A TRANSPORT PROBLEM WITH QUEUE PENALIZATION EFFECT

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Abstract. Consider a distribution of citizens in an urban area in which some services (supermarkets, post offices...) are present. Each citizen, in order to use a service, spends an amount of time which is due both to the travel time to the service and to the queue time waiting in the service. The choice of the service to be used is made by every citizen in order to be served more quickly. Two types of problems can be considered: a global optimization of the total time spent by the citizens of the whole city (we define a global *optimum* and we study it with techniques from optimal mass transportation) and an individual optimization, in which each citizen chooses the service trying to minimize just his own time expense (we define the concept of *equilibrium* and we study it with techniques from game theory). In this framework we are also able to exhibit two time-dependent strategies (based on the notions of prudence and memory respectively) which converge to the equilibrium.

1. Introduction

In this note we summarize and comment some results recently obtained by the authors in [8]. We are interested in an optimization problem coming from the modeling of the behaviour of citizens who every day need to use some services (supermarkets, post offices...) present in a city. We consider a bounded set $\Omega \subset \mathbb{R}^d$ which is the geographic reference for the city and a nonnegative function f with unit integral representing the population density. We fix k points $x_1, x_2, \dots, x_k$ at which the services are located. If every service was able to answer immediately to the demand of each customer, obviously every person would choose the one closest to his home. Namely, being $p \geq 1$ a fixed number so that the time spent to cover a distance ℓ is given by ℓ^p, then a citizen living at x would choose the service located at x_i if and only if

$$|x - x_i|^p = \min_{j=1,\dots,k} |x - x_j|^p .$$

However, in real world, if a service is crowded because a certain amount of citizens have chosen it, the satisfaction of the demand of the customer is not immediate, and some time has to be spent waiting for it. This time spent in the queue surely depends on the amount of people waiting at the service, but also on the characteristics of the service (for instance, its dimension or the ability of the employees). This can be modeled using k functions $h_1, h_2, \dots, h_k$ which express the time to be waited in dependence of the amount of people in the queue. If the amount of citizens choosing the service x_j is c_j, for $j = 1, \dots, k$, then a citizen living at x and going to the service x_i needs to use a total amount of time expressed by the quantity

$$|x - x_i|^p + h_i(c_i)$$

in order to reach the service x_i and to have his request fulfilled. In particular, the better service for the customer is not necessarily the closest one. It could be convenient for him to go a bit further away in order to be served in a bigger service, in which the queue is faster.

It is surely apparent from the above discussion that, in this modeling, the decision of each citizen depends on the choices of all the other citizens. Each customer chooses the best service according to the corresponding queue, but the queue itself depends on the choices of all the other customers.

To attack this problem, we first want to define a meaningful notion of solution, for which we aim to show existence and uniqueness, and eventually study its properties. There are in fact two notions of solution to this problem: the *global optimum* (whose treatment exploit some concepts from optimal mass transportation) and the *individual equilibrium* (which is closely related to concepts from game theory). The study of these two notions will be carried out in Sections 3 and 4. The final Section 5 will be devoted to a dynamical evolution of the problem and to some related convergence results.

We just mention a couple of (somehow) related results, due to Bouchitté, Jimenez and Rajesh [4] and Brancolini, Buttazzo, Santambrogio and Stepanov [5], regarding optimal location problems. Even though the models and the techniques present in these papers are quite different from ours, the inspiration for our work precisely came from the knowledge of their results. For some more optimization results related to economics, we refer to the references listed in [8].

We close this introduction by introducing some notation that will be extensively used in all this paper. We fix a bounded Borel set $\Omega \subset \mathbb{R}^d$ and we denote by $\mathscr{L}^d$ the d-dimensional Lebesgue measure in $\mathbb{R}^d$. We consider k points x_1, x_2, $\ldots$, x_k belonging to Ω, where k is a strictly positive integer; these points represent the location of the services in the city. The density of the citizens is given by an absolutely continuous probability measure $\mu = f\mathscr{L}^d \llcorner \Omega$, where $f : \Omega \to \mathbb{R}$ is a nonnegative function with unit integral. The measure μ defined in this way will be typically the reference measure in Ω, and we shall also write f-a.e. and f-negligible as a shortening of the more precise notation μ-a.e. and μ-negligible. For $i = 1, \ldots, k$ we consider the functions $h_i : [0, 1] \to [0, +\infty[$ which encode the amount of time to be waited in dependence of the amount of people using the service. With this we mean that, if the service located at x_i is chosen by an amount c_i of citizens, then the time that will be spent in the queue by each customer is $h_i(c_i)$. Notice that in this modeling we can also include some penalizations for the particular features of the various services, choosing the queue functions in such a way that $h_i(0) > 0$: for instance, higher prices or lower quality of the products. We do not specify a priori any condition on the functions h_i, but we will rather clarify the assumptions needed in each particular result. For every Borel set $A \subset \Omega$ we consider the indicatrix function $\mathbf{1}_A$ defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The set of probability measures on Ω and $\Omega \times \Omega$ are denoted by $\mathcal{P}(\Omega)$ and $\mathcal{P}(\Omega \times \Omega)$ respectively. When $\gamma \in \mathcal{P}(\Omega)$ is given, we denote by $\mathrm{L}^1_\gamma(\Omega)$ the vector space consisting of the Borel functions which are γ-integrable. A partition of Ω is a finite or countable family $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint (up to f-negligible sets) Borel sets $A_i \subset \Omega$ such that $\cup_{i\in\mathbb{N}} A_i$ has full measure in Ω.

2. Optimal mass transportation: a very quick review

In this section we survey some very basic results relative to optimal mass transportation, see for instance [1, 18]. Given two measures μ, $\nu \in \mathcal{P}(\Omega)$ and $p \geq 1$ we consider the following minimization problem

$$M_p(\mu, \nu) = \inf \left\{ \left(\int_\Omega |x - T(x)|^p \, d\mu(x) \right)^{1/p} \ : \ T : \Omega \to \Omega \text{ Borel and such that } T_\# \mu = \nu \right\} .$$

This problem has been first considered by Monge [16]. Due to the high nonlinearity it is extremely difficult to deal with the problem in this formulation. The following relaxed version, proposed by Kantorovich [13, 14], is thus of great utility:

$$W_p(\mu,\nu) = \inf\left\{ \left(\int_{\Omega\times\Omega} |x-y|^p \, d\gamma(x,y) \right)^{1/p} \ : \ \gamma \in \Pi(\mu,\nu) \right\},$$

where

$$\Pi(\mu,\nu) = \left\{ \gamma \in \mathcal{P}(\Omega\times\Omega) \ : \ (\pi_1)_{\#}\gamma = \mu \text{ and } (\pi_2)_{\#}\gamma = \nu \right\}.$$

It is worth noticing that W_p turns out to be a distance on $\mathcal{P}(\Omega)$, called the Wasserstein distance of order p. It metrizes the weak convergence of measures (recall that we are assuming that Ω is bounded).

We shall also make extensive use of the following dual formulation of the Monge-Kantorovich problem. For μ, $\nu \in \mathcal{P}(\Omega)$ the following equality holds:

$$W_p^p(\mu,\nu) = \sup\left\{ \int_\Omega u \, d\mu + \int_\Omega v \, d\nu \ : \ \begin{array}{c} u \in \mathrm{L}^1_\mu(\Omega)\,, \ v \in \mathrm{L}^1_\nu(\Omega) \text{ and} \\ u(x)+v(y) \le |x-y|^p \text{ for } \mu\text{-a.e. } x \text{ and } \nu\text{-a.e. } y \end{array} \right\}.$$

Moreover, there exists an optimal pair (u,v) for this dual formulation.

In the following theorem we summarize various results regarding existence and uniqueness of solutions to the Monge problem, see [6, 15, 17, 11, 7, 9, 2].

Theorem 2.1 (Existence and uniqueness of an optimal transport map)**.** *Let μ and ν be probability measures in Ω and fix $p \ge 1$. We assume that μ is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^d$. Then $M_p(\mu,\nu) = W_p(\mu,\nu)$ and there exists an optimal transport map from μ to ν, which is also unique μ-a.e. if $p > 1$.*

A small refinement of the previous uniqueness result can be obtained when the target measure ν is atomic:

Lemma 2.2. *Assume that $\mu \ll \mathscr{L}^d$ and $\nu = \sum_{i\in\mathbb{N}} b_i \delta_{y_i}$ are probability measures in Ω. Then the optimal transport map from μ to ν is unique even in the case $p = 1$.*

3. Existence, uniqueness and characterization of the optimum

A first approach to the problem is from a global point of view. The unknown is a partition $(A_i)_{i=1,\dots,k}$ of Ω, where each A_i represents the urban area where customers choosing the service located at x_i live. The total amount of people living in A_i is given by $c_i = \int_{A_i} f(x)\,dx$. Every citizen living at x and using the service located at x_i is going to spend, in order to be served, a time given by

$$|x-x_i|^p + h_i(c_i)\,.$$

In this expression, $|x-x_i|^p$ is the transport time and $h_i(c_i)$ is the time spent waiting in the queue. Consequently, the total time spent by the whole population is given by

$$\sum_{i=1}^k \int_{A_i} \Big[|x-x_i|^p + h_i(c_i) \Big] f(x)\,dx\,. \tag{3.1}$$

We want to choose the partition $(A_i)_{i=1,\dots,k}$ of Ω in such a way that the quantity in (3.1) is as small as possible; thus we are led to the following minimization problem:

$$\inf_{\substack{(A_i)_{i=1,\dots,k} \\ \text{partition of } \Omega}} \left\{ \sum_{i=1}^k \int_{A_i} \left[|x-x_i|^p \ + \ h_i\left(\int_{A_i} f(x)\ dx \right) \right] \ f(x)\ dx \right\}. \tag{3.2}$$

Definition 3.1 (Optimum)**.** *We say that a partition* $(A_i)_{i=1,\dots,k}$ *of* Ω *is an* optimum *if it is a minimizer for* (3.2)*.*

We are able to show the following result regarding the optimum. We consider for $i = 1, \dots, k$ the functions $\eta_i : [0,1] \to [0,+\infty[$ defined by

$$\eta_i(t) = t h_i(t)\,. \tag{3.3}$$

Theorem 3.2. *Assume that the functions* h_i *are lower semi-continuous. Then:*

(i) *there exists an optimum* $(\widehat{A}_i)_{i=1,\dots,k}$ *for* (3.2)*;*

(ii) *if in addition the maps* η_i *are strictly convex the optimum is unique.*

Moreover, if h_i *are differentiable in* $]0,1]$ *and continuous in* 0 *and* η_i *are convex, then*

$$\begin{cases} A_i = \Big\{ x \in \Omega \ : \ |x - x_i|^p + h_i(c_i) + c_i h_i'(c_i) < |x - x_j|^p + h_j(c_j) + c_j h_j'(c_j) \ \ \forall\, j \neq i \Big\} \\ c_i = \int_{A_i} f(x)\,dx\,, \end{cases} \tag{3.4}$$

is a necessary and sufficient condition for the optimality of the partition $(A_i)_{i=1,\dots,k}$*.*

We briefly comment on the existence and uniqueness result presented in the theorem. Let us denote by S the unit simplex in $\mathbb{R}^k$ defined by

$$S = \left\{ c = (c_1, c_2, \dots, c_k) \in \mathbb{R}^k \ : \ c_i \geq 0\,,\ \sum_{i=1}^k c_i = 1 \right\}$$

and introduce the function $F : S \to \mathbb{R}$ defined by

$$F(c_1, \dots, c_k) = W_p^p \left(f\mathscr{L}^d \llcorner \Omega, \sum_{i=1}^k c_i \delta_{x_i} \right)\,.$$

The result stated above is then an easy consequence of the two following technical observations, which permit to reduce the discussion to the study of an auxiliary minimization problem on the simplex $S \subset \mathbb{R}^k$, and whose proof relies on the theory of optimal mass transportation presented in Section 2:

- the function F is continuous and convex;
- the following equality holds:

$$\inf\,(3.2) = \inf_{c \in S} \left\{ F(c) + \sum_{i=1}^k h_i(c_i) c_i \right\}\,.$$

4. Existence, uniqueness and charaterization of the equilibrium

We observe that, even though the description given in the previous section is mathematically quite elegant and complete, the concept of optimum could be not the right one in economic terms. Indeed, it is based on the assumption that some "external authority" (for instance, the major of the city) has the right to force the decisions of all the customers. This can comport very unfair results for some particular citizens, as pointed out in the following example.

Example 4.1. Consider a small city with 1000 citizens and two bakeries. In the first bakery the shop assistant is extremely talkative, and in any case you wait 100 hours listening to his boring stories. In the second bakery the shop assistant is very organized and clever, thus typically you are served immediately, even when a lot of people is present, apart from the case in which everybody (i.e. exactly all the 1000 citizens) are in the shop; in that occasion, the shop assistant panics and slows down, so that everybody waits 1 hour.

The major of the city forces the choices of the citizens in order to minimize the global cost: in other words, he imposes with a law that the global optimum must be chosen. An unlucky customer is forced to go to the shop with the talkative guy, and all the others go to the organized place.

Consider the situation from the point of view of the unlucky customer: he is not interested in the global situation of the city. He only realizes that (because of the dictatorial decision of the major) he must wait 100 hours to buy the bread. From his point of view, a change of the bakery would imply a gain of 99 hours, and he is not so happy to waste his time in order to help the other citizens.

In this section we are going to discuss an alternative notion of solution, the equilibrium, which should be economically more realistic. In any case, we shall see that some proofs rely on the results regarding the optimum presented in Section 3, since they exploit the previous results with some modified queue functions.

4.1. **The concept of equilibrium.** As suggested from the above discussion, it is much more reasonable that a citizen living at $x \in \Omega$ and going to x_i shall be "satisfied" if and only if

$$|x - x_i|^p + h_i(c_i) \leq \min_{j=1,\dots,k} |x - x_j|^p + h_j(c_j)\,.$$

Indeed, in a free market, each customer chooses his strategy without (too many) impositions from the government, in order to minimize just his own time expense.

Definition 4.2 (Equilibrium)**.** *We say that a partition $(A_i)_{i=1,\dots,k}$ of Ω is an* equilibrium *if for every $i = 1, \dots, k$:*

$$\begin{cases} A_i = \Big\{x \in \Omega \ : \ |x - x_i|^p + h_i(c_i) < |x - x_j|^p + h_j(c_j) \quad \text{for every } j \neq i\Big\} \\[2ex] c_i = \int_{A_i} f(x)\,dx\,. \end{cases} \tag{4.1}$$

In the context of game theory this is precisely a Nash equilibrium: each player is satisfied in the sense that his strategy (i.e. the choice of the service) is the best possible, once the behaviour of the other players (in this case, the sets A_i) is fixed. This type of equilibrium is non-cooperative: each citizen chooses the service just by himself, without collaborating with the other citizens in order to choose a "better" global strategy.

We are able to show the following result concerning existence and uniqueness of the equilibrium. We notice that in general game theory the existence of a Nash equilibrium is typically a difficult task: the proof of the above result does not follow from "general game theory" arguments.

Theorem 4.3. *Assume that the functions h_i are continuous and non-decreasing. Then there exists a unique equilibrium.*

The heart of the proof relies on the following lemma, which makes a connection between the equilibrium problem and an auxiliary optimization problem, with modified queue functions.

We introduce for $i = 1, \dots, k$ the functions $g_i : [0,1] \to [0, +\infty[$ defined by

$$g_i(t) = \begin{cases} \dfrac{1}{t} \displaystyle\int_0^t h_i(s)\, ds & \text{if } 0 < t \le 1 \\ h_i(0) & \text{if } t = 0. \end{cases}$$

Lemma 4.4. *Assume that the functions h_i are continuous and non-decreasing and define the functions g_i as above. Then a partition $(A_i)_{i=1,\dots,k}$ of Ω is an equilibrium if and only if it is a minimizer of the problem*

$$\inf_{\substack{(A_i)_{i=1,\dots,k} \\ \text{partition of } \Omega}} \left\{ \sum_{i=1}^k \int_{A_i} \left[|x - x_i|^p + g_i\left(\int_{A_i} f(x)\, dx \right) \right] f(x)\, dx \right\}.$$

4.2. **A direct proof in the case $k = 2$.** We now give an alternative and more direct proof of Theorem 4.3 in the particular case $k = 2$. This has also the advantage of introducing some ideas and notation that will be important in the dynamical analysis of Section 5. In what follows, we will make extensively use of the following definitions:

$$\tau(x) = |x - x_1|^p - |x - x_2|^p, \qquad m(t) = \int_{\{x : \tau(x) < t\}} f(x)\, dx. \tag{4.2}$$

The condition in (4.1) defining the equilibrium can be read in this context as follows: *a partition (A_1, A_2) of Ω is an equilibrium when there exists $\bar t \in \mathbb{R}$ such that:*

$$A_1 = \big\{ x \in \Omega : \tau(x) < \bar t \big\}, \quad A_2 = \big\{ x \in \Omega : \tau(x) > \bar t \big\} \tag{4.3}$$

$$h_2\big(1 - m(\bar t)\big) - h_1\big(m(\bar t)\big) = \bar t. \tag{4.4}$$

Proposition 4.5. *Assume that the functions h_1 and h_2 are continuous and non-decreasing. Then there exists a unique equilibrium (A_1, A_2).*

Proof. Let us consider the map

$$U(t) = t - h_2\big(1 - m(t)\big) + h_1\big(m(t)\big).$$

It is a continuous and strictly increasing map. Notice that for $t \ge \sup_\Omega \tau$ we have $m(t) = 1$ and $U(t) = t - h_2(0) + h_1(1)$ so that $\lim_{t \to +\infty} U(t) = +\infty$. In the same way for $t \le \inf_\Omega \tau$ we have $m(t) = 0$ and $U(t) = t - h_2(1) + h_1(0)$, thus $\lim_{t \to -\infty} U(t) = -\infty$. By consequence, there exists a unique $\bar t$ such that $U(\bar t) = 0$, that is, (4.4) holds. Then the partition (A_1, A_2) associated to $\bar t$ as in (4.3) is an equilibrium and this is unique. □

4.3. **A comparison with Pareto optimum.** The situation described in Example 4.1 shows that in some cases the global optimum can really be not convenient for some citizens: in particular the optimum is better than the equilibrium for 999 of the 1000 citizens, even though it is much worse for the remaining unlucky citizen. In some classical games (like the well-known prisoner's dilemma, see for instance Section 7.8 of [3]) the equilibrium is in fact a bad strategy for *all* players, in the sense that there is a situation which is better for everybody.

On the contrary we can show that in our model this can not happen. Even more, given an equilibrium, we show that it is not possible to find a situation in which every citizen spend less time. More precisely, and introducing a further notion from game theory, we can prove that every equilibrium is a Pareto optimum for the problem. This means that, in our situation, starting from the non-cooperative Nash equilibrium it is not possible to lower the costs of all the citizens by a cooperation in the choice of the services.

Let us introduce the individual cost function

$$C(x,(B_i)_i) = \sum_{i=1}^{k} \left[|x-x_i|^p + h_i \left(\int_{B_i} f(x)\,dx \right) \right] \mathbf{1}_{B_i}(x)\,, \tag{4.5}$$

defined for $x \in \Omega$ and $(B_i)_{i=1,\dots,k}$ partition of Ω. We say that a partition $(A_i)_{i=1,\dots,k}$ of Ω is a *Pareto optimum* (see also Section 12.5 of [3]) if there exists no partition $(B_i)_{i=1,\dots,k}$ of Ω satisfying

$$C(x,(B_i)_i) \le C(x,(A_i)_i) \qquad \text{for } f\text{-a.e. } x \in \Omega$$

with strict inequality in a set of strictly positive measure. With these definitions we can show the following result.

Proposition 4.6. *If the maps h_i are strictly increasing, then every equilibrium is a Pareto optimum.*

5. Dynamical evolution and convergence to the equilibrium

We finally want to study a dynamical formulation of the problem. Assume that the citizens have some procedure to decide, day by day, where to go, using information obtained in the previous days. They know how long the queues were yesterday, and they decide which service to use today. Does this evolution converge to something? If this is the case, we expect convergence to the individual equilibrium, since the decisions of the customers are taken freely and without any cooperation. In this spirit, in this section we restrict ourselves to a case in which (thanks to the results of Section 4, i.e. we assume that the queue functions h_i are continuous and non-decreasing. Moreover, for simplicity, we consider the case of just two points x_1 and x_2.

5.1. **Standard evolution.** We first try to implement the most natural dynamical strategy. We restrict for simplicity to the case of two services x_1 and x_2 and we first introduce some notation.

We consider functions $\psi_j(x) : \Omega \to \{0,1\}$, for $j \in \mathbb{N}$, such that

$$\psi_j(x) = \begin{cases} 0 & \text{if the citizen in } x \text{ goes to } x_1 \text{ at time } j \\ 1 & \text{if the citizen in } x \text{ goes to } x_2 \text{ at time } j. \end{cases}$$

The total amount of citizens going to x_1 at time j is thus

$$m_j = \int_\Omega \big(1-\psi_j(x)\big) f(x)\,dx\,.$$

We also set

$$\tau(x) = |x-x_1|^p - |x-x_2|^p\,. \tag{5.1}$$

A natural choice of the citizens at the initial time is

$$\psi_0(x) = \begin{cases} 0 & \text{if } \tau(x) < 0 \\ 1 & \text{if } \tau(x) > 0\,; \end{cases}$$

indeed, they just choose to go to the closest service, since they have no experience of queues of the preceding days.

The most simple strategy for the time evolution we could conceive gives

$$\psi_{j+1}(x) = \begin{cases} 0 & \text{if } \tau(x) < h_2(1-m_j) - h_1(m_j) \\ 1 & \text{if } \tau(x) > h_2(1-m_j) - h_1(m_j). \end{cases}$$

Recalling the definition of $\tau(x)$ in (5.1), we see that the discriminating condition the citizen is checking in order to choose the most convenient shop is

$$|x - x_1|^p + h_1(m_j) \lesseqgtr |x - x_2|^p + h_2(1 - m_j)\,.$$

Setting

$$t_{j+1} = h_2(1 - m_j) - h_1(m_j)$$

we construct a sequence $(t_j)_j$ with the property that

$$\psi_j(x) = \begin{cases} 0 & \text{if } \tau(x) < t_j \\ 1 & \text{if } \tau(x) > t_j. \end{cases}$$

Let us consider the function $\bar\psi$ associated to the equilibrium, i.e.

$$\bar\psi(x) = \begin{cases} 0 & \text{if } x \in A_1 \\ 1 & \text{if } x \in A_2 \end{cases} = \begin{cases} 0 & \text{if } \tau(x) < \bar t \\ 1 & \text{if } \tau(x) > \bar t. \end{cases}$$

The convergence issue can then be stated as follows:

Question 5.1. *Does the convergence $\psi_j \to \bar\psi$ hold?*

Notice that it is enough to show that $t_j \to \bar t$: this implies that $\psi_j \to \bar\psi$ uniformly on the compact subsets of $\Omega \setminus \{\tau = \bar t\}$. We define

$$m(t) = \int_{\{\tau < t\}} f(x)\,dx$$

and consider the non-increasing function

$$G(t) = h_2\big(1 - m(t)\big) - h_1\big(m(t)\big)\,.$$

By the definitions previously given we have

$$t_{j+1} = G(t_j) \qquad \text{and} \qquad \bar t = G(\bar t)\,. \tag{5.2}$$

Notice that, being G non-increasing, we have a unique $\bar t$ such that $\bar t = G(\bar t)$, thus we obtain again a proof of the existence and uniqueness of the equilibrium (compare with Section 4.2).

From the properties in (5.2) and some standard fixed point arguments we deduce that we have the convergence $t_j \to \bar t$ under the assumption

$$|G'| < 1\,.$$

Observe that this assumption means that:

- the queue changes slowly as the amount of people changes;
- there are no highly populated areas.

Example 5.2. Consider a "small" city (in which all travel times are very small) with two shops in which the queue is quite consistent. Assume that the whole population lives in a small area and that everybody is closer to the first shop. At the initial time, all the citizens simply go to x_1, since it is closer. But then they experience a long queue, thus the day after they all go to the second shop. This implies that the queue is again very high, thus the subsequent day they all switch back to the first shop. This clearly gives rise to an oscillatory phenomenon, with lack of convergence.

As the above example clearly suggests, we need to find a way to damp these oscillations, which are due to the "too impulsive" decisions of the citizens. Also from the economic point of view, it is not so convincing that the customers change their mind regarding the preferred shop in such a speedy (an silly. . .) way: it is plausible that they have some resistance against too sudden decisions.

5.2. **Evolution with prudence and evolution with memory.** We propose two different models for this:

- the citizens have more prudence in the decision of the service to be used;
- the citizens have longer memory, in the sense that they remember more than a single day.

5.2.1. *Evolution with prudence.* We fix a parameter $0 < \rho < 1$ which is called the *prudence parameter* and we set

$$\psi_{j+1}(x) = \begin{cases} \rho\psi_j(x) & \text{if } \tau(x) < t_{j+1} \\ 1 - \rho\big(1 - \psi_j(x)\big) & \text{if } \tau(x) > t_{j+1}. \end{cases}$$

This can be interpreted as prudence in the decision, or as laziness or resistance against a change of habit. The introduction of this parameter has the effect of damping the oscillations.

Notice that now $\psi_j : \Omega \to [0, 1]$: this means that the decision of the citizens is no more deterministic, but it is modeled by a stochastic variable. It also corresponds to a statistical formulation of the problem, if we think that at each point there is a building with a big number of citizens. In game theory this is related to the concept of mixed strategy. This is also reminiscent of the passage from transport maps to transport plans.

The case $\rho = 0$ means no prudence, so we are again in the previous case. The case $\rho = 1$ means full prudence, thus we obtain $\psi_j = \psi_0$ for all j, and we cannot obtain convergence. We are able to show the following convergence result.

Theorem 5.3. *Assume that the population density f is bounded and that h_1 and h_2 are non-decreasing and Lipschitz continuous. Then there exists a prudence threshold $0 < \bar{\rho} < 1$ such that for every prudence parameter $\bar{\rho} < \rho < 1$ the evolution with prudence gives the convergence $\psi_j \to \bar{\psi}$.*

Moreover, we can also consider the case in which the prudence varies with time.

Theorem 5.4. *Consider a sequence of prudence parameters $(\rho_j)_j$ with*

$$\rho_j \to 1 \qquad \text{and} \qquad \sum_j (1 - \rho_j) = +\infty$$

and consider an evolution with variable prudence (i.e. $\rho = \rho_j$ in the above model). Then we have the convergence $\psi_j \to \bar{\psi}$.

5.2.2. *Evolution with memory.* In this second model we go back to a deterministic strategy, i.e. we have again $\psi_j : \Omega \to \{0, 1\}$. We fix an integer number κ, which is called the *memory coefficient*: we now assume that the population keeps track of κ previous days when deciding the strategy. This means that we consider

$$t_{j+1} = \frac{\sum_{m=j-\kappa+1}^{j} G(t_m)}{\kappa},$$

where as before

$$G(t) = h_2\big(1 - m(t)\big) - h_1\big(m(t)\big)\,.$$

The convergence in the above model is ensured by a condition of Lipschitzianity on the function G.

Theorem 5.5. *Assume that G is strictly κ-Lipschitz, i.e.*

$$|G(t_1) - G(t_2)| < \kappa |t_1 - t_2| \qquad \text{for all } t_1 \neq t_2\,.$$

Then we have the convergence $\psi_j \to \bar\psi$.

Similarly to what we have done in the case of the increasing prudence, we can consider models with increasing memory. In particular, we have the case of global memory, in which the citizens remember "the whole story of the city" in terms of queues.

Theorem 5.6. *Assume that G is Lipschitz and consider the global memory evolution*

$$t_{j+1} = \frac{\sum_{m=1}^{j} G(t_m)}{j}\,.$$

Then we have the convergence $\psi_j \to \bar\psi$.

5.3. **A proof in a simplified setting.** Even though the proofs given in [8] of all the results in Section 5.2 ultimately rely on "elementary" tools, they are at some points quite complicated. In order to make the argument of [8] more accessible to the interested reader, we give here a proof of a convergence result in the case of evolution with memory in the simplified case when $\Omega = [0,1] \subset \mathbb{R}$, $x_1 = 0$, $x_2 = 1$, $p = 1$ and $f \equiv 1$. We consider the case when the coefficient memory $\kappa = 2$.

Since we are assuming that the citizens have a memory of two days, we (arbitrarily) fix as initial data of our evolution $t_1 = t_2 = 1/2$. For $n \geq 2$, the customers are going to choose their strategy for the day $n+1$ remembering the length of the queues at time $n-1$ and n. More precisely, they are going to choose the value of t_{n+1} assuming that

- the queue in the point x_1 is $\big(h_1(t_{n-1}) + h_1(t_n)\big)/2$,
- the queue in the point x_2 is $\big(h_2(1-t_{n-1}) + h_2(1-t_n)\big)/2$.

This exactly means that we are assuming that the expected queue at each service is the average of the queues observed in the two preceding days. Then the value of t_{n+1} is determined from the condition

$$\mathrm{dist}(t_{n+1}, x_1) + \text{expected queue in } x_1 \;=\; \mathrm{dist}(t_{n+1}, x_2) + \text{expected queue in } x_2$$

(recall that we are dealing with the case $f \equiv 1$), which reads

$$t_{n+1} + \frac{h_1(t_{n-1}) + h_1(t_n)}{2} = (1 - t_{n+1}) + \frac{h_2(1-t_{n-1}) + h_2(1-t_n)}{2}\,. \tag{5.3}$$

From (5.3) we obtain

$$\begin{aligned} t_{n+1} &= \frac{h_2(1-t_{n-1}) - h_1(t_{n-1}) + h_2(1-t_n) - h_1(t_n) + 2}{4} \\ &= \frac{\Phi(t_{n-1}) + \Phi(t_n)}{2}\,, \end{aligned} \tag{5.4}$$

where we have set

$$\Phi(t) = \frac{h_2(1-t) - h_1(t) + 1}{2}\,. \tag{5.5}$$

Notice that, thanks to Proposition 4.5, there exists a unique equilibrium

$$A_1 = [0, \bar{t}[\,, \qquad A_2 =]\bar{t}, 1]\,,$$

where $\bar{t}$ is characterized by the condition

$$\bar{t} = \Phi(\bar{t})\,.$$

We want to understand the issue of the convergence of the sequence t_n defined above to the value $\bar{t}$. We are going to show the following partial case of Theorem 5.5: *Assume that the function Φ defined in* (5.5) *is L-Lipschitz, with $L < 2$. Then the sequence t_n defined in* (5.4) *converges to $\bar{t}$.*

Notice that the Lipschitz continuity of Φ is related to the Lipschitz continuity of the queue functions h_1 and h_2. Roughly speaking, we are assuming that the queue time depends nicely on the amount of people in the queue.

In order to prove the above result, we introduce the following two properties:

$$P_n(\alpha): \qquad \big|\Phi(t_n) - \Phi(\bar{t})\big| \le \alpha$$

$$Q_n(\alpha): \qquad \big|\Phi(t_n) - \Phi(\bar{t}) + \Phi(t_{n-1}) - \Phi(\bar{t})\big| \le \alpha\,.$$

Notice that $P_n(\alpha)$ is well-defined for $n \ge 1$ and $Q_n(\alpha)$ for $n \ge 2$.

Step 1. We show the validity of the implications

$$Q_n(\alpha) \quad \Longrightarrow \quad P_{n+1}\left(\frac{L}{2}\alpha\right) \quad \Longrightarrow \quad P_{n+1}(\alpha)\,. \tag{5.6}$$

Using the L-Lipschitzianity of Φ, the definitions of t_{n+1} and of $\bar{t}$ and the assumption $Q_n(\alpha)$ we immediately deduce

$$|\Phi(t_{n+1}) - \Phi(\bar{t})| \le L|t_{n+1} - \bar{t}| = L\left|\frac{\Phi(t_{n-1}) - \Phi(\bar{t}) + \Phi(t_n) - \Phi(\bar{t})}{2}\right| \le \frac{L}{2}\alpha\,.$$

Since we are assuming that $L < 2$, the second implication in (5.6) is trivial. Thus we have shown (5.6).

Step 2. We show the validity of the implication

$$\Big(P_{n-1}(\alpha) \;\; \& \;\; P_n(\alpha) \;\; \& \;\; P_{n+1}(\alpha)\Big) \quad \Longrightarrow \quad Q_{n+1}(\alpha)\,. \tag{5.7}$$

Our objective is to show that

$$\big|\Phi(t_{n+1}) - \Phi(\bar{t}) + \Phi(t_n) - \Phi(\bar{t})\big| \le \alpha\,. \tag{5.8}$$

First of all notice that (5.8) is immediate in the case when $\Phi(t_n) - \Phi(\bar{t})$ and $\Phi(t_{n+1}) - \Phi(\bar{t})$ have different sign. Indeed, thanks to the assumptions $P_n(\alpha)$ and $P_{n+1}(\alpha)$, the absolute value of both is necessarily smaller or equal to α. Thus, we assume that $\Phi(t_n) - \Phi(\bar{t})$ and $\Phi(t_{n+1}) - \Phi(\bar{t})$ have the same sign, and without loss of generality let us assume that

$$\Phi(t_n) - \Phi(\bar{t}) \le 0 \qquad \text{and} \qquad \Phi(t_{n+1}) - \Phi(\bar{t}) \le 0\,. \tag{5.9}$$

Since Φ is non-increasing, this implies that

$$t_{n+1} \ge \bar{t} \qquad \text{and} \qquad t_n \ge \bar{t}\,.$$

But this implies

$$0 \le t_{n+1} - \bar{t} = \frac{\Phi(t_{n-1}) + \Phi(t_n)}{2} - \Phi(\bar{t}) = \frac{\big[\Phi(t_{n-1}) - \Phi(\bar{t})\big] + \big[\Phi(t_n) - \Phi(\bar{t})\big]}{2}\,,$$

thus

$$\Phi(t_{n-1}) \geq \Phi(\bar{t})\,.$$

We can compute

$$\begin{aligned}\left|\Phi(t_{n+1}) - \Phi(\bar{t}) + \Phi(t_n) - \Phi(\bar{t})\right| &= \Phi(\bar{t}) - \Phi(t_{n+1}) + \Phi(\bar{t}) - \Phi(t_n)\\ &\leq L(t_{n+1} - \bar{t}) + \Phi(\bar{t}) - \Phi(t_n)\\ &= L\frac{\Phi(t_{n-1}) - \Phi(\bar{t}) + \Phi(t_n) - \Phi(\bar{t})}{2} + \Phi(\bar{t}) - \Phi(t_n)\\ &= \frac{L}{2}\big[\Phi(t_{n-1}) - \Phi(\bar{t})\big] + \left(\frac{L}{2} - 1\right)\big[\Phi(t_n) - \Phi(\bar{t})\big]\\ &\leq \frac{L\alpha}{2} + \left(1 - \frac{L}{2}\right)\alpha = \alpha\,,\end{aligned}$$

where we have used (5.9), the L-Lipschitzianity of Φ, the definition of t_{n+1} and the assumptions $P_{n-1}(\alpha)$ and $P_n(\alpha)$. This concludes the proof of the implication (5.7).

Step 3. Conclusion of the proof. From the previous steps, it is immediate to deduce that there exists a constant $C(\Phi)$ such that

$$\left|\Phi(t_n) - \Phi(\bar{t})\right| \leq C(\Phi)\left(\frac{L}{2}\right)^{\left[\frac{n}{3}\right]}, \tag{5.10}$$

where we denote with $[z]$ the largest integer which is less or equal to z. Recalling that $L < 2$ and that $\Phi(\bar{t}) = \bar{t}$, we deduce that $\Phi(t_n) \to \bar{t}$, but recalling (5.4) this also implies $t_n \to \bar{t}$, as desired.

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