REARRANGEMENTS IN METRIC SPACES AND IN THE HEISENBERG GROUP

ROBERTO MONTI

Abstract. We prove several rearrangement theorems in the setting of a metric measure space. We adapt the general scheme of the argument to the Heisenberg group, where we study Steiner and circular rearrangement for functions and sets having a suitable symmetry.

1. Introduction

Let X be a metric space with distance function d. We fix a Borel measure μ on X that is *nondegenerate*, i.e.,

$$0 < \mu(B_r(x)) < \infty, \quad \text{for all } x \in X \text{ and } r > 0. \tag{1.1}$$

Here, $B_r(x) = \{y \in X : d(x, y) < r\}$ is the ball centered at x with radius r. For any Borel set $B \subset X$ with positive and finite measure and for any function $f \in L^1(B, \mu)$ let

$$\fint_B f(x)d\mu = \frac{1}{\mu(B)} \int_B f(x)d\mu \tag{1.2}$$

denote the averaged integral of f over B.

For $1 \le p < \infty$ and $f \in L^p(X, \mu)$ we let

$$\|\nabla f\|^-_{L^p(X,\mu)} = \liminf_{r\downarrow 0} \frac{1}{r}\Big(\int_X \fint_{B_r(x)} |f(x) - f(y)|^p d\mu(y)\, d\mu(x)\Big)^{1/p}. \tag{1.3}$$

When X is $\mathbb{R}^n$ endowed with the Euclidean metric and $\mu = \mathcal{L}^n$ is the Lebesgue measure, a function $f \in L^p(\mathbb{R}^n)$ satisfies the condition $\|\nabla f\|^-_{L^p(\mathbb{R}^n,\mathcal{L}^n)} < \infty$ if and only if $f \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, the Sobolev space of functions with weak derivatives in $L^p(\mathbb{R}^n)$. In this case, the limit inferior is a limit and there is a geometric constant $0 < C_{n,p} < \infty$ depending on the dimension $n \ge 1$ and $p > 1$ such that

$$\lim_{r\downarrow 0} \frac{1}{r}\Big(\int_{\mathbb{R}^n} \fint_{B_r(x)} |f(x) - f(y)|^p dy\, dx\Big)^{1/p} = C_{n,p}\Big(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx\Big)^{1/p}.$$

For these results see [P] and [BBM]. When $p = 1$, the condition $\|\nabla f\|^-_{L^1(\mathbb{R}^n,\mathcal{L}^n)} < \infty$ is equivalent to $f \in BV(\mathbb{R}^n)$, the space of functions with bounded variation in $\mathbb{R}^n$.

Analogously, for any Borel set $E \subset X$ with $\mu(E) < \infty$ let us define the lower perimeter of E in (X, μ)

$$P^-(E; X, \mu) = \liminf_{r\downarrow 0} \frac{1}{r} \int_X \fint_{B_r(x)} |\chi_E(x) - \chi_E(y)| d\mu(y)\, d\mu(x), \tag{1.4}$$

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where $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in X \setminus E$ is the characteristic function of E. The condition $P^-(E; \mathbb{R}^n, \mathcal{L}^n) < \infty$ holds if and only if the set $E \subset \mathbb{R}^n$ has finite perimeter in the sense of De Giorgi. In this case, the limit inferior is a limit and there exists a geometric constant $0 < C_n < \infty$ depending on $n \geq 1$ such that

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{\mathbb{R}^n} \fint_{B_r(x)} |\chi_E(x) - \chi_E(y)| dy\, dx = C_n |\partial E|(\mathbb{R}^n),$$

where $|\partial E|(\mathbb{R}^n)$ is the perimeter of E in $\mathbb{R}^n$, i.e., the total variation of the characteristic function of E. For this result see [P] and [D]. In the sequel, we simplify the notation and write $P^-(E) = P^-(E; X, \mu)$.

Integral differential quotients as in (1.3)–(1.4) are a possibile definition for the "L^p-length of the gradient" of functions and for the "area of the boundary" of sets in metric measure spaces. Under weak assumptions, a function f in the Hajłasz space $M^{1,p}(X)$, see [Ha], or in the Newtonian space $N^{1,p}(X)$, see [Sh], satiesfies $\|\nabla f\|^-_{L^p(X,\mu)} < \infty$, also with limit superior in place of limit inferior. For a theory of sets with finite perimeter in metric spaces we refer to [Mi].

Based on the previous observations, in this article we address the following problem. Construct transformations of functions $f \mapsto f^\star$ and of sets $E \mapsto E^\star$ such that:

i) The function $f^\star$ and the set $E^\star$ have some "symmetry";
ii) f and $f^\star$ are μ-equimeasurable and $\mu(E^\star) = \mu(E)$;
iii) $\|\nabla f^\star\|^-_{L^p(X,\mu)} \leq \|\nabla f\|^-_{L^p(X,\mu)}$ for all $1 \leq p < \infty$ and $P^-(E^\star) \leq P^-(E)$.

We study three situations of increasing complexity: the two-points rearragement, the Steiner rearrangement, and a kind of Schwarz-type rearragement. In the last two cases, we use the same notation with the superscript $\star$. The existence of such rearrangements depends on richness and structure of the isometries of X.

The two-points rearrangement, also known as *polarization*, relies upon the existence of an involutive isometry $\varrho : X \to X$ along with a partition $X = H^- \cup H \cup H^+$ such that $\varrho H^+ = H^-$. We call the 4-tuple $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ a *reflection system* of X (see Definition 2.1 and notice the key property (2.2)). The two-points rearrangement of a function $f : X \to \mathbb{R}$ is the function $f_\mathcal{R}(x) = \max\{f(x), f(\varrho x)\}$ for $x \in H^+$ and $f_\mathcal{R}(x) = \min\{f(x), f(\varrho x)\}$ for $x \in H^-$. In Section 2, we prove several inqualities relating f and $f_\mathcal{R}$. Inequalities as (2.12) in Theorem 2.8 are called by Baernstein [Be] "master inequalities".

In Section 3, we introduce the notion of *Steiner system* in a metric space X. Roughly speaking, a *Steiner system* is a pair $(\mathcal{R}, T)$ where $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ is a reflection system of X and T is a 1-parameter group of isometries such that: $X/T \subset H$, i.e., the quotient X/T is identified with a subset of H; $\tau^{-1}x = \varrho\tau x$ for any $x \in X/T$ and $\tau \in T$. For precise and complete statements, we refer to Definition 3.2.

If the measure μ is T-invariant, then it is disintegrable along the orbits $T_x = \{\tau x \in X : \tau \in T\}$, $x \in X/T$ (see Example 3.10). Namely, there exist measures μ_x on T_x

and $\bar{\mu}$ on X/T such that for any Borel set $E \subset X$ we have

$$\mu(E) = \int_{X/T} \mu_x(E \cap T_x) d\bar{\mu}(x).$$

It is then possible to rearrange the set E along the orbits T_x obtaining a new set $E^\star$ which is ϱ-invariant and $\mu(E^\star) = \mu(E)$. The construction carries over to functions, yielding a transformation $f \mapsto f^\star$. The precedure is described at the beginning of Section 3 (see Definition 3.1). In Theorems 3.6 and 3.7, we prove Pólya-Szegö inequalities of the type iii) above. In the proof, we need several assumptions on the measure μ and on the metric space X. In particular, X is assumed to be proper in order to have a compactness theorem for functional spaces on X which is proved in Section 4.

The presentation of Section 3 is in fact more general as we consider *Schwarz systems* (see Definition 3.3). The axioms (3.12) and (3.13) of a Schwarz system make possible the "strict inequality argument" that is a crucial step in the theory of symmetrization via polarization (see [BT, p. 252] and Lemma 6.4 in [BS]). This argument appears in the proof of our Theorem 3.6, see (3.32). Condition (3.12) requires the existence of a reflection system separating, in a symmetric way, points in the same section. This property automatically holds in Steiner systems. Condition (3.13) requires a certain "metric coherence" between sections.

In the second part of the article, which has a more specific character, we prove some rearrangement theorems in the Heisenberg group $\mathbf{H}^n = \mathbb{C}^n \times \mathbb{R}$. We refer to Example 5.5 and Section 6 for the relevant definitions. Let $B_r(x)$ denote the Carnot-Carathéodory ball in $\mathbf{H}^n$ centered at $x \in \mathbf{H}^n$ and having radius $r > 0$. The following facts are proved in [P]. A function $f \in L^p(\mathbf{H}^n)$, $1 < p < \infty$, belongs to the horizontal Sobolev space $W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ if and only if

$$\liminf_{r \downarrow 0} \frac{1}{r} \Big(\int_{\mathbf{H}^n} \fint_{B_r(y)} |f(x) - f(y)|^p dx\, dy \Big)^{1/p} < \infty. \tag{1.5}$$

In this case, the limit inferior is a limit and there exists a geometric constant $0 < K_{n,p} < \infty$ depending on $p > 1$ and $n \geq 1$ such that

$$\lim_{r \downarrow 0} \frac{1}{r} \Big(\int_{\mathbf{H}^n} \fint_{B_r(y)} |f(x) - f(y)|^p dx\, dy \Big)^{1/p} = K_{n,p} \|\nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^n)}, \tag{1.6}$$

where $\nabla_{\mathbf{H}} f$ is the horizontal gradient of f.

Analogously, a Borel set $E \subset \mathbf{H}^n$ with finite measure has finite horizontal perimeter if and only if

$$\liminf_{r \downarrow 0} \frac{1}{r} \int_{\mathbf{H}^n} \fint_{B_r(y)} |\chi_E(x) - \chi_E(y)|\, dx\, dy < \infty. \tag{1.7}$$

Moreover, if E has also finite Euclidean perimeter then we have

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{\mathbf{H}^n} \fint_{B_r(y)} |\chi_E(x) - \chi_E(y)|\, dx\, dy = K_n |\partial_{\mathbf{H}} E|(\mathbf{H}^n), \tag{1.8}$$

where $0 < K_n < \infty$ is a geometric constant depending on $n \geq 1$ and $|\partial_{\mathbf{H}} E|(\mathbf{H}^n)$ denotes the horizontal perimeter of E, i.e., the horizontal total variation of the characteristic function of E.

We first study polarization in connection with formulae (1.6) and (1.8). In the related master inequalities there is an error produced by the lack in $\mathbf{H}^n$ of reflection systems satisfying (2.2). This error can be controlled assuming a suitable symmetry (see the proof of Theorem 6.1). Then we prove inequalities for the Steiner and circular rearrangement following the abstract scheme of Section 3, see Theorems 6.3 and 6.4. We illustrate here the case of sets. Let $E^\star$ denote the Steiner rearrangement of E in the t-coordinate of $\mathbf{H}^n$, i.e.,

$$E^\star = \big\{(z,t) \in \mathbb{C}^n \times \mathbb{R} : 2|t| < \mathcal{L}^1(E_z)\big\}, \tag{1.9}$$

where $E_z = \{t \in \mathbb{R} : (z,t) \in E\}$, $z \in \mathbb{C}^n$. Let $\sigma : \mathbf{H}^n \to \mathbf{H}^n$ be the mapping $\sigma(z,t) = (\bar{z},t)$, where $\bar{z} = x - iy$ is the complex conjugate of $z = x + iy$ in $\mathbb{C}^n$. A set E is σ-symmetric if $E = \sigma E$. In Section 6, Theorem 6.4, we prove that for a σ-symmetric set $E \subset \mathbf{H}^n$ of finite measure and finite horizontal perimeter there holds

$$|\partial_{\mathbf{H}} E^\star|(\mathbf{H}^n) \leq |\partial_{\mathbf{H}} E|(\mathbf{H}^n).$$

The theorem fails if we drop the σ-symmetry (see Example 6.5). We also prove some results on the circular rearrangement in a $\mathbb{C}$ component of $\mathbf{H}^n = \mathbb{C}^n \times \mathbb{R}$ (see Theorem 6.6).

These theorems seem the first results on symmetrization in the Heisenberg group. The topic has a particular interest in connection with sharp functional and geometric inequalities, such as Pansu's conjecture on the Heisenberg isoperimetric problem (see [MR] and [RR]). A theorem concerning a kind of vertical rearrangement in $\mathbf{H}^n$ is also proved by Serra Cassano and Vittone in [SCV]. The problem of rearranging sets and functions in the horizontal slices of $\mathbf{H}^n$ is more difficult. So far the only known result concerns the monotonicity of horizontal perimeter for the radial nondecreasing Steiner rearrangement of sets of $\mathbf{H}^n$ which already have a cylindrical symmetry (see [M2]).

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Let us briefly comment on the relevant literature. The principle underlying polarization can be envisaged in Chapter X of *Inequalities* by Hardy, Littlewood, and Pólya [HLP]. The method was subsequently used be Wolontis [W, p. 598] to estimate a certain conformal invariant in the complex plane under circular symmetrization. Motivated by the study of subharmonicity, Baernstein and Taylor [BT] used polarization in connection with spherical rearrangement. The same ideas were employed by Beckner [B] to establish several sharp functional inequalities on the sphere. Polarization and symmetrization are also systematically studied by Dubinin [Du] in the abstract theory of capacity. We are particularly indebted to the articles [BT] and [B], where the authors develop a unified approach to symmetrization in space forms. We axiomatize this approach in the setting of a metric measure space and we develop these ideas to prove the results in the Heisenberg group.

Steiner rearrangement was introduced in [St] to prove the isoperimetric inequality in the plane. Let $E^\star$ be the Steiner rearrangement of $E \subset \mathbb{R}^n$ w.r.t. some hyperplane. The inequality $|\partial E^\star|(\mathbb{R}^n) \leq |\partial E|(\mathbb{R}^n)$ for sets with finite perimeter was proved by De Giorgi in [DG] in his work on the isoperimetric inequality (see also [T] and [CF]). In [BS, Theorem 6.1], Brock and Solynin prove that the Steiner symmetrization in $\mathbb{R}^n$ can be obtained as the limit in the natural topology of a suitable sequence of polarizations. In fact, this sequence can be chosen in a "universal" way [VS]. Steiner rearrangement also fits hypersurface measures as Minkowski content (see [H] and [C, Chapter III.2]). Recent progress on the isoperimetric inequality deals with its quantitative version [FMP] and with the use of optimal transportation techniques to prove sharp inequalities (see e.g. [CNV]).

The Schwarz rearrangement was used in the proof [Sc] of the isoperimetric inequality in $\mathbb{R}^3$ and seems to origin in Weierstrass' lectures. The general idea consists in slicing the space in "parallel" sections and in rearranging sets and functions section by section. This is the model for our notion of *Schwarz system*.

In this research, we do not touch several important issues in the theory of rearrangemet: the study of the equality case ([BZ] and [CCF]); the continuity problem in the Sobolev setting (see [AL] and [Bu]); rearrangement inequalities for multiple integrals (see [BLL]); the connection with partial differential equations ([PS], [Ba], and [Ka]).

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A short overview of the paper is in order. In Section 2 we study polarization. Section 3 is devoted to Steiner and Schwarz rearrangement in the abstract setting. In Section 4 we prove a compactness theorem. Section 5 deals with examples, including finite dimensional Banach spaces and the hyperbolic space. In Section 6, we prove the rearrangement theorems in the Heisenberg group.

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2. Two-points rearrangement in metric spaces

Let X be a metric space with distance function d. We say that $X = H^- \cup H \cup H^+$ is a partition of X, if $H^-, H, H^+ \subset X$ are pairwise disjoint subsets of X.

Definition 2.1 (Reflection system)**.** A *reflection system* $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ of X is a partition $X = H^- \cup H \cup H^+$, with H^- and H^+ open, together with a mapping $\varrho : X \to X$ such that:

$$\text{i) } \varrho \text{ is an involutive isometry of } X \text{, i.e. } \varrho^2 = \mathrm{Id}, \text{ and } \varrho H^+ = H^-; \tag{2.1}$$

$$\text{ii) for all } x, y \in H \cup H^+ \text{ we have } d(x, y) \leq d(x, \varrho y). \tag{2.2}$$

Here and henceforth, we write for brevity $\varrho x = \varrho(x)$ and $\varrho E = \varrho(E)$ for sets $E \subset X$.

Example 2.2. Let X be a length space, $X = H^- \cup H \cup H^+$ be a partition of X such that $H = \partial H^- = \partial H^+$, and let $\varrho : X \to X$ be an involutive isometry such that $\varrho H^+ = H^-$ and $\varrho_{|H}$ =identity. Then condition (2.2) holds true.

In fact, let γ be a rectifiable curve joining $x \in H^+$ to $\varrho y \in H^-$. The curve intersects H at some point $z \in H$. We can split $\gamma = \gamma_{xz} + \gamma_{zy}$, where the sum is a concatenation of curves, γ_{xz} is the segment joining x to z and γ_{zy} is the segment joining z to ϱy. The curve $\gamma_{xz} + \varrho\gamma_{zy}$ is continuous, because ϱ is the identity on H, joins x to y, and has the same length as γ, because ϱ is an isometry. The claim $d(x, y) \le d(x, \varrho y)$ follows from the fact that X is a length space. □

Example 2.3. Let (X, d_X) be a metric space with a reflection system $\mathcal{R}$ and let (Y, d_Y) be any metric space. On the product $Z = X \times Y$ we have the product metric $d_Z = \sqrt{d_X^2 + d_Y^2}$. The reflection system $\mathcal{R}$ of X may be extended to a reflection system of Z in the natural way: the reflection ϱ is extended as the identity on the Y component; H is extended to $H \times Y$, etc.

Next, we introduce the notion of two-points rearrangement for functions and sets.

Definition 2.4 (Two-points rearrangement)**.** Let $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ be a reflection system of X and let $f : X \to \mathbb{R}$ be a function. The function $f_{\mathcal{R}} : X \to \mathbb{R}$ defined by

$$f_{\mathcal{R}}(x) = \begin{cases} \min\{f(x), f(\varrho x)\} & \text{if } x \in H^-, \\ f(x) & \text{if } x \in H, \\ \max\{f(x), f(\varrho x)\} & \text{if } x \in H^+, \end{cases} \tag{2.3}$$

is called the *two-points rearrangement* of f with respect to $\mathcal{R}$.

Example 2.5. The Lipschitz constant of a function $f : X \to \mathbb{R}$ is

$$\mathrm{Lip}(f) = \sup_{x,y\in X, x\neq y} \frac{|f(x) - f(y)|}{d(x, y)} \in [0, \infty].$$

We claim that for any reflection system $\mathcal{R}$ of X we have

$$\mathrm{Lip}(f_{\mathcal{R}}) \le \mathrm{Lip}(f). \tag{2.4}$$

Indeed, let $x, y \in X$ be such that $d(x, y) > 0$. We claim that

$$\frac{|f_{\mathcal{R}}(x) - f_{\mathcal{R}}(y)|}{d(x, y)} \le \mathrm{Lip}(f).$$

We have three cases:

1) $f_{\mathcal{R}}(x) = f(x)$ and $f_{\mathcal{R}}(y) = f(y)$;
2) $f_{\mathcal{R}}(x) \neq f(x)$ and $f_{\mathcal{R}}(y) \neq f(y)$;
3) $f_{\mathcal{R}}(x) = f(x)$ and $f_{\mathcal{R}}(y) \neq f(y)$, or viceversa.

In the first case, the claim is clear. In the second one, we have:

$$\frac{|f_{\mathcal{R}}(x) - f_{\mathcal{R}}(y)|}{d(x, y)} = \frac{|f(\varrho x) - f(\varrho y)|}{d(x, y)} = \frac{|f(\varrho x) - f(\varrho y)|}{d(\varrho x, \varrho y)} \le \mathrm{Lip}(f), \tag{2.5}$$

because ϱ is an isometry. Consider the last case. We have three subcases:

3a) $x, y \in H^+$, or $x, y \in H^-$;
3b) $x \in H^+$ and $y \in H^-$, or viceversa;
3c) $x \in H$ or $y \in H$.

Assume that $x, y \in H^+$. Then we have:

$$f(x) = f_{\mathcal{R}}(x) = \max\{f(x), f(\varrho x)\}, \quad \text{i.e., } f(x) \geq f(\varrho x),$$
$$f(y) \neq f_{\mathcal{R}}(y) = \max\{f(y), f(\varrho y)\}, \quad \text{i.e., } f(y) < f(\varrho y).$$

Thus we obtain

$$f_{\mathcal{R}}(x) - f_{\mathcal{R}}(y) = f(x) - f(\varrho y) < f(x) - f(y) \leq |f(x) - f(y)| \leq \mathrm{Lip}(f) d(x, y),$$
$$f_{\mathcal{R}}(y) - f_{\mathcal{R}}(x) = f(\varrho y) - f(x) \leq f(\varrho y) - f(\varrho x) \leq \mathrm{Lip}(f) d(\varrho x, \varrho y) = \mathrm{Lip}(f) d(x, y),$$

and the claim is proved.

Assume that $x \in H^+$ and $y \in H^-$. Because $f_{\mathcal{R}}(y) \neq f(y)$ then $f_{\mathcal{R}}(y) = f(\varrho y)$ and letting $z = \varrho y \in H^+$ we get, by (2.1) and (2.2),

$$\frac{|f_{\mathcal{R}}(x) - f_{\mathcal{R}}(y)|}{d(x, y)} = \frac{|f(x) - f(\varrho y)|}{d(x, y)} = \frac{|f(x) - f(z)|}{d(x, \varrho z)} \leq \frac{|f(x) - f(z)|}{d(x, z)} \leq \mathrm{Lip}(f).$$

The case 3c) is analogous and we leave the details to the reader.

□

The definition of two-points rearrangement for sets can be obtained specializing (2.3) to the case of characteristic functions. Namely, for any $E \subset X$ we can define the set $E_{\mathcal{R}}$ via the identity $\chi_{E_{\mathcal{R}}} = (\chi_E)_{\mathcal{R}}$. This is equivalent with the following definition.

Definition 2.6. Let $\mathcal{R}$ be a reflection system of X and let $E \subset X$ be a set. The set

$$E_{\mathcal{R}} = \big(E \cap \varrho E \cap H^-\big) \cup (E \cap H) \cup \big((E \cup \varrho E) \cap H^+\big) \tag{2.6}$$

is called the *two-points rearrangement of* E with respect to $\mathcal{R}$.

We are interested in the monotonicity of quantities as in (1.3) and (1.4) under rearrangement. To this aim, let $\phi : [0, +\infty) \to [0, +\infty)$ be a function such that:

a) ϕ is strictly increasing; (2.7)

b) ϕ is convex. (2.8)

In our case, we have $\phi(t) = t^p$ with $p \geq 1$. The basic inequality we need concerning ϕ is described in the following lemma.

Lemma 2.7. Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying (2.7) and (2.8). Then for all real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\gamma < \alpha$ and $\delta < \beta$ there holds

$$\phi(|\alpha - \beta|) + \phi(|\gamma - \delta|) \leq \phi(|\alpha - \delta|) + \phi(|\gamma - \beta|). \tag{2.9}$$

If, in addition, ϕ is strictly convex then the inequality in (2.9) is strict.

The proof of this lemma is an elementary exercise. When $\phi(t) = t^2$, inequality (2.9) reduces to $(\alpha - \gamma)(\beta - \delta) \geq 0$.

Let μ be a Borel measure on X and let $\mathcal{B}(X)$ denote the set of all Borel functions from X to $\mathbb{R}$. For any $r > 0$ let $Q_r : \mathcal{B}(X) \times \mathcal{B}(X) \to [0, \infty]$ be the functional

$$Q_r(f, g) = \int_X \fint_{B_r(x)} \phi(|f(x) - g(y)|) d\mu(y)\, d\mu(x). \tag{2.10}$$

We omit reference to ϕ in our notation Q_r. When $\phi(t) = t^p$, $1 \leq p < \infty$, we let

$$Q_{r,p}(f, g) = \int_X \fint_{B_r(x)} |f(x) - g(y)|^p d\mu(y)\, d\mu(x). \tag{2.11}$$

We also let $Q_{r,p}(f) = Q_{r,p}(f, f)$.

In the sequel, $\varrho : X \to X$ is an involutive isometry. We say that a Borel measure μ on X is ϱ-invariant if $\varrho_\sharp \mu = \mu$, i.e., $\mu(\varrho B) = \mu(B)$ for any Borel set $B \subset X$.

Theorem 2.8. Let $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ be a reflection system of X, let μ be a nondegenerate, ϱ-invariant Borel measure such that $\mu(H) = 0$, and let ϕ satisfy (2.7) and (2.8). For any $r > 0$ and for all functions $f, g \in \mathcal{B}(X)$ we have

$$Q_r(f_{\mathcal{R}}, g_{\mathcal{R}}) \leq Q_r(f, g). \tag{2.12}$$

Moreover, if ϕ is strictly convex,

$$\mu\big\{x \in H^+ : f(x) > f(\varrho x)\big\} > 0, \quad \text{and} \quad \mu\big\{y \in H^+ : g(y) < g(\varrho y)\big\} > 0, \tag{2.13}$$

then the inequality (2.12) is strict, as soon as $Q_r(f, g) < \infty$.

Proof. Let $\chi_r : X \times X \to \mathbb{R}$ be the function

$$\chi_r(x, y) = \begin{cases} \dfrac{1}{\mu(B_r(x))} & \text{if } d(x, y) < r, \\ 0 & \text{otherwise.} \end{cases}$$

As μ is ϱ-invariant, we have $\mu(B_r(\varrho x)) = \mu(\varrho B_r(x)) = \mu(B_r(x))$. Then, χ_r has the following properties:

$$\chi_r(\varrho x, \varrho y) = \chi_r(x, y) \quad \text{and} \quad \chi_r(\varrho x, y) = \chi_r(x, \varrho y). \tag{2.14}$$

We are using here the fact that ϱ is an involutive isometry. We then have

$$Q_r(f, g) = \int_{X \times X} \phi(|f(x) - g(y)|) \chi_r(x, y) d\mu \otimes \mu,$$

where we may replace the integration domain $X \times X$ with

$$(X \setminus H) \times (X \setminus H) = H^+ \times H^+ \cup H^+ \times H^- \cup H^- \times H^+ \cup H^- \times H^-.$$

In fact, we are assuming $\mu(H) = 0$. By (2.14) and $\varrho_\sharp\mu = \mu$, we obtain

$$\int_{H^-\times H^-} \phi(|f(x) - g(y)|)\chi_r(x, y)d\mu \otimes \mu = \int_{H^+\times H^+} \phi(|f(\varrho x) - g(\varrho y)|)\chi_r(x, y)d\mu \otimes \mu,$$
$$\int_{H^+\times H^-} \phi(|f(x) - g(y)|)\chi_r(x, y)d\mu \otimes \mu = \int_{H^+\times H^+} \phi(|f(x) - g(\varrho y)|)\chi_r(x, \varrho y)d\mu \otimes \mu,$$
$$\int_{H^-\times H^+} \phi(|f(x) - g(y)|)\chi_r(x, y)d\mu \otimes \mu = \int_{H^+\times H^+} \phi(|f(\varrho x) - g(y)|)\chi_r(x, \varrho y)d\mu \otimes \mu.$$

Summing up, we obtain

$$Q_r(f, g) = \iint_{H^+\times H^+} Q(f, g; x, y)\, d\mu \otimes \mu,$$

where we let

$$\begin{aligned} Q(f, g; x, y) =&\big\{\phi(|f(x) - g(y)|) + \phi(|f(\varrho x) - g(\varrho y)|)\big\}\chi_r(x, y) \\ &+ \big\{\phi(|f(x) - g(\varrho y)|) + \phi(|f(\varrho x) - g(y)|)\big\}\chi_r(x, \varrho y). \end{aligned}$$

We claim that for all $x, y \in H^+$ we have

$$Q(f_{\mathcal{R}}, g_{\mathcal{R}}; x, y) \le Q(f, g; x, y). \tag{2.15}$$

By (2.2), there are only three cases:

1) $d(x, y) \ge r$;
2) $d(x, y) \le d(x, \varrho y) < r$;
3) $d(x, y) < r \le d(x, \varrho y)$.

In the first case, there also holds $d(x, \varrho y) \ge r$, and thus $Q(f, g; x, y) = Q(f_{\mathcal{R}}, g_{\mathcal{R}}; x, y) = 0$. In the second case, we have

$$\begin{aligned} Q(f, g; x, y) &= \frac{1}{\mu(B_r(x))}\Big\{\phi(|f(x) - g(y)|) + \phi(|f(\varrho x) - g(\varrho y)|) \\ &\qquad + \phi(|f(\varrho x) - g(y)|) + \phi(|f(x) - g(\varrho y)|)\Big\} \\ &= Q(f_{\mathcal{R}}, g_{\mathcal{R}}; x, y). \end{aligned}$$

In the third and last case, inequality (2.15) is equivalent to

$$\phi(|f_{\mathcal{R}}(x) - g_{\mathcal{R}}(y)|) + \phi(|f_{\mathcal{R}}(\varrho x) - g_{\mathcal{R}}(\varrho y)|) \le \phi(|f(x) - g(y)|) + \phi(|f(\varrho x) - g(\varrho y)|). \tag{2.16}$$

If $f(x) = f(\varrho x)$ or $g(y) = g(\varrho y)$, inequality (2.16) holds as equality. Equality holds in (2.16) also in the following two cases: a) $f_{\mathcal{R}}(x) = f(x)$ and $g_{\mathcal{R}}(y) = g(y)$; b) $f_{\mathcal{R}}(x) = f(\varrho x)$ and $g_{\mathcal{R}}(y) = g(\varrho y)$.

We are left with the following two cases:

$$f(x) > f(\varrho x) \quad \text{and} \quad g(y) < g(\varrho y); \text{ or} \tag{2.17}$$
$$f(x) < f(\varrho x) \quad \text{and} \quad g(y) > g(\varrho y). \tag{2.18}$$

Possibly interchanging f and g, it is enough to consider (2.17). In this case, inequality (2.16) reduces to

$$\phi(|\alpha - \beta|) + \phi(|\gamma - \delta|) \le \phi(|\alpha - \delta|) + \phi(|\gamma - \beta|), \tag{2.19}$$

with $\alpha = f(x)$, $\beta = g(\varrho y)$, $\gamma = f(\varrho x)$, and $\delta = g(y)$. By (2.17) we have $\gamma < \alpha$ and $\delta < \beta$, and inequality (2.19) holds by Lemma 2.7.

If ϕ is strictly convex, then the inequality (2.19) is strict. If, in addition, (2.13) holds and $Q_r(f,g) < \infty$, on integrating (2.15) we get a strict inequality. $\square$

Remark 2.9. The condition (2.2) is used in the distinction of cases after (2.15). If we drop (2.2) we have a fourth case: $d(x,y) \geq r$ and $d(x,\varrho y) < r$. This produces an error term in the inequality (2.12), that no longer holds true. In some situations, it is possible to control this error term. See the proof of Theorem 6.1.

Remark 2.10. When $\phi(t) = t^2$, there is a precise version of inequality (2.12). Let

$$\Sigma_f^+ = \big\{x \in H^+ : f(x) > f(\varrho x)\big\} \quad \text{and} \quad \Sigma_f^- = \big\{x \in H^+ : f(x) < f(\varrho x)\big\} \tag{2.20}$$

denote the sets defined via the inequalities appearing in (2.17)-(2.18).

In the proof of Theorem 2.8, inequality (2.15) is an equality possibly but for the case discussed in (2.17)-(2.19). When $\phi(t) = t^2$, we may replace inequality (2.19) with the identity

$$(\alpha-\beta)^2 + (\gamma-\delta)^2 = (\alpha-\delta)^2 + (\beta-\gamma)^2 + 2(\alpha-\gamma)(\delta-\beta).$$

Now, on integrating the resulting identity, we obtain

$$Q_{r,2}(f_{\mathcal{R}}, g_{\mathcal{R}}) = Q_{r,2}(f,p) + 2\iint_{\Sigma_f^+\times\Sigma_g^-\cup\Sigma_f^-\times\Sigma_g^+,\, d(x,\varrho y)\geq r} \big(f(x)-f(\varrho x)\big)\big(g(y)-g(\varrho y)\big)\chi_r(x,y)\,d\mu\otimes\mu. \tag{2.21}$$

$\square$

The following proposition is a simplified version of Theorem 2.8. Here, the mapping ϱ does not need to be an isometry. The characterization of the strict inequality plays an important role in the proof of Theorem 3.6.

Theorem 2.11. Let $X = H^- \cup H \cup H^+$ be a Borel partition of the metric space X, let $\varrho : X \to X$ be an involutive Borel map such that $\varrho H^+ = H^-$, and let μ be a ϱ-invariant Borel measure on X such that $\mu(H) = 0$. Finally, let ϕ satisfy (2.7) and (2.8). Then for all $f, g \in \mathcal{B}(X)$ we have

$$\int_X \phi(|f_{\mathcal{R}}(x) - g_{\mathcal{R}}(x)|)\,d\mu \leq \int_X \phi(|f(x)-g(x)|)\,d\mu. \tag{2.22}$$

Moreover, if ϕ is strictly convex and

$$\mu\big\{x \in H^+ : f(x) > f(\varrho x) \text{ and } g(x) < g(\varrho x)\big\} > 0 \tag{2.23}$$

then the inequality (2.22) is strict, as soon as the right hand side of (2.22) is finite.

Proof. Using $\mu(H) = 0$ and the ϱ-invariance of μ, we obtain

$$\int_X \phi(|f(x)-g(x)|)d\mu = \int_{H^+} \big\{\phi(|f(x)-g(x)|) + \phi(|f(\varrho x) - g(\varrho x)|)\big\}d\mu.$$

It is then sufficient to establish the pointwise inequality for $x \in H^+$

$$\phi(|f_{\mathcal{R}}(x) - g_{\mathcal{R}}(x)|) + \phi(|f_{\mathcal{R}}(\varrho x) - g_{\mathcal{R}}(\varrho x)|) \leq \phi(|f(x) - g(x)|) + \phi(|f(\varrho x) - g(\varrho x)|).$$

This is inequality (2.16) and the argument is concluded as in the final part of the proof of Theorem 2.8. In fact, if $f(x) > f(\varrho x)$ and $g(x) < g(\varrho x)$ – or viceversa – the inequality is strict, provided that ϕ is strictly convex. □

Theorem 2.8 has the following corollaries.

Theorem 2.12. Let $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ be a reflection system of X, let μ be a nondegenerate, ϱ-invariant Borel measure such that $\mu(H) = 0$. For any function $f \in \mathcal{B}(X)$ and for any $1 \leq p < \infty$ there holds

$$\|f_{\mathcal{R}}\|_{L^p(X,\mu)} = \|f\|_{L^p(X,\mu)} \quad \text{and} \quad \|\nabla f_{\mathcal{R}}\|^-_{L^p(X,\mu)} \leq \|\nabla f\|^-_{L^p(X,\mu)}. \tag{2.24}$$

Moreover, if we have $\|\nabla f_{\mathcal{R}}\|_{L^2(X,\mu)} = \|\nabla f\|_{L^2(X,\mu)} < \infty$ then

$$\lim_{r\downarrow 0} \frac{1}{r^2} \int_{\Sigma_f^+} \int_{\Sigma_f^- \cap B_r(x) \setminus B_r(\varrho x)} \frac{\big(f(x) - f(\varrho x)\big)\big(f(y) - f(\varrho y)\big)}{\mu(B_r(x))} d\mu(y)\, d\mu(x) = 0, \tag{2.25}$$

where Σ_f^+ and Σ_f^- are defined in (2.20).

Proof. The equality of the L^p norms is a consequence of the ϱ-invariance of μ. By Theorem 2.8, we have $r^{-p}Q_{r,p}(f_{\mathcal{R}}) \leq r^{-p}Q_{r,p}(f)$ for any $r > 0$. On taking the liminf as $r \downarrow 0$, we get the inequality in (2.24).

Assume that both $\|\nabla f_{\mathcal{R}}\|_{L^2(X,\mu)}$ and $\|\nabla f\|_{L^2(X,\mu)}$ do exist (the limits exist), are equal and finite. Our claim (2.25) follows from (2.21) with $f = g$. □

For the perimeter we have the following theorem.

Theorem 2.13. Let $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ be a reflection system of X, let μ be a nondegenerate, ϱ-invariant Borel measure such that $\mu(H) = 0$. For any Borel set $E \subset X$ we have

$$\mu(E_{\mathcal{R}}) = \mu(E) \quad \text{and} \quad P^-(E_{\mathcal{R}}) \leq P^-(E). \tag{2.26}$$

Moreover, if $P(E_{\mathcal{R}}) = P(E) < \infty$, then

$$\begin{aligned} &\lim_{r\downarrow 0} \frac{1}{r} \int_{H^+ \cap E \setminus \varrho E} \frac{\mu((\varrho E \setminus E) \cap H^+ \cap B_r(x) \setminus B_r(\varrho x))}{\mu(B_r(x))} d\mu(x) = 0, \\ &\lim_{r\downarrow 0} \frac{1}{r} \int_{H^+ \cap \varrho E \setminus E} \frac{\mu((E \setminus \varrho E) \cap H^+ \cap B_r(x) \setminus B_r(\varrho x))}{\mu(B_r(x))} d\mu(x) = 0. \end{aligned} \tag{2.27}$$

Proof. We shortly discuss the equality case. When $f = \chi_E$, we have

$$\Sigma_f^+ = H^+ \cap E \setminus \varrho E \quad \text{and} \quad \Sigma_f^- = H^+ \cap \varrho E \setminus E.$$

Because of the identity $|\chi_E(x) - \chi_E(y)| = |\chi_E(x) - \chi_E(y)|^2$, we may use (2.21) with $f = g = \chi_E$. The claim (2.27) follows. □

We may try to extend the definition of reflection system taking into account some symmetry of the metric space, of the functions and sets (see Example 5.5 for a motivation).

We say that $\{H^-, H, H^+, \varrho, \sigma\}$ is a *reflection system with symmetry* σ of X, if $X = H^- \cup H \cup H^+$ is a partition, with H^- and H^+ open, and $\varrho, \sigma : X \to X$ are mappings such that:

$$\text{i) } \varrho \text{ is an involutive isometry and } \varrho H^+ = H^-; \tag{2.28}$$

$$\text{ii) } \varrho \text{ and } \sigma \text{ commute, } \varrho\sigma = \sigma\varrho; \tag{2.29}$$

$$\text{iii) } H^+ \text{ is } \sigma\text{-invariant, i.e., } \sigma H^+ = H^+; \tag{2.30}$$

$$\text{iv) for all } x, y \in H \cup H^+ \text{ we have } d(x, y) \leq d(x, \varrho\sigma y). \tag{2.31}$$

Notice, however, that the condition (2.31) fails to hold in the situations discussed in Example 5.5.

Theorem 2.8 holds also in the setting of a reflection system with a symmetry σ, provided that the functions and sets involved are σ-symmetric. The theory developed in Section 3 can be extended to this framework, as well.

3. Steiner and Schwarz type rearrangements

Let $\mathcal{S}(X, \mu)$ denote the set of all Borel functions $f : X \to \mathbb{R}$ such that $f \geq 0$ and $\mu\{f > t\} < \infty$ for any $t > 0$. Here and henceforth, let $\{f > t\} = \{x \in X : f(x) > t\}$ denote the t–superlevel set of f. The function $\psi_f : (0, \infty) \to [0, \infty)$, $\psi_f(t) = \mu\{f > t\}$, $t > 0$, is called *distribution function* of f. A function $g \in \mathcal{S}(X, \mu)$ is said to be a *rearrangement of* $f \in \mathcal{S}(X, \mu)$, and we write $g \sim f$, if $\psi_g = \psi_f$. Clearly, $\sim$ is an equivalence relation on $\mathcal{S}(X, \mu)$. The distribution function ψ_f is nonincreasing and lower semicontinuous. Indeed, for any $s > 0$ we have

$$\lim_{t \downarrow s} \psi_f(t) = \lim_{t \downarrow s} \mu\{f > t\} = \mu\Big(\bigcup_{t>s} \{f > t\}\Big) = \mu\{f > s\} = \psi_f(s). \tag{3.1}$$

For any $f \in \mathcal{S}(X, \mu)$ we have the representation formula

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x)\, dt, \qquad x \in X, \tag{3.2}$$

where χ_A denotes the characteristic function of $A \subset X$. A nonnegative function $f \in L^p(X, \mu)$ is in $\mathcal{S}(X, \mu)$ and, for any $1 \leq p < \infty$, we have the identity

$$\int_X f(x)^p d\mu = \int_0^\infty \mu\{f > t^{1/p}\} dt. \tag{3.3}$$

Moreover, if $g \in \mathcal{S}(X, \mu)$ is a rearrangement of f, $g \sim f$, then $g \in L^p(X, \mu)$ and $\|g\|_{L^p(X,\mu)} = \|f\|_{L^p(X,\mu)}$, by (3.3).

Let $\pi : X \to X$ be a projection, i.e., π is the identity on $\pi(X)$. The relation $x \sim y$ if and only if $\pi(x) = \pi(y)$ is an equivalence relation on X that we denote by Γ. The quotient X/Γ can be identified with $\pi(X)$ and the equivalence class of $x \in X/\Gamma$ is denoted by $\Gamma_x = \pi^{-1}(x)$. We call Γ a *foliation* of X. In fact, we have $X = \bigcup_{x \in X/\Gamma} \Gamma_x$. For a set $E \subset X$, let

$$E_x = E \cap \Gamma_x$$

denote the section of E with Γ_x.

We say that the Borel measure μ is *disintegrable along* Γ if there are Borel measures μ_x on Γ_x, $x \in X/\Gamma$, and a Borel measure $\bar{\mu}$ on X/Γ such that for any Borel set $E \subset X$ we have:

i) The function $x \mapsto \mu_x(E_x)$ is Borel measurable from X/Γ to $[0, \infty]$; (3.4)

ii) We have $\mu(E) = \displaystyle\int_{X/\Gamma} \mu_x(E_x) d\bar{\mu}(x)$. (3.5)

The existence of a disintegration satisfying (3.4)–(3.5) holds under general assumptions. It is provided by some Fubini-Tonelli type theorem – as in $\mathbb{R}^n$ – or by the disintegration theorem of measures. We discuss this issue at the end of the section.

Let μ be disintegrable along Γ. With abuse of notation, for any $x \in X/\Gamma$ we define the function $\mu_x(s) = \mu_x(B_s(x) \cap \Gamma_x)$ of the real variable $s \geq 0$. In general, we have $s \in [0, s_0(x))$, where $s_0(x) > 0$ is the minimum number, possibly $+\infty$, such that the sets $B_r(x) \cap \Gamma_x$ are stable for $r > s_0(x)$. We say that the triple $(\Gamma, (\mu_x)_{x \in X/\Gamma}, \bar{\mu})$ is a *rearrangement system of* (X, μ) if the function $s \mapsto \mu_x(s)$ is strictly increasing and continuous on $[0, s_0(x))$ for $\bar{\mu}$-a.e. $x \in X/\Gamma$.

Fix a rearrangement system of (X, μ) and let $E \subset X$ be a Borel set such that $\mu(E) < \infty$. Then we have $\mu_x(E_x) < \infty$ for $\bar{\mu}$-a.e. $x \in X/\Gamma$. We let $E_x^\star = B_s(x) \cap \Gamma_x$ where $s \in [0, s_0(x)]$ is such that $\mu_x(B_s(x) \cap \Gamma_x) = \mu_x(E_x)$. Such an s exists and is unique for $\bar{\mu}$-a.e. $x \in X/\Gamma$. We possibly let $E_x^\star = \emptyset$ for a $\bar{\mu}$-null set of $x \in X/\Gamma$.

Definition 3.1 (Rearrangement)**.** Let $(\Gamma, (\mu_x)_{x \in X/\Gamma}, \bar{\mu})$ be a rearrangement system of (X, μ).

i) For any Borel set $E \subset X$ such that $\mu(E) < \infty$ we let

$$E^\star = \bigcup_{x \in X/\Gamma} E_x^\star. \tag{3.6}$$

We call $E^\star$ the *rearrangement* of E in $(\Gamma, (\mu_x)_{x \in X/\Gamma}, \bar{\mu})$.

ii) For any $f \in \mathcal{S}(X, \mu)$, the function $f : X \to [0, \infty]$

$$f^\star(x) = \int_0^\infty \chi_{\{f>t\}^\star}(x)\, dt, \quad x \in X, \tag{3.7}$$

is called the *rearrangement of* f in $(\Gamma, (\mu_x)_{x \in X/\Gamma}, \bar{\mu})$.

Finally, we say that the rearrangement system is *regular* if $E^\star$ is a Borel set for any Borel set $E \subset X$.

The problem of determining whether the rearrangement system is regular or not is in general rather subtle. In most relevant examples, the system is indeed regular.

Let $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ be a reflection system of X and let T be a 1-parameter group of isometries of X. We fix on T the natural topology. Let $\pi : X \to X/T$ be the natural projection. As soon as we identify X/T with a subset of X, we have a foliation $X = \bigcup_{x\in X/T} T_x$, where $T_x = \{\tau x \in X : \tau \in T\}$ is the orbit of x.

Definition 3.2 (Steiner system)**.** We say that the pair $(\mathcal{R}, T)$ is a *Steiner system* of the metric space X if we have:

i) $X/T \subset H$ and $\pi : X \to X/T$ is continuous; (3.8)

ii) $\tau^{-1}x = \varrho\tau x$ for any $x \in X/T$ and $\tau \in T$; (3.9)

iii) for $x, y \in X/T$ and $z, w \in T_y$, $d(x, z) \le d(x, w)$ implies $d(y, z) \le d(y, w)$. (3.10)

By $X/T \subset H$ we mean that the quotient is identified with a subset of H. Condition (3.9) with $\tau = \mathrm{Id}$ implies $\varrho x = x$ for all $x \in X/T$. The reflection system $\mathcal{R}$ can be translated along T. Namely, for all $x \in X/T$ and $z, w \in T_x$ there exists a reflection system $\bar{\mathcal{R}} = \{\bar H^-, \bar H, \bar H^+, \bar\varrho\}$ of X such that $\bar\varrho z = w$ and $\bar\varrho : T_y \to T_y$ for any $y \in X/T$. See Example 3.4 below. Motivated by this fact, we propose the following general definition.

Definition 3.3 (Schwarz system)**.** We say that a foliation Γ of the metric space X induced by the projection $\pi : X \to X$ is a *Schwarz system* if we have:

i) $\pi : X \to \pi(X) = X/\Gamma$ is continuous; (3.11)

ii) for all $x \in X/\Gamma$ and $z, w \in \Gamma_x$ there exists a reflection system
$\mathcal{R} = \{H^-, H, H^+, \varrho\}$ such that $\varrho z = w$ and $\varrho : \Gamma_y \to \Gamma_y$ for any $y \in X/\Gamma$; (3.12)

iii) for $x, y \in X/\Gamma$ and $z, w \in \Gamma_y$, $d(x, z) \le d(x, w)$ implies $d(y, z) \le d(y, w)$. (3.13)

We can polarize the function $f^\star$ constructed starting from the foliation Γ using the reflection system given by (3.12). Condition (3.13) guarantees then the stability $f^\star_{\mathcal{R}} = f^\star$ (see the final part of the proof of Theorem 3.6). When X/Γ consists of one element, condition (3.13) is trivially satisfied. The Steiner system is a special case of Schwarz system. In the following example, we comment further on (3.12)

Example 3.4. Let $(\mathcal{R}, T)$ be a Steiner system and let G be a group of isometries of X. We denote by Γ the group generated by T and G, and we identify the quotient X/Γ with a subset of X and in fact of H, the "reflection hyperplane" of $\mathcal{R}$. Assume that $\gamma x = x$ for any $\gamma \in G$ and $x \in X/\Gamma$ and that for any $x \in X/\Gamma$ the orbits have the following representation:

$$\Gamma_x = \big\{\gamma\tau x \in X : \Gamma \in G,\, \tau \in T\big\}. \tag{3.14}$$

We claim that (3.12) holds true.

In fact, if $z_+, z_- \in \Gamma_x$, there are $\gamma_-, \gamma_+ \in G$ and $\tau_-, \tau_+ \in T$ such that $z_- = \gamma_-\tau_- x$ and $z_+ = \gamma_+\tau_+ x$. Moreover, there exist $\tau \in T$ and $\gamma \in G$ such that $\gamma\tau x = \tau_-^{-1}\gamma_-^{-1}\gamma_+\tau_+ x$.

Let $\sqrt{\tau} \in T$ be such that $\tau = \sqrt{\tau}\sqrt{\tau}$. Such a $\sqrt{\tau}$ exists, because T is a 1-parameter group. Let us define $\iota = \gamma_-\tau_-\gamma\sqrt{\tau} \in \Gamma$, and let

$$\bar H = \iota H, \quad \bar H^- = \iota H^-, \quad \bar H^+ = \iota H^+, \quad \bar\varrho = \iota\varrho\iota^{-1}.$$

We claim that $\bar\varrho z_+ = z_-$. Indeed, by (3.9) we have

$$\bar\varrho z_+ = \gamma_-\tau_-\gamma\sqrt{\tau}\varrho\sqrt{\tau}x = \gamma_-\tau_-\gamma\sqrt{\tau}\sqrt{\tau}^{-1}x = \gamma_-\tau_-\gamma x = \gamma_-\tau_-x = z_-.$$

Finally, we prove that $\bar{\mathcal{R}} = \{\bar H^-, \bar H, \bar H^+, \bar\varrho\}$ is a reflection system of X. Clearly, $\bar\varrho$ is an isometry, $\bar\varrho^2 = \mathrm{Id}$ and $\varrho\bar H^+ = \bar H^-$. Moreover, for $x, y \in \bar H^+$ we have

$$d(x, \bar\varrho y) = d(\iota^{-1}x, \varrho\iota^{-1}y) \geq d(\iota^{-1}x, \iota^{-1}y) = d(x, y).$$

The axioms (2.1)–(2.2) are satisfied. Finally, the reflection $\bar\varrho$ preserves the orbits because it is the compisition of orbits preserving isometries.

We study some qualitative properties of the rearrangement $f^\star$.

Lemma 3.5. Let $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar\mu)$ be a regular rearrangement system of (X, μ). For any $f \in \mathcal{S}(X, \mu)$, the rearrangement $f^\star$ of f enjoys the following properties:

i) $\{f^\star > t\} = \{f > t\}^\star$, $t > 0$; (3.15)

ii) $\mu_x\{f^\star > t\}_x = \mu_x\{f > t\}_x$ for $\bar\mu$-a.e. $x \in X/\Gamma$ and, in particular, $f^\star \sim f$; (3.16)

iii) $f^\star(y) \leq f^\star(z)$ if $y, z \in \Gamma_x$ for some $x \in X/\Gamma$ and $d(y, x) \geq d(z, x)$; (3.17)

iv) $f^\star(y) = f^\star(z)$ if $y, z \in \Gamma_x$ for some $x \in X/\Gamma$ and $d(y, x) = d(z, x)$. (3.18)

Proof. i) We prove that $\{f^\star > t\} \subset \{f > t\}^\star$ for any $t > 0$. Notice that the family of sets $(\{f > t\}^\star)_{t>0}$ is nonincreasing in t. For any $x \in \{f^\star > t\}$ we have

$$t < f^\star(x) = \int_0^\infty \chi_{\{f>s\}^\star}(x)ds,$$

and thus $x \in \{f > s\}^\star$ for $0 \leq s \leq t$ and the claim follows.

We preliminarly claim that

$$\{f > t\}^\star = \bigcup_{s>t}\{f > s\}^\star. \tag{3.19}$$

One inclusion is a consequence of the elementary implications

$$s > t \quad\Rightarrow\quad \{f > s\} \subset \{f > t\} \quad\Rightarrow\quad \{f > s\}^\star \subset \{f > t\}^\star.$$

We check the converse inclusion $\subset$ in (3.19). If $z \in \{f > t\}^\star$ then for some $x \in X/\Gamma$ and $r > 0$ we have $z \in \{f > t\}^\star \cap \Gamma_x = B_r(x) \cap \Gamma_x$. Thus there esists $0 < \bar r < r$ such that $z \in B_{\bar r}(x) \cap \Gamma_x$. For the function $r \mapsto \mu_x(B_r(x) \cap \Gamma_x)$ is strictly increasing for $r > 0$ and, as in (3.1), there holds

$$\begin{aligned}\lim_{s\downarrow t}\mu_x(\{f > s\}^\star \cap \Gamma_x) &= \lim_{s\downarrow t}\mu_x(\{f > s\} \cap \Gamma_x)\\ &= \mu_x(\{f > t\} \cap \Gamma_x) = \mu_x(\{f > t\}^\star \cap \Gamma_x),\end{aligned} \tag{3.20}$$

we deduce that there exists $s > t$ such that $B_{\bar r}(x) \cap \Gamma_x \subset \{f > s\}^\star \cap \Gamma_x$ and the claim (3.19) follows.

We prove the converse inclusion $\{f > t\}^\star \subset \{f^\star > t\}$. If $z \in \{f > t\}^\star$ then $z \in \{f > s\}^\star$ for some $s > t$ and thus

$$f^\star(z) = \int_0^\infty \chi_{\{f>s\}}(z)ds \geq s > t.$$

Statement ii) follows from i). Statement iii) is a consequence of the inequality

$$\int_0^\infty \chi_{\{f>s\}}(z)ds \le \int_0^\infty \chi_{\{f>s\}}(y)ds,$$

for $y, z \in \Gamma_x$ with $d(y,x) \le d(z,x)$. Statement iv) follows from iii). □

Let us introduce a few more terminology. Recall that a Borel measure μ on X is *nondegenerate* if (1.1) holds. We say that the measure μ is *diffuse* if spheres are μ-negligible, i.e.,

$$\mu\{y \in X : d(x,y) = r\} = 0, \quad \text{for all } x \in X \text{ and } r > 0. \tag{3.21}$$

We say that μ has the *Lebesgue property* if for any Borel set $A \subset X$ we have for μ-a.e. $x \in A$

$$\lim_{r\downarrow 0} \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))} = 1. \tag{3.22}$$

Finally, we say that μ is *isometric* if $\gamma_\sharp\mu = \mu$ for any isometry $\gamma : X \to X$ and $\mu(H) = 0$ for any reflection system $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ of X.

Theorem 3.6. Let X be a proper metric space, let μ be a nondegenerate, diffuse, isometric Borel measure on X with the Lebesgue property. Let $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar\mu)$ be a regular rearrangement system of (X,μ) related to the Schwarz system Γ. Then the rearrangement $f^\star$ in $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar\mu)$ of any nonnegative, compactly supported function $f \in L^p(X,\mu)$, $1 < p < \infty$, satisfies

$$\|f^\star\|_{L^p(X;\mu)} = \|f\|_{L^p(X;\mu)} \quad \text{and} \quad \|\nabla f^\star\|^-_{L^p(X;\mu)} \le \|\nabla f\|^-_{L^p(X;\mu)}. \tag{3.23}$$

Proof. The identity $\|f^\star\|_{L^p(X;\mu)} = \|f\|_{L^p(X;\mu)}$ follows from (3.3) and (3.16) in Lemma 3.5.

By assumption, the projection $\pi : X \to X/\Gamma$ is continuous. For the set $\text{supp}f = \overline{\{x \in X : f(x) \neq 0\}}$ is compact, the set $\pi(\text{supp}f)$ is compact. With the choice

$$R = 1 + \text{diam}\big(\text{supp}f \cup \pi(\text{supp}f)\big) < \infty,$$

the set

$$K = \bigcup_{x\in\pi(\text{supp}f)} B_R(x) \cap \Gamma_x$$

is bounded and thus contained in a compact set and moreover $\text{supp}f \subset K$. Because $\mu_x\{f > t\}_x \le \mu_x(B_R(x) \cap \Gamma_x)$, we also have $\text{supp}f^\star \subset K$.

Let us recall our notation

$$Q_{r,p}(f) = \int_X \fint_{B_r(x)} |f(x) - f(y)|^p d\mu(y)\, d\mu(x).$$

Let $\mathcal{A}(f)$ be the family of all nonnegative functions $g \in L^p(X;\mu)$ such that:

i) $\mu_x\{g > t\}_x = \mu_x\{f > t\}_x$ for $\bar\mu$-a.e. $x \in X/\Gamma$ and for all $t > 0$; (3.24)

ii) $g(x) = 0$ for μ-a.e. $x \in X \setminus K$; (3.25)

iii) $Q_{r,p}(g) \le Q_{r,p}(f)$ for all $0 < r \le 1$. (3.26)

The set $\mathcal{A}(f)$ is nonempty, because $f \in \mathcal{A}(f)$. We apply to $\mathcal{A}(f)$ the compactness Theorem 4.2, that is proved in Section 4. Here, we need the assumption on X to be proper and the assumption (3.21) on μ.

By (3.3), (3.5), and (3.24), we have for any $g \in \mathcal{A}(f)$

$$\int_X g(x)^p d\mu = \int_0^\infty \mu\{g > t^{1/p}\} d\mu = \int_0^\infty \mu\{f > t^{1/p}\} d\mu = \int_X f(x)^p d\mu.$$

Thus $\mathcal{A}(f)$ is uniformly bounded in $L^p(X, \mu)$. The uniform bound (4.6) holds by (3.26). In fact, we may assume that $\|\nabla f\|^-_{L^p(X;\mu)} < \infty$, otherwise there is nothing to prove. By Theorem 4.2, $\mathcal{A}(f)$ is then precompact in $L^p(X;\mu)$. $\mathcal{A}(f)$ is also closed in $L^p(X;\mu)$. Let $g_j \in \mathcal{A}(f)$, $j \in \mathbb{N}$, be a sequence such that $g_j \to g$ as $j \to \infty$ in $L^p(X;\mu)$ and μ-almost everywhere. Then g satisfies (3.25) and also (3.26), by Fatou's Lemma.

We check (3.24). We may assume that for $\bar{\mu}$-a.e. $x \in X/\Gamma$ we have $g_j(y) \to g(y)$ as $j \to \infty$ for μ_x-a.e. $y \in \Gamma_x$. Then for $\bar{\mu}$-a.e. $x \in X/\Gamma$ and for all $t > 0$ we have:

$$\begin{aligned}\lim_{j\to\infty} \mu_x(\{g > t\}_x \cap \{g_j \le t\}_x) &= \lim_{j\to\infty} \int_{\{g>t\}_x} \chi_{\{g_j \le t\}_x}(y) d\mu_x(y) \\ &= \int_{\{g>t\}_x} \lim_{j\to\infty} \chi_{\{g_j\le t\}_x} d\mu_x(y) = 0.\end{aligned} \tag{3.27}$$

Notice that for a function $g \in L^p(X, \mu)$, the set of all $t > 0$ such that $\mu\{g = t\} > 0$ is at most countable. Then we also have for all but a countable set of $t > 0$:

$$\lim_{j\to\infty} \mu_x(\{g_j > t\}_x \cap \{g \le t\}_x) = \lim_{j\to\infty} \mu_x(\{g_j > t\}_x \cap \{g < t\}_x) = 0.$$

This implies $\mu_x(\{g_j > t\}_x \Delta \{g > t\}_x) \to 0$ as $j \to \infty$, and (3.24) follows for all but a countable set of $t > 0$. By right continuity as in (3.1), (3.24) holds for all $t > 0$.

The functional $J : \mathcal{A}(f) \to [0, \infty)$

$$J(g) = \int_X |g - f^\star|^p d\mu,$$

is continuous in $L^p(X, \mu)$. By Weierstrass' Theorem, there exists $\bar{f} \in \mathcal{A}(f)$ such that

$$J(\bar{f}) = \min \Big\{ J(g) \in [0, \infty) : g \in \mathcal{A}(f) \Big\}. \tag{3.28}$$

There are two cases: 1) $J(\bar{f}) = 0$; 2) $J(\bar{f}) > 0$. In the first case, we have $\bar{f} = f^\star$, and hence, for any $0 < r \le 1$,

$$Q_{r,p}(f^\star) \le Q_{r,p}(f).$$

Dividing this inequality by r^p and taking the liminf as $r \downarrow 0$, we get $\|\nabla f^\star\|^-_{L^p(X;\mu)} \le \|\nabla f\|^-_{L^p(X;\mu)}$ and we are finished.

The case $J(\bar f) > 0$ may not occur. In this case, we have by (3.2) and (3.5)

$$\begin{aligned} 0 &< \Big(\int_X |\bar f(x) - f^\star(x)|^p d\mu(x) \Big)^{1/p} \\ &= \Big(\int_X \Big| \int_0^\infty \big(\chi_{\{\bar f > t\}}(x) - \chi_{\{f^\star > t\}}(x) \big) dt \Big|^p d\mu(x) \Big)^{1/p} \\ &\le \int_0^\infty \Big(\int_X \big| \chi_{\{\bar f > t\}}(x) - \chi_{\{f^\star > t\}}(x) \big|^p d\mu(x) \Big)^{1/p} dt \\ &= \int_0^\infty \mu(\{\bar f > t\} \Delta \{f^\star > t\})^{1/p} dt. \end{aligned}$$

Then, there exists $t > 0$ such that, letting $A = \{\bar f > t\}$ and $B = \{f^\star > t\}$, we have $\mu(A \Delta B) > 0$. As $\bar f$ and $f^\star$ are both rearrangements of f, there holds $\mu(A) = \mu(B)$. Hence, we have $\mu(A \setminus B) = \mu(B \setminus A) > 0$. By (3.22), μ-a.e. $z \in A \setminus B$ is a point of density of $A \setminus B$, i.e.,

$$\lim_{r \downarrow 0} \frac{\mu(B_r(z) \cap A \setminus B)}{\mu(B_r(z))} = 1. \tag{3.29}$$

The same holds for $B \setminus A$.

Let us define the sets

$$\begin{aligned} \Lambda_{A \setminus B} &= \big\{ x \in X/\Gamma : \text{there exists } z \in \Gamma_x \cap A \setminus B \text{ point of density of } A \setminus B \big\}, \\ \Lambda_{B \setminus A} &= \big\{ x \in X/\Gamma : \text{there exists } z \in \Gamma_x \cap B \setminus A \text{ point of density of } B \setminus A \big\}. \end{aligned}$$

We claim that $\bar\mu(\Lambda_{A\setminus B} \cap \Lambda_{B \setminus A}) > 0$. In fact, we have $\mu_x(A_x \setminus B_x) = \mu_x(B_x \setminus A_x)$ for $\bar\mu$-a.e. $x \in X/\Gamma$. This follows from the fact that both $\bar f$ and $f^\star$ sastisfy (3.24). Hence,

$$\int_{\Lambda_{A\setminus B}} \mu_x(B_x \setminus A_x) d\bar\mu(x) = \int_{\Lambda_{A\setminus B}} \mu_x(A_x \setminus B_x) d\bar\mu(x) = \mu(A \setminus B) > 0,$$

and thus there exists a set $\Lambda \subset \Lambda_{A \setminus B}$ such that $\bar\mu\Lambda > 0$ and $\mu_x(B_x \setminus A_x) > 0$ for all $x \in \Lambda$. Then, there exist $x \in X/\Gamma$, $z_- \in \Gamma_x \cap A \setminus B$ point of density of $A \setminus B$, and $z_+ \in \Gamma_x \cap B \setminus A$ point of density of $B \setminus A$.

Let $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ be the reflection system related to $z = z_+$ and $w = z_-$ given by (3.12). In particular, we may assume $z_+ \in H^+$ and $z_- \in H^-$. By (3.29) there exists a number $\eta > 0$ such that

$$\mu(B_\eta(z_-) \cap A \setminus B) > \frac{1}{2}\mu(B_\eta(z_-)), \quad \mu(B_\eta(z_+) \cap B \setminus A) > \frac{1}{2}\mu(B_\eta(z_+)). \tag{3.30}$$

Possibly choosing a smaller η we may also assume that $B_\eta(z_+) \subset H^+$ (and hence also $B_\eta(z_-) \subset H^-$).

From (3.30) we deduce that $\mu\big(H^+ \cap (B \setminus A \cap \varrho(A \setminus B))\big) > 0$. In view of $B \setminus A \cap \bar\varrho(A \setminus B) = B \setminus \varrho B \cap \varrho A \setminus A$, we eventually obtain

$$\mu\big\{ x \in H^+ : \bar f(x) < \bar f(\varrho x) \text{ and } f^\star(x) > f^\star(\varrho x) \big\} > 0. \tag{3.31}$$

This is assumption (2.23) in Theorem 2.11.

We claim that the two-points rearrangement $f^\star_{\mathcal{R}}$ satisfies $f^\star_{\mathcal{R}} = f^\star$. From $f^\star(z_+) > t \ge f^\star(z_-)$ we deduce by Lemma 3.5 that $d(z_+, x) < d(x_-, x)$. As $z_- = \bar\varrho z_+$, this implies that $x \in H^+ \cup H$, by (2.2). Now let $z, w \in \Gamma_y$, $y \in X/\Gamma$, be such that $z \in H^+$ and $w = \varrho z$. Again by (2.2), we have $d(x,z) \le d(x,w)$ and thus, by (3.13), $d(y,z) \le d(y,w)$. This yields $f^\star(z) \ge f^\star(w)$, by Lemma 3.5, and the claim is proved.

As $\phi(t) = t^p$ with $p > 1$ is strictly convex, by the statement concerning the strict inequality in Theorem 2.11 we have

$$\int_X |\bar f_{\mathcal{R}} - f^\star|^p d\mu = \int_X |\bar f_{\mathcal{R}} - f^\star_{\mathcal{R}}|^p d\mu < \int_X |\bar f - f^\star|^p d\mu. \tag{3.32}$$

This contradicts the minimality (3.28) of $\bar f$, provided that $\bar f_{\mathcal{R}} \in \mathcal{A}(f)$. We check (3.24)–(3.25) for $g = \bar f_{\mathcal{R}}$.

We start with (3.24). For a Borel set $B \subset X/\Gamma$ let h denote the function $\bar f$ restricted to $\pi^{-1}(B)$. For μ is ϱ-invariant we have $\mu\{h > t\} = \mu\{h_{\mathcal{R}} > t\}$, $t > 0$. As $\varrho : \Gamma_y \to \Gamma_y$ for any $y \in X/\Gamma$, the function $h_{\mathcal{R}}$ is also supported in $\pi^{-1}(B)$, and thus[1]

$$\int_B \mu_x\{\bar f > t\}_x d\bar\mu(x) = \int_B \mu_x\{\bar f_{\mathcal{R}} > t\}_x d\bar\mu(x)$$

for a generic B. This implies the claim (3.24).

Next, we prove that the function $\bar f_{\mathcal{R}}$ is supported in K. Let $z, w \in \Gamma_y$, $y \in X/\Gamma$, be such that $w = \varrho z$ with $z \in H^+$. Because $x \in H^+ \cup H$, there holds $d(x,z) \le d(x,w)$, by (2.2), and so $d(y,z) \le d(y,w)$, by (3.13). Now, if $f(w) > 0$ then $w \in K$. By the previous observation, this impies that also $z \in K$. This ensures that $\bar f_{\mathcal{R}}$ is supported in K.

Finally, (3.26) holds by Theorem 2.8.

□

We have an analogous theorem for the rearrangement of sets.

Theorem 3.7. Let X be a proper metric space, let μ be a nondegenerate, diffuse, isometric Borel measure on X with the Lebesgue property. Let $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar\mu)$ be a regular rearrangement system of (X,μ) related to the Schwarz system Γ. The Schwarz rearrangement $E^\star$ in $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar\mu)$ of any bounded Borel set $E \subset X$ satisfies

$$\mu(E^\star) = \mu(E) \quad \text{and} \quad P^-(E^\star) \le P^-(E). \tag{3.33}$$

Proof. The proof is analogous to the one of Theorem 3.6 and we only sketch it. First, we fix a suitable compact set $K \subset X$, as in the above proof. Then we introduce the set $\mathcal{A}(E)$ of all Borel subsets F of X such that (3.24)–(3.26) hold with $g = \chi_F$ and $f = \chi_E$ and $p = 2$ (or equivalently $p = 1$). The functional $J(F) = \mu(F\Delta E^\star)$ attains the minimum on $\mathcal{A}(E)$ at some $\bar F$. The compactness Theorem 4.2 does apply to this situation. As in the proof above, it must be $\bar F = E^\star$ and the proof is finished. □

When X/Γ consists of one point, the set $E^\star$ is a ball. Theorem 3.7 states in this case that metric balls are isoperimetric sets, within the class of bounded sets, in the metric measure space (X,μ). This is the case of space forms (Euclidean and hyperbolic space, sphere).

[1]CHECK

In the final part of this section, we address the problem of the existence of a disintegration of μ along Γ, an isometry group of X. When $X/\Gamma = \{x\}$ consists of one element there exists a trivial disintegration. In fact, we may choose $\mu_x = \mu$ and $\bar{\mu}$ =Dirac mass on X/Γ. In this case, $(\Gamma, \mu_x, \bar{\mu})$ is a rearrangement system of (X, μ) as soon as the function $s \mapsto \mu(B_s(x))$ is continuous and strictly increasing in its natural domain.

Let us recall the disintegration theorem for probability measures. A proof can be found in [DM], III.70-73. By definition, a Borel measure μ on the metric spaces X is *regular* if $\mu(E) = \sup\big\{\mu(K) : K \subset E \text{ compact}\big\}$ for any Borel set $E \subset X$.

Theorem 3.8. Let X, Y be separable metric spaces, let $\pi : X \to Y$ be a Borel map, let μ be a regular Borel probability measure on X, and let $\bar{\mu} = \pi_\sharp \mu$ be the push-forward measure of μ on Y. Then there exist Borel probability measures μ_y supported in $\pi^{-1}(y)$, $y \in Y$, such that the function $y \mapsto \mu_y(E)$ is Borel measurable and

$$\mu(E) = \int_Y \mu_y(E) d\bar{\mu}(y),$$

for any Borel set E.

We apply Theorem 3.8 to our setting in a couple of examples.

Example 3.9. Let X be a compact metric space and let Γ be a Schwarz system of X. Then any finite, regular Borel measure on X is disintegrable along Γ. This follows from Theorem 3.8 with X and $Y = X/\Gamma = \pi(X)$. In fact, Y is compact because the projection is continuous.

Though restrictive, the compact case is actually sufficient to our porpouses. In fact, in Theorem 3.6 the functions are supposed to have compact support. Then we could localize the rearrangement in some compact set and restrict the measure to this set.In the case of a Steiner system, the measure μ is assumed to be invariant with respect to a 1-parameter group of isometries. This makes possible a disintegration also in the noncompact case.

Example 3.10. Let X be a σ-compact metric space and let $(\mathcal{R}, T)$ be a Steiner system of X with $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ and $T = \{\tau_t\}_{t\in\mathbb{R}}$ such that $H^+ = \bigcup_{t>0} \tau_t(H)$, with disjoint union. Then any locally finite, T-invariant and regular Borel measure μ on X is disintegrable along T. By definition, the measure μ is T-invariant if $(\tau_t)_\sharp \mu = \mu$ for all $t \in \mathbb{R}$.

By assumption, we have $H = X/T$ and the projection $\pi : X \to X/T$ is continuous. Then H is σ-compact and w.l.g. we may assume that H is compact. For any $k \in \mathbb{Z}$ let

$$X_k = \bigcup_{t\in[k,k+1)} \tau_t(H).$$

Then we have $X = \bigcup_{k\in\mathbb{Z}} X_k$, with disjoint union. With the natural assumption that the mapping $(x, t) \mapsto \tau_t(x)$ be continuous from $H \times \mathbb{R}$ to X, the Borel set X_k is bounded. The measure $\mu_k = \mu \llcorner X_k$, the restriction of μ to X_k, is then finite and moreover the measure $\bar{\mu} = \pi_\sharp \mu_k$ is independent of $k \in \mathbb{Z}$, because μ is T invariant. By Theorem 3.8, there are probability measures μ_x^k, $x \in X/T$, supported in $T_x \cap X_k$ such that

$$\mu(E) = \int_{X/T} \mu_x^k(E \cap T_x) d\bar{\mu}(x),$$

for any Borel set $E \subset X_k$. Letting $\mu_x = \sum_{k\in\mathbb{Z}} \mu_x^k$ we obtain a disintegration of μ along T. The measures μ_x are locally finite.

We investigate whether $(T, (\mu_x)_{x\in X/T}, \bar{\mu})$ is a rearrangement system of (X, μ), i.e., whether for $\bar{\mu}$-a.e. $x \in X/T$ the function $s \mapsto \mu_x(B_s(x) \cap T_x)$ is strictly increasing and continuous for $s \geq 0$

Let $E \subset X/T$ be a Borel set and for $-\infty < r < s < \infty$ let $E_{r,s} = \bigcup_{r<t<s} \tau_t(E)$. Since μ is T-invariant we have $\mu(E_{r,s}) = \mu(E_{r+t,s+t})$ for all $t \in \mathbb{R}$. The disintegration formula (3.5) implies that

$$\int_E \mu_x(E_{r,s} \cap T_x) d\bar{\mu}(x) = \int_E \mu_x(E_{r+t,s+t} \cap T_x) d\bar{\mu}(x).$$

Because E is arbitrary, we deduce that, for fixed r, s, t, there holds $\mu_x(E_{r,s}) = \mu_x(E_{r+t,s+t})$ for $\bar{\mu}$-a.e. $x \in X/T$. Finally, this implies that there exists a set $N \subset X/T$ with $\bar{\mu}(N) = 0$ such that

$$\mu_x(E_{r,s}) = \mu_x(E_{r+t,s+t}) \tag{3.34}$$

for all $x \in (X/T) \setminus N$ and for all $r, s, t \in \mathbb{Q}$ with $r < s$. We deduce that μ_x is nonatomic, i.e., $\mu_x\{z\} = 0$ for all $z \in T_x$. In fact, if $\mu_x\{z\} = \delta > 0$ for some $z \in T_x$ then, by (3.34), this holds for all $z \in T_x$ and μ_x is not locally finite. The same argument proves that if $\mu_x(E_{r,s} \cap T_x) = 0$, $x \in E$, for some $r < s$ then $\mu_x = 0$.

We proved that for $\bar{\mu}$-a.e. $x \in X/T$ the function $s \mapsto \mu_x\big(\bigcup_{0<t<s} \tau_t(H) \cap T_x\big)$ is either identically zero or continuous and strictly increasing. This is sufficient to set up a Steiner-type rearrangement, as in Definition 3.1.

The function $s \mapsto \mu_x(B_s(x) \cap T_x)$ is strictly increasing. This follows from the discussion above. If the orbit T_x, $x \in X/T$, meets the spheres $\{y \in X : d(x, y) = s\}$ at isolated points, the function is also continuous and $(T, (\mu_x)_{x\in X/T}, \bar{\mu})$ is a rearrangement system of (X, μ) in the sense defined before Definition 3.1. □

4. Compactness

We prove the compactness theorem in $L^p(X, \mu)$ used in Section 3, Theorem 3.6. We recall that a family of functions $F \subset L^1_{\text{loc}}(X, \mu)$ is said to be *locally uniformly bounded* if for any compact set $K \subset X$ we have

$$\sup_{f\in F} \int_K |f|\, d\mu < +\infty. \tag{4.1}$$

The family F is said to be *locally uniformly absolutely continuous in* $L^1_{\text{loc}}(X, \mu)$ if for any compact set $K \subset X$ and for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any Borel set $B \subset K$ there holds

$$\mu(B) < \delta \quad \Rightarrow \quad \sup_{f\in F} \int_B |f|\, d\mu < \varepsilon. \tag{4.2}$$

A metric space is proper if closed balls are compact.

Lemma 4.1. Let X be a proper metric space with a Borel measure μ satisfying (1.1) and (3.21). Let $F \subset L^1_{\mathrm{loc}}(X,\mu)$ be a family of functions which is locally uniformly bounded and locally uniformly absolutely continuous. Then the family of functions $F_r = \{f_r \in C(X) : f \in F\}$, where

$$f_r(x) = \fint_{B_r(x)} f(y)\, d\mu(y), \tag{4.3}$$

is locally uniformly bounded in $C(X)$ and locally uniformly continuous.

Proof. Because the balls $B_r(x)$ are precompact, the functions f_r in (4.3) are well defined. Because $\mu(\{y \in X : d(x,y) = r\}) = 0$, the characteristic function of $B_r(x)$ converges μ-a.e. to the charateristic function of $B_r(x_0)$, as $x \to x_0$, for any $x_0 \in X$. By the theorem of dominated convergence, we then have

$$\lim_{x\to x_0} \int_{B_r(x)} f(y) d\mu(y) = \int_{B_r(x_0)} f(y) d\mu(y).$$

In particular, $x \mapsto \mu(B_r(x))$ is continuous (and positive). It follows that $f_r \in C(X)$.

Let $K \subset X$ be a compact set and let $K_r = \{x \in X : \mathrm{dist}(x,K) \le r\}$. The set K_r is also compact. Letting

$$C_1 = \max_{x\in K} \frac{1}{\mu(B_r(x))}, \quad C_2 = \sup_{f\in F} \int_{K_r} |f(y)|\, d\mu(y) < \infty,$$

we have $|f_r(x)| \le C_1C_2$, for any $x \in K$ and $f \in F$. Thus F_r is locally uniformly bounded.

On the other hand, for any $x, x_0 \in K$

$$\begin{aligned} |f_r(x) - f_r(x_0)| \le & \max\Big\{\frac{1}{\mu(B_r(x))}, \frac{1}{\mu(B_r(x_0))}\Big\} \int_{B_r(x)\Delta B_r(x_0)} |f(y)|\, d\mu(y) \\ & + \frac{|\mu(B_r(x)) - \mu(B_r(x_0))|}{\mu(B_r(x))\mu(B_r(x_0))} \int_{B_r(x)\cap B_r(x_0)} |f(y)|\, d\mu(y) \\ \le & C_1 \int_{B_r(x)\Delta B_r(x_0)} |f(y)|\, d\mu(y) + C_1^2 C_2 |\mu(B_r(x)) - \mu(B_r(x_0))|, \end{aligned} \tag{4.4}$$

where $B_r(x)\Delta B_r(x_0) = B_r(x) \setminus B_r(x_0) \cup B_r(x_0) \setminus B_r(x)$ denotes the symmetric difference of sets.

The function $m : X \times X \to [0,\infty)$, $m(x,x_0) = \mu(B_r(x)\Delta B_r(x_0))$ is continuous and thus uniformly continuous on $K \times K$. Moreover, $m(x_0,x_0) = 0$. Thus, for any $\delta > 0$ there is an $\eta > 0$ such that $d(x,x_0) < \eta$ implies $m(x,x_0) < \delta$. By (4.2), for any given $\varepsilon > 0$ we have

$$\sup_{f\in F} \int_{B_r(x)\Delta B_r(x_0)} |f(y)|\, d\mu(y) < \varepsilon \tag{4.5}$$

as soon as $d(x,x_0)<\eta$ and $\eta>0$ is small enough. By (4.5) and (4.4), F_r is uniformly continuous on compact sets. $\square$

Let us recall our notation, with $f\in L^p(X,\mu)$ and $r>0$,

$$Q_{r,p}(f)=\int_X \fint_{B_r(x)} |f(x)-f(y)|^p d\mu(y)\, d\mu(x).$$

Theorem 4.2 (Compactness)**.** Let (X,μ) be a proper metric measure space satisfying (1.1) and (3.21). Let $1\le p<\infty$ and let $F\subset L^p_{\mathrm{loc}}(X,\mu)$ be a set of functions such that:

i) F is uniformly bounded in $L^p_{\mathrm{loc}}(X,\mu)$; moreover, if $p=1$ assume that F is uniformly absolutely continuous;
ii) there exists a function $g\in L^p(X,\mu)$ such that $\liminf_{r\downarrow 0} r^{-p}Q_{r,p}(g)<\infty$ and for all $0<r<1$ there holds

$$\sup_{f\in F} Q_{r,p}(f)\le Q_{r,p}(g). \tag{4.6}$$

Then F is precompact in $L^p_{\mathrm{loc}}(X,\mu)$.

Proof. Let $K\subset X$ be a compact set. The assumptions in Lemma 4.1 are satisfied. The set $F_r=\big\{f_r\in C(K): f\in F\big\}$ is then equibounded and equicontinuous, and by Ascoli-Arzelà's Theorem, F_r is totally bounded with respect to the max norm and then with respect to the $L^p(K,\mu)$ norm.

We claim that

$$\liminf_{r\downarrow 0}\sup_{f\in F}\|f_r-f\|_{L^p(K,\mu)}=0. \tag{4.7}$$

This follows from ii):

$$\begin{aligned}\int_K |f_r-f|^p d\mu &= \int_K \Big|\fint_{B_r(x)} \big(f(y)-f(x)\big)d\mu(y)\Big|^p d\mu(x)\\ &\le \int_K\fint_{B_r(x)} |f(y)-f(x)|^p d\mu(y)d\mu(x)\\ &\le \int_X\fint_{B_r(x)} |g(y)-g(x)|^p d\mu(y)d\mu(x),\end{aligned}$$

the inequality holding for any $0<r<1$. This implies (4.7).

Finally, by a standard argument from (4.7) it follows that F is totally bounded in $L^p(K,\mu)$. $\square$

5. Examples

We describe some examples of reflection, Steiner, and Schwarz system.

5.1. **Banach spaces.** Let $X = Z \oplus V$ be a real vector space, where V is a 1-dimensional subspace of X. We may then decompose $x \in X$ as $x = z + v$ for unique $z \in Z$ and $v \in V$. On V we fix a total ordering. Let $\varrho : X \to X$ be the map $\varrho(x) = \varrho(z + v) = z - v$, and let $\|\cdot\|$ be a norm on X such that

$$\|\varrho x\| = \|x\| \text{ for all } x \in X. \tag{5.1}$$

Let us define the sets $H = Z$, $H^- = \{x \in X : x = z + v,\, z \in Z,\, v < 0\}$, and $H^+ = \{x \in X : x = z + v,\, z \in Z,\, v > 0\}$. We claim that $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ is a reflection system of X with the distance induced by the norm $\|\cdot\|$.

Let $v \in V$ and $z \in Z$. The function $\vartheta(t) = \|z + tv\|$ is nondecreasing for $t \geq 0$. In fact, for $0 \leq t < s$ we have

$$z + tv = \sigma(z - tv) + (1 - \sigma)(z + sv), \quad \text{with } \sigma = \frac{s - t}{s + t} \in (0, 1),$$

and therefore, also using (5.1), we have

$$\vartheta(t) = \|z + tv\| \leq \sigma\|z - tv\| + (1 - \sigma)\|z + sv\| = \sigma\vartheta(t) + (1 - \sigma)\vartheta(s),$$

that implies $\vartheta(t) \leq \vartheta(s)$.

We can now prove (2.2) and namely that $\|x - y\| \leq \|x - \varrho y\|$ for all $x, y \in H \cup H^+$. Let $x = z_1 + t_1 v$ and $y = z_2 + t_2 v$ be in $H^+ \cup H$, i.e., $t_1, t_2 \geq 0$. From the previous observation along with the trivial inequality $|t_1 - t_2| \leq t_1 + t_2$ and (5.1), it follows that

$$\|x - y\| = \|z_1 - z_2 + |t_1 - t_2|v\| \leq \|z_1 - z_2 + (t_1 + t_2)v\| = \|x - \varrho y\|.$$

We specialize to the following situation. Let us factorize $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$, for some $1 \leq m \leq n$. When $m = n$ we agree that $\mathbb{R}^{n-m} = \{0\}$. Let $G = O(m) \subset O(n)$ be the group of orthogonal transformations of $\mathbb{R}^n$ fixing the $\mathbb{R}^{n-m}$ factor. Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ such that $\|\gamma x\| = \|x\|$ for all $x \in \mathbb{R}^n$ and $\gamma \in G$. We endow $\mathbb{R}^n$ with the metric space structure induced by this norm.

Let $v \in \mathbb{R}^m \times \{0\}$, $v \neq 0$, and denote by H the hyperplane orthogonal to v. We have a natural partition $\mathbb{R}^n = H^- \cup H \cup H^+$ and a natural reflection ϱ with respect to H. As noted above, $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ is a reflection system. Let $\tau_t : \mathbb{R}^n \to \mathbb{R}^n$ be the translation $\tau_t x = x + tv$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. $T = \{\tau_t\}_{t \in \mathbb{R}}$ is a 1-parameter group of isometries. We have $\mathbb{R}^n/T = H$ and $\mathbb{R}^n/\Gamma = \{0\} \times \mathbb{R}^{n-m}$, where $\Gamma = \Gamma(T, G)$, the group generated by T and G.

We claim that the projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ induces a Schwarz system of $\mathbb{R}^n$ with the norm $\|\cdot\|$. It is elementary to check the the representation (3.14) for the orbits holds. As a consequence, condition (3.12) also holds. We check (3.13). Let $x, y \in \mathbb{R}^n/\Gamma = \{0\} \times \mathbb{R}^{n-m}$ and $z, w \in \Gamma_y = \mathbb{R}^m \times \{y\}$ be such that $z = y + t\gamma v$ and $w = y + s\xi v$, $t, s \in \mathbb{R}$ and $\gamma, \xi \in G$. If $\|x - z\| \leq \|x - w\|$ then by (5.1) we have

$$\|y - x + tv\| = \|y - x + t\gamma v\| \leq \|y - x + s\xi v\| = \|y - x + sv\|,$$

that implies $|t| \leq |s|$. This in turn implies

$$\|z - y\| = \|t\gamma v\| = |t|\|v\| \leq |s|\|v\| = \|s\xi v\| = \|w - y\|.$$

The Fubini-Tonelli theorem provides a disintegration along Γ of the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$ and a regular rearrangement system. Theorems 3.6 and 3.7 apply to the metric measure space $(\mathbb{R}^n, \|\cdot\|, \mathcal{L}^n)$. Within this setting, integral differential quotients as in (1.3) are studied in [P].

5.2. **Hyperbolic space.** Let $\mathbb{H}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the n-dimensional hyperbolic space, $n \geq 2$, given in the ball model. In the sequel, $|\cdot|$ and $\cdot$ denote the standard norm and inner product in $\mathbb{R}^n$. The metric d is defined via the identity

$$\cosh d(x,y) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}, \quad x, y \in \mathbb{H}^n. \tag{5.2}$$

In the coordinates $x = (x_1, ..., x_n)$, let

$$H^- = \big\{x \in \mathbb{H}^n : x_1 < 0\big\}, \quad H = \big\{x \in \mathbb{H}^n : x_1 = 0\big\}, \quad H^+ = \big\{x \in \mathbb{H}^n : x_1 > 0\big\}.$$

The mapping $\varrho : \mathbb{H}^n \to \mathbb{H}^n$, $\varrho(x) = (-x_1, x_2, ..., x_n)$ is an involutive isometry such that $\varrho H^+ = H^-$. By formula (5.2), condition (2.2) holds. This also follows from the remark in Example 2.2. Then $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ is a reflection system.

For any $b \in \mathbb{H}^n$, the mapping $\tau_b : \mathbb{H}^n \to \mathbb{H}^n$

$$\tau_b(x) = \frac{1-|b|^2}{|b|^2|x|^2 + 2x\cdot b + 1}x + \frac{|x|^2 + 2x\cdot b + 1}{|b|^2|x|^2 + 2x\cdot b + 1}b$$

is a hyperbolic isometry, called translation by b. With $e_1 = (1, 0, ..., 0)$, we also let

$$\tau_t(x) = \frac{1-t^2}{t^2|x|^2 + 2tx_1 + 1}x + \frac{|x|^2 + 2tx_1 + 1}{t^2|x|^2 + 2tx_1 + 1}te_1, \quad t \in (-1, 1). \tag{5.3}$$

Then $T = \{\tau_t\}_{t\in(-1,1)}$ is a 1-parameter group of isometries

$$\tau_s\tau_t = \tau_u, \quad \text{with } u = \frac{s+t}{1+st} \text{ and } s, t \in (-1, 1). \tag{5.4}$$

The quotient is $\mathbb{H}^n/T = \{x \in \mathbb{H}^n : x_1 = 0\} = H$. We claim that $(\mathcal{R}, T)$ is a Steiner system. Trivially, for any $\tau \in T$ and $x \in \mathbb{H}^n/T$ we have $\tau^{-1}x = \varrho\tau x$. This is (3.9). We check (3.10). For $x, y \in \mathbb{H}^n/T$ let $\vartheta : (-1, 1) \to [0, \infty)$ be the function

$$\vartheta(t) = \frac{|\tau_t x - y|^2}{(1-|\tau_t x|^2)(1-|y|^2)} = \frac{t^2(1 + 2x\cdot y + |x|^2|y|^2) + |x-y|^2}{(1-t^2)(1-|x|^2)(1-|y|^2)}.$$

The second identity can be checked by a short computation based on (5.3). There holds $\vartheta'(t) > 0$ for $t \in (0, 1)$ and, by (5.2), this implies (3.10).

The hyperbolic measure on $\mathbb{H}^n$ is, up to a positive multiplicative constant,

$$\mu = \frac{1}{(1-|x|^2)^n}\mathcal{L}^n,$$

where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$. A disintegration of μ along T is provided by the construction given in Example 3.10. We describe explicitly the disintegration in dimension $n = 2$. Let $f : \mathbb{H}^2 \to (-1, 1)$ be the function

$$f(x) = h\Big(\frac{2x_2}{1-|x|^2}\Big), \quad x = (x_1, x_2) \in \mathbb{H}^2,$$

where $h : \mathbb{R} \to (-1,1)$ is the function $h(s) = s/(1+\sqrt{1+s^2})$. The level sets of f are the orbits of T. Namely, for any $x = (0, x_2) \in H$ there holds $f(\tau_t x) = x_2$ for all $t \in (-1,1)$. For any Borel set $E \subset \mathbb{H}^2$, by the standard coarea formula we have

$$\mu(E) = \int_E \frac{1}{(1-|x|^2)^2} dx = \int_{-1}^{1} \int_{\{f=\sigma\}\cap E} \frac{1}{(1-|x|^2)^2 |\nabla f(x)|} d\mathcal{H}^1(x)\, d\sigma, \tag{5.5}$$

where $\mathcal{H}^1$ is the standard length measure and $|\nabla f(x)|$ is the standard length of the gradient of f. By an elementary computation, we have $|\nabla f(x)| = |f(x)/x_2|$. Using this piece of information and integrating along orbits in the set of parameters $t \in (-1,1)$, we finally obtain the disintegration

$$\mu(E) = \int_{-1}^{1} \frac{1+\sigma^2}{(1-\sigma^2)^2} \int_{E_\sigma} \frac{1}{1-t^2} dt\, d\sigma, \tag{5.6}$$

where $E_\sigma = \big\{ t \in (-1,1) : \tau_t(0,\sigma) \in E \big\}$ denotes the section of E with the orbit, at the parameters level. The measure $d\mu_\sigma = \frac{1}{1-t^2} dt$ on the orbit is in fact independent of σ: it is the Haar measure of $(-1,1)$ with the group law (5.4).

We describe now examples of Schwarz system. The arguments rely upon elementary facts of hyperbolic geometry. Let $1 \le m \le n$ and for $x \in \{0\} \times \mathbb{H}^{n-m}$, $0 \in \mathbb{H}^m$, let $\Gamma_x = \tau_x\big(\mathbb{H}^m \times \{0\}\big)$, $0 \in \mathbb{H}^{n-m}$, be the translation by x of the copy of $\mathbb{H}^m$ sitting inside $\mathbb{H}^n$. We have a foliation Γ of $\mathbb{H}^n$: $\mathbb{H}^n/\Gamma = \{0\} \times \mathbb{H}^{n-m}$ and

$$\mathbb{H}^n = \bigcup_{x \in \mathbb{H}^n/\Gamma} \Gamma_x.$$

The foliation is obviously given by a continuous projection $\pi : \mathbb{H}^n \to \{0\} \times \mathbb{H}^{n-m}$. We claim that Γ is a Schawrz system of $\mathbb{H}^n$.

Let $x, y \in \{0\} \times \mathbb{H}^{n-m}$ and $z, w \in \Gamma_y$ be such that $d(z,x) \le d(w,x)$. We claim that $d(z,y) \le d(w,y)$. As τ_{-y} maps $\{0\} \times \mathbb{H}^{n-m}$ into itself, we can without loss of generality assume that $y = 0$. The claim then follows from the equivalence

$$\frac{|z-x|^2}{(1-|x|^2)(1-|z|^2)} \le \frac{|w-x|^2}{(1-|x|^2)(1-|w|^2)} \quad \Leftrightarrow \quad |z| \le |w|$$

that holds for all $z, w \in \mathbb{H}^m \times \{0\}$ and $x \in \{0\} \times \mathbb{H}^{n-m}$. This proves (3.13).

The proof of (3.12) is elementary. Given $z, w \in \mathbb{H}^n$, $z \ne w$, let H be the "hyperplane" through 0 orthogonal to $\tau_z^{-1} w \in \mathbb{H}^n$. Let ϱ be the reflection with respect to H. We have $\tau_z^{-1} w = tv$ for some $v \in \mathbb{R}^n$, $|v| = 1$, and $t \in (0,1)$. Let $s \in (0,1)$ be such that $\frac{2s}{1+s^2} = t$. Then the conjugation of H and ϱ with $\tau_z \tau_{sv}$, namely $\bar{H} = \tau_z \tau_{sv} H \tau_{sv}^{-1} \tau_z^{-1}$ and $\bar{\varrho} = \tau_z \tau_{sv} \varrho \tau_{sv}^{-1} \tau_z^{-1}$, provides the required reflection system. Then Theorems 3.6 and 3.7 apply to the hyperbolic space with its measure.

We describe the disintegration of the hyperbolic measure μ along Γ in the case $n = 2$ and $m = 1$. This configuration represents a kind of "Steiner rearrangement" dual to the one discussed above. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x) = \frac{2x_1}{|x|^2+1}$, $x = (x_1, x_2) \in \mathbb{R}^2$. For any $s \in (-1,1)$, we have

$$\Big\{ x \in \mathbb{H}^2 : f(x) = \frac{2s}{1+s^2} \Big\} = \tau_s\big(\{0\} \times (-1,1)\big).$$

Starting from the formula (5.5), by the change of variable $\sigma = 2s/(s^2+1)$ we obtain

$$\mu(E) = 2\int_{-1}^{1} \frac{1-s^2}{1+s^2} \int_{\{f=2s/(s^2+1)\}\cap E} \frac{1}{(1-|x|^2)^2|\nabla f(x)|} d\mathcal{H}^1(x)\, ds.$$

We compute $|\nabla f(x)|$ and we express the inner integral in parametric form along the curve $t \mapsto \tau_s(0,t)$. The details are omitted. We obtain the formula

$$\mu(E) = \int_{-1}^{1} \frac{1}{1-s^2} \int_{E_s} \frac{1+t^2}{(1-t^2)^2} dt\, ds,$$

where $E_s = \big\{t \in (-1,1) : \tau_s(0,t) \in E\big\}$. This is the formula dual to (5.6).

5.3. **Sphere.** The standard sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is also rich of reflection, Steiner, and Schwarz systems. This example is well-known and motivated the general theory on polarization (see [BT] and [Be]).

5.4. **Grushin plane.** Consider the vector fields in $\mathbb{R}^2$

$$X_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_2 = |x_1|\frac{\partial}{\partial x_2}.$$

A Lipschitz curve $\gamma : [0,1] \to \mathbb{R}^2$ is admissible if $\dot\gamma = h_1 X_1(\gamma) + h_2 X_2(\gamma)$ for functions $h_1, h_2 \in L^1(0,1)$. We define the length of an admissible curve γ as

$$L(\gamma) = \int_0^1 |h(t)| dt,$$

where $h = (h_1, h_2)$. We can then define a distance d on letting, for $x, y \in \mathbb{R}^2$,

$$d(x,y) = \inf\Big\{L(\gamma) : \gamma \in \mathrm{Lip}([0,1];\mathbb{R}^2) \text{ admissible}, \gamma(0) = x, \gamma(1) = y)\Big\}.$$

Then $(\mathbb{R}^2, d)$ is a metric space known as Grushin plane and the mapping ϱ is an isometry. In fact, if γ is an admissible curve joining x to y, then $\varrho\circ\gamma$ is an admissible curve joining ϱx to ϱy and moreover $L(\gamma) = L(\varrho\circ\gamma)$.

Let $H^- = \{x \in \mathbb{R}^2 : x_2 < 0\}$, $H^+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$, and $H = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Then $\mathcal{R} = \{H^-, H, H^+, \varrho\}$ is a reflection system of $(\mathbb{R}^2, d)$. Condition (2.2) holds by the remark in Example 2.2.

Now let $T = \{\tau_t\}_{t\in\mathbb{R}}$ be the 1-parameter group of vertical translations $\tau_t x = (x_1, x_2 + t)$, $x \in \mathbb{R}^2$. We may identify $\mathbb{R}^2/T = H$ and the orbits T_x, $x \in H$, are vertical lines. We claim that $(\mathcal{R}, T)$ is a Steiner system of $(\mathbb{R}^2, d)$. We check (3.13). If $d(x,z) \le d(x,w)$ for some $z, w \in T_y$ and $x, y \in H$, then we have $|z_2| \le |w_2|$. This in turn implies that $d(y,z) \le d(y,w)$ The proof of these facts is an easy exercise.

The standard reflection with respect to the x_2-axis also defines a reflection system of $(\mathbb{R}^2, d)$. In this case, however, there is no 1-parameter group of translations compatible with the reflection system, i.e., yielding a Steiner system. For this reason, Pólya-Szegö inequalities for the x_1-rearrangement of functions and sets are more difficult. See [MM] and [M1] for some results in this direction.

5.5. **Sub-Riemannian Heisenberg group.** The following examples are of particular interest. In spite of the fact that condition (2.2) is violated, a substantial part of our rearrangement theory can be carried out in these cases, for functions and sets enjoying a suitable symmetry. This point of view is developed in Section 6.

Let $\mathbf{H}^n = \mathbb{C}^n \times \mathbb{R}$ be endowed with the group law

$$(z,t) * (\zeta,\tau) = \big(z+\zeta, t+\tau+2\mathrm{Im}(z\cdot\bar\zeta)\big),$$

where we let $z\cdot\bar\zeta = z_1\bar\zeta_2+...+z_n\bar\zeta_n$. $\mathbf{H}^n$ with this group law is known as the Heisenberg group. Let $z = x+iy$, with $x, y \in \mathbb{R}^n$. The vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial x_j} - 2x_j\frac{\partial}{\partial t}, \quad j = 1,...,n,$$

span a $2n$-dimensional left invariant distribution $\mathcal{H}$, called *horizontal distribution*. A Lipschitz curve $\gamma : [0,1] \to \mathbf{H}^n$ is *horizontal* if $\dot\gamma(s) \in \mathcal{H}(\gamma(s))$ for a.e. $s \in [0,1]$. Fix on $\mathcal{H}$ the left invariant metric that makes $X_1,...,X_n,Y_1,...,Y_n$ orthonormal, and let $|\dot\gamma|$ denote the length of $\dot\gamma$ in this metric. The length of γ is then by definition

$$L(\gamma) = \int_0^1 |\dot\gamma(s)|ds.$$

The distance between the points $(z,t),(\zeta,\tau) \in \mathbf{H}^n$ is

$$d((z,t),(\zeta,\tau)) = \inf\Big\{L(\gamma) : \gamma \in \mathrm{Lip}([0,1];\mathbf{H}^n) \text{ horizontal}, \gamma(0) = (z,t), \gamma(1) = (\zeta,\tau)\Big\}.$$

Then d is a metric on $\mathbf{H}^n$, called Carnot-Carathéodory metric.

Let us introduce two different types of what we may call "reflection system with symmetry": the horizontal reflection system and the vertical reflection system with symmetry.

We start with the horizontal reflection system. Let H, H^-, H^+ be the following subsets of $\mathbf{H}^n$:

$$H = \{(z,t) \in \mathbf{H}^n : t = 0\} \quad \text{and} \quad H^\pm = \{(z,t) \in \mathbf{H}^n : \pm t > 0\}, \tag{5.7}$$

and let $\varrho : \mathbf{H}^n \to \mathbf{H}^n$ denote the mapping

$$\varrho(z,t) = (\bar z, -t), \tag{5.8}$$

where $\bar z = \overline{x+iy} = x-iy$, with $x,y \in \mathbb{R}^n$. The mapping ϱ is an involutive isometry of $(\mathbf{H}^n, d)$ that maps H^+ to H^-. This follows from the following observation: a curve γ is horizontal if and only if $\varrho\circ\gamma$ is horizontal, and moreover $L(\gamma) = L(\varrho\circ\gamma)$. The isometry ϱ, however, does not satisfy condition (2.2).

A first attempt to overcome this problem is to consider the mapping $\varrho\circ\sigma$, where $\sigma : \mathbf{H}^n \to \mathbf{H}^n$ is the "symmetry" defined by

$$\sigma(z,t) = (\bar z, t), \quad (z,t) \in \mathbf{H}^n. \tag{5.9}$$

The mapping σ is not an isometry of $\mathbf{H}^n$ and, moreover, $\varrho\circ\sigma$ does not satisfy condition (2.2), either. In fact, we have $d((z,t),(\zeta,\tau)) \le d((z,t),(\zeta,-\tau))$ for all

$(z,t),(\zeta,\tau)\in H^+$ such that $z=\alpha\zeta$ for some $\alpha\in\mathbb{R}$. If z and ζ are not collinear, however, the inequality needs not hold.

In particular, $\mathcal{R}=\{H^-,H,H^+,\varrho\circ\sigma\}$ is not a reflection system of $\mathbf{H}^n$ with the Carnot-Carathéodory metric. However, it is a reflection system of $\mathbf{H}^n$ with the Euclidean metric. This makes possible a rearrangement argument for sets and functions that are σ-invariant (see Section 6). We call $\mathcal{R}=\{H^-,H,H^+,\varrho,\sigma\}$ a *horizontal reflection system of* $\mathbf{H}^n$ *with symmetry* σ.

Now we pass to vertical reflection systems with symmetry. In this case, we let

$$H=\{(z,t)\in\mathbf{H}^n:\mathrm{Im}(z_1)=0\},\ \ H^\pm=\{(z,t)\in\mathbf{H}^n:\pm\mathrm{Im}(z_1)>0\}.$$

The isometry ϱ is the one in (5.8). The symmetry $\sigma:\mathbf{H}^n\to\mathbf{H}^n$ is in this case

$$\sigma(z,t)=(z_1,\bar z_2,...,\bar z_n,-t),\quad (z,t)\in\mathbf{H}^n. \tag{5.10}$$

The same considerations as above apply to this situation. We call $\mathcal{R}=\{H^-,H,H^+,\varrho,\sigma\}$ a *vertical reflection system of* $\mathbf{H}^n$ *with symmetry* σ.

6. Rearrangements in the Heisenberg group

In this section, we consider the Heisenberg group $\mathbf{H}^n$ with the Carnot-Carathéodory metric introduced in Example 5.5. The notions of horizontal and vertical reflection system with symmetry σ are introduced in the same Example. The results proved in this section hold in the following more general framework: the horizontal or vertical reflection system with symmetry is conjugated by some isometry of $\mathbf{H}^n$. Precise statements can be easily deduced from the basic results.

The horizontal Sobolev space $W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$, $1\le p<\infty$, is the set of all functions $f\in L^p(\mathbf{H}^n)$ such that the distributional derivatives $X_1f,...,X_nf,Y_1f,...,Y_nf$ are in $L^p(\mathbf{H}^n)$. Moreover, we let

$$\int_{\mathbf{H}^n}|\nabla_{\mathbf{H}}f(z,t)|^p dzdt=\int_{\mathbf{H}^n}\sum_{j=1}^n\big\{(X_jf(z,t))^2+(Y_jf(z,t)^2\big\}^{p/2}dzdt.$$

For any locally integrable function $f:\mathbf{H}^n\to\mathbb{R}$ let

$$|\nabla_{\mathbf{H}}f|(\mathbf{H}^n)=\sup\Big\{\int_E f\sum_{j=1}^n\{X_j\phi_j+Y_j\psi_j\}dzdt\,:\,\phi_j,\psi_j\in C^1_c(\mathbf{H}^n),\sum_{j=1}^n\phi_j^2+\psi_j^2\le1\Big\}.$$

The space $BV_{\mathbf{H}}(\mathbf{H}^n)=\big\{f\in L^1(\mathbf{H}^n):|\nabla_{\mathbf{H}}f|(\mathbf{H}^n)<\infty\big\}$ is the space of functions with finite horizontal variation.

When $f=\chi_E$ is the characteristic function of a measurable set $E\subset\mathbf{H}^n$, we let $|\partial_{\mathbf{H}}E|(\mathbf{H}^n)=|\nabla_{\mathbf{H}}\chi_E|(\mathbf{H}^n)$. If $|\partial_{\mathbf{H}}E|(\mathbf{H}^n)<\infty$ we say that E has finite horizontal perimeter in $\mathbf{H}^n$.

The characterizations (1.6) and (1.8) of Sobolev and BV norms with infinitesimal integral difference quotients are proved in [P]. For $\mathcal{R}=\{H^-,H,H^+,\varrho,\sigma\}$ vertical or horizontal reflection system with symmetry σ, the two-points rearrangement of a function $f:\mathbf{H}^n\to\mathbb{R}$ is defined in (2.3). The function f is σ-symmetric if $f=f\circ\sigma$.

For σ-symmetric functions, the two-points rearrangement reads as follows. When $\mathcal{R}$ is the horizontal reflection system we have

$$f_{\mathcal{R}}(z,t) = \begin{cases} \max\{f(z,t), f(z,-t)\} & \text{if } t \geq 0, \\ \min\{f(z,t), f(z,-t)\} & \text{if } t \leq 0. \end{cases}$$

When $\mathcal{R}$ is the vertical reflection system we have

$$f_{\mathcal{R}}(z,t) = \begin{cases} \max\{f(z,t), f(\bar{z}_1, z_2, ..., z_n, t)\} & \text{if } \mathrm{Im}(z_1) \geq 0, \\ \min\{f(z,t), f(\bar{z}_1, z_2, ..., z_n, t)\} & \text{if } \mathrm{Im}(z_1) \leq 0. \end{cases}$$

6.1. **Two-points rearrangement.**

Theorem 6.1. Let $\mathcal{R}$ be either a horizontal or a vertical reflection system of $\mathbf{H}^n$ with symmetry σ and let $1 \leq p < \infty$. For any σ-symmetric function $f \in C^1_c(\mathbf{H}^n)$ we have $f_{\mathcal{R}} \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ and moreover

$$\int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f_{\mathcal{R}}(z,t)|^p dzdt \leq \int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f(z,t)|^p dzdt. \tag{6.1}$$

Proof. With abuse of notation, we denote points of $\mathbf{H}^n$ by x, y. For any $0 < r < 1$ let

$$Q_{r,p}(f) = \int_{\mathbf{H}^n \times \mathbf{H}^n} |f(x) - f(y)|^p \chi_r(x,y) dxdy,$$

where

$$\chi_r(x,y) = \begin{cases} \dfrac{1}{\mathcal{L}^{2n+1}(B_r(x))} & \text{if } d(x,y) < r, \\ 0 & \text{otherwise.} \end{cases}$$

Here, d stands for the Carnot-Carathéodory metric and $B_r(x)$ denote Carnot-Carathéodory balls. Notice that $\mathcal{L}^{2n+1}(B_r(x)) = r^{2n+2}\mathcal{L}^{2n+1}(B_1(0))$ is independent of x.

Let L denote the Lipschitz constant of f with respect to the Euclidean metric:

$$L = \mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|}. \tag{6.2}$$

Let

$$K = \big\{(z,t) \in \mathbf{H}^n : |\mathrm{Re}(z_i)| \leq R,\, |\mathrm{Im}(z_i)| \leq R,\, |t| \leq R,\, i = 1, ..., n\big\} \tag{6.3}$$

be a compact cube centered at 0 and with axes parallel to the coordinate axes and such that

$$\mathrm{dist}_{\mathbf{H}}(\mathbf{H}^n \setminus K, \mathrm{supp}(f)) \geq 1. \tag{6.4}$$

Here, $\mathrm{dist}_{\mathbf{H}}$ stands for Carnot-Carathéodory distance. Condition (6.4) holds if $R > 0$ is large enough. By a well known estimate, there exists a constant $C_K > 0$ such that

$$|x - y| \leq C_K d(x,y) \text{ for all } x, y \in K. \tag{6.5}$$

Let H be the reflection hyperplane of $\mathcal{R} = \{H^-, H, H^+, \varrho, \sigma\}$ and let $(H \cap K)_r$ denote the $C_K r$-neighborhood in the Euclidean metric of $H \cap K$ in $\mathbf{H}^n$, and namely:

$$(H \cap K)_r = \big\{x \in \mathbf{H}^n : \mathrm{dist}(x, H \cap K) < rC_K\big\}.$$

Here and hereafter, dist stands for the Euclidean distance. We claim that for any $0 < r < 1$ we have

$$Q_{r,p}(f_{\mathcal{R}}) \le Q_{r,p}(f) + 2L^p C_K^p r^p \mathcal{L}^{2n+1}(H \cap K)_r. \tag{6.6}$$

Because

$$\lim_{r \downarrow 0} \mathcal{L}^{2n+1}(H \cap K)_r = 0,$$

the claim (6.1) follows from (6.6), by formula (1.6) (which also holds for $p = 1$ for smooth enough functions).

We prove (6.6). As in the proof of Theorem 2.8, we have

$$\begin{aligned} Q_{r,p}(f) = & \iint_{H^+ \times H^+} \big\{ |f(x) - f(y)|^p + |f(\varrho x) - f(\varrho y)|^p \big\} \chi_r(x, y)\, dxdy \\ & + \iint_{H^+ \times H^+} \big\{ |f(x) - f(\varrho y)|^p + |f(\varrho x) - f(y)|^p \big\} \chi_r(x, \varrho y)\, dxdy. \end{aligned}$$

In the latter integral we perform the change of variable $y = \sigma z$. Using the symmetries $f(\varrho\sigma y) = f(\varrho y)$ and $f(\sigma y) = f(y)$ we obtain

$$Q_{r,p}(f) = \iint_{H^+ \times H^+} Q(f; x, y)\, dxdy,$$

where we let

$$\begin{aligned} Q(f; x, y) = & \big\{ |f(x) - f(y)|^p + |f(\varrho x) - f(\varrho y)|^p \big\} \chi_r(x, y) \\ & + \big\{ |f(x) - f(\varrho y)|^p + |f(\varrho x) - f(y)|^p \big\} \chi_r(x, \varrho\sigma y). \end{aligned}$$

Let $x, y \in H^+$. We have the following four cases:

1) $d(x, y) \ge r$ and $d(x, \varrho\sigma y) \ge r$;
2) $d(x, y) < r$ and $d(x, \varrho\sigma y) < r$;
3) $d(x, y) < r \le d(x, \varrho\sigma y)$;
4) $d(x, \varrho\sigma y) < r \le d(x, y)$.

In the proof of Theorem 2.8, we had no case 4). In the cases 1), 2), and 3) we have

$$Q(f_{\mathcal{R}}; x, y) \le Q(f; x, y). \tag{6.7}$$

The proof is the same as in Theorem 2.8. We study the case 4). Let

$$E_r = \big\{ (x, y) \in H^+ \times H^+ : d(x, \varrho\sigma y) < r \le d(x, y) \big\}.$$

If $(x, y) \in E_r$, we have

$$Q(f; x, y) = \big\{ |f(x) - f(\varrho y)|^p + |f(\varrho x) - f(y)|^p \big\} \chi_r(x, \varrho\sigma y).$$

The function $f_{\mathcal{R}}$ is σ-symmetric. Moreover, $f_{\mathcal{R}}$ is the Euclidean two-points rearrangement of f with respect to the hyperplane H. By (2.4), we have $\mathrm{Lip}(f_{\mathcal{R}}) \le \mathrm{Lip}(f) = L$. In particular, we have $f_{\mathcal{R}} \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$, trivially. By (6.5), we have

$$|f_{\mathcal{R}}(x) - f_{\mathcal{R}}(\varrho y)| = |f_{\mathcal{R}}(x) - f_{\mathcal{R}}(\varrho\sigma y)| \le L|x - \varrho\sigma y| \le LC_K d(x, \varrho\sigma y) < LC_K r,$$

and analogously $|f_{\mathcal{R}}(\varrho x) - f_{\mathcal{R}}(y)| < LC_K r$.

By (6.4) we may assume that $x, y \in K$. In fact, if $x \in \mathbf{H}^n \setminus K$ (or $y \in \mathbf{H}^n \setminus K$) and $d(x, \varrho\sigma y) < r < 1$, we have

$$f(x) = f(\varrho x) = f(\varrho y) = f(y) = 0,$$

and thus $Q(f_{\mathcal{R}}; x, y) = 0$. On the other hand, if $d(x, \sigma\varrho y) < r$ and $x, y \in H^+ \cap K$, we have

$$\mathrm{dist}(\varrho x, H \cap K) = \mathrm{dist}(x, H \cap K) < |x - \sigma\varrho y| \le C_K d(x, \sigma\varrho y) \le C_K r.$$

Then we have

$$\begin{aligned}
\int_{E_r} Q(f_{\mathcal{R}}; x, y) dx dy &\le \int_{(H\cap K)_r} \int_{H^+} \big\{ |f_{\mathcal{R}}(x) - f_{\mathcal{R}}(\varrho y)|^p + |f_{\mathcal{R}}(\varrho x) - f_{\mathcal{R}}(y)|^p \big\} \chi_r(x, \varrho\sigma y) dy\, dx \\
&\le 2L^p C_K^p r^p \int_{H^+ \cap (H\cap K)_r} \int_{H^+} \chi_r(x, \varrho\sigma y) dy\, dx \\
&\le 2L^p C_K^p r^p \mathcal{L}^{2n+1}(H \cap K)_r,
\end{aligned}$$

and finally

$$\begin{aligned}
\int_{H^+ \times H^+} Q(f_{\mathcal{R}}; x, y) dx dy &= \int_{H^+ \times H^+ \setminus E_r} Q(f_{\mathcal{R}}; x, y) dx dy + \int_{E_r} Q(f_{\mathcal{R}}; x, y) dx dy \\
&\le \int_{H^+ \times H^+} Q(f; x, y) dx dy + 2L^p C_K^p r^p \mathcal{L}^{2n+1}(H \cap K)_r.
\end{aligned}$$

This is (6.6). □

We extend Theorem 6.1 to the case of Sobolev functions in $W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ and to sets with finite horizontal perimeter. We need the following density theorems which are proved in [FSSC], in a more general framework. For any $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$, $1 \le p < \infty$, there exists a sequence $f_h \in C^1(\mathbf{H}^n) \cap W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$, $h \in \mathbb{N}$, such that

$$\lim_{h\to\infty} \|f_h - f\|_p = \lim_{h\to\infty} \|\nabla_{\mathbf{H}} f_h - \nabla_{\mathbf{H}} f\|_p = 0. \tag{6.8}$$

This is Theorem 1.2.3 in [FSSC]. The functions f_h are obtained as convolutions of the form

$$f_\varepsilon(x) = \int_{\mathbf{H}^n} f(y) J_\varepsilon(x - y) dy, \quad \varepsilon > 0,\ x \in \mathbf{H}^n,$$

where $J_\varepsilon(x) = \varepsilon^{2n+1} J(|x|/\varepsilon)$ is a standard approximation of the identity and $|x|$ denotes the Euclidean norm of $x \in \mathbf{H}^n = \mathbb{R}^{2n+1}$. If f is σ-symmetric with σ as in (5.9) or (5.10), then also f_ε is σ-symmetric. Multiplying each f_h by a suitable cut-off function, we may then assume that the functions f_h in (6.8) are compactly supported and σ-symmetric, if f is σ-symmetric.

Analogously, for any $f \in BV_{\mathbf{H}}(\mathbf{H}^n)$ there exists a sequence $f_h \in C^1(\mathbf{H}^n) \cap BV_{\mathbf{H}}(\mathbf{H}^n)$, $h \in \mathbb{N}$, such that

$$\lim_{h\to\infty} \|f_h - f\|_1 = 0 \quad \text{and} \quad \lim_{h\to\infty} |\nabla_{\mathbf{H}} f_h|(\mathbf{H}^n) = |\nabla_{\mathbf{H}} f|(\mathbf{H}^n). \tag{6.9}$$

This is Theorem 2.2.2 in [FSSC]. The functions f_h may be assumed to be compactly supported and σ-symmetric, if f is σ-symmetric.

Corollary 6.2. Let $\mathcal{R}$ be either a horizontal or a vertical reflection system of $\mathbf{H}^n$ with symmetry σ.

i) Let $1 < p < \infty$. For any σ-symmetric function $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ we have $f_{\mathcal{R}} \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ and moreover

$$\int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f_{\mathcal{R}}(z,t)|^p dzdt \le \int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f(z,t)|^p dzdt. \tag{6.10}$$

ii) For any σ-symmetric function $f \in BV_{\mathbf{H}}(\mathbf{H}^n)$ we have $f_{\mathcal{R}} \in BV_{\mathbf{H}}(\mathbf{H}^n)$ and moreover

$$|\nabla_{\mathbf{H}} f_{\mathcal{R}}|(\mathbf{H}^n) \le |\nabla_{\mathbf{H}} f|(\mathbf{H}^n), \tag{6.11}$$

where $f_{\mathcal{R}}$ is the two-points rearrangement of f.

Proof. This follows from Theorem 6.1, on using the approximation in (6.8) and (6.9). The proof is elementary and we skip the details. □

6.2. **Vertical Steiner rearrangement.** Let $\mathcal{R} = \{H^-, H, H^+, \varrho, \sigma)$ be the horizontal reflection system with symmetry introduced in Example 5.5, in particular we have $H = \{(z,t) \in \mathbf{H}^n : t = 0\}$. We prove some theorems on the Steiner rearrangement of sets and functions in the t-coordinate. We call this procedure *vertical* Steiner rearrangement. So the vertical rearrangement corresponds to the horizontal reflection system.

The mappings $\tau_s : \mathbf{H}^n \to \mathbf{H}^n$, $s \in \mathbb{R}$, $\tau_s(z,t) = (z, t+s)$ form a 1-parameter group $T = \{\tau_s\}_{s\in\mathbb{R}}$ of isometries of $\mathbf{H}^n$. We may identify the reflection hyperplane H with $\mathbf{H}^n/T$. The orbits are vertical lines $T_z = \{(z,t) \in \mathbf{H}^n : t \in \mathbb{R}\}$, $z \in \mathbb{C}^n$, and the projection $\pi : \mathbf{H}^n \to \mathbf{H}^n/T = H$, $\pi(z,t) = (z,0)$, is continuous.

The natural disintegration of the Lebesgue measure $\mathcal{L}^{2n+1}$ along the orbits T_z is given by Fubini-Tonelli theorem. For any measurable set $E \subset \mathbf{H}^n$ we have

$$\mathcal{L}^{2n+1}(E) = \int_H \mathcal{L}^1(E_z) d\mathcal{L}^{2n}(z),$$

where $E_z = \{t \in \mathbb{R} : (z,t) \in E\}$, $z \in \mathbb{C}^n$. For any measurable set $E \subset \mathbf{H}^n$ with finite measure, we call the set

$$E^\star = \big\{(z,t) \in \mathbb{C}^n \times \mathbb{R} : 2|t| < \mathcal{L}^1(E_z)\big\},$$

the *vertical Steiner rearrangement of* E. Analogously, for any measurable, rearrangeable function $f : \mathbf{H}^n \to [0,\infty)$, we call the function

$$f^\star(z,t) = \int_0^\infty \chi_{\{f>s\}^\star}(z,t)\, ds, \quad (z,t) \in \mathbf{H}^n,$$

the *vertical Steiner rearrangement of* f.

Theorem 6.3. Let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$, $p > 1$, be a nonnegative, σ-symmetric function and let $f^\star$ be the vertical Steiner rearrangement of f. Then $f^\star \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ and

$$\int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f^\star(z,t)|^p dzdt \leq \int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f(z,t)|^p dzdt. \tag{6.12}$$

Proof. We follow closely the proof of Theorem 3.6. For the reader's convenience, we repeat some of the details.

Step 1. Let us first assume that $f \in C^1_c(\mathbf{H}^n)$.

As in (6.3), let $K \subset \mathbf{H}^n$ be a compact cube centered at the origin with axes parallel to the coordinate axes and containing the support of f. Let $\mathcal{A}(f)$ be the family of all nonnegative, σ-symmetric functions $g \in L^p(\mathbf{H}^n)$ such that:

$$\text{i) } \mathcal{L}^1\{g > s\}_z = \mathcal{L}^1\{f > s\}_z \text{ for } \mathcal{L}^{2n}\text{-a.e. } z \in \mathbb{C}^n,\ s > 0; \tag{6.13}$$

$$\text{ii) } g(z,t) = 0 \text{ for all } (z,t) \in \mathbf{H}^n \setminus K; \tag{6.14}$$

$$\text{iii) } \|\nabla_{\mathbf{H}} g\|_p \leq \|\nabla_{\mathbf{H}} f\|_p. \tag{6.15}$$

The set $\mathcal{A}(f)$ is uniformly bounded in $W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ and boundedly supported. By the compactness Theorem in [GN] (see Section 4), $\mathcal{A}(f)$ is compact in $L^p(K)$. The closure of $\mathcal{A}(f)$ can be shown as in the proof of Theorem 3.6 and we skip the details.

The functional $J : \mathcal{A}(f) \to [0,\infty)$

$$J(g) = \int_{\mathbf{H}^n} |g(z,t) - f^\star(z,t)|^p dzdt, \tag{6.16}$$

achieves the minimum at some $\bar f \in \mathcal{A}(f)$. There are two cases: 1) $J(\bar f) = 0$; 2) $J(\bar f) > 0$. In the first case, we are finished. On the other hand, the case $J(\bar f) > 0$ may not occur. The proof of this fact is the same as in Theorem 3.6. We sketch the argument below.

If $J(\bar f) > 0$ there exists a $\delta > 0$ such that, letting $A = \{\bar f > \delta\}$ and $B = \{f^\star > \delta\}$, we have $\mathcal{L}^{2n+1}(A \Delta B) > 0$. The same argument after (3.29) proves then the existence of some $z \in \mathbb{C}^n$, $t_-, t_+ \in \mathbb{R}$, and $\eta > 0$ such that $(z,t_-) \in A \setminus B$, $(z,t_+) \in B \setminus A$ and

$$\begin{aligned} \mathcal{L}^{2n+1}(U_\eta(z,t_-) \cap A \setminus B) &> \frac{1}{2}\mathcal{L}^{2n+1}(U_\eta(z,t_-)), \\ \mathcal{L}^{2n+1}(U_\eta(z,t_+) \cap B \setminus A) &> \frac{1}{2}\mathcal{L}^{2n+1}(U_\eta(z,t_+)). \end{aligned} \tag{6.17}$$

Here, $U_\eta(z,t)$ is the Euclidean ball centered at $(z,t) \in \mathbf{H}^n = \mathbb{C}^n \times \mathbb{R}$ with radius η.

Let $t = (t_- + t_+)/2$, $\bar H = \tau_t H$, $\bar H^- = \tau_t H^-$, $\bar H^+ = \tau_t H^+$, and let $\bar\varrho$ denote the standard (Euclidean) reflection w.r.t. $\bar H$. Let $\bar f_{\bar{\mathcal{R}}}$ be the function defined as follows

$$\bar f_{\bar{\mathcal{R}}}(z,t) = \begin{cases} \max\{\bar f(z,t), \bar f(\bar\varrho(z,t))\} & \text{if } (z,t) \in \bar H^+, \\ \min\{\bar f(z,t), \bar f(\bar\varrho(z,t))\} & \text{if } (z,t) \in \bar H^-, \end{cases}$$

and $\bar f_{\bar{\mathcal{R}}} = \bar f$ on $\bar H$. By a straightforward generalization of Corollary 6.2, this function satisfies $\|\nabla_{\mathbf{H}} \bar f_{\bar{\mathcal{R}}}\|_p \leq \|\nabla_{\mathbf{H}} \bar f\|_p \leq \|\nabla_{\mathbf{H}} f\|_p$. Moreover, the function $g = \bar f_{\bar{\mathcal{R}}}$ is σ-symmetric and satisfies (6.13)-(6.14). Thus $\bar f_{\bar{\mathcal{R}}}$ is an element of $\mathcal{A}(f)$. Finally, assumption (2.23) of Theorem 2.11 holds by (6.17) and $f^\star_{\bar{\mathcal{R}}} = f^\star$. The proof of the last statement is as in Theorem 3.6. Therefore we have

$$J(\bar{f}_{\bar{\mathcal{R}}}) = \int_{\mathbf{H}^n} |\bar{f}_{\bar{\mathcal{R}}} - f^\star|^p dzdt = \int_{\mathbf{H}^n} |\bar{f}_{\bar{\mathcal{R}}} - f^\star_{\bar{\mathcal{R}}}|^p dzdt < \int_{\mathbf{H}^n} |\bar{f} - f^\star|^p dzdt = J(\bar{f}).$$

This contradicts the minimality of $\bar{f}$.

Step 2. We prove the theorem for any nonnegative, σ-symmetric function $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$.

The proof is by approximation. Let $(f_h)_{h\in\mathbb{N}}$ be a sequence of nonnegative, σ-symmetric functions $f_h \in C^1_c(\mathbf{H}^n)$ such that (6.8) holds. We can also assume that $f_h(z,t) \to f(z,t)$ as $h \to \infty$ for $\mathcal{L}^{2n+1}$-a.e. $(z,t) \in \mathbf{H}^n$. It follows that for $\mathcal{L}^{2n}$-a.e. $z \in \mathbb{C}^n$ and for all but a countable set of $s > 0$ there holds

$$\lim_{h\to\infty} \mathcal{L}^1\big(\{f_h > s\}_z \Delta \{f > s\}_z\big) = 0. \tag{6.18}$$

The proof of this claim is analogous to one in Theorem 3.6. We skip the details.

By the Step 1, we have

$$\int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f^\star_h(z,t)|^p dzdt \le \int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f_h(z,t)|^p dzdt.$$

It follows that, up to a subsequence, the sequence $(f^\star_h)_{h\in\mathbb{N}}$ converges weakly in $W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ to a function g such that

$$\int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} g(z,t)|^p dzdt \le \liminf_{h\to\infty} \int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f^\star_h(z,t)|^p dzdt.$$

We may also assume that $f^\star_h \to g$ $\mathcal{L}^{2n+1}$-a.e. in $\mathbf{H}^n$. We claim that $g = f^\star$.

We preliminarly prove that for $\mathcal{L}^{2n}$-a.e. $z \in \mathbb{C}^n$ and for all but a countable sets of $s > 0$ there holds

$$\lim_{h\to\infty} \mathcal{L}^1\big(\{f^\star_h > s\}_z \Delta \{g > s\}_z\big) = 0. \tag{6.19}$$

First notice that for $\mathcal{L}^{2n}$-a.e. $z \in \mathbb{C}^n$ and for all but a countable set of $s > 0$ we have $\mathcal{L}^1\{g = s\}_z = 0$. Moreover, the functions $t \mapsto f^\star_h(z,t)$ and $t \mapsto g(z,t)$ are even and nonincreasing for $t \ge 0$. The sets $I_h(z,s) = \{f^\star_h > s\}_z$ and $I(z,s) = \{g(z,t) > s\}_z$ are therefore essentially symmetric intervals, $I_h(z,s) = (-a_h, a_h)$ and $I(z,s) = (-a,a)$ for some $a, a_h \ge 0$. Using the pointwise convergence and $\mathcal{L}^1\{g = s\}_z = 0$, it is elementary to show that $a_h \to a$ as $h \to \infty$. This proves (6.19).

From (6.18) and (6.19), we deduce that

$$\mathcal{L}^1\{g > s\}_z = \lim_{h\to\infty} \mathcal{L}^1\{f^\star_h > s\}_z = \lim_{h\to\infty} \mathcal{L}^1\{f_h > s\}_z = \mathcal{L}^1\{f > s\}_z.$$

This implies that $g = f^\star$, and the proof is finished. □

In the sequel, let $|\partial_{\mathbf{H}} E| = |\partial_{\mathbf{H}} E|(\mathbf{H}^n)$ denote the horizontal perimeter of E in $\mathbf{H}^n$.

Theorem 6.4. Let $E \subset \mathbf{H}^n$ be a σ-symmetric set of finite measure and finite $\mathbf{H}$-perimeter and let $E^\star$ be the vertical Steiner rearrangement of E. Then $E^\star$ is of finite $\mathbf{H}$-perimeter and

$$|\partial_{\mathbf{H}} E^\star| \le |\partial_{\mathbf{H}} E|. \tag{6.20}$$

Proof. The proof is a repetition of the one of Theorem 6.3. We sketch a few details.

Step 1. We claim that for any $f \in C^1_c(\mathbf{H}^n)$ with $\|f\|_\infty \le 1$ we have $|\nabla_{\mathbf{H}} f^\star|(\mathbf{H}^n) \le |\nabla_{\mathbf{H}} f|(\mathbf{H}^n)$.

As in (6.3), let $K \subset \mathbf{H}^n$ be a cube centered at the origin and containing the support of f. Let $\mathcal{A}(f)$ be the set of all nonnegative, σ-symmetric functions $g \in L^1(\mathbf{H}^n)$ with $\|g\|_\infty \le 1$ and such that

i) $\mathcal{L}^1\{g > s\}_z = \mathcal{L}^1\{f > s\}_z$ for $\mathcal{L}^{2n}$-a.e. $z \in \mathbb{C}^n$, $s > 0$;

ii) $g(z,t) = 0$ for all $(z,t) \in \mathbf{H}^n \setminus K$;

iii) $|\nabla_{\mathbf{H}} g|(\mathbf{H}^n) \le |\nabla_{\mathbf{H}} f|(\mathbf{H}^n)$.

The set $\mathcal{A}(f)$ is compact in $L^2(K)$. The functional $J : \mathcal{A}(f) \to [0,\infty)$

$$J(g) = \int_{\mathbf{H}^n} |g(z,t) - f^\star(z,t)|^2 dzdt,$$

achieves the minimum at some $\bar f \in \mathcal{A}(f)$. The case $J(\bar f) > 0$ may not occur. The proof is the same as in Theorem 6.3. Here, we use Corollary 6.2, part ii).

Step 2. The proof of the theorem is a line-by-line repetition of the *Step 2* of Theorem 6.3. Here, we use the approximation (6.9) of the characteristic function of E with a sequence of smooth functions for which we may use *Step 1.* $\square$

Example 6.5. Theorem 6.4 does not hold if we drop the σ-symmetry of the set. We construct a set $E \subset \mathbf{H}^1$ such that its vertical Steiner rearrangement $E^\star$ satisfies

$$|\partial_{\mathbf{H}} E^\star| > |\partial_{\mathbf{H}} E|.$$

The set E is the left translation of a cylinder.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and define the horizontal area of the graph of a Lipschitz function $f : D \to \mathbb{R}$ as

$$A_{\mathbf{H}}(f) = \int_D \sqrt{\Big(\frac{\partial f}{\partial x} - 2y\Big)^2 + \Big(\frac{\partial f}{\partial y} + 2x\Big)^2} dxdy. \tag{6.21}$$

This area is the horizontal perimeter of the epigraph of f inside the cylinder $D \times \mathbb{R}$. Formula (6.21) is a special case of the formula

$$|\partial_{\mathbf{H}} E| = \int_{\partial E} \sqrt{(X\cdot\nu)^2 + (Y\cdot\nu)^2} d\mathcal{H}^2, \tag{6.22}$$

for a bounded open set $E \subset \mathbf{H}^1 = \mathbb{R}^3$ with Lipschitz boundary. Here, ν is the unit normal to the boundary, $\cdot$ is the standard scalar product of $\mathbb{R}^3$, and $\mathcal{H}^2$ stands for the standard 2-dimensional Hausdorff measure in $\mathbb{R}^3$.

Fix a real number $c > 0$ and, for $a, b \in \mathbb{R}$, let $f_{a,b}$ be the affine function $f_{a,b}(x,y) = ax + by + c$. The horizontal area of the graph of this function depends only on the parameter $s = \sqrt{a^2+b^2}$, and namely, by (6.21),

$$A(s) = A_{\mathbf{H}}(f) = \int_0^1 \int_0^{2\pi} \sqrt{s^2 + 4rs\sin\vartheta + 4r^2} d\vartheta\, rdr, \quad s \ge 0.$$

The derivative in s of the function A is

$$A'(s) = \int_0^1 \int_0^{2\pi} \frac{s + 2r\sin\vartheta}{\sqrt{s^2 + 4rs\sin\vartheta + 4r^2}} d\vartheta\, rdr, \quad s \geq 0,$$

and, in particular, we have $A'(0) = 0$. The second derivative is

$$A''(s) = \int_0^1 \int_0^{2\pi} \frac{4r^3\cos^2\vartheta}{(s^2 + 4rs\sin\vartheta + 4r^2)^{3/2}} d\vartheta\, dr > 0, \quad s \geq 0.$$

Then A' is strictly increasing and thus also A is strictly increasing for $s \geq 0$.

Now let $C_{a,b} \subset \mathbf{H}^1$ be the cylinder

$$C_{a,b} = \big\{(x+iy, t) \in \mathbf{H}^1 : x + iy \in D,\, |t - ax - by| < c\big\}.$$

We claim that for all $a, b \in \mathbb{R}$ we have

$$|\partial_{\mathbf{H}} C_{0,0}| \leq |\partial_{\mathbf{H}} C_{a,b}|$$

with equality if and only if $a = b = 0$.

We start from the following identity

$$|\partial_{\mathbf{H}} C_{a,b}| = 2A_{\mathbf{H}}(f_{a,b}) + \mathcal{H}^2(\partial D \times \mathbb{R} \cap \partial C_{a,b}), \tag{6.23}$$

which easily follows from (6.22). By Fubini-Tonelli theorem, $\mathcal{H}^2(\partial D \times \mathbb{R} \cap \partial C_{a,b}) = 4\pi c$ is independent of a, b. The claim follows.

Now let $p = (z_0, 0) \in \mathbf{H}^1$ be a point such that $z_0 \neq 0$ and let

$$E = p * C_{0,0} = \big\{(z,t) \in \mathbf{H}^1 : |z - z_0| < 1,\, |t - 2\mathrm{Im}(z_0\bar{z})| < c\big\}.$$

The vertical Steiner rearrangement of E is the cylinder

$$E^\star = \big\{(z,t) \in \mathbf{H}^1 : |z - z_0| < 1,\, |t| < c\big\} = p * C_{a,b}$$

for suitable $a, b \in \mathbb{R}$ that satisfy $a^2 + b^2 \neq 0$, because $z_0 \neq 0$. By the left invariance of the horizontal perimeter and by the discussion of the equality case in (6.23), we have

$$|\partial_{\mathbf{H}} E^\star| = |\partial_{\mathbf{H}} C_{a,b}| > |\partial_{\mathbf{H}} C_{0,0}| = |\partial_{\mathbf{H}} E|.$$

□

6.3. **Circular rearrangement.** Let $\mathcal{R} = \{H^-, H, H^+, \varrho, \sigma\}$ be the vertical reflection system with symmetry introduced in Example 5.5. In particular, we have $H = \big\{(z,t) \in \mathbf{H}^1 : \mathrm{Im}(z_1) = 0\big\}$, $\varrho(z,t) = (\bar{z}, -t)$, and $\sigma(z,t) = (z_1, \bar{z}_2, ..., \bar{z}_n, -t)$, $(z,t) \in \mathbf{H}^n$.

For any $s \in \mathbb{S}^1 = \{s \in \mathbb{C} : |s| = 1\}$ let $\tau_s : \mathbf{H}^n \to \mathbf{H}^n$ be the mapping $\tau_s(z,t) = (sz_1, z_2, ..., z_n, t)$. Then $T = \{\tau_s\}_{s \in \mathbb{S}^1}$ is a 1-parameter group of isometries of $\mathbf{H}^n$ with the Carnot-Carathéodory metric. We may identify

$$\mathbf{H}^n/T = \{(z,t) \in \mathbf{H}^n :\ \mathrm{Im}(z_1) = 0, \mathrm{Re}(z_1) \geq 0\} \subset H.$$

The orbits $T_{z,t} = \{(sz_1, z_2, ..., z_n, t) \in \mathbf{H}^n :\ s \in \mathbb{S}^1\}$ are circles and the projection $\pi : \mathbf{H}^n \to \mathbf{H}^n/T$ is $\pi(z,t) = (|z_1|, z_2, ..., z_n, t)$.

The natural disintegration of the Lebesgue measure $\mathcal{L}^{2n+1}$ is given by cylindrical coordinates. For any Borel set $E \subset \mathbf{H}^n$ we have

$$\mathcal{L}^{2n+1}(E) = \int_{\mathbf{H}^n/T} \mathcal{H}^1(E_{z,t}) d\mathcal{L}^{2n}(z,t),$$

where $E_{z,t} = E \cap T_{z,t}$ is the section of E with the orbit of $(z,t) \in \mathbf{H}^n/T$. Here, $\mathcal{H}^1$ is the standard 1-dimensional Hausdorff measure in $\mathbb{R}^{2n+1}$.

For any $(z,t) \in \mathbf{H}^n/T$ we let

$$E^\star_{z,t} = \big\{ (sz_1, z_2, ..., z_n, t) \in \mathbf{H}^n : s \in \mathbb{S}^1, \, \mathrm{Re}(s) > s_0 \big\}$$

where $s_0 \in [-1,1]$ is the unique real number such that $\mathcal{H}^1(E^\star_{z,t}) = \mathcal{H}^1(E_{z,t})$. We call the set

$$E^\star = \bigcup_{(z,t)\in \mathbf{H}^n/T} E^\star_{z,t} \tag{6.24}$$

the *circular rearrangement of* E. This definition is a special case of (3.6). Analogously, for any nonnegative, measurable (rearrangeable) function $f : \mathbf{H}^n \to \mathbb{R}$, we call the function $f^\star$ defined in (3.7) the *circular rearrangement of* f.

Theorem 6.6. i) Let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$, $p > 1$, be a nonnegative, σ-symmetric function and let $f^\star$ be the circular rearrangement of f. Then $f^\star \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^n)$ and

$$\int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f^\star(z,t)|^p dzdt \leq \int_{\mathbf{H}^n} |\nabla_{\mathbf{H}} f(z,t)|^p dzdt. \tag{6.25}$$

ii) Let $E \subset \mathbf{H}^n$ be a σ-symmetric set of finite measure and finite $\mathbf{H}$-perimeter and let $E^\star$ be the circular rearrangement of E. Then $E^\star$ is of finite $\mathbf{H}$-perimeter and

$$|\partial_{\mathbf{H}} E^\star| \leq |\partial_{\mathbf{H}} E|. \tag{6.26}$$

Proof. The proof is a straightforward adaptation of the proof of Theorems 6.3 and 6.4. A repetition of the details is not necessary, here. □

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Roberto Monti: Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Trieste, 63, 35121 Padova, Italy

E-mail address: monti@math.unipd.it