta_{i_k j_k} \qquad \text{where } \delta_{ij} = \begin{cases} 1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}.$$

The induced **norm** is denoted by $\|\cdot\|$. An **m-vector field** is a map $\xi:\mathbb{R}^n\to\Lambda_m\mathbb{R}^n$.

The dual Hilbert space to $\Lambda_m\mathbb{R}^n$, denoted by $\Lambda^m\mathbb{R}^n$, is the space of **m-covectors**. The element dual to $\mathrm{e}_{i_1\dots i_m}$ is denoted by $d\mathrm{e}_{i_1\dots i_m}$. A **differential m-form** is a map $\omega:\mathbb{R}^n\to\Lambda^m\mathbb{R}^n$.

We denote with $\langle\cdot,\,\cdot\rangle$ the duality pairing between m-vectors and m-covectors. Moreover, the same symbol denotes in this paper the bilinear **pairing**, which is a map $\Lambda^p\mathbb{R}^n\times\Lambda_q\mathbb{R}^n\to\Lambda^{p-q}\mathbb{R}^n$ for $p>q$ and $\Lambda^p\mathbb{R}^n\times\Lambda_q\mathbb{R}^n\to\Lambda_{q-p}\mathbb{R}^n$ for $q>p$ whose non-vanishing images on a basis are

$$\mathrm{de}_{i_1\dots i_\ell}=\big\langle \mathrm{de}_{i_1\dots i_\ell}\wedge\mathrm{de}_{i_{\ell+1}\dots i_{\ell+m}},\,\mathrm{e}_{i_{\ell+1}\dots i_{\ell+m}}\big\rangle \qquad \text{if } p=\ell+m>m=q$$

$$\mathrm{e}_{i_{\ell+1}\dots i_{\ell+m}}=\big\langle \mathrm{de}_{i_1\dots i_\ell},\,\mathrm{e}_{i_1\dots i_\ell}\wedge\mathrm{e}_{i_{\ell+1}\dots i_{\ell+m}}\big\rangle \qquad \text{if } p=\ell<\ell+m=q.$$

Consider any differential m-form

$$\omega=\sum_{i_1\dots i_m}\omega_{i_1\dots i_m}\,\mathrm{de}_{i_1\dots i_m}$$

which is differentiable. The **exterior derivative** $d\omega$ of ω is the differential $(m+1)$-form

$$d\omega=\sum_{i_1\dots i_m}\sum_{j=1}^n\frac{\partial\omega_{i_1\dots i_m}}{\partial x_j}\,\mathrm{de}_j\wedge\mathrm{de}_{i_1\dots i_m}.$$

If $\omega\in\mathcal{C}^i(\mathbb{R}^n;\Lambda^m\mathbb{R}^n)$, the i-th exterior derivative is denoted with $d^i\omega$.
Consider any m-vector field

$$\xi=\sum\xi_{i_1\dots i_m}\,\mathrm{e}_{i_1\dots i_m}$$

which is differentiable. The **pointwise divergence** $(\operatorname{div}\xi)_{\text{a.c.}}$ of ξ is the $(m-1)$-vector field

$$(\operatorname{div}\xi)_{\text{a.c.}}=\sum_{i_1\dots i_m}\sum_{j=1}^n\frac{\partial\xi_{i_1\dots i_m}}{\partial x_j}\,\big\langle\mathrm{de}_j,\,\mathrm{e}_{i_1\dots i_m}\big\rangle.$$

Consider the space $\mathscr{D}^m$ of $\mathcal{C}^\infty$-differential m-form with compact support. The topology is generated by the seminorms

$$\nu^i_K(\phi)=\sup_{x\in K,\,0\le j\le i}\|d^j\phi(x)\|\qquad\text{with } K \text{ compact subset of } \mathbb{R}^n,\ i\in\mathbb{N}.$$

The dual space to $\mathscr{D}^m$, endowed with the weak topology, is called the space of m**-dimensional currents** and it is denoted by $\mathscr{D}_m$. The support of a current $T\in\mathscr{D}_m$ is the smallest close set $K\subset\mathbb{R}^n$ such that $T(\omega)=0$ whenever $\omega\in\mathscr{D}^m$ vanishes out of K. The **mass** of a current $T\in\mathscr{D}_m$ is defined as

$$\mathbf{M}(T)=\sup\Big\{T(\omega):\ \omega\in\mathscr{D}^m,\ \sup_{x\in\mathbb{R}^n}\|\omega(x)\|\le1\Big\}.$$

The **flat norm** of a current $T\in\mathscr{D}_m$ is defined as

$$\mathbf{F}(T)=\sup\Big\{T(\omega):\ \omega\in\mathscr{D}^m,\ \sup_{x\in\mathbb{R}^n}\|\omega(x)\|\le1,\ \sup_{x\in\mathbb{R}^n}\|d\omega(x)\|\le1\Big\}.$$

An m-dimensional current $T\in\mathscr{D}_m$ is **representable by integration**, and we denote it by $T=\mu\wedge\xi$, if there exists a Radon measure μ over $\mathbb{R}^n$ and a μ-locally integrable m-vector field ξ such that

$$T(\omega)=\int_{\mathbb{R}^n}\big\langle\omega,\,\xi\big\rangle\,d\mu\qquad\forall\omega\in\mathscr{D}^m.$$

If $m\ge1$, the **boundary** of an m-dimensional current T is defined as

$$\partial T\in\mathscr{D}_{m-1},\qquad\big(\partial T\big)(\omega)=T(d\omega)\text{ whenever }\omega\in\mathscr{D}^{m-1}.$$

If either $m=0$, or both T and ∂T are representable by integration, then we will call T **locally normal**. If T is locally normal and compactly supported, then T is called **normal**. The **F**-closure, in $\mathscr{D}_m$, of the normal currents is the space of **locally flat chains**. Its subspace of currents with finite mass is the **M**-closure, in $\mathscr{D}_m$, of the normal currents.

To each $\mathscr{L}^n$-measurable m-vector field ξ such that $\|\xi\|$ is locally integrable there corresponds the current $\mathscr{L}^n\wedge\xi\in\mathscr{D}_m(\mathbb{R}^n)$. If ξ is of class $\mathcal{C}^1$, then this current is locally normal and the divergence of ξ is related to the boundary of the corresponding current by

$$-\partial\big(\mathscr{L}^n\wedge\xi\big)=\mathscr{L}^n\wedge(\operatorname{div}\xi)_{\text{a.c.}},$$

Moreover, if Ω is an open set with $\mathcal{C}^1$ boundary, $\hat n$ is its outer unit normal and $d\hat n$ the dual of $\hat n$, then

$$\partial\big(\mathscr{L}^n\wedge(\mathbb{1}_\Omega\xi)\big)=-(\mathscr{L}^n\llcorner\Omega)\wedge(\operatorname{div}\xi)_{\text{a.c.}}+(\mathscr{H}^{n-1}\llcorner\partial\Omega)\wedge\big\langle d\hat n,\,\xi\big\rangle.\tag{5.10}$$

In the next subsection, we are going to find the analogue of the Green-Gauss Formula (5.10) for the k-dimensional current associated to k-faces, restricted to $\mathcal{D}$-cylinders. In order to do this, we will re-define the function $(\operatorname{div}\xi)_{\text{a.c.}}$ for a less regular k-vector field and this definition will be an extension of the above one.

5.2.2. *Divergence of the Current of k-Faces on $\mathcal{D}$-cylinders.* As a preliminary study, restrict again the attention to a $\mathcal{D}$-cylinder as in Subsection 5.1.1, and keep the notation we had there.

The k-faces, restricted to $\mathscr{C}^k$, define a k-vector field

$$\xi(x) = \mathbb{1}_{\mathscr{C}^k}\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_k.$$

In general, this vector field does not enjoy much regularity. Nevertheless, as a consequence of the study of Section 4, one can find a representation of $\partial(\mathscr{L}^n \wedge \xi)$ like the one in a regular setting, (5.10). This involves the density α of the push-forward with σ which was studied before, see (4.67).

Lemma 5.9. *Consider a function λ such that it is continuously differentiable on each face and assume $\mathscr{C}^k$ bounded.*
Then, the k-dimensional current $(\mathscr{L}^n \wedge \lambda\xi)$ is normal and the following formula holds

$$\partial(\mathscr{L}^n \wedge \lambda\xi) = -\mathscr{L}^n \wedge (\operatorname{div}\lambda\xi)_{\text{a.c.}} + (\mathscr{H}^{n-1} \llcorner \mathfrak{d}\mathscr{C}^k) \wedge \langle d\hat{n}, \lambda\xi\rangle,$$

where $\mathfrak{d}\mathscr{C}^k$, $\hat{n}$ are defined in (5.4), $d\hat{n}$ is the differential 1-form at each point dual to the vector field $\hat{n}$, and $(\operatorname{div}\lambda\xi)_{\text{a.c.}}$ is defined here as

$$(\operatorname{div}\lambda\xi)_{\text{a.c.}} := \sum_{i=1}^{k}(-1)^{i+1}(\operatorname{div}\lambda\mathrm{v}_i)_{\text{a.c.}}\,\mathrm{v}_1 \wedge \cdots \wedge \widehat{\mathrm{v}}_i \wedge \cdots \wedge \mathrm{v}_k$$

with the functions $(\operatorname{div}\mathrm{v}_i)_{\text{a.c.}}$ of (5.5):

$$(\operatorname{div}\lambda\mathrm{v}_i)_{\text{a.c.}}(x) = \left(\lambda(x)\frac{\partial_{t_i}\alpha\big(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\ldots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i\mathrm{v}_i(x)\big)}{\alpha\big(\pi_{\langle \mathrm{e}_1,\ldots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i\mathrm{v}_i(x)\big)} + \partial_{\mathrm{v}_i}\lambda(x)\right)\mathbb{1}_{\mathscr{C}^k}(x).$$

Proof. Actually, this is consequence of Corollary 5.4 in Subsection 5.1.1, reducing to computations in coordinates. One has to verify the equality of the two currents on a basis.

For simplicity, consider first

$$\omega = \phi\, d\mathrm{e}_2 \wedge \cdots \wedge d\mathrm{e}_k.$$

with $\phi \in \mathcal{C}^1(\mathbb{R}^n)$. Then

$$d\omega = \partial_{x_1}\phi\, d\mathrm{e}_1 \wedge \cdots \wedge d\mathrm{e}_k + \sum_{i=k+1}^{n}\partial_{x_i}\phi\, d\mathrm{e}_i \wedge \cdots \wedge d\mathrm{e}_n,$$

$$\langle d\omega, \xi\rangle = \nabla\phi\cdot \mathrm{v}_1 \qquad \big\langle \omega,\, (\operatorname{div}\lambda\xi)_{\text{a.c.}}\big\rangle = (\operatorname{div}\lambda\mathrm{v}_1)_{\text{a.c.}}\phi \qquad \big\langle \omega,\, \langle d\hat{n},\, \xi\rangle\big\rangle = \phi\,\hat{n}\cdot \mathrm{e}_1$$

and the thesis reduces exactly to Lemma 5.2, and Remark 5.3:

$$\partial(\mathscr{L}^n \wedge \lambda\xi)(\omega) := \int_{\mathscr{C}^k}\langle d\omega, \lambda\xi\rangle\, d\mathscr{L}^n \overset{5.2}{=} -\int_{\mathscr{C}^k}\big\langle \omega,\, (\operatorname{div}\lambda\xi)_{\text{a.c.}}\big\rangle\, d\mathscr{L}^n + \int_{\mathfrak{d}\mathscr{C}^k}\big\langle \omega,\, \langle d\hat{n},\, \lambda\xi\rangle\big\rangle\, d\mathscr{H}^{n-1}$$
$$=: -\mathscr{L}^n \wedge (\operatorname{div}\lambda\xi)_{\text{a.c.}} + (\mathscr{H}^{n-1} \llcorner \mathfrak{d}\mathscr{C}^k) \wedge (\hat{n} \wedge \lambda\xi).$$

The same lemma applies with $(-1)^{i+1}\mathrm{v}_i$ instead of v_1 if

$$\omega = \phi\, d\mathrm{e}_1 \wedge \cdots \wedge \widehat{d\mathrm{e}_i} \wedge \cdots \wedge d\mathrm{e}_k,$$

since the following formulas hold:

$$\langle d\omega, \xi\rangle = (-1)^{i+1}\nabla\phi\cdot \mathrm{v}_i \qquad \big\langle \omega,\, (\operatorname{div}\lambda\xi)_{\text{a.c.}}\big\rangle = (-1)^{i+1}(\operatorname{div}\lambda\mathrm{v}_i)_{\text{a.c.}}\phi \qquad \big\langle \omega,\, \langle d\hat{n},\, \xi\rangle\big\rangle = (-1)^{i+1}\phi\,\hat{n}\cdot \mathrm{e}_i.$$

Let us show the equality more in general. By a direct computation, one can verify that

$$\mathrm{v}_1 \wedge \cdots \wedge \widehat{\mathrm{v}}_i \wedge \cdots \wedge \mathrm{v}_k = \sum_{h=0}^{k-1}\ \sum_{\substack{k<i_{h+1}<\ldots\\ \cdots<i_{k-1}\le n}}\ \sum_{\substack{\sigma\in S(1\ldots\hat{\imath}\ldots k-1)\\ \sigma(1)<\cdots<\sigma(h)}} \operatorname{sgn}\sigma\, \mathrm{v}_{\sigma(h+1)}^{i_{h+1}}\cdots \mathrm{v}_{\sigma(k-1)}^{i_{k-1}}\, \mathrm{e}_{\sigma(1)\ldots\sigma(h)i_{h+1}\ldots i_{k-1}},$$

where v_i^j is the j-th component of v_i, $S(1\ldots\hat{\imath}\ldots k)$ denotes the group of permutation of the integers $\{1,\ldots,\hat{\imath},\ldots,k\}$, with i is missing, and, if $\sigma \in S(1\ldots\hat{\imath}\ldots k)$, $\operatorname{sgn}\sigma$ is 1 if the permutation is even, -1 otherwise.

On the other hand, consider now a $(k-1)$ form $\omega = \phi\, de_{i_1\dots i_h} \wedge de_{i_{h+1}\dots i_{k-1}}$, where $1 \le i_1 < \dots < i_h \le k$, and $k < i_{h+1} < \dots < i_{k-1} \le n$. Then, again by direct computation,

$$\langle d\omega, \xi\rangle = \sum_{\substack{\sigma\in S(1\dots k)\\ \sigma(2)=i_1,\dots,\sigma(h+1)=i_h}} (\nabla\phi \cdot \mathrm{v}_{\sigma(1)}) \operatorname{sgn}\sigma\, \mathrm{v}^{i_{h+1}}_{\sigma(h+2)} \dots \mathrm{v}^{i_{k-1}}_{\sigma(k)},$$

$$\begin{aligned}\langle \omega, (\operatorname{div}\lambda\xi)_{\text{a.c.}}\rangle &= \phi \sum_{i=1}^{k} (-1)^{i+1} (\operatorname{div}\lambda \mathrm{v}_i)_{\text{a.c.}} \sum_{\substack{\sigma\in S(1\dots\hat{\imath}\dots k-1)\\ \sigma(1)=i_1,\dots,\sigma(h)=i_h}} \operatorname{sgn}\sigma\, \mathrm{v}^{i_{h+1}}_{\sigma(h+1)} \dots \mathrm{v}^{i_{k-1}}_{\sigma(k-1)}\\ &= \sum_{\substack{\sigma\in S(1\dots k)\\ \sigma(2)=i_1,\dots,\sigma(h+1)=i_h}} (\phi \cdot (\operatorname{div}\lambda \mathrm{v}_{\sigma(1)})_{\text{a.c.}}) \operatorname{sgn}\sigma\, \mathrm{v}^{i_{h+1}}_{\sigma(h+2)} \dots \mathrm{v}^{i_{k-1}}_{\sigma(k)},\end{aligned}$$

and finally

$$\begin{aligned}\langle \omega, \langle d\hat{n}, \xi\rangle\rangle &= \sum_{i=1}^{k} (-1)^{i+1} (\hat{n}\cdot \mathrm{e}_i) \langle \omega, \mathrm{v}_1 \wedge \dots \wedge \widehat{\mathrm{v}}_i \wedge \dots \wedge \mathrm{v}_k\rangle\\ &= \sum_{\substack{\sigma\in S(1\dots k)\\ \sigma(2)=i_1,\dots,\sigma(h+1)=i_h}} (\phi\, \hat{n}\cdot \mathrm{v}_{\sigma(1)}) \operatorname{sgn}\sigma\, \mathrm{v}^{i_{h+1}}_{\sigma(h+2)} \dots \mathrm{v}^{i_{k-1}}_{\sigma(k)}\end{aligned}$$

Therefore the thesis reduces to Corollary 5.4, being each v^i_j constant on each face. □

5.2.3. *Divergence of the current of k-faces in the whole space.* In the previous section, we considered a k-dimensional current $(\mathscr{L}^n \llcorner \mathscr{C}^k) \wedge \xi$ identified by the restriction to a $\mathcal{D}$-cylinder $\mathscr{C}^k$ of the k-faces of f, projected on $\mathbb{R}^n$. We established the formula analogous to (5.10) for the border of this current, which is representable by integration w.r.t. the measures $\mathscr{L}^n \llcorner \mathscr{C}^k$ and $\mathscr{H}^{n-1} \llcorner \mathfrak{d}\mathscr{C}^k$. In particular, when $\mathscr{C}^k$ is bounded it is a normal current.
Moreover, we have related the density of the absolutely continuous part to the function α by

$$(\operatorname{div}\xi)_{\text{a.c.}} = \sum_{i=1}^{k} (-1)^{i+1} \frac{\partial_{t_i}\alpha\big(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\big)}{\alpha\big(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, \sum_{i=1}^k x - x\cdot \mathrm{e}_i \mathrm{v}_i(x)\big)} \mathbb{1}_{\mathscr{C}^k}(x)\, \mathrm{v}_1 \wedge \dots \wedge \widehat{\mathrm{v}}_i \wedge \dots \wedge \mathrm{v}_k.$$

We observe now that the partition we of $\mathbb{R}^n$ into the sets $\{F^k\}_{k=1}^n$, and the remaining set that we call now $\widetilde{F}^0$, define a $(n+1)$-uple of currents. The elements of this $(n+1)$-uple are described by the following statement, which is basically Corollary 5.6 when rephrased in this setting.

Corollary 5.10. *Let $\{\mathscr{C}^k_\ell\}_{\ell\in\mathbb{N}}$ be a countable partition of E^k in $\mathcal{D}$-cylinders as in Lemma 4.6 and, up to a refinement of the partition, assume moreover that the $\mathcal{D}$-cylinders are bounded.*
Consider a k-vector field $\xi_k \in L^1(\mathbb{R}^n; \Lambda_k\mathbb{R}^n)$ corresponding, at each point $x \in E^k$, to the k-plane $\langle \mathcal{D}(x)\rangle$, and vanishing elsewhere. Assume moreover that it is continuously differentiable if restricted to any set $E^k_{\nabla f(x)}$, with locally integrable derivatives, meaning more precisely that $\xi_k \circ \sigma^{\mathrm{w}_\ell + \mathrm{t}}(z)$ belongs to $L^1_{\mathscr{H}^{n-k}(z)}(Z^k_\ell; \mathcal{C}^1_{\mathrm{t}}(C^k; \Lambda_k\mathbb{R}^n))$ for each ℓ.
Then, the k-dimensional current $\mathscr{L}^n \wedge \xi_k$ is a locally flat chain, since it is the limit in the flat norm of normal currents: indeed, for $k > 0$ one has

$$\partial\big(\mathscr{L}^n \wedge \xi_k\big) = \mathbf{F}\text{-}\lim_{\ell} \sum_{i=1}^{\ell} \Big\{ -\mathscr{L}^n \wedge (\operatorname{div}(\mathbb{1}_{\mathscr{C}^k_i}\xi_k))_{\text{a.c.}} + \big(\mathscr{H}^{n-1} \llcorner \mathfrak{d}\mathscr{C}^k_i\big) \wedge \langle d\hat{n}_i, \xi_k\rangle \Big\},$$

where $(\operatorname{div} \mathbb{1}_{\mathscr{C}^k_i}\xi_k)_{\text{a.c.}}$ is the one of Lemma 5.9, $\mathfrak{d}\mathscr{C}^k_i$, the border of $\mathscr{C}^k_i$ transversal to $\mathcal{D}$, and $\hat{n}_i$, the outer unit normal, are defined in Formula (5.4), and $d\hat{n}_i$ is the dual to $\hat{n}_i$.

Notice finally that the current $\mathscr{L}^n \wedge \xi_k$ is itself locally normal if restricted to the interior of E^k. However, in general E^k can have empty interior. If $\partial\big(\mathscr{L}^n \wedge \xi_k\big)$ is representable by integration, then the density of its absolutely continuous part w.r.t. $\mathscr{L}^n$, at any point $x \in \mathscr{C}^k_\ell$, is given by $\operatorname{div}(\mathbb{1}_{\mathscr{C}^k_\ell}\xi_k)_{\text{a.c.}}(x)$.

Table of Notations

The following table collects some of the notations in the article.

$\mathscr{B}(\mathbb{R}^d)$	Borel sets in $\mathbb{R}^d$
$\mathscr{L}^d$	d-dimensional Lebesgue measure
$\mathscr{H}^d$	d-dimensional Hausdorff outer measure
(X, Σ, μ)	$\Sigma = \sigma$-algebra of subsets of X and $\mu =$ measure on Σ, i.e. $\mu : \Sigma \to [0, +\infty]$, $\mu(\emptyset) = 0$ and μ is countably additive on disjoint sets of Σ
$L^1_{(\mathrm{loc})}(\mu)$	(locally) integrable functions (w.r.t. μ)
$L^\infty_{(\mathrm{loc})}$	(locally) essentially bounded functions
$\mathcal{C}^k_{(\mathrm{c})}$	k-times continuously differentiable functions (with compact support)
$\mathbb{1}_A$	$\mathbb{1}_A(x) = 1$ if $x \in A$, $\mathbb{1}_A(x) = 0$ otherwise
$\mu \llcorner A$	restriction of a measure μ to a set A
$\mu = \int \mu_\alpha \, d\nu$	disintegration of μ, see Definition 2.1
$\mu \ll \nu$	$\mu(A) = 0$ whenever $\nu(A) = 0$ (absolute continuity of a measure μ w.r.t. ν)
equivalent	μ is eqivalent to ν if $\mu \ll \nu$ and $\nu \ll \mu$
separated	two sets A and B sets are separated if each is disjoint from the other's closure
perpendicular	A set A is perpendicular to an affine plane H of $\mathbb{R}^d$ if $\exists\, w \in H$ s.t. $\pi_H(A) = w$
$v \cdot w$	Euclidean scalar product in $\mathbb{R}^n$
$\lVert\cdot\rVert$	Euclidean norm in $\mathbb{R}^n$
$\mathbf{S}^{n-1}, \mathbf{B}^n$	$\{x \in \mathbb{R}^n : \lVert x\rVert = 1\}$, $\{x \in \mathbb{R}^n : \lVert x\rVert \leq 1\}$
$\mathbf{G}(k, n)$	Grassmaniann of k-dimensional vector spaces in $\mathbb{R}^n$
π_L	orthogonal projection from $\mathbb{R}^d$ to the affine plane $L \subset \mathbb{R}^d$
$\langle \cdot, \cdot \rangle$	pairing, see Subsection 5.2.1
$\langle \mathrm{v}_1, \dots, \mathrm{v}_k \rangle$	linear span of vectors $\{\mathrm{v}_1, \dots, \mathrm{v}_k\}$ in $\mathbb{R}^n$
$\mathrm{aff}(A)$	affine hull of A, the smallest affine plane containing A
$\mathrm{conv}(A)$	convex envelope of A, the smallest convex set containing A
$\dim(A)$	linear dimension of $\mathrm{aff}(A)$
$\mathrm{ri}(C)$	relative interior of C, the interior of C w.r.t. the topology of $\mathrm{aff}(C)$
$\mathrm{rb}(C)$	relative boundary of C, the boundary of C w.r.t. the topology of $\mathrm{aff}(C)$
R-face	see Definition 4.11
extreme points	zero-dimensional R-faces
$\mathrm{ext}(C)$	extreme points of a convex set C
$\mathrm{dom}\, g$	the domain of a function g
$\mathrm{graph}\, g$	$\{(x, g(x)) : x \in \mathrm{dom}\, g\}$ (graph)
$\mathrm{epi}\, g$	$\{(x, t) : x \in \mathrm{dom}\, g,\ t \geq g(x)\}$ (epigraph)

∇g	gradient of g
$\partial^- g$	subdifferential of g, see Page 6
$g\lvert_a$	evaluation of g at the point a
$g\lvert_b^a$	the difference $g(b) - g(a)$
$g_{\lvert A}$	the restriction of g to a subset A of $\mathrm{dom}\, g$
f	a fixed convex function $\mathbb{R}^n \to \mathbb{R}$
$\mathrm{dom}\, \nabla f$	a fixed σ-compact set where f is differentiable, see Subsection 3.1
$\mathrm{Im}\, \nabla f$	$\{\nabla f(x) : x \in \mathrm{dom}\, \nabla f\}$, see Subsection 3.1
face of f	intersection of graph $f_{\lvert \mathrm{dom}\, \nabla f}$ with a tangent hyperplane
k-face of f	k-dimensional face of f
F_y	$\nabla f^{-1}(y) = \{x \in \mathrm{dom}\, \nabla f : \nabla f(x) = y\}$
F_y^k	F_y when $\dim(F_y) = k$, $\quad k = 0, \dots, n$
E_y, E_y^k	the sets, respectively, $\mathrm{ri}(F_y)$ and $\mathrm{ri}(F_y^k)$
E^k, F^k	the sets, respectively, $\underset{y}{\cup} E_y^k$ and $\underset{y}{\cup} F_y^k$
$\mathcal{P}(x)$	see Formula (4.10)
$\mathcal{R}(x)$	$F_{\nabla f(x)}$, for every $x \in \mathrm{dom}\, \nabla f$
$\mathcal{T}$	$\{x \in \mathrm{dom}\, \nabla f : \mathcal{R}(x) \neq \{x\}\}$
$\mathcal{D}$	multivalued map of unit faces directions, see Formula (4.11)
Z^k	section of a sheaf set, see Definition 4.3
$\mathscr{Z}^k$	sheaf set, see Definition 4.3
$[\mathrm{v}, \mathrm{w}]$	segment that connects v to w, i.e. $\{(1 - \lambda)\mathrm{v} + \lambda \mathrm{w} : \lambda \in [0, 1]\}$
$\Pi_{i=1}^k [\mathrm{v}_i, \mathrm{w}_i]$	k-dimensional rectangle in $\mathbb{R}^n$ with sides parallel to $\{[\mathrm{v}_i, \mathrm{w}_i]\}_{i=1}^k$, equal to the convex envelope of $\{\mathrm{v}_i, \mathrm{w}_i\}_{i=1}^k$
$\mathscr{C}^k(\mathscr{Z}^k, C^k)$	k-dimensional $\mathcal{D}$-cylinder $\mathscr{C}^k$, see Definition 4.5
$\mathfrak{d}\mathscr{C}^k$, $\hat{n}_{\lvert \mathfrak{d}\mathscr{C}^k}$	border of $\mathscr{C}^k$ transversal to $\mathcal{D}$ and outer unit normal, see Formula (5.4)
$\sigma^{\mathrm{w}+t\mathrm{e}}$	a map which parametrizes a $\mathcal{D}$-cylinder $\mathscr{C}^k(\mathscr{Z}^k, C^k)$, see Formula (4.28)
$\sigma^{t\mathrm{e}}$	$\sigma^{t\mathrm{e}} = \sigma^{0+t\mathrm{e}}$, where $\mathrm{e} \in \mathbf{S}^{n-1}$, $t \in \mathbb{R}$
σ^{t}	if we write $\mathrm{t} = t\mathrm{e}$ with e a unit direction, then $\sigma^{\mathrm{t}} = \sigma^{0+t\mathrm{e}}$
$\alpha(t, s, x)$	see Formula (4.67)
$\mathrm{div}\, \mathrm{v}$	if $\mathrm{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, its divergence is the distribution $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R}^n) \ni \varphi \mapsto -\int \mathrm{v} \cdot \nabla \varphi$
$(\mathrm{div}\, \mathrm{v})_{\mathrm{a.c.}}$	see Notation 5.1.1, Formula (5.8)
v_i	see Definition 5.1
$(\mathrm{div}\, \mathrm{v}_i)_{\mathrm{a.c.}}$	see Formula (5.5)

We avoid to recall here the notation on tensors and currents, which is the matter of Subsection 5.2.1.

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Acknowledgements

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Laura Caravenna, S.I.S.S.A., via Beirut 4, 34014 Trieste, Italy
E-mail address: l.caravenna@sissa.it

Sara Daneri, S.I.S.S.A., via Beirut 4, 34014 Trieste, Italy
E-mail address: daneri@sissa.it