THE SHARP QUANTITATIVE SOBOLEV INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract. The classical Sobolev embedding theorem of the space of functions of bounded variation $BV(\mathbb{R}^n)$ into $L^{n'}(\mathbb{R}^n)$ is proved in a sharp quantitative form.

1. Introduction

A fundamental tool in the study of partial differential equations is the *Sobolev inequality*

$$n\omega_n^{1/n}\|f\|_{L^{n'}} \le \int_{\mathbb{R}^n} |\nabla f(x)|\,dx\,,\ \forall f \in C_c^\infty(\mathbb{R}^n)\,. \tag{1.1}$$

Here $n \ge 2$, $n' = n/(n-1)$ and ω_n denotes the measure of the unit ball B of $\mathbb{R}^n$. As well-known, (1.1) is equivalent to the classical *isoperimetric inequality* (see e.g. [4], sect. 5.6). This last inequality states that the perimeter $P(E)$ of a Borel set $E \subseteq \mathbb{R}^n$ of finite measure is necessarily larger than the perimeter of a ball with the same measure, namely that

$$n\omega_n^{1/n}|E|^{1/n'} \le P(E)\,. \tag{1.2}$$

A sharp quantitative form of the isoperimetric inequality has been established in our recent paper [6], following the line of research started by Bernstein [1] and Bonnesen [2], and later carried over by Osserman [11], [12], Fuglede [5], Hall, Hayman and Weitsman [8] and Hall [7]. More precisely, we have proved the following strengthening of (1.2),

$$n\omega_n^{1/n}|E|^{1/n'}\left(1 + \frac{\lambda^*(E)^2}{K(n)}\right) \le P(E)\,,\ 0 < |E| < \infty\,, \tag{1.3}$$

where $K(n)$ is a constant depending only on the dimension and $\lambda^*(E)$ is the *Fraenkel asymmetry of* E,

$$\lambda^*(E) := \min\left\{\frac{|E\Delta(x+rB)|}{|E|} : r^n\omega_n = |E|\,, x \in \mathbb{R}^n\right\}\,,$$

which measures how far E is from being a ball. Inequality (1.3) is sharp in the sense that the term $\lambda^*(E)^2$ cannot be replaced by $\lambda^*(E)^q$ for any $q < 2$ (as one can check by taking E to be an ellipsoid arbitrarily close to a ball).

In this paper we extend the quantitative inequality (1.3) to the functional case. To this aim, if $f \in BV(\mathbb{R}^n)$, we denote by $\|Df\| = |Df|(\mathbb{R}^n)$ the total variation of the distributional derivative Df. Thus, if $f \in W^{1,1}(\mathbb{R}^n)$, $\|Df\| = \int_{\mathbb{R}^n}|\nabla f|$. Moreover, simple approximation arguments show that (1.1) is equivalent to

$$n\omega_n^{1/n}\|f\|_{L^{n'}} \le \|Df\|,\ \forall f \in BV(\mathbb{R}^n)\,. \tag{1.4}$$

It is well-known that equality holds in (1.4) if and only if $f = a\chi_{x+rB}$ for some $a \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $r > 0$. Therefore, in analogy with the Fraenkel asymmetry, it is natural to introduce the *functional asymmetry*, a quantity measuring how far a generic function $f \in BV(\mathbb{R}^n)$ is from being optimal for (1.4), setting

$$\lambda(f) := \min\left\{\frac{\|f - a\,\chi_{x+rB}\|_{L^{n'}}^{n'}}{\|f\|_{L^{n'}}^{n'}} : |a|^{n'}r^n\omega_n = \|f\|_{L^{n'}}^{n'}\,, a \in \mathbb{R}\,, x \in \mathbb{R}^n\right\} \tag{1.5}$$

(the fact that the above minimum exists is proved in Lemma B.1). Notice that $\lambda(f)$ is invariant with respect to both rescaling of the coordinates, and multiplication by a constant; moreover, $\lambda(f) \le 2^{n'}$ for any f. The main result of the paper is the following.

Theorem 1.1. *There exists a dimensional constant $C = C(n)$ such that*

$$n\omega_n^{1/n}\|f\|_{L^{n'}}\left(1+\frac{\lambda(f)^2}{C(n)}\right)\le\|Df\| \tag{1.6}$$

for every $f \in BV(\mathbb{R}^n)$.

Andrea Cianchi [3] has recently obtained an estimate similar to (1.6) in which the exponent 2 in $\lambda(f)^2$ is replaced by a slightly larger one depending on the dimension. Notice, however, that –as for the quantitative isoperimetric inequality (1.3)– the exponent 2 is sharp, as one can check testing the inequality on characteristic functions of ellipsoids.

When (1.6) is evaluated at $f = \chi_E$ it furnishes an alternative version of (1.3), where the Fraenkel asymmetry $\lambda^*(E)$ of the set E is replaced by the functional asymmetry of its characteristic function $\lambda(\chi_E)$. The two inequalities are however equivalent as on the one hand we have trivially that $\lambda(\chi_E) \le \lambda^*(E)$, while on the other hand it holds $\lambda^*(E) \le 2^{n'+1}\lambda(\chi_E)$, as shown in Lemma B.2.

Inequality (1.6) can be restated equivalently as

$$\lambda(f)\le C(n)\sqrt{\delta(f)}\,,\ \forall f\in BV(\mathbb{R}^n)\,, \tag{1.7}$$

where $\delta(f)$ is the *Sobolev deficit of f*,

$$\delta(f):=\frac{\|Df\|}{n\omega_n^{1/n}\|f\|_{L^{n'}}}-1\,.$$

Notice that $\delta(f)$ has the same scaling invariant properties as $\lambda(f)$.

As shown in section 4, in order to prove (1.7) it is not restrictive to assume that f is nonnegative. In this case, the underlying strategy is to replace the generic function f with a more symmetric function g; to be effective, this reduction has to be done in such a way that the inequality (1.7) for f follows by establishing the same inequality for the more symmetric function g. For instance, this could be done if

$$\lambda(f)\le C(n)\lambda(g), \qquad\qquad \delta(g)\le C(n)\delta(f).$$

These two requirements are obviously in competition: transforming f into a more symmetric function g lowers its deficit $\delta(f)$, but also its asymmetry $\lambda(f)$. Therefore, keeping this remark in mind, it is clear that one has to choose how to symmetrize f in such a way that $\lambda(f)$ does not become too small. The first step in this symmetrization process is to pass from f to a $n-$symmetric function g, (i.e., a function which is symmetric with respect to n orthogonal hyperplanes), in such a way that

$$\lambda(f)\le C(n)\big(\lambda(g)+\sqrt{\delta(f)}\big)\,, \qquad\qquad \delta(g)\le C(n)\delta(f)\,, \tag{1.8}$$

see Theorem 2.1. We remark that a similar starting point was carried over in [6] when dealing with sets; however, in the present case the proof is harder.

The second step is to pass from a $n-$symmetric function to a spherically symmetric one; in fact, it can be proved (Theorem 2.2) that if g is n-symmetric and $g^\star$ denotes its symmetric decreasing rearrangement, then a particularly convenient quantitative version of the Polya-Szegö inequality $\int_{\mathbb{R}^n}|\nabla g|\ge\int_{\mathbb{R}^n}|\nabla g^\star|$ holds true. More precisely we prove that, for a $n-$symmetric g, whenever $\|g\|_{L^{n'}}=1$ and $\delta(g)\le 1$ we have

$$\int_{\mathbb{R}^n}|g-g^\star|^{n'}\le C(n)\sqrt{\|Dg\|-\|Dg^\star\|}\,. \tag{1.9}$$

We remark that the assumption that g is $n-$symmetric is needed in order to obtain an estimate of $\|g-g^\star\|_{L^{n'}}$ as in (1.9); indeed, in the general case, the right hand side can be 0 and yet $g\ne g^\star$.

Since the Polya-Szegö inequality implies that $\delta(g^\star)\le\delta(g)$, (1.8) and (1.9) can be combined to reduce the proof of Theorem 1.1 to the case when f is spherically symmetric and decreasing. In this particular case, a delicate geometrical construction shows that (1.7) holds indeed in the stronger form

$$\lambda(f)\le C(n)\delta(f)\,,$$

see Theorem 3.1.

2. Reduction to the spherically symmetric case

We start by specifying the natural context in which (1.1) should be considered. The space $BV(\mathbb{R}^n)$ of the functions of bounded variation is defined as the space of those functions f in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that there exists a sequence $f_h \in C^\infty_c(\mathbb{R}^n)$ with $f_h \to f$ a.e. and $\sup \int_{\mathbb{R}^n} |\nabla f_h| < +\infty$. Notice that, from this definition, it follows that Df is a $\mathbb{R}^n-$valued Radon measure and that $f \in L^{n'}$; moreover, one can always construct the sequence f_h in such a way that f_h converges to f in $L^{n'}$ and $\int_{\mathbb{R}^n} |\nabla f_h| \to |Df|(\mathbb{R}^n) =: \|Df\|$. This shows in particular that (1.1) implies (1.4). We shall also consider $BV^+(\mathbb{R}^n) = \{f \in BV(\mathbb{R}^n) : \ f \geq 0\}$.

Let us define the *symmetric decreasing rearrangement* $f^\star$ of a function $f \in BV^+(\mathbb{R}^n)$ as the unique spherically symmetric function $f^\star$ such that for all $t > 0$ the set $\{x : \ f^\star(x) > t\}$ is the ball centered at the origin with

$$|\{x : \ f^\star(x) > t\}| = |\{x : \ f(x) > t\}|.$$

We say that $f \in BV^+(\mathbb{R}^n)$ is a *spherically symmetric decreasing function* if $f = f^\star$.

Given a function $f \in BV(\mathbb{R}^n)$, we will say that f is *$k-$symmetric*, with $1 \leq k \leq n$, if f is symmetric with respect to the first k coordinate hyperplanes, i.e. if $f(x) = f(x - 2x_i e_i)$ for every $x \in \mathbb{R}^n$ and $1 \leq i \leq k$. Here and in the following e_i, $1 \leq i \leq n$, are the elements of the canonical basis of $\mathbb{R}^n$. The goal of this section is to prove the two following theorems. Here and in the sequel, by $C = C(n)$ we denote a constant depending only on the dimension n whose value may increase from line to line.

Theorem 2.1. *Let $f \in BV^+(\mathbb{R}^n)$. There exists a n-symmetric function $g \in BV^+(\mathbb{R}^n)$ such that*

$$\lambda(f) \leq C(n)(\lambda(g) + \sqrt{\delta(f)})\,, \qquad\qquad \delta(g) \leq 2^n \delta(f)\,.$$

Theorem 2.2. *Let $f \in BV^+(\mathbb{R}^n)$ be n-symmetric. Then*

$$\int_{\mathbb{R}^n} |f - f^\star|^{n'} \leq C(n) \|Df\|^{n'-1/2} \sqrt{\|Df\| - \|Df^\star\|}\,. \tag{2.1}$$

Moreover,

$$\lambda(f) \leq C(n)(\lambda(f^\star) + \sqrt{\delta(f)}). \tag{2.2}$$

We point out that, though our strategy for improving (1.6) will be to pass first to a $n-$symmetric function, and then to its symmetric rearrangement, for technical reasons we shall prove first Theorem 2.2 and then Theorem 2.1.

The first result we need in the proof of the above theorems is an elementary lemma that expresses the L^q distance of two functions in terms of the distance of their level sets.

Lemma 2.3. *Let $f, g \in L^q(\mathbb{R}^n)$ with $q \geq 1$. Then*

$$\int_{\mathbb{R}^n} |f(x) - g(x)|^q \, dx = \int_{-\infty}^{+\infty} \int_{\{f>t\}\Delta\{g>t\}} |f(x) - g(x)|^{q-1} \, dx \, dt\,, \tag{2.3}$$

and moreover

$$\|f - g\|_{L^q(\mathbb{R}^n)} \leq \int_{-\infty}^{+\infty} \big|\{f > t\}\Delta\{g > t\})\big|^{1/q} \, dt\,. \tag{2.4}$$

Proof. By Fubini's Theorem,

$$\begin{aligned}
\int_{\{f>g\}} |f(x) - g(x)|^q \, dx &= \iint_{\{(x,t):\ f(x)>t\geq g(x)\}} |f(x) - g(x)|^{q-1} \, dx \, dt \\
&= \int_{-\infty}^{+\infty} \int_{\{f>t\geq g\}} |f(x) - g(x)|^{q-1} \, dx \, dt \\
&= \int_{-\infty}^{+\infty} \int_{\{f>t\}\setminus\{g>t\}} |f(x) - g(x)|^{q-1} \, dx \, dt,
\end{aligned}$$

from which (2.3) follows. To conclude, set $E_t = \{f > t\}\Delta\{g > t\}$; then

$$\int_{\mathbb{R}^n} |f(x) - g(x)|^q \, dx = \int_{-\infty}^{+\infty} \int_{E_t} |f(x) - g(x)|^{q-1} \, dx \, dt \le \int_{-\infty}^{+\infty} |E_t|^{1/q} \Big(\int_{E_t} |f(x) - g(x)|^q \, dx \Big)^{1-1/q} dt$$
$$\le \Big(\int_{\mathbb{R}^n} |f(x) - g(x)|^q \, dx \Big)^{1-1/q} \int_{-\infty}^{+\infty} |E_t|^{1/q} \, dt \,,$$

and (2.4) follows. □

We now prove Theorem 2.2.

Proof. (of Theorem 2.2). Thanks to (2.4) we have

$$\|f - f^\star\|_{L^{n'}} \le \int_0^\infty d(t)^{1/n'} \, dt \,, \tag{2.5}$$

where we put for brevity $d(t) = |\{f > t\}\Delta\{f^\star > t\}|$. Let $\mu(t) = |\{f > t\}|$ be the distribution function of f. By the Coarea Formula and by (1.3)

$$\|Df\| = \int_0^\infty P(\{f > t\}) \, dt \ge \int_0^\infty n\omega_n^{1/n} \mu(t)^{1/n'} \left(1 + \frac{\lambda^*(\{f > t\})^2}{K} \right) dt \,. \tag{2.6}$$

Since $\{f^\star > t\}$ is a ball of measure $\mu(t)$,

$$\int_0^\infty n\omega_n^{1/n} \mu(t)^{1/n'} \, dt = \int_0^\infty P(\{f^\star > t\}) \, dt = \|Df^\star\| \,. \tag{2.7}$$

Furthermore $\{f > t\}$ is a n-symmetric set, so by Lemma 2.2 in [6], which is the analogue of Lemma 2.4 below with λ^* in place of λ, we find

$$2^n \lambda^*(\{f > t\}) \ge \frac{d(t)}{\mu(t)} \,. \tag{2.8}$$

In conclusion we have proved that, by (2.6) and (2.7),

$$\int_0^\infty \left(\frac{d(t)}{\mu(t)} \right)^2 \mu(t)^{1/n'} \, dt \le \frac{4^n K}{n\omega_n^{1/n}} \big(\|Df\| - \|Df^\star\| \big) \,.$$

We apply the Hölder inequality with the exponent $2n'$ to get

$$\int_0^\infty d(t)^{1/n'} \, dt = \int_0^\infty \left(\frac{d(t)}{\mu(t)} \right)^{1/n'} \mu(t)^{1/n'} \, dt$$
$$\le \left(\int_0^\infty \left(\frac{d(t)}{\mu(t)} \right)^2 \mu(t)^{1/n'} \, dt \right)^{1/2n'} \left(\int_0^\infty \mu(t)^{1/n'} \, dt \right)^{1/(2n')'} .$$

Since again by (2.6)

$$\int_0^\infty \mu(t)^{1/n'} \, dt \le \frac{1}{n\omega_n^{1/n}} \|Df\|,$$

gathering (2.5) and (2.8) we prove (2.1).
Since $\lambda(f)$ and $\delta(f)$ are invariant by multiplication by constants, to prove (2.2) we may assume $\|f\|_{L^{n'}} = 1$. By definition $\lambda(f) \le 2^{n'}$, so inequality (2.2) is trivial if $\delta(f) \ge 1$. Otherwise, from (1.5), from the triangular inequality, and from the general fact that $(a + b)^{n'} \le 2^{n'-1}(a^{n'} + b^{n'})$ for any a, $b \ge 0$, it follows that

$$\lambda(f) \le 2^{n'-1} \Big(\|f - f^\star\|_{L^{n'}}^{n'} + \lambda(f^\star) \Big).$$

Inequality (2.2) then follows because, by (2.1),

$$\int_{\mathbb{R}^n} |f - f^\star|^{n'} \le C \|Df\|^{n'-1/2} \sqrt{\|Df\| - \|Df^\star\|} \le C|1 + \delta(f)|^{n'-1/2} \sqrt{\delta(f)}.$$

□

Next we pass to prove that in evaluating the asymmetry of a $k-$symmetric function, we can restrict to the class of the optimal functions for (1.1) with the same symmetries. More precisely, for every $H \subset \mathbb{R}^n$ we set

$$\lambda(f|H) := \min\left\{ \frac{\|f - a\,\chi_{x+rB}\|_{L^{n'}}^{n'}}{\|f\|_{L^{n'}}} : |a|^{n'} r^n \omega_n = \|f\|_{L^{n'}}^{n'}, a \in \mathbb{R}, x \in H \right\}, \tag{2.9}$$

and then prove the following lemma. Notice that the same argument used in the proof of Lemma B.1 shows that also the minimization problem on the right hand side of (2.9) has a solution.

Lemma 2.4. *Let $H = \{x : x_i = 0,\, 1 \le i \le k\}$ be the intersection of the first k coordinate hyperplanes and let f be a k-symmetric function. Then*

$$\lambda(f|H) \le 3^{n'} \lambda(f).$$

Proof. Without loss of generality we may assume that $\|f\|_{L^{n'}} = 1$; let then $x \in \mathbb{R}^n$, $a \in \mathbb{R}$ and $r > 0$ be such that $\lambda(f) = \|f - a\chi_{x+rB}\|_{L^{n'}}^{n'}$. We consider the projection z of x over H, and denote by y the point obtained by reflecting x with respect to H. We also set, for any $w \in \mathbb{R}^n$, $g_w = a\chi_{w+rB}$. By the symmetry of f we deduce that

$$\lambda(f) = \|f - g_x\|_{L^{n'}}^{n'} = \|f - g_y\|_{L^{n'}}^{n'}.$$

Therefore

$$\lambda(f|H) \le \|f - g_z\|_{L^{n'}}^{n'} \le \Big(\|f - g_x\|_{L^{n'}} + \|g_x - g_z\|_{L^{n'}}\Big)^{n'}.$$

By construction

$$\|g_x - g_z\|_{L^{n'}} \le \|g_x - g_y\|_{L^{n'}} \le \|g_x - f\|_{L^{n'}} + \|f - g_y\|_{L^{n'}} = 2\lambda(f)^{1/n'}.$$

Then the result follows. □

In the sequel we will often make use of the following continuity result, which is a non-quantitative version of Theorem 1.1.

Lemma 2.5. *For every $\varepsilon > 0$ there is $\delta > 0$ such that, if $\delta(f) \le \delta$, then $\lambda(f) \le \varepsilon$.*

Proof. Let $\{f_h\} \subseteq BV(\mathbb{R}^n)$ be such that $\delta(f_h) \to 0$. Without loss of generality, we may assume that $\|f_h\|_{L^{n'}} = 1$; by Theorem A.1 we may also assume, up to a rescaling and a translation as in (A.1), which does not change neither $\|f_h\|_{L^{n'}}$ nor $\delta(f_h)$, nor $\lambda(f_h)$, that $f_h \to f$ in $L^{n'}(\mathbb{R}^n)$ for some $f \in BV(\mathbb{R}^n)$ with $\delta(f) = 0$ and so $\lambda(f) = 0$. Since $g \mapsto \lambda(g)$ is continuous in $L^{n'}$, we conclude that $\lambda(f_h) \to 0$ as claimed. □

We are now going to prove Theorem 2.1. The main idea is that given two orthogonal directions we can always modify our function so to make it symmetric with respect to at least one of the two directions, in such a way that the required estimates hold true; this immediately leads to the construction of a $(n-1)-$symmetric function. The same strategy was adopted in the case of sets [6]; however, the passage from $n-1$ to n symmetries, that was somehow trivial in the case of sets, becomes now more delicate, and will be treated by a different argument.

We start by proving the following lemma.

Lemma 2.6. *For every $f \in BV^+(\mathbb{R}^n)$, there exists a $(n-1)$-symmetric function $g \in BV^+(\mathbb{R}^n)$ such that*

$$\lambda(f) \le C(n)\lambda(g), \qquad \delta(g) \le 2^{n-1}\delta(f).$$

Proof. Without loss of generality we assume that $\|f\|_{L^{n'}} = 1$. For every $k \in \mathbb{N}$, $1 \le k \le n$ we consider an hyperplane H_k of the form $\{x : x_k = s_k\}$ for some $s_k \in \mathbb{R}$, such that, denoting $H_k^+ = \{x : x_k > s_k\}$ and $H_k^- = \{x : x_k < s_k\}$, one has

$$\int_{H_k^+} f^{n'} = \int_{H_k^-} f^{n'} = \frac{1}{2}.$$

We let $T_k : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection with respect to H_k and define

$$f_k^+(x) := \begin{cases} f(x), & x \in H_k^+ \,, \\ f(T_k(x)), & x \in H_k^- \,, \end{cases} \qquad f_k^-(x) := \begin{cases} f(T_k(x))\,, & x \in H_k^+ \,, \\ f(x)\,, & x \in H_k^- \,. \end{cases}$$

Then $f_k^+, f_k^- \in BV^+(\mathbb{R}^n)$ and they are both symmetric with respect to the hyperplane H_k. Moreover

$$\|Df\| \geq \frac{\|Df_k^+\| + \|Df_k^-\|}{2} \,.$$

This inequality is indeed an equality if $f \in C_c^\infty(\mathbb{R}^n)$; the general case of a BV function can be obtained by approximation. Therefore, we always have

$$\max\{\delta(f_k^+), \delta(f_k^-)\} \leq 2\delta(f) \,. \tag{2.10}$$

We now consider a pair of functions g_k^+ and g_k^- optimal for (2.9), so that

$$\lambda(f_k^+|H_k) = \int_{\mathbb{R}^n} |f_k^+ - g_k^+|^{n'} \,, \qquad \lambda(f_k^-|H_k) = \int_{\mathbb{R}^n} |f_k^- - g_k^-|^{n'} \,.$$

Then

$$\begin{aligned} \lambda(f) &\leq \int_{\mathbb{R}^n} |f - g_k^+|^{n'} = \int_{H_k^+} |f_k^+ - g_k^+|^{n'} + \int_{H_k^-} |f_k^- - g_k^+|^{n'} \\ &\leq 2^{n'-1}\left(\int_{H_k^+} |f_k^+ - g_k^+|^{n'} + \int_{H_k^-} |f_k^- - g_k^-|^{n'} + \int_{H_k^-} |g_k^+ - g_k^-|^{n'} \right). \end{aligned} \tag{2.11}$$

We state the following two claims:

Claim 1. *There exist two constants C_1 and $\bar\delta$ depending only on n such that, if $\delta(f) \leq \bar\delta$, then*

$$\int_{\mathbb{R}^n} |g_i^\sigma - g_j^\tau|^{n'} \leq C_1 \int_{H_i^\sigma \cap H_j^\tau} |g_i^\sigma - g_j^\tau|^{n'} \,, \tag{2.12}$$

whenever $1 \leq i < j \leq n$, $\sigma, \tau \in \{+,-\}$.

Claim 2. *If $\delta(f) \leq \bar\delta$, for every pair of indexes $1 \leq i < j \leq n$, there exists $k \in \{i,j\}$ such that*

$$\int_{H_k^-} |g_k^+ - g_k^-|^{n'} \leq 4^{n'} C_1 \left(\int_{H_k^+} |f_k^+ - g_k^+|^{n'} + \int_{H_k^-} |f_k^- - g_k^-|^{n'} \right). \tag{2.13}$$

The proof is now divided in three steps: in the first one we show how the second claim implies the thesis; in the second step we show how the second claim descends from the first one; in the third step we eventually prove the first claim.

Step I. *The second claim implies the thesis.*

Assume that the second claim holds and $\delta(f) \leq \bar\delta$. Then for every i, j with $1 \leq i < j \leq n$ there is $k \in \{i,j\}$ such that (2.13) holds. By (2.11) we find

$$\begin{aligned} \lambda(f) &\leq C\left(\int_{H_k^+} |f_k^+ - g_k^+|^{n'} + \int_{H_k^-} |f_k^- - g_k^-|^{n'} \right) = C\left(\frac{\lambda(f_k^+|H_k)}{2} + \frac{\lambda(f_k^-|H_k)}{2} \right) \\ &\leq C\big(\lambda(f_k^+) + \lambda(f_k^-)\big) \,, \end{aligned} \tag{2.14}$$

where in the last inequality we have applied Lemma 2.4 together with the fact that f_k^+ and f_k^- are symmetric with respect to the hyperplane H_k. By (2.10) and (2.14) we conclude that whenever $1 \leq i < j \leq n$ there exist $k \in \{i,j\}$ and $\sigma \in \{+,-\}$ such that

$$\lambda(f) \leq C\lambda(f_k^\sigma) \,, \qquad \delta(f_k^\sigma) \leq 2\delta(f) \,.$$

Furthermore the function f_k^σ is symmetric with respect to the hyperplane H_k, hence it is $1-$ symmetric up to a translation and a relabelling of the axes. The Theorem then follows iterating this basic procedure $n-1$ times.

Finally, if $\delta(f) \geq \bar\delta$ it suffices to take a $(n-1)-$symmetric function g such that $\delta(g) \leq 2^{n-1}\bar\delta$, $\lambda(g) > 0$.

Step II. *The first claim implies the second one.*
Let us assume that the first claim holds, take $i=1$, $j=2$ for simplicity and introduce

$$h_k := g_k^+ \chi_{H_k^+} + g_k^- \chi_{H_k^-}.$$

Thanks to the first claim,

$$\int_{\mathbb{R}^n} |h_1(x)-h_2(x)|^{n'}\,dx \ge \int_{H_1^+\cap H_2^+} |h_1(x)-h_2(x)|^{n'}\,dx = \int_{H_1^+\cap H_2^+} |g_1^+(x)-g_2^+(x)|^{n'}\,dx$$
$$\ge \frac{1}{C_1}\int_{\mathbb{R}^n} |g_1^+(x)-g_2^+(x)|^{n'}\,dx.$$

Similarly,

$$\int_{\mathbb{R}^n} |h_1(x)-h_2(x)|^{n'}\,dx \ge \int_{H_1^-\cap H_2^+} |h_1(x)-h_2(x)|^{n'}\,dx = \int_{H_1^-\cap H_2^+} |g_1^-(x)-g_2^+(x)|^{n'}\,dx$$
$$\ge \frac{1}{C_1}\int_{\mathbb{R}^n} |g_1^-(x)-g_2^+(x)|^{n'}\,dx.$$

Therefore,

$$\int_{\mathbb{R}^n} |g_1^+(x)-g_1^-(x)|^{n'}\,dx \le 2^{n'}C_1\int_{\mathbb{R}^n} |h_1(x)-h_2(x)|^{n'}\,dx\,,$$

and similarly

$$\int_{\mathbb{R}^n} |g_2^+(x)-g_2^-(x)|^{n'}\,dx \le 2^{n'}C_1\int_{\mathbb{R}^n} |h_1(x)-h_2(x)|^{n'}\,dx\,.$$

On the other hand,

$$\int_{\mathbb{R}^n} |h_1(x)-h_2(x)|^{n'}\,dx \le 2^{n'-1}\biggl(\int_{\mathbb{R}^n} |h_1(x)-f(x)|^{n'}\,dx + \int_{\mathbb{R}^n} |h_2(x)-f(x)|^{n'}\,dx\biggr)$$
$$= 2^{n'-1}\biggl(\int_{H_1^+} |g_1^+(x)-f_1^+(x)|^{n'}\,dx + \int_{H_1^-} |g_1^-(x)-f_1^-(x)|^{n'}\,dx$$
$$+\int_{H_2^+} |g_2^+(x)-f_2^+(x)|^{n'}\,dx + \int_{H_2^-} |g_2^-(x)-f_2^-(x)|^{n'}\,dx\biggr).$$

Therefore there exists $i\in\{1,2\}$ such that

$$\int_{H_i^+} |g_i^+(x)-f_i^+(x)|^{n'}\,dx + \int_{H_i^-} |g_i^-(x)-f_i^-(x)|^{n'}\,dx \ge \frac{1}{2^{n'}}\int_{\mathbb{R}^n} |h_1(x)-h_2(x)|^{n'}\,dx$$
$$\ge \frac{1}{4^{n'}C_1}\int_{\mathbb{R}^n} |g_i^+(x)-g_i^-(x)|^{n'}\,dx,$$

so that (2.13) follows.

Step III. *The first claim is true.*
To show the first claim, we first state the following simple estimate, whose proof is very similar to that of Lemma 2.4 in [6].
There exist two dimensional constants σ and C_1 such that, if $F=a\chi_{x+rB}$ and $G=b\chi_{y+sB}$ and the following assumptions hold,

- $\int_{\mathbb{R}^n} F^{n'} = \int_{\mathbb{R}^n} G^{n'} = 1$;
- $x\in\partial I$ and $y\in\partial J$, where I and J are two orthogonal half-spaces;
- $\int_{I\cap J} F^{n'} \ge 1/8$ and $\int_{I\cap J} G^{n'} \ge 1/8$;
- $\|F-G\|_{L^{n'}} \le \sigma$;

(2.15)

then

$$\int_{\mathbb{R}^n} |F-G|^{n'} \le C_1\int_{I\cap J} |F-G|^{n'}\,. \tag{2.16}$$

Inequality (2.12) follows from (2.16) with $F=g_i^\sigma$ and $G=g_j^\tau$, $I=H_i^\sigma$ and $J=H_j^\tau$. So we only need to check that assumptions (2.15) hold; in fact, the first two are obvious by construction. Concerning the third one, we observe that it follows easily whenever $\lambda(f)$ is sufficiently small; and this is true, by Lemma 2.5, if $\delta \le \bar\delta$.

In order to check the last condition, we begin by showing that $\lambda(f|H_i)$ is controlled by $\lambda(f)$. To this aim, let $g = d\chi_{z+tB}$ a function such that $\|g\|_{L^{n'}} = 1$ and $\lambda(f) = \|f-g\|_{L^{n'}}^{n'}$. Denote by $\hat g_i = d\chi_{z^i+tB}$, where z^i is the projection of z on H_i, and assume, to fix the ideas, that $z \in H_i^+$. Then

$$\lambda(f|H_i) \le \|f-\hat g_i\|_{L^{n'}}^{n'} \le \Big(\|f-g\|_{L^{n'}} + \|g-\hat g_i\|_{L^{n'}}\Big)^{n'} = \Big(\lambda(f)^{1/n'} + \|g-\hat g_i\|_{L^{n'}}\Big)^{n'}. \tag{2.17}$$

Since g and $\hat g_i$ are characteristic functions of equal balls multiplied by the same constant d, simple geometric arguments show that

$$\begin{aligned}\|g-\hat g_i\|_{L^{n'}}^{n'} &\le C\int_{H_i^+} |g-\hat g_i|^{n'}\,dx = C\int_{H_i^+} |g^{n'} - \hat g_i^{n'}|\,dx \le C'\bigg|\int_{H_i^+} g^{n'} - \hat g_i^{n'}\,dx\bigg| \\ &= C'\big|\,\|g\|_{L^{n'}(H_i^+)}^{n'} - \|f\|_{L^{n'}(H_i^+)}^{n'}\,\big| \le C''\|g-f\|_{L^{n'}(H_i^+)} \le C''\lambda(f)^{1/n'}.\end{aligned}$$

This inequality together with (2.17) implies that $\lambda(f|H_i) \le C\lambda(f)^{1/n'}$; clearly, the same is true for $\lambda(f|H_j)$. Notice now that if $\tilde g_i$ realizes the minimum in the definition of $\lambda(f|H_i)$,

$$\begin{aligned}\|f_i^+ - f\|_{L^{n'}}^{n'} &= \int_{H_i^-} |f_i^+ - f|^{n'}\,dx \le 2^{n'-1}\bigg(\int_{H_i^-} |f_i^+ - \tilde g_i|^{n'}\,dx + \int_{H_i^-} |\tilde g_i - f|^{n'}\,dx\bigg) \\ &= 2^{n'-1}\bigg(\int_{H_i^+} |f-\tilde g_i|^{n'}\,dx + \int_{H_i^-} |\tilde g_i - f|^{n'}\,dx\bigg) = 2^{n'-1}\lambda(f|H_i) \le C\lambda(f)^{1/n'}.\end{aligned}$$

A similar estimate holds replacing f_i^+ by f_i^- or $f_j^\pm$. As a consequence of the above estimate we have also that

$$\lambda(f_i^\sigma|H_i) \le 2^{n'-1}\big(\lambda(f|H_i) + \|f_i^\sigma - f\|_{L^{n'}}^{n'}\big) \le C\lambda(f)^{1/n'}.$$

Finally from the last two estimates we have

$$\|g_i^\sigma - g_j^\tau\|_{L^{n'}} \le \|g_i^\sigma - f_i^\sigma\|_{L^{n'}} + \|f_i^\sigma - f\|_{L^{n'}} + \|f - f_j^\tau\|_{L^{n'}} + \|f_j^\tau - g_j^\tau\|_{L^{n'}} \le C\lambda(f)^{1/n'^2},$$

proving –provided $\delta(f) \le \bar\delta$– the last assumption in (2.15), hence the first claim. □

Proof. (of Theorem 2.1). As we have already done before, we may assume without loss of generality that $\|f\|_{L^{n'}} = 1$ and $\delta(f) \le 1$. Moreover, by Lemma 2.6, it is also possible to assume that f is $(n-1)$-symmetric. Let H_n, $H_n^\pm$, and $f_n^\pm$ be defined as in the proof of that Lemma, however in the sequel to simplify the notation we suppress the index n. Up to a translation we can assume that $H = \{x : x_n = 0\}$, so that f^+ and f^- are n-symmetric. We recall that by (2.10) $\delta(f^\pm) \le 2\delta(f)$; hence, the theorem will be achieved once we show that

$$\lambda(f) \le C\big(\lambda(f^+) + \sqrt{\delta(f)}\big)\,. \tag{2.18}$$

To this end, let $g^\pm$ be two functions realizig $\lambda(f^\pm|\{0\})$. Then by Lemma 2.4

$$\begin{aligned}\lambda(f) &\le \int_{\mathbb{R}^n} |f-g^+|^{n'} = \frac{1}{2}\Big(\|f^+ - g^+\|_{L^{n'}}^{n'} + \|f^- - g^+\|_{L^{n'}}^{n'}\Big) \\ &\le \frac{1}{2}\|f^+ - g^+\|_{L^{n'}}^{n'} + 2^{n'-2}\Big(\|f^- - f^+\|_{L^{n'}}^{n'} + \|f^+ - g^+\|_{L^{n'}}^{n'}\Big) \\ &\le C\Big(\lambda(f^+|\{0\}) + \|f^- - f^+\|_{L^{n'}}^{n'}\Big) \le C\Big(\lambda(f^+) + \|f^- - f^+\|_{L^{n'}}^{n'}\Big).\end{aligned} \tag{2.19}$$

Let $f^{+\star}$ and $f^{-\star}$ be the symmetric decreasing rearrangements of f^+ and f^- respectively, then by (2.1) in Theorem 2.2, $\delta(f) \le 1$ and (2.10), we have

$$\int_{\mathbb{R}^n} |f^{+\star} - f^+|^{n'} \le C\sqrt{\delta(f^+)} \le C\sqrt{\delta(f)}$$

and similarly for f^-. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} |f^+ - f^-|^{n'} &\le C\Big(\int_{\mathbb{R}^n} |f^{+\star} - f^{-\star}|^{n'} + \int_{\mathbb{R}^n} |f^{+\star} - f^+|^{n'} + \int_{\mathbb{R}^n} |f^{-\star} - f^-|^{n'}\Big)\\ &\le C\Big(\int_{\mathbb{R}^n} |f^{+\star} - f^{-\star}|^{n'} + \sqrt{\delta(f)}\Big).\end{aligned}$$

By this last estimate and by (2.19) we deduce

$$\lambda(f) \le C\Big(\lambda(f^+) + \sqrt{\delta(f)} + \int_{\mathbb{R}^n} |f^{+\star} - f^{-\star}|^{n'}\Big).$$

Therefore, in order to prove (2.18), and hence the theorem, we just need to show that

$$\int_{\mathbb{R}^n} |f^{+\star} - f^{-\star}|^{n'} \le C\sqrt{\delta(f)}\,. \tag{2.20}$$

By (2.4) we find

$$\int_{\mathbb{R}^n} |f^{+\star} - f^{-\star}|^{n'} \le \left(\int_0^\infty \big|\{f^{+\star} > t\}\Delta\{f^{-\star} > t\}\big|^{1/n'}\,dt\right)^{n'}. \tag{2.21}$$

Since $\{f^{\pm\star} > t\}$ are balls of center 0 and measure $2|\{f > t\} \cap H^\pm| =: 2\mu^\pm(t)$, we have

$$\big|\{f^{+\star} > t\}\Delta\{f^{-\star} > t\}\big| = 2|\mu^+(t) - \mu^-(t)|$$

On the other hand $\mu(t) = \mu^+(t) + \mu^-(t)$; thus, defining $\mu_{\max}(t) = \max\{\mu^+(t), \mu^-(t)\}$, by the Coarea Formula and the isoperimetric inequality we find

$$\begin{aligned}\|Df\| = |Df|(\mathbb{R}^n) &= \int_0^\infty P(f > t)\,dt \ge n\omega_n^{1/n}\int_0^\infty \mu(t)^{1/n'}\,dt\\ &= n\omega_n^{1/n}\int_0^\infty 2^{1/n'}\left(\frac{\mu^+(t) + \mu^-(t)}{2}\right)^{1/n'} dt \ge n\omega_n^{1/n}\int_0^\infty \mu_{\max}(t)^{1/n'}\,dt\,.\end{aligned} \tag{2.22}$$

As $q = 1/n' \in (0,1)$, by an easy computation we have for every $0 < a < b$

$$\left(\frac{a+b}{2}\right)^q \ge \frac{a^q + b^q}{2} + \frac{q(1-q)}{8}\,b^{q-2}(b-a)^2;$$

so,

$$\|Df\| \ge n\omega_n^{1/n}\int_0^\infty 2^q\,\frac{(\mu^+(t))^q + (\mu^-(t))^q}{2}\,dt + \frac{1}{C}\int_0^\infty \mu_{\max}(t)^{q-2}|\mu^+(t) - \mu^-(t)|^2\,dt\,.$$

Recalling again that $|\{f^{\pm\star} > t\}| = 2\mu^\pm(t)$ and that these level sets are balls, we find

$$n\omega_n^{1/n}\int_0^\infty 2^q(\mu^+(t))^q = \int_0^\infty P(f^{\pm\star} > t)\,dt = \|Df^{\pm\star}\|\,.$$

Therefore we conclude that

$$\|Df\| \ge \frac{\|Df^{+\star}\| + \|Df^{-\star}\|}{2} + \frac{1}{C}\int_0^\infty \mu_{\max}(t)^{q-2}|\mu^+(t) - \mu^-(t)|^2\,dt\,,$$

i.e.,

$$\delta(f) \ge \frac{\delta(f^{+\star}) + \delta(f^{-\star})}{2} + \frac{1}{C}\int_0^\infty \mu_{\max}(t)^{q-2}|\mu^+(t) - \mu^-(t)|^2\,dt\,.$$

The first two terms are non negative, thus

$$C\delta(f) \ge \int_0^\infty \mu_{\max}(t)^{q-2}|\mu^+(t) - \mu^-(t)|^2\,dt\,.$$

Eventually, by (2.21) and recalling (2.22) it is

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|f^{+\star}-f^{-\star}|^{n'}\right)^q &\le 2^q\int_0^\infty|\mu^+(t)-\mu^-(t)|^q\,dt = 2^q\int_0^\infty\left(\frac{|\mu^+(t)-\mu^-(t)|}{\mu_{\max}(t)}\right)^q\mu_{\max}(t)^q\,dt\\ &\le 2^q\left(\int_0^\infty\mu_{\max}(t)^q\,dt\right)^{1-q/2}\left(\int_0^\infty\frac{|\mu^+(t)-\mu^-(t)|^2}{\mu_{\max}(t)^2}\mu_{\max}(t)^q\,dt\right)^{q/2}\\ &\le C\|Df\|^{1-q/2}\left(\int_0^\infty|\mu^+(t)-\mu^-(t)|^2\mu_{\max}(t)^{q-2}\,dt\right)^{q/2}\le C\delta(f)^{q/2}\end{aligned}$$

and the proof of (2.20), therefore of the theorem, is achieved. □

Remark 2.7. *Notice that in (2.18) $\lambda(f^+)$ can be replaced by $\lambda(f^-)$, hence the $n-$symmetric function g in the statement of Theorem 2.1 can be arbitrarily chosen equal to f^+ or f^-.*

3. The spherically symmetric case

In this section we are concerned with the case of spherically symmetric decreasing functions.

Theorem 3.1. *Let $f\in BV^+(\mathbb{R}^n)$ be a spherically symmetric decreasing function. Then*

$$\lambda(f)\le C(n)\delta(f)\,.$$

Proof. Without loss of generality we may assume that $\delta(f)\le\delta^*$ for a suitable small constant δ^*, and that $f\in C_c^\infty(\mathbb{R}^n)$, $f\ge 0$, with $f(x)=u(|x|)$ for $u:[0,\infty)\to[0,\infty)$ such that $u(r)=0$ if $r\ge M$ and $u'(r)<0$ on $(0,M)$. By a rescaling and a multiplication by a constant we can also require that

$$\int_B f^{n'}=\int_{\mathbb{R}^n\setminus B}f^{n'}=\frac12\int_{\mathbb{R}^n}f^{n'}\,,\qquad\qquad u(1)=1\,,\tag{3.1}$$

so that

$$\|f\|_{L^{n'}}^{n'}=2\int_B f^{n'}\ge 2\omega_n\,.\tag{3.2}$$

By Lemma 2.5 we may assume $\lambda(f)\le\varepsilon(n)$, for $\varepsilon(n)$ as small as we wish. We claim that, provided $\varepsilon(n)$ is small enough, then

$$\int_{\mathbb{R}^n}f^{n'}\le C(n)\,,\tag{3.3}$$

for a constant $C(n)$ independent from f. To show this, let $m=\|f\|_{L^{n'}}^{n'}$ and let $h=\alpha\chi_{rB}$ be such that $\|h\|_{L^{n'}}=\|f\|_{L^{n'}}$, and

$$\frac{\|f-h\|_{L^{n'}}^{n'}}{m}=\lambda(f|\{0\})\le 3^{n'}\lambda(f)\le 3^{n'}\varepsilon(n)\,.\tag{3.4}$$

If $\alpha\le 2$, then (3.4) implies $3^{n'}\varepsilon(n)m\ge\int_{B\cap\{f>2\}}(f-2)^{n'}$, therefore

$$\begin{aligned}\frac m2=\int_B f^{n'}&\le 2^{n'}\omega_n+\int_{B\cap\{f>2\}}f^{n'}\le 2^{n'}\omega_n+2^{n'-1}\left(2^{n'}\omega_n+\int_{B\cap\{f>2\}}(f-2)^{n'}\right)\\ &\le 4^{n'}\omega_n+2^{n'-1}3^{n'}\varepsilon(n)m\,,\end{aligned}$$

and in conclusion $m\le C(n)$ provided $\varepsilon(n)$ is small enough. On the other hand the case $\alpha\ge 2$ can be excluded as soon as $\varepsilon(n)$ is small enough, for by (3.4), if $\alpha\ge 2$, we have

$$3^{n'}\varepsilon(n)m\ge\int_{\mathbb{R}^n\setminus B}|h-f|^{n'}\ge\int_{\mathbb{R}^n\setminus B}f^{n'}=\frac m2\,.$$

Therefore (3.3) is proved. Consider now the function $g_0=\chi_{cB}$, where c is such that $\int_{\mathbb{R}^n}g_0^{n'}=\int_{\mathbb{R}^n}f^{n'}$; note that, by (3.2), $c\ge 2^{1/n}$ and moreover

$$\lambda(f)\le\frac{1}{\|f\|_{L^{n'}}}\int_{\mathbb{R}^n}|f-g_0|^{n'}\le C\int_{\mathbb{R}^n}|f-g_0|^{n'}\,.$$

Therefore the theorem is proved by showing that

$$\int_{\mathbb{R}^n} |f - g_0|^{n'} \le C\delta(f)\,. \tag{3.5}$$

To this end we now introduce a step function g related to f in a special way. We first define $v : [0,\infty) \to [0,\infty)$ by

$$v(r) = (1+a)\chi_{[0,1]}(r) + \chi_{(1,b]}(r)\,,\ r \ge 0\,,$$

where $a \ge 0$ and $b \ge 1$ are chosen so that

$$(1+a)^{n'}\omega_n = (b^n - 1)\omega_n = \frac{1}{2}\int_{\mathbb{R}^n} f^{n'}\,; \tag{3.6}$$

note that, in fact, $b \ge 2^{1/n}$ by (3.2). We set $g(x) = v(|x|)$, so $g \in BV^+(\mathbb{R}^n)$, it is spherically symmetric, decreasing and, as an immediate consequence of (3.6), it satisfies

$$\int_B g^{n'} = \int_{\mathbb{R}^n\setminus B} g^{n'} = \frac{1}{2}\int_{\mathbb{R}^n} f^{n'}\,. \tag{3.7}$$

Thanks to (3.7) it is more convenient to compare f with g rather than with g_0. The proof of (3.5), hence of the theorem, will be achieved by proving the following two estimates:

$$\int_{\mathbb{R}^n} |g_0 - g|^{n'} \le C\delta(f)\,, \tag{3.8}$$

$$\int_{\mathbb{R}^n} |f - g|^{n'} \le C\delta(f)\,. \tag{3.9}$$

We now split the proof into several steps.

Step I. $\|Dg\| = n\omega_n(a + b^{n-1}) \le \|Df\|$, *hence* $\delta(g) \le \delta(f)$.

The first equality is a simple computation; concerning the inequality $\|Dg\| \le \|Df\|$, notice first that, by (3.1) and (3.7), $\chi_B f$ and $\chi_B g$ have the same $L^{n'}$ norm. Recalling that characteristic functions of ball are optimal for the Sobolev inequality, we deduce

$$|Df|(B) = \|D\chi_B f\| - n\omega_n \ge \|D(\chi_B g)\| - n\omega_n = n\omega_n a.$$

Similarly

$$|Df|(\mathbb{R}^n \setminus B) = |D(\chi_B + \chi_{\mathbb{R}^n\setminus B} f)|(\mathbb{R}^n) \ge |D(\chi_B + \chi_{\mathbb{R}^n\setminus B} g)|(\mathbb{R}^n) = n\omega_n b^{n-1}\,,$$

and thus $\|Dg\| \le \|Df\|$ follows. The fact that $\delta(g) \le \delta(f)$ is immediate as $\|f\|_{L^{n'}} = \|g\|_{L^{n'}}$.

Step II. $0 \le a \le C\delta(f)$ *and* $2^{1/n} \le b \le c \le 2^{1/n} + C\delta(f)$.

First of all notice that since f is bounded in $L^{n'}$ by (3.3), (3.6) yields that a and b are bounded. If $\{f_h\}$ is a sequence such that $\delta(f_h) \to 0$, hence $\delta(g_h) \to 0$, then the functions $\{g_h\}$ are equibounded in $L^\infty(\mathbb{R}^n)$, have equibounded supports and $\|Dg_h\|$ are bounded. Therefore, we may assume that g_h converges strongly in $L^{n'}(\mathbb{R}^n)$ to some function $\hat{g}$ such that $\delta(\hat{g}) = 0$, thus $\hat{g} = \alpha\chi_{rB}$. Recalling that $b_h \ge 2^{1/n}$, we have $\alpha = 1$, hence $a_h \to 0$. This proves that a can be taken as small as we wish provided $\delta(f)$ is small enough. Next we note that, as $a \ge 0$, the inequality $b \le c$ follows from

$$\omega_n c^n = \int_{\mathbb{R}^n} g_0^{n'} = \int_{\mathbb{R}^n} f^{n'} = \int_{\mathbb{R}^n} g^{n'} = \omega_n(1+a)^{n'} + \omega_n(b^n - 1) \ge \omega_n b^n\,.$$

Moreover

$$\omega_n c^n = \int_{\mathbb{R}^n} g_0^{n'} = \int_{\mathbb{R}^n} f^{n'} = 2\int_B g^{n'} = 2\omega_n(1+a)^{n'}\,.$$

Thus the inequality $c \le 2^{1/n} + C\delta(f)$ will follow immediately from $a \le C\delta(f)$. To prove this last estimate, we apply (3.6) to deduce

$$b = (1 + (1+a)^{n'})^{1/n} = 2^{1/n} + \frac{a}{(n-1)2^{1/n'}} + o(a)\,.$$

On the one hand, if we let $\gamma = 2^{-1/n'} + 2^{-1} > 1$, then

$$\|Dg\| = n\omega_n(a + b^{n-1}) = n\omega_n\left(a + 2^{1/n'}\left(1 + \frac{a}{2} + o(a)\right)\right) = n\omega_n 2^{1/n'}\left(1 + \gamma a + o(a)\right);$$

on the other hand, by (3.6) $\|g\|_{L^{n'}} = (2\omega_n)^{1/n'}(1+a)$. Therefore, by Step I,

$$\delta(f) \geq \delta(g) = \frac{1 + \gamma a + o(a)}{(1+a)} - 1 = \frac{(\gamma - 1)a}{1+a} + o(a).$$

This proves that $a \leq C\delta(f)$, and therefore concludes the proof of Step II.

Step III. *Proof of (3.8).*
This is a pretty easy consequence of Step II, as

$$\int_{\mathbb{R}^n} |g_0 - g|^{n'} = \omega_n a^{n'} + \omega_n(c^n - b^n).$$

The remaining steps are now devoted to the proof of (3.9).

Step IV. $\int_{r_0 B} |f-g|^{n'} \leq C\delta(f)$, *where* $r_0 = \min\{r \geq 0 : u(r) \leq 2\}$.
This is trivial if $r_0 = 0$; otherwise, we claim

$$r_0^n \leq \frac{2n'}{2^{n'} - 1} a \leq C\delta(f). \tag{3.10}$$

Indeed, on the one hand we have, thanks to (3.1), (3.7) and Step II,

$$\int_B f^{n'} = \int_B g^{n'} = \omega_n(1+a)^{n'} \leq \omega_n(1 + 2n'a),$$

while on the other hand, by definition of r_0,

$$\int_B f^{n'} \geq |r_0 B| 2^{n'} + |B \setminus r_0 B| \geq \omega_n 2^{n'} r_0^n + \omega_n(1 - r_0^n) = \omega_n\big((2^{n'} - 1) r_0^n + 1\big),$$

so that (3.10) follows. Therefore, as $f \geq 2 \geq 1 + a = g$ on $r_0 B$,

$$\int_{r_0 B} |f - g|^{n'} \leq \int_{r_0 B} f^{n'} = \int_B f^{n'} - \int_{B \setminus r_0 B} f^{n'} \leq \omega_n(1 + 2n'a) - \omega_n(1 - r_0^n) = \omega_n(2n'a + r_0^n).$$

The proof of Step IV is then concluded by applying Step II and (3.10).

Step V. $\int_B |f - g|^{n'} \leq C\delta(f)$.
As $\int_B f^{n'} = \int_B g^{n'}$ and $f \geq g$ on $r_0 B$, it follows that

$$\int_{B \setminus r_0 B} f^{n'} \leq \int_{B \setminus r_0 B} g^{n'} \leq (1 + 2n'a)|B \setminus r_0 B|.$$

Since $1 \leq f \leq 2$ on $B \setminus r_0 B$,

$$\int_{B \setminus r_0 B} f^{n'} = \int_{B \setminus r_0 B} (1 + (f-1))^{n'} \geq \int_{B \setminus r_0 B} (1 + n'(f-1)) \geq \int_{B \setminus r_0 B} (1 + n'(f-1)^{n'}).$$

By the last two estimates

$$\int_{B \setminus r_0 B} (f - g_0)^{n'} = \int_{B \setminus r_0 B} (f-1)^{n'} \leq 2a|B \setminus r_0 B| \leq C\delta(f).$$

We conclude the proof of Step V thanks to (3.8) and to Step IV.

Step VI. *Defining* r_1 *by* $u(r_1) = 1/2$, *one has*

$$\int_{\mathbb{R}^n \setminus B} |f - g|^{n'} \leq \int_{\mathbb{R}^n \setminus r_1 B} f^{n'} + \int_{r_1 B \setminus B} (1 - f^{n'}) + \omega_n(\max\{r_1^n, b^n\} - \min\{r_1^n, b^n\}). \tag{3.11}$$

Let us begin with $r_1 B \setminus B$. If $r_1 \leq b$ then

$$\int_{r_1 B \setminus B} |f - g|^{n'} = \int_{r_1 B \setminus B} (1 - f)^{n'} \leq \int_{r_1 B \setminus B} (1 - f^{n'}),$$

as $0 \le f \le 1$ on $\mathbb{R}^n \setminus B$ and therefore $(1-f)^{n'} \le 1 - f \le 1 - f^{n'}$. On the other hand if $r_1 \ge b$ then

$$\int_{r_1B\setminus B} |f-g|^{n'} = \int_{r_1B\setminus bB} f^{n'} + \int_{bB\setminus B} (1-f)^{n'} = \int_{r_1B\setminus bB} \Big(f^{n'} - (1-f)^{n'}\Big) + \int_{r_1B\setminus B} (1-f)^{n'}$$
$$\le \omega_n (r_1^n - b^n) + \int_{r_1B\setminus B} (1 - f^{n'})\,.$$

Therefore

$$\int_{r_1B\setminus B} |f-g|^{n'} \le \omega_n(\max\{r_1^n, b^n\} - b^n\}) + \int_{r_1B\setminus B} (1-f^{n'})\,. \tag{3.12}$$

Passing now to $\mathbb{R}^n \setminus r_1B$, again if $b \le r_1$, we have

$$\int_{\mathbb{R}^n\setminus r_1B} |f-g|^{n'} = \int_{\mathbb{R}^n\setminus r_1B} f^{n'}\,.$$

If else $b \ge r_1$ then

$$\int_{\mathbb{R}^n\setminus r_1B} |f-g|^{n'} = \int_{\mathbb{R}^n\setminus bB} f^{n'} + \int_{bB\setminus r_1B} (1-f)^{n'} = \int_{\mathbb{R}^n\setminus r_1B} f^{n'} + \int_{bB\setminus r_1B} \Big((1-f)^{n'} - f^{n'}\Big)$$
$$\le \int_{\mathbb{R}^n\setminus r_1B} f^{n'} + \omega_n(b^n - r_1^n)\,.$$

Therefore

$$\int_{\mathbb{R}^n\setminus r_1B} |f-g|^{n'} \le \omega_n(\max\{r_1^n, b^n\} - r_1^n\}) + \int_{\mathbb{R}^n\setminus r_1B} f^{n'}\,. \tag{3.13}$$

Adding up (3.12) and (3.13) we find (3.11).

Step VII. *Proof of (3.9).*

This step will conclude the proof of the theorem. Thanks to Step V, in order to prove (3.9) it suffices to estimate the right hand side of (3.11) by $C\delta(f)$. To this end, we modify f into a new function $f_1(x) = u_1(|x|)$, with

$$u_1(r) = u(r)\chi_{[0,1]} + \chi_{[1,s_1]} + \frac{1}{2}\chi_{[s_1,s_2]}\,,$$

where $s_1 \le r_1 \le s_2$ are uniquely determined by

$$\int_{r_1B} f^{n'} = \int_{r_1B} f_1^{n'}, \qquad \int_{\mathbb{R}^n\setminus r_1B} f^{n'} = \int_{\mathbb{R}^n\setminus r_1B} f_1^{n'}. \tag{3.14}$$

As in Step I it can be easily checked that $\|Df\| \ge \|Df_1\|$, hence $\delta(f) \ge \delta(f_1)$. Moreover, arguing exactly as in Step II and recalling Step V, we have that if $\delta(f) \to 0$ then f_1 converges strongly in $L^{n'}$ to $\chi_{2^{1/n}B}$, hence s_1, $s_2 \to 2^{1/n}$, so that $t_1 = b - s_1$ and $t_2 = s_2 - b$ converge to 0. By (3.14)

$$\int_{\mathbb{R}^n\setminus r_1B} f^{n'} = \int_{\mathbb{R}^n\setminus r_1B} f_1^{n'} = \omega_n \frac{s_2^n - r_1^n}{2^{n'}} \le \omega_n \frac{s_2^n - s_1^n}{2^{n'}} \le C(t_1+t_2) + o(t_1+t_2)\,. \tag{3.15}$$

Similarly, again by (3.14),

$$\int_{r_1B\setminus B} f^{n'} = \int_{r_1B\setminus B} f_1^{n'} = \omega_n(s_1^n - 1) + \omega_n\frac{r_1^n - s_1^n}{2^{n'}} = \omega_n(r_1^n - 1) - \omega_n\left(1 - \frac{1}{2^{n'}}\right)(r_1^n - s_1^n)$$

so that

$$\int_{r_1B\setminus B} 1 - f^{n'} = \omega_n\left(1 - \frac{1}{2^{n'}}\right)(r_1^n - s_1^n) \le C(s_2^n - s_1^n) \le C(t_1+t_2) + o(t_1+t_2)\,. \tag{3.16}$$

Therefore, by (3.15) and by (3.16), thanks to Step VI it follows

$$\int_{\mathbb{R}^n\setminus B} |f-g|^{n'} \le C(t_1+t_2) + o(t_1+t_2)\,. \tag{3.17}$$

Now we show that

$$t_1 = \eta t_2 + o(t_2) \qquad \text{with } \eta = \frac{1}{2^{n'}-1} < 1. \tag{3.18}$$

Indeed by (3.1), (3.6) and (3.14)

$$\omega_n(b^n-1)=\int_{\mathbb{R}^n\setminus B} g^{n'}=\int_{\mathbb{R}^n\setminus B} f^{n'}=\int_{\mathbb{R}^n\setminus B} f_1^{n'}=\omega_n(s_1^n-1)+\omega_n\frac{s_2^n-s_1^n}{2^{n'}};$$

hence, adding on both sides ω_n and dividing by $\omega_n b^n$, since $t_1,t_2\to 0$ as $\delta(f)\to 0$,

$$1=\left(1-\frac{t_1}{b}\right)^n+\frac{1}{2^{n'}}\left(\left(1+\frac{t_2}{b}\right)^n-\left(1-\frac{t_1}{b}\right)^n\right)=1-n\frac{t_1}{b}+\frac{n(t_2+t_1)}{2^{n'}b}+o(t_1+t_2)\,.$$

Therefore

$$t_1=\frac{t_2}{2^{n'}-1}+o(t_1+t_2)\,,$$

and (3.18) easily follows. In order to prove (3.9), thanks to (3.17) and to (3.18), it remains to show that $t_2\le C\delta(f)$.
To this aim, we have

$$\begin{aligned}|Df_1|(\mathbb{R}^n\setminus\overline{B})-|Dg|(\mathbb{R}^n\setminus\overline{B})&=n\omega_n\frac{s_1^{n-1}+s_2^{n-1}}{2}-n\omega_n b^{n-1}\\&=\frac{n\omega_n b^{n-1}}{2}\left(\left(1-\frac{t_1}{b}\right)^{n-1}+\left(1+\frac{t_2}{b}\right)^{n-1}-2\right)\\&=\frac{n(n-1)\omega_n b^{n-2}}{2}(t_2-t_1)+o(t_2)=\frac{n(n-1)\omega_n b^{n-2}(1-\eta)}{2}\,t_2+o(t_2)\,.\end{aligned}$$

Therefore

$$\begin{aligned}t_2+o(t_2)&\le C\big(|Df_1|(\mathbb{R}^n\setminus\overline{B})-|Dg|(\mathbb{R}^n\setminus\overline{B})\big)\le C\big(\|Df_1\|-\|Dg_0\|+n\omega_n(c^{n-1}-b^{n-1})\big)\\&\le C\big(\|Df\|-\|Dg_0\|+\delta(f)\big)\le C\delta(f)\,.\end{aligned}$$

where we have used Step II and the fact that

$$\|Df\|-\|Dg_0\|=n\omega_n^{1/n}\|f\|_{L^{n'}}\delta(f)\le C\delta(f).$$

The proof is then achieved. □

4. Proof of Theorem 1.1

We gather the results from the previous sections in order to prove Theorem 1.1.

Proof. (of Theorem 1.1). Let $f\in BV(\mathbb{R}^n)$ and, without loss of generality, let us assume that $\|f\|_{L^{n'}}=1$. We want to further reduce to the case in which f does not change sign showing that f is "close" either to $|f|$ or to $-|f|$, in the sense that

$$\min\left\{\int_{\mathbb{R}^n}|f+|f||^{n'},\int_{\mathbb{R}^n}|f-|f||^{n'}\right\}\le\frac{2^{n'-1}}{2^{1/n}-1}\,\delta(f)\,,\qquad \delta(|f|)=\delta(-|f|)\le\delta(f)\,. \tag{4.1}$$

The second estimate is an immediate consequence of the elementary inequality $\|D(\pm|f|)\|\le\|Df\|$. Concerning the first estimate, we remark that it can be rewritten as

$$\min\left\{\int_{\{f>0\}}f^{n'},1-\int_{\{f>0\}}f^{n'}\right\}\le\frac{1}{2(2^{1/n}-1)}\,\delta(f)\,,$$

and this is true since, by the Sobolev inequality and the concavity of $t\mapsto t^{1/n'}+(1-t)^{1/n'}-1$, one has

$$\begin{aligned}\delta(f)&=\frac{1}{n\omega_n^{1/n}}\Big(\|Df^+\|+\|Df^-\|\Big)-1\ge\left(\int_{\{f>0\}}|f|^{n'}\right)^{1/n'}+\left(1-\int_{\{f>0\}}|f|^{n'}\right)^{1/n'}-1\\&\ge\frac{2^{1/n}-1}{1/2}\min\left\{\int_{\{f>0\}}f^{n'},1-\int_{\{f>0\}}f^{n'}\right\}.\end{aligned}$$

By (4.1), to show the theorem it is admissible to assume that $f \in BV^+(\mathbb{R}^n)$, with $\|f\|_{L^{n'}} = 1$; we then apply Theorem 2.1, finding a n-symmetric function $g \in BV^+(\mathbb{R}^n)$ such that $\lambda(f) \leq C\big(\lambda(g) + \sqrt{\delta(f)}\big)$ and $\delta(g) \leq C\delta(f)$. By Theorem 2.2 and by Theorem 3.1 we deduce

$$\lambda(f) \leq C\big(\lambda(g^\star) + \sqrt{\delta(f)}\big) \leq C\big(\delta(g^\star) + \sqrt{\delta(f)}\big)\,;$$

finally, thanks to the Polya-Szegö inequality $\delta(g^\star) \leq \delta(g)$ the thesis follows as $\delta(g) \leq C\delta(f)$. □

Appendix A. A compactness result

In this section we prove a compactness result for minimizing sequences of the Sobolev imbedding Theorem for $p = 1$. This extends an analogous result stated, for $p > 1$, in [9]. The proof given here follows very closely the one of [13, Theorem 4.9] and is presented for the sake of completeness.

Theorem A.1. *Let $f_h \in BV(\mathbb{R}^n)$ with $\|f_h\|_{L^{n'}} = 1$ and $\delta(f_h) \to 0$. There exist $x_h \in \mathbb{R}^n$ and $r_h > 0$ such that, up to a subsequence, the rescaled-translated functions*

$$g_h(x) := r_h^{n-1} f_h\big(r_h(x - x_h)\big) \tag{A.1}$$

converge strongly in $L^{n'}(\mathbb{R}^n)$ to some function $f \in BV(\mathbb{R}^n)$ with $\|Df\| = n\omega_n^{1/n}$.

Note that the above rescaling does not change neither the $L^{n'}$ norm nor the total variation; moreover, the functions g_h in (A.1) converge weakly to $f \in BV(\mathbb{R}^n)$ and the total variations $\|Dg_h\|$ converge to $\|Df\|$.

Proof. (of Theorem A.1). Let us consider the sequence of probability measures $\nu_h = |f_h|^{n'}\,dx$. For every $h \in \mathbb{N}$ we can find $x_h \in \mathbb{R}^n$, $r_h > 0$ such that

$$\nu_h(x_h + r_h B) = \frac{1}{2}\,, \qquad \qquad \nu_h(x + rB) = \frac{1}{2} \Longrightarrow r \geq r_h.$$

Therefore, still denoting by f_h the rescaled-translated functions defined by (A.1), we may assume that, for any h,

$$\nu_h(B) = \frac{1}{2}\,, \qquad \qquad \nu_h(x + rB) < \frac{1}{2}\ \forall\, r < 1,\ x \in \mathbb{R}^n\,. \tag{A.2}$$

We apply now Lions' *concentration-compactness* lemma (see [9, pag. 115ff], [13, Lemma 4.3]) to the sequence ν_h. Up to a subsequence, there are three possibilities:

(i) *vanishing:* for every $r > 0$

$$\lim_{h\to\infty} \sup_{x\in\mathbb{R}^n} \nu_h(x + rB) = 0;$$

(ii) *dichotomy:* there exists $\lambda \in (0, 1)$ such that for any $\varepsilon > 0$ there exist $R > 0$ and a sequence y_h such that, for any $R' > R$, we have, definitively,

$$\nu_h(y_h + RB) > \lambda - \varepsilon\,, \qquad \qquad \nu_h(\mathbb{R}^n \setminus (y_h + R'B)) > 1 - \lambda - \varepsilon\,.$$

(iii) *compactness:* there exist y_h such that for any $\varepsilon > 0$ there is $R > 0$ with the property that for all h

$$\nu_h(y_h + RB) \geq 1 - \varepsilon\,.$$

By the equality in (A.2) we can exclude case (i).
In order to exclude also case (ii), notice that it would imply the existence of a sequence ψ_h obtained by suitably rescaling-translating the f_h's such that $\|\psi_h\|_{L^{n'}} = 1$, $\delta(\psi_h) \to 0$ and

$$\int_B |\psi_h|^{n'} > \lambda - \varepsilon, \qquad \qquad \int_{\mathbb{R}^n\setminus 2B} |\psi_h|^{n'} > 1 - \lambda - \varepsilon$$

for h large enough. Let now $\varphi \in C_c^\infty(2B)$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in a neighborhood of B. Then by the Sobolev inequality we find

$$\begin{aligned}\|D\psi_h\| &= |D(\varphi\psi_h)|(B) + |D\psi_h|(2B\setminus B) + |D((1-\varphi)\psi_h)|(\mathbb{R}^n \setminus 2B)\\ &= \|D(\varphi\psi_h)\| - |D(\varphi\psi_h)|(2B\setminus B) + |D\psi_h|(2B\setminus B) + \|D((1-\varphi)\psi_h)\| - |D((1-\varphi)\psi_h)|(2B\setminus B)\\ &\ge n\omega_n^{1/n}\Big(\|\varphi\psi_h\|_{L^{n'}} + \|(1-\varphi)\psi_h\|_{L^{n'}}\Big) - 2\int_{2B\setminus B}\psi_h|\nabla\varphi|\,.\end{aligned}$$

If we let $h \to \infty$ we deduce

$$n\omega_n^{1/n} \ge n\omega_n^{1/n}\Big((\lambda-\varepsilon)^{1/n'} + (1-\lambda-\varepsilon)^{1/n'}\Big) - 2(2\varepsilon)^{1/n'}\|\nabla\varphi\|_{L^n(2B\setminus B)}\,;$$

letting $\varepsilon \to 0$, by the strict concavity of $t \mapsto t^{1/n'}$ and $0 < \lambda < 1$ we find a contradiction. We have then shown that the sequence f_h can be assumed to satisfy (A.2), as well as to be in case (iii).

Note that, when $\varepsilon < 1/2$ in (iii), it must be $B \cap (y_h + RB) \neq \emptyset$ and therefore, up to replacing R with $1 + 2R$, we can restate (iii) in the following way: for every ε there is $R > 0$ such that

$$\nu_h(RB) \ge 1 - \varepsilon\,. \tag{A.3}$$

By this estimate, applying the usual compactness Theorem for BV functions on bounded open sets and a standard diagonalization argument, we get the existenxe of $f \in BV(\mathbb{R}^n)$ such that (up to subsequences) $f_h \to f$ in $L^q_{\rm loc}(\mathbb{R}^n)$ for every $q < n'$ and weakly in $L^{n'}$. We can also assume that $\nu_h \stackrel{*}{\rightharpoonup} \nu$ for some measure ν and, by (A.3), $\nu(\mathbb{R}^n) = 1$. Notice that (A.2) gives

$$\nu(x + rB) \le \frac{1}{2} \qquad \text{for any } x \in \mathbb{R}^n,\ r < 1. \tag{A.4}$$

We can now introduce the measures $\mu_h = |Df_h|$ and assume $\mu_h \stackrel{*}{\rightharpoonup} \mu$. Notice that since $f_h \to f$ in $L^1_{\rm loc}(\mathbb{R}^n)$, $|Df| \le \mu$, hence $\|Df\| \le \|\mu\| \le n\omega_n^{1/n}$. We will conclude once we prove that $\|f\|_{L^{n'}} = 1$. Indeed, this fact will imply at once that $f_h \to f$ strongly in $L^{n'}(\mathbb{R}^n)$ and that $\|Df\| = n\omega_n^{1/n}$. From [10, Lemma 1.1] (see also [13, Lemma 4.8]) there exist $z_j \in \mathbb{R}^n$ and $\alpha_j \ge 0$ such that

(a) $\nu = |f|^{n'}\,dx + \sum_{j\in\mathbb{N}}\alpha_j\delta_{z_j}$;
(b) $\mu \ge |Df| + n\omega_n^{1/n}\sum_{j\in\mathbb{N}}\alpha_j^{1/n'}\delta_{z_j}$.

By the Sobolev inequality and the assumption $\delta(f_h) \to 0$,

$$\begin{aligned}n\omega_n^{1/n} &= \lim_{h\to\infty}\|Df_h\| \ge \|\mu\| \ge \|Df\| + n\omega_n^{1/n}\sum_{j\in\mathbb{N}}\alpha_j^{1/n'} \ge n\omega_n^{1/n}\Big(\|f\|_{L^{n'}} + \sum_{j\in\mathbb{N}}\alpha_j^{1/n'}\Big)\\ &\ge n\omega_n^{1/n}\Big(\|f\|^{n'}_{L^{n'}} + \sum_{j\in\mathbb{N}}\alpha_j\Big)^{1/n'} = n\omega_n^{1/n}\,.\end{aligned}$$

Therefore equality holds, and in particular exactly one element of $\{\alpha_j\} \cup \{\|f\|^{n'}_{L^{n'}}\}$ is different from 0, i.e. it is equal to 1. However, by (A.4), $\alpha_j \le 1/2$ for any j, thus $\alpha_j = 0$ for all j and $\|f\|^{n'}_{L^{n'}} = 1$ as desired. □

Appendix B. Elementary properties of $\lambda(f)$

We start by proving that in the definition (1.5) of $\lambda(f)$ the infimum on the right-hand side is attained.

Lemma B.1. *Let $f \in L^{n'}(\mathbb{R}^n)$, $f \neq 0$. Then the minimization problem*

$$\lambda(f) := \inf\left\{\frac{\|f - a\,\chi_{x+rB}\|^{n'}_{L^{n'}}}{\|f\|^{n'}_{L^{n'}}} : |a|^{n'}r^n\omega_n = \|f\|^{n'}_{L^{n'}}\,, a \in \mathbb{R}\,, x \in \mathbb{R}^n\right\},$$

has a solution.

Proof. We may assume without loss of generality that $\|f\|_{L^{n'}} = 1$. We claim that $\lambda(f) < 2$. To this aim, notice that since $f \neq 0$ we may always assume (up to a translation and a change of sign) that $f(0) > 0$ and that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{rB} |f(x) - f(0)|^{n'} dx = 0.$$

Making use of the above equation, we fix a sequence $r_h \to 0$ so that the functions $f_h(y) = f(r_h y)$ converge in $L^{n'}(B)$ and a.e. in B to the constant value $f(0)$. For all h denote by a_h the positive number such that $a_h^{n'} r_h^n \omega_n = 1$. Let us show that there exists h such that $\|f - a_h \chi_{r_h B}\|_{L^{n'}}^{n'} < 2$. In fact, if this is not true we have in particular that $\|f - a_h\|_{L^{n'}(r_h B)}^{n'} \geq 1$ for all h and, by rescaling,

$$r_h^n \int_B |f_h(y) - r_h^{1-n} \omega_n^{-1/n'}|^{n'} dy \geq 1 \quad \text{for all } h\,,$$

which in turn gives immediately that

$$\int_B \left(|1 - \omega_n^{1/n'} r_h^{n-1} f_h(y)|^{n'} - 1 \right) dy \geq 0 \quad \text{for all } h\,.$$

Notice that

$$\frac{\left| |1 - \omega_n^{1/n'} r_h^{n-1} f_h(y)|^{n'} - 1 \right|}{r_h^{n-1}} \leq C(n) \max\{|f_h(y)|, r_h |f_h(y)|^{n'}\}\,.$$

Therefore, the functions on the left hand side of this inequality are equi-integrable and thus, since $r_h^{n-1} f_h(y) \to 0$ for a.e. $y \in B$, we easily get

$$-f(0) n' \omega_n^{(n'+1)/n'} = \lim_{h\to\infty} \int_B \frac{|1 - \omega_n^{1/n'} r_h^{n-1} f_h(y)|^{n'} - 1}{r_h^{n-1}} dy \geq 0\,,$$

which is a contradiction, since $f(0) > 0$. Hence, $\lambda(f) < 2$.

Let us now consider a minimizing sequence $a_h \chi_{x_h + r_h B}$ for $\lambda(f)$. Up to a subsequence, we may assume that r_h converge to some value $r \in [0, \infty]$. Let us prove that $0 < r < \infty$. In fact,

$$\begin{aligned} \int_{\mathbb{R}^n} |f - a_h \chi_{x_h + r_h B}|^{n'} &= \int_{x_h + r_h B} |f - a_h|^{n'} + \int_{\mathbb{R}^n \setminus (x_h + r_h B)} |f|^{n'} \\ &\geq \left[1 - \left(\int_{x_h + r_h B} |f|^{n'} \right)^{1/n'} \right]^{n'} + \int_{\mathbb{R}^n \setminus (x_h + r_h B)} |f|^{n'}\,; \end{aligned} \tag{B.1}$$

then, if $r = 0$, since $|(x_h + r_h B)| \to 0$, passing to the limit in this inequality we would conclude that $\lambda(f) \geq 2$, which is impossible. Similarly, if $r = \infty$, we have that for every $R > 0$

$$\begin{aligned} \int_{\mathbb{R}^n} |f - a_h \chi_{x_h + r_h B}|^{n'} &= \int_{RB} |f - a_h \chi_{x_h + r_h B}|^{n'} + \int_{\mathbb{R}^n \setminus RB} |f - a_h \chi_{x_h + r_h B}|^{n'} \\ &\geq \left| \left(\int_{RB} |f|^{n'} \right)^{1/n'} - |a_h| |(x_h + r_h B) \cap RB|^{1/n'} \right|^{n'} \\ &\quad + \left| |a_h| |(x_h + r_h B) \setminus RB|^{1/n'} - \left(\int_{\mathbb{R}^n \setminus RB} |f|^{n'} \right)^{1/n'} \right|^{n'}\,. \end{aligned}$$

Since $|a_h||(x_h + r_h B) \cap RB| \to 0$, letting first $h \to \infty$ and then $R \to \infty$, from the previous inequality we get again $\lambda(f) \geq 2$. This proves that $0 < r < \infty$ and this in turn implies that a_h converges to a finite value $a \neq 0$. Finally, if the sequence x_h were unbounded, from (B.1) we would get again $\lambda(f) \geq 2$. This shows that also x_h converges (up to a subsequence) to some $x \in \mathbb{R}^n$ and that $\lambda(f) = \|f - a\, \chi_{x + rB}\|_{L^{n'}}^{n'}$. □

We conclude by proving the next comparison lemma.

Lemma B.2. *Let E be a Borel subset of $\mathbb{R}^n$ with finite measure. Then $\lambda(\chi_E) \leq \lambda^*(E) \leq 2^{n'+1} \lambda(\chi_E)$.*

Proof. The first inequality is trivial. Concerning the second one, without loss of generality we can assume that $|E| = 1$ and that

$$\lambda(\chi_E) = \int |\chi_E - a\chi_{rB}|^{n'} .$$

If $a \leq 1/2$ then

$$\lambda(\chi_E) \geq \int_E |\chi_E - a\chi_{rB}|^{n'} \geq \int_E \frac{1}{2^{n'}} = \frac{2}{2^{n'+1}} \geq \frac{\lambda^*(E)}{2^{n'+1}} .$$

Otherwise let $s > 0$ be such that $\omega_n s^n = |E| = 1$; if $a \in [1/2, 1]$, by $a^{n'}\omega_n r^n = 1$ we deduce $s \leq r$, and therefore

$$\lambda^*(E) \leq |E \Delta sB| = 2|sB \setminus E| \leq 2|rB \setminus E| = \frac{2}{a^{n'}} \int_{rB\setminus E} a^{n'} \leq 2^{n'+1}\lambda(\chi_E) .$$

In the end, if $a \geq 1$ then $s \geq r$ and therefore

$$\lambda^*(E) \leq |E \Delta sB| = 2|E \setminus sB| \leq 2|E \setminus rB| = 2 \int_{E\setminus rB} |\chi_E - a\chi_{rB}|^{n'} \leq 2\lambda(\chi_E) .$$

□

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