A VARIATIONAL APPROACH TO THE ISOPERIMETRIC INEQUALITY FOR THE ROBIN EIGENVALUE PROBLEM

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Abstract. The isoperimetric inequality for the first eigenvalue of the Laplace operator with Robin boundary conditions was recently proved by Daners in the context of Lipschitz sets. This paper introduces a new approach to the isoperimetric inequality, based on the theory of special functions of bounded variation (SBV). We extend the notion of the first eigenvalue λ_1 for general domains with finite volume (possibly unbounded and with irregular boundary), and we prove that the balls are the unique minimizers of λ_1 among domains with prescribed volume.

Keywords : Robin eigenvalue, non-smooth domains, isoperimetric inequality, SBV-spaces.

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1. Introduction

If $\Omega \subset \mathbb{R}^N$ is open, bounded and with a sufficiently smooth boundary ($N \geq 2$), the Robin eigenvalue problem for the Laplace operator is given by

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\beta > 0$ and ν denotes the outer normal on $\partial\Omega$. It is well known that the eigenvalue problem (1.1) admits an increasing and divergent sequence of solutions $(\lambda_n(\Omega))_{n\geq 1}$. Moreover the first eigenvalue $\lambda_1(\Omega)$ is characterized variationally as

$$\lambda_1(\Omega) = \min_{\substack{u \in W^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} u^2\,d\mathcal{H}^{N-1}}{\int_\Omega u^2\,dx}, \tag{1.2}$$

where the boundary term is intended in the sense of traces.

The minimum of λ_1 among the class of bounded Lipschitz sets with prescribed volume is achieved on balls, i.e.,

$$\lambda_1(B) \leq \lambda_1(\Omega), \tag{1.3}$$

where B is a ball such that $|B| = |\Omega|$. The isoperimetric inequality (1.3) is sometimes referred to as the Faber-Krahn inequality for Robin problems. It has been proved by Bossel [3] for planar smooth domains, and by Daners [9] for Lipschitz domains in higher dimensions. Concerning the uniqueness of minimizers, Daners and Kennedy [10] have shown that the balls are the unique minimizers within the class of C^2-domains, and the same result among the family of Lipschitz domains has been obtained by Bucur and Daners [5] (also for the case of the p-Laplacian operator, see moreover [7]). These results rely upon a direct comparison between B and Ω which is carried out by means of a level set representation for the first eigenvalue together with rearrangement techniques.

Concerning domains with non-smooth boundary, the problem of defining λ_1 has been dealt by Daners in [8]. Clearly, both the PDE version (1.1) and the Rayleigh quotient (1.2) involve boundary terms which are not easily defined for general domains. Daners employs the abstract completion $V(\Omega)$ of the space

$$V_0(\Omega) := W^{1,2}(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega)$$

under the norm

$$\|u\|_V := \|u\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(\partial\Omega;\mathcal{H}^{N-1})}$$

to provide a Hilbert-space functional setting for a generalization of the boundary value problem (1.1) (here $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure). The boundary terms are determined under the completion by those of the functions in $V_0(\Omega)$, which are simply given by the restriction on $\partial\Omega$ in view of their continuity. The spectrum of the generalized problem has the same features of the classical one, so that the lowest eigenvalue is taken by definition as the first Robin eigenvalue of Ω. The validity of (1.3) within this generalized context is not obvious.

The aim of the paper is to establish variationally inequality (1.3), where Ω is an open domain with prescribed volume $m > 0$, and $\lambda_1(\Omega)$ is suitably defined if Ω is unbounded or not smooth (see Definition 4.2 and relation (1.5) below). The associated uniqueness result is that if Ω is optimal for λ_1, then Ω is a ball up to negligible sets (Theorem 4.4). Moreover, if $(\Omega_n)_{n\in\mathbb{N}}$ is minimizing sequence for λ_1, i.e., $\lambda_1(\Omega_n) \to \lambda_1(B)$, then up to translating Ω_n the symmetric difference between Ω_n and B has a volume which tends to zero (Proposition 6.5).

The starting point of our approach is the following observation. Let Ω be bounded and with Lipschitz boundary, and let us extend $u \in W^{1,2}(\Omega)$ to the entire space by setting $u = 0$ outside Ω. The extended function, still denoted by u, could present discontinuities across $\partial\Omega$, and can be studied in the framework of the theory of *functions of bounded variation*. More precisely we have that u^2 belongs to the space of *special functions of bounded variation* $SBV(\mathbb{R}^N)$ introduced by De Giorgi and Ambrosio [11] to deal with free discontinuity problems (see Section 2). The Rayleigh quotient for $u \in W^{1,2}(\Omega)$ can be rewritten in SBV-terms as

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}[(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\,dx}, \tag{1.4}$$

where ∇u is the approximate gradient of u, J_u is the set of discontinuity points of u, and $u^\pm$ are the traces of u on J_u from the two sides (see Section 2 for the precise definitions of ∇u, J_u and $u^\pm$).

The optimization of λ_1 leads thus naturally to the minimization of (1.4) on a suitable class of functions of SBV-type which we denote by $SBV^{\frac{1}{2}}(\mathbb{R}^N)$, and whose support has volume less or equal than m. More precisely we let u vary in the space

$$SBV^{\frac{1}{2}}(\mathbb{R}^N) := \{u : \mathbb{R}^N \to [0,+\infty[\text{ is measurable and } u^2 \in SBV(\mathbb{R}^N)\}.$$

The above analysis of the Rayleigh quotients suggests to extend the notion of first eigenvalue for a general open set Ω with finite volume by setting (see Definition 4.2)

$$\lambda_1(\Omega) := \min\left\{\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta\int_{J_u}[(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\,dx} : u \in SBV^{\frac{1}{2}}(\mathbb{R}^N), u \neq 0,\right.$$
$$\left. |supp(u) \setminus \Omega| = 0, \mathcal{H}^{N-1}(J_u \setminus \partial\Omega) = 0\right\}, \tag{1.5}$$

where $supp(u) = \{u \neq 0\}$ denotes the support of u. If Ω is bounded and Lipschitz regular, then $\lambda_1(\Omega)$ coincides with the usual first eigenvalue. For bounded but irregular sets, we have that $\lambda_1(\Omega)$ is in general lower than the first eigenvalue proposed by Daners in [8] (see Section 4). Clearly if λ

denotes the infimum of (1.4) on the functions in $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ whose support has a volume less or equal than m, we have

$$\lambda \le \lambda_1(\Omega)$$

for every open set with $|\Omega| = m$.

In Section 5 we prove that the infimum λ is achieved on a function $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ (see Theorem 4.5): this is done by means of a concentration-compactness argument together with compactness and lower semicontinuity properties in $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ which we establish in Section 3. In Section 6, on the basis of an Euler-Lagrange equation (Proposition 6.1), we prove (Theorem 4.6) that such a minimizer has a support which is equal to a ball B of volume m, and after a suitable renormalization it coincides with the first eigenfunction of B inside the ball. Then the isoperimetric inequality (1.3) follows since $\lambda = \lambda_1(B)$, together with the associated uniqueness result.

We conclude the Introduction noting that by means of the same arguments (suitably changing the functional spaces of SBV-type), we can prove the isoperimetric inequality (1.3) also for the first eigenvalue of the p-Laplacian operator with $p \in]1, +\infty[$, generalizing to arbitrary domains the result of [5].

The paper is organized as follows. In Section 2 we fix the notation and recall some basic facts from the theory of functions of bounded variation. Section 3 is devoted to the study of the space $SBV^{\frac{1}{2}}(\mathbb{R}^N)$. In Section 4 we define the first Robin eigenvalue for an open set with finite measure by means of our SBV-approach, and we state the corresponding isoperimetric inequality. The optimization of the Rayleigh quotient (1.4) is carried out in Section 5, while the structure of minimizers and the convergence of a minimizing sequence to a ball are studied in Section 6.

2. Notation and preliminaries

Throughout the paper, $B_r(x)$ will denote the open ball of center $x \in \mathbb{R}^N$ and radius $r > 0$. If $E \subset \mathbb{R}^N$, we will denote its volume by $|E|$, and 1_E will stand for its characteristic function, i.e., $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$. For $A \subseteq \mathbb{R}^N$ open set and $p \ge 1$, $L^p(A)$ will denote the usual Lebesgue space of p-summable functions, while $W^{1,p}(A)$ will denote the Sobolev space of functions in $L^p(A)$ whose derivative in the sense of distributions is p-summable. Moreover $\|u\|_\infty$ will stand for the sup-norm of u, while *supp*(u) will denote the set $\{u \ne 0\}$, well defined up to zero Lebesgue measure. Finally we will use the following notation: for $a, b \in \mathbb{R}$

$$a \wedge b := \min\{a, b\} \qquad \text{and} \qquad a \vee b := \max\{a, b\}.$$

In the rest of the section we recall some basic facts about fine properties of measurable maps and about functions of bounded variation.

Approximate continuity and differentiability. Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and let $x \in \mathbb{R}^N$. We set

$$u^+(x) := \inf\{t \in \mathbb{R} : x \in \{u > t\}^0\}$$

and

$$u^-(x) := \sup\{t \in \mathbb{R} : x \in \{u < t\}^0\}.$$

Here E^0 for E measurable set in $\mathbb{R}^N$ stands for the set of points $y \in \mathbb{R}^N$ with zero density with respect to E, that is such that

$$\limsup_{r \to 0^+} \frac{|E \cap B_r(y)|}{r^N} = 0.$$

The values $u^+(x)$ and $u^-(x)$ are called the *upper* and *lower essential limits* of u at x. If they coincide and are equal to l, we say that l is the *essential limit* of u at x and we write

$$l = \operatorname*{ap\,lim}_{y \to x} u(y).$$

The function u is said *approximately continuous* at x if the essential limit of u at x exists and is equal to $u(x)$, that is if $u^+(x) = u^-(x) = u(x)$. Notice that if $u \in L^1_{loc}(\mathbb{R}^N)$ and x is a Lebesgue point for u with Lebesgue value l, then l is also the approximate limit of u at x.

The set

$$J_u := \{x \in \mathbb{R}^N \, : \, u^-(x) < u^+(x)\}$$

is called the *jump set* of u.

We say that u is *approximately differentiable* at x if there exists a vector $\xi \in \mathbb{R}^N$ such that

$$\operatorname*{ap\,lim}_{y \to x} \frac{u(y) - u(x) - \xi \cdot (y - x)}{|y - x|} = 0.$$

If such a vector ξ exists, it turns out that it is unique and it is called the *approximate gradient* of u at x, and is denoted by $\nabla u(x)$. It is easily seen that approximate differentiability implies approximate continuity.

It follows from the definition that if two measurable functions u and v coincide on a measurable subset E, then u and v share the same upper and lower essential limits on any point of density 1 of E, that is at almost every point of E in view of Lebesgue density theorem. Moreover u is approximately differentiable at a point of density 1 of E if and only if v is approximately differentiable, and their approximate gradient are equal. In particular it follows that if $u = c$ on a measurable set E with $c \in \mathbb{R}$, then u is approximately differentiable for a.e. $x \in E$ with $\nabla u(x) = 0$.

We will use the following stability properties which easily follow from the above definitions. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then

$$(g \circ u)^+(x) = g(u^+(x)), \qquad (g \circ u)^-(x) = g(u^-(x))$$

if g is increasing, while

$$(g \circ u)^+(x) = g(u^-(x)), \qquad (g \circ u)^-(x) = g(u^+(x))$$

if g is decreasing. In particular we infer $J_{g \circ u} \subseteq J_u$. If g is differentiable and u is approximately differentiable at x we have the following chain rule formula:

$$\nabla(g \circ u)(x) = g'(u(x)) \nabla u(x).$$

Functions of bounded variation. Let $A \subseteq \mathbb{R}^N$ be an open set. We say that $u \in BV(A)$ if $u \in L^1(A)$ and its derivative in the sense of distributions is a finite Radon measure on A, i.e., $Du \in \mathcal{M}_b(A; \mathbb{R}^N)$. $BV(A)$ is called the space of *functions of bounded variation* on A. $BV(A)$ is a Banach space under the norm $\|u\|_{BV(A)} := \|u\|_{L^1(A)} + \|Du\|_{\mathcal{M}_b(A;\mathbb{R}^N)}$. We refer the reader to [1] for an exhaustive treatment of the space BV.

Concerning the fine properties, a function $u \in BV(A)$ (or better every representative of u) is a.e. approximately differentiable on A, with approximate gradient $\nabla u \in L^1(A; \mathbb{R}^N)$. Moreover, the jump set J_u is a $(N-1)$-rectifiable set, i.e., $J_u \subseteq \cup_{i \in \mathbb{N}} M_i$ up to a $\mathcal{H}^{N-1}$-negligible set, with M_i a C^1-hypersurface in $\mathbb{R}^N$. The measure Du admits the following representation for every Borel set $B \subseteq A$:

$$Du(B) = \int_B \nabla u \, dx + \int_{J_u \cap B} (u^+ - u^-) \nu_u \, d\mathcal{H}^{N-1} + D^c u(B),$$

where $\nu_u(x)$ is the normal to J_u at x, and $D^c u$ is singular with respect to the Lebesgue measure and concentrated outside J_u. $D^c u$ is usually referred to as the *Cantor part* of Du. The normal ν_u coincides $\mathcal{H}^{N-1}$-a.e. on J_u with the normal to the hypersurfaces M_i. The direction of $\nu_u(x)$ is chosen in such a way that $u^\pm(x)$ is the approximate limit of u at x on the sets $\{y \in \mathbb{R}^N : \nu_u(x) \cdot (y - x) \gtrless 0\}$. Moreover, $u^\pm$ coincide $\mathcal{H}^{N-1}$-almost everywhere on J_u with the traces $\gamma^\pm(u)$ of u on J_u which are defined by the following Lebesgue-type limit quotient relation

$$\lim_{r \to 0} \frac{1}{r^N} \int_{B_r^\pm(x)} |u(x) - \gamma^\pm(u)(x)| \, dx = 0$$

where $B_r^\pm(x) := \{y \in B_r(x) : \nu_u(x) \cdot (y - x) \gtrless 0\}$ (see [1, Remark 3.79]).

If A is bounded and with a Lipschitz boundary, then $BV(A) \hookrightarrow L^{N/N-1}(A)$. Moreover, the following compactness result holds: if $(u_n)_{n \in \mathbb{N}}$ is bounded in $BV(A)$, there exist $u \in BV(A)$ and a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ such that

$$u_{n_k} \to u \qquad \text{strongly in } L^1(A)$$

and

$$Du_{n_k} \to Du \qquad \text{weakly* in the sense of measures.}$$

We say in this case that $u_{n_k} \overset{*}{\rightharpoonup} u$ weakly* in $BV(A)$.

We say that $u \in SBV(A)$ if $u \in BV(A)$ and $D^c u = 0$. $SBV(A)$ is called the space of *special function of bounded variation* on A. This space is very useful in free discontinuity problems in view of the following compactness and lower-semicontinuity result due to L. Ambrosio (see [1, Theorems 4.7-4.8]).

Theorem 2.1. *Let $A \subset \mathbb{R}^N$ be open and bounded, $p \in]1, +\infty[$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(A)$ such that*

$$\sup_n \int_A |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(J_{u_n}) + \|u_n\|_\infty < +\infty. \tag{2.1}$$

Then there exist $u \in SBV(A)$ with $\nabla u \in L^p(A;\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$u_{n_k} \to u \qquad \text{strongly in } L^1(A),$$

$$\nabla u_{n_k} \to \nabla u \qquad \text{weakly in } L^p(A;\mathbb{R}^N)$$

and

$$\mathcal{H}^{N-1}(J_u) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(J_{u_n}).$$

From the proof of Ambrosio's Theorem, we infer that the conclusions still hold provided that in place of (2.1) we have $u_n \overset{*}{\rightharpoonup} u$ weakly* in $BV(A)$, $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L^p(A;\mathbb{R}^N)$ and $\sup_n \mathcal{H}^{N-1}(J_{u_n}) < +\infty$. In this case we say that $u_n \rightharpoonup u$ weakly in $SBV(A)$.

3. The space $SBV^{\frac{1}{2}}(\mathbb{R}^N)$: compactness and lower semicontinuity results

In this section we define and prove the principal properties of the function space we will use to study the optimization of the first eigenvalue for the Robin boundary value problem.

Definition 3.1. *We say that $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ if u is nonnegative almost everywhere and $u^2 \in SBV(\mathbb{R}^N)$.*

We note that the exponent $1/2$ is not connected to any concept of fractional derivative. Moreover, the exponent does not refer to the summability of ∇u as for the spaces SBV^p, $p > 1$, frequently employed in problems arising in fracture mechanics, for which ∇u is p-summable.

The following lemma concerns some fine properties of functions in $SBV^{\frac{1}{2}}(\mathbb{R}^N)$.

Lemma 3.2. *Let $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$. Then the following facts hold.*

(a) *u is a.e. approximately differentiable with approximate gradient ∇u such that*

$$\nabla u^2 = 2u\nabla u \qquad \text{a.e. in } \mathbb{R}^N. \tag{3.1}$$

(b) *The jump set J_u is $\mathcal{H}^{N-1}$-rectifiable and a normal ν_u can be chosen in such a way that*

$$D^j(u^2) = [(u^+)^2 - (u^-)^2]\, \nu_u \, d\mathcal{H}^{N-1} \llcorner J_u.$$

(c) *For every $\varepsilon > 0$ we have $u \vee \varepsilon \in SBV(\Omega)$ for every bounded open set $\Omega \subset \mathbb{R}^N$.*

Proof. Let $\varepsilon > 0$ and let Ω be a bounded open set in $\mathbb{R}^N$. By the chain rule in BV [1, Theorem 3.96], we have that $v := u^2 \vee \varepsilon^2 = (u \vee \varepsilon)^2 \in SBV(\Omega)$. By taking the square root, again by the chain rule in BV, we deduce that $u \vee \varepsilon \in SBV(\Omega)$. Point (c) is thus proved.

Since $u \in L^2(\mathbb{R}^N)$, almost all points in $\mathbb{R}^N$ are Lebesgue points of u, and in particular points of approximate continuity. Let us fix a Lebesgue representative of u, i.e., let $u(x)$ be equal to the Lebesgue value if x is a Lebesgue point, and let $u(x) = 0$ otherwise. Let $E_n := \{x \in \Omega : u(x) > 1/n\}$. Since $u = u \vee (1/n)$ on E_n, and $u \vee (1/n)$ is a.e. approximately differentiable on Ω since it belongs to $SBV(\Omega)$, we get that u is a.e. approximately differentiable on E_n. Moreover, u is a.e. approximately differentiable on $E_0 := \{x \in \Omega : u(x) = 0\}$ with zero approximate gradient. Since $\Omega = E_0 \cup \bigcup_{n\ge1} E_n$, we conclude that u is a.e. approximately differentiable on Ω. As Ω is arbitrary, we get the approximate differentiability on $\mathbb{R}^N$, and equality (3.1) follows by composition arguments (see Section 2), so that point (a) is proved.

Point (b) follows again by composition arguments (see Section 2): since u is positive, $J_u = J_{u^2}$ so that the $\mathcal{H}^{N-1}$-rectifiability and we can choose $\nu_u = \nu_{u^2}$ so that the representation formula for $D^j(u^2)$ holds. The proof of the lemma is now complete. $\square$

The main result of the section is the following theorem.

Theorem 3.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ and let $C > 0$ be such that for every $n \in \mathbb{N}$*

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx + \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2]\, d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N} u_n^2\, dx \le C. \tag{3.2}$$

Then there exist $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that the following facts hold.

(a) *Compactness: $u_{n_k} \to u$ strongly in $L^2_{loc}(\mathbb{R}^N)$.*

(b) *Lower semicontinuity: for every open set $A \subseteq \mathbb{R}^N$ we have*

$$\int_A |\nabla u|^2\, dx \le \liminf_{k\to\infty} \int_A |\nabla u_{n_k}|^2\, dx \tag{3.3}$$

and

$$\int_{J_u\cap A} [(u^+)^2 + (u^-)^2]\, d\mathcal{H}^{N-1} \le \liminf_{k\to\infty} \int_{J_{u_{n_k}}\cap A} [(u_{n_k}^+)^2 + (u_{n_k}^-)^2]\, d\mathcal{H}^{N-1}. \tag{3.4}$$

Proof. The idea is to consider $v_n := u_n^2 \in SBV(\mathbb{R}^N)$, and to infer the result from compactness in BV and the lower semicontinuity properties in SBV. Let us divide the proof in several steps. In the following, $\Omega \subset \mathbb{R}^N$ will be an open bounded set.

Step 1. Recall that by Lemma 3.2

$$\nabla v_n = 2u_n \nabla u_n \tag{3.5}$$

and

$$D^j v_n = [(u_n^+)^2 - (u_n^-)^2]\nu_{u_n} \mathcal{H}^{N-1} \llcorner J_{u_n}. \tag{3.6}$$

Notice that $(v_n)_{n\in\mathbb{N}}$ is bounded in $BV(\mathbb{R}^N)$. In fact this follows from (3.5) and (3.6) in view of the bound (3.2). We deduce that there exists $v \in BV(\mathbb{R}^N)$ and a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ such that

$$v_{n_k} \overset{*}{\rightharpoonup} v \qquad \text{locally weakly}^* \text{ in } BV(\mathbb{R}^N). \tag{3.7}$$

By the compact embedding of BV into L^1 for bounded and Lipschitz domains we get in particular

$$v_{n_k} \to v \qquad \text{strongly in } L^1_{loc}(\mathbb{R}^N).$$

By setting $u := \sqrt{v}$ (recall that $v \ge 0$ on $\mathbb{R}^N$), we have $u \in L^2(\mathbb{R}^N)$ and

$$u_{n_k} \to u \qquad \text{strongly in } L^2_{loc}(\mathbb{R}^N). \tag{3.8}$$

Step 2. For every $\varepsilon > 0$ we have

$$u_{n_k}^\varepsilon \to u \vee \varepsilon \qquad \text{strongly in } L^2(\Omega), \tag{3.9}$$

where $u_{n_k}^\varepsilon := u_{n_k} \vee \varepsilon$. Notice that by composition (see Section 2) we have

$$\nabla u_{n_k}^\varepsilon = \nabla u_{n_k} 1_{\{u_{n_k} > \varepsilon\}} \qquad \text{in } \Omega, \tag{3.10}$$

so that $(\nabla u_{n_k}^\varepsilon)_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega;\mathbb{R}^N)$. Moreover, by (3.2) and Lemma 3.2 we deduce that

$$\mathcal{H}^{N-1}(J_{u_n^\varepsilon}) \le \frac{1}{\varepsilon^2} \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2]\, d\mathcal{H}^{N-1} \le \frac{C}{\varepsilon^2}.$$

As a consequence, concerning the jump part $D^j u^\varepsilon_{n_k}$ of the derivative of $u^\varepsilon_{n_k}$, we have that there exists $C_\varepsilon > 0$ such that for every k

$$|D^j u^\varepsilon_{n_k}|(\Omega) = \int_{J_{u^\varepsilon_{n_k}}\cap\Omega} |(u^\varepsilon_{n_k})^+ - (u^\varepsilon_{n_k})^-| \, d\mathcal{H}^{N-1}$$
$$\le \int_{J_{u^\varepsilon_{n_k}}\cap\Omega} \{C_\varepsilon + [(u^\varepsilon_{n_k})^+]^2 + [(u^\varepsilon_{n_k})^-]^2\} \, d\mathcal{H}^{N-1} \le \frac{C_\varepsilon C}{\varepsilon^2} + 2C.$$

By (3.9) and the remark after Ambrosio's Theorem 2.1, we deduce that $u \vee \varepsilon \in SBV(\Omega)$ for every $\varepsilon > 0$, and

$$u^\varepsilon_{n_k} \rightharpoonup u \vee \varepsilon \qquad \text{weakly in } SBV(\Omega), \tag{3.11}$$

i.e., separate convergence for the absolutely continuous and the jump part of the derivative holds.

Step 3. In view of the bound (3.2), up to a further subsequence we may assume that

$$\nabla u_{n_k} \rightharpoonup \Phi \qquad \text{weakly in } L^2(\mathbb{R}^N; \mathbb{R}^N). \tag{3.12}$$

Let ∇u be the approximate gradient of u. By (3.8) and (3.10) we deduce that for a.e. $\varepsilon > 0$

$$\nabla u^\varepsilon_{n_k} \rightharpoonup \Phi 1_{\{u>\varepsilon\}} \qquad \text{weakly in } L^2(\Omega; \mathbb{R}^N).$$

In view of (3.11) we deduce that for a.e. $\varepsilon > 0$

$$\Phi 1_{\{u>\varepsilon\}} = \nabla u 1_{\{u>\varepsilon\}} \qquad \text{in } \Omega.$$

Since Ω is arbitrary, we conclude that

$$\nabla u = \begin{cases} \Phi & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u = 0\}. \end{cases} \tag{3.13}$$

By lower semicontinuity we get

$$\int_\Omega |\nabla u|^2 \, dx \le \int_\Omega |\Phi|^2 \, dx \le \liminf_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, dx$$

so that, since Ω is arbitrary, (3.3) follows.

Step 4. We claim that $v = u^2 \in SBV(\mathbb{R}^N)$ with

$$\nabla v = 2u\nabla u \tag{3.14}$$

and

$$D^j v = [(u^+)^2 - (u^-)^2]\nu_u \, d\mathcal{H}^{N-1} \llcorner J_u. \tag{3.15}$$

Indeed, since $v \in BV(\mathbb{R}^N)$, (3.15) follows directly from the definition of $D^j v$ and by the relation $v = u^2$. In particular, we get

$$\int_{J_u} [(u^+)^2 - (u^-)^2] \, d\mathcal{H}^{N-1} < +\infty. \tag{3.16}$$

Let us set $v_\varepsilon := v \vee \varepsilon^2$ for $\varepsilon > 0$. By the chain rule in BV [1, Theorem 3.96] we have that $v_\varepsilon \in BV(\Omega)$, and $v_\varepsilon \xrightharpoonup{*} v$ weakly* in $BV(\Omega)$ as $\varepsilon \to 0$. Since $v_\varepsilon = (u \vee \varepsilon)^2$, and $u \vee \varepsilon \in SBV(\Omega)$, again by the chain rule in BV we get $v_\varepsilon \in SBV(\Omega)$ with

$$\nabla v_\varepsilon = 2(u \vee \varepsilon)\nabla u 1_{\{u>\varepsilon\}} \qquad \text{in } \Omega \tag{3.17}$$

and

$$D^j v_\varepsilon = [(u^+ \vee \varepsilon)^2 - (u^- \vee \varepsilon)^2]\nu_u \, d\mathcal{H}^{N-1} \llcorner (J_u \cap \Omega). \tag{3.18}$$

Since $\nabla u \in L^2(\Omega; \mathbb{R}^N)$ by (3.13), from (3.17) we deduce that

$$\nabla v_\varepsilon \to 2u\nabla u \qquad \text{strongly in } L^1(\Omega; \mathbb{R}^N)$$

as $\varepsilon \to 0$. By (3.18), (3.16) and the monotone convergence theorem we have

$$D^j v^\varepsilon \to [(u^+)^2 - (u^-)^2]\nu_u \, d\mathcal{H}^{N-1} \llcorner (J_u \cap \Omega) \qquad \text{strongly in } \mathcal{M}_b(\Omega; \mathbb{R}^N)$$

as $\varepsilon \to 0$. This entails that Dv is the sum of an absolutely continuous measure and of a measure concentrated on J_u, which is an $\mathcal{H}^{N-1}$-rectifiable set in view of Lemma 3.2. We infer that the Cantor part $D^c v$ of Dv vanishes, since $D^c v$ is singular with respect to the Lebesgue measure and cannot charge sets which are σ-finite with respect to $\mathcal{H}^{N-1}$. We conclude that $v \in SBV(\Omega)$. Since Ω is arbitrary, we deduce that the claim holds. As a consequence $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$, so that point (a) follows in view of (3.8).

Step 5. By (3.7), (3.14) and (3.15), we have that

$$Dv_{n_k} \overset{*}{\rightharpoonup} Dv = 2u\nabla u\, dx + [(u^+)^2 - (u^-)^2]\nu_u\, d\mathcal{H}^{N-1} \llcorner J_u$$

locally weakly* in the sense of measures in $\mathbb{R}^N$. Notice that

$$D^a v_{n_k} = 2u_{n_k} \nabla u_{n_k} \rightharpoonup 2u\nabla u \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^N).$$

Indeed, for every $\varphi \in L^\infty(\Omega; \mathbb{R}^N)$ we get using (3.8),(3.12) and (3.13)

$$\lim_{k\to\infty} \int_\Omega 2u_{n_k} \nabla u_{n_k} \cdot \varphi\, dx = \int_\Omega 2u\Phi \cdot \varphi\, dx = \int_\Omega 2u\nabla u \cdot \varphi\, dx.$$

As a consequence we conclude that

$$D^j v_{n_k} \overset{*}{\rightharpoonup} D^j v \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega; \mathbb{R}^N).$$

Let $v^\varepsilon_{n_k}$ and v^ε be equal to the positive part of $(v_{n_k} - \varepsilon)$ and $(v - \varepsilon)$ respectively, where $\varepsilon > 0$. By the chain rule in BV, and in view also of the bound (3.2), we can apply the lower semicontinuity result [4, Theorem 2.12] to the sequence $(v^\varepsilon_{n_k})_{k\in\mathbb{N}}$ obtaining

$$\int_{J_{v^\varepsilon}\cap\Omega} |(v^\varepsilon)^+| + |(v^\varepsilon)^-|\, d\mathcal{H}^{N-1} \le \liminf_{k\to\infty} \int_{J_{v^\varepsilon_{n_k}}\cap\Omega} |(v^\varepsilon_{n_k})^+| + |(v^\varepsilon_{n_k})^-|\, d\mathcal{H}^{N-1}.$$

Since ε is arbitrary we get

$$\int_{J_v\cap\Omega} |v^+| + |v^-|\, d\mathcal{H}^{N-1} \le \liminf_{k\to\infty} \int_{J_{v_{n_k}}\cap\Omega} |v^+_{n_k}| + |v^-_{n_k}|\, d\mathcal{H}^{N-1}$$

so that inequality (3.4) follows as Ω is also arbitrary. The proof of the theorem is now complete. □

An immediate consequence of Theorem 3.3 is given by the following proposition which will be used in connection with the definition of the first Robin eigenvalue for arbitrary domains.

Proposition 3.4. *Let $\Omega \subseteq \mathbb{R}^N$ be open with $|\Omega| < +\infty$, and let $\beta > 0$. Then the minimum problem*

$$\min \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2]\, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\, dx} : u \in SBV^{\frac{1}{2}}(\mathbb{R}^N), u \neq 0, \right.$$
$$\left. |supp(u) \setminus \Omega| = 0, \mathcal{H}^{N-1}(J_u \setminus \partial\Omega) = 0 \right\}$$

admits a solution.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence. By rescaling we can assume that $\|u_n\|_{L^2(\mathbb{R}^N)} = 1$ for every $n \in \mathbb{N}$. By comparing u_n with a smooth function with compact support in Ω we get that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx + \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2]\, d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N} u_n^2\, dx \le C$$

for some $C > 0$. By Theorem 3.3 we deduce that there exists $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ such that up to a subsequence

$$u_n \to u \qquad \text{strongly in } L^2_{loc}(\mathbb{R}^N) \tag{3.19}$$

and

$$\int_\Omega |\nabla u|^2\,dx + \beta \int_{J_u\cap\Omega} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1} \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx + \beta \int_{J_{u_n}\cap\Omega} [(u_n^+)^2 + (u_n^-)^2]\,d\mathcal{H}^{N-1}. \tag{3.20}$$

Since $|supp(u_n)\setminus\Omega| = 0$, we deduce that $|supp(u)\setminus\Omega| = 0$. Moreover, in view of the Sobolev-Gagliardo-Nirenberg inequality in $BV(\mathbb{R}^N)$ applied to u_n^2 we get

$$\begin{aligned}\|u_n^2\|_{L^{N/N-1}(\mathbb{R}^N)} &\le C_N|D(u_n^2)|(\mathbb{R}^N) = C_N\left[\int_{\mathbb{R}^N} 2|u_n\nabla u_n|\,dx + \int_{J_{u_n}} |(u_n^+)^2 - (u_n^-)^2|\,d\mathcal{H}^{N-1}\right]\\ &\le C_N\left[\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2]\,d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N} u_n^2\,dx\right] \le C_N C < +\infty,\end{aligned}$$

where C_N depends only on the dimension N. We infer that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^{2N/N-1}(\mathbb{R}^N)$. Similarly we get that $u \in L^{2N/N-1}(\mathbb{R}^N)$. Since for every ball $B_R(0)$ with $R > 0$ we have in view of Hölder inequality

$$\begin{aligned}\|u_n - u\|_{L^2(\mathbb{R}^N)} = \|u_n - u\|_{L^2(\Omega)} &\le \|u_n - u\|_{L^2(\Omega\cap B_R(0))} + \|u_n - u\|_{L^2(\Omega\setminus B_R(0))}\\ &\le \|u_n - u\|_{L^2(\Omega\cap B_R(0))} + \tilde{C}|\Omega\setminus B_R(0)|^{1/N}\end{aligned}$$

for some $\tilde{C} > 0$, by (3.19) and the assumption $|\Omega| < +\infty$ we conclude that

$$u_n \to u \qquad \text{strongly in } L^2(\mathbb{R}^N).$$

In particular $\|u\|_{L^2(\mathbb{R}^N)} = 1$, so that $u \neq 0$. Moreover, by the lower semicontinuity (3.4) applied to $A := \mathbb{R}^N \setminus \partial\Omega$ we get that $\mathcal{H}^{N-1}(J_u \setminus \Omega) = 0$. We conclude that u is an admissible function and (3.20) entails that u is a minimum point for the problem.

□

We conclude the section by commenting on the connection between our space $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ and the function space which Daners [8] uses for his approach to the study of the Robin boundary value problem on arbitrary domains.

Let A be an open and bounded set in $\mathbb{R}^N$. Let us consider

$$V_0(A) := \{u \in W^{1,2}(A) \cap C(\bar{A}) \cap C^\infty(A) \,:\, \|u\|_V < +\infty\},$$

where

$$\|u\|_V^2 := \|\nabla u\|^2_{L^2(A;\mathbb{R}^N)} + \|u_{|\partial A}\|^2_{L^2(\partial A;\mathcal{H}^{N-1})}.$$

Daners considers in his paper [8] the completion $V(A)$ of $V_0(A)$ under the $\|\cdot\|_V$ norm. An important feature of this space is that it is compactly embedded into $L^2(A)$. The proof of this fact is due to Maz'ja [15, Corollary 4.11.1/2] and relies on the embedding $V(A) \hookrightarrow L^{2N/N-1}(A)$, which he proves using the coarea formula and the classical isoperimetric inequality, and on the compact embedding of $W^{1,2}(A)$ into $L^2_{loc}(A)$.

The space $V(A)$ is linked to our space $SBV^{\frac{1}{2}}(\mathbb{R}^N)$. The key point is to observe that if $u \in V_0(A)$, and we extend u to zero outside A (still denoting the function by u) we obtain *essentially* an element of $SBV^{\frac{1}{2}}(\mathbb{R}^N)$. More precisely, since u is not necessarily positive, we have $v := u^2 \in SBV(\mathbb{R}^N)$, with $\nabla v = 2u\nabla u 1_A$ (we think ∇u extended to zero outside A for example) and $J_v \subset \partial A$. The embedding into $L^{2N/N-1}(A)$ is then quite easy to obtain using the standard BV-theory. Indeed, thank to the continuity of u on $\bar{A}$, the points of J_v have positive density with respect to $\bar{A}$ and $\mathbb{R}^N \setminus \bar{A}$, and we have for $x \in J_v$

$$v^+(x) = u^2(x) \qquad \text{and} \qquad v^-(x) = 0.$$

It follows that

$$(3.21)\quad |Dv|(\mathbb{R}^N) = \|2u\nabla u 1_A\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} + \int_{J_v} (v^+)^2\, d\mathcal{H}^{N-1} \\ \le \|2u\nabla u\|_{L^1(A;\mathbb{R}^N)} + \int_{\partial A} u^2\, d\mathcal{H}^{N-1} < +\infty.$$

From the embedding of $BV(\mathbb{R}^N)$ into $L^{N/N-1}(\mathbb{R}^N)$ we deduce that

$$\|u^2\|_{L^{N/N-1}(A)} \le C_N \left[\|2u\nabla u\|_{L^1(A;\mathbb{R}^N)} + \int_{\partial A} u^2\, d\mathcal{H}^{N-1}\right] \\ \le C_N \left[b\|u\|^2_{L^2(A)} + \frac{1}{b}\|\nabla u\|^2_{L^2(A;\mathbb{R}^N)} + \int_{\partial A} u^2\, d\mathcal{H}^{N-1}\right]$$

where $b > 0$. Hölder inequality and the choice of a sufficiently small b entail the embedding of $V_0(A)$ into $L^{2N/N-1}(A)$. By density, the embedding extends to the whole space $V(A)$.

Let us prove the claim on $v = u^2$. Since we can work separately with the positive and the negative part of u, we can assume $u \ge 0$. Let us fix $\varepsilon > 0$. Since $\|u\|_V < +\infty$, we deduce that

$$\mathcal{H}^{N-1}(\partial_\varepsilon A) \le \frac{4}{\varepsilon^2}\|u_{|\partial A}\|^2_{L^2(\partial A;\mathcal{H}^{N-1})} < +\infty,$$

where ∂A_ε is the compact set given by $\partial_\varepsilon A := \{x \in \partial A : u(x) \ge \varepsilon/2\}$. Let $\{U_\eta\}_{\eta>0}$ be a family of open neighborhoods of $\partial_\varepsilon A$ with $\cap_{\eta>0} U_\eta = \partial_\varepsilon A$, and let $\{B^\eta_{r_i}(x_i)\}_{i=1,\dots,n}$ be a finite family of balls covering ∂A_ε, with $\cup_{i=1}^n B_{r_i}(x_i) \subseteq U_\eta$ and such that

$$(3.22)\qquad \sum_{i=1}^n \omega_{N-1} r_i^{N-1} \le (1+\eta)\mathcal{H}^{N-1}(\partial_\varepsilon A).$$

Let u_ε be the positive part of $(u - \varepsilon)$, and v_η be defined as

$$v_\eta := \begin{cases} u_\varepsilon^2 & \text{in } A \setminus \bigcup_{i=1}^n \bar{B}^\eta_{r_i}(x_i) \\ 0 & \text{otherwise in } \mathbb{R}^N. \end{cases}$$

It turns out easily that $v_\eta \in SBV(\mathbb{R}^N)$ with support compactly contained in A, and with

$$\nabla v_\eta = 2u_\varepsilon \nabla u 1_{A\setminus \bigcup_{i=1}^n \bar{B}^\eta_{r_i}(x_i)}$$

and $J_{v_\eta} \subseteq \bigcup_{i=1}^n \partial B^\eta_{r_i}(x_i)$. In view of (3.22), we can apply Ambrosio's compactness theorem to $v_\eta \to w_\varepsilon := u_\varepsilon^2$ as $\eta \to 0$, deducing that $w_\varepsilon \in SBV(\mathbb{R}^N)$ with $\nabla w_\varepsilon = 2u_\varepsilon \nabla u 1_A$ and $J_{w_\varepsilon} \subseteq \partial_\varepsilon A$. Moreover, since u is continuous on $\bar{A}$, the points of J_{w_ε} have positive density with respect to A and $\mathbb{R}^N \setminus \bar{A}$ and

$$D^j w_\varepsilon = u_\varepsilon^2 \nu_{w_\varepsilon}\, \mathcal{H}^{N-1} \llcorner J_{w_\varepsilon}.$$

In particular we get that

$$|D^j w_\varepsilon| \le \mu := u^2 \mathcal{H}^{N-1} \llcorner \partial A.$$

Since μ is concentrated on $\cup_{n\ge 1}\partial_{1/n} A$ which is σ-finite with respect to $\mathcal{H}^{N-1}$, by letting $\varepsilon \to 0$ we deduce that $v \in SBV(\mathbb{R}^N)$, so that the claim follows.

Inequality (3.21) shows that in order to bound the BV norm of v it suffices to control the integral of u^2 only on the points of ∂A which have positive density with respect to $\bar{A}$ and $\mathbb{R}^N \setminus \bar{A}$. The information carried by the integral of u^2 on the rest of ∂A is irrelevant for the embedding of $V_0(A)$ into $L^{2N/N-1}(A)$, and with this respect taking the *abstract completion* of $V_0(A)$ with respect to $\|\cdot\|_V$ (which gives an Hilbert space isometric to a subspace of $W^{1,2}(A) \times L^2(\partial A, \mathcal{H}^{N-1})$) is redundant. This is connected to a remark by Daners [8, Remark 3.2, point (d)]. He notices that the embedding $V(A) \hookrightarrow L^{2N/N-1}(A)$ could fail to be injective, because of the *possible* existence of a sequence $u_n \in W^{1,2}(A) \cap C(\bar{A}) \cap C^\infty(A)$ with $u_n \to 0$ strongly in $W^{1,2}(A)$ and $\int_{\partial A} u_n^2\, d\mathcal{H}^{N-1} \not\to 0$. Such a phenomenon can indeed occur as was shown by Arendt and Warma in [2] by employing arguments from capacity theory. Our analysis indicates that the problem can be caused by the points of ∂A which have not positive density with respect to $\bar{A}$ or $\mathbb{R}^N \setminus \bar{A}$, as the following example shows.

Let us consider $A \subset \mathbb{R}^3$ defined in the following way. Let $R_k := [a_k, b_k] \times [0,1] \times [0, c_k]$ with $a_0 = 0$, $a_{k+1} = b_k$ and $\sum_k |b_k - a_k| = 1$. For each k, let us consider the set C_k given by the union of $(k-1)$ closed cylinders with axis parallel to the z-axis, whose height is 1, and with a circular base with radius $r_k < \frac{b_k - a_k}{2}$ centered at the points $(\frac{a_k+b_k}{2}, \frac{i}{k}, 0)$ with $i = 1, \dots, k-1$. Let $v_n(x,y,z) := \frac{1}{1-x_n}(x - x_n)^+$, where $x_n := \sum_{k=0}^n (b_k - a_k)$. We choose c_k and r_k so small that by setting

$$A := \operatorname{int} \bigcup_{k=1}^{\infty} (R_k \cup C_k)$$

we have $\sum_k [\mathcal{H}^{N-1}(\partial R_k) + \mathcal{H}^{N-1}(\partial C_k)] < +\infty$ and $\|\nabla v_n\|_{L^2(A;\mathbb{R}^3)} \to 0$ as $n \to \infty$. Notice that the square $Q := \{1\} \times [0,1] \times [0,1]$ is contained in ∂A, since the points of Q are accumulation points for the family of cylinders given by the C_k's. The construction can be carried out in such a way that the points of Q have zero density with respect to $\bar{A}$. Moreover we have $v_n = 1$ on Q. As a consequence $\int_{\partial A} u_n^2 \, d\mathcal{H}^{N-1} \geq 1$ for every n, and so it cannot converge to 0 as $n \to \infty$. Since by general density results we can smooth out v_n to obtain u_n with the same boundary data and with vanishing $W^{1,2}$ norm, the result follows.

4. Isoperimetric inequality for Robin problems on arbitrary open sets

Let us fix $\beta > 0$. Given $\Omega \subset \mathbb{R}^N$ open, bounded and with Lipschitz boundary, the first eigenvalue $\lambda_1(\Omega)$ for the Laplacian on Ω with Robin boundary conditions is given by the minimization of the following Rayleigh quotient

$$\lambda_1(\Omega) = \min_{\substack{u \in W^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1}}{\int_\Omega u^2 \, dx}, \tag{4.1}$$

where the boundary term is intended in the sense of traces. The minimum is achieved on the linear space generated by an eigenfunction ψ_1 which is positive in Ω: we assume that ψ_1 has unitary norm in $L^2(\Omega)$, and we refer to ψ_1 as *the first eigenfunction.*

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and with Lipschitz boundary, and let ψ_1 be the associated first eigenfunction of the Robin boundary value problem. Then the function*

$$u_1(x) := \begin{cases} \psi_1(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

belongs to $SBV^{\frac{1}{2}}(\mathbb{R}^N)$. Moreover $J_{u_1} \subseteq \partial\Omega$ and

$$\frac{\int_{\mathbb{R}^N} |\nabla u_1|^2 \, dx + \beta \int_{J_{u_1}} [(u_1^+)^2 + (u_1^-)^2] \, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u_1^2 \, dx} = \frac{\int_\Omega |\nabla \psi_1|^2 \, dx + \beta \int_{\partial\Omega} \psi_1^2 \, d\mathcal{H}^{N-1}}{\int_\Omega \psi_1^2 \, dx} = \lambda_1(\Omega).$$

Proof. Since $\psi_1^2 \in W^{1,1}(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$ and Ω has a Lipschitz boundary, we deduce from [1, Theorem 3.87] that $u_1^2 \in SBV(\mathbb{R}^N)$ with $J_{u_1^2} \subseteq \partial\Omega$ and

$$(u_1^2)^+ = \psi_1^2 \qquad \text{and} \qquad (u_1^2)^- = 0.$$

Since $u_1 \geq 0$, we deduce that $u_1 \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ together with the result on the Rayleigh quotients. □

The preceding Lemma together with Proposition 3.4 suggests the following definition for the first eigenvalue of a general open domain with finite volume (possibly unbounded and irregular).

Definition 4.2 (The first Robin eigenvalue for a general domain). *Let $\Omega \subseteq \mathbb{R}^N$ be open with $|\Omega| < +\infty$. We set*

$$\lambda_1(\Omega) := \min \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2] \, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2 \, dx} : u \in SBV^{\frac{1}{2}}(\mathbb{R}^N), u \neq 0, \right.$$
$$\left. |supp(u) \setminus \Omega| = 0, \mathcal{H}^{N-1}(J_u \setminus \partial\Omega) = 0 \right\}. \tag{4.2}$$

If Ω is bounded and with Lipschitz boundary, the definition reduces to the classical one given in (4.1). If Ω has an irregular boundary, definition (4.2) permits to overcome the problem that the trace on the boundary for Sobolev functions is not easily defined.

Remark 4.3. As mentioned in the Introduction, Daners [8] defines the first eigenvalue of Ω by taking the infimum of the Rayleigh quotient appearing in (4.1) on the space

$$V_0(\Omega) := W^{1,2}(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega).$$

(The boundary term is simply the restriction of the function on $\partial\Omega$.) The connection with our definition has been exploited in the comments after Theorem 3.3 in Section 3: our arguments show that $\lambda_1(\Omega)$ is lower or equal than the one proposed by Daners.

We notice that our definition of $\lambda_1(\Omega)$ permits to deal with richer geometrical situations: for example, if Ω contains a compact smooth crack K, our definition of $\lambda_1(\Omega \setminus K)$ involves functions whose traces from the two sides of K can be different, as it is natural by looking at $\Omega \setminus K$ as limit of Lipschitz domains with holes shrinking on K. On the contrary, the definition of Daners involves functions which extend continuously on K.

We conclude the remark by noting that Daners approach is shaped in such a way to provide a functional framework in which a generalized boundary value problem of Robin's type can be settled, allowing to speak of higher order eigenvalues. Our SBV-approach is well suited (and natural) just for defining the first eigenvalue because the functional space involved in the definition of λ_1 is not a Hilbert space (if $\partial\Omega$ is "too large").

Let $m > 0$ be given, and let us set

$$\mathcal{A}_m(\mathbb{R}^N) := \{\Omega \subset \mathbb{R}^N \,:\, \Omega \text{ is open with } |\Omega| = m\}.$$

The main result of the paper is the following.

Theorem 4.4 (Isoperimetric inequality for Robin problems). *Let B be a ball of volume $m > 0$. Then*

$$\lambda_1(B) \le \lambda_1(\Omega) \tag{4.3}$$

for every $\Omega \in \mathcal{A}_m(\mathbb{R}^N)$. Moreover, equality holds if and only if Ω is a ball up to a negligible set.

The proof of Theorem 4.4 will be achieved by studying the variational problem

$$\inf_{\substack{u \in \mathcal{F}_m(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2]\, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\, dx} \tag{4.4}$$

where

$$\mathcal{F}_m(\mathbb{R}^N) := \{u \in SBV^{\frac{1}{2}}(\mathbb{R}^N) \,:\, |supp(u)| \le m\} \tag{4.5}$$

and where the space $SBV^{\frac{1}{2}}(\mathbb{R}^N)$ is given by Definition 3.1. Directly from the very definition of λ we get that

$$\lambda \le \inf_{\Omega \in \mathcal{A}_m(\mathbb{R}^N)} \lambda_1(\Omega). \tag{4.6}$$

The proof of the isoperimetric inequality for λ_1 relies on the following results which we prove in Section 5 and 6 respectively.

Theorem 4.5 (Minimization of the Rayleigh quotient). *The minimum problem*

$$\min_{\substack{u \in \mathcal{F}_m(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2]\, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\, dx} \tag{4.7}$$

admits a solution.

Theorem 4.6 (Minimizers are balls). *Let $u \in \mathcal{F}_m(\mathbb{R}^N)$ be a minimizer of the Rayleigh quotient* (4.7) *with $\|u\|_{L^2(\mathbb{R}^N)} = 1$, and let $\lambda > 0$ be the associated minimum value. Then there exists a ball $B \subset \mathbb{R}^N$ of volume m such that $\lambda = \lambda_1(B)$ and*

$$u = \psi_1 1_B, \tag{4.8}$$

where $\lambda_1(B)$ is the first eigenvalue and ψ_1 is the associated first eigenfunction of the classical Robin problem (1.1).

In view of the preceding results, the proof of Theorem 4.4 is straightforward.

Proof of Theorem 4.4. Inequality (4.3) follows from (4.6) and the fact that $\lambda = \lambda_1(B)$ in view of Theorem 4.6. Concerning the uniqueness, let Ω be such that $\lambda_1(\Omega) = \lambda_1(B)$. Let $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ be a function whose Rayleigh quotient is equal to $\lambda_1(\Omega)$. Then u is a minimizer of (4.7) on $\mathcal{F}_m(\mathbb{R}^N)$. By Theorem 4.6 we infer that it coincides with the first positive eigenfunction of a ball $\tilde{B}$ of volume m, extended to zero outside $\tilde{B}$. Since $\tilde{B} \subseteq \Omega$ (the first eigenfunction is positive) and $|\Omega| = |\tilde{B}| = m$, we conclude that $\Omega = \tilde{B}$ up to a negligible set, so that the proof is concluded. □

Remark 4.7. Since the minimum λ of (4.7) is attained on a function $u \in \mathcal{F}_m(\mathbb{R}^N)$ in view of Theorem 4.5, by employing the density result [6] we could prove directly that

$$\lambda = \inf\{\lambda_1(\Omega) : \Omega \subset \mathbb{R}^N \text{ is open, bounded, with Lipschitz boundary, and } |\Omega| \le m\}$$

without invoking the relation $\lambda = \lambda_1(B)$ entailed by Theorem 4.6. The preceding equality shows that problem (4.4) is in a certain sense a *relaxation* of the optimization problem for λ_1 on the class of Lipschitz domains with a volume constraint.

5. Optimization of the Rayleigh quotient

This section is devoted to the proof of Theorem 4.5. In order to prove the result, we need some preliminary lemmas. Recall that $\mathcal{F}_m(\mathbb{R}^N)$ is defined in (4.5).

Lemma 5.1. *Let $u \in \mathcal{F}_m(\mathbb{R}^N)$ and $t \in]0,1[$. By setting $v(x) := u(tx)$ for every $x \in \mathbb{R}^N$, we have $v \in \mathcal{F}_{m/t^N}(\mathbb{R}^N)$ and*

$$\begin{aligned} &\frac{\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \beta \int_{J_v} [(v^+)^2 + (v^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} v^2\,dx} \\ &\qquad \le t \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\,dx}. \end{aligned} \tag{5.1}$$

Proof. Clearly $v \in \mathcal{F}_{m/t^N}(\mathbb{R}^N)$. Moreover, straightforward calculations show that

$$\int_{\mathbb{R}^N} |\nabla v|^2\,dx = t^{2-N} \int_{\mathbb{R}^N} |\nabla u|^2\,dx,$$

$$\int_{\mathbb{R}^N} v^2\,dx = t^{-N} \int_{\mathbb{R}^N} u^2\,dx,$$

and

$$\int_{J_v} [(v^+)^2 + (v^-)^2]\,d\mathcal{H}^{N-1} = t^{1-N} \int_{J_u} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1}.$$

Then we obtain

$$\begin{aligned} \frac{\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \beta \int_{J_v} [(v^+)^2 + (v^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} v^2\,dx} &= \frac{t^2 \int_{\mathbb{R}^N} |\nabla u|^2\,dx + t\beta \int_{J_u} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\,dx} \\ &\le t \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\,dx} \end{aligned}$$

so that the proof of (5.1) is concluded. □

Lemma 5.2. *Let $u \in \mathcal{F}_m(\mathbb{R}^N)$ be such that $\int_{\mathbb{R}^N} u^2\,dx = L > 0$ and*

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{J_u} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1} = M > 0.$$

Then there exist $y \in \mathbb{R}^N$ such that

$$|\mathrm{supp}(u) \cap Q_1(y)| \geq \left(\frac{S}{M(1+1/L)+2} \right)^N \tag{5.2}$$

where S depends only on N, and $Q_1(y)$ is an open unit cube centered at y.

Proof. Let us adapt to our SBV-context an idea by H. Brezis reported in E.H. Lieb's paper [12, Section 3]. Since J_u is $\mathcal{H}^{N-1}$-rectifiable in view of Lemma 3.2, we can find $z \in \mathbb{R}^N$ such that the grid $z + \xi + \partial Q_1$, with $\xi \in \mathbb{Z}^N$ and Q_1 the unit cube centered at the origin, intersects J_u on a $\mathcal{H}^{N-1}$-negligible set. We clearly can find $y = z + \bar{\xi} \in \mathbb{R}^N$ such that

$$\int_{Q_1(y)} u^2\,dx \neq 0 \tag{5.3}$$

and

$$\int_{Q_1(y)} |\nabla u|^2\,dx + \int_{J_u \cap Q_1(y)} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1} \leq K \int_{Q_1(y)} u^2\,dx, \tag{5.4}$$

where $K = M\,(1 + 1/L)$. Since by definition $u^2 \in SBV(\mathbb{R}^N)$, the immersion of $BV(Q_1(y))$ into $L^{N/N-1}(Q_1(y))$ entails that there exists $S > 0$ depending only on N such that

$$\begin{aligned} S\|u^2\|_{L^{N/N-1}(Q_1(y))} &\leq \|u^2\|_{BV(Q_1(y))} \\ &= 2\int_{Q_1(y)} |u||\nabla u|\,dx + \int_{J_u \cap Q_1(y)} [(u^+)^2 - (u^-)^2]\,d\mathcal{H}^{N-1} + \int_{Q_1(y)} u^2\,dx \\ &\leq \int_{Q_1(y)} [|\nabla u|^2 + u^2]\,dx + \int_{J_u \cap Q_1(y)} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1} + \int_{Q_1(y)} u^2\,dx \end{aligned}$$

so that in view of (5.4) and by applying Hölder inequality we get

$$\begin{aligned} S\left(\int_{Q_1(y)} u^{\frac{2N}{N-1}}\,dx \right)^{N/N-1} &\leq (K+2)\int_{Q_1(y)} u^2\,dx \\ &\leq (K+2)\left(\int_{Q_1(y)} u^{\frac{2N}{N-1}}\,dx \right)^{N/N-1} |\mathrm{supp}(u) \cap Q_1(y)|^{1/N}. \end{aligned}$$

In view of (5.3) we conclude that

$$|\mathrm{supp}(u) \cap Q_1(y)|^{1/N} \geq \frac{S}{K+2}$$

so that (5.2) follows. □

Lemma 5.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}_m(\mathbb{R}^N)$ such that*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2\,dx > 0.$$

Then

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2]\,d\mathcal{H}^{N-1} > 0.$$

Proof. By contradiction, let $(u_{n_k})_{k\in\mathbb{N}}$ be a subsequence such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} u_{n_k}^2\,dx = L > 0 \tag{5.5}$$

and

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}|\nabla u_{n_k}|^2\,dx+\beta\int_{J_{u_{n_k}}}[(u_{n_k}^+)^2+(u_{n_k}^-)^2]\,d\mathcal{H}^{N-1}=0.$$

Since by definition $v_n := u_n^2 \in SBV(\mathbb{R}^N)$, by Sobolev inequality for the embedding of $BV(\mathbb{R}^N)$ into $L^{N/N-1}(\mathbb{R}^N)$ we get for k large

$$\begin{aligned}C\|u_{n_k}^2\|_{L^{N/N-1}(\mathbb{R}^N)} &\le |Du_{n_k}^2|(\mathbb{R}^N)=2\int_{\mathbb{R}^N}|u_{n_k}||\nabla u_{n_k}|\,dx+\int_{J_{u_{n_k}}}[(u_{n_k}^+)^2-(u_{n_k}^-)^2]\,d\mathcal{H}^{N-1}\\ &\le 2(\sqrt{L}+1)\left(\int_{\mathbb{R}^N}|\nabla u_{n_k}|^2\,dx\right)^{1/2}+\int_{J_{u_{n_k}}}[(u_{n_k}^+)^2+(u_{n_k}^-)^2]\,d\mathcal{H}^{N-1}\to 0\end{aligned}$$

so that

$$u_{n_k}\to 0\qquad\text{strongly in } L^{2N/N-1}(\mathbb{R}^N).$$

Since $|supp(u_{n_k})|\le m$ for every $k\in\mathbb{N}$, by Hölder inequality we deduce that $u_{n_k}\to 0$ strongly in $L^2(\mathbb{R}^N)$, which is against (5.5). □

We are now in a position to prove Theorem 4.5.

Proof of Theorem 4.5. The proof of the Theorem involves a *compactness-vanishing-dichotomy alternative* in the framework of the concentration-compactness principle of P.L. Lions [13, 14].

Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence in $\mathcal{F}_m(\mathbb{R}^N)$ for problem (4.7), and let λ be the associated value of the infimum. By Lemma 5.1 we may assume that for every $n\in\mathbb{N}$

$$|supp(u_n)|=m,$$

and (renormalizing)

$$\int_{\mathbb{R}^N}u_n^2\,dx=1. \tag{5.6}$$

In view of Lemma 5.3, we get $\lambda>0$.

Following [14], for every $R>0$ let us set

$$Q_n(R):=\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}u_n^2\,dx.$$

Clearly $Q_n:[0,+\infty[\to[0,+\infty[$ is a sequence of monotone functions. Up to a subsequence, in view of Helly's theorem, we may assume that

$$Q_n\to Q\qquad\text{pointwise on }[0,+\infty[$$

for a suitable monotone increasing function $Q:[0,+\infty[\to[0,+\infty[$. We will have different alternatives depending on the behaviour of $Q(R)$. Following Lions, if $Q\equiv 0$, we say that *vanishing* occurs, while if $\lim_{R\to+\infty}Q(R)=\alpha$ with $0<\alpha<1$, we say that *dichotomy* occurs. In the following we show that vanishing and dichotomy cannot occur, so that $\lim_{R\to+\infty}Q(R)=1$, i.e., we have *compactness*, and this leads to a solution of the problem.

We divide the proof in three steps.

Step 1: Vanishing cannot occur. We claim that it is not possible that $Q(R)\equiv 0$, i.e., for every $R>0$

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}u_n^2\,dx=0. \tag{5.7}$$

Since

$$\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx+\beta\int_{J_{u_n}}[(u_n^+)^2+(u_n^-)^2]\,d\mathcal{H}^{N-1}=:M_n\to\lambda>0,$$

by Lemma 5.2 we can find $y_n\in\mathbb{R}^N$ and $C>0$ such that

$$\liminf_{n\to\infty}|\mathrm{supp}(u_n)\cap Q_1(y_n)|^{1/N}\ge C. \tag{5.8}$$

Since by (5.7) we have

$$\lim_{n\to\infty}\int_{Q_1(y_n)} u_n^2\,dx = 0, \tag{5.9}$$

it is not restrictive (up to reducing $Q_1(y_n)$ if necessary, which is possible in view of (5.8)) to assume that the trace $\gamma(u_n^2)$ of $u_n^2 \in SBV(\mathbb{R}^N)$ on the boundary of $Q_1(y_n)$ converges to zero strongly in $L^1(\partial Q_1(y_n))$, i.e.,

$$\lim_{n\to\infty}\int_{\partial Q_1(y_n)} \gamma(u_n^2)\,dx = 0. \tag{5.10}$$

Let us consider $w_n := u_n 1_{\mathbb{R}^N\setminus Q_1(y_n)} \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$. Notice that by (5.9) we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} w_n^2\,dx = 1.$$

Moreover we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w_n|^2\,dx + \beta\int_{J_{w_n}} [(w_n^+)^2 + (w_n^-)^2]\,d\mathcal{H}^{N-1} \\ = \int_{\mathbb{R}^N\setminus Q_1(y_n)} |\nabla u_n|^2\,dx + \beta\int_{J_{u_n}\setminus \bar{Q}_1(y_n)} [(u_n^+)^2 + (u_n^-)^2]\,d\mathcal{H}^{N-1} + \beta\int_{\partial Q_1(y_n)} \gamma(u_n^2)\,d\mathcal{H}^{N-1} \\ \le M_n + \beta\int_{\partial Q_1(y_n)} \gamma(u_n^2)\,d\mathcal{H}^{N-1}.\end{aligned}$$

By (5.10) we conclude that $(w_n)_{n\in\mathbb{N}}$ is a minimizing sequence for the main problem (4.7). But this cannot be the case because in view of (5.8) we have that

$$|supp(w_n)| = (1-\varepsilon)m$$

with $\varepsilon > 0$, so that using the rescaling property (5.1) we obtain a contradiction.

Step 2: Dichotomy cannot occur. Let us assume that $\lim_{R\to+\infty} Q(R) = \alpha$ with $0 < \alpha < 1$. Following [14], for every $\varepsilon > 0$ we can find two radial cut off functions $\varphi_n, \psi_n : \mathbb{R}^N \to [0,1]$ with $\|\nabla\varphi_n\|_\infty, \|\nabla\psi_n\|_\infty \le \varepsilon$,

$$\lim_{n\to\infty} \mathrm{dist}(\mathrm{supp}(\varphi_n), \mathrm{supp}(1-\psi_n)) = +\infty,$$

and we can find $y_n \in \mathbb{R}^N$ such that by setting $v_n := \varphi_n(\cdot + y_n)u_n$ and $w_n := (1-\psi_n)(\cdot + y_n)u_n$ we have

$$\|u_n^2 - v_n^2 - w_n^2\|_{L^1(\mathbb{R}^N)} \le \varepsilon,$$

$$\left|\int_{\mathbb{R}^N} v_n^2\,dx - \alpha\right| \le \varepsilon \qquad \text{and} \qquad \left|\int_{\mathbb{R}^N} w_n^2\,dx - (1-\alpha)\right| \le \varepsilon. \tag{5.11}$$

A straightforward calculation shows that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N} [|\nabla u_n|^2 - |\nabla v_n|^2 - |\nabla w_n|^2]\,dx \ge e(\varepsilon), \tag{5.12}$$

where $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. On the other hand, since $J_{v_n} \subseteq J_{u_n}$, $J_{w_n} \subseteq J_{u_n}$ and $v_n^\pm, w_n^\pm \le u_n^\pm$, we get

$$\begin{aligned}\liminf_{n\to\infty}\int_{J_{u_n}} (u_n^+)^2 + (u_n^-)^2\,d\mathcal{H}^{N-1} - \int_{J_{v_n}} (v_n^+)^2 + (v_n^-)^2\,d\mathcal{H}^{N-1} \\ - \int_{J_{w_n}} (w_n^+)^2 + (w_n^-)^2\,d\mathcal{H}^{N-1} \ge 0.\end{aligned} \tag{5.13}$$

We conclude that for n large

$$\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u_n^2\,dx} \geq e(\varepsilon) + (1-2\varepsilon)\cdot$$
$$\cdot \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} |\nabla w_n|^2\,dx + \beta \int_{J_{v_n}} [(v_n^+)^2 + (v_n^-)^2]\,d\mathcal{H}^{N-1} + \beta \int_{J_{w_n}} [(w_n^+)^2 + (w_n^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} v_n^2\,dx + \int_{\mathbb{R}^N} w_n^2\,dx}.$$

Up to a subsequence we thus may assume that for n large

$$\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u_n^2\,dx}$$
$$\geq e(\varepsilon) + (1-2\varepsilon)\frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \beta \int_{J_{v_n}} [(v_n^+)^2 + (v_n^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} v_n^2\,dx}.$$

By passing to the limit we obtain

$$\frac{\lambda - e(\varepsilon)}{1-2\varepsilon} \geq \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \beta \int_{J_{v_n}} [(v_n^+)^2 + (v_n^-)^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} v_n^2\,dx}. \tag{5.14}$$

If ε is small enough, inequalities (5.12), (5.13) and (5.11) together with Lemmas 5.3 and 5.2 imply that for n large

$$|\mathrm{supp}(w_n)| \geq \left(\frac{S}{\lambda(1+1/(1-\alpha+\varepsilon))+2}\right)^N =: \eta_\varepsilon m,$$

where $\eta_\varepsilon \in]0,1[$ and S depends only on N. Clearly, since a similar lower bound holds also for $supp(v_n)$, we get $\eta_\varepsilon \to \eta$, with $\eta \in]0,1[$ depending only on α, λ, and m. As a consequence we get that for n large

$$|\mathrm{supp}(v_n)| \leq (1-\eta_\varepsilon)m,$$

so that in view of Lemma 5.1 and (5.14) we deduce that

$$(1-\eta_\varepsilon)\frac{\lambda - e(\varepsilon)}{1-2\varepsilon} \geq \lambda,$$

which is impossible if ε is small enough. This proves that dichotomy cannot occur, and the proof of Step 2 is concluded.

Step 3: Compactness and conclusion. In view of Steps 1 and 2, we have that

$$\lim_{R\to\infty} Q(R) = 1,$$

i.e., we are in the *compact case*. Following [14], there exist $y_n \in \mathbb{R}^N$ such that by setting

$$v_n(x) := u_n(x+y_n)$$

we have up to a subsequence

$$v_n \rightharpoonup u \qquad \text{weakly in } L^2(\mathbb{R}^N) \tag{5.15}$$

for some $u \in L^2(\mathbb{R}^N)$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} v_n^2\,dx \leq \varepsilon(R) \tag{5.16}$$

with $\varepsilon(R) \to 0$ as $R \to +\infty$.

Since v_n satisfies the assumptions of Theorem 3.3, we deduce that $u \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ and $v_n \to u$ strongly in $L^2_{loc}(\mathbb{R}^N)$. Let us fix $R > 0$. By (5.16) we deduce

$$\int_{B_R(0)} u^2\,dx = \lim_{n\to\infty} \int_{B_R(0)} v_n^2\,dx \geq 1 - \varepsilon(R).$$

Sending $R \to +\infty$ we conclude

$$\int_{\mathbb{R}^N} u^2\,dx = 1$$

so that by (5.15) and (5.6)

$$v_n \to u \qquad \text{strongly in } L^2(\mathbb{R}^N). \tag{5.17}$$

This entails $u \in \mathcal{F}_m(\mathbb{R}^N)$. Moreover, by the lower semicontinuity result given in Theorem 3.3 we deduce

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2]\,d\mathcal{H}^{N-1}$$
$$\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \beta \int_{J_{u_n}} [(u_n^+)^2 + (u_n^-)^2]\,d\mathcal{H}^{N-1} = \lambda,$$

so that, in view of (5.17) and (5.6) we infer that $u \in \mathcal{F}_m(\mathbb{R}^N)$ is a minimizer of the main problem (4.7). The proof of Theorem 4.5 is now concluded. ☐

Remark 5.4. From the proof of Theorem 4.5 we deduce the following result: if $u_n \in \mathcal{F}_m(\mathbb{R}^N)$ is a minimizing sequence for (4.7), then up to a subsequence and to a translation $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$ for some minimizer $u \in \mathcal{F}_m(\mathbb{R}^N)$ of (4.7).

6. Structure of minimizers

This section is devoted to the proof of Theorem 4.6. In order to get the result, we will follow the strategy employed in [5] which we adapt to our SBV-context.

Let $u \in \mathcal{F}_m(\mathbb{R}^N)$ be a minimizer of (4.7) such that $\|u\|_{L^2(\mathbb{R}^N)} = 1$, and let $\lambda > 0$ be the associated minimum value. In what follows we will identify functions $v \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ with their approximately continuous representatives, i.e., we let $v(x)$ be equal to the approximate limit at x if $x \notin J_v$, and let $v(x) = 0$ if $x \in J_v$. Moreover, we will consider $\nabla v = 0$ on the negligible set where v is not approximately differentiable.

Let us set for $t \geq 0$

$$U_t := \{x \in \mathbb{R}^N : u(x) > t\}.$$

By Coarea Formula in BV [1, Theorem 3.40] applied to u^2, it turns out that for a.e. $t > 0$ the set U_t has finite perimeter: we denote by $\partial^* U_t$ its reduced boundary. Moreover, U_t has also finite volume for every $t > 0$.

Proposition 6.1 (Euler-Lagrange equation). *Let $t > 0$ and $v \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ be such that $|supp(v) \setminus U_t| = 0$, $\|v\|_\infty < +\infty$, $J_v \subseteq J_u$ and*

$$\int_{J_u \setminus J_v} v^2\,d\mathcal{H}^{N-1} < +\infty.$$

Then the following equality holds:

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v\,dx + \beta \int_{J_u} [u^+\gamma^+(v) + u^-\gamma^-(v)]\,d\mathcal{H}^{N-1} = \lambda \int_{\mathbb{R}^N} uv\,dx, \tag{6.1}$$

where $\gamma^\pm(v)$ are the traces of v on the rectifiable set J_u oriented by the normal ν_u.

Proof. Let $s \in \mathbb{R}$ be so small that the variation $u + sv$ belongs to $\mathcal{F}_m(\mathbb{R}^N)$. Since $J_{u+sv} \subseteq J_u$, the Rayleigh quotient for $u + sv$ may be written as follows

$$\frac{\int_{\mathbb{R}^N} |\nabla u + s\nabla v|^2\,dx + \beta \int_{J_{u+sv}} [(\gamma^+(u) + s\gamma^+(v))^2 + (\gamma^-(u) + s\gamma^-(v))^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} (u+sv)^2\,dx}. \tag{6.2}$$

Notice that there exists a countable set B such that for $s \notin B$

$$\mathcal{H}^{N-1}(J_u \setminus J_{u+sv}) = 0. \tag{6.3}$$

Indeed, the sets

$$J_u \qquad \text{and} \qquad D := \{x \in J_u : \gamma^+(u)(x) \neq \gamma^-(u)(x)\}$$

coincide up to a $\mathcal{H}^{N-1}$-negligible set, and the same holds for

$$J_{u+sv} \qquad \text{and} \qquad D_s := \{x \in J_u : \gamma^+(u+sv)(x) \neq \gamma^-(u+sv)(x)\}.$$

Notice that the sets $D \setminus D_s$ are disjoint for every $s \neq 0$: indeed if $x \in D \setminus D_s$ we have

$$\gamma^+(u) \neq \gamma^-(u) \qquad \text{and} \qquad \gamma^+(u+sv)(x) = \gamma^-(u+sv)(x),$$

so that $\gamma^+(v)(x) \neq \gamma^-(v)(x)$ and

$$s = -\frac{\gamma^+(u)(x) - \gamma^-(u)(x)}{\gamma^+(v)(x) - \gamma^-(v)(x)},$$

i.e., s has a precise value. By general arguments of measure theory, we conclude that only for a countable number of $s \neq 0$ the sets $D \setminus D_s$ have positive $\mathcal{H}^{N-1}$-measure. If denote by B such a countable set of values, relation (6.3) follows.

For $s \notin B$ we deduce that (6.2) can be rewritten in the following form

$$\frac{\int_{\mathbb{R}^N} |\nabla u + s\nabla v|^2\,dx + \beta \int_{J_u} [(\gamma^+(u) + s\gamma^+(v))^2 + (\gamma^-(u) + s\gamma^-(v))^2]\,d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} (u+sv)^2\,dx} := g(s).$$

The Euler-Lagrange equation (6.1) follows now by exploiting the fact that $g(s) \geq g(0)$ with $s \notin B$, and recalling that $\gamma^+(u)(x) = u^+(x)$ and $\gamma^-(u)(x) = u^-(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$. □

For every Borel function $\varphi : \mathbb{R}^N \to [0, +\infty[$ we set for those t for which U_t has finite perimeter and has finite positive volume

$$H(U_t, \varphi) := \frac{1}{|U_t|}\Bigg[-\int_{U_t} \varphi^2\,dx + \int_{\partial^* U_t \setminus J_u} \varphi\,d\mathcal{H}^{N-1} \\ + \beta\left(\mathcal{H}^{N-1}(J_u \cap \{u^+ > t\}) + \mathcal{H}^{N-1}(J_u \cap \{u^- > t\})\right)\Bigg],$$

whenever the expression is well defined. The functional H is similar to the functional H_Ω employed in [9] and in [5]: in our context also "jump terms" appear. Notice that only the values of φ on U_t and $\partial^* U_t$ are involved in the definition.

For every $t > 0$ we have

$$\frac{|\nabla u|}{u} \in L^2(U_t).$$

We need some lemmas.

Lemma 6.2. *For a.e. $t > 0$ the following equality holds*

$$\lambda = H\left(U_t, \frac{|\nabla u|}{u}\right). \tag{6.4}$$

Proof. Following [5], we derive (6.4) from the Euler-Lagrange equation (6.1) by choosing suitable variations v. For every $\varepsilon > 0$ let us consider

$$v_\varepsilon := \frac{1}{u}\min\left\{1, \frac{\max\{0, u-t\}}{\varepsilon}\right\},$$

where $t > 0$ is such that U_t has finite perimeter and has finite positive volume: we know that almost every t matches these requirements. As $\varepsilon \to 0$ we obtain

$$v_\varepsilon \to \frac{1}{u}1_{U_t} \qquad \text{pointwise a.e. on } \mathbb{R}^N.$$

Since v_ε is a good test for the Euler-Lagrange equation of u, we get

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\varepsilon\,dx + \beta \int_{J_u} [u^+\gamma^+(v_\varepsilon) + u^-\gamma^-(v_\varepsilon)]\,d\mathcal{H}^{N-1} = \lambda \int_{\mathbb{R}^N} u v_\varepsilon\,dx. \tag{6.5}$$

Notice that by monotone convergence

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} u v_\varepsilon\,dx = |U_t|. \tag{6.6}$$

Let us consider the first term on the left-hand side. In the following calculations we will repeatedly use the fact that if ψ is an approximately differentiable function and E is a level set of ψ, then $\nabla\psi = 0$ almost everywhere on E. By the chain rule in BV [1, Theorem 3.96] we get

$$\begin{aligned}(6.7)\qquad \int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\varepsilon \, dx &= \int_{U_{t+\varepsilon}} \nabla u \nabla \left(\frac{1}{u}\right) dx + \frac{1}{\varepsilon} \int_{U_t \setminus U_{t+\varepsilon}} \nabla u \nabla \left(1 - \frac{t}{u}\right) dx \\ &= -\int_{U_{t+\varepsilon}} \frac{|\nabla u|^2}{u^2} \, dx + \frac{t}{\varepsilon} \int_{U_t \setminus U_{t+\varepsilon}} \frac{|\nabla u|^2}{u^2} \, dx.\end{aligned}$$

Notice that in view of the coarea formula in BV [1, Theorem 3.40] (applied for example to $u \vee t \in SBV_{loc}(\mathbb{R}^N)$) we have

$$\begin{aligned}(6.8)\qquad \int_{U_t \setminus U_{t+\varepsilon}} \frac{|\nabla u|^2}{u^2} \, dx &= \int_{U_t \setminus U_{t+\varepsilon}} \frac{|\nabla u|}{u^2} |\nabla u| \, dx = \int_{(U_t \setminus U_{t+\varepsilon}) \setminus J_u} \frac{|\nabla u|}{u^2} d|Du| \\ &= \int_t^{t+\varepsilon} \int_{\partial^* U_s \setminus J_u} \frac{|\nabla u|}{u^2} \, d\mathcal{H}^{N-1} \, ds.\end{aligned}$$

We deduce that for a.e. $t > 0$

$$(6.9)\qquad \lim_{\varepsilon \to 0^+} \frac{t}{\varepsilon} \int_{U_t \setminus U_{t+\varepsilon}} \frac{|\nabla u|^2}{u^2} \, dx = t \int_{\partial^* U_t \setminus J_u} \frac{|\nabla u|}{u^2} \, d\mathcal{H}^{N-1} = \int_{\partial^* U_t \setminus J_u} \frac{|\nabla u|}{u} \, d\mathcal{H}^{N-1}$$

and

$$(6.10)\qquad \lim_{\varepsilon \to 0^+} \int_{U_{t+\varepsilon}} \frac{|\nabla u|^2}{u^2} \, dx = \int_{U_t} \frac{|\nabla u|^2}{u^2} \, dx.$$

Coming back to the jump-term in (6.5) we have by the property of traces under composition

$$\begin{aligned}&\int_{J_u} [u^+ \gamma^+(v_\varepsilon) + u^- \gamma^-(v_\varepsilon)] \, d\mathcal{H}^{N-1} \\ &\qquad = \int_{J_u} \left[u^+ \frac{1}{u^+} \min\left\{1, \frac{\max\{0, u^+ - t\}}{\varepsilon}\right\} \right. \\ &\qquad\qquad \left. + u^- \frac{1}{u^-} \min\left\{1, \frac{\max\{0, u^- - t\}}{\varepsilon}\right\} \right] d\mathcal{H}^{N-1} \\ &\qquad = \int_{J_u} \left[\min\left\{1, \frac{\max\{0, u^+ - t\}}{\varepsilon}\right\} + \min\left\{1, \frac{\max\{0, u^- - t\}}{\varepsilon}\right\} \right] d\mathcal{H}^{N-1}.\end{aligned}$$

By monotone convergence we conclude that

$$\begin{aligned}(6.11)\qquad &\lim_{\varepsilon \to 0^+} \int_{J_u} [u^+ \gamma^+(v_\varepsilon) + u^- \gamma^-(v_\varepsilon)] \, d\mathcal{H}^{N-1} \\ &\qquad = \mathcal{H}^{N-1}(J_u \cap \{u^+ > t\}) + \mathcal{H}^{N-1}(J_u \cap \{u^- > t\}).\end{aligned}$$

By taking the limit as $\varepsilon \to 0$ in (6.5) in view of (6.6)-(6.11) we obtain precisely (6.4) for a.e. $t > 0$, so that the proof is concluded. □

Lemma 6.3. *Let $\varphi : \mathbb{R}^N \to [0, +\infty[$ be a Borel function such that $\varphi \in L^2(U_t)$ for every $t > 0$. Set*

$$(6.12)\qquad w := \varphi - \frac{|\nabla u|}{u} \qquad \text{and} \qquad F(t) := \int_{U_t} w \frac{|\nabla u|}{u} \, dx.$$

Then $F :]0, +\infty[\to \mathbb{R}$ is locally absolutely continuous and

$$(6.13)\qquad H(U_t, \varphi) \le \lambda - \frac{1}{t|U_t|} \frac{d}{dt}(t^2 F(t))$$

for a.e. $t \in]0, +\infty[$. Moreover, there is strict inequality (6.13) if and only if $\varphi \neq \frac{|\nabla u|}{u}$ in U_t on a set of positive measure.

Proof. By the level set representation (6.4) of λ we get for a.e. $t > 0$ (using also the inequality $a^2 - b^2 \geq 2b(a-b)$ for $a, b \in \mathbb{R}$)

$$\begin{aligned} H(U_t, \varphi) &= \lambda + \frac{1}{|U_t|}\left[\int_{\partial^* U_t \setminus J_u} w \, d\mathcal{H}^{N-1} - \int_{U_t}\left(\varphi^2 - \frac{|\nabla u|^2}{u^2}\right) dx\right] \\ &\leq \lambda + \frac{1}{|U_t|}\left[\int_{\partial^* U_t \setminus J_u} w \, d\mathcal{H}^{N-1} - 2\int_{U_t} w\frac{|\nabla u|}{u} \, dx\right] \\ &= \lambda + \frac{1}{|U_t|}\left[\int_{\partial^* U_t \setminus J_u} w \, d\mathcal{H}^{N-1} - 2F(t)\right]. \end{aligned}$$

Moreover strict inequality holds if and only if $\varphi \neq \frac{|\nabla u|}{u}$ in U_t on a set of positive measure. We claim that $F(t)$ is locally absolutely continuous and for a.e. $t > 0$

$$F'(t) = -\frac{1}{t}\int_{\partial^* U_t \setminus J_u} w \, d\mathcal{H}^{N-1}. \tag{6.14}$$

Then it follows that

$$H(U_t, \varphi) \leq \lambda + \frac{1}{|U_t|}(-tF'(t) - 2F(t)) = \lambda - \frac{1}{t|U_t|}\frac{d}{dt}(t^2 F(t))$$

which is precisely (6.13).

In order to conclude the proof we have to check claim (6.14). By the coarea formula in BV [1, Theorem 3.40] we get for every $t > 0$

$$\begin{aligned} F(t) = \int_{U_t} w\frac{|\nabla u|}{u} \, dx = \int_{U_t \setminus J_u} \frac{w}{u} d|Du| &= \int_t^{+\infty}\int_{\partial^* U_s \setminus J_u} \frac{w}{u} \, d\mathcal{H}^{N-1} \, ds \\ &= \int_t^{+\infty}\frac{1}{s}\int_{\partial^* U_s \setminus J_u} w \, d\mathcal{H}^{N-1} \, ds, \end{aligned}$$

so that $F(t)$ is locally absolutely continuous and claim (6.14) holds. $\square$

Lemma 6.4. *Let $\varphi : \mathbb{R}^N \to [0, +\infty[$ be a Borel function such that $\varphi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and*

$$\varphi \neq \frac{|\nabla u|}{u} \qquad \text{on a subset of positive measure of } supp(u).$$

Then there exists $S \subset]0, +\infty[$ of positive measure such that for every $t \in S$

$$\lambda > H(U_t, \varphi).$$

Proof. By contradiction let

$$\lambda \leq H(U_t, \varphi)$$

for a.e. $t > 0$. By (6.13) we infer

$$\frac{d}{dt}(t^2 F(t)) \leq 0$$

where $F(t)$ is defined in (6.12). Moreover we have strict inequality for t small enough since the union of U_t exhausts *supp*(u). We claim that

$$\lim_{t \to +\infty} t^2 F(t) = 0. \tag{6.15}$$

From this we infer that $\lim_{t \to 0^+} t^2 F(t) > 0$. But since

$$\begin{aligned} F(t) = \int_{U_t} w\frac{|\nabla u|}{u} \, dx = \int_{U_t} \varphi\frac{|\nabla u|}{u} \, dx - \int_{U_t} \frac{|\nabla u|^2}{u^2} \, dx \leq \int_{U_t} \varphi\frac{|\nabla u|}{u} \, dx &\leq \frac{1}{t}\int_{U_t} \varphi|\nabla u| \, dx \\ &\leq \frac{1}{t}\|\varphi\|_{L^2(\mathbb{R}^N)}\|\nabla u\|_{L^2(\mathbb{R}^N)} \end{aligned}$$

we get $\lim_{t \to 0^+} t^2 F(t) \leq 0$, which is absurd.

In order to conclude the proof we need to check claim (6.15). Notice that for every $t > 0$

$$t^2 \int_{U_t} \varphi \frac{|\nabla u|}{u}\, dx \le \int_{U_t} u\varphi |\nabla u|\, dx \le \|\varphi\|_\infty \|u\|_{L^2(U_t)} \|\nabla u\|_{L^2(U_t;\mathbb{R}^N)}$$

and

$$t^2 \int_{U_t} \frac{|\nabla u|^2}{u^2}\, dx \le \int_{U_t} |\nabla u|^2\, dx \le \|\nabla u\|^2_{L^2(U_t;\mathbb{R}^N)}.$$

From

$$t^2 F(t) = t^2 \int_{U_t} \varphi \frac{|\nabla u|}{u}\, dx - t^2 \int_{U_t} \frac{|\nabla u|^2}{u^2}\, dx$$

we infer that claim (6.15) holds, so that the proof is concluded. □

We are now in a position to prove Theorem 4.6.

Proof of Theorem 4.6. Let $B(0)$ be a ball centered at 0 with volume m and radius R, and let λ_1 be the first eigenvalue and ϕ_1 the first eigenfunction of the associated Robin problem: clearly λ_1 is invariant under translations of $B(0)$. Moreover $\lambda_1 \ge \lambda$ by inequality (4.6).

The proof will follow by *dearranging* $|\nabla\phi_1|/\phi_1$ on the support of u to get a function φ to be employed in connection with Lemma 6.4. We divide the proof in several steps.

Step 1. The function ϕ_1 is smooth up to the boundary of $B(0)$, and it is radial decreasing with

$$\lim_{|x|\to R} \phi_1(x) > 0.$$

Moreover it turns out that (see for instance [5])

$$x \mapsto \beta(x) := \frac{|\nabla\phi_1(x)|}{\phi_1(x)}$$

is radial increasing with

$$\beta(0) = 0 \qquad \text{and} \qquad \beta(x) = \beta \text{ for } |x| = R.$$

Let

$$\beta_r := \beta(x) \qquad \text{for } |x| = r.$$

Let $r(t)$ be the radius of the ball centered at 0 with volume $|U_t|$. Let us consider $\varphi : \mathbb{R}^N \to]0, +\infty[$ defined as

$$\varphi(x) := \beta_{r(t)} \qquad \text{if} \qquad u(x) = t > 0,$$

and $\varphi = 0$ otherwise. The function φ is the *dearrangement* of β on the support of u in the sense that the radial rearrangement of φ gives back β. By construction φ is Borel measurable with $\varphi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Step 2. We claim that

$$H(U_t, \varphi) \ge \lambda_1 \ge \lambda \tag{6.16}$$

for $t > 0$ outside a countable set. Indeed, concerning the volume terms appearing in $H(U_t, \varphi)$ we have

$$\int_{U_t} \varphi^2\, dx = \int_{B_{r(t)}(0)} \frac{|\nabla\phi_1|^2}{\phi_1^2}\, dx, \tag{6.17}$$

and

$$\int_{\partial^* U_t \setminus J_u} \varphi\, d\mathcal{H}^{N-1} = \int_{\partial^* U_t \setminus J_u} \beta_{r(t)}\, d\mathcal{H}^{N-1} = \beta_{r(t)} \mathcal{H}^{N-1}(\partial^* U_t \setminus J_u). \tag{6.18}$$

Equality (6.17) follows by general properties of the radial decreasing rearrangement. The first equality in (6.18) follows by the definition of φ.

The jump term appearing in $H(U_t, \varphi)$, for t outside a countable set, can be expressed as

$$\beta \left(\mathcal{H}^{N-1}(J_u \cap \{u^+ \ge t\}) + \mathcal{H}^{N-1}(J_u \cap \{u^- \ge t\}) \right).$$

Recalling that $|U_t| = |B_{r(t)}(0)|$ and $\beta \geq \beta_{r(t)}$, we infer

$$\begin{aligned} (6.19) \quad H(U_t, \varphi) &= \frac{1}{|B_{r(t)}|}\Big[-\int_{B_{r(t)}(0)} \frac{|\nabla \phi_1|^2}{\phi_1^2}\,dx + \beta_{r(t)}\mathcal{H}^{N-1}(\partial^* U_t \setminus J_u) \\ &\qquad + \beta\left(\mathcal{H}^{N-1}(J_u \cap \{u^+ \geq t\}) + \mathcal{H}^{N-1}(J_u \cap \{u^- \geq t\})\right)\Big] \\ &\geq \frac{1}{|B_{r(t)}|}\Big[-\int_{B_{r(t)}(0)} \frac{|\nabla \phi_1|^2}{\phi_1^2}\,dx + \beta_{r(t)}\mathcal{H}^{N-1}(\partial^* U_t \setminus J_u) + \beta_{r(t)}\mathcal{H}^{N-1}(\partial^* U_t \cap J_u)\Big] \\ &= \frac{1}{|B_{r(t)}|}\Big[-\int_{B_{r(t)}(0)} \frac{|\nabla \phi_1|^2}{\phi_1^2}\,dx + \beta_{r(t)}\mathcal{H}^{N-1}(\partial^* U_t)\Big] \end{aligned}$$

By the classical isoperimetric inequality we have that

$$\mathcal{H}^{N-1}(\partial^* U_t) \geq \mathcal{H}^{N-1}(\partial B_{r(t)}(0))$$

so that we deduce

$$\begin{aligned} (6.20) \quad H(U_t, \varphi) &\geq \frac{1}{|B_{r(t)}|}\Big[-\int_{B_{r(t)}(0)} \frac{|\nabla \phi_1|^2}{\phi_1^2}\,dx + \beta_{r(t)}\mathcal{H}^{N-1}(\partial B_{r(t)}(0))\Big] \\ &= \frac{1}{|B_{r(t)}|}\Big[-\int_{B_{r(t)}(0)} \frac{|\nabla \phi_1|^2}{\phi_1^2}\,dx + \int_{\partial B_{r(t)}(0)} \frac{|\nabla \phi_1|}{\phi_1}\,d\mathcal{H}^{N-1}\Big] = \lambda_1, \end{aligned}$$

where the last equality follows by the level set representation of the first Robin eigenvalue for regular domains in the special case of the ball [5, Proposition 2.3]. Since $\lambda_1 \geq \lambda$, inequality (6.16) follows.

Step 3. Let us assume that $\varphi \neq \frac{|\nabla u|}{u}$ on a subset of positive measure of *supp*(u). By Lemma 6.4 we deduce that

$$\lambda > H(U_t, \varphi)$$

for $t \in S$ with $|S| > 0$. But for a.e. $t \in S$ we have by Step 2 that

$$\lambda > H(U_t, \varphi) \geq \lambda_1 \geq \lambda$$

which is absurd. We infer that

$$\varphi = \frac{|\nabla u|}{u} \qquad \text{a.e. on } supp(u).$$

In view of Lemma 6.2 we deduce that $\lambda_1 = \lambda$. Moreover, the inequalities appearing in (6.19) and (6.20) are equalities for $t > 0$ outside a negligible set E. In particular for $t \notin E$

$$\mathcal{H}^{N-1}(\partial^* U_t) = \mathcal{H}^{N-1}(\partial B_{r(t)}).$$

As $|U_t| = |B_{r(t)}|$, we conclude by the classical isoperimetric inequality that U_t is a ball up to a negligible set. As a consequence, since

$$supp(u) = \bigcup_{t \notin E} U_t,$$

we conclude that *supp*(u) is a ball B up to a negligible set (t can vary in a countable set of values).

Referring again to the inequalities in (6.19) which are now equalities, for $t \notin E$ such that $\beta > \beta_{r(t)}$ we get that

$$\mathcal{H}^{N-1}(\partial^* U_t \cap J_u) = 0.$$

For $t \notin E$ such that $\beta = \beta_{r(t)}$, we have that $|U_t| = |B|$, so that $U_t = B = supp(u)$ up to a negligible set: in particular we have $\partial^* U_t = \partial B$. Since by general results on the jump set of a BV function $J_u \subset \bigcup_{s \in Q} \partial^* U_s$ with Q at most countable, we deduce that

$$\mathcal{H}^{N-1}(J_u \setminus \partial B) = 0.$$

As a consequence the restriction of u to B belongs to $W^{1,2}(B)$ because the jump part of the derivative of u is supported on ∂B. Since

$$\lambda_1 = \lambda = \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{J_u} [(u^+)^2 + (u^-)^2]\, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\, dx} = \frac{\int_B |\nabla u|^2\, dx + \int_{\partial B} u^2\, d\mathcal{H}^{N-1}}{\int_B u^2\, dx},$$

where the surface integral is intended in the usual sense of traces in Sobolev spaces, we conclude that $u = \psi_1$ on B, where ψ_1 is the first eigenfunction associated to λ_1. The representation (4.8) is thus proved and the proof is complete. □

The following result is a consequence of Theorem 4.6.

Proposition 6.5. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_m(\mathbb{R}^N)$ such that $\lambda_1(\Omega_n) \to \lambda_1(B)$, where B is a ball such that $|B| = m$. Then up to translating Ω_n we have*

$$1_{\Omega_n} \to 1_B \qquad \text{strongly in } L^1(\mathbb{R}^N), \tag{6.21}$$

that is the symmetric difference between Ω_n and B has a volume which tends to zero.

Proof. Let $u_n \in SBV^{\frac{1}{2}}(\mathbb{R}^N)$ be a minimizer for the problem (4.2) defining $\lambda_1(\Omega_n)$. In view of Remark 5.4 and of Theorem 4.6, we have that up to a translation (the passage to a subsequence is not necessary in view of the structure of the minimizers)

$$u_n \to \psi_1 1_B \qquad \text{strongly in } L^2(\mathbb{R}^N),$$

where ψ_1 is the first (classical) eigenfunction for the ball B. For a.e. $0 < t < \min_{\bar{B}} \psi_1$ we have

$$1_{\{u_n > t\}} \to 1_B \qquad \text{strongly in } L^2(\mathbb{R}^N).$$

Since $\{u_n > t\}$ is contained in Ω_n up to a negligible set, and $|\Omega_n| = |B| = m$, we infer that (6.21) holds, so that the proof is concluded. □

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