A Γ-CONVERGENCE APPROACH TO STABILITY OF UNILATERAL MINIMALITY PROPERTIES IN FRACTURE MECHANICS AND APPLICATIONS

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Abstract. We prove a stability result for a large class of unilateral minimality properties which arise naturally in the theory of crack propagation proposed by Francfort and Marigo in [22]. Then we give an application to the quasistatic evolution of cracks in composite materials.

The main tool in the analysis is a Γ-convergence result for energies of the type

$$\mathcal{E}_n(u,K) := \int_{\Omega\setminus K} f_n(x,\nabla u(x))\,dx + \int_{S(u)\setminus K_n} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x),$$

where $S(u)$ is the jump set of u and $(K_n)_{n\in\mathbb{N}}$ is a sequence of rectifiable sets with $\mathcal{H}^{N-1}(K_n) \le M$. We prove that no interaction occur in the Γ-limit process between the bulk and the surface part of the energy, and relying on this result, we introduce a new notion of convergence for $(N-1)$-rectifiable sets called σ-convergence, which is useful in the study of the stability of unilateral minimality properties.

Keywords : variational models, energy minimization, free discontinuity problems, Γ-convergence, quasistatic crack propagation, homogenization, composite materials.

2000 Mathematics Subject Classification: 35R35, 35J85, 35J25, 74R10, 35B27, 74E30.

Introduction

In this paper we deal with the problem of stability of *unilateral minimality properties* with varying volume and surface energies, and we give an application to the study of crack propagation in composite materials.

Let K be a $(N-1)$-dimensional set contained in $\Omega \subseteq \mathbb{R}^N$, and let u be a possibly vector valued function on Ω whose discontinuities are contained in K and which is sufficiently regular outside K. We say that the pair (u,K) is a *unilateral minimizer* with respect to the energy densities f and g if

$$\int_{\Omega\setminus K} f(x,\nabla u(x))\,dx + \int_K g(x,\nu)\,d\mathcal{H}^{N-1}(x) \le \int_{\Omega\setminus H} f(x,\nabla v(x))\,dx + \int_H g(x,\nu)\,d\mathcal{H}^{N-1}(x), \tag{0.1}$$

for every $(N-1)$-dimensional set H containing K, and for every function v whose discontinuities are contained in H and which is sufficiently regular outside H. Here ν stands for the normal vector to K and H at the point x, while $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure. (u,K) is said to be *unilateral* minimizer because it is a minimum only among pairs (v,H) with H larger than K.

The unilateral minimality property (0.1) is a key point in the theory of quasistatic crack evolution in elastic bodies proposed by Francfort and Marigo in [22], which is inspired by the classical Griffith's criterion of crack propagation. In the framework of [22], Ω represents an hyperelastic body in the reference configuration, u is its deformation, and K represents a crack inside Ω across which the deformation u may jump. The total energy of the configuration (u,K) is given by

$$\mathcal{E}(u,K) := \int_{\Omega\setminus K} f(x,\nabla u(x))\,dx + \int_K g(x,\nu)\,d\mathcal{H}^{N-1}(x). \tag{0.2}$$

The first term is referred to as the *bulk energy* of the body, while the second term is referred to as the *surface energy* of the crack. The presence of x in f and g takes into account possible inhomogeneities, while the presence of the normal ν in g takes into account a possible anisotropy of the body.

Following [22], if Ω is subject to a time dependent loading process, a quasistatic crack evolution can be described by a pair $(u(t), K(t))$ where the crack $K(t)$ grows in time, $(u(t), K(t))$ satisfies the unilateral minimality property (0.1) at each time t, and the total energy (0.2) evolves in relation with the power of external loads in such a way that no dissipation occurs.

The unilateral minimality property (0.1) can be interpreted as a static equilibrium property along the irreversible process of crack growth. In fact an immediate consequence of (0.1) is that $u(t)$ is the elastic deformation in $\Omega \setminus K(t)$ associated to the external load. As for the crack $K(t)$, (0.1) states a minimality condition only among enlarged cracks (unilateral minimality), taking thus into account the irreversibility of the process. Together with non dissipation, and under some regularity assumptions on the cracks, the unilateral minimality property implies that the Griffith's criterion is satisfied along the evolution (see [19]).

In [22] Francfort and Marigo suggest that the quasistatic evolution $(u(t), K(t))$ during the loading process can be obtained as a limit of a discretized in time evolution $(u_n(t), K_n(t))$ which by construction satisfies at each time the unilateral minimality property (0.1). We are thus led to a problem of *stability* for unilateral minimizers, i.e., if the minimality property (0.1) is conserved in the passage from $(u_n(t), K_n(t))$ to $(u(t), K(t))$.

The first mathematical result of stability for unilateral minimality properties has been obtained by Dal Maso and Toader [19] in a two dimensional setting under a topological restriction on the admissible cracks. They consider compact cracks with a bound on the number of their connected components, and converging with respect to the Hausdorff metric. An extension of this result to unilateral minimality properties involving the symmetrized gradient of planar elasticity is due to Chambolle [16], while an extension to higher order minimality properties in connection to quasistatic crack growth in a plate has been proved by Acanfora and Ponsiglione in [1].

A second result of stability for unilateral minimality properties has been obtained by Francfort and Larsen in [21], where they give an existence result for quasistatic crack evolutions in the context of SBV functions. In the framework of *generalized antiplanar shear* (i.e., $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$), the authors consider cracks K which are rectifiable sets in $\overline{\Omega}$, and associated displacements u in $SBV(\Omega)$ with jump set $S(u)$ contained in K. A key point for their result is the stability for unilateral minimizers of the form $(u_n, S(u_n))$ with bulk energy given by $f(x,\xi) = |\xi|^2$ and surface energy given by $g(x,\nu) \equiv 1$. More precisely, writing the unilateral minimality property in the equivalent form

$$\int_\Omega |\nabla u_n|^2 \, dx \leq \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S(v) \setminus S(u_n)) \qquad \text{for all } v \in SBV(\Omega)$$

(which corresponds to (0.1) with $H = S(u_n) \cup S(v)$), they prove that if $u_n \rightharpoonup u$ weakly in $SBV(\Omega)$ (see Section 1 for a definition), then u satisfies the same minimality property. The main tool for proving this stability result is a geometrical construction which they called Transfer of Jump Sets [21, Theorem 2.1].

The case in which $S(u_n)$ is replaced by a rectifiable set K_n has been treated by Dal Maso, Francfort and Toader in [18], where they consider also a Carathéodory bulk energy $f(x,\xi)$ quasiconvex and with p growth estimates in ξ, and a Borel surface energy $g(x,\nu)$ bounded and bounded away from zero. They employ a variational notion of convergence for rectifiable sets which they called σ^p-convergence to recover a crack K in the limit (see Section 5), and they prove a Transfer of Jump Sets theorem for $(K_n)_{n\in\mathbb{N}}$ satisfying $\mathcal{H}^{N-1}(K_n) \leq C$ [18, Theorem 5.1] in order to prove that minimality is preserved.

In this paper we provide a different approach to the problem of stability of unilateral minimality properties based on Γ-convergence which permits also to treat the case of varying bulk and surface energy densities f_n and g_n. We restrict our analysis to the scalar case. Our approach is based on the observation that the problem has a variational character. In fact, considering for a while the case of fixed energy densities f and g with f convex in ξ, we have that if (u_n, K_n) is a unilateral minimizer for the energy (0.2), then u_n is a minimum for the functional

$$\mathcal{E}_n(v) := \int_\Omega f(x, \nabla v)\,dx + \int_{S(v)\setminus K_n} g(x,\nu)\,d\mathcal{H}^{N-1}(x).$$

Then the problem of stability of unilateral minimizers can be treated in the framework of Γ-convergence which ensures the convergence of minimizers. In Section 4, using an abstract representation result by Bouchitté, Fonseca, Leoni and Mascarenhas [10], we prove that the Γ-limit (up to a subsequence) of the functional $\mathcal{E}_n$ can be represented as

$$\mathcal{E}(v) := \int_\Omega f(x, \nabla v)\,dx + \int_{S(v)} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x),$$

where g^- is a suitable function defined on $\Omega \times S^{N-1}$ determined only by g and $(K_n)_{n\in\mathbb{N}}$, and such that $g^- \le g$. If we assume that $u_n \rightharpoonup u$ weakly in $SBV(\Omega)$, then by Γ-convergence we get that u is a minimizer for $\mathcal{E}$. Suppose now that K is a rectifiable set in Ω such that $S(u) \subseteq K$ and

$$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K. \tag{0.3}$$

Then we have immediately that the pair (u, K) is a unilateral minimizer for f and g because for all pairs (v, H) with $S(v) \subseteq H$ and $K \subseteq H$ we have

$$\begin{aligned}\int_\Omega f(x, \nabla u(x))\,dx = \mathcal{E}(u) \le \mathcal{E}(v) = \int_\Omega f(x,\nabla v(x))\,dx + \int_{S(v)} g^-(x,\nu)\,d\mathcal{H}^{N-1} \\ = \int_\Omega f(x,\nabla v(x))\,dx + \int_{S(v)\setminus K} g^-(x,\nu) \le \int_\Omega f(x,\nabla v(x))\,dx + \int_{H\setminus K} g(x,\nu).\end{aligned} \tag{0.4}$$

The rectifiable set K satisfying (0.3) is constructed in Section 5, where we define a new variational notion of convergence for rectifiable sets which we call *σ-convergence*, and which departs from the notion of σ^p-convergence given in [18]. The σ-limit K of a sequence of rectifiable sets $(K_n)_{n\in\mathbb{N}}$ is constructed looking for the Γ-limit $\mathcal{H}^-$ in the strong topology of $L^1(\Omega)$ of the functionals

$$\mathcal{H}_n^-(u) := \begin{cases} \mathcal{H}^{N-1}(S(u)\setminus K_n) & u \in P(\Omega), \\ +\infty & \text{otherwise,}\end{cases}$$

where $P(\Omega)$ is the space of piecewise constant function in Ω (see (1.1)). Roughly, the σ-limit K is the maximal rectifiable set on which the density h^- representing $\mathcal{H}^-$ vanishes. By the growth estimate on g it turns out that K is also the maximal rectifiable set on which the density g^- vanishes, so that K is the natural limit candidate for K_n in order to preserve the unilateral minimality property. The definition of σ-convergence involves only the surface energies $\mathcal{H}_n^-$, and as a consequence it does not depend on the exponent p and it is stable with respect to infinitesimal perturbations in length (see Remark 5.2). Moreover it turns out that the σ-limit K contains the σ^p-limit points of $(K_n)_{n\in\mathbb{N}}$, so that our Γ-convergence approach improves also the minimality property given by the previous approaches.

Our method naturally extends to the case of varying bulk and surface energy densities f_n and g_n, and this is indeed the main motivation for which we developed our Γ-convergence approach. The key point to recover the effective energy densities f and g for the minimality property in the limit and to repeat the chain of inequalities (0.4) is a Γ-convergence result for functionals of the form

$$\int_\Omega f_n(x, \nabla u_n(x))\,dx + \int_{S(u_n)\setminus K_n} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x), \tag{0.5}$$

where $(K_n)_{n\in\mathbb{N}}$ is a sequence of rectifiable sets such that $\mathcal{H}^{N-1}(K_n) \le C$. In Section 4, we prove that the Γ-limit has the form

$$\int_\Omega f(x, \nabla u(x))\,dx + \int_{S(u)} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x), \tag{0.6}$$

where f is determined only by $(f_n)_{n\in\mathbb{N}}$, and g^- is determined only by $(g_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$, that is no interaction occurs between the bulk and the surface part of the functionals in the Γ-convergence process. A result of this type has been proved in the case of periodic homogenization (in the vectorial case, and with dependence on the trace of u in the surface part of the energy, but without removing K_n from $S(u_n)$) by Braides, Defranceschi and Vitali [12].

In order to prove the integral representation (0.6), we use the result of Bouchitté, Fonseca, Leoni and Mascarenhas [10] to represent the Γ-limit of (0.5) in an integral form through a volume energy density f_∞ and a surface energy density g_∞ (see (4.1)). Then employing a blow-up analysis we prove that f_∞ is the density of the Γ-limit of the functionals (0.5) restricted to Sobolev functions, while g_∞ is the density of the Γ-limit of the functionals (0.5) restricted to characteristic functions of sets with finite perimeter. This characterization immediately implies the non-interaction between bulk and surface energies. In the blow-up analysis we need to replace SBV-functions with vanishing jump set with Sobolev functions (Step 2 in the proof of Theorem 4.1), and SBV-functions with vanishing gradient with characteristic functions of sets with finite perimeter (Step 4 in the proof of Theorem 4.1). The first operation is done using suitable variants of Lusin Approximation Theorem for SBV-function [7, Theorem 5.36] which can be found, e.g., in [25]. The second one requires a careful use of Coarea formula for BV functions and of Fubini's Theorem. Coarea formula has been largely employed in the proof of lower semicontinuity results for functionals on SBV since the pioneering paper by Ambrosio [3] (in connection with the notion of BV-ellipticity for surface energy densities). Our use of Fubini's Theorem (we need it to achieve precise boundary conditions) is inspired by the proof of the Transfer of Jump Sets Theorem [21, Theorem 2.3] by Francfort and Larsen.

We notice that an approach to stability in the line of Dal Maso, Francfort and Toader in the case of varying energies needs a Transfer of Jump Sets for f_n, g_n and f, g, which seems difficult to be derived without any Γ-convergence argument. Our approach also provides this result (Theorem 6.4).

In Section 8 we deal with the study of quasistatic crack evolution in composite materials. More precisely we study the asymptotic behavior of a quasistatic evolution $t \to (u_n(t), K_n(t))$ relative to the bulk energy density f_n and the surface energy density g_n. Using our stability result we prove (Theorem 8.1) that $t \to (u_n(t), K_n(t))$ converges to a quasistatic evolution $t \to (u(t), K(t))$ relative to the effective bulk and surface energy densities f and g. Moreover convergence for bulk and surface energies at every time holds. This analysis applies to the case of composite materials, i.e., materials obtained through a fine mixture of different phases. The model case is that of periodic homogenization, i.e., materials with total energy given by

$$\mathcal{E}_\varepsilon(u, K) := \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx + \int_K g\left(\frac{x}{\varepsilon}, \nu\right) d\mathcal{H}^{N-1}(x),$$

where ε is a small parameter giving the size of the mixture, and f, g are periodic in x. Our result implies that a quasisistatic crack evolution $t \to (u_\varepsilon(t), K_\varepsilon(t))$ for ε small is very near to a quasistatic evolution for the homogeneous material having bulk and surface energy densities $f_{\rm hom}$ and $g_{\rm hom}$, which are obtained from f and g through periodic homogenization formulas available in the literature (see for example [12]).

The paper is organized as follows. In Section 1 we make precise the functional setting of the problem. In Section 2 we prove a blow up result for Γ-limits which will be employed in the proof of the main results. In Section 3 we prove some representation results which we use in Section 4 where we deal with the Γ-convergence of free discontinuity functionals (0.5). The notion of σ-convergence for rectifiable sets is contained in Section 5, while the main result on stability for unilateral minimizers is contained in Section 6. In Section 7 we prove a stability result for unilateral minimality properties with boundary conditions which will be employed in Section 8 for the study of quasistatic crack evolution in composite materials.

1. The functional setting of the problem

In this section we introduce the precise functional setting for the study of the unilateral minimality property (0.1). Throughout the paper we suppose that Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary, and we denote by $\mathcal{A}(\Omega)$ the family of its open subsets.

In the unilateral minimality property (0.1), we consider $(N-1)$-dimensional sets which are rectifiable, i.e., contained up to a set of $\mathcal{H}^{N-1}$-measure zero in the union of a sequence of C^1-hypersurfaces of $\mathbb{R}^N$. We will use the following notation: given K_1, K_2 rectifiable sets in $\mathbb{R}^N$, we say that $K_1 \tilde{\subseteq} K_2$ if $K_1 \subseteq K_2$ up to a set of $\mathcal{H}^{N-1}$-measure zero; similarly we say that $K_1 \tilde{=} K_2$ if $K_1 = K_2$ up to a set of $\mathcal{H}^{N-1}$-measure zero.

Given $1 < p < +\infty$, the functions in (0.1) belong to the space $SBV^p(\Omega)$ defined as

$$SBV^p(\Omega) := \{u \in SBV(\Omega) \,:\, \nabla u \in L^p(A, \mathbb{R}^N), \mathcal{H}^{N-1}(S(u)) < +\infty\}.$$

For the notations and the general theory concerning the function space $SBV(\Omega)$ (*special functions of bounded variation*), we refer the reader to [7]. We will consider *weak convergence* in $SBV^p(\Omega)$ defined in the following way: $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$ if

$$\begin{aligned} &u_n \to u \quad \textit{strongly in } L^1(\Omega),\\ &\nabla u_n \rightharpoonup \nabla u \quad \textit{weakly in } L^p(\Omega; \mathbb{R}^N),\\ &\mathcal{H}^{N-1}(S(u_n)) \le C. \end{aligned}$$

We indicate by $P(\Omega)$ the family of sets with finite perimeter in Ω, that is the class of sets $E \subseteq \Omega$ such that $1_E \in SBV(\Omega)$. In view of the applications of Sections 3, 4 and 5, it will be useful to look at $P(\Omega)$ in term of functions, that is to use the following equivalent description:

$$P(\Omega) = \{u \in SBV(\Omega) \,:\, u(x) \in \{0,1\} \text{ for a.e. } x \in \Omega\}. \tag{1.1}$$

2. Blow-up for Γ-limits

In this section we state some blow-up results for Γ-convergent sequences of integral functionals $\mathcal{F}_n(u)$ defined in (2.2) which will be used in Section 4. Moreover under additional hypothesis on $\mathcal{F}_n$, we obtain a regularity result for the density of the Γ-limit $\mathcal{F}$ which will be employed in Section 8. For the definition and the basic properties of Γ-convergence, we refer the reader to [17].

Let $1 < p < +\infty$ and let $f : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function such that

$$a_1(x) + \alpha|\xi|^p \le f(x, \xi) \le a_2(x) + \beta|\xi|^p, \tag{2.1}$$

where $a_1, a_2 \in L^1(\Omega)$ and $\alpha, \beta > 0$. Let us assume that

$$\xi \to f(x, \xi) \quad \text{is convex for a.e. } x \in \Omega.$$

Let B_1 be the unit ball in $\mathbb{R}^N$ with center 0 and radius 1. The following blow up result in the sense of Γ-convergence is a direct consequence of the Scorza-Dragoni theorem for Carathéodory functions and of [17, Theorem 5.14].

Lemma 2.1. *There exists $N \subseteq \Omega$ with $|N| = 0$ such that for every $x \in \Omega \setminus N$ and for every sequence $(\rho_k)_{k \in \mathbb{N}}$ converging to zero, the functionals*

$$F_k(u) := \begin{cases} \int_{B_1} f(x + \rho_k y, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \textit{otherwise in } L^1(B_1) \end{cases}$$

Γ-converge in the strong topology of $L^1(B_1)$ to the functional

$$F(u) := \begin{cases} \int_{B_1} f(x, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \textit{otherwise in } L^1(B_1). \end{cases}$$

Let us consider now $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ Carathéodory functions satisfying the growth estimate (2.1) uniformly in n, and let $\mathcal{F}_n : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be defined as

$$\mathcal{F}_n(u, A) := \begin{cases} \int_A f_n(x, \nabla u(x))\, dx & u \in W^{1,p}(A), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

Let us assume (and this is always true up to a subsequence, see Proposition 3.1) that for all $A \in \mathcal{A}(\Omega)$ $\mathcal{F}_n(\cdot, A)$ Γ-converge with respect to the strong topology of $L^1(\Omega)$ to a functional $\mathcal{F}(\cdot, A)$ such that for all $u \in W^{1,p}(\Omega)$

$$\mathcal{F}(u, A) := \int_A f(x, \nabla u(x))\, dx \tag{2.3}$$

for some Carathéodory function f (independent of u and A) which satisfies estimate (2.1). Using Lemma 2.1 and a diagonal argument we conclude that the following proposition holds.

Proposition 2.2. *There exists $N \subseteq \Omega$ with $|N| = 0$ such that for every $x \in \Omega \setminus N$ and for every sequence $(\rho_k)_{k\in\mathbb{N}}$, there exists $(n_k)_{k\in\mathbb{N}}$ (possibly depending on x) such that the functionals*

$$F_k(u) := \begin{cases} \int_{B_1} f_{n_k}(x + \rho_k y, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1) \end{cases}$$

Γ-converge in the strong topology of $L^1(B_1)$ to the functional

$$F(u) := \begin{cases} \int_{B_1} f(x, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1). \end{cases}$$

Remark 2.3. In the case of periodic homogenization, i.e., in the case in which $f_n(x, \xi) := f(nx, \xi)$ with f periodic in x, it is sufficient to choose n_k in such a way that $n_k \rho_k \to +\infty$. In fact for $x = 0$ we have

$$F_k(u) := \begin{cases} \int_{B_1} f((n_k\rho_k)y, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1) \end{cases}$$

which still Γ-converges to (see for instance [17])

$$F(u) := \begin{cases} \int_{B_1} f_{\text{hom}}(\nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1). \end{cases}$$

In the rest of the section we prove a regularity result for the density f defined in (2.3) under additional hypothesis on f_n which will be employed in Section 8. Let us assume that for a.e. $x \in \Omega$

(a) $f_n(x, \cdot)$ is convex;

(b) $f_n(x, \cdot)$ is of class C^1;

(c) for all $M \geq 0$ and for all ξ_n^1, ξ_n^2 such that $|\xi_n^1| \leq M$, $|\xi_n^2| \leq M$, $|\xi_n^1 - \xi_n^2| \to 0$ we have

$$|\nabla_\xi f_n(x, \xi_n^1) - \nabla_\xi f_n(x, \xi_n^2)| \to 0. \tag{2.4}$$

Notice that for instance $f_n(x, \xi) := a_n(x)|\xi|^p$ with $\alpha \leq a_n(x) \leq \beta$ satisfies the assumptions above. Notice moreover that by lower semicontinuity of Γ-limits $\xi \to f(x, \xi)$ is convex for a.e. $x \in \Omega$.

We need the following lemma which is a straightforward variant of [18, Lemma 4.9].

Lemma 2.4. *Let (X, A, μ) be a finite measure space, $p > 1$, $N \geq 1$, and let $H_n : X \times \mathbb{R}^N \to \mathbb{R}$ be a sequence of Carathéodory functions which satisfy the following properties: there exist a positive constant $a \geq 0$ and a nonnegative function $b \in L^{p'}(X)$, with $p' := p/(p-1)$ such that*

(1) *$|H_n(x, \xi)| \leq a|\xi|^{p-1} + b(x)$ for every $x \in X$, $\xi \in \mathbb{R}^N$;*

(2) *for all $M \geq 0$ and for a.e. $x \in \Omega$, for all ξ_n^1, ξ_n^2 such that $|\xi_n^1| \leq M$, $|\xi_n^2| \leq M$, $|\xi_n^1 - \xi_n^2| \to 0$ we have*

$$|H_n(x, \xi_n^1) - H_n(x, \xi_n^2)| \to 0.$$

Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is bounded in $L^p(X, \mathbb{R}^N)$ and that $(\Psi_n)_{n\in\mathbb{N}}$ converges to 0 strongly in $L^p(X, \mathbb{R}^N)$. Then

$$\int_X [H_n(x, \Phi_n(x) + \Psi_n(x)) - H_n(x, \Phi_n(x))]\Phi(x)\, d\mu(x) \to 0,$$

for every $\Phi \in L^p(X, \mathbb{R}^N)$.

The following regularity result on f holds.

Proposition 2.5. *For a.e. $x \in \Omega$ the function $\xi \to f(x,\xi)$ is of class C^1.*

Proof. According to Proposition 2.2, let $x \in \Omega$, $\rho_k \to 0$ and $(n_k)_{k\in\mathbb{N}}$ be such that $(F_k)_{k\in\mathbb{N}}$ Γ-converges with respect to the strong topology of $L^1(B_1)$ to F.

Let $(\phi_k)_{k\in\mathbb{N}}$ be a recovering sequence for the affine function $y \to \xi\cdot y$ with $\xi \in \mathbb{R}^N$. Up to a further subsequence, we can always assume that there exists $\psi \in \mathbb{R}^N$ such that

$$\frac{1}{|B_1|}\int_{B_1} \nabla_\xi f_{n_k}(x+\rho_k y, \nabla\phi_k(y))\,dy \to \psi. \tag{2.5}$$

Let $t_j \searrow 0$ and let $\eta \in \mathbb{R}^N$. By the convexity of f_{n_k} in the second variable, we have

$$\int_{B_1} f_{n_k}(x+\rho_k y, \nabla\phi_k(y)+t_j\eta) - f_{n_k}(x+\rho_k y, \nabla\phi_k(y))\,dy$$
$$\le t_j \int_{B_1} \nabla_\xi f_{n_k}(x+\rho_k y, \nabla\phi_k(y)+t_j\eta)\eta\,dy.$$

By Γ-convergence we can find k_j such that

$$\frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} - \frac{1}{j} \le \frac{1}{|B_1|}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j}y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\,dy,$$

so that we have

$$\limsup_{j\to+\infty} \frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} \le \frac{1}{|B_1|}\limsup_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j}y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\,dy. \tag{2.6}$$

Notice that by Lemma 2.4 and by (2.5) we have that

$$\lim_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j}y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\,dy$$
$$= \lim_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j}y, \nabla\phi_{k_j}(y))\eta\,dy = |B_1|\psi\eta,$$

and so for every subgradient ζ of $f(x,\cdot)$ at ξ by (2.6) we have

$$\zeta\eta \le \limsup_{j\to+\infty}\frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} \le \psi\eta.$$

We deduce that $\zeta = \psi$, so that $f(x,\cdot)$ is Gateaux differentiable at ξ with $\nabla_\xi f(x,\xi) = \psi$: since $f(x,\cdot)$ is convex, we get that $f(x,\cdot)$ is of class C^1. □

Remark 2.6. An hypothesis of *equiuniform continuity* for $(\nabla_\xi f_n(x,\xi))_{n\in\mathbb{N}}$ like (2.4) is needed in order to preserve C^1-regularity in the passage from f_n to f. Otherwise it is easy to provide a counterexample considering $\xi \to f_n(\xi)$ smooth convex functions uniformly converging to a non differentiable convex function $\xi \to f(\xi)$, and noting that the associated functionals Γ-converge.

3. Some integral representation lemmas

Let $a_1, a_2 \in L^1(\Omega)$, $1 < p < +\infty$, and let $\alpha, \beta > 0$. For all $n \in \mathbb{N}$ let $f_n : \Omega\times\mathbb{R}^N \to [0,+\infty[$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$

$$a_1(x) + \alpha|\xi|^p \le f_n(x,\xi) \le a_2(x) + \beta|\xi|^p, \tag{3.1}$$

and let $g_n : \Omega \times S^{N-1} \to [0,+\infty[$ be a Borel function such that for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ and for all $\nu \in S^{N-1} := \{\eta \in \mathbb{R}^N : |\eta| = 1\}$

$$\alpha \le g_n(x,\nu) \le \beta. \tag{3.2}$$

In Section 4 we will be interested in the functionals on $L^1(\Omega)\times\mathcal{A}(\Omega)$

$$\mathcal{E}_n(u,A) := \begin{cases} \int_A f_n(x,\nabla u(x))\,dx + \int_{A\cap(S(u)\setminus K_n)} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x) & u \in SBV^p(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.3}$$

where $\mathcal{A}(\Omega)$ denotes the family of open subsets of Ω, and $(K_n)_{n\in\mathbb{N}}$ is a sequence of rectifiable sets in Ω such that

$$\mathcal{H}^{N-1}(K_n) \le C.$$

In particular we will be interested in the Γ-limit in the strong topology of $L^1(\Omega)$ of $(\mathcal{E}_n(\cdot,A))_{n\in\mathbb{N}}$ for every $A \in \mathcal{A}(\Omega)$. To this extend we consider the functionals $\mathcal{F}_n : L^1(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$

$$\mathcal{F}_n(u,A) := \begin{cases} \int_A f_n(x,\nabla u(x))\,dx & u \in W^{1,p}(A), \\ +\infty & \text{otherwise}, \end{cases} \tag{3.4}$$

and the functionals $\mathcal{G}_n^- : P(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty[$

$$\mathcal{G}_n^-(u,A) := \int_{A\cap(S(u)\setminus K_n)} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{3.5}$$

defined respectively on Sobolev and piecewise constant functions with values in $\{0,1\}$ (see (1.1)) respectively. We will reconstruct the Γ-limit of $(\mathcal{E}_n(\cdot,A))_{n\in\mathbb{N}}$ through the Γ-limits of $(\mathcal{F}_n(\cdot,A))_{n\in\mathbb{N}}$ and $(\mathcal{G}_n^-(\cdot,A))_{n\in\mathbb{N}}$.

For the results of Section 6, we will need also the functionals $\mathcal{G}_n : P(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty[$

$$\mathcal{G}_n(u,A) := \int_{A\cap S(u)} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x). \tag{3.6}$$

In the following, given $\mathcal{H}$ defined on $X\times\mathcal{A}(\Omega)$ with values in $[0,+\infty]$, where $X = L^1(\Omega)$ or $X = P(\Omega)$, following [10] we set for every $\psi \in L^1(A)$ and $A \in \mathcal{A}(\Omega)$

$$\mathbf{m}_{\mathcal{H}}(\psi,A) = \inf_{u\in X}\{\mathcal{H}(u,A) \,:\, u=\psi \text{ in a neighborhood of } \partial A\}. \tag{3.7}$$

Moreover for all $x \in \mathbb{R}^N$, $a,b \in \mathbb{R}$ and $\nu \in S^{N-1}$ let $u_{x,a,b,\nu} : B_1(x) \to \mathbb{R}$ be defined by

$$u_{x,a,b,\nu}(y) := \begin{cases} b & \text{if } (y-x)\nu \ge 0, \\ a & \text{if } (y-x)\nu < 0, \end{cases} \tag{3.8}$$

where $B_1(x)$ is the ball of center x and radius 1.

The following Γ-convergence and representation result for the functionals $\mathcal{F}_n$ holds (see Buttazzo and Dal Maso [15], Bouchitté, Fonseca, Leoni and Mascarenhas [10, Theorem 2]).

Proposition 3.1. *There exists $\mathcal{F} : L^1(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$ such that up to a subsequence the functionals $\mathcal{F}_n(\cdot,A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot,A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover for all $u \in W^{1,p}(\Omega)$ we have that*

$$\mathcal{F}(u,A) = \int_A f(x,\nabla u(x))\,dx, \tag{3.9}$$

where

$$f(x,\xi) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\xi(z-x),B_\rho(x))}{\omega_N\rho^N}, \tag{3.10}$$

$\mathbf{m}_{\mathcal{F}}$ is defined in (3.7)*, and ω_N is the volume of the unit ball in $\mathbb{R}^N$. Finally f is a Carathéodory function satisfying the growth conditions* (3.1).

Let us come to the functionals $\mathcal{G}_n$ defined in (3.6). The following proposition holds (see Ambrosio and Braides [5, Theorem 3.2], Bouchitté, Fonseca, Leoni and Mascarenhas [10, Theorem 3]).

Proposition 3.2. *There exists $\mathcal{G} : P(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty[$ such that up to a subsequence $\mathcal{G}_n(\cdot,A)$ Γ-converges with respect to the strong topology of $L^1(\Omega)$ to $\mathcal{G}(\cdot,A)$ for all $A \in \mathcal{A}(\Omega)$. Moreover for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that*

$$\mathcal{G}(u,A) = \int_{A\cap S(u)} g(x,\nu)\,dx, \tag{3.11}$$

with

$$g(x,\nu) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{G}}(u_{x,0,1,\nu}, B_\rho(x))}{\omega_{N-1}\rho^{N-1}}, \tag{3.12}$$

where $\mathbf{m}_{\mathcal{G}}$ *is defined in* (3.7) *and* $u_{x,0,1,\nu}$ *is as in* (3.8).

Let us come to the functionals $\mathcal{G}_n^-$ defined in (3.5). The following proposition holds.

Proposition 3.3. *There exists* $\mathcal{G}^- : P(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty[$ *such that up to a subsequence* $\mathcal{G}_n^-(\cdot,A)$ Γ*-converges with respect to the strong topology of* $L^1(\Omega)$ *to* $\mathcal{G}^-(\cdot,A)$ *for all* $A\in\mathcal{A}(\Omega)$. *Moreover for all* $u\in P(\Omega)$ *and* $A\in\mathcal{A}(\Omega)$ *we have that*

$$\mathcal{G}^-(u,A) = \int_{A\cap S(u)} g^-(x,\nu)\, d\mathcal{H}^{N-1}(x), \tag{3.13}$$

with

$$g^-(x,\nu) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{G}^-}(u_{x,0,1,\nu}, B_\rho(x))}{\omega_{N-1}\rho^{N-1}}, \tag{3.14}$$

where $\mathbf{m}_{\mathcal{G}^-}$ *is defined in* (3.7) *and* $u_{x,0,1,\nu}$ *is as in* (3.8).

Proof. The Γ-convergence result for $\mathcal{G}_n^-(\cdot,A)$ is given by the result of Ambrosio and Braides [5, Theorem 3.2]. For the sequel we need also the explicit formula (3.14) for the density g^- which is not given directly by the results of [5] and [10] because of a lack of coercivity from below (we are removing K_n from $S(u)$). So in what follows we modify the concrete approximation $\mathcal{G}_n^-(\cdot,A)$ for $\mathcal{G}^-(\cdot,A)$ in order to get the coerciveness we need, and to obtain in the end formula (3.14).

Let us consider the functionals

$$\mathcal{G}_n^\varepsilon(u,A) := \begin{cases} \int_{A\cap S(u)} g_n^\varepsilon(x,\nu)\, d\mathcal{H}^{N-1}(x) & u\in PC(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{3.15}$$

where

$$g_n^\varepsilon(x,\nu) := \begin{cases} \varepsilon & \text{if } x\in K_n, \nu=\nu_{K_n}(x), \\ g_n(x,\nu) & \text{otherwise.} \end{cases} \tag{3.16}$$

Let us denote by $\mathcal{G}^\varepsilon(\cdot,A)$ the Γ-limit (up to a subsequence) of $\mathcal{G}_n^\varepsilon(\cdot,A)$ for all $A\in\mathcal{A}(\Omega)$. Since $\mathcal{G}^\varepsilon$ is such that for ε small

$$\varepsilon\mathcal{H}^{N-1}(S(u)\cap A) \le \mathcal{G}^\varepsilon(u,A) \le \beta\mathcal{H}^{N-1}(S(u)\cap A),$$

by Proposition 3.2 we have that

$$\mathcal{G}^\varepsilon(u,A) = \int_{S(u)\cap A} g^\varepsilon(x,\nu)\, d\mathcal{H}^{N-1}(x),$$

where $g^\varepsilon : \Omega\times S^{N-1}\to[0,+\infty]$ is given by

$$g^\varepsilon(x,\nu) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{G}^\varepsilon}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}. \tag{3.17}$$

Let $\mu_n := \mathcal{H}^{N-1}\llcorner K_n$. Since $\mathcal{H}^{N-1}(K_n)\le C$, up to a subsequence we have

$$\mu_n \overset{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in the sense of measures} \tag{3.18}$$

for some finite Radon measure μ. Notice that for all $u\in PC(\Omega)$ and $A\in\mathcal{A}(\Omega)$ we have

$$\mathcal{G}_n^\varepsilon(u,A) \le \mathcal{G}_n^-(u,A) + \varepsilon\mu_n(A),$$

so that by Γ-convergence and by (3.18) we get for $n\to+\infty$

$$\mathcal{G}^\varepsilon(u,A) \le \mathcal{G}^-(u,A) + \varepsilon\mu(\bar{A}). \tag{3.19}$$

Up to a set of $\mathcal{H}^{N-1}$-measure zero we have

$$H(x) := \limsup_{\rho\to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}} < +\infty. \tag{3.20}$$

In fact if by contradiction $H(x) = +\infty$ on a Borel set B with $\mathcal{H}^{N-1}(B) > 0$, then by [7, Theorem 2.56] we deduce that $\mu(B) = \infty$. But this is against $\mu(\Omega) < +\infty$.

Let us prove that for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ we have

$$g^-(x,\nu) = \lim_{\varepsilon\to 0} g^\varepsilon(x,\nu), \tag{3.21}$$

where $g^-(x,\nu)$ is defined in (3.14). In fact, notice that $\{g^\varepsilon\}_\varepsilon$ is monotone decreasing in ε and that $g^- \le g^\varepsilon$ for all $\varepsilon > 0$, so that for all x and ν

$$g^-(x,\nu) \le \lim_{\varepsilon\to 0} g^\varepsilon(x,\nu).$$

On the other hand, by (3.19) we have that

$$\frac{\mathbf{m}_{\mathcal{G}^\varepsilon}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}} \le \frac{\mathbf{m}_{\mathcal{G}^-}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}} + \varepsilon\frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}}.$$

Taking the lim sup for $\rho \to 0^+$ we have

$$g^\varepsilon(x,\nu) \le g^-(x,\nu) + \varepsilon H(x),$$

and so letting $\varepsilon \to 0$ we obtain for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$

$$\lim_{\varepsilon\to 0} g^\varepsilon(x,\nu) \le g^-(x,\nu)$$

which gives (3.21). Since for all $u \in PC(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have $\mathcal{G}^\varepsilon(u,A) \to \mathcal{G}^-(u,A)$ as $\varepsilon \to 0$, we conclude that

$$\mathcal{G}^-(u,A) = \lim_{\varepsilon\to 0}\mathcal{G}^\varepsilon(u,A) = \lim_{\varepsilon\to 0}\int_{S(u)\cap A} g^\varepsilon(x,\nu)\,d\mathcal{H}^{N-1}(x) = \int_{S(u)\cap A} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x), \tag{3.22}$$

so that the representation formulas (3.13) and (3.14) hold. □

Remark 3.4. It is immediate to check that if we replace $P(\Omega)$ in Proposition 3.3 by the space $P_{a,b}(\Omega) := \{u \in SBV(\Omega) : u(x) \in \{a,b\} \text{ for a.e. } x \in \Omega\}$, with $a, b \in \mathbb{R}$, then the Γ-limit in the strong topology of $L^1(\Omega)$ of $\mathcal{G}_n^-(\cdot,A)$ can still be represented by the density g^- defined in (3.14).

Let us finally come to the functionals $\mathcal{E}_n$ defined in (3.3). Using the growth estimates (3.1) and (3.2) on f_n and g_n (see [12]), there exists $\mathcal{E} : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty[$ such that up to a subsequence $\mathcal{E}_n(\cdot,A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot,A)$ for all $A \in \mathcal{A}(\Omega)$. For every $\varepsilon > 0$ let us set

$$\mathcal{E}_\varepsilon(u,A) := \mathcal{E}(u,A) + \varepsilon\int_{S(u)\cap A} 1 + |[u]|\,d\mathcal{H}^{N-1}, \tag{3.23}$$

where $[u](x)$ denotes the jump of u at x, i.e., $[u](x) := u^+(x) - u^-(x)$. By the representation result of Bouchitté, Fonseca, Leoni and Mascarenhas [10, Theorem 1] we get that for all $u \in SBV^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$

$$\mathcal{E}_\varepsilon(u,A) = \int_A f^\varepsilon_\infty(x,\nabla u(x))\,dx + \int_{A\cap S(u)} g^\varepsilon_\infty(x,u^-(x),u^+(x),\nu)\,d\mathcal{H}^{N-1}(x) \tag{3.24}$$

with f^ε_∞ and g^ε_∞ satisfying the following formulas

$$f^\varepsilon_\infty(x,\xi) := \limsup_{\rho\to 0^+}\frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(\xi(z-x), B_\rho(x))}{\omega_N\rho^N}, \tag{3.25}$$

$$g^\varepsilon_\infty(x,a,b,\nu) := \limsup_{\rho\to 0^+}\frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(u_{x,a,b,\nu}, B_\rho(x))}{\omega_{N-1}\rho^{N-1}}, \tag{3.26}$$

where $\mathbf{m}_{\mathcal{E}_\varepsilon}$ is defined in (3.7) and $u_{x,a,b,\nu}$ is as in (3.8).

Notice that f^ε_∞ and g^ε_∞ are monotone decreasing in ε, and that $\mathcal{E}_\varepsilon(\cdot,A)$ converges pointwise to $\mathcal{E}(\cdot,A)$ as $\varepsilon \to 0$ for every $A \in \mathcal{A}(\Omega)$. We conclude that the representation result for $\mathcal{E}_\varepsilon$ implies a representation result for the functional $\mathcal{E}$.

Summarizing we have that the following proposition holds.

Proposition 3.5. *There exists $\mathcal{E} : L^1(\Omega)\times\mathcal{A}(\Omega) \to [0,+\infty]$ such that up to a subsequence $\mathcal{E}_n(\cdot, A)$ Γ-converges in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover, for every $u \in SBV^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that*

$$\mathcal{E}(u, A) = \int_A f_\infty(x, \nabla u(x))\, dx + \int_{A\cap S(u)} g_\infty(x, u^-(x), u^+(x), \nu)\, d\mathcal{H}^{N-1}(x)$$

with

$$f_\infty(x, \xi) := \lim_{\varepsilon\to 0} f^\varepsilon_\infty(x, \xi) \qquad \text{and} \qquad g_\infty(x, a, b, \nu) := \lim_{\varepsilon\to 0} g^\varepsilon_\infty(x, a, b, \nu), \tag{3.27}$$

where f^ε_∞ and g^ε_∞ are defined in (3.25) *and* (3.26) *respectively.*

Remark 3.6. In the rest of the paper we will often make use the following property which is implied by the fact that $\mathcal{E}(u, \cdot)$ is a Radon measure for every $u \in SBV^p(\Omega)$. If $(u_n)_{n\in\mathbb{N}}$ is a recovering sequence for u with respect to $\mathcal{E}_n(\cdot, \Omega)$, then $(u_n)_{n\in\mathbb{N}}$ is optimal for u with respect to $\mathcal{E}_n(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$ such that the measure $\mathcal{E}(u, \cdot)$ vanishes on ∂A.

4. A Γ-convergence result for free discontinuity problems

The main result of this section is the following Γ-convergence theorem concerning the functionals $\mathcal{E}_n$ defined in (3.3).

Theorem 4.1. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that $\mathcal{H}^{N-1}(K_n) \le C$ for all $n \in \mathbb{N}$. Let us assume that for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{F}_n(\cdot, A)$ and $\mathcal{G}^-_n(\cdot, A)$ defined in* (3.4) *and* (3.5) *Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ and $\mathcal{G}^-(\cdot, A)$ respectively. Then for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{E}_n(\cdot, A)$ defined in* (3.3) *Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot, A)$ such that for all $u \in SBV^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$*

$$\mathcal{E}(u, A) = \int_A f(x, \nabla u(x))\, dx + \int_{A\cap S(u)} g^-(x, \nu)\, d\mathcal{H}^{N-1}(x),$$

where f and g^- are the densities of $\mathcal{F}$ and $\mathcal{G}^-$ according to Propositions 3.1 and 3.3.

Proof. By Proposition 3.5 we know that up to a subsequence the functionals $\mathcal{E}_n(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to a functional $\mathcal{E}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$, which for all $u \in SBV^p(\Omega)$ and for all $A \in \mathcal{A}(\Omega)$ can be represented as

$$\mathcal{E}(u, A) = \int_A f_\infty(x, \nabla u)\, dx + \int_{S(u)\cap A} g_\infty(x, u^-(x), u^+(x), \nu)\, d\mathcal{H}^{N-1}(x), \tag{4.1}$$

where f_∞ and g_∞ satisfy formula (3.27). The theorem will be proved if we show that for all $u \in SBV^p(\Omega)$ we have

$$f_\infty(x, \nabla u(x)) = f(x, \nabla u(x)) \qquad \text{for a.e. } x \in \Omega,$$

and

$$g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) = g^-(x, \nu_{S(u)}(x)) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in S(u),$$

where $\nu_{S(u)}(x)$ is the normal to $S(u)$ at x.

We will use the following relations between the functionals $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}^-$ which follow immediately by Γ-convergence. If $u \in W^{1,p}(A)$, then

$$\mathcal{E}(u, A) \le \mathcal{F}(u, A), \tag{4.2}$$

while if $u \in P(A)$ we have

$$\mathcal{E}(u, A) \le \int_A a_2\, dx + \mathcal{G}^-(u, A), \tag{4.3}$$

where a_2 appears in the growth estimate (3.1) for f_n.

The proof will be divided into four steps.

Step 1: $\mathbf{f_\infty(x, \nabla u(x)) \le f(x, \nabla u(x))}$ for a.e. $\mathbf{x \in \Omega}$.

This inequality can be derived using the explicit formulas for f_∞ and f. Let $x \in \Omega$, $\xi \in \mathbb{R}^N$, and let us fix $\varepsilon > 0$. For every $\rho > 0$ let $u_{\varepsilon,\rho} \in W^{1,p}(B_\rho(x))$ be such that $u_{\varepsilon,\rho}(z) = \xi(z-x)$ in a neighborhood of $\partial B_\rho(x)$ and

$$\mathcal{F}(u_{\varepsilon,\rho}, B_\rho(x)) \le \mathbf{m}_{\mathcal{F}}(\xi(z-x), B_\rho(x)) + \varepsilon\omega_N\rho^N, \tag{4.4}$$

where $\mathbf{m}_{\mathcal{F}}$ is defined in (3.7). Since $u_{\varepsilon,\rho} \in W^{1,p}(B_\rho(x))$ by (4.2) we have

$$\mathcal{E}(u_{\varepsilon,\rho}, B_\rho(x)) \le \mathcal{F}(u_{\varepsilon,\rho}, B_\rho(x))$$

so that by definition of $\mathcal{E}_\varepsilon$ (see (3.23)) we have

$$\mathcal{E}_\varepsilon(u_{\varepsilon,\rho}, B_\rho(x)) \le \mathcal{F}(u_{\varepsilon,\rho}, B_\rho(x)).$$

In view of the explicit formulas (3.25) and (3.10) for f^ε_∞ and f, and taking into account (4.4), we get

$$\begin{aligned} f^\varepsilon_\infty(x,\xi) &= \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(\xi(z-x), B_\rho(x))}{\omega_N\rho^N} \le \limsup_{\rho\to 0^+} \frac{\mathcal{E}_\varepsilon(u_{\varepsilon,\rho}, B_\rho(x))}{\omega_N\rho^N} \\ &\le \limsup_{\rho\to 0^+} \frac{\mathcal{F}(u_{\varepsilon,\rho}, B_\rho(x))}{\omega_N\rho^N} \le \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(\xi(z-x), B_\rho(x))}{\omega_N\rho^N} + \varepsilon = f(x,\xi) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, by (3.27) we obtain that $f_\infty(x,\xi) \le f(x,\xi)$, so that the step is concluded.

Step 2: $\mathbf{f_\infty(x, \nabla u(x)) \ge f(x, \nabla u(x))}$ for a.e. $\mathbf{x \in \Omega}$.

In view of the representation (4.1) for $\mathcal{E}$, and since $\mathcal{E}(u,\Omega) < +\infty$, by Radon-Nikodým Theorem we get that for a.e. $x \in \Omega$

$$f_\infty(x, \nabla u(x)) = \lim_{\rho\to 0^+} \frac{\mathcal{E}(u, B_\rho(x))}{\omega_N\rho^N} < +\infty. \tag{4.5}$$

Let $(u_n)_{n\in\mathbb{N}}$ be a recovering sequence for $\mathcal{E}(u,\Omega)$: by the growth estimate (3.2) on g_n, and since $\mathcal{H}^{N-1}(K_n) \le C$, we have that $\mathcal{H}^{N-1}(S(u_n))$ is bounded so that, up to a subsequence,

$$\mu_n := \mathcal{H}^{N-1} \llcorner S(u_n) \overset{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in the sense of measures} \tag{4.6}$$

for some finite positive Radon measure μ. Notice that for a.e. $x \in \Omega$ we have

$$H(x) := \limsup_{\rho\to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\rho^{N-1}} = 0. \tag{4.7}$$

In fact, if by contradiction there exists a Borel set B with positive Lebesgue measure and $t > 0$ such that

$$H(x) \ge t \qquad \text{for every } x \in B,$$

then (see for instance [7, Theorem 2.56]) we deduce that

$$\mu \llcorner B \ge t\mathcal{H}^{N-1} \llcorner B,$$

so that $\mu(B) = \infty$. But this is against the fact that μ is finite.

Since the main inequality of the step should hold a.e. in Ω, we can assume that u is approximatively differentiable at x, $x \notin N$ with N defined in the blow-up result for Γ-limit given by Proposition 2.2, and that (4.5) and (4.7) hold.

Let us choose $\rho_i \searrow 0$ in such a way that $\mathcal{E}(u, \partial B_{\rho_i}(x)) = 0$: notice that this is possible since $\mathcal{E}(u,\cdot)$ is a Radon measure and so the family of radii ρ such that $\mathcal{E}(u, \partial B_\rho(x)) > 0$ is at most countable.

In view of Remark 3.6, for every i there exists n_i such that for $n \ge n_i$

$$\begin{aligned} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\omega_N\rho_i^N} &\ge \frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\omega_N\rho_i^N} - \frac{1}{i} \\ &\ge \frac{\int_{B_{\rho_i}(x)} f_n(x, \nabla u_n(x))\,dx}{\omega_N\rho_i^N} - \frac{1}{i} = \frac{1}{\omega_N}\int_{B_1} f_n(x+\rho_i y, \nabla v^i_n(y))\,dy - \frac{1}{i}, \end{aligned} \tag{4.8}$$

where

$$v_n^i(y) := \frac{u_n(x+\rho_i y) - u(x)}{\rho_i}.$$

We can choose $(n_i)_{i\in\mathbb{N}}$ is such a way that

$$v_{n_i}^i \to \nabla u(x)\cdot y \qquad \text{strongly in } L^1(B_1) \text{ for } i \to +\infty, \tag{4.9}$$

and

$$\lim_{i\to+\infty} \mathcal{H}^{N-1}(S(v_{n_i}^i)) = 0. \tag{4.10}$$

In fact we have

$$v_n^i(y) \to \frac{u(x+\rho_i y) - u(x)}{\rho_i} \qquad \text{strongly in } L^1(B_1) \text{ as } n \to +\infty$$

and since u is approximately differentiable at x

$$\frac{u(x+\rho_i y) - u(x)}{\rho_i} \to \nabla u(x)\cdot y \qquad \text{strongly in } L^1(B_1) \text{ as } i \to +\infty.$$

Moreover by (4.6) we have

$$\mathcal{H}^{N-1}(S(v_n^i)) = \frac{\mathcal{H}^{N-1}(S(u_n)\cap B_{\rho_i}(x))}{\rho_i^{N-1}} \le \frac{\mu_n(\bar{B}_{\rho_i}(x))}{\rho_i^{N-1}},$$

so that we get

$$\limsup_{n\to+\infty} \mathcal{H}^{N-1}(S(v_n^i)) \le \frac{\mu(\bar{B}_{\rho_i}(x))}{\rho_i^{N-1}} \to 0 \qquad \text{as } i \to +\infty.$$

As a consequence, by a diagonal argument we can achieve (4.9) and (4.10).

By (4.8) and (4.5) we finally deduce

$$f_\infty(x,\nabla u(x)) = \lim_{i\to+\infty} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\omega_N \rho_i^N} \ge \liminf_{i\to+\infty} \frac{1}{\omega_N}\int_{B_1} f_{n_i}(x+\rho_i y, \nabla v_{n_i}^i(y))\,dy. \tag{4.11}$$

Using a truncation argument we can assume that $(v_{n_i}^i)_{i\in\mathbb{N}}$ is uniformly bounded in $L^\infty(B_1)$, so that in view of the growth estimate on f_n, by (4.11) and (4.10) we get

$$\|\nabla v_{n_i}^i\|^p_{L^p(B_1,\mathbb{R}^N)} \le C \quad \text{and} \quad \int_{S(v_{n_i}^i)} |[v_{n_i}^i]|\,d\mathcal{H}^{N-1} \to 0.$$

By [25, Lemma 2.1] we get that there exists $w_i \in W^{1,p}(B_1)$ such that $w_i \to \nabla u(x)\cdot y$ strongly in $L^1(B_1)$ as $i \to +\infty$ and such that

$$\liminf_{i\to+\infty} \int_{B_1} f_{n_i}(x+\rho_i y, \nabla v_{n_i}^i(y))\,dy = \liminf_{i\to+\infty} \int_{B_1} f_{n_i}(x+\rho_i y, \nabla w_i(y))\,dy.$$

If n_i is chosen such that the blow-up for Γ-limits given by Proposition 2.2 holds, we get that

$$\liminf_{i\to+\infty} \int_{B_1} f_{n_i}(x+\rho_i y, \nabla w_i(y))\,dy \ge \omega_N f(x,\nabla u(x)),$$

so that in view of (4.11) we obtain

$$f_\infty(x,\nabla u(x)) \ge f(x,\nabla u(x)).$$

Step 3: $\mathbf{g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) \le g^-(x, \nu_{S(u)}(x))}$ for $\mathcal{H}^{\mathbf{N-1}}$-a.e. $\mathbf{x \in S(u)}$.

Since $\mathcal{H}^{N-1}(K_n) \le C$, up to a subsequence we have that

$$\mu_n := \mathcal{H}^{N-1} \llcorner K_n \stackrel{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in the sense of measures} \tag{4.12}$$

for some finite positive Radon measure μ. We have that for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$

$$H(x) := \limsup_{\rho\to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}} < +\infty. \tag{4.13}$$

In fact if by contradiction $H(x) = +\infty$ on a Borel set B with $\mathcal{H}^{N-1}(B) > 0$, then we deduce that $\mu(B) = \infty$ (see for instance [7, Theorem 2.56]). But this is against $\mu(\Omega) < +\infty$.

We claim that for all $v \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$ such that $\bar{A} \subseteq \Omega$

$$\alpha \mathcal{H}^{N-1}(S(v) \cap A) \leq \mathcal{G}^{-}(v, A) + \alpha\mu(\bar{A}), \tag{4.14}$$

where α is the positive constant appearing in the growth estimate (3.2) for g_n. In fact, considering the functional $\mathcal{G}_n^-$ defined in (3.5), and taking into account the growth estimate on g_n we have that for all $n \in \mathbb{N}$

$$\alpha \mathcal{H}^{N-1}\left((S(v) \setminus K_n) \cap A\right) \leq \mathcal{G}_n^-(v, A)$$

so that we deduce

$$\alpha \mathcal{H}^{N-1}\left(S(v) \cap A\right) \leq \mathcal{G}_n^-(v, A) + \alpha\mu_n(A).$$

Passing to the Γ-limit for $n \to +\infty$, and using the weak* convergence of μ_n to μ, we obtain that (4.14) holds.

Since the main inequality of the step should hold for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$, we can choose $x \in S(u)$ in such a way that u has an approximate jump at x, (4.13) holds and such that

$$\limsup_{\rho \to 0^+} \frac{\int_{B_\rho(x)} a_2 \, dx}{\rho^{N-1}} = 0, \tag{4.15}$$

where a_2 appears in the growth estimate (3.1) for f_n. Let us indicate $u^-(x), u^+(x)$ and $\nu_{S(u)}(x)$ simply by u^-, u^+ and ν. Let us moreover set $[u] := u^+ - u^-$.

Following Remark 3.4, let us consider the functionals $\mathcal{G}_n^-$ defined in (3.5) acting on the space $P_{u^-,u^+}(\Omega) := \{u \in SBV(\Omega) \,:\, u(y) \in \{u^-, u^+\} \text{ for a.e. } y \in \Omega\}$.

Let us fix $\varepsilon > 0$. For every $\rho > 0$, let $u_{\varepsilon,\rho} \in P_{u^-,u^+}(B_\rho(x))$ be such that $u_{\varepsilon,\rho} = u_{x,u^-,u^+,\nu}$ in a neighborhood of $\partial B_\rho(x)$ and

$$\mathcal{G}^-(u_{\varepsilon,\rho}, B_\rho(x)) \leq \mathbf{m}_{\mathcal{G}^-}(u_{x,u^-,u^+,\nu}, B_\rho(x)) + \varepsilon\omega_{N-1}\rho^{N-1}, \tag{4.16}$$

where $\mathbf{m}_{\mathcal{G}^-}$ is defined in (3.7), and $u_{x,a,b,\nu}$ is defined in (3.8). Since $u_{\varepsilon,\rho} \in P_{u^-,u^+}(B_\rho(x))$, we have that

$$\mathcal{E}_\varepsilon(u_{\varepsilon,\rho}, B_\rho(x)) = \mathcal{E}(u_{\varepsilon,\rho}, B_\rho(x)) + \varepsilon(1 + |[u]|)\mathcal{H}^{N-1}(S(u_{\varepsilon,\rho}) \cap B_\rho(x)),$$

and by (4.3)

$$\mathcal{E}(u_{\varepsilon,\rho}, B_\rho(x)) \leq \int_{B_\rho(x)} a_2 \, dx + \mathcal{G}^-(u_{\varepsilon,\rho}, B_\rho(x)).$$

By (4.14) and (4.16) we deduce

$$\begin{aligned}
&\mathcal{E}_\varepsilon(u_{\varepsilon,\rho}, B_\rho(x)) = \mathcal{E}(u_{\varepsilon,\rho}, B_\rho(x)) + \varepsilon(1 + |[u]|)\mathcal{H}^{N-1}(S(u_{\varepsilon,\rho}) \cap B_\rho(x)) \\
&\leq \int_{B_\rho(x)} a_2 \, dx + \mathcal{G}^-(u_{\varepsilon,\rho}, B_\rho(x)) + \varepsilon(1 + |[u]|)\mathcal{H}^{N-1}(S(u_{\varepsilon,\rho}) \cap B_\rho(x)) \\
&\leq \int_{B_\rho(x)} a_2 \, dx + \mathcal{G}^-(u_{\varepsilon,\rho}, B_\rho(x)) + \frac{\varepsilon}{\alpha}(1 + |[u]|)\left(\mathcal{G}^-(u_{\varepsilon,\rho}, B_\rho(x)) + \alpha\mu(\bar{B}_\rho(x))\right) \\
&= \int_{B_\rho(x)} a_2 \, dx + \left(1 + \frac{\varepsilon}{\alpha}(1 + |[u]|)\right)\mathcal{G}^-(u_{\varepsilon,\rho}, B_\rho(x)) + \varepsilon(1 + |[u]|)\mu(\bar{B}_\rho(x)) \\
&\leq \int_{B_\rho(x)} a_2 \, dx + \left(1 + \frac{\varepsilon}{\alpha}(1 + |[u]|)\right)\left(\mathbf{m}_{\mathcal{G}^-}(u_{x,u^-,u^+,\nu}, B_\rho(x)) + \varepsilon\omega_{N-1}\rho^{N-1}\right) \\
&+ \varepsilon(1 + |[u]|)\mu(\bar{B}_\rho(x))
\end{aligned}$$

Dividing by $\omega_{N-1}\rho^{N-1}$, in view of the explicit formulas (3.26) and (3.14) for g_∞^ε and g^-, and taking into account (4.15) and (4.13) we deduce

$$\begin{aligned}
g_\infty^\varepsilon(x, u^-, u^+, \nu) &= \limsup_{\rho \to 0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(u_{x,u^-,u^+,\nu}, B_\rho(x))}{\omega_{N-1}\rho^{N-1}} \leq \limsup_{\rho \to 0^+} \frac{\mathcal{E}_\varepsilon(u_{\varepsilon,\rho}, B_\rho(x))}{\omega_{N-1}\rho^{N-1}} \\
&\leq \left(1 + \frac{\varepsilon}{\alpha}(1 + |[u]|)\right)(g^-(x, \nu) + \varepsilon) + \varepsilon(1 + |[u]|)H(x).
\end{aligned}$$

Letting $\varepsilon \to 0$, by (3.27) we obtain $g_\infty(x, u^-, u^+, \nu) \le g^-(x, \nu)$, so that the step is concluded.

Step 4: $\mathbf{g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) \ge g^-(x, \nu_{S(u)}(x))}$ **for** $\mathcal{H}^{N-1}$**-a.e.** $\mathbf{x \in S(u)}$**.**

The proof of this step requires a careful use of Coarea formula for BV functions and of Fubini's Theorem in order to modify an SBV function with small gradient into a piecewise constant function. As mentioned in the Introduction, the construction follows some steps in the proof the Transfer of Jump Sets Theorem [21, Theorem 2.3] by Francfort and Larsen.

In view of the representation (4.1) for $\mathcal{E}$, and since $\mathcal{E}(u, \Omega) < +\infty$, by Radon-Nikodým Theorem we have that for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$

$$g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) = \lim_{\rho \to 0^+} \frac{\mathcal{E}(u, B_\rho(x))}{\omega_{N-1}\rho^{N-1}} < +\infty. \tag{4.17}$$

Moreover for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$ we have also

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} |a_1(y)|\, dy}{\rho^{N-1}} = 0, \tag{4.18}$$

where a_1 appears in the growth estimate (3.1) for f_n.

Let $(u_n)_{n\in\mathbb{N}}$ be a recovering sequence for $\mathcal{E}(u, \Omega)$. By the growth estimate (3.2) on g_n, and since $\mathcal{H}^{N-1}(K_n) \le C$, up to a subsequence we have

$$\mu_n := \mathcal{H}^{N-1} \llcorner S(u_n) \overset{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in the sense of measures}$$

for some finite positive Radon measure μ. Moreover for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$ we have (see for instance [7, Theorem 2.56])

$$H(x) := \limsup_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\rho^{N-1}} < +\infty. \tag{4.19}$$

Since the main inequality of the step should hold for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$, we can choose $x \in S(u)$ such that x is a point of approximate jump for u, and (4.17), (4.18) and (4.19) hold.

Let $\rho_i \searrow 0$ be such that $\mathcal{E}(u, \partial B_{\rho_i}(x)) = 0$ and such that the representation formula (3.14) for g^- holds along ρ_i, i.e.,

$$g^-(x, \nu) = \lim_{i \to +\infty} \frac{\mathbf{m}_{\mathcal{G}^-}(u_{x,0,1,\nu_{S(u)}(x)}, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}}, \tag{4.20}$$

where $u_{x,a,b,\nu}$ is defined in (3.8).

In view of Remark 3.6 for every $i \in \mathbb{N}$ there exists $n_i \in \mathbb{N}$ such that for $n \ge n_i$ we have

$$\begin{aligned}
\frac{\mathcal{E}(u, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} &\ge \frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \frac{1}{i} \\
&\ge \frac{\int_{B_{\rho_i}(x) \cap [S(u_n) \setminus K_n]} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x)}{\omega_{N-1}\rho_i^{N-1}} + \frac{\int_{B_{\rho_i}(x)} a_1(y)\, dy}{\omega_{N-1}\rho_i^{N-1}} - \frac{1}{i} \\
&= \frac{1}{\omega_{N-1}} \int_{B_1 \cap [S(v_n^i) \setminus K_n^i]} g_n(x + \rho_i y, \nu)\, d\mathcal{H}^{N-1}(y) + \frac{\int_{B_{\rho_i}(x)} a_1(y)\, dy}{\omega_{N-1}\rho_i^{N-1}} - \frac{1}{i},
\end{aligned} \tag{4.21}$$

where

$$v_n^i(y) := u_n(x + \rho_i y) \quad \text{and} \quad K_n^i := \frac{\{K_n \cap B_{\rho_i}(x)\} - x}{\rho_i}.$$

We claim that we can find w_n^i piecewise constant in B_1 such that for $n \to +\infty$

$$w_n^i \to w^i \qquad \text{strongly in } L^1(B_1), \tag{4.22}$$

where w^i is piecewise constant and $w^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of ∂B_1, and such that for n large

$$\mathcal{H}^{N-1}\left(S(w_n^i) \setminus S(v_n^i)\right) \le e_i, \tag{4.23}$$

with $e_i \to 0$ as $i \to +\infty$. By the growth estimate (3.2) on g_n and claim (4.23), we deduce

$$\int_{B_1\cap[S(v_n^i)\setminus K_n^i]} g_n(x+\rho_i y,\nu)\,d\mathcal{H}^{N-1}(y) \geq \int_{B_1\cap[S(w_n^i)\setminus K_n^i]} g_n(x+\rho_i y,\nu)\,d\mathcal{H}^{N-1}(y) - \hat{e}_i, \tag{4.24}$$

with $\hat{e}_i \to 0$ for $i \to +\infty$. By (4.21), in view of (4.17), (4.18) and (4.24) we have that for n large

$$g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x)) \geq \int_{B_1\cap[S(w_n^i)\setminus K_n^i]} g_n(x+\rho_i y,\nu)\,d\mathcal{H}^{N-1}(y) - \hat{e}_i,$$

where $\hat{e}_i \to 0$. Rescaling to the ball $B_{\rho_i}(x)$ we get

$$\begin{aligned} g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x)) &\geq \frac{\int_{B_{\rho_i}(x)\cap[S(z_n^i)\setminus K_n]} g_n(\zeta,\nu)\,d\mathcal{H}^{N-1}(\zeta)}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i \\ &= \frac{\mathcal{G}_n^-(z_n^i, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i, \end{aligned} \tag{4.25}$$

where

$$z_n^i(\zeta) := w_n^i\left(\frac{\zeta - x}{\rho_i}\right) \to z^i(\zeta) := w^i\left(\frac{\zeta-x}{\rho_i}\right) \qquad \text{strongly in } L^1(B_{\rho_i}(x))$$

and $\mathcal{G}_n^-$ is defined in (3.5). Since $\mathcal{G}_n^-(\cdot,B_{\rho_i}(x))$ Γ-converges to $\mathcal{G}^-(\cdot,B_{\rho_i}(x))$, using Γ-liminf inequality by (4.25) we have that

$$g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x)) \geq \frac{\mathcal{G}^-(z^i,B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i \geq \frac{\mathbf{m}_{\mathcal{G}^-}(u_{x,0,1,\nu_{S(u)}(x)}, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i.$$

Letting $i \to +\infty$, and recalling the representation formula (4.20) for $g^-(x,\nu)$, we have that the result is proved.

In order to complete the proof of the step, we have to prove the claims (4.22) and (4.23). Since

$$\nabla v_n^i(y) = \rho_i \nabla u_n(x+\rho_i y),$$

we get by the growth estimate (3.1) on f_n

$$\begin{aligned} \int_{B_1} |\nabla v_n^i(y)|^p\,dy = \rho_i^p \int_{B_1} |\nabla u_n(x+\rho_i y)|^p\,dy &= \rho_i^p \frac{\int_{B_{\rho_i}(x)} |\nabla u_n(z)|^p\,dz}{\rho_i^N} \\ &\leq \frac{\rho_i^{p-1}}{\alpha}\left(\frac{\mathcal{E}_n(u_n,B_{\rho_i}(x))}{\rho_i^{N-1}} - \frac{\int_{B_{\rho_i}(x)} a_1(y)\,dy}{\rho_i^{N-1}}\right). \end{aligned} \tag{4.26}$$

Since u_n is optimal for u, and using (4.17) we have that

$$\frac{\mathcal{E}_n(u_n,B_{\rho_i}(x))}{\rho_i^{N-1}} \overset{n\to+\infty}{\longrightarrow} \frac{\mathcal{E}(u,B_{\rho_i}(x))}{\rho_i^{N-1}} \overset{i\to+\infty}{\longrightarrow} \omega_{N-1} g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x)) < +\infty.$$

In view also of (4.18), from (4.26) we conclude that we can choose n_i such that for $n \geq n_i$

$$\int_{B_1} |\nabla v_n^i(y)|^p\,dy \leq C\rho_i^{p-1}$$

for some constant $C \geq 0$. By Coarea formula for BV functions (see [7, Theorem 3.40]) we get

$$\int_{u^-(x)}^{u^+(x)} \mathcal{H}^{N-1}\left(\partial^* E_n^i(t) \setminus S(v_n^i)\right)\,dt \leq \int_{B_1} |\nabla v_n^i|\,dy \leq \tilde{C}\rho_i^{1-\frac{1}{p}},$$

for a suitable constant $\tilde{C}$, where

$$E_n^i(t) := \{x \in B_1 \,:\, x \text{ is a Lebesgue point for } v_n^i \text{ and } v_n^i(x) > t\}$$

and ∂^* denotes the reduced boundary. By the Mean Value Theorem there exists $t_n^i \in [u^-(x),u^+(x)]$ such that

$$\mathcal{H}^{N-1}\left(\partial^* E_n^i(t_n^i) \setminus S(v_n^i)\right) \leq \frac{\tilde{C}}{u^+(x)-u^-(x)}\rho_i^{1-\frac{1}{p}}. \tag{4.27}$$

We now employ a construction similar to that employed by Francfort and Larsen in their Transfer of Jump Sets Theorem [21, Theorem 2.3]. Since x is a jump point for u we have that for $i \to +\infty$

$$u(x+\rho_i y) \to u_{0,u^-(x),u^+(x),\nu_{S(u)}(x)} \qquad \text{strongly in } L^1(B_1), \tag{4.28}$$

where $u_{x,a,b,\nu}$ is defined in (3.8). Since

$$v_n^i(y) \to u(x+\rho_i y) \qquad \text{strongly in } L^1(B_1) \text{ for } n \to +\infty,$$

by (4.28) we have that for n large

$$|B_1^+ \triangle E_n^i(t_n^i)| \le e_i, \tag{4.29}$$

where $B_1^+ := \{y \in B_1 \,:\, y \cdot \nu_{S(u)}(x) \ge 0\}$, $A \triangle B := (A \setminus B) \cup (B \setminus A)$, and $e_i \to 0$ for $i \to +\infty$. By Fubini's Theorem and by (4.29) we have

$$\begin{aligned}\int_0^{\sqrt{e_i}} \mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H(s)\right) ds &\le \int_0^1 \mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H(s)\right) ds \\ &= |B_1^+ \setminus E_n^i(t_n^i)| \le e_i,\end{aligned}$$

where $H(s) := \{y \in B_1 \,:\, y \cdot \nu_{S(u)}(x) = s\}$. By the Mean Value Theorem we get that there exists $0 < s_n^{i,+} < \sqrt{e_i}$ such that setting $H_n^{i,+} := H(s_n^{i,+})$ we have

$$\mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H_n^{i,+}\right) \le \sqrt{e_i}. \tag{4.30}$$

Similarly we obtain $-\sqrt{e_i} < s_n^{i,-} < 0$ such that setting $H_n^{i,-} := H(s_n^{i,-})$ we have

$$\mathcal{H}^{N-1}\left((E_n^i(t_n^i) \setminus B_1^+) \cap H_n^{i,-}\right) \le \sqrt{e_i}. \tag{4.31}$$

Let us write $y = (y', y_N)$, where y_N is the coordinate along $\nu_{S(u)}(x)$ and y' the coordinates in the hyperplane orthogonal to $\nu_{S(u)}(x)$. Let l_i be such that for every $y \in B_1$

$$|y_N| \ge 2\sqrt{e_i} \Longrightarrow |y'| \le 1 - l_i.$$

Clearly $l_i \searrow 0$ as $i \to +\infty$. Let us set

$$D_n^i := (E_n^i(t_n^i) \cup \{y \in B_1 \,:\, y_N \ge s_n^{i,+}\}) \setminus \{y \in B_1 \,:\, y_N \le s_n^{i,-}\}$$

and

$$w_n^i(y) := \begin{cases} 1 & |y'| \ge 1 - l_i, y_N \ge 0, \\ 0 & |y'| \ge 1 - l_i, y_N < 0, \\ 1 & |y'| \le 1 - l_i, y \in D_n^i, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that w_i^n is piecewise constant, with $w_n^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of ∂B_1 independent of n, and that (up to a $\mathcal{H}^{N-1}$-negligible set)

$$S(w_n^i) \subseteq \partial^* E_n^i(t_n^i) \cup \left((B_1^+ \setminus E_n^i(t_n^i)) \cap H_n^{i,+}\right) \cup \left((E_n^i(t_n^i) \setminus B_1^+) \cap H_n^{i,-}\right) \cup C_i \cup A_i, \tag{4.32}$$

where C_i is the cylinder given by

$$C_i := \{(y', y_N) \,:\, |y'| = 1 - l_i, |y_N| \le 2\sqrt{e_i}\},$$

and A_i is the annulus

$$A_i := \{(y', 0) \,:\, |y'| \ge 1 - l_i\}.$$

Clearly $\mathcal{H}^{N-1}(C_i) \to 0$ and $\mathcal{H}^{N-1}(A_i) \to 0$ as $i \to +\infty$.

By (4.27), (4.30) and (4.31), from (4.32) we get that for n large

$$\mathcal{H}^{N-1}\left(S(w_n^i) \setminus S(v_n^i)\right) \le \tilde{e}_i \tag{4.33}$$

where $\hat{e}_i \to 0$ as $i \to +\infty$, so that claim (4.23) is proved.

In order to prove claim (4.22), notice that

$$\mathcal{H}^{N-1}(S(v_n^i)) = \frac{\mathcal{H}^{N-1}(S(u_n) \cap B_{\rho_i}(x))}{\rho_i^{N-1}} \le \frac{\mu_n(\bar{B}_{\rho_i}(x))}{\rho_i^{N-1}},$$

and so, since $\mu_n \overset{*}{\rightharpoonup} \mu$ weakly* in the sense of measure, by assumption (4.19) we get

$$\limsup_{n\to+\infty} \mathcal{H}^{N-1}(S(v_n^i)) \le \frac{\mu(\bar{B}_{\rho_i}(x))}{\rho_i^{N-1}} \to H(x) < +\infty \quad \text{for } i \to +\infty.$$

By (4.33) we conclude that for i and n large enough

$$\mathcal{H}^{N-1}(S(w_n^i)) \le H(x) + 1.$$

By compactness for sets with finite perimeter (see for example [7, Theorem 3.39]) we get for $n \to +\infty$

$$w_n^i \to w^i \qquad \text{strongly in } L^1(B_1),$$

where $w^i \in P(B_1)$. Since by construction $w_n^i = u_{0,0,1,\nu_{S(u)}(x)}$ on a neighborhood of ∂B_1 which is independent of n, we conclude that $w^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of ∂B_1, so that claim (4.22) is proved. □

Remark 4.2. Theorem 4.1 states that in the Γ-limit process there is no interaction between bulk and surface energies, since they are constructed looking at Γ-convergence problems in Sobolev space and in the space of piecewise constant functions respectively. As a consequence, considering bulk and surface energy densities of the form $c_1 f_n$ and $c_2 g_n$ with $c_1, c_2 > 0$, we get in the limit $c_1 f$ and $c_2 g$ as bulk and surface energy densities. We remark that a key assumption for non interaction is given by equi-boundedness of $\mathcal{H}^{N-1}(K_n)$: dropping this assumption, interaction can occur even in the case of constant densities, for example $f(\xi) := |\xi|^p$ and $g(x,\nu) \equiv 1$ (if we consider in $]0,1[$ the set $K_n := \{\frac{i}{n} : i = 1, \dots, n-1\}$, we get as Γ-limit the zero functional).

We conclude the section establishing a lower semicontinuity result for SBV functions in the case of varying bulk and surface energies which is a generalization of Ambrosio's lower semicontinuity theorems [3].

Proposition 4.3. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that $\mathcal{H}^{N-1}(K_n) \le C$ for all $n \in \mathbb{N}$. Let us assume that for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{F}_n(\cdot, A)$ and $\mathcal{G}_n^-(\cdot, A)$ defined in* (3.4) *and* (3.5) *Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ and $\mathcal{G}^-(\cdot, A)$ respectively. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$.*

Then for all $A \in \mathcal{A}(\Omega)$ we have

$$\int_A f(x, \nabla u(x))\, dx \le \liminf_{n\to+\infty} \int_A f_n(x, \nabla u_n(x))\, dx, \tag{4.34}$$

and

$$\int_{S(u)\cap A} g^-(x,\nu)\, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{(S(u_n)\setminus K_n)\cap A} g_n(x,\nu)\, d\mathcal{H}^{N-1}, \tag{4.35}$$

where f and g^- are the densities of $\mathcal{F}$ and $\mathcal{G}^-$ respectively.

In particular if $K_n = \emptyset$ we have

$$\int_{S(u)\cap A} g(x,\nu)\, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{S(u_n)\cap A} g_n(x,\nu)\, d\mathcal{H}^{N-1}, \tag{4.36}$$

where g is the density of $\mathcal{G}$ defined in Proposition 3.2.

Proof. By Theorem 4.1, we have that for all $h, k \in \mathbb{N}$ and for all $A \in \mathcal{A}(\Omega)$ the functionals

$$\mathcal{E}_n^{h,k}(u, A) := h \int_A f_n(x, \nabla u(x))\, dx + k \int_{(S(u)\setminus K_n)\cap A} g_n(x,\nu)\, d\mathcal{H}^{N-1}$$

Γ-converge in the strong topology of $L^1(\Omega)$ to

$$\mathcal{E}^{h,k}(u, A) := h \int_A f(x, \nabla u(x))\, dx + k \int_{S(u)\cap A} g^-(x,\nu)\, d\mathcal{H}^{N-1}.$$

In particular by Γ-liminf inequality we have

$$\mathcal{E}^{h,k}(u, A) \le \liminf_{n\to+\infty} \mathcal{E}_n^{h,k}(u_n, A).$$

By the growth estimate (3.2) on g_n we get

$$\int_A f(x,\nabla u(x))\,dx \le \liminf_{n\to+\infty}\int_A f_n(x,\nabla u_n(x))\,dx + \frac{k}{h}\int_{(S(u_n)\setminus K_n)\cap A} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x)$$
$$\le \liminf_{n\to+\infty}\int_A f_n(x,\nabla u_n(x))\,dx + \frac{k}{h}C,$$

for some constant C independent of h and k. Since h,k are arbitrary we get that (4.34) holds. The proof of (4.35) is analogous. □

5. A new variational convergence for rectifiable sets

In this section we use the Γ-convergence results of Sections 3 and 4 in order to introduce a variational notion of convergence for rectifiable sets which will be employed in the study of stability of unilateral minimality properties.

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω, and let us assume following Ambrosio and Braides [5, Theorem 3.2] that the functionals $\mathcal{H}_n^- : P(\Omega)\times\mathcal{A}(\Omega)\to[0,+\infty)$ defined by

$$\mathcal{H}_n^-(u,A) := \mathcal{H}^{N-1}\left((S(u)\setminus K_n)\cap A\right) \tag{5.1}$$

Γ-converge with respect to the strong topology of $L^1(\Omega)$ for every $A\in\mathcal{A}(\Omega)$ to a functional $\mathcal{H}^-(\cdot,A)$, which by the representation result of Bouchitté, Fonseca, Leoni and Mascarenhas [10, Theorem 3] is of the form

$$\mathcal{H}^-(u,A) := \int_{S(u)\cap A} h^-(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{5.2}$$

for some function $h^- : \Omega\times S^{N-1}\to[0,+\infty)$. Notice that we do not use directly Proposition 3.3 (with $g_n\equiv 1$) because we are not assuming that $\mathcal{H}^{N-1}(K_n)\le C$ for some $C>0$.

Definition 5.1 (**σ-convergence of rectifiable sets**). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω. We say that K_n σ-converges in Ω to K if the functionals $(\mathcal{H}_n^-)_{n\in\mathbb{N}}$ defined in* (5.1) *Γ-converge in the strong topology of $L^1(\Omega)$ to the functional $\mathcal{H}^-$ defined in* (5.2)*, and K is the (unique) rectifiable set in Ω such that*

$$h^-(x,\nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x\in K, \tag{5.3}$$

and such that for every rectifiable set $H\subseteq\Omega$ we have

$$h^-(x,\nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x\in H \Longrightarrow H\,\tilde{\subseteq}\,K, \tag{5.4}$$

where $H\,\tilde{\subseteq}\,K$ means that $H\subseteq K$ up to a set of $\mathcal{H}^{N-1}$-measure zero.

Remark 5.2. From Definition 5.1 it comes directly that σ-convergence of rectifiable sets is stable under infinitesimal perturbation in length. More precisely, let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K, and let $(\tilde K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that $\mathcal{H}^{N-1}(\tilde K_n \triangle K_n)\to 0$, where $\triangle$ denotes the symmetric difference of sets. Then $\tilde K_n$ σ-converges in Ω to K.

Let us now come to the main properties of σ-convergence for rectifiable sets. By compactness of Γ-convergence, we deduce the following compactness result for σ-convergence.

Proposition 5.3 (**compactness**). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n)\le C$. Then there exists a subsequence $(n_h)_{h\in\mathbb{N}}$ and a rectifiable set K in Ω such that K_{n_h} σ-converges in Ω to K. Moreover*

$$\mathcal{H}^{N-1}(K) \le \liminf_{n\to+\infty}\mathcal{H}^{N-1}(K_n). \tag{5.5}$$

Proof. By Proposition 3.3, up to a subsequence we have that for all $A\in\mathcal{A}(\Omega)$ the functionals $\mathcal{H}_n^-(\cdot,A)$ defined in (5.1) Γ-converge in the strong topology of $L^1(\Omega)$ to a functional $\mathcal{H}^-(\cdot,A)$ which can be represented through a density h^- according to (5.2).

Let us consider the class

$$\mathcal{K} := \{H\subseteq\Omega : H \text{ is rectifiable and } h^-(x,\nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x\in H\}.$$

Notice that $\mathcal{K}$ contains at least the empty set. Let us prove that for all $H \in \mathcal{K}$ we have

$$\mathcal{H}^{N-1}(H) \leq L := \liminf_{n\to+\infty} \mathcal{H}^{N-1}(K_n). \tag{5.6}$$

In fact let $H \in \mathcal{K}$. Since $H = \cup_i H_i$ with H_i compact and rectifiable with $\mathcal{H}^{N-1}(H_i) < +\infty$, it is not restrictive to consider $\mathcal{H}^{N-1}(H) < +\infty$. Given $\varepsilon > 0$, by a covering argument we can find an open set U and a piecewise constant function $v \in P(\Omega)$ such that

$$\mathcal{H}^{N-1}(H \setminus U) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}\left((S(v) \triangle H) \cap U\right) < \varepsilon, \tag{5.7}$$

where $\triangle$ denotes the symmetric difference of sets. Since $h^- \leq 1$ we have

$$\mathcal{H}^-(v, U) = \int_{S(v)\cap U} h^-(x,\nu)\, d\mathcal{H}^{N-1}(x) = \int_{(S(v)\setminus H)\cap U} h^-(x,\nu)\, d\mathcal{H}^{N-1}(x) < \varepsilon. \tag{5.8}$$

Let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to $\mathcal{H}^-(\cdot, U)$. Then by (5.8) we have that

$$\limsup_{n\to+\infty} \mathcal{H}^{N-1}\left((S(v_n) \setminus K_n) \cap U\right) < \varepsilon. \tag{5.9}$$

By Ambrosio's Theorem, (5.7), and (5.9) we deduce that

$$\begin{aligned} \mathcal{H}^{N-1}(H) &= \mathcal{H}^{N-1}(H \cap U) + \mathcal{H}^{N-1}(H \setminus U) \leq \mathcal{H}^{N-1}(S(v) \cap U) + 2\varepsilon \\ &\leq \liminf_{n\to+\infty} \mathcal{H}^{N-1}(S(v_n) \cap U) + 2\varepsilon \leq \liminf_{n\to+\infty} \mathcal{H}^{N-1}(K_n) + 3\varepsilon = L + 3\varepsilon. \end{aligned}$$

Since ε is arbitrary we get that (5.6) holds.

Let us now consider

$$\tilde{L} := \sup\{\mathcal{H}^{N-1}(H) \,:\, H \in \mathcal{K}\} < +\infty,$$

and let $(H_k)_{k\in\mathbb{N}}$ be a maximizing sequence for $\tilde{L}$. We set $K := \bigcup_{k=1}^{\infty} H_k$. Clearly (5.5) and (5.3) hold. Moreover, since $\mathcal{H}^{N-1}(K) = \tilde{L}$ we have that (5.4) holds, and the proof is concluded. □

Remark 5.4. Let $\Omega := (-1,1) \times (-1,1)$ in $\mathbb{R}^2$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of closed sets with $K_n \to K := \{(-1,1)\} \times \{0\}$ in the Hausdorff metric and such that $\mathcal{H}^1 \llcorner K_n \overset{*}{\rightharpoonup} a\mathcal{H}^1 \llcorner K$ weakly* in the sense of measures. If $a < 1$ by (5.5) we deduce that K_n σ-converges in Ω to the empty set. We stress that the condition $a \geq 1$ is not enough to guarantee that K is the σ-limit of $(K_n)_{n\in\mathbb{N}}$. In fact considering

$$K_n := \bigcup_{i=-n}^{n} \left\{\frac{i}{n}\right\} \times \left[-\frac{1}{n}, \frac{1}{n}\right]$$

we have $\mathcal{H}^1 \llcorner K_n \overset{*}{\rightharpoonup} 2\mathcal{H}^1 \llcorner K$ weakly* in the sense of measures. However also in this case we have that K_n σ-converges in Ω to the empty set. In fact let us consider $u \in P(\Omega)$ such that $u = 1$ in $\Omega^+ := (-1,1) \times (0,1)$ and $u = 0$ in $\Omega^- := (-1,1) \times (-1,0)$, and let u_n be a sequence in $P(\Omega)$ such that $u_n \to u$ strongly in $L^1(\Omega)$ and with $\mathcal{H}^{N-1}(S(u_n)) \leq C$. Let (e_1, e_2) be the canonical base of $\mathbb{R}^2$. By Ambrosio's theorem we get that

$$\nu[u_n]\mathcal{H}^1 \llcorner S(u_n) \overset{*}{\rightharpoonup} e_2 \mathcal{H}^1 \llcorner S(u) \quad \text{weakly* in the sense of measures.}$$

Considering the vector field φe_2 with $\varphi \in C_c^\infty(\Omega)$ we get

$$\int_{S(u_n)\setminus K_n} \varphi e_2 \cdot \nu[u_n]\, d\mathcal{H}^1 = \int_{S(u_n)} \varphi e_2 \cdot \nu[u_n]\, d\mathcal{H}^1 \to \int_K \varphi\, d\mathcal{H}^1.$$

If $0 \leq \varphi \leq 1$ we have that

$$\int_K \varphi\, d\mathcal{H}^1 \leq \liminf_n \mathcal{H}^1(S(u_n) \setminus K_n) = \liminf_{n\to+\infty} \mathcal{H}_n^-(u_n)$$

so that we deduce $\liminf_{n\to+\infty} \mathcal{H}_n^-(u_n) = 2$. By Γ-liminf we conclude that $\mathcal{H}^-(u) \geq 2$ that is $h^-(x, e_2) = 1$ for $\mathcal{H}^1$-a.e. $x \in K$. Since the σ-limit of $(K_n)_{n\in\mathbb{N}}$ can be only contained in K, we deduce that the σ-limit is the empty set.

The following proposition shows that the σ-limit is a natural limit candidate for a sequence of rectifiable sets in connection with unilateral minimality properties (see the Introduction).

Proposition 5.5. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$ and K_n σ-converging in Ω to K. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of Borel functions satisfying the estimates* (3.2), *and let g^- be the energy density of the Γ-limit in the strong topology of $L^1(\Omega)$ of the functionals $(\mathcal{G}_n^-)_{n\in\mathbb{N}}$ defined in* (3.5)*. Then we have*

$$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K,$$

and for every rectifiable set $H \subseteq \Omega$

$$g^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H \Longrightarrow H \tilde{\subseteq} K.$$

Proof. Notice that in view of the growth estimate (3.2) for g_n we get

$$\alpha\mathcal{H}^-(u, A) \le \mathcal{G}^-(u, A) \le \beta\mathcal{H}^-(u, A) \tag{5.10}$$

for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Moreover by the assumption $\mathcal{H}^{N-1}(K_n) \le C$, by Proposition 3.3 we have the explicit formula (3.14) for g^- and an analogous one for h^- (with $\mathcal{H}^-$ in place of $\mathcal{G}^-$). From (5.10) we deduce that

$$\alpha h^-(x, \nu) \le g^-(x, \nu) \le \beta h^-(x, \nu)$$

for all $x \in \Omega$ and $\nu \in S^{N-1}$. Then the thesis follows from the definition of σ-convergence. □

Remark 5.6. Notice that Proposition 5.5 holds also without requiring that $\mathcal{H}^{N-1}(K_n) \le C$ (which is however natural for applications to stability of unilateral minimality properties arising in fracture mechanics, see the Introduction). In fact it is sufficient to use the inequality (5.10) and a covering argument as in the proof of Proposition 5.3.

The following lower semicontinuity result for surface energies along sequences of rectifiable sets converging in the sense of σ-convergence will be employed in Section 8.

Proposition 5.7 (**lower semicontinuity**)**.** *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of Borel functions satisfying the estimates* (3.2), *and let g be the associated function according to Proposition 3.2. Then we have*

$$\int_K g(x, \nu)\, d\mathcal{H}^{N-1}(x) \le \liminf_{n\to+\infty} \int_{K_n} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x).$$

Proof. Let $H \tilde{\subseteq} K$ with $\mathcal{H}^{N-1}(H) < +\infty$. Given $\varepsilon > 0$, by a covering argument we can find an open set U and a piecewise constant function $v \in P(\Omega)$ such that

$$\mathcal{H}^{N-1}(H \setminus U) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}\left((S(v) \triangle H) \cap U\right) < \varepsilon, \tag{5.11}$$

where $\triangle$ denotes the symmetric difference of sets. If $(v_n)_{n\in\mathbb{N}}$ is a recovering sequence for v with respect to $\mathcal{H}^-(\cdot, U)$ defined in (5.2) we have

$$\limsup_{n\to+\infty} \mathcal{H}^{N-1}\left((S(v_n) \setminus K_n) \cap U\right) < \varepsilon. \tag{5.12}$$

By (5.11), (5.12), by the growth estimate on g_n, and by Γ-convergence we have that

$$\begin{aligned}
\int_H g(x, \nu)\, d\mathcal{H}^{N-1}(x) &= \int_{H\cap U} g(x, \nu)\, d\mathcal{H}^{N-1}(x) + \int_{H\setminus U} g(x, \nu)\, d\mathcal{H}^{N-1}(x) \\
&\le \int_{S(v)\cap U} g(x, \nu)\, d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \le \liminf_{n\to+\infty} \int_{S(v_n)\cap U} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \\
&\le \liminf_{n\to+\infty} \int_{K_n} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) + 3\beta\varepsilon.
\end{aligned}$$

Since ε is arbitrary we deduce

$$\int_H g(x, \nu)\, d\mathcal{H}^{N-1}(x) \le \liminf_{n\to+\infty} \int_{K_n} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x),$$

and since H is arbitrary in K the proof is concluded. □

The following proposition is essential in the study of stability of unilateral minimality properties.

Proposition 5.8. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $1 < p < +\infty$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$ and $\mathcal{H}^{N-1}(S(u_n)\setminus K_n) \to 0$. Then $S(u) \tilde{\subseteq} K$.*

Proof. Let us consider $\tilde{K}_n := S(u_n) \cap K_n$. By compactness, up to a further subsequence we have that $\tilde{K}_n$ σ-converges in Ω to a rectifiable set $\tilde{K} \tilde{\subseteq} K$. Let $\tilde{h}^-$ be the density associated to $(\tilde{K}_n)_{n\in\mathbb{N}}$ according to Definition 5.1. By lower semicontinuity given by Proposition 4.3 we have

$$\int_{S(u)} \tilde{h}^-(x,\nu)\, d\mathcal{H}^{N-1}(x) \le \liminf_{n\to+\infty} \mathcal{H}^{N-1}\left(S(u_n)\setminus \tilde{K}_n\right) \le \liminf_{n\to+\infty} \mathcal{H}^{N-1}\left(S(u_n)\setminus K_n\right) = 0.$$

We deduce that

$$\tilde{h}^-(x,\nu_{S(u)}(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in S(u),$$

so that by definition of σ-limit we conclude $S(u) \tilde{\subseteq} \tilde{K} \tilde{\subseteq} K$. □

The next corollary shows that our σ-limit of rectifiable sets always contains the σ^p-limit introduced by Dal Maso, Francfort and Toader in [18] in order to study quasistatic crack growth in nonlinear elasticity. We recall that K_n σ^p-converges in Ω to K if the following hold:

(1) if $u_h \rightharpoonup u$ weakly in $SBV^p(\Omega)$ with $S(u_h) \subseteq K_{n_h}$, then $S(u) \subseteq K$;
(2) $K = S(u)$ for some $u \in SBV^p(\Omega)$, and there exists $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$ with $S(u_n) \subseteq K_n$.

Corollary 5.9. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $1 < p < +\infty$, and let us assume that K_n σ^p-converges in Ω to some rectifiable set $\tilde{K}$. Then $\tilde{K} \tilde{\subseteq} K$.*

Proof. The proof readily follows from Proposition 5.8 and point (2) of the definition of σ^p-convergence. □

Remark 5.10. Notice that in general we can have that the σ^p-limit $\tilde{K}$ of $(K_n)_{n\in\mathbb{N}}$ is strictly contained in K. In fact we can consider $\Omega := (-1,1)\times(-1,1)$ in $\mathbb{R}^2$, and

$$K_n := \{(-1,1)\setminus L_n\} \times \{0\}$$

with $L_n \subseteq (-1,1)$ and $|L_n| \to 0$. In this case we get $K = (-1,1)\times\{0\}$, while if L_n is chosen in such a way that its c_p-*capacity* is big enough (see the celebrated example of the Neumann sieve, we refer to [26]) we get $\tilde{K} = \emptyset$.

This example is based on the fact that the σ^p-limit is influenced by infinitesimal perturbations of the K_n while the set K is not, as pointed out in Remark 5.2.

In Sections 7 and 8, we will need a definition of σ-convergence in the closed set $\overline{\Omega}$.

Definition 5.11 (σ**-convergence in** $\overline{\Omega}$)**.** *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$. We say that K_n σ-converges in $\overline{\Omega}$ to $K \tilde{\subseteq} \overline{\Omega}$ if K_n σ-converges in Ω' to K for every open bounded set Ω' such that $\overline{\Omega} \subseteq \Omega'$.*

Notice that to check the σ-convergence in $\overline{\Omega}$ of rectifiable sets, it is enough check σ-convergence in Ω' for just one Ω' with $\overline{\Omega} \subseteq \Omega'$.

6. Stability of unilateral minimality properties

In this section we apply the results of Section 4 and Section 5 to obtain the stability result of unilateral minimality properties under Γ-convergence for bulk and surface energies.

Definition 6.1 (unilateral minimizers). *Let $f : \Omega \times \mathbb{R}^N \to [0,+\infty[$ be a Carathéodory function and let $g : \Omega \times S^{N-1} \to [0,+\infty[$ be a Borel function satisfying the growth estimates* (3.1) *and* (3.2)*. We say that the pair (u,K) with $u \in SBV^p(\Omega)$ and K rectifiable set in Ω is a unilateral minimizer with respect to f and g if $S(u) \tilde{\subseteq} K$, and*

$$\int_\Omega f(x,\nabla u(x))\, dx \le \int_\Omega f(x,\nabla v(x))\, dx + \int_{H\setminus K} g(x,\nu),$$

for all pairs (v,H) with $v \in SBV^p(\Omega)$, and H rectifiable set in Ω such that $S(v) \tilde{\subseteq} H$. Here $\tilde{\subseteq}$ means "contained up to a set of $\mathcal{H}^{N-1}$-measure zero".

As in the previous sections, let $f_n : \Omega \times \mathbb{R}^N \to [0,+\infty[$ be Carathéodory functions and let $g_n : \Omega \times S^{N-1} \to [0,+\infty[$ be Borel functions satisfying the growth estimates (3.1) and (3.2).

Let us assume that the functionals $(\mathcal{F}_n(\cdot,A))_{n\in\mathbb{N}}$ and $(\mathcal{G}_n(\cdot,A))_{n\in\mathbb{N}}$ defined in (3.4) and (3.6) Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot,A)$ and $\mathcal{G}(\cdot,A)$ for every $A \in \mathcal{A}(\Omega)$ respectively. Let f be the density of $\mathcal{F}$ according to Proposition 3.1 and let g be the density of $\mathcal{G}$ according to Proposition 3.2.

The main result of the paper is the following stability result for unilateral minimality properties under σ-convergence of rectifiable sets (see Definition 5.1), and Γ-convergence of bulk and surface energies.

Theorem 6.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$ and such that K_n σ-converges in Ω to K. Let us assume that the pair $(u_n,K_n)_{n\in\mathbb{N}}$ is a unilateral minimizer for f_n and g_n.*

Then (u,K) is a unilateral minimizer for f and g. Moreover we have

$$\lim_{n\to+\infty} \int_\Omega f_n(x,\nabla u_n(x))\,dx = \int_\Omega f(x,\nabla u(x))\,dx. \tag{6.1}$$

Proof. By Theorem 4.1 we have that the functionals

$$\mathcal{E}_n(u) := \begin{cases} \int_\Omega f_n(x,\nabla u(x))\,dx + \int_{S(u)\setminus K_n} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x) & u \in SBV^p(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge with respect to the strong topology of $L^1(\Omega)$ to the functional

$$\mathcal{E}(u) := \begin{cases} \int_\Omega f(x,\nabla u(x))\,dx + \int_{S(u)} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x) & u \in SBV^p(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where f and g^- are defined in (3.10) and (3.14) respectively, with $g^- \le g$.

By Proposition 5.8 we have $S(u) \tilde{\subseteq} K$, so that u is admissible for K, while by Proposition 5.5 we have that

$$g^-(x,\nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K. \tag{6.2}$$

Then the unilateral minimality of the pair (u,K) easily follows. In fact, by Γ-convergence we have that u is a minimizer for $\mathcal{E}$ and $\mathcal{E}_n(u_n) \to \mathcal{E}(u)$. By (6.2) and since $g^- \le g$, for all pairs (v,H) with $S(v) \tilde{\subseteq} H$ we have

$$\begin{aligned}\int_\Omega f(x,\nabla u(x))\,dx = \mathcal{E}(u) \le \mathcal{E}(v) = \int_\Omega f(x,\nabla v(x))\,dx + \int_{S(v)} g^-(x,\nu)\,d\mathcal{H}^{N-1} \\ = \int_\Omega f(x,\nabla v(x))\,dx + \int_{S(v)\setminus K} g^-(x,\nu) \le \int_\Omega f(x,\nabla v(x))\,dx + \int_{H\setminus K} g(x,\nu),\end{aligned}$$

so that the unilateral minimality property holds. The convergence of bulk energies (6.1) is given by the convergence $\mathcal{E}_n(u_n) \to \mathcal{E}(u)$. □

Remark 6.3 (stability under σ^p-convergence). In the case of fixed bulk and surface energy densities f and g, Dal Maso, Francfort and Toader [18] proved the stability of the unilateral minimality property under σ^p-convergence for the rectifiable sets K_n (see Section 5 just before Corollary 5.9 for the definition). The analogue result in the case of varying energies readily follows by Theorem 6.2. In fact by Corollary 5.9 we have that if K_n σ^p-converges in Ω to $\tilde{K}$, then $\tilde{K}$ is contained in the σ-limit of $(K_n)_{n\in\mathbb{N}}$. Since $S(u) \tilde{\subseteq} \tilde{K}$, we get that the unilateral minimality of the pair $(u,\tilde{K})$ is implied by the unilateral minimality of (u,K).

As mentioned in the Introduction, a method for proving stability of unilateral minimality properties nearer to the approach of [18] would be to prove a generalization of the Transfer of Jump Sets by Francfort and Larsen [21, Theorem 2.1] to the case of varying energies. The following theorem based on the arguments of Section 4 provides such a generalization.

Theorem 6.4 (Transfer of Jump Sets). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$ and K_n σ-converging in Ω to K. For every $v \in SBV^p(\Omega)$ there exists $(v_n)_{n\in\mathbb{N}}$ sequence in $SBV^p(\Omega)$ with $v_n \rightharpoonup v$ weakly in $SBV^p(\Omega)$ and such that*

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla v_n(x))\, dx = \int_\Omega f(x, \nabla v(x))\, dx$$

and

$$\limsup_{n\to+\infty} \int_{S(v_n)\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) \le \int_{S(v)\setminus K} g(x,\nu)\, d\mathcal{H}^{N-1}(x).$$

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to $(\mathcal{E}_n)_{n\in\mathbb{N}}$ defined in (3.3) which Γ-converge to $\mathcal{E}$ defined in Theorem 4.1. By growth estimates on f_n and g_n, and since $\mathcal{H}^{N-1}(K_n) \le C$, we get $v_n \rightharpoonup v$ weakly in $SBV^p(\Omega)$. Taking into account the lower semicontinuity result of Proposition 4.3, we get that

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla v_n(x))\, dx = \int_\Omega f(x, \nabla v(x))\, dx$$

and

$$\lim_{n\to+\infty} \int_{S(v_n)\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1} = \int_{S(v)} g^-(x,\nu)\, d\mathcal{H}^{N-1} \le \int_{S(v)\setminus K} g(x,\nu)\, d\mathcal{H}^{N-1}$$

because $g^- = 0$ on K, and $g^- \le g$. □

7. Stability of unilateral minimality properties with boundary conditions

In view of the application of Section 8, we need a stability result for unilateral minimality properties with boundary conditions.

Let $\partial_D\Omega \subseteq \partial\Omega$. In order to take into account a boundary datum on $\partial_D\Omega$, we will use the following notation: if $u, \psi \in SBV(\Omega)$ we set

$$S^\psi(u) := S(u) \cup \{x \in \partial_D\Omega \,:\, u(x) \neq \psi(x)\}, \tag{7.1}$$

where the inequality on $\partial_D\Omega$ is intended in the sense of traces.

In order to set the problem, let $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function satisfying the growth estimate (3.1), and let $g_n : \overline{\Omega} \times S^{N-1} \to [0, +\infty[$ be a Borel function satisfying the growth estimate (3.2). We consider unilateral minimality properties of the form

$$\int_\Omega f_n(x, \nabla u_n)\, dx \le \int_\Omega f_n(x, \nabla v)\, dx + \int_{H\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) \tag{7.2}$$

for every $v \in SBV^p(\Omega)$ and for every rectifiable set H in $\overline{\Omega}$ such that $S^{\psi_n}(v) \tilde{\subseteq} H$. Here $(K_n)_{n\in\mathbb{N}}$ is a sequence of rectifiable sets in $\overline{\Omega}$ with $\mathcal{H}^{N-1}(K_n) \le C$, $(u_n)_{n\in\mathbb{N}}$ is a sequence in $SBV^p(\Omega)$ with $S^{\psi_n}(u_n) \tilde{\subseteq} K_n$, $\psi_n \in W^{1,p}(\Omega)$ with

$$\psi_n \to \psi \qquad \text{strongly in } W^{1,p}(\Omega),$$

and $S^{\psi_n}(\cdot)$ is defined in (7.1).

In order to treat $S^{\psi_n}(\cdot)$ as an internal jump and in order to recover the surface energy on $\partial_D\Omega$ for the minimality property in the limit, let us consider an open bounded set Ω' such that $\overline{\Omega} \subset \Omega'$ and let us consider $g'_n : \Omega' \times S^{N-1} \to [0, +\infty[$ such that

$$g'_n(x,\nu) := \begin{cases} g_n(x,\nu) & \text{if } x \in \overline{\Omega}, \\ \beta + 1 & \text{otherwise.} \end{cases}$$

Let us consider the functionals $\mathcal{G}'_n : P(\Omega') \times \mathcal{A}(\Omega') \to [0, +\infty]$ defined by

$$\mathcal{G}'_n(v, A) := \int_{S(v)\cap A} g'_n(x,\nu)\, d\mathcal{H}^{N-1}(x)$$

and let $\mathcal{G}' : P(\Omega') \times \mathcal{A}(\Omega') \to [0,+\infty]$ be their Γ-limit in the strong topology of $L^1(\Omega')$, which according to Proposition 3.3 is of the form

$$\mathcal{G}'(v,A) := \int_{S(v)\cap A} g'(x,\nu)\, d\mathcal{H}^{N-1}(x). \tag{7.3}$$

We clearly have $g'(x,\nu) = g(x,\nu)$ for $x \in \Omega$, where g is the surface energy density defined in (3.12), while it turns out that (see Remark 7.2) the surface energy given by the restriction of g' to $\partial\Omega \times S^{N-1}$ is completely determined by the functions g_n.

Let us set

$$f'_n(x,\xi) := \begin{cases} f_n(x,\xi) & \text{if } x \in \Omega, \\ \alpha|\xi|^p & \text{otherwise,} \end{cases}$$

and let f' be the energy density of the Γ-limit of the functionals on $W^{1,p}(\Omega')$ associated to f'_n according to Proposition 3.1. We easily have that

$$f'(x,\xi) := \begin{cases} f(x,\xi) & \text{if } x \in \Omega, \\ \alpha|\xi|^p & \text{otherwise,} \end{cases}$$

where f is defined in (3.10). Since Ω is Lipschitz, we can assume using an extension operator that $\psi_n, \psi \in W^{1,p}(\mathbb{R}^N)$ and that $\psi_n \to \psi$ strongly in $W^{1,p}(\mathbb{R}^N)$.

Before stating our stability result, we need the following Γ-convergence result, which is a version of Theorem 4.1 that takes into account boundary data.

Lemma 7.1. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$ such that $\mathcal{H}^{N-1}(K_n) \le C$. Let us assume that the functionals*

$$\mathcal{E}'_n(v) := \begin{cases} \int_{\Omega'} f'_n(x,\nabla v(x))\,dx + \int_{S(v)\setminus K_n} g'_n(x,\nu)\,d\mathcal{H}^{N-1}(x) & \text{if } v \in SBV^p(\Omega'), \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge in the strong topology of $L^1(\Omega')$ according to Theorem 4.1 to

$$\mathcal{E}'(v) := \begin{cases} \int_{\Omega'} f'(x,\nabla v(x))\,dx + \int_{S(v)} g'^{-}(x,\nu)\,d\mathcal{H}^{N-1}(x) & \text{if } v \in SBV^p(\Omega'), \\ +\infty & \text{otherwise.} \end{cases}$$

Then we have that the functionals

$$\tilde{\mathcal{E}}'_n(v) := \begin{cases} \mathcal{E}'_n(v) & \text{if } v = \psi_n \text{ on } \Omega' \setminus \overline{\Omega}, \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge in the strong topology of $L^1(\Omega')$ to

$$\tilde{\mathcal{E}}'(v) := \begin{cases} \mathcal{E}'(v) & \text{if } v = \psi \text{ on } \Omega' \setminus \overline{\Omega}, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Let $v \in SBV^p(\Omega')$ with $v = \psi$ on $\Omega' \setminus \overline{\Omega}$, and let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to the functionals $\mathcal{E}'_n$. We have that

$$\nabla v_n \to \nabla\psi \qquad \text{strongly in } L^p(\Omega' \setminus \overline{\Omega}; \mathbb{R}^N), \tag{7.4}$$

and

$$\mathcal{H}^{N-1}(S(v_n) \cap (\Omega' \setminus \overline{\Omega})) \to 0. \tag{7.5}$$

In fact we have that for all $U \in \mathcal{A}(\Omega')$ such that $\overline{U} \subseteq \Omega' \setminus \overline{\Omega}$ and $\mathcal{E}'(v, \partial U) = 0$

$$\nabla v_n \to \nabla\psi \qquad \text{strongly in } L^p(U; \mathbb{R}^N), \tag{7.6}$$

and

$$\mathcal{H}^{N-1}(S(v_n) \cap U) \to 0. \tag{7.7}$$

Let $\varepsilon > 0$ and let us consider an open set $V \in \mathcal{A}(\Omega')$ such that $\partial\Omega \subseteq V$, $\mathcal{E}'(v, \partial V) = 0$,

$$\int_{V\cap\Omega} |a_1|\,dx < \varepsilon, \quad \int_V f'(x,\nabla v(x))\,dx < \varepsilon, \quad \text{and} \int_V f'(x,\nabla\psi(x))\,dx < \varepsilon, \tag{7.8}$$

where a_1 appears in the growth estimate (3.1) for f_n. Then for n large (no interaction between bulk and surface part occurs) we have

$$\int_V f'_n(x, \nabla v_n(x))\, dx < \varepsilon. \tag{7.9}$$

Notice that

$$\int_{\Omega'\setminus\overline{\Omega}} |\nabla v_n - \nabla\psi|^p\, dx = \int_{\Omega'\setminus(\Omega\cup V)} |\nabla v_n - \nabla\psi|^p\, dx + \int_{V\setminus\overline{\Omega}} |\nabla v_n - \nabla\psi|^p\, dx$$
$$\leq \int_{\Omega'\setminus(\Omega\cup V)} |\nabla v_n - \nabla\psi|^p\, dx + \frac{2^{p-1}}{\alpha}\int_V f'_n(x, \nabla v_n(x)) + f'(x, \nabla\psi(x))\, dx + \frac{2^{p-1}}{\alpha}\int_{V\cap\Omega} 2|a_1|\, dx.$$

By (7.6) we have $\nabla v_n \to \nabla\psi$ strongly in $L^p(\Omega' \setminus (\Omega \cup V); \mathbb{R}^N)$. In view of (7.8) and (7.9), and since ε is arbitrary, we get that (7.4) holds.

The idea we follow in order to prove (7.5) is the following: if (7.5) does not hold, we can modify the sequence v_n moving $S(v_n)$ inside Ω, where the surface energy is much less than in $\Omega' \setminus \overline{\Omega}$, getting a sequence which is more convenient in energy. But this is against the fact that v_n is a recovering sequence. Let us come to the details. Up to a subsequence we have

$$\mu_n := \mathcal{H}^{N-1} \llcorner (S(v_n) \cap (\Omega' \setminus \overline{\Omega})) \overset{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega').$$

In view of (7.7), in order to prove (7.5) it is sufficient to show that $\mu(\partial\Omega) = 0$. Let us assume by contradiction that $\mu(\partial\Omega) \neq 0$: then there exists a cube Q_ρ of center $x \in \partial\Omega$ and edge 2ρ such that $\mathcal{E}'(v, \partial Q_\rho) = 0$ and

$$\mu(Q_\rho) > \sigma > 0. \tag{7.10}$$

Up to a translation we may assume that $x = 0$, and moreover we can assume that

$$\Omega \cap Q_\rho = \{(x', y) : x' \in (-\rho, \rho), y \in (-\rho, h(x'))\},$$

where (x', y) is a suitable orthogonal coordinate system and h is a Lipschitz function. Let $\eta > 0$ be such that setting

$$V_\eta := \{(x', y) : x' \in (-\rho, \rho),\ y \in (h(x') - \eta, h(x') + \eta)\}$$

we have $V_\eta \subseteq Q_\rho$, and $\mathcal{E}'(v, \partial V_\eta) = 0$. Let us set

$$V_\eta^- := \{(x', y) \in V_\eta : y < h(x')\} \quad \text{and} \quad V_\eta^+ := \{(x', y) \in V_\eta : y > h(x')\}.$$

By (7.10) we have that for n large

$$\mathcal{H}^{N-1}(S(v_n) \cap V_\eta^+) > \sigma. \tag{7.11}$$

Let $\hat{v}$ be the function defined on V_η obtained reflecting $v_{|V_\eta^+}$ to V_η^-: more precisely let us set

$$\hat{v} = \begin{cases} v(x', y) & \text{if } (x', y) \in V_\eta^+, \\ v(x', 2h(x') - y) & \text{if } (x', y) \in V_\eta^-. \end{cases}$$

We clearly have $v \in W^{1,p}(V_\eta)$. Let $\hat{v}_n$ be obtained in the same way from $(v_n)_{|V_\eta^+}$. Let us consider

$$w_n := v_n + \hat{v} - \hat{v}_n.$$

We have $w_n \rightharpoonup v$ weakly in $SBV^p(V_\eta)$ so that by lower semicontinuity given by Proposition 4.3 we get

$$\int_{S(v)\cap V_\eta} g'^{-}(x, \nu)\, d\mathcal{H}^{N-1}(x) \leq \liminf_{n\to+\infty} \int_{(S(w_n)\setminus K_n)\cap V_\eta} g'_n(x, \nu)\, d\mathcal{H}^{N-1}(x). \tag{7.12}$$

On the other hand, since $\mathcal{E}'(v, \partial V_\eta) = 0$, we have that v_n is a recovering sequence for v in V_η. In particular we get that

$$\int_{S(v)\cap V_\eta} g'^{-}(x, \nu)\, d\mathcal{H}^{N-1}(x) = \lim_{n\to+\infty} \int_{(S(v_n)\setminus K_n)\cap V_\eta} g'_n(x, \nu)\, d\mathcal{H}^{N-1}(x). \tag{7.13}$$

Formulas (7.12) and (7.13) give a contradiction because for n large by (7.11) and since $K_n \tilde{\subseteq} \overline{\Omega}$ and $S(w_n) \tilde{\subseteq} \overline{\Omega} \cap Q_\rho$ (recall that $g'_n(x,\nu) = \beta + 1$ for $x \in \Omega' \setminus \overline{\Omega}$)

$$\int_{(S(v_n)\setminus K_n)\cap V_\eta} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) - \int_{(S(w_n)\setminus K_n)\cap V_\eta} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) > \sigma.$$

We conclude that (7.5) holds.

We are now in a position to prove the Γ-limsup inequality for $\tilde{\mathcal{E}}'_n$ and $\tilde{\mathcal{E}}'$ (the Γ-liminf is immediate from the Γ-convergence of $\mathcal{E}'_n$ to $\mathcal{E}'$ and the fact that the constraint is closed under the strong topology of $L^1(\Omega)$). Let $\varepsilon > 0$, and let $U \in \mathcal{A}(\Omega')$ be such that $\partial\Omega \subseteq U$, $\mathcal{E}'(v, \partial U) = 0$, and

$$\int_U f(x, \nabla v)\, dx < \varepsilon. \tag{7.14}$$

In view of (7.4) and (7.5) we can find $\varphi_n \in SBV^p(\Omega')$ such that $\varphi_n = \psi_n - v_n$ on $\Omega' \setminus \overline{\Omega}$, $\varphi_n = 0$ on $\Omega \setminus U$ and

$$\begin{aligned} &\varphi_n \to 0 \qquad \text{strongly in } L^1(\Omega'),\\ &\nabla\varphi_n \to 0 \qquad \text{strongly in } L^p(\Omega'; \mathbb{R}^N),\\ &\mathcal{H}^{N-1}(S(\varphi_n)) \to 0. \end{aligned}$$

Let us consider

$$\tilde{v}_n := v_n + \varphi_n.$$

We have $\tilde{v}_n = \psi_n$ on $\Omega' \setminus \overline{\Omega}$. Moreover

$$\limsup_{n\to+\infty} \int_{S(\tilde{v}_n)\setminus K_n} g'_n(x,\nu)\, d\mathcal{H}^{N-1} = \limsup_{n\to+\infty} \int_{S(v_n)\setminus K_n} g'_n(x,\nu)\, d\mathcal{H}^{N-1},$$

and using the growth estimate on f'_n

$$\begin{aligned} \limsup_{n\to+\infty} &\left| \int_{\Omega'} f'_n(x, \nabla\tilde{v}_n(x))\, dx - \int_{\Omega'} f'_n(x, \nabla v_n(x))\, dx \right| \\ &\le \limsup_{n\to+\infty} \int_{U\cap\Omega} f_n(x, \nabla\tilde{v}_n(x)) + f_n(x, \nabla v_n(x))\, dx \\ &\le \limsup_{n\to+\infty} \int_U a_2(x)\, dx + \left(\frac{2^{p-1}}{\alpha} + 1\right) \int_U f_n(x, \nabla v_n(x))\, dx \\ &\qquad + \frac{2^{p-1}}{\alpha} \int_U |a_1|\, dx + 2^{p-1} \int_U |\nabla\varphi_n|^p\, dx. \end{aligned}$$

By (7.14) we get

$$\limsup_{n\to+\infty} \int_U f_n(x, \nabla v_n(x))\, dx < \varepsilon.$$

Then we conclude

$$\limsup_{n\to+\infty} \left| \int_{\Omega'} f'_n(x, \nabla\tilde{v}_n(x))\, dx - \int_{\Omega'} f'_n(x, \nabla v_n(x))\, dx \right| \le e(\varepsilon),$$

with $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. We deduce that

$$\limsup_{n\to+\infty} \tilde{\mathcal{E}}'(\tilde{v}_n) \le \tilde{\mathcal{E}}'(v) + e(\varepsilon),$$

with $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since ε is arbitrary, using a diagonal argument we have that the Γ-limsup inequality is proved. □

Remark 7.2. In view of Lemma 7.1 we can prove that the surface energy determined by the restriction of g' to $\partial\Omega$ is actually independent of the choice of Ω' and of the constant value c' of g'_n on $\Omega' \setminus \overline{\Omega}$ provided that $c' > \beta$. In fact g' is the surface energy density of the Γ-limit in the strong topology of $L^1(\Omega')$ of the functionals on $SBV^p(\Omega')$ defined as

$$\hat{\mathcal{E}}'_n(v) := \int_{\Omega'} f'_n(x, \nabla v(x))\, dx + \int_{S(v)} g'_n(x,\nu)\, d\mathcal{H}^{N-1}(x).$$

Following the proof of Lemma 7.1 (for the functionals $\mathcal{E}'_n$ with $K_n = \emptyset$), if $v = \psi$ outside $\overline{\Omega}$, we can find $(v_n)_{n\in\mathbb{N}}$ recovering sequence for v with respect to $(\hat{\mathcal{E}}'_n, \Omega', c')$ such that $v_n = \psi_n$ outside $\overline{\Omega}$. Since $\hat{\mathcal{E}}'(v, \Omega') = \lim_{n\to+\infty} \hat{\mathcal{E}}'_n(v_n, \Omega')$ and

$$\lim_{n\to+\infty} \hat{\mathcal{E}}'_n(v_n, \Omega' \setminus \overline{\Omega}) = \lim_{n\to+\infty} \int_{\Omega'\setminus\overline{\Omega}} \alpha|\nabla\psi_n|^p \, dx = \int_{\Omega'\setminus\overline{\Omega}} \alpha|\nabla\psi|^p \, dx = \hat{\mathcal{E}}'(v, \Omega' \setminus \overline{\Omega})$$

we have

$$\hat{\mathcal{E}}'(v, \overline{\Omega}) = \lim_{n\to+\infty} \int_\Omega f_n(x, \nabla v_n) \, dx + \int_{S(v_n)} g_n(x, \nu) \, d\mathcal{H}^{N-1}(x). \tag{7.15}$$

If Ω'' is an open set such that $\overline{\Omega} \subseteq \Omega''$, we have that $(v_n)_{|\Omega'\cap\Omega''}$ is a recovering sequence also for $(\hat{\mathcal{E}}'_n, \Omega'' \cap \Omega', c'')$. In fact if this is not the case, we can find $(\tilde{v}_n)_{n\in\mathbb{N}}$ recovering sequence for v with respect to $(\hat{\mathcal{E}}'_n, \Omega' \cap \Omega'', c')$ with $\tilde{v}_n = \psi_n$ outside $\overline{\Omega}$ and such that

$$\begin{aligned}\liminf_{n\to+\infty} \int_\Omega f_n(x, \nabla\tilde{v}_n) \, dx &+ \int_{S(\tilde{v}_n)} g_n(x, \nu) \, d\mathcal{H}^{N-1}(x) \\ &< \liminf_{n\to+\infty} \int_\Omega f_n(x, \nabla v_n) \, dx + \int_{S(v_n)} g_n(x, \nu) \, d\mathcal{H}^{N-1}(x) = \hat{\mathcal{E}}'(v, \overline{\Omega}).\end{aligned}$$

But this implies that

$$\hat{v}_n := \begin{cases} \tilde{v}_n & \text{in } \overline{\Omega} \\ \psi_n & \text{in } \Omega' \setminus \overline{\Omega} \end{cases}$$

is such that $\liminf_{n\to+\infty} \hat{\mathcal{E}}'_n(\hat{v}_n, \Omega') < \hat{\mathcal{E}}'(v, \Omega')$ which is absurd.

By (7.15), in view of the non interaction between bulk and surface energy, and taking into account the lower semicontinuity result of Proposition 4.3, we deduce

$$\int_{S(v)} g'(x, \nu) \, d\mathcal{H}^{N-1} = \lim_{n\to+\infty} \int_{S(v_n)} g_n(x, \nu) \, d\mathcal{H}^{N-1}.$$

We conclude that the surface energy given by the restriction of g' to $\overline{\Omega} \times S^{N-1}$ is determined only by the $g_n : \overline{\Omega} \times S^{N-1} \to [0, +\infty]$.

The stability result for unilateral minimality properties with boundary conditions under σ-convergence in $\overline{\Omega}$ for rectifiable sets (see Definition 5.11) and Γ-convergence of bulk and surface energies is the following.

Theorem 7.3. *Let $\psi_n \in W^{1,p}(\Omega)$ with $\psi_n \to \psi$ strongly in $W^{1,p}(\Omega)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$ with $\mathcal{H}^{N-1}(K_n) \le C$, such that K_n σ-converges in $\overline{\Omega}$ to K, and $S^{\psi_n}(u_n) \tilde{\subseteq} K_n$.*

Let us assume that the pair (u_n, K_n) satisfies the unilateral minimality property (7.2) with respect to f_n, g_n and ψ_n. Then (u, K) satisfies the unilateral minimality property with respect to f, g and ψ, where f is defined in (3.10) and g is the restriction of g' defined in (7.3) to $\overline{\Omega} \times S^{N-1}$. Moreover we have

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla u_n(x)) \, dx = \int_\Omega f(x, \nabla u(x)) \, dx.$$

Proof. Since the boundary datum ψ_n is imposed only on $\partial_D\Omega$, we can consider $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$ as part of the cracks, that is we can replace in the unilateral minimality properties K_n with $K'_n := K_n \cup \partial_N\Omega$.

It is easy to prove that K'_n σ-converges in $\overline{\Omega}$ to $K \cup \partial_N\Omega$. Then the proof follows that of Theorem 6.2 employing the functionals $(\tilde{\mathcal{E}}'_n)_{n\in\mathbb{N}}$ defined in Lemma 7.1 with K'_n in place of K_n. □

8. Quasistatic evolution of cracks in composite materials

The aim of this section is to apply the stability results for unilateral minimality properties of Section 7 to the study the asymptotic behavior of crack evolutions relative to varying bulk and surface energy densities f_n and g_n. As mentioned in the Introduction, this problem is inspired by the problem of crack propagation in composite materials. We restrict our analysis to the case of antiplanar shear, where the elastic body is an infinite cylinder.

Let us recall the result of Dal Maso, Francfort and Toader [18] about quasistatic crack evolution in nonlinear elasticity. It is a very general existence and approximation result concerning a variational theory of crack propagation inspired by the model introduced by Francfort and Marigo in [22]. As already said, we consider the antiplanar case and for simplicity we neglect body and traction forces, and so we adapt the mathematical tools employed in [18] to this scalar setting.

As in the previous sections, let $\Omega \subset \mathbb{R}^N$ (which for $N = 2$ represents a section of the cylindrical hyperelastic body) be an open bounded set with Lipschitz boundary. The family of admissible cracks is the class of rectifiable subsets of $\overline{\Omega}$, while the class of admissible displacements is given by the functional space $SBV^p(\Omega)$, where $1 < p < +\infty$. Let $\partial_D\Omega$ be a subset of $\partial\Omega$. Given $\psi \in W^{1,p}(\Omega)$, we say that the displacement u is admissible for the fracture K and the boundary datum ψ and we write $u \in AD(\psi, K)$ if $S(v) \,\tilde{\subseteq}\, K$ and $v = \psi$ on $\partial_D\Omega \setminus K$. This can be summarized by the notation $S^\psi(u) \,\tilde{\subseteq}\, K$, where $S^\psi(\cdot)$ is defined in (7.1).

Let $f(x, \xi) : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function which is convex and C^1 in ξ for a.e. $x \in \Omega$, $f(x, 0) = 0$ for a.e. $x \in \Omega$, and satisfies the growth estimate

$$a_1(x) + \alpha|\xi|^p \leq f(x, \xi) \leq a_2(x) + \beta|\xi|^p, \tag{8.1}$$

where $a_1, a_2 \in L^1(\Omega)$ and $\alpha, \beta > 0$. Let moreover $g : \overline{\Omega} \times S^{N-1} \to [0, +\infty[$ be a Borel function such that

$$\alpha \leq g(x, \nu) \leq \beta. \tag{8.2}$$

The total energy of a configuration (u, K) is given by

$$\mathcal{E}(u, K) := \int_\Omega f(x, \nabla u(x))\, dx + \int_K g(x, \nu) d\mathcal{H}^{N-1}(x).$$

We will usually refer to the first term as the *bulk energy* of u and we write

$$\mathcal{E}^b(u) := \int_\Omega f(x, \nabla u(x))\, dx,$$

while we will refer to the second term as the *surface energy* of K and we write

$$\mathcal{E}^s(K) := \int_K g(x, \nu) d\mathcal{H}^{N-1}(x).$$

Let us consider now a time dependent boundary datum $\psi \in W^{1,1}\big([0, T]; W^{1,p}(\Omega)\big)$ (i.e., the function $t \to \psi(t)$ is absolutely continuous from $[0, T]$ to the Banach space $W^{1,p}(\Omega)$, with summable time derivative, see for instance [13]), such that for all $t \in [0, T]$

$$\|\psi(t)\|_{L^\infty(\Omega)} \leq C. \tag{8.3}$$

In [18] Dal Maso, Francfort and Toader proved the existence (indeed in a much more general setting) of an *irreversible quasistatic crack evolution* in Ω relative to the boundary displacement ψ, i.e., the existence of a map $t \to (u(t), K(t))$ where $u(t) \in AD(\psi(t), K(t))$, $\|u(t)\|_{L^\infty(\Omega)} \leq \|\psi(t)\|_\infty$ and such that the following three properties hold:

1. *irreversibility:* $K(t_1) \,\tilde{\subseteq}\, K(t_2)$ for all $0 \leq t_1 \leq t_2 \leq T$;
2. *global stability:* for all $K(t) \,\tilde{\subseteq}\, K$ and $v \in AD(\psi(t), K)$

$$\mathcal{E}(u(t), K(t)) \leq \mathcal{E}(v, K); \tag{8.4}$$

3. *energy balance:* the function $t \to \mathcal{E}(u(t), K(t))$ is absolutely continuous and

$$\frac{d}{dt}\mathcal{E}(u(t), K(t)) = \int_\Omega \nabla_\xi f(x, \nabla u(t)) \nabla\dot{\psi}(t)\, dx, \tag{8.5}$$

where $\dot{\psi}$ denotes the time derivative of $t \to \psi(t)$.

For every $n \in \mathbb{N}$ let us consider admissible bulk and surface energy densities $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ and $g_n : \Omega \times S^{N-1} \to [0, +\infty[$ for the model of [18] satisfying the growth estimates (8.1) and (8.2) uniformly in n. Let us moreover assume that f_n is such that for a.e. $x \in \Omega$ and for all $M \geq 0$

$$|\nabla_\xi f_n(x, \xi_n^1) - \nabla_\xi f_n(x, \xi_n^2)| \to 0 \tag{8.6}$$

for all ξ_n^1, ξ_n^2 such that $|\xi_n^1| \leq M$, $|\xi_n^2| \leq M$ and $|\xi_n^1 - \xi_n^2| \to 0$. Notice that for instance $f_n(x, \xi) := a_n(x)|\xi|^p$ with $\alpha \leq a_n(x) \leq \beta$ satisfies (8.6). We denote by $\mathcal{E}_n$, $\mathcal{E}_n^b$ and $\mathcal{E}_n^s$ the total, bulk and surface energies associated to f_n and g_n.

Let f and g be the effective energy densities associated to f_n and g_n in the sense of Theorem 7.3, i.e., let f be given by Proposition 3.1 and let g be the restriction to $\overline{\Omega} \times S^{N-1}$ of the function g' defined in (7.3). Notice that by Proposition 2.5 we have that the function $f(x, \cdot)$ is C^1; moreover it is convex in ξ and satisfies the growth estimate (8.1). On the other hand, g satisfies the growth estimate (8.2), so that f and g are admissible bulk and surface energy densities for the model of [18].

Let $t \to \psi_n(t)$ be a sequence of admissible time dependent boundary displacements satisfying (8.3) and such that

$$\psi_n \to \psi \qquad \text{strongly in } W^{1,1}\Big([0, T], W^{1,p}(\Omega)\Big).$$

Let $t \to (u_n(t), K_n(t))$ be a quasistatic evolution for the boundary datum ψ_n relative to the energy densities f_n and g_n according to [18]. Let us assume moreover the following stronger form of global stability at time $t = 0$

$$\mathcal{E}_n(u_n(0), K_n(0)) \leq \mathcal{E}_n(v, K) \tag{8.7}$$

for all (v, K) such that $v \in AD(\psi(0), K)$. The minimality (8.7) readily implies that $K_n(0) = S^{\psi_n(0)}(u_n(0))$. Notice that the existence of a quasistatic crack evolution satisfying (8.7) is easily achieved by performing a global minimization at time $t = 0$ in the step by step procedure employed in [18], as done for example in [21]: see also Remark 8.2.

The main result of this section is the following Theorem which asserts that the σ-limit in $\overline{\Omega}$ of $K_n(t)$ (see Definition 5.11) still determines a quasistatic crack growth with respect to the effective energy densities f and g.

Theorem 8.1. *There exists a quasistatic crack growth $t \to (u(t), K(t))$ relative to the energy densities f and g and the boundary datum ψ such that up to a subsequence (not rebelled) the following hold:*

(1) *for all $t \in [0, T]$*

$$K_n(t) \ \sigma\text{-converges in } \overline{\Omega} \text{ to } K(t),$$

and there exists a further subsequence n_k (depending possibly on t) such that

$$u_{n_k}(t) \rightharpoonup u(t) \qquad \text{weakly in } SBV^p(\Omega);$$

(2) *for every $t \in [0, T]$ we have convergence of total energies*

$$\mathcal{E}_n(u_n(t), K_n(t)) \to \mathcal{E}(u(t), K(t)),$$

and in particular separate convergence for bulk and surface energies, i.e.,

$$\mathcal{E}_n^b(u_n(t)) \to \mathcal{E}^b(u(t)) \quad \text{and} \quad \mathcal{E}^s(K_n(t)) \to \mathcal{E}^s(K(t)).$$

Proof. By the global stability conditions (8.7) and (8.4), by the energy balance condition (8.5), by the growth estimates on f_n and g_n, and by the L^∞-bound $\|u_n(t)\|_\infty \leq \|\psi_n(t)\|_\infty \leq C$, we have that there exists a constant C such that for all $t \in [0, T]$ and for all $n \in \mathbb{N}$

$$\|\nabla u_n(t)\|^p + \mathcal{H}^{N-1}(K_n(t)) + \|u_n(t)\|_{L^\infty(\Omega)} \leq C. \tag{8.8}$$

We divide the proof into three steps.

Step 1: Compactness for the cracks. In view of (8.8), using a variant of Helly's theorem for increasing functions (see for instance [19, Theorem 6.3] for the case of Hausdorff converging compact sets), we can find a subsequence (not rebelled) of $(K_n(\cdot))_{n\in\mathbb{N}}$ and an increasing map $t \to K(t)$ such that $K_n(t)$ σ-converges in $\overline{\Omega}$ to $K(t)$ for all $t \in [0,T]$.

Step 2: Compactness for the displacements. Notice that the sequence $(u_n(t))_{n\in\mathbb{N}}$ is relatively compact in $SBV^p(\Omega)$ by (8.8). We now want to select a particular limit point of this sequence. With this aim, let us consider

$$\vartheta_n(t) := \int_\Omega \nabla_\xi f_n(x, \nabla u_n(t))\nabla\dot{\psi}_n(t)\,dx \qquad \text{and} \qquad \vartheta(t) := \limsup_{n\to+\infty} \vartheta_n(t). \tag{8.9}$$

Let us see that there exists $u(t) \in SBV^p(\Omega)$ such that

$$\vartheta(t) = \int_\Omega \nabla_\xi f(x, \nabla u(t))\nabla\dot{\psi}(t)\,dx \tag{8.10}$$

and

$$u_{n_k}(t) \rightharpoonup u(t) \qquad \text{weakly in } SBV^p(\Omega) \tag{8.11}$$

for a suitable subsequence n_k depending on t. In fact let us consider a subsequence n_k such that

$$\vartheta(t) = \lim_{k\to+\infty} \int_\Omega \nabla_\xi f(x, \nabla u_{n_k}(t))\nabla\dot{\psi}_{n_k}(t)\,dx,$$

and

$$u_{n_k}(t) \rightharpoonup u \qquad \text{weakly in } SBV^p(\Omega).$$

By Proposition 5.8 we get $S^{\psi(t)}(u)\,\tilde{\subseteq}\,K(t)$, so that $u \in AD(\psi(t), K(t))$. By global stability for $(u_n(t), K_n(t))$ we have that

$$\int_\Omega f_{n_k}(x, \nabla u_{n_k}(t))\,dx \le \int_\Omega f_{n_k}(x, \nabla v)\,dx + \int_{H\setminus K_{n_k}(t)} g_{n_k}(x,\nu)\,d\mathcal{H}^{N-1}(x)$$

for all $v \in AD(\psi_{n_k}(t), H)$ with $K_{n_k}(t)\,\tilde{\subseteq}\,H$. Then by the stability result of Theorem 7.3 we get that

$$\int_\Omega f(x, \nabla u)\,dx \le \int_\Omega f(x, \nabla v)\,dx + \int_{H\setminus K(t)} g(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{8.12}$$

for all $v \in AD(\psi(t), H)$ with $K(t)\,\tilde{\subseteq}\,H$ and

$$\int_\Omega f_{n_k}(x, \nabla u_{n_k}(t))\,dx \to \int_\Omega f(x, \nabla u)\,dx. \tag{8.13}$$

We claim that

$$\lim_{k\to+\infty} \int_\Omega \nabla_\xi f_{n_k}(x, \nabla u_{n_k}(t))\nabla\Phi\,dx = \int_\Omega \nabla_\xi f(x, \nabla u)\nabla\Phi\,dx \tag{8.14}$$

for all $\Phi \in W^{1,p}(\Omega)$. This has been done in [18, Lemma 4.11] in the case of fixed bulk energy, and our proof is just a variant based on the Γ-convergence results of Section 4 and on assumption (8.6) which permit to deal with varying energies. Let us consider $s_j \searrow 0$ and $k_j \to +\infty$: we have by Lagrange Theorem

$$\int_\Omega \frac{f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t) + s_j\nabla\Phi) - f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t))}{s_j}\,dx = \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t) + \tilde{s}_j\nabla\Phi)\nabla\Phi\,dx$$

where $\tilde{s}_j \in [0, s_j]$. Up to a further subsequence for k_j, by (8.13) and by the lower semicontinuity given by Proposition 4.3 we can assume that for j large enough

$$\int_\Omega \frac{f(x, \nabla u + s_j\nabla\Phi) - f(x, \nabla u)}{s_j}\,dx - \frac{1}{j} \le \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t) + \tilde{s}_j\nabla\Phi)\nabla\Phi\,dx. \tag{8.15}$$

By Lemma 2.4 we have

$$\liminf_{j\to+\infty} \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t) + \tilde{s}_j\nabla\Phi)\nabla\Phi\,dx = \liminf_{j\to+\infty} \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t))\nabla\Phi\,dx,$$

so that by (8.15) we get

$$\int_\Omega \nabla_\xi f(x,\nabla u)\nabla\Phi\,dx \le \liminf_{j\to+\infty}\int_\Omega \nabla_\xi f_{n_{k_j}}(x,\nabla u_{n_{k_j}}(t))\nabla\Phi\,dx.$$

Changing Φ with $-\Phi$, we get that (8.14) is proved. In particular since $\nabla\dot\psi_{n_k}(t)\to\dot\psi(t)$ strongly in $L^p(\Omega,\mathbb{R}^N)$ we have

$$\begin{aligned}\vartheta(t) &= \lim_{k\to+\infty}\int_\Omega \nabla_\xi f(x,\nabla u_{n_k}(t))\nabla\dot\psi_{n_k}(t)\,dx\\ &= \lim_{k\to+\infty}\int_\Omega \nabla_\xi f(x,\nabla u_{n_k}(t))\nabla\dot\psi(t)\,dx + \lim_{k\to+\infty}\int_\Omega \nabla_\xi f(x,\nabla u_{n_k}(t))(\nabla\dot\psi_{n_k}(t)-\nabla\dot\psi(t))\,dx\\ &= \int_\Omega \nabla_\xi f(x,\nabla u)\nabla\dot\psi(t)\,dx.\end{aligned}$$

Setting $u(t):=u$ we deduce that (8.10) and (8.11) hold, $u(t)\in AD(\psi(t),K(t))$ and

$$\int_\Omega f(x,\nabla u(t))\,dx \le \int_\Omega f(x,\nabla v)\,dx + \int_{H\setminus K(t)} g(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{8.16}$$

for all $v\in AD(\psi(t),H)$ with $K(t)\,\tilde\subseteq\, H$. Notice that from the global stability (8.7) at time $t=0$, which implies $K_n(0)=S^{\psi_n(0)}(u_n(0))$, by Lemma 7.1 (where we take $K_n=\emptyset$) we deduce

$$\int_\Omega f(x,\nabla u(t))\,dx + \int_{K(0)} g(x,\nu)\,d\mathcal{H}^{N-1}(x) \le \int_\Omega f(x,\nabla v)\,dx + \int_H g(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{8.17}$$

for all $v\in AD(\psi(t),H)$ and

$$\mathcal{E}(u(0),K(0)) = \lim_{n\to+\infty}\mathcal{E}_n(u_n(0),K_n(0)). \tag{8.18}$$

Step 3: Conclusion. Let us consider $t\to(u(t),K(t))$ with $u(t)$ and $K(t)$ defined in Step 2 and Step 1 respectively. In order to see that $t\to(u(t),K(t))$ is a quasistatic crack evolution with respect to f and g, we have to check the energy balance condition, since admissibility, irreversibility and global stability (see (8.16) and (8.17)) hold by construction in view of Steps 1 and 2.

The inequality

$$\mathcal{E}(u(t),K(t)) \ge \mathcal{E}(u(0),K(0)) + \int_0^t\int_\Omega \nabla_\xi f(x,\nabla u(\tau))\nabla\dot\psi(\tau)\,dx\,d\tau, \qquad t\in[0,T] \tag{8.19}$$

is a consequence of the global stability condition of $(u(t),K(t))$ (see [18, Lemma 7.1] for details).

On the other hand by lower semicontinuity given by Proposition 4.3 and by Proposition 5.7 (applied to g' from which g is obtained by restriction) we have for all $t\in[0,T]$

$$\mathcal{E}(u(t),K(t)) \le \liminf_{n\to+\infty}\mathcal{E}_n(u_n(t),K_n(t)). \tag{8.20}$$

By (8.9), (8.10), (8.18), (8.19) and (8.20), and by applying also Fatou's Lemma in the limsup version, we get for all $t\in[0,T]$

$$\begin{aligned}\mathcal{E}(u(t),K(t)) &\le \liminf_{n\to+\infty}\mathcal{E}_n(u_n(t),K_n(t)) \le \limsup_{n\to+\infty}\mathcal{E}_n(u_n(t),K_n(t))\\ &= \limsup_{n\to+\infty}\left[\mathcal{E}_n(u_n(0),K_n(0)) + \int_0^t \vartheta_n(s)\,ds\right] \le \mathcal{E}(u(0),K(0)) + \int_0^t \vartheta(s)\,ds\\ &= \mathcal{E}(u(0),K(0)) + \int_0^t\int_\Omega \nabla_\xi f(x,\nabla u(\tau))\nabla\dot\psi(\tau)\,dx\,d\tau \le \mathcal{E}(u(t),K(t)).\end{aligned}$$

We conclude that

$$\mathcal{E}(u(t),K(t)) = \mathcal{E}(u(0),K(0)) + \int_0^t\int_\Omega \nabla_\xi f(x,\nabla u(\tau))\nabla\dot\psi(\tau)\,dx\,d\tau$$

and

$$\lim_{n\to+\infty}\mathcal{E}_n(u_n(t),K_n(t)) = \mathcal{E}(u(t),K(t)). \tag{8.21}$$

Finally by lower semicontinuity for the bulk and surface energies under weak convergence for the displacements and σ-convergence in $\overline{\Omega}$ for the cracks, from (8.21) we conclude that

$$\lim_{n\to+\infty} \mathcal{E}^b_n(u_n(t)) = \mathcal{E}^b(u(t)) \qquad \text{and} \qquad \lim_{n\to+\infty} \mathcal{E}^s_n(K_n(t)) = \mathcal{E}^s(K(t)),$$

so that the theorem is proved. □

Remark 8.2. Following the arguments of preceding proof, it turns out that Theorem 8.1 also holds in the following *discretized in time* version, which is closer in spirit to the approach of Francfort and Marigo [22] to quasistatic crack propagation, and of the subsequent papers on the subject ([1], [16], [18], [19], [21], and [23]).

Let $0 = t^\delta_0 < \cdots < t^\delta_h = T$ be a subdivision of $[0, T]$ with step $\delta > 0$, and let $(u^i_{\delta,n}, K^i_{\delta,n})$ be such that

$$(u^i_{\delta,n}, K^i_{\delta,n}) \in \text{ argmin } \{\mathcal{E}^b_n(u) + \mathcal{E}^s_n(K) \,:\, u \in AD(\psi(t^\delta_i), K),\, K^{i-1}_{\delta,n} \tilde{\subseteq} K\},$$

where we set $K^{-1}_{\delta,n} := \emptyset$. Let $\delta_n \to 0$, and let $t \to (u_n(t), K_n(t))$ be the *discretized in time* evolution defined as

$$u_n(t) := u^i_{\delta_n,n}, \qquad K_n(t) := K^i_{\delta_n,n}, \qquad t^{\delta_n}_{i-1} \le t < t^{\delta_n}_i,$$

with $u_n(T) := u^{h_n}_{\delta_n,n}$ and $K_n(T) := K^{h_n}_{\delta_n,n}$.

Then there exists a quasistatic crack growth $t \to (u(t), K(t))$ relative to the energy densities f and g and the boundary datum ψ such that, up to a subsequence (not rebelled), points (1) and (2) of Theorem 8.1 hold.

Remark 8.3. Notice that for all $t \in [0, T]$ $K_n(t)$ converges to $K(t)$ also in the sense of σ^p-convergence by Dal Maso, Francfort and Toader [18] (see Section 5 just before Corollary 5.9 for a definition). In fact, by compactness of σ^p-convergence, up to a further subsequence we have that $K_n(t)$ σ^p-converges to some $\tilde{K}(t)$; by Corollary 5.9 $\tilde{K}(t)$ is contained in $K(t)$ so that the pair $(u(t), \tilde{K}(t))$ is a unilateral minimizer with respect to f and g. Following Step 3 we obtain that $\mathcal{E}^s_n(K_n(t)) \to \mathcal{E}^s(\tilde{K}(t))$, which together with $\mathcal{E}^s_n(K_n(t)) \to \mathcal{E}^s(K(t))$ implies $K(t) \tilde{=} \tilde{K}(t)$ for all $t \in [0, T]$.

We conclude that in order to deal with the study of the asymptotic behavior of quasistatic crack evolutions, the notion of σ-convergence and σ^p-convergence of rectifiable sets are equivalent. Notice however that, as pointed out in the Introduction, in order to handle the problem using directly the tool of σ^p-convergence, one needs a Transfer of Jump Sets like our Theorem 6.4, which seems difficult to be derived without any Γ-convergence argument.

Acknowledgments. This work began while the authors were visiting the Laboratoire J.-L. Lions of the University of Paris 6 and the L.P.M.T.M. of the University of Paris 13 under the support of the Università Italo-francese. The authors wish to thank G. Dal Maso and G.A. Francfort for several interesting discussions.

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