AMBROSIO-TORTORELLI APPROXIMATION OF QUASI-STATIC EVOLUTION OF BRITTLE FRACTURES

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ABSTRACT. We define a notion of quasistatic evolution for the elliptic approximation of the Mumford-Shah functional proposed by Ambrosio and Tortorelli. Then we prove that this regular evolution converges to a quasi static growth of brittle fractures in linearly elastic bodies.

Keywords : variational models, energy minimization, free discontinuity problems, elliptic approximation, crack propagation, quasistatic evolution, brittle fracture.

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CONTENTS

1. INTRODUCTION

In 1998 Francfort and Marigo [15] proposed a model of quasistatic growth of brittle fractures in linearly elastic bodies inspired to the classical Griffith criterion.

Let $\Omega \subseteq \mathbb{R}^3$ be an elastic body, $\partial_D\Omega$ a part of its boundary, and let $g : \partial_D\Omega \to \mathbb{R}^3$ be the spatial displacement of Ω at the points of $\partial_D\Omega$. Following [15], given a preexisting crack $\Gamma_1 \subseteq \overline{\Omega}$, the new crack Γ and the displacement $u : \Omega \setminus \Gamma \to \mathbb{R}^3$ associated to g at the equilibrium minimizes the following total energy

$$\mathcal{E}(v, g, \Gamma) := \int_\Omega \mu|Ev|^2 + \lambda|trEu|^2\, dx + k\mathcal{H}^2(\Gamma), \tag{1.1}$$

among all cracks Γ with $\Gamma_1 \subseteq \Gamma$ and all displacements $v : \Omega \setminus \Gamma \to \mathbb{R}^3$ with $v = g$ on $\partial_D\Omega \setminus \Gamma$. Here Ev denotes the symmetric part of the gradient of v, tr denotes the trace of the matrix, and $\mathcal{H}^2$ denotes the two dimensional Hausdorff measure. The coefficients μ, λ and k depend on the material. Thus the theory of [15] determines the growth of the crack (as the formation of its new components) through a competition between the *bulk energy* given by $\int_\Omega \mu|Eu|^2 + \lambda|trEu|^2\, dx$ and the *surface energy* given by $k\mathcal{H}^2(\Gamma)$. The boundary condition is required only on $\partial_D\Omega \setminus \Gamma$ because the displacement in a fractured region is supposed to be not transmitted. We indicate by $\mathcal{E}(g, \Gamma)$ the minimum value of (1.1) among all $v : \Omega \setminus \Gamma \to \mathbb{R}^3$ with $v = g$ on $\partial_D\Omega \setminus \Gamma$.

Suppose that the boundary displacement g varies with the time $t \in [0, 1]$. The quasistatic evolution $t \to \Gamma(t)$ proposed in [15] requires that:

(1) $\Gamma(t)$ is increasing in time, i.e., $\Gamma(t_1) \subseteq \Gamma(t_2)$ for all $0 \leq t_1 \leq t_2 \leq 1$;

(2) $\mathcal{E}(g(t), \Gamma(t)) \leq \mathcal{E}(g(t), \Gamma)$ for all cracks Γ such that $\cup_{s<t}\Gamma(s) \subseteq \Gamma$;

(3) the total energy $\mathcal{E}(g(t), \Gamma(t))$ is absolutely continuous in time, and its derivative is equal to the power of external forces.

Condition (1) stands for the *irreversibility* of the evolution (fracture can only increase); condition (2) states that each time t is of *static equilibrium*, while condition (3) stands for the *nondissipativity* of the process.

The problem of giving a precise mathematical formulation of the preceding model has been the object of several recent papers. In 2000, Dal Maso and Toader [12] dealt with the case of antiplanar shear in dimension two: the authors consider a cylindric elastic body $\Omega = \Omega' \times \mathbb{R}$ with $\Omega' \subseteq \mathbb{R}^2$ subject to displacements of the form $u(\pi x)e_3$ where e_3 is the unit vector of the x_3-axis, and π is the projection on Ω'. The boundary antiplanar displacement is assigned on $\partial_D\Omega' \times \mathbb{R}$ while the admissible cracks are of the form $K \times \mathbb{R}$ with K compact subset of $\overline{\Omega'}$ with a prescribed number of connected components and with finite $\mathcal{H}^1$-length. A generalization to non-isotropic surface energies is contained in [16].

Recently Francfort and Larsen [14] proposed a mathematical formulation which involves the space SBV of special functions of bounded variation (see Section 2). Their approach permits to treat antiplanar shear in a N-dimensional setting, and allows to consider fractures with a possibly infinite number of connected components. To be precise, they consider displacements of the form $u(x)e_{N+1}$, where $u \in SBV(\Omega)$ and e_{N+1} denotes the unitary vector of the $(N+1)$-axis. The crack at time t is defined as $\Gamma(t) \times \mathbb{R}$ where $\Gamma(t) := \cup_{s<t}\left[S_{u(s)} \cup (\partial_D\Omega \cap \{u(s) \neq g(s)\})\right]$, and the pair $(u(t), \Gamma(t))$ is such that:

(a) for all $z \in SBV(\Omega)$

$$\int_\Omega |\nabla u(t)|^2 + \mathcal{H}^{N-1}(\Gamma(t)) \leq \int_\Omega |\nabla z|^2 + \mathcal{H}^{N-1}(S_z \cup (\partial_D\Omega \cap \{z \neq g(t)\}) \cup \Gamma(t)); \tag{1.2}$$

(b) the total energy $\mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2 + \mathcal{H}^{N-1}(\Gamma(t))$ is absolutely continuous and

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau.$$

Numerical computations concerning this model of evolution (see [7]) are performed using a discretization in time procedure and an approximation of the total energy proposed in 1990 by Ambrosio and Tortorelli (see [5],[6]). Being the new energy elliptic, the difficulties arising in the discretization of the free discontinuity term given by the fracture are avoided. Supposing to have determined the displacement u_i and the fracture K_i at the time t_i, one minimizes the Ambrosio-Tortorelli functional in the domain $\Omega \setminus K_i$ under the boundary conditions $g(t_{i+1})$, and hence reconstruct the couple (u_{i+1}, K_{i+1}). In this way, errors due to the discretization in time and to the approximation of the energy are introduced. In order to study the convergence of the procedure, one is led to formulate a natural notion of quasistatic evolution for the Ambrosio-Tortorelli functional. The aim of this paper is to prove the convergence of this regular evolution to an evolution of brittle fractures in the sense of [14].

The Ambrosio-Tortorelli functional is given by

$$F_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v)^2\,dx$$

where $(u,v) \in H^1(\Omega) \times H^1(\Omega)$, $0 \leq v \leq 1$, $0 < \eta_\varepsilon << \varepsilon$. F_ε contains an *elliptic part*

$$\int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2\,dx \tag{1.3}$$

and a *surface part*

$$MM_\varepsilon(v) := \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v)^2\,dx \tag{1.4}$$

which is a term of Modica-Mortola type (see [17]).

If a sequence $(u_\varepsilon, v_\varepsilon)$ is such that $F_\varepsilon(u_\varepsilon, v_\varepsilon) + ||u_\varepsilon||_\infty \le C$, then $v_\varepsilon \to 1$ strongly in $L^2(\Omega)$, and it turns out that, up to a subsequence, $u_\varepsilon \to u$ in measure for some $u \in SBV(\Omega)$; roughly speaking, the gradient of u_ε becomes larger and larger in the thick regions in which v_ε approaches zero, possibly creating some jumps in the limit. We conclude that the function u_ε has to be considered as a regularization of the displacement u, while the function v_ε has to be intended as a function which tends to 0 in the region where S_u will appear, and to 1 elsewhere. Moreover (1.3) and (1.4) have to be interpreted as regularizations of the bulk and surface energy of u.

In the regular context of the Ambrosio and Tortorelli functional, we define through a variational argument the following notion of quasistatic evolution (Theorem 3.1): for every $\varepsilon > 0$ we find a map $t \to (u_\varepsilon(t), v_\varepsilon(t))$ from $[0,1]$ to $H^1(\Omega) \times H^1(\Omega)$, $0 \le v_\varepsilon(t) \le 1$, $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ such that:

(a) for all $0 \le s < t \le 1$: $v_\varepsilon(t) \le v_\varepsilon(s)$;

(b) for all $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ with $u = g(t)$, $v = 1$ on $\partial_D\Omega$, $0 \le v \le v_\varepsilon(t)$:

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v); \tag{1.5}$$

(c) the energy $\mathcal{E}_\varepsilon(t) := F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ is absolutely continuous and for all $t \in [0,1]$

$$\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0) + 2\int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla \dot g(\tau)\,dx\,d\tau;$$

(d) there exists a constant C depending only on g such that $\mathcal{E}_\varepsilon(t) \le C$ for all $t \in [0,1]$.

Condition (a) permits to recover in this regular context the fact that the fracture is increasing in time: in fact, as $v_\varepsilon(t)$ determines the fracture in the regions where it is near zero, the condition $v_\varepsilon(t) \le v_\varepsilon(s)$ ensures that existing cracks are preserved at subsequent times. Condition (b) reproduces the minimality condition at each time with respect to larger fractures, while condition (c) describes the evolution in time of the total energy. Condition (d) gives the necessary compactness in order to let $\varepsilon \to 0$. In the particular case in which $||g(t)||_\infty \le C_1$ for all $t \in [0,1]$, it turns out that, using truncation arguments, $||u_\varepsilon(t)||_\infty \le C_1$ for all t so that a uniform L^∞ bound is available at any time. The requirement $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ for all $t \in [0,1]$ is made in such a way that, letting $\varepsilon \to 0$, the surface energy of the fracture in the limit is the usual one also for the part touching the boundary $\partial_D\Omega$.

The main result of the paper (Theorem 3.2) is that, as $\varepsilon \to 0$, the quasistatic evolution $t \to (u_\varepsilon(t), v_\varepsilon(t))$ for the Ambrosio-Tortorelli functional converges to a quasistatic evolution for brittle fracture in the sense of [14]. More precisely, there exists a quasistatic evolution $t \to (u(t), \Gamma(t))$, $u(t) \in SBV(\Omega)$, relative to the boundary data g and a sequence $\varepsilon_n \to 0$ such that for all $t \in [0,1]$ which are not discontinuity points of $\mathcal{H}^{N-1}(\Gamma(\cdot))$ we have

$$v_{\varepsilon_n}(t)\nabla u_{\varepsilon_n}(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^N),$$

$$\int_\Omega (\eta_{\varepsilon_n} + v_{\varepsilon_n}(t)^2)|\nabla u_{\varepsilon_n}(t)|^2\,dx \to \int_\Omega |\nabla u(t)|^2\,dx,$$

and

$$MM_{\varepsilon_n}(v_{\varepsilon_n}(t)) \to \mathcal{H}^{N-1}(\Gamma(t)).$$

Moreover $\mathcal{E}_{\varepsilon_n}(t) \to \mathcal{E}(t)$ for all $t \in [0,1]$. We thus obtain an approximation of the total energy at any time, and an approximation of the strain, of the bulk and the surface energy at all time up to a countable set. The main step in the proof is to derive the unilateral minimality property (1.2) from its regularized version (1.5). Given $z \in SBV(\Omega)$, a natural way consists in constructing $z_n \in H^1(\Omega)$ and $v_n \in H^1(\Omega)$ with $z_n = g(t)$, $v_n = 1$ on $\partial_D\Omega$, $0 \le v_n \le v_n(t)$ and such that

$$\lim_n \int_\Omega (\eta_{\varepsilon_n} + v_n^2)|\nabla z_n|^2\,dx = \int_\Omega |\nabla z|^2\,dx, \tag{1.6}$$

and

$$\limsup_n \left[MM_{\varepsilon_n}(v_n) - MM_{\varepsilon_n}(v_{\varepsilon_n}(t))\right] \le \mathcal{H}^{N-1}\left(S_z \setminus \Gamma(t)\right). \tag{1.7}$$

We thus need a recovery sequence both for the displacement and the fracture: moreover we have to take into account the boundary conditions and the constraint $v_n \le v_n(t)$. Density results on z, such that of considering S_z polyhedral, cannot be directly applied since the set $S_z \setminus \Gamma(t)$ could increase too much; on the other hand it is not possible to work in $\Omega \setminus \Gamma(t)$ since no regularity results are available for $\Gamma(t)$ apart from its rectifiability. It turns out that $S_z \cap \Gamma(t)$ is the part of the fracture more difficult to be regularized, and in fact all the problems in the construction of (z_n, v_n) are already present in the particular case $S_z \subseteq \Gamma(t)$. In order to fix ideas, let us suppose to be in this situation; we solve the problem in two steps. We firstly construct $\tilde{z}_n \in SBV(\Omega)$ with $\nabla\tilde{z}_n \to \nabla z$ strongly in $L^2(\Omega;\mathbb{R}^N)$ and such that $S_{\tilde{z}_n}$ is related to $u_n(t)$ and $v_n(t)$ with precise energy estimates: this is done following the ideas of [14, Theorem 2.1], that is using local reflections and gluing along the boundaries of suitable upper levels of $u_n(t)$, but we have to choose the upper levels in a more precise way. In a second time, we regularize $S_{\tilde{z}_n}$ using not only $v_n(t)$, which is quite natural, but also $u_n(t)$, so that (1.6) and (1.7) hold.

The plan of the paper is the following. We introduce in Section 2 the notation and the main tools employed in the rest of the paper. Section 3 contains the statements of the main theorems. In Section 4 we treat the quasistatic evolution for the Ambrosio-Tortorelli functional, while in Section 5 we prove its convergence to a quasistatic growth of brittle fractures. The derivation of the minimality property (1.2) is contained in Section 6.

2. Notation and Preliminaries

In this section we state the notations and introduce the main tools used in the rest of the paper.

Basic notation. In the rest of the paper, we will employ the following basic notations:

- Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary;
- $\partial_D\Omega$ is a subset on $\partial\Omega$ open in the relative topology;
- $L^p(\Omega;\mathbb{R}^m)$ with $1 \le p \le +\infty$ and $m \ge 1$ is the Lebesgue space of p-summable $\mathbb{R}^m$-valued functions;
- $H^1(\Omega)$ is the Sobolev spaces of functions in $L^2(\Omega)$ with distributional derivative in $L^2(\Omega;\mathbb{R}^N)$;
- if $u \in H^1(\Omega)$, ∇u is its gradient;
- if $u, v \in H^1(\Omega)$, $u \le v$ in Ω means that $u(x) \le v(x)$ for a.e. $x \in \Omega$;
- $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure;
- $\|\cdot\|_\infty$ is the sup-norm;
- $\mathbb{1}_A$ is the characteristic function of A;
- if $\sigma \in]0, +\infty[$, $o(\sigma)$ is such that $\lim_{\sigma\to 0^+} o(\sigma) = 0$.

Special functions of bounded variation. For the general theory of functions of bounded variation, we refer to [4]; here we recall some basic definitions and theorems we need in the sequel.

Let A be an open subset of $\mathbb{R}^N$, and let $u : A \to \mathbb{R}^n$. We say that $u \in BV(A;\mathbb{R}^n)$ if $u \in L^1(A;\mathbb{R}^n)$, and its distributional derivative is a vector-valued Radon measure on A.

We say that $u \in SBV(A;\mathbb{R}^n)$ if $u \in BV(A;\mathbb{R}^n)$ and its distributional derivative can be represented as

$$Du(B) = \int_B \nabla u(x)\,dx + \int_{B\cap S_u} (u^+(x) - u^-(x)) \otimes \nu_x \, d\mathcal{H}^{N-1}(x)$$

where ∇u denotes the approximate gradient of u, S_u denotes the set of approximate jumps of u, u^+ and u^- are the traces of u on S_u, and ν_x is the normal to S_u at x. The space $SBV(A;\mathbb{R}^n)$ is called the space of special functions of bounded variation. Note that if $u \in SBV(A;\mathbb{R}^n)$, then the singular part of Du is concentrated on S_u which turns out to be countably $\mathcal{H}^{N-1}$-rectifiable. We set $SBV(A) := SBV(A;\mathbb{R})$.

We say that $u \in GSBV(A)$ if for every $M \geq 0$ we have $(u \wedge M) \vee (-M) \in SBV(A)$. For every $p \in [1,\infty]$, let us set $SBV^p(A,\mathbb{R}^n) := \{u \in SBV(A,\mathbb{R}^n) \,:\, \nabla u \in L^p(A; M^{N\times n})\}$, and $GSBV^p(A) := \{u \in GSBV(A) \,:\, \nabla u \in L^p(A;\mathbb{R}^N)\}$.

The space SBV is very useful when dealing with variational problems involving volume and surface energies because of the following compactness and lower semicontinuity result due to L.Ambrosio (see [1], [3]).

Theorem 2.1. *Let (u_k) be a sequence in $SBV(A;\mathbb{R}^n)$ such that there exist $q > 1$ and $c \geq 0$ with*

$$\int_A |\nabla u_k|^q \,dx + \mathcal{H}^{N-1}(S_{u_k}) + ||u_k||_{\infty,A} \leq c$$

for every $k \in \mathbb{N}$. Then there exist a subsequence (u_{k_h}) and a function $u \in SBV(A;\mathbb{R}^n)$ such that

$$\begin{aligned} u_{k_h} &\to u \quad \text{strongly in } L^1(A;\mathbb{R}^n),\\ \nabla u_{k_h} &\rightharpoonup \nabla u \quad \text{weakly in } L^1(A;M^{N\times n}),\\ \mathcal{H}^{N-1}(S_u) &\leq \liminf_h \mathcal{H}^{N-1}(S_{u_{k_h}}). \end{aligned} \tag{2.1}$$

In the rest of the paper, we will say that $u_k \to u$ in $SBV(A;\mathbb{R}^n)$ if u_k and u satisfy (2.1).

Quasi-static evolution of brittle fractures. Let $g : [0,1] \to H^1(\Omega)$ be absolutely continuous; we indicate the gradient of g at time t by $\nabla g(t)$, and the time derivative of g at time t by $\dot{g}(t)$. Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with Lipshitz boundary, and let $\partial_D\Omega \subseteq \partial\Omega$. The main result of [14] is the following theorem.

Theorem 2.2. *There exists a crack $\Gamma(t) \subseteq \overline{\Omega}$ and a field $u(t) \in SBV(\Omega)$ such that*

(a) *$\Gamma(t)$ increases with t;*

(b) *$u(0)$ minimizes*

$$\int_\Omega |\nabla v|^2 \,dx + \mathcal{H}^{N-1}(S_v \cup \{x \in \partial_D\Omega : v(x) \neq g(0)(x)\})$$

among all $v \in SBV(\Omega)$ (inequalities on $\partial_D\Omega$ are intended for the traces of v and g);

(c) *for $t > 0$, $u(t)$ minimizes*

$$\int_\Omega |\nabla v|^2 \,dx + \mathcal{H}^{N-1}\left([S_v \cup \{x \in \partial_D\Omega : v(x) \neq g(t)(x)\}] \setminus \Gamma(t)\right)$$

among all $v \in SBV(\Omega)$;

(d) *$S_{u(t)} \cup \{x \in \partial_D\Omega \,:\, u(t)(x) \neq g(t)(x)\} \subseteq \Gamma(t)$, up to a set of $\mathcal{H}^{N-1}$–measure 0.*

Furthermore, the total energy

$$\mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2 \,dx + \mathcal{H}^{N-1}(\Gamma(t))$$

is absolutely continuous and is given by

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau.$$

Finally, for any countable, dense set $I \subseteq [0,1]$, the crack $\Gamma(t)$ and the field $u(t)$ can be chosen such that

$$\Gamma(t) = \bigcup_{\tau \in I, \tau \le t} \left(S_{u(\tau)} \cup \{x \in \partial_D \Omega \,:\, u(\tau)(x) \neq g(\tau)(x)\} \right)$$

The Ambrosio-Tortorelli functional. In [5] and [6], Ambrosio and Tortorelli proposed an elliptic approximation of the Mumford-Shah functional in the sense of Γ-convergence. Their result has been extended in the vectorial case in [13], where non-isotropic surface energies are also considered. Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. For every $u \in GSBV(\Omega)$ let

$$F(u) := \int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{N-1}(S_u)$$

the well known Mumford-Shah functional; for every $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ the Ambrosio-Tortorelli functional is defined by

$$F_\varepsilon(u,v) := \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2\, dx + \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2\, dx + \frac{1}{2\varepsilon}\int_\Omega (1-v)^2\, dx$$

where $\eta_\varepsilon > 0$ and $\eta_\varepsilon << \varepsilon$. Let us indicate the space of Borel functions on Ω by $\mathcal{B}(\Omega)$ and let us consider on $\mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$ the functionals

$$\mathcal{F}(u,v,\Omega) := \begin{cases} F(u) & u \in GSBV(\Omega), v \equiv 1 \text{ a.e. on } \Omega \\ \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\mathcal{F}_\varepsilon(u,v,\Omega) := \begin{cases} F_\varepsilon(u,v) & (u,v) \in H^1(\Omega), 0 \le v \le 1 \\ \\ +\infty & \text{otherwise.} \end{cases}$$

The Ambrosio-Tortorelli result can be expressed in the following way.

Theorem 2.3. *The functionals $(\mathcal{F}_\varepsilon)$ on $\mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$ Γ-converge to $\mathcal{F}$ with respect to the convergence in measure.*

In particular, we will use several times the following fact: if $u^i_\varepsilon \in H^1(\Omega)$, $i = 1, \dots, n$, and $v_\varepsilon \in H^1(\Omega)$ are such that $\sum_{i=1}^n F_\varepsilon(u^i_\varepsilon, v_\varepsilon) + \|u^i_\varepsilon\|_\infty \le C$, there exist $u^i \in SBV(\Omega)$, $i = 1, \dots, n$ and a sequence $\varepsilon_k \to 0$ such that $u^i_{\varepsilon_k} \to u^i$ a.e., and

$$\int_\Omega |\nabla u^i|^2\, dx \le \liminf_{\varepsilon \to 0} \int_\Omega (\eta_\varepsilon + {v_\varepsilon}^2)|\nabla u^i_\varepsilon|^2\, dx, \tag{2.2}$$
$$\mathcal{H}^{N-1}\left(\bigcup_{i=1}^n S_{u^i}\right) \le \liminf_{\varepsilon \to 0} \left(\frac{\varepsilon}{2}\int_\Omega |\nabla v_\varepsilon|^2\, dx + \frac{1}{2\varepsilon}\int_\Omega (1-v_\varepsilon)^2\, dx\right).$$

A density result. Let $A \subseteq \mathbb{R}^N$ be open. We say that $K \subseteq A$ is polyhedral (with respect to A), if it is the intersection of A with the union of a finite number of $(N-1)$-dimensional simplexes of S.

The following density result is proved in [9].

Theorem 2.4. *Assume that ∂A is locally Lipschitz, and let $u \in GSBV^p(A)$. For every $\varepsilon > 0$, there exists a function $v \in SBV^p(A)$ such that*

(a) *S_v is essentially closed, i.e., $\mathcal{H}^{N-1}(\overline{S_v} \setminus S_v) = 0$;*

(b) $\overline{S_v}$ *is a polyhedral set;*

(c) $v \in W^{k,\infty}(A \setminus \overline{S_v})$ *for every* $k \in \mathbb{N}$*;*

(d) $||v - u||_{L^p(A)} < \varepsilon$*;*

(e) $||\nabla v - \nabla u||_{L^p(A;\mathbb{R}^N)} < \varepsilon$*;*

(f) $|\mathcal{H}^{N-1}(S_v) - \mathcal{H}^{N-1}(S_u)| < \varepsilon$.

Theorem 2.4 has been generalized to non-isotropic surface energies in [10]. In Section 6, we will use the following result.

Proposition 2.5. *Let* $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$*,* B *an open ball such that* $\overline{\Omega} \subseteq B$*, and let* $\Omega' := B \setminus \partial_N\Omega$*. Given* $g \in H^1(B)$ *and* $u \in SBV(\Omega')$ *with* $u = g$ *on* $\Omega' \setminus \overline{\Omega}$*, there exists* $u_h \in SBV(\Omega')$ *such that*

(a) $u_h = g$ *in* $\Omega' \setminus \overline{\Omega}$ *and in a neighborhood of* $\partial_D\Omega$*;*

(b) $\overline{S_{u_h}}$ *is polyhedral and* $\overline{S_{u_h}} \subseteq \Omega$ *for all* h*;*

(c) $\nabla u_h \to \nabla u$ *strongly in* $L^2(\Omega';\mathbb{R}^N)$*;*

(d) *for all* A *open subset of* Ω' *with* $\mathcal{H}^{N-1}(\partial A \cap S_u) = 0$*, we have*

$$\lim_h \mathcal{H}^{N-1}(A \cap S_{u_h}) = \mathcal{H}^{N-1}(A \cap S_u).$$

Proof. Using a partition of unity, we may prove the result in the case $\Omega' := Q \times]-1,1[$, $\Omega := \{(x,y) \in Q \times]-1,1[:\ y > f(x)\}$, $\partial_D\Omega := \{(x,y) \in Q \times]-1,1[:\ y = f(x)\}$, where Q is unit cube in $\mathbb{R}^{N-1}$ and $f : Q \to \mathbb{R}$ is a Lipshitz function with values in $]-\frac{1}{2},\frac{1}{2}[$. Let $g \in H^1(\Omega')$, and let $u \in SBV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$.

Let $w_h := u(x - he_N)$ where e_N is the versor of the N-axis, and let φ_h be a cut off function with $\varphi_h = 1$ on $\{y \le f(x) + \frac{h}{3}\}$, $\varphi_h = 0$ on $\{y \ge f(x) + \frac{h}{2}\}$, and $||\nabla\varphi_h||_\infty \le \frac{1}{h}$. Let us set $v_h := \varphi_h g + (1-\varphi_h)w_h$. We have that $v_h = g$ in $\Omega' \setminus \overline{\Omega}$ and in a neighborhood of $\partial_D\Omega$; moreover we have

$$\nabla v_h = \varphi_h \nabla g + (1-\varphi_h)\nabla w_h + \nabla\varphi_h(g - w_h).$$

Since $\nabla\varphi_h(g-w_h) \to 0$ strongly in $L^2(\Omega';\mathbb{R}^N)$, we have $\nabla v_h \to \nabla u$ strongly in $L^2(\Omega';\mathbb{R}^N)$. Finally, for all A open subset of Ω' with $\mathcal{H}^{N-1}(\partial A \cap S_u) = 0$, we have

$$\lim_h \mathcal{H}^{N-1}(A \cap S_{v_h}) = \mathcal{H}^{N-1}(A \cap S_u).$$

In order to conclude the proof, let us apply Theorem 2.4 obtaining $\tilde{v}_h$ with polyhedral jumps such that $||v_h - \tilde{v}_h||_{L^2(\Omega')} + ||\nabla v_h - \nabla\tilde{v}_h||_{L^2(\Omega';\mathbb{R}^N)} \le h^2$, $|\mathcal{H}^{N-1}(S_{v_h}) - \mathcal{H}^{N-1}(S_{\tilde{v}_h})| \le h$. If we set $u_h := \varphi_h g + (1-\varphi_h)\tilde{v}_h$, we obtain the thesis. □

3. The Main Results

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with Lipshitz boundary, and let $\partial_D\Omega \subseteq \partial\Omega$. If $g \in W^{1,1}([0,1];H^1(\Omega))$, we indicate the gradient of g at time t by $\nabla g(t)$, and the time derivative of g at time t by $\dot{g}(t)$.

Concerning the Ambrosio-Tortorelli functional, the following theorem holds.

Theorem 3.1. *Let* $g \in W^{1,1}([0,1];H^1(\Omega))$*. Then for all* $\varepsilon > 0$ *there exists a strongly measurable map*

$$\begin{array}{ccc} [0,1] & \longrightarrow & H^1(\Omega) \times H^1(\Omega) \\ t & \longmapsto & (u_\varepsilon(t), v_\varepsilon(t)) \end{array}$$

such that $0 \le v_\varepsilon(t) \le 1$ *in* Ω, $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ *on* $\partial_D\Omega$ *for all* $t \in [0,1]$, *and:*

(a) *for all* $0 \le s \le t \le 1 : v_\varepsilon(t) \le v_\varepsilon(s)$;

(b) *for all* $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ *with* $u = g(0)$, $v = 1$ *on* $\partial_D\Omega$
$$F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) \le F_\varepsilon(u,v);$$

(c) *for all* $t \in]0,1]$ *and for all* $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ *with* $0 \le v \le v_\varepsilon(t)$ *on* Ω, *and* $u = g(t)$, $v = 1$ *on* $\partial_D\Omega$
$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v);$$

(d) *the function* $t \to F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ *is absolutely continuous and*
$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau.$$

The main result of the paper is the following theorem.

Theorem 3.2. *Let* $g \in W^{1,1}([0,1]; H^1(\Omega))$ *be such that* $\|g(t)\|_\infty \le C$ *for all* $t \in [0,1]$, *and let* $g_h \in W^{1,1}([0,1]; H^1(\Omega))$ *be a sequence of absolutely continuous functions with* $\|g_h(t)\|_\infty \le C$, $g_h(t) \in C(\overline{\Omega})$ *for all* $t \in [0,1]$ *and such that* $g_h \to g$ *strongly in* $W^{1,1}([0,1]; H^1(\Omega))$. *For all* $\varepsilon > 0$, *let* $t \to (u_{\varepsilon,h}(t), v_{\varepsilon,h}(t))$ *be a quasistatic evolution for the Ambrosio-Tortorelli functional* F_ε *with boundary data* g_h *given by Theorem 3.1.*

Then there exists a quasistatic evolution $t \to (u(t), \Gamma(t))$, $u(t) \in SBV(\Omega)$, *relative to the boundary data* g *in the sense of Theorem 2.2, and two sequences* $\varepsilon_n \to 0$ *and* $h_n \to +\infty$ *such that, setting* $u_n := u_{\varepsilon_n,h_n}$ *and* $v_n := v_{\varepsilon_n,h_n}$, *the following hold:*

(a) *for all* $t \in [0,1]$ *we have*
$$F_{\varepsilon_n}(u_n(t), v_n(t)) \to \mathcal{E}(t);$$

(b) *if* $\mathcal{N}$ *denotes the point of discontinuity of* $\mathcal{H}^{N-1}(\Gamma(\cdot))$, *for all* $t \in [0,1] \setminus \mathcal{N}$ *we have*
$$v_n(t)\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^N),$$
$$\lim_n \int_\Omega (\eta_n + v_n^2(t))|\nabla u_n(t)|^2\, dx = \int_\Omega |\nabla u(t)|^2\, dx,$$
and
$$\lim_n \frac{\varepsilon_n}{2}\int_\Omega |\nabla v_n(t)|^2\, dx + \frac{1}{2\varepsilon_n}\int_\Omega (1 - v_n(t))^2\, dx = \mathcal{H}^{N-1}(\Gamma(t)).$$

Theorem 3.1 concerning the quasistatic evolution for the Ambrosio-Tortorelli functional is proved in Section 4. In Section 5 we prove the compactness and approximation result given by Theorem 3.2. An important step in the proof is given by Theorem 5.6 to which is dedicated the entire Section 6.

4. Quasi-static evolution for the Ambrosio-Tortorelli functional

This section is devoted to the proof of Theorem 3.1 where a suitable notion of quasistatic evolution in a regular context is proposed. The evolution will be obtained through a discretization in time procedure: each step will be performed using a variational argument which will give the minimality property stated in points (b) and (c).

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with Lipshitz boundary, and let $\partial_D\Omega \subseteq \partial\Omega$. Let $g \in W^{1,1}([0,1]; H^1(\Omega))$. Given $\delta > 0$, let N_δ be the largest integer such that $\delta N_\delta \le 1$; for $i \ge 0$ we set $t_i^\delta = i\delta$ and for $0 \le i \le N_\delta$ we set $g_i^\delta = g(t_i^\delta)$. Define u_0^δ and v_0^δ as a minimum for the problem

$$\min\{F_\varepsilon(u,v) : (u,v) \in H^1(\Omega) \times H^1(\Omega), 0 \le v \le 1 \text{ in } \Omega, u = g_0^\delta, v = 1 \text{ on } \partial_D\Omega\}, \tag{4.1}$$

and let $(u^\delta_{i+1}, v^\delta_{i+1})$ be a minimum for the problem

$$\min\{F_\varepsilon(u,v) : (u,v) \in H^1(\Omega)\times H^1(\Omega), 0 \le v \le v^\delta_i \text{ in } \Omega, u = g^\delta_{i+1}, v = 1 \text{ on } \partial_D\Omega\}. \tag{4.2}$$

Problems (4.1) and (4.2) are well posed: in fact, referring for example to problem (4.2), let (u_n, v_n) be a minimizing sequence. Since $(g^\delta_{i+1}, v^\delta_i)$ is an admissible pair, we obtain that there exists a constant $C > 0$ such that for all n

$$F_\varepsilon(u_n, v_n) \le C.$$

Since $\varepsilon, \eta_\varepsilon > 0$, we deduce that (u_n, v_n) is bounded in $H^1(\Omega)\times H^1(\Omega)$ so that up to a subsequence $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$. We get immediately that $u = g^\delta_{i+1}$ and $v = 1$ on $\partial_D\Omega$ since $u_n = g^\delta_{i+1}$ and $v_n = 1$ on $\partial_D\Omega$ for all n; on the other hand, since $v_n \to v$ strongly in $L^2(\Omega)$, we obtain that $0 \le v \le v^\delta_i$. By semicontinuity, we have

$$F_\varepsilon(u,v) \le \liminf_n F_\varepsilon(u_n, v_n)$$

so that (u, v) is a minimum point for problem (4.2).

We note that by minimality of the pair $(u^\delta_{i+1}, v^\delta_{i+1})$, we may write

$$\begin{aligned}(4.3)\quad & F_\varepsilon(u^\delta_{i+1}, v^\delta_{i+1}) \le F_\varepsilon(u^\delta_i + g^\delta_{i+1} - g^\delta_i, v^\delta_i) = \\ &= F_\varepsilon(u^\delta_i, v^\delta_i) + 2\int_\Omega (\eta_\varepsilon + (v^\delta_i)^2)\nabla u^\delta_i \nabla(g^\delta_{i+1} - g^\delta_i)\,dx + \int_\Omega (\eta_\varepsilon + (v^\delta_i)^2)|\nabla(g^\delta_{i+1} - g^\delta_i)|^2\,dx \le \\ &\le F_\varepsilon(u^\delta_i, v^\delta_i) + 2\int_{t^\delta_i}^{t^\delta_{i+1}}\int_\Omega (\eta_\varepsilon + (v^\delta_i)^2)\nabla u^\delta_i \nabla\dot g(\tau)\,dx\,d\tau + e(\delta)\int_{t^\delta_i}^{t^\delta_{i+1}} ||\nabla \dot g(\tau)||_{L^2(\Omega;\mathbb{R}^N)}\,d\tau,\end{aligned}$$

where

$$e(\delta) := (1+\eta_\varepsilon)\max_{0\le r\le N_\delta - 1}\int_{t^\delta_r}^{t^\delta_{r+1}} ||\nabla \dot g(\tau)||_{L^2(\Omega;\mathbb{R}^N)}\,d\tau$$

is infinitesimal as $\delta \to 0$.

We now make a piecewise constant interpolation defining

$$u^\delta_\varepsilon(t) = u^\delta_i, \quad v^\delta_\varepsilon(t) = v^\delta_i, \quad g^\delta(t) = g^\delta_i \quad \text{for } t^\delta_i \le t < t^\delta_{i+1}. \tag{4.4}$$

Note that by construction the map $t \to v^\delta_\varepsilon(t)$ is decreasing from $[0, 1]$ to $L^2(\Omega)$. Moreover, iterating the estimate (4.3), we obtain

$$\begin{aligned} F_\varepsilon(u^\delta_\varepsilon(t), v^\delta_\varepsilon(t)) \le\; & F_\varepsilon(u^\delta_\varepsilon(s), v^\delta_\varepsilon(s)) + 2\int_{s^\delta}^{t^\delta}\int_\Omega (\eta_\varepsilon + v^\delta_\varepsilon(\tau)^2)\nabla u^\delta_\varepsilon(\tau)\nabla \dot g(\tau)\,dx\,d\tau + \\ (4.5)\qquad & + e(\delta)\int_{s^\delta}^{t^\delta} ||\nabla \dot g(\tau)||_{L^2(\Omega;\mathbb{R}^N)}\,d\tau\end{aligned}$$

where $s^\delta := t^\delta_i$ and $t^\delta := t^\delta_j$ are the step discretization points such that $t^\delta_i \le s < t^\delta_{i+1}$ and $t^\delta_j \le t < t^\delta_{j+1}$.

Note that by minimality of the pair $(u^\delta_\varepsilon(t), v^\delta_\varepsilon(t))$, we have

$$F_\varepsilon(u^\delta_\varepsilon(t), v^\delta_\varepsilon(t)) \le F_\varepsilon(g^\delta(t), v^\delta_\varepsilon(t))$$

so that

$$\int_\Omega (\eta_\varepsilon + v^\delta_\varepsilon(t)^2)|\nabla u^\delta_\varepsilon(t)|^2\,dx \le \int_\Omega (\eta_\varepsilon + v^\delta_\varepsilon(t)^2)|\nabla g^\delta(t)|^2\,dx \le C_1 \tag{4.6}$$

with $C_1 > 0$ independent of δ and t. In particular by (4.6) we have that

$$||\nabla u^\delta_\varepsilon(t)||^2_{L^2(\Omega;\mathbb{R}^N)} \le \frac{C_1}{\eta_\varepsilon}.$$

Since $u^\delta_\varepsilon(t) = g^\delta(t)$ on $\partial_D\Omega$, and $g^\delta(t)$ is uniformly bounded in $H^1(\Omega)$ for all t and δ, we get by a variant of Poincaré inequality that $u^\delta_\varepsilon(t)$ is uniformly bounded in $H^1(\Omega)$ for all t and δ.

Now we come to v_ε^δ in order to obtain some coerciveness in the space $H^1(\Omega)$. Notice that

$$\begin{aligned} 2 & \left| \int_0^{t^\delta} \int_\Omega (\eta_\varepsilon + v_\varepsilon^\delta(\tau)^2) \nabla u_\varepsilon^\delta(\tau) \nabla \dot g(\tau)\, dx\, d\tau \right| \le \\ \le & \; 2 \int_0^{t^\delta} \sqrt{\eta_\varepsilon + 1} \left(\int_\Omega (\eta_\varepsilon + v_\varepsilon^\delta(t)^2) |\nabla u_\varepsilon^\delta(t)|^2\, dx \right)^{\frac{1}{2}} ||\nabla \dot g(\tau)||_{L^2(\Omega;\mathbb{R}^N)}\, d\tau, \end{aligned}$$

and by (4.6), we obtain

$$\left| 2 \int_0^{t^\delta} \int_\Omega (\eta_\varepsilon + v_\varepsilon^\delta(\tau)^2) \nabla u_\varepsilon^\delta(\tau) \nabla \dot g(\tau)\, dx\, d\tau \right| \le C_2 \tag{4.7}$$

with $C_2 > 0$ independent of t and δ.

By (4.5) with $s = 0$, and (4.7), we deduce

$$\begin{aligned} \frac{\varepsilon}{2} & \int_\Omega |\nabla v_\varepsilon^\delta(t)|^2\, dx + \frac{1}{2\varepsilon} \int_\Omega (1 - v_\varepsilon^\delta(t))^2\, dx \le \\ \le & \; F_\varepsilon(u_\varepsilon^\delta(0), v_\varepsilon^\delta(0)) + 2 \int_0^{t^\delta} \int_\Omega (\eta_\varepsilon + v_\varepsilon^\delta(\tau)^2) \nabla u_\varepsilon^\delta(\tau) \nabla \dot g(\tau)\, dx\, d\tau + \\ & + e(\delta) \int_{s^\delta}^{t^\delta} ||\nabla \dot g(\tau)||_{L^2(\Omega;\mathbb{R}^N)}\, d\tau \le \\ \le & \; F_\varepsilon(u_\varepsilon^\delta(0), v_\varepsilon^\delta(0)) + C_2 + e(\delta) \int_0^1 ||\nabla \dot g(\tau)||_{L^2(\Omega}\, d\tau. \end{aligned}$$

We conclude that there exists $C > 0$ independent of t and δ such that for all $t \in [0,1]$

$$||v_\varepsilon^\delta(t)||_{H^1(\Omega)} \le C. \tag{4.8}$$

We now want to pass to the limit in δ as $\delta \to 0$.

Lemma 4.1. *There exists a sequence $\delta_n \to 0$ and a strongly measurable map $v_\varepsilon : [0,1] \to H^1(\Omega)$ such that $v_\varepsilon^{\delta_n}(t) \rightharpoonup v_\varepsilon(t)$ weakly in $H^1(\Omega)$ for all $t \in [0,1]$. Moreover, v_ε is decreasing from $[0,1]$ to $L^2(\Omega)$, and $0 \le v_\varepsilon(t) \le 1$ in Ω, $v_\varepsilon(t) = 1$ on $\partial_D \Omega$ for all $t \in [0,1]$.*

Proof. Since the map $t \to v_\varepsilon^\delta(t)$ is monotone decreasing from $[0,1]$ to $L^2(\Omega)$, and $0 \le v_\varepsilon^\delta(t) \le 1$ for all t, we deduce by a variant of Helly's compactness theorem for sequences of monotone real functions, that there exists a subsequence $\delta_n \to 0$ and a decreasing map $v_\varepsilon : [0,1] \to L^2(\Omega)$ such that for all $t \in [0,1]$ we have $v_\varepsilon^{\delta_n}(t) \to v_\varepsilon(t)$ strongly in $L^2(\Omega)$. In particular we deduce $0 \le v_\varepsilon(t) \le 1$ in Ω. By (4.8), we have that for all $t \in [0,1]$, up to a subsequence, $v_\varepsilon^{\delta_n}(t) \rightharpoonup w$ weakly in $H^1(\Omega)$; since $v_\varepsilon^{\delta_n}(t) \to v_\varepsilon(t)$ strongly in $L^2(\Omega)$, we deduce that $w = v_\varepsilon(t)$ so that $v_\varepsilon(t) \in H^1(\Omega)$, and $v_\varepsilon^{\delta_n}(t) \rightharpoonup v_\varepsilon(t)$ weakly in $H^1(\Omega)$. As a consequence, $v_\varepsilon(t) = 1$ on $\partial_D \Omega$ for all $t \in [0,1]$. Finally, v_ε is strongly measurable from $[0,1]$ to $H^1(\Omega)$ because it is weakly measurable and separably valued (see [18, Chapter V, Section 4]). ☐

Let us consider the sequence δ_n, and the map v_ε given by Lemma 4.1. We indicate $u_\varepsilon^{\delta_n}$, $v_\varepsilon^{\delta_n}$ and g^{δ_n} simply by u_ε^n, v_ε^n and g_n.

Lemma 4.2. *There exists a strongly measurable map $u_\varepsilon : [0,1] \to H^1(\Omega)$ such that $u_\varepsilon^n(t) \to u_\varepsilon(t)$ strongly in $H^1(\Omega)$ for all $t \in [0,1]$. In particular, $u_\varepsilon(t) = g(t)$ on $\partial_D \Omega$ for all $t \in [0,1]$.*

Proof. Let $t \in [0,1]$. We note that $u_\varepsilon^n(t)$ is the minimum of the following problem

$$\min \left\{ \int_\Omega (\eta_\varepsilon + v_\varepsilon^n(t)^2) |\nabla z|^2\, dx \,:\, z \in H^1(\Omega), z = g_n(t) \,\text{on}\, \partial_D \Omega \right\}.$$

Since by Lemma 4.1 $v_\varepsilon^n(t) \to v_\varepsilon(t)$ strongly in $L^2(\Omega)$, and $g_n(t) \to g(t)$ strongly in $H^1(\Omega)$, we deduce by standard results on Γ-convergence (see [11]), that $u_\varepsilon^n(t) \rightharpoonup u_\varepsilon(t)$ weakly in $H^1(\Omega)$ where $u_\varepsilon(t)$ is the solution of the problem

$$\min\left\{\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t))|\nabla z|^2\,dx \,:\, z \in H^1(\Omega), z = g(t) \text{ on } \partial_D\Omega\right\}.$$

Moreover, we have also convergence of energies, that is

$$\lim_n \int_\Omega (\eta_\varepsilon + v_\varepsilon^n(t)^2)|\nabla u_\varepsilon^n(t)|^2\,dx = \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t))|\nabla u_\varepsilon(t)|^2\,dx. \tag{4.9}$$

Since $v_\varepsilon^n(t)\nabla u_\varepsilon^n(t) \rightharpoonup v_\varepsilon(t)\nabla u_\varepsilon(t)$ weakly in $L^2(\Omega;\mathbb{R}^N)$, we obtain

$$\int_\Omega v_\varepsilon^2(t)|\nabla u_\varepsilon(t)|^2\,dx \le \liminf_n \int_\Omega v_\varepsilon^n(t)^2|\nabla u_n(t)|^2\,dx,$$

so that by (4.9) we deduce $\nabla u_\varepsilon^n(t) \to \nabla u_\varepsilon(t)$ strongly in $L^2(\Omega;\mathbb{R}^N)$. We conclude that $u_\varepsilon^n(t) \to u_\varepsilon(t)$ strongly in $H^1(\Omega)$ for all $t \in [0,1]$, and so the map $t \to u_\varepsilon(t)$ is strongly measurable from $[0,1]$ to $H^1(\Omega)$. Finally $u_\varepsilon(t) = g(t)$ on $\partial_D\Omega$ and the proof is complete. □

The following minimality property for the pair $(u_\varepsilon(t), v_\varepsilon(t))$ holds.

Proposition 4.3. *If $t \in]0,1]$, for every $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ such that $0 \le v \le v_\varepsilon(t)$ in Ω, and $u = g(t)$, $v = 1$ on $\partial_D\Omega$, we have*

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v).$$

Moreover, for all $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ such that $u = g(0)$, $v = 1$ on $\partial_D\Omega$, we have

$$F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) \le F_\varepsilon(u,v).$$

Proof. Let us set

$$u_n := u + g_n(t) - g(t),$$

and

$$v_n := \min\{v_\varepsilon^n(t), v\};$$

we have $u_n \to u$ strongly in $H^1(\Omega)$, and $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$. Since $0 \le v_n \le v_\varepsilon^n(t)$ in Ω, and $u_n = g_n(t)$, $v_n = 1$ on $\partial_D\Omega$, by the minimality property of the pair $(u_\varepsilon^n(t), v_\varepsilon^n(t))$ we get

$$F_\varepsilon(u_\varepsilon^n(t), v_\varepsilon^n(t)) \le F_\varepsilon(u_n, v_n),$$

that is

$$\begin{aligned}\int_\Omega (\eta_\varepsilon + v_\varepsilon^n(t)^2)|\nabla u_\varepsilon^n(t)|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla v_\varepsilon^n(t)|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1 - v_\varepsilon^n(t))^2\,dx \le \\ \le \int_\Omega (\eta_\varepsilon + {v_n}^2)|\nabla u_n|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla v_n|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1 - v_n)^2\,dx.\end{aligned} \tag{4.10}$$

Notice that

$$\frac{\varepsilon}{2}\int_\Omega |\nabla v_n|^2\,dx = \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t) < v\}} |\nabla v_\varepsilon^n(t)|^2\,dx + \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t) \ge v\}} |\nabla v|^2\,dx$$

so that (4.10) becomes

$$\begin{aligned}\int_\Omega (\eta_\varepsilon + v_\varepsilon^n(t)^2)|\nabla u_\varepsilon^n(t)|^2\,dx + \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t) \ge v\}} |\nabla v_\varepsilon^n(t)|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1 - v_\varepsilon^n(t))^2\,dx \le \\ \le \int_\Omega (\eta_\varepsilon + {v_n}^2)|\nabla u_n|^2\,dx + \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t) \ge v\}} |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1 - v_n)^2\,dx.\end{aligned}$$

For $n \to \infty$, the right hand side is less than $F_\varepsilon(u,v)$. Let us consider the left hand side. By semicontinuity we have

$$\liminf_n \frac{\varepsilon}{2} \int_{\{v_\varepsilon^n(t) \geq v\}} |\nabla v_\varepsilon^n(t)|^2 \, dx \geq \frac{\varepsilon}{2} \int_\Omega |\nabla v_\varepsilon(t)|^2 \, dx,$$

and so we conclude that $F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \leq F_\varepsilon(u,v)$.

For the case $t = 0$, by lower semicontinuity we get immediately the result. □

In order to obtain the proof of Theorem 3.1, we need the following proposition.

Proposition 4.4. *For all $0 \leq s \leq t \leq 1$, we have that*

$$\begin{aligned} F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) - F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \quad &\geq \quad 2\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t)) \nabla u_\varepsilon(t)(\nabla g(t) - \nabla g(s)) \, dx + \\ &\quad -\sigma(t-s) \int_s^t ||\nabla \dot g(\tau)||_{L^2(\Omega;\mathbb{R}^N)} \, d\tau \end{aligned}$$

where σ is an increasing positive function with $\sigma(r) \to 0$ as $r \to 0^+$.

Proof. By Proposition 4.3, we have

$$F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \leq F_\varepsilon(u_\varepsilon(t) - g(t) + g(s), v_\varepsilon(t))$$

so that

$$\begin{aligned} F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \quad &\leq \quad F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) - 2\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t)) \nabla u_\varepsilon(t)(\nabla g(t) - \nabla g(s)) \, dx + \\ &\quad + \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t)) |\nabla g(t) - \nabla g(s)|^2 \, dx. \end{aligned}$$

Then we conclude that

$$\begin{aligned} F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) - F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \quad &\geq \quad 2\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t)) \nabla u_\varepsilon(t)(\nabla g(t) - \nabla g(s)) \, dx + \\ &\quad -\sigma(t-s) \int_s^t ||\nabla \dot g(\tau)||_{L^2} \, d\tau \end{aligned}$$

where

$$\sigma(r) := (1 + \eta_\varepsilon) \max_{t-s=r} \int_s^t ||\nabla \dot g(\tau)||_{L^2(\Omega;\mathbb{R}^N)} \, d\tau,$$

and so the proof is complete. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Let us consider the sequence $\delta_n \to 0$ given by Lemma 4.1, and let us indicate the discrete evolutions $u_\varepsilon^{\delta_n}$ and $v_\varepsilon^{\delta_n}$ defined in (4.4) simply by u_ε^n and v_ε^n. Let us denote also by $u_\varepsilon(t)$ and $v_\varepsilon(t)$ their limits at time t according to Lemma 4.1 and Lemma 4.2. We have that the maps $t \to u_\varepsilon(t)$ and $t \to v_\varepsilon(t)$ are strongly measurable from $[0,1]$ to $H^1(\Omega)$; moreover for all $t \in [0,1]$ we have $0 \leq v_\varepsilon(t) \leq 1$ in Ω, $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on $\partial_D \Omega$ and $t \to v_\varepsilon(t)$ is decreasing from $[0,1]$ to $L^2(\Omega)$ so that point (a) is proved. By construction we get point (b) and by Proposition 4.3 we get point (c).

Let us come to condition (d). Let us fix $t \in [0,1]$, and let us divide the interval $[0,t]$ in k subintervals with endpoints $s_j^k := \frac{jt}{k}$ where $j = 0, 1, \cdots, k$. Let us define $\tilde u_k(s) := u_\varepsilon(s_{j+1}^k)$, and $\tilde v_k(s) := v_\varepsilon(s_{j+1}^k)$ for $s_j^k < s \leq s_{j+1}^k$. Then, applying Proposition 4.4, we have

$$\begin{aligned} F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \quad &\geq \quad F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_\Omega (\eta_\varepsilon + \tilde v_k^2(\tau)) \nabla \tilde u_k(\tau) \nabla \dot g(\tau) \, dx \, d\tau + \\ &\quad -\sigma\left(\frac{t}{k}\right) \int_0^t ||\nabla \dot g(\tau)||_{L^2} \, d\tau. \end{aligned} \tag{4.11}$$

Since $t \to v_\varepsilon(t)$ is monotone decreasing from $[0,1]$ to $L^2(\Omega)$, we have that $\tilde{v}_k(s) \to v_\varepsilon(s)$ strongly in $L^2(\Omega)$ for a.e. $s \in [0,t]$; consequently, we have that $\tilde{u}_k(s) \to u_\varepsilon(s)$ strongly in $H^1(\Omega)$ as noted in Lemma 4.2. We conclude by the Dominated Convergence Theorem that

$$\lim_k \int_0^t \int_\Omega (\eta_\varepsilon + \tilde{v}_k^2(\tau)) \nabla \tilde{u}_k(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau = \int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau.$$

By (4.11) we deduce that

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \geq F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau. \tag{4.12}$$

On the other hand, from (4.5), and since $F_\varepsilon(u_\varepsilon^n(0), v_\varepsilon^n(0)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0))$ for all n, we deduce

$$\limsup_n F_\varepsilon(u_\varepsilon^n(t), v_\varepsilon^n(t)) \leq F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau. \tag{4.13}$$

Since by semicontinuity we have for all $t \in [0,1]$

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \leq \liminf_n F_\varepsilon(u_\varepsilon^n(t), v_\varepsilon^n(t)),$$

by (4.12) and (4.13), we conclude that

$$\lim_n F_\varepsilon(u_\varepsilon^n(t), v_\varepsilon^n(t)) = F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)). \tag{4.14}$$

In particular

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau,$$

and this proves point (d). □

Remark 4.5. The map $\{t \to v_\varepsilon(t),\, t \in [0,1]\}$ is decreasing from $[0,1]$ to $L^2(\Omega)$, so that v_ε is continuous with respect to the strong topology of $L^2(\Omega)$ at all points except a countable set. Since

$$\lambda(t) := \frac{\varepsilon}{2} \int_\Omega |\nabla v_\varepsilon(t)|^2\, dx + \frac{1}{2\varepsilon} \int_\Omega (1 - v_\varepsilon(t))^2\, dx$$

is monotone increasing (see Proposition 5.8), we conclude that $v_\varepsilon : [0,1] \to H^1(\Omega)$ is continuous with respect to the strong topology at all points except a countable set. Then we have $v_\varepsilon \in L^\infty([0,1], H^1(\Omega))$. Moreover, we have that $u_\varepsilon : [0,1] \to H^1(\Omega)$ is continuous at the continuity points of v_ε as observed in Lemma 4.2. We conclude that $u_\varepsilon \in L^\infty([0,1], H^1(\Omega))$.

Remark 4.6. The minimality property of point (c) of Theorem 3.1 holds indeed in this stronger form: if $t \in]0,1]$, for all $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ with $0 \leq v \leq v_\varepsilon(s)$ on Ω for all $s < t$, and $u = g(t)$, $v = 1$ on $\partial_D \Omega$, we have

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \leq F_\varepsilon(u, v).$$

In fact, if $0 \leq v \leq v_\varepsilon(s)$, by the minimality property of $(u_\varepsilon(s), v_\varepsilon(s))$ we have

$$F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \leq F_\varepsilon(u + g(s) - g(t), v),$$

so that, letting $s \to t$ and using the continuity of $F_\varepsilon(u_\varepsilon(\cdot), v_\varepsilon(\cdot))$ we get the result.

This stronger minimality property is the reformulation in the context of the Ambrosio-Tortorelli functional of the minimality of the cracks required in [15] (see the Introduction).

5. Quasi-static growth of brittle fracture

In this section, we prove that the evolution for the Ambrosio-Tortorelli functional F_ε converges as $\varepsilon \to 0$ to a quasistatic evolution of brittle fractures in linearly elastic bodies in the sense of [14].

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with Lipshitz boundary. Let $\partial_D\Omega \subseteq \partial\Omega$, and let su set $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Let $g \in W^{1,1}([0,1]; H^1(\Omega))$. In order to treat in a convenient way the boundary condition as $\varepsilon \to 0$, let B be an open ball such that $\overline{\Omega} \subset B$, and let us set $\Omega' := B \setminus \partial_N\Omega$ and $\Omega_D := \Omega' \setminus \overline{\Omega}$. Let E be an extension operator from $H^1(\Omega)$ to $H^1(B)$: we indicate $Eg(t)$ still by $g(t)$ for all $t \in [0,1]$. In this enlarged context, the following proposition holds.

Proposition 5.1. *Let us consider the evolution $t \to (u_\varepsilon(t), v_\varepsilon(t))$ from $[0,1]$ to $H^1(\Omega) \times H^1(\Omega)$ given by Theorem 3.1, and let us extend $u_\varepsilon(t)$ and $v_\varepsilon(t)$ to Ω' setting $u_\varepsilon(t) = g(t)$ and $v_\varepsilon(t) = 1$ on Ω_D respectively. Then the map*

$$\begin{array}{ccc} [0,1] & \longrightarrow & H^1(\Omega') \times H^1(\Omega') \\ t & \longmapsto & (u_\varepsilon(t), v_\varepsilon(t)) \end{array}$$

is strongly measurable and the following facts hold:

(a) *for all $0 \le s \le t \le 1 : v_\varepsilon(t) \le v_\varepsilon(s)$;*

(b) *for all $(u,v) \in H^1(\Omega') \times H^1(\Omega')$ with $u = g(0)$, $v = 1$ on Ω_D:*

$$F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) \le F_\varepsilon(u,v); \tag{5.1}$$

(c) *for $t \in]0,1]$ and for all $(u,v) \in H^1(\Omega') \times H^1(\Omega')$ with $0 \le v \le v_\varepsilon(t)$ on Ω', and $u = g(t)$, $v = 1$ on Ω_D:*

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v); \tag{5.2}$$

(d) *the function $t \to F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ is absolutely continuous and*

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_{\Omega'} (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau. \tag{5.3}$$

Proof. Recall that for all $t \in [0,1]$ we have $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$, and $0 \le v_\varepsilon(t) \le 1$ in Ω. The extensions to $H^1(\Omega')$ are thus well defined. We obtain a strongly measurable map $t \to (u_\varepsilon(t), v_\varepsilon(t))$ from $[0,1]$ to $H^1(\Omega') \times H^1(\Omega')$ such that $0 \le v_\varepsilon(t) \le 1$ in Ω', $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on Ω_D, and such that

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v)$$

for all $(u,v) \in H^1(\Omega') \times H^1(\Omega')$ with $0 \le v \le v_\varepsilon(t)$ on Ω', $u = g(t)$, $v = 1$ on Ω_D; note in fact that the integrations on Ω_D which appear in both sides are the same. By the same reason, we get the minimality property at time $t = 0$ and deduce that the function $t \to F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ is absolutely continuous with

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_{\Omega'} (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau.$$

□

From now on, we assume that there exists a constant $C > 0$ such that for all $t \in [0,1]$, $\|g(t)\|_\infty \le C$, and that there exists $g_h \in W^{1,1}([0,1], H^1(\Omega'))$ such that $\|g_h\|_\infty \le C$, $g_h \in C(\overline{\Omega'})$, and $g_h \to g$ strongly in $W^{1,1}([0,1], H^1(\Omega'))$. For every $\varepsilon > 0$ we indicate by $(u_{\varepsilon,h}, v_{\varepsilon,h})$ the evolution for the Ambrosio-Tortorelli functional relative to the boundary data g_h given by Proposition 5.1. The bound on the sup-norm is made in order to apply Ambrosio's compactness theorem in SBV when $\varepsilon \to 0$. Notice that we may assume by a truncation argument that $||u_{\varepsilon,h}(t)||_\infty \le ||g_h(t)||_\infty$, that is

$$||u_{\varepsilon,h}(t)||_\infty \le C. \tag{5.4}$$

We conclude that $u_{\varepsilon,h}(t)$ is uniformly bounded in $L^\infty(\Omega')$ as ε, h and t vary. Moreover we have that the following holds.

Lemma 5.2. *There exists a constant $C_1 \ge 0$ depending only on g such that for all $t \in [0,1]$, ε, h*

$$F_\varepsilon(u_{\varepsilon,h}(t), v_{\varepsilon,h}(t)) + \|u_{\varepsilon,h}(t)\|_\infty \le C_1. \tag{5.5}$$

Proof. Notice that $F_\varepsilon(u_{\varepsilon,h}(0), v_{\varepsilon,h}(0)) \le F_\varepsilon(g_h(0), 1)$ so that the term $F_\varepsilon(u_{\varepsilon,h}(0), v_{\varepsilon,h}(0))$ is bounded as ε and h vary. We now derive an estimate for the derivative of the total energy. Since $0 \le v_{\varepsilon,h}(\tau) \le 1$ and $\eta_\varepsilon \to 0$, by Hölder inequality we get

$$\left| \int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) \nabla u_{\varepsilon,h}(\tau) \nabla \dot{g}_h(\tau)\, dx \right| \le$$
$$\le 2 \left(\int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) |\nabla u_{\varepsilon,h}(\tau)|^2\, dx \right)^{\frac{1}{2}} ||\nabla \dot{g}_h(\tau)||_{L^2(\Omega';\mathbb{R}^N)};$$

since by the minimality property (5.2)

$$\int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) |\nabla u_{\varepsilon,h}(\tau)|^2\, dx \le \int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) |\nabla g_h(\tau)|^2\, dx,$$

we get the conclusion by (5.3) and (5.4). □

As a consequence of (5.5), we have

$$\int_{\Omega'} (1 - v_{\varepsilon,h}(t)) |\nabla v_{\varepsilon,h}(t)|\, dx \le \frac{\varepsilon}{2} \int_\Omega |\nabla v_{\varepsilon,h}(t)|^2\, dx + \frac{1}{2\varepsilon} \int_\Omega (1 - v_{\varepsilon,h}(t))^2\, dx \le C_1,$$

so that the functions $w_{\varepsilon,h}(t) := (1 - v_{\varepsilon,h}(t))^2$ have uniformly bounded variation.

By coarea formula for BV-functions (see [4, Theorem 3.40]), we have that

$$\int_0^1 \mathcal{H}^{N-1} \left(\partial^* \{ v_{\varepsilon,h}(t) > s \} \right) ds = \int_{\Omega'} (1 - v_{\varepsilon,h}(t)) |\nabla v_{\varepsilon,h}(t)|\, dx$$

(∂^* denotes the essential boundary) so that by the Mean Value theorem, for all $j \ge 1$ there exists $b^j_{\varepsilon,h}(t) \in [\frac{1}{2^{j+1}}, \frac{1}{2^j}]$ with

$$\frac{1}{2^{j+1}} \mathcal{H}^{N-1} \left(\partial^* \{ v_{\varepsilon,h}(t) > b^j_{\varepsilon,h}(t) \} \right) \le C_1. \tag{5.6}$$

Let us set

$$B_{\varepsilon,h}(t) := \left\{ b^j_{\varepsilon,h}(t) \,:\, j \ge 1 \right\}. \tag{5.7}$$

We now let $\varepsilon \to 0$. Let D be countable and dense in $[0,1]$ with $0 \in D$.

Lemma 5.3. *There exists a sequence ε_n such that for all $t \in D$ there exists $u_h(t) \in SBV(\Omega')$, $u_h(t) = g_h(t)$ on Ω_D, with*

$$u_{\varepsilon_n,h}(t) 1\!\!1_{\{ v_{\varepsilon_n,h}(t) > b^1_{\varepsilon_n,h}(t) \}} \to u_h(t) \quad \text{in } SBV(\Omega').$$

In particular for all $t \in D$ we have

$$\int_{\Omega'} |\nabla u_h(t)|^2\, dx + \mathcal{H}^{N-1}(S_{u_h(t)}) + \|u_h(t)\|_\infty \le C_1. \tag{5.8}$$

Proof. For all $t \in [0,1]$ we may apply Ambrosio's compactness Theorem 2.1 to the function $z_n(t) := u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^1_{\varepsilon_n,h}(t)\}}$: in fact $z_n(t)$ is bounded in $L^\infty(\Omega')$ and $\nabla z_n(t)$ is bounded in $L^2(\Omega')$ by (5.5), and $S_{z_n(t)} \subseteq \partial_*\{v_{\varepsilon_n,h}(t) > b^1_{\varepsilon_n,h}(t)\}$ so that $\mathcal{H}^{N-1}(S_{z_n(t)})$ is uniformly bounded in n by (5.6). Using a diagonal argument, there exists a subsequence such that for all $t \in D$, $z_n(t) \to u_h(t)$ in $SBV(\Omega')$; in particular, we have that $u_h(t) = g_h(t)$ on Ω_D, and by (5.5) and the Γ-liminf inequality for the Ambrosio-Tortorelli functional (2.2), we get (5.8). □

The following lemma deals with the possibility of truncating at other levels given by the elements of $B_{\varepsilon_n,h}(t)$.

Lemma 5.4. *Let $t \in D$ and $j \geq 1$. For every $b^j_{\varepsilon_n,h}(t) \in B_{\varepsilon_n,h}(t)$ we have that*

$$u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} \to u_h(t) \quad \text{in } SBV(\Omega').$$

Proof. Note that, up to a subsequence, $u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} \to z$ in $SBV(\Omega')$ because of Ambrosio's Theorem 2.1. By (5.4), we have that

$$\|u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^1_{\varepsilon_n,h}(t)\}}\|_{L^2(\Omega')} \leq$$
$$\leq C\left|\left\{b^j_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(t) \leq b^1_{\varepsilon_n,h}(t)\right\}\right|.$$

Since $v_{\varepsilon_n,h}(t) \to 1$ strongly in $L^2(\Omega')$, we conclude that

$$\left|\left\{b^j_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(t) \leq b^1_{\varepsilon_n,h}(t)\right\}\right| \to 0,$$

so that

$$||z - u_h(t)||_{L^2(\Omega')} =$$
$$= \lim_n ||u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^1_{\varepsilon_n,h}(t)\}}||_{L^2(\Omega')} = 0,$$

that is $z = u_h(t)$ and the proof is complete. □

The following lemma deals with the possibility of truncating at time s using the function $v_{\varepsilon_n,h}(t)$ for $t \geq s$.

Lemma 5.5. *Let $s, t \in D$ with $s \leq t$, and $j \geq 1$. Then for every $b^j_{\varepsilon_n,h}(t) \in B_{\varepsilon_n,h}(t)$ we have that*

$$u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} \to u_h(s) \quad \text{in } SBV(\Omega').$$

Proof. Up to a subsequence, by Ambrosio's Theorem, we have that

$$u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} \to z \quad \text{in } SBV(\Omega').$$

Since $v_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(s)$, we have that $\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\} \subseteq \{v_{\varepsilon_n,h}(s) > b^{j+1}_{\varepsilon_n,h}(s)\}$. Then we have

$$\|u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(s)>b^{j+1}_{\varepsilon_n,h}(s)\}}\|_{L^2(\Omega')} \leq C\left|\left\{v_{\varepsilon_n,h}(t) \leq b^j_{\varepsilon_n,h}(t)\right\}\right|.$$

Since $v_{\varepsilon_n,h}(t) \to 1$ strongly in $L^2(\Omega')$, we conclude that $\left|\left\{v_{\varepsilon_n,h}(t) \leq b^j_{\varepsilon_n,h}(t)\right\}\right| \to 0$. By Lemma 5.4 we have

$$u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(s)>b^{j+1}_{\varepsilon_n,h}(s)\}} \to u_h(s) \quad \text{in } SBV(\Omega'),$$

so that $z = u_h(s)$ and the proof is complete. □

We now pass to the analysis of $u_h(t)$ with $t \in D$. The following minimality property for the functions $u_h(t)$ with $t \in D$ is crucial for the subsequent results.

Theorem 5.6. *Let $t \in D$. Then for every $z \in SBV(\Omega')$ with $z = g_h(t)$ on Ω_D, we have that*

$$\int_{\Omega'} |\nabla u_h(t)|^2 \, dx \le \int_{\Omega'} |\nabla z|^2 \, dx + \mathcal{H}^{N-1}\left(S_z \setminus \bigcup_{s \le t, s \in D} S_{u_h(s)} \right).$$

The proof is quite technical, and it is postponed to Section 6. We now let $h \to \infty$.

Proposition 5.7. *There exists $h_n \to \infty$ such that for all $t \in D$ there exists $u(t) \in SBV(\Omega')$ with $u(t) = g(t)$ on Ω_D such that $u_{h_n}(t) \to u(t)$ in $SBV(\Omega')$. Moreover, $\nabla u_{h_n}(t) \to \nabla u(t)$ strongly in $L^2(\Omega'; \mathbb{R}^N)$ and for all $z \in SBV(\Omega')$ with $z = g(t)$ on Ω_D we have*

$$\int_{\Omega'} |\nabla u(t)|^2 \, dx \le \int_{\Omega'} |\nabla z|^2 \, dx + \mathcal{H}^{N-1}\left(S_z \setminus \bigcup_{s \le t, s \in D} S_{u(s)} \right).$$

Proof. The compactness is given by Ambrosio's Theorem in view of (5.8). The strong convergence of the gradients and the minimality property is a consequence of the minimality property of Theorem 5.6 and of [14, Theorem 2.1]. □

We can now deal with ε and h at the same time.

Proposition 5.8. *There exists $\varepsilon_n \to 0$ and $h_n \to +\infty$ such that for all $t \in D$ there exists $u(t) \in SBV(\Omega')$ with $u(t) = g(t)$ on Ω_D such that for all $j \ge 1$*

$$u_{\varepsilon_n, h_n}(t) 1_{\{v_{\varepsilon_n, h_n}(t) > b^j_{\varepsilon_n, h_n}(t)\}} \to u(t) \quad \text{in } SBV(\Omega').$$

Furthermore for all $z \in SBV(\Omega')$ with $z = g(t)$ on Ω_D we have

$$\int_{\Omega'} |\nabla u(t)|^2 \, dx \le \int_{\Omega'} |\nabla z|^2 \, dx + \mathcal{H}^{N-1}\left(S_z \setminus \bigcup_{s \le t, s \in D} S_{u(s)} \right),$$

and we may suppose that the functions $\lambda_{\varepsilon_n, h_n}$ converge pointwise on $[0,1]$ to an increasing function λ such that for all $t \in D$

$$\lambda(t) \ge \mathcal{H}^{N-1}\left(\bigcup_{s \le t, s \in D} S_{u(s)} \right). \tag{5.9}$$

Finally, we have that for all $t \in D$

$$\int_{\Omega'} |\nabla u(t)|^2 \, dx + \mathcal{H}^{N-1}(S_{u(t)}) + \|u(t)\|_\infty \le C_1. \tag{5.10}$$

Proof. We find ε_n and h_n combining Lemma 5.3 and Proposition 5.7, and using a diagonal argument. Passing to the second part of the proposition, notice that the functions $\lambda_{\varepsilon_n, h_n}$ are monotone increasing. In fact if $s \le t$, since $v_{\varepsilon_n, h_n}(t) \le v_{\varepsilon_n, h_n}(s)$, and $v_{\varepsilon_n, h_n}(t) = 1$ on Ω_D, by the minimality property (5.2), we have that

$$F_{\varepsilon_n}(u_{\varepsilon_n, h_n}(s), v_{\varepsilon_n, h_n}(s)) \le F_{\varepsilon_n}(u_{\varepsilon_n, h_n}(s), v_{\varepsilon_n, h_n}(t)),$$

so that

$$\lambda_{\varepsilon_n, h_n}(t) - \lambda_{\varepsilon_n, h_n}(s) \ge$$
$$\ge \int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n, h_n}(s)^2) |\nabla u_{\varepsilon_n, h_n}(s)|^2 \, dx - \int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n, h_n}(t)^2) |\nabla u_{\varepsilon_n, h_n}(s)|^2 \, dx \ge 0.$$

Moreover by (5.5) we have $0 \le \lambda_{\varepsilon_n, h_n} \le C_1$. Applying Helly's theorem, we get that there exists an increasing function λ up to a subsequence $\lambda_{\varepsilon_n, h_n} \to \lambda$ pointwise in $[0,1]$. In order to prove (5.9), let us fix $s_1, \dots, s_m \in D \cap [0,t]$; we want to prove that

$$\lambda(t) = \lim_n \lambda_{\varepsilon_n, h_n}(t) \geq \mathcal{H}^{N-1}\left(\bigcup_{i=1}^m S_{u(s_i)} \right). \tag{5.11}$$

Then taking the sup over all possible $s_1, \dots, s_m$, we can deduce (5.9). Consider $z_n \in SBV(\Omega', \mathbb{R}^m)$ defined as

$$z_n(x) := (u_{\varepsilon_n,h_n}(s_1), \dots, u_{\varepsilon_n,h_n}(s_m)).$$

Notice that by (5.5), and the fact that $t \to v_{\varepsilon_n,h_n}(t)$ is decreasing in $L^2(\Omega')$, we obtain that there exists $C' > 0$ such that for all n

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n,h_n}(t)^2)|\nabla z_n(t)|^2\,dx + \frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla v_{\varepsilon_n,h_n}(t)|^2\,dx + \frac{1}{2\varepsilon_n}\int_{\Omega'} (1 - v_{\varepsilon_n,h_n}(t))^2\,dx \leq C'.$$

Then we may apply (2.2) obtaining (5.11). Finally (5.10) is a consequence of (5.5) and the lower semicontinuity (2.2). The proof is now concluded. □

Let us extend the evolution $\{t \to (u(t), \Gamma(t)) : t \in D\}$ of Proposition 5.8 to the entire interval $[0,1]$. Let us set for every $t \in [0,1]$

$$\Gamma(t) := \bigcup_{s \in D, s \leq t} S_{u(s)}. \tag{5.12}$$

Proposition 5.9. *For every $t \in [0,1]$ there exists $u(t) \in SBV(\Omega')$ with $u(t) = g(t)$ on Ω_D such that $\nabla u \in L^\infty([0,1], L^2(\Omega'; \mathbb{R}^N))$, ∇u is left continuous in $[0,1] \setminus D$ with respect to the strong topology, and such that, if Γ is as in* (5.12)*, the following hold:*

(a) *for all $t \in [0,1]$*

$$S_{u(t)} \subseteq \Gamma(t) \text{ up to a set of } \mathcal{H}^{N-1}-\text{measure } 0, \tag{5.13}$$

and if λ is as in Proposition 5.8

$$\lambda(t) \geq \mathcal{H}^{N-1}(\Gamma(t)); \tag{5.14}$$

(b) *for all $z \in SBV(\Omega')$ with $z = g(0)$ on Ω_D*

$$\int_{\Omega'} |\nabla u(0)|^2\,dx + \mathcal{H}^{N-1}\left(S_{u(0)}\right) \leq \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}\left(S_z\right). \tag{5.15}$$

(c) *for all $t \in]0,1]$ and for all $z \in SBV(\Omega')$ with $z = g(t)$ on Ω_D*

$$\int_{\Omega'} |\nabla u(t)|^2\,dx \leq \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}\left(S_z \setminus \Gamma(t)\right). \tag{5.16}$$

Finally,

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau \tag{5.17}$$

where

$$\mathcal{E}(t) := \int_{\Omega'} |\nabla u(t)|^2\,dx + \mathcal{H}^{N-1}(\Gamma(t)). \tag{5.18}$$

Proof. Let $t \in [0,1] \setminus D$ and let $t_n \in D$ with $t_n \nearrow t$; by (5.10) we can apply Ambrosio's Theorem obtaining $u \in SBV(\Omega')$ with $u = g(t)$ on Ω_D such that $u(t_n) \to u$ in $SBV(\Omega')$ up to subsequences. Let us set $u(t) := u$. By [14, Lemma 3.7], we have that (5.13) and (5.16) hold, and that the convergence $\nabla u(t_n) \to \nabla u$ is strong in $L^2(\Omega'; \mathbb{R}^N)$. Notice that $\nabla u(t)$ is uniquely determined by (5.13) and (5.16) since the gradient of the solutions of the minimum problem

$$\min\left\{ \int_{\Omega'} |\nabla u|^2\,dx \,:\, u = g(t) \text{ on } \Omega_D, S_u \subseteq \Gamma(t) \text{ up to a set of } \mathcal{H}^{N-1}-\text{measure } 0 \right\}$$

is unique by the strict convexity of the functional. We conclude that $\nabla u(t)$ is well defined. The argument above proves that ∇u is left continuous at all the points of $[0,1] \setminus D$. It turns out that ∇u is continuous in $[0,1]$ up to a countable set. In fact let us consider $t \in [0,1] \setminus (D \cup \mathcal{N})$ where $\mathcal{N}$ is the set of discontinuities of the function $\mathcal{H}^{N-1}(\Gamma(\cdot))$. Let $t_n \searrow t$. By Ambrosio's Theorem, we have that there exists $u \in SBV(\Omega')$ with $u = g(t)$ on Ω_D such that, up to a subsequence, $u(t_n) \to u$ in $SBV(\Omega')$. Since t is a continuity point of $\mathcal{H}^1(\Gamma(\cdot))$, we deduce that $S_u \subseteq \Gamma(t)$ up to a set of $\mathcal{H}^{N-1}$-measure 0. Moreover by [14, Lemma 3.7] we have that u satisfies the minimality property (5.16), and $\nabla u(t_n) \to \nabla u$ strongly in $L^2(\Omega';\mathbb{R}^N)$. We deduce that $\nabla u = \nabla u(t)$, and so $\nabla u(\cdot)$ is continuous in $[0,1] \setminus (D \cup \mathcal{N})$. We conclude that $\nabla u(\cdot)$ is continuous in $[0,1]$ up to a countable set, so that $\nabla u \in L^\infty([0,1]; L^2(\Omega';\mathbb{R}^N))$.

We have that (5.14) is a direct consequence of (5.9), while (5.15) is a consequence of (5.1) and the Γ-convergence result of Ambrosio and Tortorelli [5] and [6].

Finally, in order to prove (5.17), we can reason in the following way. Given $t \in [0,1]$ and $m > 0$, let $s_i^m := \frac{i}{m} t$ for all $i = 0, \dots, m$. Let us set $u^m(s) := u(s_{i+1}^m)$ for $s_i^m < s \le s_{i+1}^m$. By (5.16) we have

$$\mathcal{E}(t) \ge \mathcal{E}(0) + 2 \int_0^t \int_{\Omega'} \nabla u^m(\tau) \nabla \dot{g}(\tau) \, d\tau \, dx + o_m \tag{5.19}$$

where $o_m \to 0$ for $m \to +\infty$ because g is absolutely continuous. Since ∇u is continuous with respect to the strong topology of $L^2(\Omega';\mathbb{R}^N)$ in $[0,1]$ up to a countable set, passing to the limit for $m \to +\infty$ we deduce that (5.17) holds, and the proof is concluded. □

We are now in a position to prove our convergence result. We need the following lemma.

Lemma 5.10. *Let $\tilde{\mathcal{N}}$ be the set of discontinuity points of the function λ given by Proposition 5.8. Then for every $t \in [0,1] \setminus \tilde{\mathcal{N}}$, and $j \ge 1$ we have that*

$$\nabla u_{\varepsilon_n,h_n}(t) 1\!\!1_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} \rightharpoonup \nabla u(t) \quad \text{weakly in } L^2(\Omega';\mathbb{R}^N).$$

Proof. Let $t \in [0,1] \setminus \tilde{\mathcal{N}}$: we may suppose that $t \notin D$, since otherwise the result has already been established. Let $s \in D$ with $s < t$. We set

$$J := \inf \left\{ \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) |\nabla z|^2 \, dx \, : \, z = g_{h_n}(s) \text{ on } \Omega_D \right\},$$

and we indicate by $w_n(s,t)$ the minimum point of this problem. Notice that $u_{\varepsilon_n,h_n}(t) - w_n(s,t)$ is the minimum for

$$K := \inf \left\{ \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) |\nabla z|^2 \, dx \, : \, z = g_{h_n}(t) - g_{h_n}(s) \text{ on } \Omega_D \right\}.$$

Comparing $u_{\varepsilon_n,h_n}(t) - w_n(s,t)$ with $g_{h_n}(t) - g_{h_n}(s)$, we have

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) |\nabla u_{\varepsilon_n,h_n}(t) - \nabla w_n(s,t)|^2 \, dx \le \\ \le \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) |\nabla g_{h_n}(t) - \nabla g_{h_n}(s)|^2 \, dx. \tag{5.20}$$

Since $u_{\varepsilon_n,h_n}(s) - w_n(s,t)$ is a good test for J, we have

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) \nabla w_n(s,t) (\nabla u_{\varepsilon_n,h_n}(s) - \nabla w_n(s,t)) \, dx = 0,$$

and so the following equality holds

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))(|\nabla u_{\varepsilon_n,h_n}(s)|^2 - |\nabla w_n(s,t)|^2)\, dx = \\ = \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))(|\nabla u_{\varepsilon_n,h_n}(s) - \nabla w_n(s,t)|^2)\, dx.$$

Since $v_{\varepsilon_n,h_n}(t) \le v_{\varepsilon_n,h_n}(s)$ and by minimality of $u_{\varepsilon_n,h_n}(s)$ we have

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))|\nabla u_{\varepsilon_n,h_n}(s)|^2\, dx + \lambda_{\varepsilon_n,h_n}(s) \le \\ \le \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(s))|\nabla u_{\varepsilon_n,h_n}(s)|^2\, dx + \lambda_{\varepsilon_n,h_n}(s) \le \\ \le \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))|\nabla w_n(s,t)|^2\, dx + \lambda_{\varepsilon_n,h_n}(t).$$

so that

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))(|\nabla u_{\varepsilon_n,h_n}(s) - \nabla w_n(s,t)|^2)\, dx = \\ = \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))(|\nabla u_{\varepsilon_n,h_n}(s)|^2 - |\nabla w_n(s,t)|^2)\, dx \le \\ \le \lambda_{\varepsilon_n,h_n}(t) - \lambda_{\varepsilon_n,h_n}(s). \tag{5.21}$$

By (5.20) and (5.21), we conclude that there exists $C' > 0$ with

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))(|\nabla u_{\varepsilon_n,h_n}(t) - \nabla u_{\varepsilon_n,h_n}(s)|^2)\, dx \le \\ \le C'\|\nabla g_{h_n}(t) - \nabla g_{h_n}(s)\| + (\lambda_{\varepsilon_n,h_n}(t) - \lambda_{\varepsilon_n,h_n}(s)). \tag{5.22}$$

Then we conclude that for $b^j_{\varepsilon_n,h_n}(t) \in B_{\varepsilon_n,h_n}(t)$

$$||\nabla u_{\varepsilon_n,h_n}(t)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} - \nabla u_{\varepsilon_n,h_n}(s)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}}||_{L^2(\Omega';\mathbb{R}^N)} \le \\ \le o(t-s) \tag{5.23}$$

since $\lambda_{\varepsilon_n,h_n} \to \lambda$ pointwise, and t is a continuity point for λ. Recall that by Lemma 5.5

$$\nabla u_{\varepsilon_n,h_n}(s)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} \rightharpoonup \nabla u(s) \quad \text{weakly in } L^2(\Omega';\mathbb{R}^N).$$

Since

$$\nabla u_{\varepsilon_n,h_n}(t)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} - \nabla u(t) = \\ = (\nabla u_{\varepsilon_n,h_n}(t)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} - \nabla u_{\varepsilon_n,h_n}(s)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}}) + \\ + (\nabla u_{\varepsilon_n,h_n}(s)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} - \nabla u(s)) + (\nabla u(s) - \nabla u(t)),$$

by (5.23) and the left continuity of $\{\tau \to \nabla u(\tau)\}$ at the points of $[0,1] \setminus D$, we have that

$$\nabla u_{\varepsilon_n,h_n}(t)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} \rightharpoonup \nabla u(t) \quad \text{weakly in } L^2(\Omega';\mathbb{R}^N),$$

so that the lemma is proved. □

We are now in a position to prove the main theorem of the paper.

Proof of Theorem 3.2. By Proposition 5.1, we may extend $(u_{\varepsilon,h}(t), v_{\varepsilon,h}(t))$ to Ω' setting $u_{\varepsilon,h}(t) = g_h(t)$ and $v_{\varepsilon,h}(t) = 1$ on Ω_D, obtaining a quasistatic evolution in Ω'. In this context, the points of $\partial_D\Omega$ where the boundary condition is violated in the limit simply become discontinuity points of the extended function. Thus we prove the result in this equivalent setting involving Ω'.

Let $\varepsilon_n \to 0$ and $h_n \to +\infty$ be the sequences determined by Proposition 5.8. Let us indicate $u_{\varepsilon_n,h_n}(t), v_{\varepsilon_n,h_n}(t)$ and F_{ε_n} by $u_n(t), v_n(t)$ and F_n. Moreover, let us write $B_n(t)$ and $b_n^j(t)$ for $B_{\varepsilon_n,h_n}(t)$ and $b^j_{\varepsilon_n,h_n}(t)$. Let $\{t \to (u(t),\Gamma(t)) \in SBV(\Omega')\,, t\in[0,1]\}$ be the evolution relative to the boundary data g given by Proposition 5.9; up to a subsequence, we have that $u_n(t)1\!\!1_{\{v_n(t)>b_n^j(t)\}} \to u(t)$ in $SBV(\Omega')$ for all $j \geq 1$ and for all t in a countable and dense subset $D \subseteq [0,1]$ with $0 \in D$. Moreover for all $t \in [0,1]$ we have that

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau, \tag{5.24}$$

where $\mathcal{E}(t) := \int_{\Omega'} |\nabla u(t)|^2\,dx + \mathcal{H}^{N-1}(\Gamma(t))$ and $\Gamma(t)$ is as in (5.12).

By point (b) of Proposition 5.1 and the Ambrosio-Tortorelli Theorem 2.3 we have

$$\lim_n F_n(u_n(0), v_n(0)) = \mathcal{E}(0). \tag{5.25}$$

For $m \geq 1$, notice that

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\nabla \dot{g}_{h_n}(\tau)\,dx = \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)1\!\!1_{\{v_n(\tau)>b_n^m(\tau)\}}\nabla \dot{g}_{h_n}(\tau)\,dx +$$
$$+ \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)1\!\!1_{\{v_n(\tau)\leq b_n^m(\tau)\}}\nabla \dot{g}_{h_n}(\tau)\,dx.$$

If $\tau \in [0,1]$, we have the estimate

$$\left|\int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)1\!\!1_{\{v_n(\tau)\leq b_k^m(\tau)\}}\nabla \dot{g}_{h_n}(\tau)\,dx\right| \leq$$
$$\leq \sqrt{\eta_{\varepsilon_n} + \frac{1}{2^{2m}}}\left(\int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))|\nabla u_n(\tau)|^2\,dx\right)^{\frac{1}{2}} ||\nabla \dot{g}_{h_n}(\tau)||_{L^2(\Omega';\mathbb{R}^N)} \leq$$
$$\leq \sqrt{\eta_{\varepsilon_n} + \frac{1}{2^{2m}}}C \to \frac{C}{2^m}.$$

Moreover, by Lemma 5.10 we have that for a.e. $\tau \in [0,1]$

$$\lim_n \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)1\!\!1_{\{v_n(\tau)>b_n^m(\tau)\}}\nabla \dot{g}_{h_n}(\tau)\,dx = \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx,$$

and we deduce that for such τ

$$\limsup_n \left|\int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\nabla \dot{g}_{h_n}(\tau)\,dx - \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\right| \leq \frac{C}{2^m}.$$

Since m is arbitrary, we have that for a.e. $\tau \in [0,1]$

$$\lim_n \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\nabla \dot{g}_{h_n}(\tau)\,dx = \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx. \tag{5.26}$$

By (5.3), (5.25), (5.26) and the Dominated Convergence Theorem, we conclude that for all $t \in [0,1]$

$$\lim_n F_n(u_n(t), v_n(t)) = \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau. \tag{5.27}$$

Since by Proposition 5.8 we have $\liminf_n F_n(u_n(t), v_n(t)) \geq \mathcal{E}(t)$ for all $t \in D$, by (5.24) we have for all $t \in D$

$$\lim_n F_n(u_n(t), v_n(t)) = \mathcal{E}(t).$$

In particular we get for all $t \in D$

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau, \tag{5.28}$$

and since by Proposition 5.9 $\nabla u(\cdot)$ and $\mathcal{H}^{N-1}(\Gamma(\cdot))$ are left continuous at $t \notin D$ and so $\mathcal{E}(\cdot)$ is, we conclude that the equality holds for all $t \in [0,T]$. Recalling all the properties stated in Proposition 5.9, we deduce that $\{t \to (u(t), \Gamma(t)) : t \in [0,1]\}$ is a quasistatic evolution relative to the boundary data g. In order to prove point (a), it is sufficient to see that $\liminf_n F_n(u_n(t), v_n(t)) \geq \mathcal{E}(t)$ holds for all $t \in [0,1]$. Considering $s \geq t$ with $s \in D$, we have

$$F_n(u_n(s), v_n(s)) = F_n(u_n(t), v_n(t)) + 2\int_s^t \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\nabla \dot{g}_{h_n}(\tau)\, dx$$

so that

$$\liminf_n F_n(u_n(t), v_n(t)) \geq \mathcal{E}(s) - 2\int_s^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau.$$

Letting $s \searrow t$, since $\mathcal{E}(\cdot)$ is continuous and by (5.28), we obtain $\liminf_n F_n(u_n(t), v_n(t)) \geq \mathcal{E}(t)$, and so point (a) is now completely proved.

Let us come to point (b). By Lemma 5.10, we know that if $\tilde{\mathcal{N}}$ is the set of discontinuity points of λ, for all $t \in [0,1] \setminus \tilde{\mathcal{N}}$ and for all $j \geq 1$ we have $\nabla u_n(t)\mathbb{1}_{\{v_n(t) > b_k^j(t)\}} \rightharpoonup \nabla u(t)$ weakly in $L^2(\Omega', \mathbb{R}^N)$. Since

$$v_n(t)\nabla u_n(t) = v_n(t)\nabla u_n(t)\mathbb{1}_{\{v_n(t) > b_n^j(t)\}} + v_n(t)\nabla u_n(t)\mathbb{1}_{\{v_n(t) < b_n^j(t)\}},$$

we get immediately that $v_n(t)\nabla u_n(t) \rightharpoonup \nabla u(t)$ weakly in $L^2(\Omega', \mathbb{R}^N)$. For all such t, we have that

$$\liminf_n \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(t))|\nabla u_n(t)|^2\, dx \geq \int_{\Omega'} |\nabla u(t)|^2\, dx,$$

and by (5.14)

$$\liminf_n \frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla v_n(t)|^2\, dx + \frac{1}{2\varepsilon_n}\int_{\Omega'} (1 - v_n(t))^2\, dx \geq \mathcal{H}^{N-1}(\Gamma(t)).$$

By point (a), we have that the two preceding inequalities are equalities. In particular, λ and $\mathcal{H}^{N-1}(\Gamma(\cdot))$ coincide up to a countable set in $[0,1]$. We deduce that λ and $\mathcal{H}^{N-1}(\Gamma(\cdot))$ have the same continuity points, that is $\tilde{\mathcal{N}} = \mathcal{N}$. We conclude that for all $t \in [0,1] \setminus \mathcal{N}$ we have $v_n(t)\nabla u_n(t) \to \nabla u(t)$ strongly in $L^2(\Omega', \mathbb{R}^N)$,

$$\lim_n \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(t))|\nabla u_n(t)|^2\, dx = \int_{\Omega'} |\nabla u(t)|^2\, dx,$$

and

$$\lim_n \frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla v_n(t)|^2\, dx + \frac{1}{2\varepsilon_n}\int_{\Omega'} (1 - v_n(t))^2\, dx = \mathcal{H}^{N-1}(\Gamma(t)),$$

so that point (b) is proved, and the proof of the theorem is complete. □

6. Proof of Theorem 5.6

In this section we give the proof of Theorem 5.6 which is an essential step in the analysis of Section 5. For simplicity of notation, for all $t \in D$ we write $u(t)$, $u_n(t)$ and $v_n(t)$ for $u_h(t)$, $u_{\varepsilon_n,h}(t)$ and $v_{\varepsilon_n,h}(t)$ respectively. Moreover, let us write $B_n(t)$, $b_n^j(t)$ for $B_{\varepsilon_n,h}(t)$ and $b^j_{\varepsilon_n,h}(t)$, where $B_{\varepsilon_n,h}(t)$ is defined as in (5.7).

Given $z \in SBV(\Omega')$ with $z = g_h(t)$ on Ω_D, we want to see that

$$\int_{\Omega'} |\nabla u(t)|^2\, dx \leq \int_{\Omega'} |\nabla z|^2\, dx + \mathcal{H}^{N-1}\left(S_z \setminus \Gamma(t)\right), \tag{6.1}$$

where $g_h(t) \in H^1(\Omega') \cap C(\overline{\Omega'})$ and $\Gamma(t) = \bigcup_{s \leq t, s \in D} S_{u(s)}$.

The plan is to use the minimality property (5.2) of the approximating evolution, so that the main point is to construct a sequence $(z_n, v_n) \in H^1(\Omega') \times H^1(\Omega')$ such that $z_n = g_h(t)$, $v_n = 1$ on Ω_D, $0 \leq v_n \leq v_n(t)$, and such that

$$\lim_n \int_{\Omega'} (\eta_n + v_n^2)|\nabla z_n|^2\, dx = \int_{\Omega'} |\nabla z|^2\, dx$$

and

$$\limsup_n \left[MM_n(v_n) - MM_n(v_n(t))\right] \leq \mathcal{H}^{N-1}\left(S_z \setminus \Gamma(t)\right),$$

where we use the notation

$$MM_n(w) := \frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla w|^2\, dx + \frac{1}{2\varepsilon_n}\int_{\Omega'} (1-w)^2\, dx.$$

If a sequence with these properties exists, then by property (5.2) we get the result. The following lemma contains the main ideas in order to prove Theorem 5.6.

Lemma 6.1. *Let $t \in D$; given $z \in SBV(\Omega')$ with $z = g_h(t)$ on Ω_D we have that*

$$\int_{\Omega'} |\nabla u(t)|^2\, dx \leq \int_{\Omega'} |\nabla z|^2\, dx + \mathcal{H}^{N-1}\left(S_z \setminus S_{u(t)}\right). \tag{6.2}$$

In order to prove Lemma 6.1, we need several preliminary results. Let $z \in SBV(\Omega')$ be such that $z = g_h(t)$ on Ω_D. Given $\sigma > 0$, let U be a neighborhood of $S_{u(t)}$ such that $|U| \leq \sigma$, and $||\nabla z||_{L^2(U;\mathbb{R}^N)} \leq \sigma$. Let $C := \{x \in \partial_D\Omega : \partial_D\Omega \text{ is not differentiable at } x\}$. We recall that there exists a countable and dense set $A \subseteq \mathbb{R}$ such that up to a set of $\mathcal{H}^{N-1}$-measure zero

$$S_{u(t)} = \bigcup_{a,b\in A} \partial^* E_a \cap \partial^* E_b$$

where $E_a := \{x \in \Omega' : u(t)(x) \geq a\}$ and ∂^* denotes the essential boundary. Consider

$$J_j := \left\{x \in S_{u(t)} \setminus C : [u(t)(x)] \geq \frac{1}{j}\right\},$$

with j chosen in such a way that $\mathcal{H}^{N-1}(S_{u(t)} \setminus J_j) \leq \sigma$. For $x \in J_j$, let $a_1(x), a_2(x) \in A$ be such that $u^-(t)(x) < a_1(x) < a_2(x) < u^+(t)(x)$ and $a_2(x) - a_1(x) \geq \frac{1}{2j}$. Following [14, Theorem 2.1], we consider a finite disjoint collection of closed cubes $\{Q_i\}_{i=1,\dots,k}$ with center $x_i \in J_j$, radius r_i and with normal $\nu(x_i)$ such that $\bigcup_{i=1}^k Q_i \subseteq U$, $\mathcal{H}^{N-1}(J_j \setminus \bigcup_{i=1}^k Q_i) \leq \sigma$, and for all $i = 1,\dots,k$, $j = 1,2$

1. $\mathcal{H}^{N-1}\left(S_{u(t)} \cap \partial Q_i\right) = 0$;
2. $r_i^{N-1} \leq 2\mathcal{H}^{N-1}\left(S_{u(t)} \cap Q_i\right)$;
3. $\mathcal{H}^{N-1}\left(\left[S_{u(t)} \setminus \partial^* E_{a_j(x_i)}\right] \cap Q_i\right) \leq \sigma r_i^{N-1}$;
4. $\mathcal{H}^{N-1}\left(\left\{y \in \partial^* E_{a_j(x_i)} \cap Q_i : \operatorname{dist}(y, H_i) \geq \frac{\sigma}{2} r_i\right\}\right) < \sigma r_i^{N-1}$ where H_i denotes the intersection of Q_i with the hyperplane through x_i orthogonal to $\nu(x_i)$;
5. $\mathcal{H}^{N-1}\left(\left(S_z \setminus S_{u(t)}\right) \cap Q_i\right) < \sigma r_i^{N-1}$ and $\mathcal{H}^{N-1}(S_z \cap \partial Q_i) = 0$.

Note that we may suppose that $Q_i \subseteq \Omega$ if $x_i \in \Omega$. Moreover we may require that (see [14, Theorem 2.1] and references therein) for all $i = 1,\dots,k$ and $j = 1,2$

$$\|1_{E_{a_j(x_i)} \cap Q_i} - 1_{Q_i^+}\|_{L^1(\Omega')} \leq \sigma^2 r_i^N. \tag{6.3}$$

Let us indicate by R_i the rectangle given by the intersection of Q_i with the strip centered at H_i with width $2\sigma r_i$, and let us set $V_i := \{y + s\nu(x_i) : y \in \partial Q_i, s \in \mathbb{R}\} \cap R_i$. Note that up to changing the strip, we can suppose $\mathcal{H}^{N-1}(\partial R_i \cap (S_u \cup S_z)) = \emptyset$.

If $x_i \in \partial_D\Omega$, since $x_i \notin C$, we may require that

$$\partial\Omega \cap Q_i \subseteq \{x \,:\, |(x - x_i)\cdot \nu(x_i)| < \sigma r_i\}; \tag{6.4}$$

moreover, if $(Q_i^+ \setminus R_i) \subseteq \Omega$, we can assume that $g_h(t) < a_1(x_i)$ on $\partial\Omega\cap Q_i$ because $g_h(t)$ is continuous and $g_h(t)(x_i) = u^-(x_i) < a_1(x_i)$. Similarly we may require that $g_h(t) > a_2(x_i)$ on $\partial\Omega \cap Q_i$ in the case $(Q_i^- \setminus R_i) \subseteq \Omega$.

Since we can reason up to subsequences of ε_n, we may suppose that $\sum_n \varepsilon_n \le \frac{1}{8}$. Since by (5.5) we have that $||u_n(t)||_\infty < C_1$ and $v_n(t) \to 1$ strongly in $L^2(\Omega')$, by Lemma 5.4 we deduce that $u_n(t) \to u(t)$ in measure. By (6.3), we deduce that for n large enough

$$|Q_i^+ \setminus E^n_{a_2(x_i)}| \le 2\sigma^2 r_i^N, \tag{6.5}$$

where we use the notation $E^n_a := \{x \in \Omega' \,:\, u_n(t)(x) \ge a\}$. Let $G_n \subseteq]\frac{\sigma}{4}r_i, \frac{\sigma}{2}r_i[$ be the set of all s such that

$$\int_{H_i(s)} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, d\mathcal{H}^{N-1} \ge \frac{C_1}{\sigma r_i \varepsilon_n};$$

we get immediately by (5.5) that

$$|G_n| \le \sigma r_i \varepsilon_n,$$

so that, setting $G := \bigcup_n G_n$, we have $|G| \le \frac{\sigma}{8}r_i$ and $|]\frac{\sigma}{4}r_i, \frac{\sigma}{2}r_i[\setminus G| \ge \frac{\sigma}{8}r_i$. From (6.5), applying Fubini's Theorem we obtain

$$\int_{]\frac{\sigma}{4}r_i, \frac{\sigma}{2}r_i[\setminus G} \mathcal{H}^{N-1}\left(H_i(s) \setminus E^n_{a_2(x_i)}\right) ds \le 2\sigma^2 r_i^N,$$

so that there exists $\overline{s} \in]\frac{\sigma}{4}r_i, \frac{\sigma}{2}r_i[\setminus G$ such that, setting $H_i^+ := H_i(\overline{s})$, we have

$$\mathcal{H}^{N-1}\left(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}}\right) \le 16\sigma r_i^{N-1}. \tag{6.6}$$

Moreover we have by construction

$$\int_{H_i^+} (\eta_n + v_n^2(t))|\nabla u_n|^2 \, d\mathcal{H}^{N-1} \le K_n, \tag{6.7}$$

where K_n is of the order of $\frac{1}{\varepsilon_n}$. In a similar way, there exists $H_i^- := H_i(\tilde{s})$ with $\tilde{s} \in]-\frac{\sigma}{2}r_i, -\frac{\sigma}{4}r_i[$ and

$$\mathcal{H}^{N-1}\left(H_i^- \cap E^n_{a_1(x_i)+\frac{\delta}{2}}\right) \le 16\sigma r_i^{N-1}, \tag{6.8}$$

and

$$\int_{H_i^-} (\eta_n + v_n^2(t))|\nabla u_n|^2 \, d\mathcal{H}^{N-1} \le K_n \tag{6.9}$$

where K_n is of the order of $\frac{1}{\varepsilon_n}$. We indicate by $\tilde{R}_i$ the intersection of Q_i with the strip determined by H_i^+ and H_i^-.

A similar argument prove that, up to reducing Q_i (preserving the estimates previously stated), we may suppose that

$$\int_{V_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, d\mathcal{H}^{N-1} \le K_n, \tag{6.10}$$

where K_n is of the order of $\frac{1}{\varepsilon_n}$.

In order to prove Lemma 6.1, we claim that we can suppose $z = g_h(t)$ on Ω_D and in a neighborhood $\mathcal{V}$ of $\partial_D\Omega \setminus \bigcup_{i=1}^k Q_i$, $S_z \setminus \bigcup_{i=1}^k R_i$ polyhedral with closure contained in Ω, and $\mathcal{H}^{N-1}((S_z \setminus S_{u(t)}) \cap Q_i) \le \sigma r_i^{N-1}$ for all $i = 1, \dots, k$. In fact, by Proposition 2.5, there exists $w_m \in SBV(\Omega')$ with $w_m = g_h(t)$ in $\Omega' \setminus \overline{\Omega}$ and in a neighborhood $\mathcal{V}_m$ of $\partial_D\Omega$ such that $w_m \to z$ strongly in $L^2(\Omega')$, $\nabla w_m \to \nabla z$ strongly in $L^2(\Omega';\mathbb{R}^N)$, $\overline{S_{w_m}} \subseteq \Omega$ polyhedral, and such that for all A open subset of Ω' with $\mathcal{H}^{N-1}(\partial A \cap S_z) = 0$, we have

$$\lim_m \mathcal{H}^{N-1}(A \cap S_{w_m}) = \mathcal{H}^{N-1}(A \cap S_z).$$

Let us fix $\sigma' > 0$ and let us consider for all $i = 1, \dots, k$ a rectangle R_i' centered in x_i, oriented as R_i and such that $R_i' \subseteq \mathrm{int}(R_i)$, $\mathcal{H}^{N-1}(\partial R_i' \cap S_z) = 0$, $\mathcal{H}^{N-1}(S_z \cap (\mathrm{int}(R_i) \setminus R_i')) \leq \sigma' r_i^{N-1}$, where $\mathrm{int}(R_i)$ denotes the interior part of R_i. Let ψ_i be a smooth function such that $0 \leq \psi_i \leq 1$, $\psi_i = 1$ on R_i' and $\psi_i = 0$ outside R_i. Setting $\psi := \sum_{i=1}^k \psi_i$, let us consider $z_m := \psi z + (1-\psi) w_m$. Note that $z_m \to z$ strongly in $L^2(\Omega')$, $\nabla z_m \to \nabla z$ strongly in $L^2(\Omega';\mathbb{R}^N)$, $z_m = g_h(t)$ in Ω_D and in a neighborhood $\mathcal{V}_m'$ of $\partial_D \Omega \setminus \bigcup_{i=1}^k R_i$, $S_{w_m} \setminus \bigcup_{i=1}^k R_i$ is polyhedral with closure contained in Ω. Finally, for $m \to +\infty$, we have $\mathcal{H}^{N-1}(S_{z_m} \setminus \bigcup_{i=1}^k Q_i) \to \mathcal{H}^{N-1}(S_z \setminus \bigcup_{i=1}^k Q_i))$ and $\limsup_m \mathcal{H}^{N-1}(S_{z_m} \cap (\mathrm{int}(R_i) \setminus R_i')) \leq 2\mathcal{H}^{N-1}(S_z \cap (\mathrm{int}(R_i) \setminus R_i')) \leq 2\sigma' r_i^{N-1}$. So, if (6.2) holds for z_m, we obtain for $m \to +\infty$ that (6.2) holds also for z since σ' is arbitrary, and so the claim is proved.

We begin with the following lemma.

Lemma 6.2. *Let $B_n(t)$ be as in (5.7), and let us consider $b_n^2 := b_n^{j_2}(t), b_n^3 := b_n^{j_3}(t) \in B_n(t)$ with $j_2 > j_3 > 1$. Suppose that $k_n := \frac{b_n^3}{b_n^2} > 1$ and let k, b be such that $1 < k \leq k_n$, $b_n^3 \leq b$ for all n. Then setting*

$$w_n := \begin{cases} \frac{k_n}{k_n-1}(v_n(t) - b_n^3) + b_n^3 & \text{in } \{b_n^2 \leq v_n(t) \leq b_n^3\} \\ 0 & \text{in } \{v_n(t) \leq b_n^2\} \\ v_n(t) & \text{in } \{v_n(t) \geq b_n^3\} \end{cases}$$

we have that $w_n \in H^1(\Omega')$ with $w_n = 1$ on Ω_D, $0 \leq w_n \leq v_n(t)$ in Ω' and

$$\limsup_n \left(MM_n(w_n) - MM_n(v_n(t))\right) \leq \frac{2C_1 k}{(k-1)^2} + \frac{C_1}{(k-1)(1-b)^2} + \frac{C_1 b}{(1-b)^2}, \tag{6.11}$$

where C_1 is given by (5.5). Moreover there exist $b_n^1 := b_n^{j_1}(t) \in B_n(t)$ with $j_1 > j_2 + 1$ and a cut-off function $\varphi_n \in H^1(\Omega')$ with $\varphi_n = 0$ in $\{v_n(t) \leq b_n^1\}$, $\varphi_n = 1$ on $\{v_n(t) \geq b_n^2\}$ (in particular on Ω_D) and such that

$$\lim_n \eta_n \int_{\Omega'} |\nabla \varphi_n|^2 \, dx = 0 \tag{6.12}$$

Proof. w_n is well defined in $H^1(\Omega')$, and by construction $w_n = 1$ on Ω_D and $0 \leq w_n \leq v_n(t)$ in Ω'. Let us estimate $MM_n(w_n) - MM_n(v_n)$. Since

$$\frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla w_n|^2 \, dx = \frac{\varepsilon_n}{2}\int_{\{v_n(t) \geq b_n^3\}} |\nabla v_n(t)|^2 \, dx + \frac{\varepsilon_n}{2}\int_{\{b_n^2 \leq v_n(t) \leq b_n^3\}} |\nabla w_n|^2 \, dx,$$

and $MM_n(v_n(t)) \leq C_1$ by (5.5), we have that

$$\begin{aligned} & \frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla w_n|^2 \, dx - \frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla v_n(t)|^2 \, dx \leq \\ \leq \; & \frac{\varepsilon_n}{2}\int_{\{b_n^2 \leq v_n(t) \leq b_n^3\}} \left(\frac{k_n^2}{(k_n-1)^2} - 1\right) |\nabla v_n(t)|^2 \, dx - \frac{\varepsilon_n}{2}\int_{\{v_n(t) \leq b_n^2\}} |\nabla v_n(t)|^2 \, dx \leq \\ \leq \; & C_1 \left(\frac{k_n^2}{(k_n-1)^2} - 1\right) = \frac{C_1(2k_n - 1)}{(k_n-1)^2} \leq \frac{2C_1 k}{(k-1)^2}. \end{aligned}$$

Moreover we have that

$$\begin{aligned}
&\frac{1}{2\varepsilon_n}\int_{\Omega'}(1-w_n)^2\,dx-\frac{1}{2\varepsilon_n}\int_{\Omega'}(1-v_n(t))^2\,dx=\\
&= \frac{1}{2\varepsilon_n}\int_{\Omega'}\left[(1-w_n)^2-(1-v_n(t))^2\right]\,dx=\\
&= \frac{1}{2\varepsilon_n}\int_{\Omega'}(v_n(t)-w_n)(2-v_n(t)-w_n)\,dx=\\
&= \frac{1}{2\varepsilon_n}\int_{\{b_n^2\le v_n(t)\le b_n^3\}}\left(v_n(t)-\frac{k_n}{k_n-1}(v_n(t)-b_n^3)-b_n^3\right)(2-v_n(t)-w_n)\,dx+\\
&\quad+\frac{1}{2\varepsilon_n}\int_{\{v_n(t)\le b_n^2\}}v_n(t)(2-v_n(t))\,dx=\\
&= \frac{1}{2\varepsilon_n}\int_{\{b_n^2\le v_n(t)\le b_n^3\}}\frac{1}{k_n-1}(b_n^3-v_n(t))(2-v_n(t)-w_n)\,dx+\\
&\quad+\frac{1}{2\varepsilon_n}\int_{\{v_n(t)\le b_n^2\}}v_n(t)(2-v_n(t))\,dx\le\\
&\le \frac{C_1}{(k_n-1)(1-b_n^3)^2}+\frac{C_1b_n^2}{(1-b_n^2)^2}\le\frac{C_1}{(k-1)(1-b)^2}+\frac{C_1b}{(1-b)^2}
\end{aligned}$$

because $\frac{|\{v_n(t)\le s\}|}{\varepsilon_n}\le\frac{C_1}{(1-s)^2}$. We conclude that

$$\limsup_n\,(MM_n(w_n)-MM_n(v_n(t)))\le\frac{2C_1k}{(k-1)^2}+\frac{C_1}{(k-1)(1-b)^2}+\frac{C_1b}{(1-b)^2}.$$

Let $j_1>j_2+1$: we have that $b_n^1:=b_n^{j_1}$ and b_n^2 are not in adjacent intervals, and so there exists $l>0$ with $0<l\le b_n^2-b_n^1$. Let us divide the interval $[b_n^1,b_n^2]$ in h_n intervals of the same size $I_j, j=1,\dots,h_n$, with h_n such that $\frac{\eta_n}{\varepsilon_n}h_n\to 0$. Since

$$\sum_{j=1}^{h_n}\frac{\varepsilon_n}{2}\int_{\{v_n(t)\in I_j\}}|\nabla v_n(t)|^2\,dx\le\frac{\varepsilon_n}{2}\int_{\Omega'}|\nabla v_n(t)|^2\,dx\le C_1,$$

we deduce that there exists I_n such that

$$\frac{\varepsilon_n}{2}\int_{\{v_n(t)\in I_n\}}|\nabla v_n(t)|^2\,dx\le\frac{C_1}{h_n}. \tag{6.13}$$

Let α_n,β_n be the extremes of I_n. Let us set

$$\varphi_n:=\frac{1}{\beta_n-\alpha_n}(v_n-\alpha_n)^+\wedge 1. \tag{6.14}$$

Then $\varphi_n\in H^1(\Omega)$, $\varphi_n=0$ in $\{v_n(t)\le b_n^1\}$, $\varphi_n=1$ on $\{v_n(t)\ge b_n^2\}$ (in particular on Ω_D) and by (6.13) and the choice of h_n we have that

$$\eta_n\int_{\Omega'}|\nabla\varphi_n|^2\,dx = \eta_n\int_{\{\alpha_n\le v_n(t)\le\beta_n\}}\frac{1}{(\beta_n-\alpha_n)^2}|\nabla v_n(t)|^2\,dx \le \frac{\eta_n}{\varepsilon_n}\frac{2C_1}{h_n}\frac{h_n^2}{l^2}\to 0,$$

so that the proof is complete. □

In the following lemmas, we will use the following notation: for all measurable set $B\subseteq\Omega'$ we set

$$MM_n(w,B):=\frac{\varepsilon_n}{2}\int_B|\nabla w|^2\,dx+\frac{1}{2\varepsilon_n}\int_B(1-w)^2\,dx. \tag{6.15}$$

Let b_n^1 be as in Lemma 6.2 and let $\delta:=\frac{1}{8j}$ so that for all $i=1,\dots,k$

$$a_1(x_i)<a_1(x_i)+\delta<a_2(x_i)-\delta<a_2(x_i).$$

Lemma 6.3. *For each $i = 1, \dots, k$, there exists $w_n^{2,i} \in H^1(Q_i)$ and $[\gamma_n^i - \tau_n^i, \gamma_n^i + \tau_n^i] \subseteq [a_1(x_i) + \delta, a_2(x_i) - \delta]$ such that $0 \le w_n^{2,i} \le 1$, $w_n^{2,i} = 0$ in $\{\gamma_n^i - \tau_n^i \le u_n(t) \le \gamma_n^i + \tau_n^i\} \cap Q_i$, $w_n^{2,i} = 1$ on $\left[\left\{u_n(t) \le a_1(x_i) + \frac{3}{4}\delta\right\} \cup \left\{u_n(t) \ge a_2(x_i) - \frac{3}{4}\delta\right\}\right] \cap Q_i$, and*

$$\limsup_n \sum_{i=1}^k MM_n(w_n^{2,i}, \{v_n(t) > b_n^1\}) \le o(\sigma). \tag{6.16}$$

Moreover there exists $\varphi_n^{2,i} \in H^1(Q_i)$ such that $0 \le \varphi_n^{2,i} \le 1$, $\varphi_n^{2,i} = 0$ on $\{\gamma_n^i - \frac{\tau_n^i}{2} \le u_n(t) \le \gamma_n^i + \frac{\tau_n^i}{2}\} \cap Q_i$, $\varphi_n^{2,i} = 1$ on $\left[\{u_n(t) \le \gamma_n^i - \tau_n^i\} \cup \{u_n(t) \ge \gamma_n^i + \tau_n^i\}\right] \cap Q_i$, and

$$\lim_n \eta_n \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{2,i}|^2 \, dx = 0. \tag{6.17}$$

Proof. For each i let us consider the strip

$$S_n^i := E^n_{a_1(x_i)+\delta} \setminus E^n_{a_2(x_i)-\delta}.$$

Let $h_n \in \mathbb{N}$ and let us divide $[a_1(x_i)+\delta, a_2(x_i)-\delta]$ in h_n intervals of the same size: there exists a subinterval with extremes α_n^i and β_n^i such that, setting $\tilde{S}_n^i := \{x \in \Omega' : \alpha_n^i \le u_n(t) \le \beta_n^i\}$,

$$\begin{aligned}\int_{\tilde{S}_n^i \cap Q_i} &\left[\sigma(\eta_n + v_n^2(t))|\nabla u_n(t)|^2 + (1-\sigma)\right] dx \le \\ &\le \frac{1}{h_n} \int_{S_n^i \cap Q_i} \left[\sigma(\eta_n + v_n^2(t))|\nabla u_n(t)|^2 + (1-\sigma)\right] dx.\end{aligned} \tag{6.18}$$

Let $\gamma_n^i := \frac{\alpha_n^i + \beta_n^i}{2}$ and $\tau_n^i := \frac{a_2(x_i) - a_1(x_i) - 2\delta}{4h_n}$. We set

$$w_n^{2,i} := \begin{cases} \frac{1}{\beta_n^i - \gamma_n^i - \tau_n^i}(u_n(t) - \gamma_n^i - \tau_n^i)^+ \wedge 1 & \text{in } \{u_n(t) \ge \gamma_n^i + \tau_n^i\} \cap Q_i \\ 0 & \text{in } \{\gamma_n^i - \tau_n^i \le u_n(t) \le \gamma_n^i + \tau_n^i\} \cap Q_i \\ \frac{1}{\gamma_n^i - \tau_n^i - \alpha_n^i}(u_n(t) - \gamma_n^i + \tau_n^i)^- \wedge 1 & \text{in } \{u_n(t) \le \gamma_n^i - \tau_n^i\} \cap Q_i. \end{cases}$$

We have that

$$\begin{aligned}\frac{\varepsilon_n}{2} \int_{Q_i \cap \{v_n(t) > b_n^1\}} &|\nabla w_n^{2,i}|^2 \, dx + \frac{1}{2\varepsilon_n} \int_{Q_i \cap \{v_n(t) > b_n^1\}} (1 - w_n^{2,i})^2 \, dx \le \\ \le \frac{\varepsilon_n}{2} \left(\frac{4h_n^2}{\delta^2} \int_{\tilde{S}_n^i \cap (Q_i \cap \{v_n(t) > b_n^1\})} |\nabla u_n(t)|^2 \, dx \right) &+ \frac{1}{2\varepsilon_n} |\tilde{S}_n^i \cap (Q_i \cap \{v_n(t) > b_n^1\})|.\end{aligned}$$

Since by (6.18)

$$\int_{\tilde{S}_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx \le \frac{1}{h_n} \left[\int_{S_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx + \frac{1-\sigma}{\sigma} |S_n^i \cap Q_i| \right]$$

and

$$|\tilde{S}_n^i \cap Q_i| \le \frac{1}{h_n} \left[\frac{\sigma}{1-\sigma} \int_{S_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx + |S_n^i \cap Q_i| \right]$$

we have

$$\begin{aligned} MM_n(w_n^{2,i}, &\{v_n(t) > b_n^1\}) \le \\ &\le \frac{2h_n\varepsilon_n}{\delta^2(\eta_n + (b_n^1)^2)} \left[\int_{S_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx + \frac{1-\sigma}{\sigma} |S_n^i \cap Q_i| \right] + \\ &\qquad + \frac{1}{2\varepsilon_n h_n} \left[\frac{\sigma}{1-\sigma} \int_{S_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx + |S_n^i \cap Q_i| \right]. \end{aligned}$$

Summing on $i=1,\dots,k$, recalling (5.5) and letting $d\in]0,1]$ with $\eta_n+(b_n^1)^2\ge d^2$ for all n, we obtain

$$\sum_{i=1}^k MM_n(w_n^{2,i},\{v_n(t)>b_n^1\})\le$$
$$\le h_n\varepsilon_n\frac{2}{\delta^2d^2}\left[C_1+\frac{1-\sigma}{\sigma}|\cup Q_i|\right]+\frac{1}{\varepsilon_nh_n}\frac{1}{2}\left[\frac{\sigma}{1-\sigma}C_1+|\cup Q_i|\right].$$

We choose h_n in such a way that the preceding quantity is less than (recall that $|\cup Q_i|\le |U|<\sigma$)

$$\sqrt{\frac{1}{\delta^2d^2}\left(C_1+1-\sigma\right)\left(\frac{\sigma}{1-\sigma}C_1+\sigma\right)}.$$

Then we obtain

$$\sum_{i=1}^k MM_n(w_n^{2,i},\{v_n(t)>b_n^1\})\le\sqrt{\frac{1}{\delta^2d^2}(C_1+1-\sigma))\left(\frac{\sigma}{1-\sigma}C_1+\sigma\right)}=o(\sigma).$$

This prove the first part of the lemma.

Let us define $\varphi_n^{2,i}$ as $w_n^{2,i}$ but operating with the levels $\gamma_n^i-\tau_n^i\le\gamma_n^i-\frac{\tau_n^i}{2}$ and $\gamma_n^i+\frac{\tau_n^i}{2}\le\gamma_n^i+\tau_n^i$. Reasoning as above we obtain

$$\eta_n\sum_{i=1}^k\int_{Q_i\cap\{v_n(t)>b_n^1\}}|\nabla\varphi_n^{2,i}|^2\,dx\le\frac{16\eta_nh_n}{\delta^2d^2}(C_1+1-\sigma)\to 0$$

since h_n has been chosen of the order of $\frac{1}{\varepsilon_n}$. □

Lemma 6.4. *Let $Q_i\subseteq\Omega$. Then there exists $w_n^{3,i}\in H^1(Q_i)$ such that $0\le w_n^{3,i}\le 1$, $w_n^{3,i}=0$ in a neighborhood of $H_i^+\setminus E^n_{a_2(x_i)-\frac{3}{4}\delta}$ and of $H_i^-\cap E^n_{a_1(x_i)+\frac{3}{4}\delta}$, $w_n^{3,i}=1$ on $Q_i\setminus R_i$ for n large, and*

$$\limsup_n\sum_{Q_i\subseteq\Omega}MM_n(w_n^{3,i},\{v_n(t)>b_n^1\})\le o(\sigma). \tag{6.19}$$

Moreover there exists a cut-off function $\varphi_n^{3,i}\in H^1(Q_i)$ such that $\varphi_n^{3,i}=0$ in a neighborhood of $H_i^+\setminus E^n_{a_2(x_i)-\delta}$ and of $H_i^-\cap E^n_{a_1(x_i)+\delta}$, $\varphi_n^{3,i}=1$ on $Q_i\setminus R_i$ for n large, $\mathrm{supt}(\nabla\varphi_n^{3,i})\subseteq\{w_n^{3,i}=0\}$, *and*

$$\lim_n\eta_n\int_{Q_i\cap\{v_n(t)>b_n^1\}}|\nabla\varphi_n^{3,i}|^2\,dx=0. \tag{6.20}$$

Proof. Let $\pi_i^\pm$ be the planes which contain $H_i^\pm$, and for $x\in\Omega'$, let $\pi_i^\pm x$ be its projection on $\pi_i^\pm$. Let us now consider $(u_n(t))_{|H_i^+}$: we set

$$\psi_n^{i,+}(y):=\frac{4}{\delta}\left(u_n(y)-a_2(x_i)+\frac{3}{4}\delta\right)^+\wedge 1$$

Note that $\psi_n^{i,+}$ is equal to zero on $H_i^+\setminus E^n_{a_2(x_i)-\frac{3}{4}\delta}$ and so on $\{x\in H_i^+\,:\,u_n(t)(x)=\gamma_n^i\}$ where γ_n^i is defined as in Lemma 6.3. Moreover, $\psi_n^{i,+}=1$ on $H_i^+\cap E^n_{a_2(x_i)-\frac{\delta}{2}}$. If $d\in]0,1]$ is such that $\eta_n+(b_n^1)^2\ge d^2$, by (6.7) we have

$$\int_{H_i^+\cap\{v_n(t)>b_n^1\}}|\nabla\psi_n^{i,+}|^2\,d\mathcal{H}^{N-1}\le\frac{16}{\delta^2}\int_{H_i^+\cap\{v_n(t)>b_n^1\}}|\nabla u_n|^2\,d\mathcal{H}^{N-1}\le\frac{16K_n}{\delta^2d^2}. \tag{6.21}$$

Let us define

$$\tilde\psi_n^{i,+}(y):=\frac{4}{\delta}(u_n(y)-a_2(x_i)+\delta)^+\wedge 1$$

which is null on $H_i^+\setminus E^n_{a_2(x_i)-\delta}$.

In a similar way we construct $\psi_n^{i,-}$ and $\tilde{\psi}_n^{i,-}$ on H_i^- which are null on $H_i^- \cap E^n_{a_1(x_i)+\frac{3}{4}\delta}$ and on $H_i^- \cap E^n_{a_1(x_i)+\delta}$ respectively. Let us set

$$w_n^{3,i,\pm}(x) := \left[\psi_n^{i,\pm}(\pi_i^\pm x) + \frac{1}{\varepsilon_n}(d_{H_i^\pm}(x) - l_n^i)^+\right] \wedge 1$$

with $\frac{l_n^i}{\varepsilon_n} \to 0$ and $\frac{\eta_n}{(l_n^i)^2} \to 0$. This is possible since $\eta_n << \varepsilon_n$. Let $A_n^i := (H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}}) \times]-\varepsilon_n - l_n^i, \varepsilon_n + l_n^i[\cap \{v_n(t) > b_n^1\}$. Then we have by definition of $\psi_n^{i,\pm}$, by (6.6), (6.21) and the fact that $K_n\varepsilon_n$ is bounded in n

$$\begin{aligned}
&\limsup_n MM_n(w_n^{3,i,+}, \{v_n(t) > b_n^1\}) = \\
&= \limsup_n \left\{ \frac{\varepsilon_n}{2} \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla w_n^{3,i,+}|^2 \, dx + \frac{1}{2\varepsilon_n} \int_{Q_i \cap \{v_n(t) > b_n^1\}} (1 - w_n^{3,i,+})^2 \, dx \right\} \le \\
&\le \limsup_n \left\{ \frac{\varepsilon_n}{2} \int_{A_n^i} \left(|\nabla \psi_n(\pi_i^+ x)|^2 + \frac{1}{\varepsilon_n^2} \right) dx + \right. \\
&\qquad \left. + \frac{1}{2\varepsilon_n} \left(2\mathcal{H}^{N-1}(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}})(\varepsilon_n + l_n^i) \right) \right\}
\end{aligned}$$

so that we get

$$\begin{aligned}
&\limsup_n MM_n(w_n^{3,i,+}, \{v_n(t) > b_n^1\}) = \\
&\le \limsup_n \left\{ \frac{\varepsilon_n}{2} \frac{16K_n}{\delta^2 d^2} 2(\varepsilon_n + l_n^i) + \frac{\varepsilon_n}{2} \frac{2}{\varepsilon_n^2} (\varepsilon_n + l_n^i) \mathcal{H}^{N-1}(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}}) + \right. \\
&\qquad \left. + \frac{\varepsilon_n + l_n^i}{\varepsilon_n} \mathcal{H}^{N-1}(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}}) \right\} \le \\
&\qquad \le 2 \limsup_n \mathcal{H}^{N-1}(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}}) \le 4\sigma r_i^{N-1}.
\end{aligned}$$

Similar calculations hold for $w_n^{3,i,-}$. Let us set $w_n^{3,i} := w_n^{3,i,+} \wedge w_n^{3,i,-}$. Then $0 \le w_n^{3,i} \le 1$, $w_n^{3,i} = 0$ in a neighborhood of $H_i^+ \setminus E^n_{a_2(x_i)-\frac{3}{4}\delta}$ and of $H_i^- \cap E^n_{a_1(x_i)+\frac{3}{4}\delta}$, $w_n^{3,i} = 1$ on $Q_i \setminus R_i$ for n large, and we have that

$$\limsup_n \sum_{Q_i \subseteq \Omega} MM_n(w_n^{3,i}, \{v_n(t) > b_n^1\}) \le o(\sigma),$$

which prove the first part of the lemma.

We define

$$\varphi_n^{3,i}(x) := \left[\tilde{\psi}_n^{i,+}(\pi_i^+ x) + \frac{1}{l_n^i} \left(d_{H_i^+}(x) - \frac{l_n^i}{2} \right)^+ \right] \wedge \left[\tilde{\psi}_n^{i,-}(\pi_i^- x) + \frac{1}{l_n^i} \left(d_{H_i^-}(x) - \frac{l_n^i}{2} \right)^+ \right] \wedge 1.$$

The previous calculations prove that

$$\lim_n \eta_n \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{3,i}|^2 \, dx = 0$$

since $\dfrac{\eta_n}{(l_n^i)^2} \to 0$. Moreover $\varphi_n^{3,i} = 1$ on $Q_i \setminus R_i$ for n large. □

Lemma 6.5. *Suppose that $Q_i \subseteq \Omega$; then there exists $w_n^{4,i} \in H^1(\Omega')$ such that $0 \le w_n^{4,i} \le 1$, $w_n^{4,i} = 0$ in a neighborhood of V_i, $w_n^{4,i} = 1$ on Ω_D for n large and*

$$\limsup_n \sum_{Q_i \subseteq \Omega} MM_n(w_n^{4,i}) \le o(\sigma). \tag{6.22}$$

Moreover there exists a cut-off function $\varphi_n^{4,i}$ *such that* $\varphi_n^{4,i} = 0$ *in a neighborhood of* V_i, $\varphi_n^{4,i} = 1$ *on* Ω_D *for* n *large,* $\text{supt}(\nabla\varphi_n^{4,i}) \subseteq \{w_n^{4,i} = 0\}$*, and*

$$\lim_n \eta_n \int_{\Omega'} |\nabla\varphi_n^{4,i}|^2 \, dx = 0. \tag{6.23}$$

Proof. Let us set

$$w_n^{4,i}(x) := \frac{1}{\varepsilon_n}(d_{V_i}(x) - l_n^i)^+ \wedge 1,$$

and

$$\varphi_n^{4,i}(x) := \frac{1}{l_n^i}\left(d_{V_i}(x) - \frac{l_n^i}{2}\right)^+ \wedge 1$$

where $\frac{l_n^i}{\varepsilon_n} \to 0$ and $\frac{\eta_n}{(l_n^i)^2} \to 0$. We have immediately (since $\sum_{Q_i \subseteq \Omega} \mathcal{H}^{N-1}(V_i) \leq o(\sigma)$)

$$\limsup_n \sum_{Q_i \subseteq \Omega} MM_n(w_n^{4,i}) \leq o(\sigma)$$

while

$$\lim_n \eta_n \int_{\Omega'} |\nabla\varphi_n^{4,i}|^2 \, dx = 0$$

since $\frac{\eta_n}{(l_n^i)^2} \to 0$. For n large enough, $w_n^{4,i} = 1$, $\varphi_n^{4,i} = 1$ on Ω_D and the proof is complete. □

We recall that $z = g_h(t)$ in a neighborhood $\mathcal{V}$ of $\partial\Omega \setminus \cup Q_i$.

Lemma 6.6. *Let* $Q_i \cap \partial_D\Omega \neq \emptyset$ *with* $Q_i^+ \setminus R_i \subseteq \Omega$*. Then* $E^n_{a_1(x_i)+\frac{\delta}{2}} \cap Q_i \subseteq \Omega$ *for all* n*, and there exists* $w_n^{b,i,+} \in H^1(\Omega')$ *with* $0 \leq w_n^{b,i,+} \leq 1$, $w_n^{b,i,+} = 1$ *on* Ω_D, $w_n^{b,i,+} = 0$ *in a neighborhood of*

$$V_i^{n,+} := \left[V_i \cap E^n_{a_1(x_i)+\delta}\right] \cup \left[(V_i \cap Q_i^+) \setminus \mathcal{V}\right],$$

and such that

$$\limsup_n \sum_{Q_i \cap \partial_D\Omega \neq \emptyset} MM_n(w_n^{b,i,+}, \{v_n(t) > b_n^1\}) \leq o(\sigma). \tag{6.24}$$

Moreover there exists a cut-off function $\varphi_n^{b,i,+}$ *such that* $\varphi_n^{b,i,+} = 1$ *on* Ω_D, $\varphi_n^{b,i,+} = 0$ *in a neighborhood of* $V_i^{n,+}$, $\text{supt}(\nabla\varphi_n^{b,i,+}) \subseteq \{w_n^{b,i,+} = 0\}$*, and*

$$\lim_n \eta_n \int_{\Omega' \cap \{v_n(t) > b_n^1\}} |\nabla\varphi_n^{b,i,+}|^2 \, dx = 0 \tag{6.25}$$

Proof. Note that by construction, $E^n_{a_1(x_i)+\frac{\delta}{2}} \cap Q_i \subseteq \Omega$ since $u_n(t)$ is continuous and $u_n(t) = g_h(t)$ on Ω_D. It is now sufficient to operate as in Lemma 6.4 and in Lemma 6.5. In fact, in view of (6.10), we may construct $\tilde{w}_n^{b,i,+} \in H^1(\Omega')$ such that $0 \leq \tilde{w}_n^{b,i,+} \leq 1$, $\tilde{w}_n^{b,i,+} = 0$ in a neighborhood of $V_i \cap E^n_{a_1(x_i)+\delta}$, $\tilde{w}_n^{b,i,+} = 1$ on Ω_D and on $V_i \setminus E^n_{a_1(x_i)+\frac{\delta}{2}}$, and such that $\limsup_n MM_n(\tilde{w}_n^{b,i,+}, \{v_n(t) > b_n^1\}) \leq o(\sigma) r_i^{N-1}$. Referring to $(V_i \cap Q_i^+) \setminus \mathcal{V}$, we can reason as in Lemma 6.5 getting $\overline{w}_n^{b,i,+}$, such that $0 \leq \overline{w}_n^{b,i,+} \leq 1$, $\overline{w}_n^{b,i,+} = 0$ in a neighborhood of $(V_i \cap Q_i^+) \setminus \mathcal{V}$, $\overline{w}_n^{b,i,+} = 1$ on Ω_D, and such that $\limsup_n MM_n(\overline{w}_n^{b,i,+}) \leq o(\sigma) r_i^{N-1}$.

Setting $w_n^{b,i,+} := \tilde{w}_n^{b,i,+} \wedge \overline{w}_n^{b,i,+}$, we get the first part of the thesis. Similarly, we may construct $\varphi_n^{b,i,+}$ which satisfies (6.25). □

In a similar way we can prove the following lemma.

Lemma 6.7. *Let* $Q_i \cap \partial_D\Omega \neq \emptyset$ *with* $Q_i^- \setminus R_i \subseteq \Omega$*. Then* $Q_i \setminus E^n_{a_2(x_i)-\frac{\delta}{2}} \subseteq \Omega$ *for all* n*, and there exists* $w_n^{b,i,-} \in H^1(\Omega')$ *with* $0 \leq w_n^{b,i,-} \leq 1$, $w_n^{b,i,-} = 1$ *on* Ω_D, $w_n^{b,i,-} = 0$ *in a neighborhood of*

$$V_i^{n,-} := \left[V_i \setminus E^n_{a_2(x_i)-\delta}\right] \cup \left[(V_i \cap Q_i^-) \setminus \mathcal{V}\right],$$

and such that

$$\limsup_n \sum_{Q_i \cap \partial_D \Omega \neq \emptyset} MM_n(w_n^{b,i,-}, \{v_n(t) > b_n^1\}) \leq o(\sigma). \tag{6.26}$$

Moreover there exists a cut-off function $\varphi_n^{b,i,-}$ *such that* $\varphi_n^{b,i,-} = 1$ *on* Ω_D*,* $\varphi_n^{b,i,-} = 0$ *in a neighborhood of* $V_i^{n,-}$*,* $\mathrm{supt}(\nabla \varphi_n^{b,i,-}) \subseteq \{w_n^{b,i,-} = 0\}$*, and*

$$\lim_n \eta_n \int_{\Omega' \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{b,i,-}|^2 \, dx = 0 \tag{6.27}$$

We can now prove Lemma 6.1.

Proof of Lemma 6.1. We employ the notation of the preceding lemmas. Following [14, Theorem 2.1], for each i let us define z_i^+ on $Q_i^+ \cup R_i$ to be equal to z on $Q_i^+ \setminus R_i$ and to the symmetrization of z with respect to $H_i(\sigma)$ on R_i. Similarly we define z_i^-.

For each $Q_i \subseteq \Omega$, let us set z_n^i to be equal to z_i^+ on $(Q_i^+ \setminus \tilde{R}_i) \cup (E^n_{\gamma_n^i} \cap \tilde{R}_i)$, and to z_i^- in the rest of Q_i.

If $Q_i \cap \partial_D \Omega \neq \emptyset$ with $Q_i^+ \setminus R_i \subseteq \Omega$, by Lemma 6.3 and Lemma 6.6 we have $E^n_{\gamma_n^i - \tau_n^i} \cap Q_i \subseteq Q_i^+$ for all n, and its closure does not intersect $\partial\Omega$. We define z_n^i to be equal to z_i^+ on $(Q_i^+ \setminus \tilde{R}_i) \cup (E^n_{\gamma_n^i} \cap \tilde{R}_i)$, and to $g_h(t)$ in the rest of Q_i. If $Q_i^- \setminus R_i \subseteq \Omega$, by Lemma 6.3 and Lemma 6.7 we have $Q_i \setminus E^n_{\gamma_n^i + \tau_n^i} \subseteq \Omega$, and its closure does not intersect $\partial\Omega$. We define z_n^i to be equal to z_i^- on $(Q_i^- \setminus \tilde{R}_i) \cup (\tilde{R}_i \setminus E^n_{\gamma_n^i})$, and to $g_h(t)$ in the rest of Q_i.

Let us now define $\tilde{z}_n$ to be equal to z outside $\bigcup_{i=1}^k R_i$, and to z_n^i inside each R_i. We have $\tilde{z}_n = g_h(t)$ on Ω_D. Note that if $Q_i \subseteq \Omega$, $H_i^+ \setminus E^n_{\gamma_n^i}$, $H_i^- \cap E^n_{\gamma_n^i}$, $V_i^{\pm}$, and $\partial^* E^n_{\gamma_n^i} \cap Q_i$ could be contained in $S_{\tilde{z}_n}$. Similarly, if $Q_i \cap \partial\Omega \neq \emptyset$ and $Q_i^+ \setminus R^i \subseteq \Omega$ (the other case being similar), then $H_i^+ \setminus E^n_{\gamma_n^i}$, $V_i^{n,\pm}$ and $\partial^* E^n_{\gamma_n^i} \cap Q_i$ could be contained in $S_{\tilde{z}_n}$.

By assumption on U, we have that

$$||\tilde{z}_n - z||_{L^2(\Omega')} + ||\nabla \tilde{z}_n - \nabla z||_{L^2(\Omega';\mathbb{R}^N)} \leq o(\sigma); \tag{6.28}$$

moreover, besides the possible jumps previously individuated, $\tilde{z}_n$ has in R_i polyhedral jumps which are a reflected version of the polyhedral jumps of z in Q_i. By assumption on z, we conclude that the union of these polyhedral sets $P_i(S_z)$ has $\mathcal{H}^{N-1}$ measure which is of the order of σ that is $\mathcal{H}^{N-1}(P(S_z)) \leq o(\sigma)$ where $P(S_z) := \bigcup_{i=1}^k P_i(S_z)$.

Let $\tilde{w}_n$ be optimal for the Ambrosio-Tortorelli approximation of $[S_z \setminus (\bigcup Q_i)] \cup P(S_z)$ (as we can find for example in [13, Lemma 3.3]), that is $\tilde{w}_n$ is null in a neighborhood of $[S_z \setminus (\bigcup Q_i)] \cup P(S_z)$ and

$$\begin{aligned} \limsup_n MM_n(\tilde{w}_n) &\leq \mathcal{H}^{N-1}(S_z \setminus (\cup Q_i) \cup P(S_z)) \leq \\ &\leq \mathcal{H}^{N-1}(S_z \setminus S_{u(t)}) + o(\sigma). \end{aligned} \tag{6.29}$$

As in [13], let $\tilde{\varphi}_n$ be a cut-off function associated to $\tilde{w}_n$, such that

$$\lim_n \eta_n \int_{\Omega'} |\nabla \tilde{\varphi}_n|^2 \, dx = 0. \tag{6.30}$$

Let us set for all $Q_i \subseteq \Omega$

$$w_n^i := \begin{cases} \min\{\tilde{w}_n, w_n^{2,i}, w_n^{3,i}, w_n^{4,i}\} & \text{in } \tilde{R}_i \\ \min\{\tilde{w}_n, w_n^{3,i}, w_n^{4,i}\} & \text{in } R_i \setminus \tilde{R}_i \\ \min\{\tilde{w}_n, w_n^{4,i}\} & \text{outside } R_i, \end{cases}$$

and

$$\varphi_n^i := \begin{cases} \min\{\tilde{\varphi}_n, \varphi_n^{2,i}, \varphi_n^{3,i}, \varphi_n^{4,i}\} & \text{in } \tilde{R}_i \\ \min\{\tilde{\varphi}_n, \varphi_n^{3,i}, \varphi_n^{4,i}\} & \text{in } R_i \setminus \tilde{R}_i \\ \min\{\tilde{\varphi}_n, \varphi_n^{4,i}\} & \text{outside } R_i. \end{cases}$$

For all Q_i such that $Q_i \cap \partial_D \Omega \neq \emptyset$ with $Q_i^+ \setminus R_i \subseteq \Omega$, let us set

$$w_n^i := \begin{cases} \min\{\tilde{w}_n, w_n^{2,i}, w_n^{3,i,+}, w_n^{b,i,+}\} & \text{in } \tilde{R}_i \cap E^n_{\gamma_n^i} \\ \min\{w_n^{2,i}, w_n^{b,i,+}\} & \text{in } (\tilde{R}_i \setminus E^n_{\gamma_n^i}) \cup Q_i^- \\ \min\{\tilde{w}_n, w_n^{3,i,+}, w_n^{b,i,+}\} & \text{in } R_i \setminus (\tilde{R}_i \cup Q_i^-) \\ 1 & \text{in } \Omega_D \\ \min\{\tilde{w}_n, w_n^{b,i,+}\} & \text{otherwise} \end{cases}$$

and

$$\varphi_n^i := \begin{cases} \min\{\tilde{\varphi}_n, \varphi_n^{2,i}, \varphi_n^{3,i,+}, \varphi_n^{b,i,+}\} & \text{in } \tilde{R}_i \cap E^n_{\gamma_n^i} \\ \min\{\varphi_n^{2,i}, \varphi_n^{b,i,+}\} & \text{in } (\tilde{R}_i \setminus E^n_{\gamma_n^i}) \cup Q_i^- \\ \min\{\tilde{\varphi}_n, \varphi_n^{3,i,+}, \varphi_n^{b,i,+}\} & \text{in } R_i \setminus (\tilde{R}_i \cup Q_i^-) \\ 1 & \text{in } \Omega_D \\ \min\{\tilde{\varphi}_n, \varphi_n^{b,i,+}\} & \text{otherwise} \end{cases}$$

Similarly we reason for the case $Q_i^- \setminus R_i \subseteq \Omega$. By construction, for all $i = 1, \dots, k$ we have that $w_n^i, \varphi_n^i \in H^1(\Omega')$, $0 \le w_n^i, \varphi_n^i \le 1$ and $w_n^i, \varphi_n^i = 1$ on Ω_D for n large.

Note that by Lemmas 6.3, 6.4, 6.5, 6.6 and 6.7, and by (6.29) and (6.30), we have that

$$\limsup_n \sum_{i=1}^k MM_n(w_n^i, \{v_n(t) > b_n^1\}) \le \mathcal{H}^{N-1}(S_z \setminus S_{u(t)}) + o(\sigma), \tag{6.31}$$

and

$$\lim_n \eta_n \sum_{i=1}^k \int_{\Omega' \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^i(x)|^2 \, dx = 0. \tag{6.32}$$

We are now in a position to conclude the proof. We set

$$v_n := \min\{w_n, w_n^i, i = 1, \dots, k\}, \quad \varphi_n := \min\{\varphi_n, \varphi_n^i, i = 1, \dots, k\}.$$

Note that $\varphi_n = 0$ in a neighborhood of $S_{\tilde{z}_n}$, and $\varphi_n = 1$ on Ω_D for n large. Moreover $0 \le v_n \le w_n \le v_n(t)$ in Ω' and $v_n = 1$ on Ω_D. Let $z_n := \varphi_n \tilde{z}_n$; we have $z_n \in H^1(\Omega')$ with $z_n = g_h(t)$ on Ω_D. By (5.2), we have that

$$F_{\varepsilon_n}(u_n(t), v_n(t)) \le F_{\varepsilon_n}(z_n, v_n),$$

and so

$$\int_\Omega (\eta_n + v_n(t)^2) |\nabla u_n(t)|^2 \, dx \le \int_{\Omega'} (\eta_n + v_n^2) |\nabla(\varphi_n \tilde{z}_n|^2 \, dx + MM_n(v_n) - MM_n(v_n(t)).$$

We may write

$$\int_{\Omega'} (\eta_n + v_n(t)^2)|\nabla u_n(t)|^2\,dx \le$$
$$\le \int_{\Omega'} (\eta_n+1)|\nabla \tilde z_n|^2\,dx + \int_{\Omega'} (\eta_n + v_n^2)(2\nabla\varphi_n\nabla\tilde z_n + \tilde z_n|\nabla\varphi_n|^2)\,dx +$$
$$+ MM_n(w_n) - MM_n(v_n(t)) + \sum_{i=1}^k MM_n(w_n^i, \{v_n(t) > b_n^1\}).$$

Taking into account (6.28), (6.12), (6.32), (6.11), and (6.31), we have that passing to the limit

$$\int_{\Omega'} |\nabla u|^2\,dx \le \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}(S_z \setminus S_{u(t)}) +$$
$$+ \frac{2Ck}{(k-1)^2} + \frac{C}{(k-1)(1-b)^2} + \frac{Cb}{(1-b)^2} + o(\sigma),$$

so that, letting $\sigma \to 0$ and then $b \to 0$, $k \to \infty$ (which is permitted choosing appropriately j_2 and j_3), we obtain the thesis. □

We can now pass to the proof of Theorem 5.6. Given $0 = t_1 \le t_2 \le \ldots \le t_k = t$, it is sufficient to prove that

$$\int_{\Omega'} |\nabla u(t)|^2\,dx \le \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}\left(S_z \setminus \left(\bigcup_{i=1}^k S_{u(t_i)}\right)\right). \tag{6.33}$$

Passing to the sup on $t_1, \ldots, t_k$, we deduce in fact the thesis. We obtain (6.33) using the same arguments of Lemma 6.1; defining

$$J_j := \left\{ x \in \bigcup_{m=1,\ldots,k} \left(\bigcup_{a_1,a_2\in A^k} \left[\partial^* E^k_{a_1} \cap \partial^* E^k_{a_2}\right]\right) : \min_{l=1,\ldots,k} [u_l(x)] > \frac{1}{j} \right\},$$

where E^k_a and A_k are defined as the corresponding sets for $u(t)$, following [14], we cover J_j in such a way that for all $x_i \in J_j$ there exists l with $x_i \in S_{u(t_l)}$ and

$$\mathcal{H}^{N-1}\left(\left[\bigcup_{r=1,\ldots,k} S_{u(t_r)} \setminus S_{u(t_l)}\right] \cap Q_i\right) \le \sigma r_i^{N-1}.$$

So in each Q_i there exists $u(t_l)$ such that $\bigcup_{r=1}^k S_{u(t_r)} \cap Q_i$ is essentially (with respect to the measure $\mathcal{H}^{N-1}$) $S_{u(t_l)} \cap Q_i$. Recalling that $v_n(t) \le v_n(t_l)$ for all $l = 1, \ldots, k$, we have

$$\int_{\Omega'} (\eta_n + v_n(t)^2)|\nabla u_n(t_l)|^2\,dx \le \int_{\Omega'} (\eta_n + v_n(t_l)^2)|\nabla u_n(t_l)|^2\,dx \le C,$$

and so it is readily seen that the arguments of Lemma 6.1 can be adapted to prove (6.33).

7. A final remark

The previous results can be extended to recover the case of non isotropic surface energies, i.e., energies of the form

$$\int_\Omega |\nabla u|^2\,dx + \int_\Gamma \varphi(\nu_x)\,d\mathcal{H}^{N-1}(x) \tag{7.1}$$

where ν_x is the normal to Γ at x, and φ is a norm on $\mathbb{R}^N$. In fact all the previous arguments are based on Theorem 2.3 concerning the elliptic approximation and on Theorem 2.4 about the density of piecewise smooth functions with respect to the total energy. An elliptic approximation of Ambrosio-Tortorelli type of (7.1) has been proved in [13], while a density result of piecewise smooth functions with respect to non-isotropic surface energies has been proved in [10]. We conclude that all the previous theorems can be modified in order to treat the more general energy (7.1).

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