Finite-Difference Approximation of Energies in Fracture Mechanics

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Abstract

We provide a variational approximation by finite-difference energies of functionals of the type

$$\mu \int_{\Omega} |\mathcal{E}u|^2 \, dx + \frac{\lambda}{2} \int_{\Omega} |\mathrm{div} u(x)|^2 \, dx + \int_{J_u} \Phi\left([u], \nu_u\right) \, d\mathcal{H}^{n-1},$$

defined for $u \in SBD(\Omega)$, which are related to variational models in fracture mechanics for linearly-elastic materials. We perform this approximation in dimension 2 via both discrete and continuous functionals. In the discrete scheme we treat also boundary value problems and we give an extension of the approximation result to dimension 3.

Mathematics Subject Classification (2000): 49J45, 74R10, 49M25.

1 Introduction

In this paper we provide a variational approximation by discrete energies of functionals of the type

$$\mu \int_{\Omega \setminus K} |\mathcal{E}u(x)|^2 \, dx + \frac{\lambda}{2} \int_{\Omega \setminus K} |\mathrm{div}\, u(x)|^2 \, dx + \int_K \Phi([u], \nu) \, d\mathcal{H}^{n-1} \tag{1.1}$$

defined for every closed hypersurface $K \subseteq \Omega$ with normal ν and $u \in \mathcal{C}^1(\Omega \setminus K; \mathbb{R}^n)$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain of $\mathbb{R}^n$. Here $\mathcal{E}u = \frac{1}{2}(\nabla u + \nabla^t u)$ denotes the symmetric part of the gradient of u, $[u]$ is the jump of u through K along ν and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. These functionals are related to variational models in fracture mechanics for linearly elastic materials in the framework of Griffith's theory of brittle fracture (see [33]). In this context u represents the displacement field of the body, with Ω as a reference configuration. The volume term in (1.1) represents the bulk energy of the body in the "solid region", where linear elasticity is supposed to hold, μ, λ being the Lamé constants of the material. The surface term is the energy necessary to produce the fracture, proportional to the crack surface K in the isotropic case and, in general, depending on the normal ν to K and on the jump $[u]$.

The weak formulation of the problem leads to functionals of the type

$$\mu \int_\Omega |\mathcal{E}u(x)|^2\,dx + \frac{\lambda}{2}\int_\Omega |\mathrm{div}\,u(x)|^2\,dx + \int_{J_u} \Phi([u],\nu_u)\,d\mathcal{H}^{n-1} \tag{1.2}$$

defined on the space $SBD(\Omega)$ of integrable functions u whose symmetrized distributional derivative Eu is a bounded Radon measure with density $\mathcal{E}u$ with respect to the Lebesgue measure and with singular part concentrated on an $(n-1)$-dimensional set J_u, on which it is possible to define a normal ν_u in a weak sense and one-sided traces.

The description of continuum models in Fracture Mechanics as variational limits of discrete systems has been the object of recent research (see [17],[19],[20], [15] and [36]). In particular, in [19] an asymptotic analysis has been performed for discrete energies of the form

$$\mathcal{H}_\varepsilon(u) = \sum_{x,y\in R_\varepsilon,\ x\neq y} \Psi_\varepsilon(u(x)-u(y), x-y), \tag{1.3}$$

where R_ε is the portion of the lattice $\varepsilon\mathbb{Z}^n$ of step size $\varepsilon > 0$ contained in Ω and $u : R_\varepsilon \to \mathbb{R}^n$ may be interpreted as the displacement of a particle parameterized by $x \in R_\varepsilon$. In this model the energy of the system is obtained by superposition of energies which take into account pairwise interactions, according to the classical theory of crystalline structures. Upon identifying u in (1.3) with the function in L^1 constant on each cell of the lattice $\varepsilon\mathbb{Z}^n$, the asymptotic behaviour of functionals $\mathcal{H}_\varepsilon$ can be studied in the framework of Γ-convergence of energies defined on L^1 (see [25],[23]). A complete theory has been developed when u is scalar-valued; in this case the proper space where the limit energies are defined is the space of SBV functions (see for instance [24]). An important model case is when $\Psi_\varepsilon(z,w) = \rho(\frac{w}{\varepsilon})\varepsilon^{n-1}f(\frac{|z|^2}{\varepsilon})$. In this case we may rewrite $\mathcal{H}_\varepsilon$ as

$$\sum_{\xi\in\mathbb{Z}^n} \rho(\xi) \sum_{\alpha\in R_\varepsilon^\xi} \varepsilon^{n-1} f\left(\varepsilon|D_\varepsilon^\xi u(\alpha)|^2\right),$$

where R_ε^ξ is a suitable portion of R_ε and $D_\varepsilon^\xi u(x)$ denotes the difference quotient $\frac{1}{\varepsilon}\left(u(x+\varepsilon\xi)-u(x)\right)$. Functionals of this type have been studied also in [22] in the framework of computer vision. In [22] and, in a general framework, in [19] it has been proved that, if $f(t) = \min\{t,1\}$ and ρ is a positive function with suitable summability and symmetry properties, then $\mathcal{H}_\varepsilon$ approximates functionals of the type

$$c\int_\Omega |\nabla u(x)|^2\,dx + \int_{J_u} \Phi([u],\nu_u)\,d\mathcal{H}^{n-1} \tag{1.4}$$

defined for $u \in SBV(\Omega)$, which are formally very similar to that in (1.2). A similar result holds by replacing $\min\{t,1\}$ by any increasing function f with $f(0) = 0$, $f'(0) = a > 0$ and $f(\infty) = b < +\infty$.

Following this approach, in order to approximate (1.2), one may think to "symmetrize" the effect of the difference quotient by considering the family of functionals

$$\sum_{\xi\in\mathbb{Z}^n}\rho(\xi)\sum_{\alpha\in R^\xi_\varepsilon}\varepsilon^{n-1}f\left(\varepsilon|\langle D^\xi_\varepsilon u(\alpha),\xi\rangle|^2\right).$$

By letting ε tend to 0, we obtain as limit a proper subclass of functionals (1.2). Indeed, the two coefficients μ and λ of the limit functionals are related by a fixed ratio. This limitation corresponds to the well-known fact that pairwise interactions produce only particular choices of the Lamé constants.

To overcome this difficulty we are forced to take into account in the model non-central interactions. The idea underlying this paper is to introduce a suitable discretization of the divergence, call it $\mathrm{div}^\xi_\varepsilon u$, that takes into account also interactions in directions orthogonal to ξ, and to consider functionals of the form

$$\sum_{\xi\in\mathbb{Z}^n}\rho(\xi)\sum_{\alpha\in R^\xi_\varepsilon}\varepsilon^{n-1}f\left(\varepsilon\left(|D^\xi_\varepsilon u(\alpha)|^2+\theta|\mathrm{div}^\xi_\varepsilon u(\alpha)|^2\right)\right),\tag{1.5}$$

with θ a strictly positive parameter (for more precise definitions see Sections 3 and 7). In Theorem 3.1 we prove that with suitable choices of f,ρ and θ we can approximate functionals of type (1.2) in dimension 2 and 3 with arbitrary μ,λ and Φ satisfying some symmetry properties due to the geometry of the lattice. Actually, the general form of the limit functional is the following

$$\int_\Omega W(\mathcal{E}u(x))\,dx+c\int_\Omega|\mathrm{div}\,u(x)|^2\,dx+\int_{J_u}\Phi([u],\nu_u)\,d\mathcal{H}^{n-1},\tag{1.6}$$

with W explicitly given; in particular we may choose $W(\mathcal{E}u(x))=\mu|\mathcal{E}u(x)|^2$ and $c=\frac{\lambda}{2}$. We underline that the energy density of the limit surface term is always anisotropic due to the simmetries of the lattices $\varepsilon\mathbb{Z}^n$. The dependence on $[u]$, ν_u arises in a natural way from the discretizations chosen and the vectorial framework of the problem. To drop the anisotropy of the limit surface energy we consider as well a continuous version of the approximating functionals (1.5) given by

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{\varepsilon}f\left(\varepsilon\left(|\langle D^\xi_\varepsilon u(x),\xi\rangle|^2+\theta|\mathrm{div}^\xi_\varepsilon u(x)|^2\right)\right)\rho(\xi)\,dx\,d\xi,$$

where in this case ρ is a symmetric convolution kernel which corresponds to a polycrystalline approach. By varying f,ρ and θ, as stated in Theorem 3.8, we obtain as limit functionals of the form

$$\mu\int_\Omega|\mathcal{E}u(x)|^2\,dx+\frac{\lambda}{2}\int_\Omega|\mathrm{div}\,u(x)|^2\,dx+\gamma\mathcal{H}^{n-1}(J_u)\tag{1.7}$$

for any choice of positive constants μ,λ and γ. This continuous model generalizes the one proposed by E. De Giorgi and studied by M. Gobbino in [31], to approximate the Mumford-Shah functional

$$c \int_\Omega |\nabla u(x)|^2\, dx + \mathcal{H}^{n-1}(J_u)$$

defined for $u \in SBV(\Omega)$.

The main technical issue of the paper is that, in the proof of both the discrete and the continuous approximation, we cannot reduce to the 1-dimensional case by an integral-geometric approach as in [19],[22],[31], due to the presence of the divergence term. For a deeper insight of the techniques used we refer to Sections 4 and 5; we just underline that the proofs of the two approximations (discrete and continuous) are strictly related.

Analogously to [19], in Section 7 we treat boundary value problems in the discrete scheme for the 2-dimensional case and a convergence result for such problems is derived (see Proposition 6.3 and Theorem 6.4).

2 Notation and Preliminaries

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$; $|\cdot|$ will be the usual euclidean norm. For $x, y \in \mathbb{R}^n$, $[x, y]$ denotes the segment between x and y. If $a, b \in \mathbb{R}$ we write $a \wedge b$ and $a \vee b$ for the minimum and maximum between a and b, respectively. If $\xi = (\xi^1, \xi^2) \in \mathbb{R}^2$, we denote by $\xi^\perp$ the vector in $\mathbb{R}^2$ orthogonal to ξ defined by $\xi^\perp := (-\xi^2, \xi^1)$.

If Ω is a bounded open subset of $\mathbb{R}^n$, $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ are the families of open and Borel subsets of Ω, respectively. If μ is a Borel measure and B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. If $B \subseteq \mathbb{R}^n$ is a Borel set, we will also use the notation $|B|$ for $\mathcal{L}^n(B)$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified. We use standard notation for Lebesgue spaces.

We recall also the notion of convergence in measure on the space $L^1(\Omega; \mathbb{R}^n)$. We say that a sequence u_n converges to u in measure if for every $\eta > 0$ we have $\lim_n |\{x \in \Omega : |u_n(x) - u(x)| > \eta\}| = 0$. The space $L^1(\Omega; \mathbb{R}^n)$, when endowed with this convergence, is metrizable, an example of metric being

$$d(u, v) := \int_\Omega \frac{|u(x) - v(x)|}{1 + |u(x) - v(x)|}\, dx$$

for $u, v \in L^1(\Omega; \mathbb{R}^n)$.

2.1 BV and BD functions

Let Ω be a bounded open set of $\mathbb{R}^n$. If $u \in L^1(\Omega;\mathbb{R}^N)$, we denote by S_u the complement of the Lebesgue set of u, i.e. $x \notin S_u$ if and only if

$$\lim_{\rho\to 0} \rho^{-n} \int_{B_\rho(x)} |u(y) - z| \, dy = 0$$

for some $z \in \mathbb{R}^N$. If z exists then it is unique and we denote it by $\bar{u}(x)$. The set S_u is Lebesgue-negligible and $\bar{u}$ is a Borel function equal to u $\mathcal{L}^n$ a.e. in Ω.

Moreover, we say that $x \in \Omega$ is a *jump point* of u, and we denote by J_u the set of all such points for u, if there exist $a, b \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and

$$\lim_{\rho\to 0} \rho^{-n} \int_{B^+_\rho(x,\nu)} |u(y) - a| \, dy = 0, \quad \lim_{\rho\to 0} \rho^{-n} \int_{B^-_\rho(x,\nu)} |u(y) - b| \, dy = 0, \tag{2.1}$$

where $B^\pm_\rho(x,\nu) := \{y \in B_\rho(x) : \ \pm\langle y - x, \nu(x)\rangle > 0\}$.

The triplet (a, b, ν), uniquely determined by (2.1) up to a permutation of (a, b) and a change of sign of ν, will be denoted by $(u^+(x), u^-(x), \nu_u(x))$. Notice that J_u is a Borel subset of S_u.

We say that u is *approximately differentiable* at a Lebesgue point x if there exists $L \in \mathbb{R}^{N\times n}$ such that

$$\lim_{\rho\to 0} \rho^{-n-1} \int_{B_\rho(x)} |u(y) - \bar{u}(x) - L(y - x)| \, dy = 0. \tag{2.2}$$

If u is approximately differentiable at a Lebesgue point x, then L, uniquely determined by (2.2), will be denoted by $\nabla u(x)$ and will be called the *approximate gradient* of u at x.

Eventually, given a Borel set $J \subset \mathbb{R}^n$, we say that J is $\mathcal{H}^{n-1}$*-rectifiable* if

$$J = N \cup \bigcup_{i\geq 1} K_i$$

where $\mathcal{H}^{n-1}(N) = 0$ and each K_i is a compact subset of a C^1 $(n-1)$ dimensional manifold. Thus, for a $\mathcal{H}^{n-1}$-rectifiable set J it is possible to define $\mathcal{H}^{n-1}$ a.e. a unitary normal vector field ν.

2.1.1 BV functions

We recall some definitions and basic results on functions with bounded variation. For a detailed study of the properties of these functions we refer to [9] (see also [26], [30]).

Definition 2.1 *Let $u \in L^1(\Omega;\mathbb{R}^N)$; we say that u is a* function with bounded variation *in Ω, and we write $u \in BV(\Omega;\mathbb{R}^N)$, if the distributional derivative Du of u is a $N \times n$ matrix-valued measure on Ω with finite total variation.*

If $u \in BV(\Omega;\mathbb{R}^N)$, then u is approximately differentiable $\mathcal{L}^n$ a.e. in Ω and J_u turns out to be $\mathcal{H}^{n-1}$-rectifiable.

Let us consider the Lebesgue decomposition of Du with respect to $\mathcal{L}^n$, i.e., $Du = D^a u + D^s u$, where $D^a u$ is the absolutely continuous part and $D^s u$ is the singular one. The density of $D^a u$ with respect to $\mathcal{L}^n$ coincides $\mathcal{L}^n$ a.e. with the approximate gradient ∇u of u. Define the *jump part* of Du, $D^j u$, to be the restriction of $D^s u$ to J_u, and the *Cantor part*, $D^c u$, to be the restriction of $D^s u$ to $\Omega \setminus J_u$, thus we have

$$Du = D^a u + D^j u + D^c u.$$

Moreover, it holds $D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u$, where if $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ $a \otimes b$ denotes the matrix whose entries are $a_i b_j$ with $1 \leq i \leq N$ and $1 \leq j \leq n$.

Definition 2.2 *Let $u \in BV(\Omega;\mathbb{R}^N)$; we say that u is a* special function with bounded variation *in Ω, and we write $u \in SBV(\Omega;\mathbb{R}^N)$, if $D^c u = 0$.*

Functionals involved in free-discontinuity problems are often not coercive in $SBV(\Omega;\mathbb{R}^N)$, then it is useful to consider the following wider class (see [24],[5]).

Definition 2.3 *Given $u \in L^1(\Omega;\mathbb{R}^N)$, we say that u is a* generalized special function with bounded variation *in Ω, and we write $u \in GSBV(\Omega;\mathbb{R}^N)$, if $g(u) \in SBV(\Omega)$ for every $g \in C^1(\mathbb{R}^N)$ such that ∇g has compact support.*

Notice that $GSBV(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) = SBV(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$. Functions $u \in GSBV(\Omega,\mathbb{R}^N)$ are approximately differentiable a.e. in Ω, and J_u turns out to be $\mathcal{H}^{n-1}$-rectifiable (see [5]).

In the space $GSBV(\Omega;\mathbb{R}^N)$ the following closure and lower semicontinuity theorem holds (see [4], [6]).

Theorem 2.4 *Let $(u_h) \subset GSBV\left(\Omega;\mathbb{R}^N\right)$ and assume that*

$$\sup_h \left(\int_\Omega |\nabla u_h|^2 \, dx + \mathcal{H}^{n-1}(J_{u_h}) \right) < +\infty.$$

If $u_h \to u$ in measure in Ω, then $u \in GSBV\left(\Omega;\mathbb{R}^N\right)$ and $\nabla u_h \to \nabla u$ weakly in $L^2\left(\Omega;\mathbb{R}^{N\times n}\right)$. Moreover

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_h \mathcal{H}^{n-1}(J_{u_h}).$$

2.1.2 BD functions

We recall some definitions and basic results on functions with bounded deformation. For a general treatment of this subject we refer to [8] (see also [12],[35]).

Definition 2.5 *Let $u \in L^1(\Omega;\mathbb{R}^n)$; we say that u is a* function with bounded deformation *in Ω, and we write $u \in BD(\Omega)$, if the symmetric part of the distributional derivative of u, $Eu := \frac{1}{2}(Du + D^t u)$, is a $n \times n$ matrix-valued measure on Ω with finite total variation.*

If $u \in BD(\Omega)$, then u is approximately differentiable a.e. in Ω and J_u turns out to be $\mathcal{H}^{n-1}$-rectifiable.

As in the BV case, we can decompose Eu as

$$Eu = E^a u + E^j u + E^c u,$$

where $E^a u$ is the absolutely continuous part of Eu with respect to $\mathcal{L}^n$ with density

$$\frac{1}{2}\left(\nabla u + \nabla^t u\right) =: \mathcal{E}u,$$

$E^j u$ is the restriction of Eu to J_u, and $E^c u$ is the restriction of $E^s u$ to $\Omega \setminus J_u$. Moreover, $E^j u$ turns out to be equal to $(u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u$, where $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$.

Definition 2.6 *Let $u \in BD(\Omega)$; we say that u is a* special function with bounded deformation *in Ω, and we write $u \in SBD(\Omega)$, if $E^c u = 0$.*

Before stating the Slicing Theorem (see [8]) we fix some notation. Let $\xi \in \mathbb{R}^n \setminus \{0\}$ and let $\Pi^\xi = \{y \in \mathbb{R}^n : \langle \xi, y\rangle = 0\}$. If $y \in \Pi^\xi$ and $B \subset \mathbb{R}^n$ we set $B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\}$. Moreover, given $u : B \to \mathbb{R}^n$, we define $u^{\xi,y} : B^\xi_y \to \mathbb{R}$ by $u^{\xi,y}(t) := \langle u\left(y + t\xi\right), \xi\rangle$.

Theorem 2.7 *Let $u \in L^1(\Omega;\mathbb{R}^n)$ and let $\{\xi_1, ..., \xi_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then the following two conditions are equivalent:*

(i) *For every $\xi = \xi_i + \xi_j$, $1 \le i, j \le n$, there holds*

$$u^{\xi,y} \in SBV(\Omega^\xi_y) \text{ for } \mathcal{H}^{n-1} \text{ a.e. } y \in \Pi^\xi,$$
$$\int_{\Pi^\xi} \left|Du^{\xi,y}\right|(\Omega^\xi_y)\, d\mathcal{H}^{n-1}(y) < +\infty;$$

(ii) $u \in SBD(\Omega)$.

Moreover, if $u \in SBD(\Omega)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ the following properties hold:

(a) $\dot{u}^{\xi,y}(t) = \langle \mathcal{E}u\left(y + t\xi\right)\xi, \xi\rangle$ *for a.e. $t \in \Omega^\xi_y$;*

(b) $J_{u^{\xi,y}} = \left(J^\xi_u\right)^\xi_y$ *for $\mathcal{H}^{n-1}$ a.e. $y \in \Pi^\xi$, where*

$$J^\xi_u := \left\{x \in J_u : \ \langle u^+(x) - u^-(x), \xi\rangle \ne 0\right\};$$

(c) $\mathcal{H}^{n-1}\left(J_u \setminus J^\xi_u\right) = 0$ *for $\mathcal{H}^{n-1}$ a.e. $\xi \in \mathbb{S}^{n-1}$.*

The following compactness result in $SBD(\Omega)$ is due to Bellettini, Coscia and Dal Maso (see [12]) and its proof is based on slicing techniques and on the characterization of $SBD(\Omega)$ provided by Theorem 2.7.

Theorem 2.8 *Let $(u_j) \subset SBD(\Omega)$ be such that*

$$\sup_j \left(\int_\Omega |\mathcal{E}u_j|^2 \, dx + \mathcal{H}^{n-1}(J_{u_j}) + \|u_j\|_{L^\infty} \right) < +\infty.$$

Then there exists a subsequence (not relabelled) (u_j) converging in $L^1_{loc}(\Omega;\mathbb{R}^n)$ to a function $u \in SBD(\Omega)$. Moreover $\mathcal{E}u_j$ weakly converge to $\mathcal{E}u$ in $L^2(\Omega;\mathbb{R}^{n^2})$ and

$$\liminf_{j\to+\infty} \mathcal{H}^{n-1}(J_{u_j}) \geq \mathcal{H}^{n-1}(J_u).$$

We state now a lower semicontinuity result in SBD that can be proved by following the same ideas and strategy of the proof of Theorem 2.8.

Theorem 2.9 *Let $u_j, u \in SBD(\Omega)$ be such that $u_j \to u$ in $L^1(\Omega;\mathbb{R}^n)$ and*

$$\sup_j \int_\Omega |\langle \mathcal{E}u_j(x)\xi, \xi\rangle|^2 \, dx + \int_{J^\xi_{u_j}} |\langle \nu_{u_j}, \xi\rangle| \, d\mathcal{H}^{n-1} < +\infty \tag{2.3}$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$. Then $\langle \mathcal{E}u_j(x)\xi, \xi\rangle \to \langle \mathcal{E}u(x)\xi, \xi\rangle$ weakly in $L^2(\Omega)$ and

$$\int_{J^\xi_u} |\langle \nu_u, \xi\rangle| \, d\mathcal{H}^{n-1} \leq \liminf_j \int_{J^\xi_{u_j}} |\langle \nu_{u_j}, \xi\rangle| \, d\mathcal{H}^{n-1}.$$

In particular, if (2.3) holds for every $\xi \in \{\xi_1, \dots, \xi_n\}$ orthogonal basis in $\mathbb{R}^n$, then $\operatorname{div} u_j \to \operatorname{div} u$ *weakly in $L^2(\Omega)$.*

Finally, we introduce the following subspace of $SBD(\Omega)$

$$SBD^2(\Omega) := \left\{ u \in SBD(\Omega) : \int_\Omega |\mathcal{E}u(x)|^2 \, dx + \mathcal{H}^1(J_u) < +\infty \right\}.$$

2.2 Γ-convergence

We recall the notion of Γ-convergence (see [25]). Let (X, d) be a metric space. A family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $F_\varepsilon : X \to [0, +\infty]$ is said to *Γ-converge* to a functional $F : X \to [0, +\infty]$ at $u \in X$, and we write $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$, if for every sequence (ε_j) of positive numbers decreasing to 0 the following two conditions hold:

(i) (*lower semicontinuity inequality*) for all sequences (u_j) converging to u in X we have $F(u) \leq \liminf_j F_{\varepsilon_j}(u_j)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u in X such that $F(u) \geq \limsup_j F_{\varepsilon_j}(u_j)$.

We say that F_ε Γ-converges to F if $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$ at all points $u \in X$ and that F is the *Γ-limit* of F_ε. If we define the *lower and upper Γ-limits* by

$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon):\ u_\varepsilon \to u\},$$

respectively, then the conditions (i) and (ii) are equivalent to $F'(u) = F''(u) = F(u)$. Note that the functions F' and F'' are lower semicontinuous.

In the sequel we will denote by $\Gamma(meas)$-$\liminf$, $\Gamma(meas)$-$\limsup$ and $\Gamma(L^1)$-$\liminf$, $\Gamma(L^1)$-$\limsup$, the lower and upper Γ-limits on the space L^1 endowed with the metric of the L^1 strong convergence and the convergence in measure, respectively.

The reason for the introduction of this notion is explained by the following fundamental theorem.

Theorem 2.10 *Let $F = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon$, and let a compact set $K \subset X$ exist such that $\inf_X F_\varepsilon = \inf_K F_\varepsilon$ for all ε. Then*

$$\exists \min_X F = \lim_{\varepsilon\to 0^+} \inf_X F_\varepsilon. \tag{2.4}$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_{\varepsilon_j}(u_j) = \lim_j \inf_X F_{\varepsilon_j}$ then its limit is a minimum point for F.

We refer to [23] for an exposition of the main properties of Γ-convergence (see also [18]).

2.3 Preliminary lemmas

In this section we state and prove some preliminary results, that will be used in the sequel.

Let $\mathcal{B} := \{\xi_1, \dots, \xi_n\}$ an orthogonal basis of $\mathbb{R}^n$. Then for any measurable function $u : \mathbb{R}^n \to \mathbb{R}^n$ and $y \in \mathbb{R}^n \setminus \{0\}$ define

$$T_y^{\varepsilon,\mathcal{B}} u(x) := u\left(\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right) \tag{2.5}$$

where $[z]_{\mathcal{B}} := \sum_{i=1}^n \left[\frac{\langle z,\xi_i\rangle}{|\xi_i|^2}\right] \xi_i$.

Notice that $T_y^{\varepsilon,\mathcal{B}} u$ is constant on each cell $\alpha + \varepsilon Q_{\mathcal{B}}$, $\alpha \in \varepsilon \bigoplus_{i=1}^n \xi_i \mathbb{Z}$, where $Q_{\mathcal{B}} := \{x \in \mathbb{R}^n : 0 < \langle x, \xi_i\rangle \le |\xi_i|^2\}$. The following result generalizes Lemma 3.36 in [14].

Lemma 2.11 *Let $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}^n;\mathbb{R}^n)$, then for any compact set K of $\mathbb{R}^n$ it holds*

$$\lim_{\varepsilon\to 0} \int_{Q_{\mathcal{B}}} \|T_y^{\varepsilon,\mathcal{B}} u_\varepsilon - u\|_{L^1(K,\mathbb{R}^n)}\, dy = 0. \tag{2.6}$$

Proof. For the sake of simplicity we assume $\mathcal{B} = \{e_1, \dots, e_n\}$. With fixed a compact set K, call I_ε the double integral in (2.6). By Fubini Theorem and the change of variable $\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon}\right]_{\mathcal{B}} \to y$ we get

$$\begin{aligned} I_\varepsilon &= \int_K \int_{(0,1)^n} \left| u_\varepsilon \left(\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon} \right]_{\mathcal{B}} \right) - u(x) \right| dy\, dx \\ &\leq \int_K \frac{1}{\varepsilon^n} \int_{x+\varepsilon(-1,1)^n} |u_\varepsilon(y) - u(x)|\, dy\, dx \\ &\leq \int_K \frac{1}{\varepsilon^n} \int_{x+\varepsilon(-1,1)^n} (|u_\varepsilon(y) - u_\varepsilon(x)| + |u_\varepsilon(x) - u(x)|)\, dy\, dx. \end{aligned}$$

The further change of variable $y \to x + \varepsilon y$ and Fubini Theorem yield

$$I_\varepsilon \leq \int_{(-1,1)^n} \int_K |u_\varepsilon(x + \varepsilon y) - u_\varepsilon(x)|\, dx\, dy + 2^n \int_K |u_\varepsilon(x) - u(x)|\, dx,$$

thus the conclusion follows by the uniform continuity of the translation operator for strongly converging families in $L^1_{loc}(\mathbb{R}^n; \mathbb{R}^n)$. $\square$

Remark 2.12 Let $C_\varepsilon \subset Q_{\mathcal{B}}$ a family of sets such that

$$\liminf_{\varepsilon \to 0^+} |C_\varepsilon| \geq c > 0. \tag{2.7}$$

Then under the hypothesis of the previous lemma, for any compact set K of $\mathbb{R}^n$ we can choose $y_\varepsilon \in C_\varepsilon$ such that $T^{\varepsilon,\mathcal{B}}_{y_\varepsilon} u_\varepsilon \to u$ in $L^1(K; \mathbb{R}^n)$. Indeed, by the Mean Value Theorem, there exist $y_\varepsilon \in C_\varepsilon$ such that

$$\begin{aligned} \|T^{\varepsilon,\mathcal{B}}_{y_\varepsilon} u_\varepsilon - u\|_{L^1(K,\mathbb{R}^n)} |C_\varepsilon| &\leq \int_{C_\varepsilon} \|T^{\varepsilon,\mathcal{B}}_{y} u_\varepsilon - u\|_{L^1(K,\mathbb{R}^n)}\, dy \\ &\leq \int_{Q_{\mathcal{B}}} \|T^{\varepsilon,\mathcal{B}}_{y} u_\varepsilon - u\|_{L^1(K,\mathbb{R}^n)}\, dy. \end{aligned}$$

Then the conclusion easily follows from (2.6) and (2.7). This property will be used in the proof of Propositions 4.4 and 5.1.

In the sequel for $n = 2$, $\xi \in \mathbb{R}^2 \setminus \{0\}$ and $\mathcal{B} = \{\xi, \xi^\perp\}$, we will denote the operators $T^{\varepsilon,\mathcal{B}}_y$ and $[\,\cdot\,]_{\mathcal{B}}$ by $T^{\varepsilon,\xi}_y$ and $[\,\cdot\,]_\xi$, respectively.

Lemma 2.13 *Let J be a $\mathcal{H}^{N-1}$-rectifiable set and define*

$$J^\xi_\varepsilon := \{x \in \mathbb{R}^n \,:\, x = y + t\xi \text{ with } t \in (-\varepsilon, \varepsilon) \text{ and } y \in J\} \tag{2.8}$$

for $\xi \in \mathbb{R}^n$ and

$$J^{\xi_1,\dots,\xi_r}_\varepsilon := \bigcup_{i=1}^r J^{\xi_i}_\varepsilon \tag{2.9}$$

for $\xi_1, \ldots, \xi_r \in \mathbb{R}^n$, r being a positive integer. Then, if $\mathcal{H}^{N-1}(J) < +\infty$

$$\limsup_{\varepsilon \to 0} \frac{\mathcal{L}^n \left(J_\varepsilon^{\xi_1,..,\xi_r} \right)}{\varepsilon} \leq 2 \int_J \sup_i |\langle \nu, \xi_i \rangle| \, d\mathcal{H}^{n-1}, \tag{2.10}$$

where $\nu(x)$ is the unitary normal vector to J at x.

Proof. First note that by Fubini Theorem and the Generalized Coarea Formula (see [9])

$$\mathcal{L}^n \left(J_\varepsilon^\xi \right) \leq 2\varepsilon \int_{\Pi^\xi} \# \left(J_\varepsilon^\xi \right)_y^\xi \, d\mathcal{H}^{n-1}(y) = 2\varepsilon \int_J |\langle \nu, \xi \rangle| \, d\mathcal{H}^{n-1}, \tag{2.11}$$

hence

$$\mathcal{L}^n \left(J_\varepsilon^{\xi_1,..,\xi_r} \right) \leq 2\varepsilon \int_J \sum_{i=1}^r |\langle \nu, \xi_i \rangle| \, d\mathcal{H}^{n-1} \leq 2r\varepsilon \sup_i |\xi_i| \mathcal{H}^{N-1}(J). \tag{2.12}$$

By the very definition of rectifiability there exist countably many compact subsets K_i of C^1 graphs such that

$$\mathcal{H}^{N-1} \left(J \setminus \bigcup_{i \geq 1} K_i \right) = 0,$$

and $\mathcal{H}^{N-1}(K_i \cap K_j) = 0$ for $i \neq j$. Thus, by (2.12) for any $M \in \mathbb{N}$ we have

$$\frac{\mathcal{L}^n \left(J_\varepsilon^{\xi_1,..,\xi_r} \right)}{\varepsilon} \leq \sum_{1 \leq i \leq M} \frac{\mathcal{L}^n \left((K_i)_\varepsilon^{\xi_1,..,\xi_r} \right)}{\varepsilon} + 2r \sup_i |\xi_i| \mathcal{H}^{N-1} \left(J \setminus \bigcup_{1 \leq i \leq M} K_i \right),$$

hence, first letting $\varepsilon \to 0$ and then $M \to +\infty$ it follows

$$\limsup_{\varepsilon \to 0} \frac{\mathcal{L}^n \left(J_\varepsilon^{\xi_1,..,\xi_r} \right)}{\varepsilon} \leq \sum_{i \geq 1} \limsup_{\varepsilon \to 0} \frac{\mathcal{L}^n \left((K_i)_\varepsilon^{\xi_1,..,\xi_r} \right)}{\varepsilon}.$$

Thus, it suffices to prove (2.10) for J compact subset of a C^1 graph. Up to an outer approximation with open sets we may assume J open. Furtherly, splitting J into its connected components, we can reduce ourselves to prove the inequality for J connected. For such a J (2.10) follows by an easy computation. $\square$

Lemma 2.14 *Let $\lambda : \mathcal{A}(\Omega) \to [0, +\infty)$ be a superadditive function on disjoint open sets, let μ be a positive measure on Ω and let $\psi_h : \Omega \to [0, +\infty]$ be a countable family of Borel functions such that $\lambda(A) \geq \int_A \psi_h \, d\mu$ for every $A \in \mathcal{A}(\Omega)$.*

Set $\psi = \sup_{h \in \mathbb{N}} \psi_h$, then

$$\lambda(A) \geq \int_A \psi \, d\mu$$

for every $A \in \mathcal{A}(\Omega)$.

The proof of this Lemma can be found in [14].

3 The main result

In this section all the results are set in $\mathbb{R}^2$. A generalization to higher dimension will be given in Section 6.

We introduce first a discretization of the divergence. Fix $\xi, \zeta \in \mathbb{R}^2 \setminus \{0\}$; for $\varepsilon > 0$ and for any $u : \mathbb{R}^2 \to \mathbb{R}^2$ define

$$\begin{aligned}
D_\varepsilon^\xi u(x) &:= \langle u(x+\varepsilon\xi) - u(x), \xi \rangle, \\
\mathrm{div}_\varepsilon^{\xi,\zeta} u(x) &:= D_\varepsilon^\xi u(x) + D_\varepsilon^\zeta u(x), \\
|D_{\varepsilon,\xi} u(x)|^2 &:= |D_\varepsilon^\xi u(x)|^2 + |D_\varepsilon^{-\xi} u(x)|^2, \\
|\mathrm{Div}_{\varepsilon,\xi} u(x)|^2 &:= |\mathrm{div}_\varepsilon^{\xi,\xi^\perp} u(x)|^2 + |\mathrm{div}_\varepsilon^{\xi,-\xi^\perp} u(x)|^2 \\
&\quad + |\mathrm{div}_\varepsilon^{-\xi,\xi^\perp} u(x)|^2 + |\mathrm{div}_\varepsilon^{-\xi,-\xi^\perp} u(x)|^2.
\end{aligned} \tag{3.1}$$

Starting from this definition we will provide discrete and continuous approximation results for functionals of type (1.2) and (1.6). We underline that this is only one possible definition of discretized divergence that seems to agree with mechanical models of neighbouring atomic interactions.

We can give also the following alternative definition

$$\begin{aligned}
D_\varepsilon^\xi u(x) &:= \langle u(x+\varepsilon\xi) - u(x-\varepsilon\xi), \xi \rangle, \\
|D_{\varepsilon,\xi} u(x)|^2 &:= \frac{1}{2} |D_\varepsilon^\xi u(x)|^2 \\
|\mathrm{Div}_{\varepsilon,\xi} u(x)|^2 &:= |D_\varepsilon^\xi u(x) + D_\varepsilon^{\xi^\perp} u(x)|^2.
\end{aligned} \tag{3.2}$$

This second definition can be motivated by the fact that from a numerical point of view it gives a more accurate approximation of the divergence as $\varepsilon \to 0$, although the centered differences usually have other drawbacks.

For the sake of simplicity, in the proofs of Sections 4 and 5, we will assume that the "finite-difference terms" involved in the approximating functionals are defined by (3.1). The arguments we will use can be easily adapted when considering definition (3.2).

3.1 Discrete approximation result

Let Ω be a bounded open set of $\mathbb{R}^2$ and, for $\varepsilon > 0$, define

$$\mathcal{A}_\varepsilon(\Omega) := \{u : \Omega \to \mathbb{R}^2 \ : \ u \equiv \mathit{const} \text{ on } (\alpha + [0,\varepsilon)^2) \cap \Omega \text{ for any } \alpha \in \varepsilon\mathbb{Z}^2\}.$$

Let $f : [0, +\infty) \to [0, +\infty)$ be an increasing function, such that $a, b > 0$ exist with

$$a := \lim_{t \to 0+} \frac{f(t)}{t}, \quad b := \lim_{t \to +\infty} f(t) \tag{3.3}$$

and $f(t) \le (at) \wedge b$ for any $t \ge 0$. For $u \in \mathcal{A}_\varepsilon(\Omega)$ and $\xi \in \mathbb{Z}^2$, set

$$\mathcal{F}_\varepsilon^{d,\xi}(u) := \sum_{\alpha \in R_\varepsilon^\xi} \varepsilon f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi}u(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(\alpha)|^2\right)\right), \tag{3.4}$$

where θ is a strictly positive parameter and

$$R_\varepsilon^\xi := \{\alpha \in \varepsilon\mathbb{Z}^2 : \ [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\xi^\perp, \alpha + \varepsilon\xi^\perp] \subset \Omega\}.$$

Then consider the functional $F_\varepsilon^d : L^1(\Omega;\mathbb{R}^2) \to [0,+\infty]$ defined as

$$F_\varepsilon^d(u) = \begin{cases} \sum\limits_{\xi \in \mathbb{Z}^2} \rho(\xi)\mathcal{F}_\varepsilon^{d,\xi}(u) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{3.5}$$

where $\rho : \mathbb{Z}^2 \to [0,+\infty)$ is such that $\sum\limits_{\xi \in \mathbb{Z}^2} |\xi|^4 \rho(\xi) < +\infty$ and $\rho(\xi) > 0$ for $\xi = e_1, e_1 + e_2$.

Theorem 3.1 *Let Ω be a convex bounded open set of $\mathbb{R}^2$. Then F_ε^d Γ-converges on $L^\infty(\Omega;\mathbb{R}^2)$ to the functional $F^d : L^\infty(\Omega;\mathbb{R}^2) \to [0,+\infty]$ given by*

$$F^d(u) = \begin{cases} \sum\limits_{\xi \in \mathbb{Z}^2} \rho(\xi)\mathcal{F}^\xi(u) & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{3.6}$$

with respect to both the $L^1(\Omega;\mathbb{R}^2)$-convergence and the convergence in measure, where

$$\begin{aligned} \mathcal{F}^\xi(u) := \ & 2a\int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2\,dx + 4a\theta|\xi|^4 \int_\Omega |\mathrm{div}\, u(x)|^2\,dx \\ & +2b\int_{J_u} \Phi^\xi(u^+ - u^-, \nu_u)\,d\mathcal{H}^1, \end{aligned}$$

with the function $\Phi^\xi : \mathbb{R}^2 \to [0,+\infty)$ defined by

$$\Phi^\xi(z,\nu) := \psi^\xi(z,\nu) \vee \psi^{\xi^\perp}(z,\nu),$$

where for $\eta \in \mathbb{R}^2$

$$\psi^\eta(z,\nu) := \begin{cases} |\langle \nu, \eta\rangle| & \text{if } \langle z, \eta\rangle \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The proof of the theorem above will be given in Section 4 as a consequence of some propositions, which deal with lower and upper Γ-limits separately.

Remark 3.2 Notice that the surface term can be written explicitly as

$$\int_{J_u} \Phi^\xi(u^+ - u^-, \nu_u)\, d\mathcal{H}^1 = \int_{J_u^\xi \setminus J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle|\, d\mathcal{H}^1 \tag{3.7}$$

$$+ \int_{J_u^{\xi^\perp} \setminus J_u^\xi} |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^1 + \int_{J_u^\xi \cap J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^1.$$

Remark 3.3 We point out that the assumption Ω convex will be used only in the proof of the Γ-$\limsup$ inequality. This assumption can be weakened (see Remark 4.5).

Remark 3.4 Notice that the domain of F^d is $L^\infty(\Omega;\mathbb{R}^2) \cap SBD^2(\Omega)$. Indeed, taking into account the assumption on ρ, an easy computation shows that

$$F^d(u) \geq \sum_{\xi = e_1, e_1+e_2} \rho(\xi)\mathcal{F}^\xi(u) \geq c\left(\int_\Omega |\mathcal{E}u(x)|^2\, dx + \mathcal{H}^1(J_u)\right).$$

Remark 3.5 Note that, by a suitable choice of the discrete function ρ, the limit functional is isotropic in the volume term, i.e.,

$$F^d(u) = \mu_1 \int_\Omega |\mathcal{E}u(x)|^2\, dx + \lambda_1 \int_\Omega |\mathrm{div}\, u(x)|^2\, dx + \int_{J_u} \Phi(u^+ - u^-, \nu_u)\, d\mathcal{H}^1. \tag{3.8}$$

Choose, for example, $\rho(e_1) = \rho(e_2) = 2\rho(e_2 \pm e_1) \neq 0$ and $\rho(\xi) = 0$ elsewhere. Moreover, for suitable choices of f and θ, it is possible to approximate functionals of type (3.8) for any strictly positive μ_1, λ_1.

By dropping the divergence term in (3.4) (i.e. $\theta = 0$), one can consider the functional $G_\varepsilon^d : L^1(\Omega;\mathbb{R}^2) \to [0,+\infty]$ defined as

$$G_\varepsilon^d(u) = \begin{cases} \sum\limits_{\xi\in\mathbb{Z}^2} \rho(\xi) \sum\limits_{\alpha \in \widetilde{R}_\varepsilon^\xi} \varepsilon f\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi}u(\alpha)|^2\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.9}$$

where $\widetilde{R}_\varepsilon^\xi := \{\alpha \in \varepsilon\mathbb{Z}^2 : [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \subset \Omega\}$ and ρ is as above and satisfies also the condition $\rho(e_2) \neq 0$.

Theorem 3.6 *Let Ω be a convex bounded open set of $\mathbb{R}^2$. Then G_ε^d Γ-converges on $L^\infty(\Omega;\mathbb{R}^2)$ to the functional $G^d : L^\infty(\Omega;\mathbb{R}^2) \to [0,+\infty]$ given by*

$$G^d(u) = \begin{cases} \sum\limits_{\xi\in\mathbb{Z}^2} \rho(\xi) G^\xi(u) & \textit{if } u \in SBD(\Omega) \\ +\infty & \textit{otherwise} \end{cases} \tag{3.10}$$

with respect to both the $L^1(\Omega;\mathbb{R}^2)$-convergence and the convergence in measure, where

$$G^\xi(u) := 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2\, dx + 2b \int_{J_u^\xi} |\langle \nu_u, \xi\rangle|\, d\mathcal{H}^1.$$

The proof of this theorem can be recovered from the proof of Theorem 3.1, up to slight modifications. We only remark that the further hypothesis $\rho(e_2) \neq 0$ is needed in order to have good coercivity properties of the family G^d_ε.

Remark 3.7 Notice that, although the definition of G^d_ε corresponds in some sense to taking $\theta = 0$ in (3.5), its Γ-limit G^d differs from F^d for $\theta = 0$ in the surface term.

3.2 Continuous approximation result

Let Ω be a bounded open set of $\mathbb{R}^2$ and let $f : [0,+\infty) \to [0,+\infty)$ be as in the previous section.

For $\varepsilon > 0$, define $F^c_\varepsilon : L^1(\Omega;\mathbb{R}^2) \to [0,+\infty]$ as

$$F^c_\varepsilon(u) := \int_{\mathbb{R}^2} \rho(\xi)\mathcal{F}^{c,\xi}_\varepsilon(u)\, d\xi,$$

where

$$\mathcal{F}^{c,\xi}_\varepsilon(u) := \frac{1}{\varepsilon}\int_{\Omega^\xi_\varepsilon} f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi}u(x)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(x)|^2\right)\right) dx \tag{3.11}$$

with

$$\Omega^\xi_\varepsilon := \{x \in \mathbb{R}^2 : [x-\varepsilon\xi, x+\varepsilon\xi] \cup [x-\varepsilon\xi^\perp, x+\varepsilon\xi^\perp] \subset \Omega\},$$

$\theta > 0$ and $\rho(\xi) = \psi(|\xi|)$ where $\psi : [0,+\infty) \to [0,+\infty)$ is such that for some $M > 0$ $\operatorname{ess\,inf}_{|t|\le M} \psi(t) > 0$ and $\int_0^{+\infty} t^5\psi(t)\, dt < +\infty$.

Theorem 3.8 *F^c_ε Γ-converges on $L^\infty(\Omega;\mathbb{R}^2)$ with respect to the $L^1(\Omega;\mathbb{R}^2)$-convergence to the functional $F^c : L^\infty(\Omega;\mathbb{R}^2) \to [0,+\infty]$ given by*

$$F^c(u) := \begin{cases} \mu\int_\Omega |\mathcal{E}u(x)|^2\, dx + \lambda\int_\Omega |\operatorname{div} u(x)|^2\, dx \; + \; \gamma\mathcal{H}^1(J_u) & \\ & \textit{if } u \in SBD(\Omega) \\ +\infty & \textit{otherwise,} \end{cases}$$

where

$$\mu := a\int_{\mathbb{R}^2} \rho(y)\left(|y|^4 - 4y_1^2y_2^2\right)\, dy,$$
$$\lambda := a\int_{\mathbb{R}^2} \rho(y)\left(4\theta|y|^4 + 2y_1^2y_2^2\right)\, dy,$$
$$\gamma := 2b\int_{\mathbb{R}^2} \rho(y)\left(|y_1| \vee |y_2|\right)\, dy.$$

Moreover, F^c_ε converges to F^c pointwise on $L^\infty(\Omega;\mathbb{R}^2)$.

The proof of the theorem above will be consequence of propositions in Sections 4 and 5.

Remark 3.9 Notice that $\mu = a \int_{\mathbb{R}^2} \rho(y) \left(y_1^2 - y_2^2\right)^2 \, dy$, so that μ, λ and γ are all positive. Moreover, the summability assumption on ψ easily yields the finiteness of such constants.

Remark 3.10 We underline that for any positive coefficients μ, λ and γ, we can choose f, ρ and θ such that the limit functional has the form

$$\mu \int_\Omega |\mathcal{E}u(x)|^2 \, dx + \lambda \int_\Omega |\operatorname{div} u(x)|^2 \, dx + \gamma \mathcal{H}^1(J_u).$$

Analogously to the discrete case, we may drop in (3.11) the divergence term and consider the sequence of functionals $G_\varepsilon^c : L^1(\Omega;\mathbb{R}^2) \to [0,+\infty]$ defined by

$$G_\varepsilon^c(u) := \int_{\mathbb{R}^2} \rho(\xi) \frac{1}{\varepsilon} \int_{\widetilde{\Omega}_\varepsilon^\xi} f\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi} u(x)|^2\right) \, dx \, d\xi$$

with $\widetilde{\Omega}_\varepsilon^\xi := \{x \in \mathbb{R}^2 : [x - \varepsilon\xi, x + \varepsilon\xi] \subset \Omega\}$ and ρ as above. By applying the same slicing techniques of [31] it can be proved the following result.

Theorem 3.11 *G_ε^c Γ-converges on $L^\infty(\Omega;\mathbb{R}^2)$ with respect to the $L^1(\Omega;\mathbb{R}^2)$-convergence to the functional $G^c : L^\infty(\Omega;\mathbb{R}^2) \to [0,+\infty]$ given by*

$$G^c(u) := \begin{cases} \mu' \int_\Omega |\mathcal{E}u(x)|^2 \, dx + \lambda' \int_\Omega |\operatorname{div} u(x)|^2 \, dx \; + \gamma' \mathcal{H}^1(J_u) & \\ & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \mu' &:= a \int_{\mathbb{R}^2} \rho(y) \left(|y|^4 - 4y_1^2 y_2^2\right) \, dy, \\ \lambda' &:= 2a \int_{\mathbb{R}^2} \rho(y) y_1^2 y_2^2 \, dy, \\ \gamma' &:= 2b \int_{\mathbb{R}^2} \rho(y) |y_1| \, dy. \end{aligned}$$

Remark 3.12 As in the discrete case, the Γ-limit G^c does not correspond to F^c for $\theta = 0$.

Remark 3.13 The restriction to $L^\infty(\Omega;\mathbb{R}^2)$ in Theorems 3.1 and 3.8 is technical in order to characterize the Γ-limit. For a function u in $L^1(\Omega;\mathbb{R}^2) \setminus L^\infty(\Omega;\mathbb{R}^2)$, by following the procedure of the proof of Proposition 4.1 below, one can deduce from the finiteness of the Γ-limits that the one dimensional sections of u belong to $SBV(\Omega^\xi_y)$. Anyway, since condition (i) of Theorem 2.7 is not in general satisfied, one cannot conclude that $u \in SBD(\Omega)$. On the other hand this condition is satisfied if $u \in BD(\Omega)$, so that Theorems 3.1, 3.6, 3.8, 3.11 still hold if we replace $L^\infty(\Omega;\mathbb{R}^2)$ by $BD(\Omega)$.

4 The discrete case

In this section we will prove Theorem 3.1. In the sequel we need to "localize" the functionals $\mathcal{F}^{d,\xi}_\varepsilon$ as

$$\mathcal{F}^{d,\xi}_\varepsilon(u,A) := \sum_{\alpha\in R^\xi_\varepsilon(A)} \varepsilon f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi}u(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(\alpha)|^2\right)\right),$$

for any $A \in \mathcal{A}(\Omega)$ and $u \in \mathcal{A}_\varepsilon(\Omega)$, where

$$R^\xi_\varepsilon(A) := \{\alpha \in \varepsilon\mathbb{Z}^2 : \ [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\xi^\perp, \alpha + \varepsilon\xi^\perp] \subset A\}.$$

Proposition 4.1 *For any $u \in L^\infty(\Omega;\mathbb{R}^2)$,*

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} F^d_\varepsilon(u) \geq F^d(u).$$

Proof. *Step* 1 Let us first prove the inequality in the case $f(t) = (at) \wedge b$. Let $\varepsilon_j \to 0$, $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ and $u \in L^\infty(\Omega;\mathbb{R}^2)$ be such that $u_j \to u$ in measure. We can suppose that $\liminf_j F^d_{\varepsilon_j}(u_j) = \lim_j F^d_{\varepsilon_j}(u_j) < +\infty$. In particular, for any $\xi \in \mathbb{Z}^2$ such that $\rho(\xi) \neq 0$, $\liminf_j \mathcal{F}^{d,\xi}_{\varepsilon_j}(u_j) < +\infty$. Using this estimate for $\xi \in \{e_1, e_1 + e_2\}$, we will deduce that $u \in SBD(\Omega)$ and we will obtain the required inequality by proving that, for any $\xi \in \mathbb{Z}^2$ such that $\rho(\xi) \neq 0$,

$$\liminf_j \mathcal{F}^{d,\xi}_{\varepsilon_j}(u_j) \geq \mathcal{F}^{d,\xi}(u). \tag{4.1}$$

To this aim, as in Theorem 4.1 of [19], we will proceed by splitting the lattice $\mathbb{Z}^2$ into similar sub-lattices and reducing ourselves to study the limit of functionals defined on one of these sub-lattices. Indeed, fixed $\xi \in \mathbb{Z}^2$ such that $\rho(\xi) \neq 0$, we split $\mathbb{Z}^2$ into an union of disjoint copies of $|\xi|\mathbb{Z}^2$ as

$$\mathbb{Z}^2 = \bigcup_{i=1}^{|\xi|^2}(z_i + \mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp),$$

where

$$\{z_i : i = 1,\dots,|\xi|^2\} := \{\alpha \in \mathbb{Z}^2 \,:\, 0 \leq \langle\alpha,\xi\rangle < |\xi|,\, 0 \leq \langle\alpha,\xi^\perp\rangle < |\xi|\}.$$

Then, for any $A \in \mathcal{A}(\Omega)$, we write

$$\mathcal{F}^{d,\xi}_{\varepsilon_j}(u_j, A) = \sum_{i=1}^{|\xi|^2} \mathcal{F}^{d,\xi,i}_{\varepsilon_j}(u_j, A)$$

where

$$\mathcal{F}^{d,\xi,i}_{\varepsilon_j}(u_j,A) := \sum_{R^{\xi,i}_{\varepsilon_j}(A)} \varepsilon_j f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u_j(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon_j,\xi}u_j(\alpha)|^2\right)\right)$$

with $R^{\xi,i}_{\varepsilon_j}(A) := R^{\xi}_{\varepsilon_j}(A) \cap \varepsilon_j(z_i + \mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp)$. We split as well the lattice $\mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp$ into an union of disjoint sub-lattices as

$$\mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp = Z^\xi \cup (Z^\xi + \xi) \cup (Z^\xi + \xi^\perp) \cup (Z^\xi + (\xi + \xi^\perp))$$

where $Z^\xi := 2\mathbb{Z}\xi \oplus 2\mathbb{Z}\xi^\perp$. We confine now our attention to the sequence

$$\mathcal{F}_j(A) := \sum_{\alpha\in Z_j(A)} \varepsilon_j f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u_j(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon_j,\xi}u_j(\alpha)|^2\right)\right)$$

where $Z_j(A) := R^{\xi}_{\varepsilon_j}(A) \cap \varepsilon_j Z^\xi$. Set

$$I_j := \left\{\alpha \in R^{\xi}_{\varepsilon_j} : |D_{\varepsilon_j,\xi}u_j(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon_j,\xi}u_j(\alpha)|^2 > \frac{b}{a}\varepsilon_j\right\}$$

and let (v_j) be the sequence in $SBV(\Omega;\mathbb{R}^2)$, whose components are piecewise affine, uniquely determined by

$$\langle v_j(x),\xi\rangle := \begin{cases} \langle u_j(\alpha - \varepsilon_j\xi),\xi\rangle & x \in (\alpha + \varepsilon_j Q_\xi)\cap\Omega \\ & \alpha \in \varepsilon_j Z^\xi \cap I_j \\ \langle u_j(\alpha),\xi\rangle + \frac{1}{\varepsilon_j|\xi|^2} D^{\xi}_{\varepsilon_j}u_j(\alpha)\langle x-\alpha,\xi\rangle & x \in (\alpha + \varepsilon_j Q_{\xi,+})\cap\Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j \\ \langle u_j(\alpha),\xi\rangle + \frac{1}{\varepsilon_j|\xi|^2} D^{-\xi}_{\varepsilon_j}u_j(\alpha)\langle x-\alpha,\xi\rangle & x \in (\alpha + \varepsilon_j Q_{\xi,-})\cap\Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j \end{cases}$$

$$\langle v_j(x),\xi^\perp\rangle := \begin{cases} \langle u_j(\alpha - \varepsilon_j\xi^\perp),\xi^\perp\rangle & \\ & x \in (\alpha + \varepsilon_j Q_{\xi^\perp})\cap\Omega \\ & \alpha \in \varepsilon_j Z^\xi \cap I_j \\ \langle u_j(\alpha),\xi^\perp\rangle + \frac{1}{\varepsilon_j|\xi|^2} D^{\xi^\perp}_{\varepsilon_j}u_j(\alpha)\langle x-\alpha,\xi^\perp\rangle & \\ & x \in (\alpha + \varepsilon_j Q_{\xi^\perp,+})\cap\Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j \\ \langle u_j(\alpha),\xi^\perp\rangle + \frac{1}{\varepsilon_j|\xi|^2} D^{-\xi^\perp}_{\varepsilon_j}u_j(\alpha)\langle x-\alpha,\xi^\perp\rangle & \\ & x \in (\alpha + \varepsilon_j Q_{\xi^\perp,-})\cap\Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j, \end{cases}$$

where

$$Q_\xi := \{x \in \mathbb{R}^2 \,:\, |\langle x, \xi\rangle| \leq |\xi|^2,\, |\langle x, \xi^\perp\rangle| \leq |\xi|^2\}$$
$$Q_{\xi,\pm} := \{x \in Q_\xi \,:\, \pm\langle x, \xi\rangle \geq 0\}.$$

In order to clarify this construction, we note that, in the case $\xi = e_1$, $v_j = (v_j^1, v_j^2)$ is the sequence whose component v_j^i is piecewise affine along the direction e_i and piecewise constant along the orthogonal direction, for $i = 1, 2$.
It is easy to check that v_j still converges to u in measure. Let us fix $\eta > 0$ and consider $A_\eta := \{x \in A \,:\, \mathrm{dist}\,(x, \mathbb{R}^2 \setminus A) > \eta\}$. Note that, by construction, for j large we have

$$\begin{aligned}\sum_{\alpha \in Z_j(A)\setminus I_j} & a\left(|D_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u_j(\alpha)|^2\right) \\ & \geq \frac{a}{2|\xi|^2}\int_{A_\eta} |\langle \mathcal{E} v_j(x)\xi, \xi\rangle|^2\, dx + a\theta|\xi|^2 \int_{A_\eta} |\mathrm{div}\, v_j(x)|^2\, dx\end{aligned}$$

and

$$\begin{aligned}& b\varepsilon_j \#\{Z_j(A) \cap I_j\} \\ & \quad \geq \frac{b}{2|\xi|^2}\max\left\{\int_{J^\xi_{v_j}\cap A_\eta} |\langle \nu_{v_j}(y), \xi\rangle|\, d\mathcal{H}^1(y), \int_{J^{\xi^\perp}_{v_j}\cap A_\eta} |\langle \nu_{v_j}(y), \xi^\perp\rangle|\, d\mathcal{H}^1(y)\right\}\end{aligned}$$

Then, for j large and for any fixed $\delta \in [0,1]$,

$$\begin{aligned}\mathcal{F}_j(A) \quad \geq \quad & \sum_{\alpha \in Z_j(A)\setminus I_j} a\left(|D_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u_j(\alpha)|^2\right) \\ & + b\varepsilon_j \#\{Z_j(A) \cap I_j\} \\ \geq \quad & \frac{a}{2|\xi|^2}\int_{A_\eta} |\langle \mathcal{E} v_j(x)\xi, \xi\rangle|^2\, dx + a\theta|\xi|^2 \int_{A_\eta} |\mathrm{div}\, v_j(x)|^2\, dx \\ & + \frac{b}{2|\xi|^2}\delta \int_{J^\xi_{v_j}\cap A_\eta} |\langle \nu_{v_j}(y), \xi\rangle|\, d\mathcal{H}^1(y) \\ & + \frac{b}{2|\xi|^2}(1-\delta)\int_{J^{\xi^\perp}_{v_j}\cap A_\eta} |\langle \nu_{v_j}(y), \xi^\perp\rangle|\, d\mathcal{H}^1(y). \end{aligned} \tag{4.2}$$

In particular by applying a slicing argument and taking into account the notation used in Theorem 2.7, by Fatou Lemma, we get

$$\begin{aligned} +\infty \quad > \quad & \liminf_j \mathcal{F}_j(A) \\ \geq \quad & \frac{1}{2|\xi|^2}\int_{\Pi^\xi} \liminf_j \left(a\int_{(A_\eta)^\xi_y} |\dot{v}_j^{\xi,y}|^2\, dt + b\#\left(J\left(v_j^{\xi,y}\right)\right)\right) d\mathcal{H}^1(y). \end{aligned} \tag{4.3}$$

Note that, even if $\rho(\xi^\perp) = 0$, taking into account also the divergence term and the second surface term in (4.2), we can obtain an analogue of the inequality (4.3) for $\xi^\perp$. By the closure and lower semicontinuity Theorem 2.4 we deduce that $u^{\zeta,y} \in SBV\big((A_\eta)^\zeta_y\big)$, and since $u \in L^\infty(\Omega;\mathbb{R}^2)$ we get

$$c \geq \int_{\Pi^\xi} |Du^{\zeta,y}|\big((A_\eta)^\zeta_y\big)\, d\mathcal{H}^1(y) \tag{4.4}$$

for $\zeta = \xi, \xi^\perp$. Recall that by assumption $\rho(e_1), \rho(e_1+e_2) \neq 0$, thus (4.4) holds in particular for $\zeta = e_1, e_2, e_1+e_2$. Then by Theorem 2.7, we get that $u \in SBD(A_\eta)$ for any $\eta > 0$. Moreover, since the estimate in (4.4) is uniform with respect to η, we conclude that $u \in SBD(A)$.

Going back to (4.2), by applying Theorem 2.9 and then letting $\eta \to 0$, we get

$$\begin{aligned}\liminf_j \mathcal{F}_j(A) \geq\ & \frac{a}{2|\xi|^2}\int_A |\langle \mathcal{E}u(x)\xi,\xi\rangle|^2\,dx + a\theta|\xi|^2\int_A |\operatorname{div} u(x)|^2\,dx \\ & + \frac{b}{2|\xi|^2}\left(\delta\int_{J_u^\xi\cap A}|\langle \nu_u,\xi\rangle|\,d\mathcal{H}^1 + (1-\delta)\int_{J_u^{\xi^\perp}\cap A}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\right),\end{aligned}$$

for any $\delta \in [0,1]$.

Note that, using the inequality above with $A = \Omega$, $\xi = e_1, e_1+e_2$, it can be easily checked that $\mathcal{E}u \in L^2(\Omega;\mathbb{R}^{2\times 2})$ and $\mathcal{H}^1(J_u) < +\infty$. Then, by Lemma 2.14 applied with

$$\lambda(A) = \liminf_j \mathcal{F}_j(A),$$

$$\mu = \frac{a}{2|\xi|^2}\mathcal{L}^2 \llcorner \Omega + \frac{b}{2|\xi|^2}\mathcal{H}^1 \llcorner J_u,$$

$$\psi_h(x) = \begin{cases} \big(|\langle \mathcal{E}u(x)\xi,\xi\rangle|^2 + \theta|\xi|^2|\operatorname{div} u(x)|^2\big) & \text{on } \Omega\setminus J_u \\ \delta_h|\langle\nu_u,\xi\rangle| & \text{on } J_u^\xi\setminus J_u^{\xi^\perp} \\ (1-\delta_h)|\langle\nu_u,\xi^\perp\rangle| & \text{on } J_u^{\xi^\perp}\setminus J_u^\xi \\ \delta_h|\langle\nu_u,\xi\rangle| + (1-\delta_h)|\langle\nu_u,\xi^\perp\rangle| & \text{on } J_u^\xi \cap J_u^{\xi^\perp}, \end{cases}$$

with $\delta_h \in \mathbb{Q}\cap[0,1]$, we get

$$\begin{aligned}\liminf_j \mathcal{F}_j(\Omega) \geq\ & \frac{a}{2|\xi|^2}\int_\Omega |\langle \mathcal{E}u(x)\xi,\xi\rangle|^2\,dx + a\theta|\xi|^2\int_\Omega |\operatorname{div} u(x)|^2\,dx \\ & + \frac{b}{2|\xi|^2}\Big(\int_{J_u^\xi\setminus J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1 + \int_{J_u^{\xi^\perp}\setminus J_u^\xi}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1 \\ & + \int_{J_u^\xi\cap J_u^{\xi^\perp}} |\langle\nu_u,\xi\rangle| \vee |\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\Big).\end{aligned}$$

Finally, since the argument above is not affected by the choice of the sub-lattices in which $\mathbb{Z}^2$ has been split with respect to ξ, we obtain (4.1). The thesis follows by summing over $\xi \in \mathbb{Z}^2$.

Step 2 If f is any increasing positive function satisfying (3.3), we can find two sequences of positive numbers (a_i) and (b_i) such that $\sup_i a_i = a$, $\sup_i b_i = b$ and $f(t) \geq (a_i t) \wedge b_i$ for any $t \geq 0$. By Step 1 we have that $\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} F^d_\varepsilon(u)$ is finite only if $F^d(u)$ is finite and

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} F^d_\varepsilon(u) \geq \sum_{\xi\in\mathbb{Z}^2} \rho(\xi)\left(2a_i \int_\Omega |\langle \mathcal{E}u(x)\xi,\xi\rangle|^2\,dx\right.$$
$$\left. +4a_i\theta|\xi|^4 \int_\Omega |\mathrm{div}\, u(x)|^2\,dx + 2b_i \int_{J_u} \Phi^\xi(u^+ - u^-,\nu_u)\,d\mathcal{H}^1\right).$$

Then the thesis follows as above from Lemma 2.14. □

The following proposition will be crucial for the proof of the Γ-limsup inequality in both the discrete and the continuous case.

Proposition 4.2 *Let $u \in SBD^2(\Omega) \cap L^\infty(\Omega,\mathbb{R}^2)$, then*

$$\limsup_{\varepsilon\to 0} \mathcal{F}^{c,\xi}_\varepsilon(u) \leq \mathcal{F}^\xi(u).$$

Proof. Using the notation of Lemma 2.13, set

$$J^\varepsilon_u := \left(J^\xi_u \setminus J^{\xi^\perp}_u\right)^\xi_\varepsilon \cup \left(J^{\xi^\perp}_u \setminus J^\xi_u\right)^{\xi^\perp}_\varepsilon \cup \left(J^\xi_u \cap J^{\xi^\perp}_u\right)^{\xi,\xi^\perp}_\varepsilon.$$

Since $f(t) \leq b$, by Lemma 2.13 there follows

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{F}^{c,\xi}_\varepsilon(u) &\leq \limsup_{\varepsilon\to 0} \mathcal{F}^{c,\xi}_\varepsilon\left(u,\Omega^\xi_\varepsilon \setminus J^\varepsilon_u\right) + b\limsup_{\varepsilon\to 0}\frac{\mathcal{L}^2\left(J^\varepsilon_u\right)}{\varepsilon}\\
&\leq \limsup_{\varepsilon\to 0} \mathcal{F}^{c,\xi}_\varepsilon\left(u,\Omega^\xi_\varepsilon \setminus J^\varepsilon_u\right)\\
&\quad +2b\left(\int_{J^\xi_u\setminus J^{\xi^\perp}_u} |\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1 + \int_{J^{\xi^\perp}_u\setminus J^\xi_u} |\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\right.\\
&\quad \left.+\int_{J^\xi_u\cap J^{\xi^\perp}_u} |\langle\nu_u,\xi\rangle|\vee|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\right).\end{aligned}$$

Let us prove that for a.e. $x \in \Omega^\xi_\varepsilon \setminus J^\varepsilon_u$ and for $\zeta \in \{\pm\xi, \pm\xi^\perp\}$

$$D^\zeta_\varepsilon u(x) = \langle u(x+\varepsilon\zeta) - u(x),\zeta\rangle = \int_0^\varepsilon \langle \mathcal{E}u(x+s\zeta)\zeta,\zeta\rangle\,ds. \tag{4.5}$$

Let, for instance, $\zeta = \xi$, then using the notation of Theorem 2.7 if $x \in \Omega^\xi_\varepsilon \setminus J^\varepsilon_u$ and $x = y + t\xi$, with $y \in \Pi^\xi$, we get

$$\langle u(x+\varepsilon\xi) - u(x),\xi\rangle = u^{\xi,y}(t+\varepsilon) - u^{\xi,y}(t).$$

Since $u \in SBD\,(\Omega)$, for $\mathcal{H}^1$-a.e. $y \in \Pi^\xi$ we have that $u^{\xi,y} \in SBV\big(\big(\Omega^\xi_\varepsilon\big)^\xi_y\big)$, $\dot u^{\xi,y}(t) = \langle \mathcal{E}u(y+t\xi)\xi,\xi\rangle$ for $\mathcal{L}^1$ a.e. $t \in \big(\Omega^\xi_\varepsilon\big)^\xi_y$ and $J_{u^{\xi,y}} = \big(J^\xi_u\big)^\xi_y$. Thus

$$
\begin{aligned}
u^{\xi,y}(t+\varepsilon) - u^{\xi,y}(t) \quad &= \quad \int_t^{t+\varepsilon} \langle \mathcal{E}u(y+s\xi)\xi,\xi\rangle\, ds \\
&\quad + \sum_{s \in \left(J^\xi_u\right)^\xi_y} \Big(\big(u^{\xi,y}\big)^+(s) - \big(u^{\xi,y}\big)^-(s)\Big)\ \mathrm{sgn}\,\langle \xi, \nu_u\rangle
\end{aligned}
\tag{4.6}
$$

and, since $\big(J^\xi_u\big)^\xi_y \cap [t,t+\varepsilon] = \emptyset$, (4.5) follows.

Moreover, Jensen's inequality, Fubini Theorem and (4.5) yield

$$
\begin{aligned}
\frac{1}{\varepsilon^2}\int_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u} \big|D^\zeta_\varepsilon u(x)\big|^2\, dx \quad &= \quad \int_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u} \frac{1}{\varepsilon^2}\left|\int_0^\varepsilon \langle \mathcal{E}u(x+s\zeta)\zeta,\zeta\rangle\, ds\right|^2\, dx \\
&\leq \quad \int_\Omega |\langle \mathcal{E}u(x)\zeta,\zeta\rangle|^2\, dx,
\end{aligned}
\tag{4.7}
$$

for $\zeta = \pm\xi$.

Let us also prove that

$$
\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega^\xi_\varepsilon\setminus J^\varepsilon_u} |\mathrm{div}^{\xi,\xi^\perp}_\varepsilon u|^2\, dx \leq |\xi|^4 \int_\Omega |\mathrm{div}\, u(x)|^2\, dx. \tag{4.8}
$$

Setting

$$
g(x) := |\xi|^2 \mathrm{div}\, u(x)
$$

and

$$
g_\varepsilon(x) := \frac{1}{\varepsilon}\mathrm{div}^{\xi,\xi^\perp}_\varepsilon u(x)\mathcal{X}_{\Omega^\xi_\varepsilon\setminus J^\varepsilon_u}(x),
$$

(4.8) follows if we prove prove that

$$
\|g - g_\varepsilon\|_{L^2(\Omega)} \to 0. \tag{4.9}
$$

Note that

$$
g(x) = \langle \mathcal{E}u(x)\xi,\xi\rangle + \langle \mathcal{E}u(x)\xi^\perp,\xi^\perp\rangle, \tag{4.10}
$$

and that by (4.5) on $\Omega^\xi_\varepsilon \setminus J^\varepsilon_u$ we have

$$
\mathrm{div}^{\xi,\xi^\perp}_\varepsilon u(x) = \int_0^\varepsilon \langle \mathcal{E}u(x+s\xi)\xi,\xi\rangle + \langle \mathcal{E}u(x+s\xi^\perp)\xi^\perp,\xi^\perp\rangle\, ds. \tag{4.11}
$$

Thus, by absolute continuity and Jensen's inequality we get

$$
\begin{aligned}
\|g-g_\varepsilon\|^2_{L^2(\Omega)} \quad &\leq \quad o(1) + 2|\xi|^4 \int_{\Omega^\xi_\varepsilon} \frac{1}{\varepsilon}\int_0^\varepsilon |\mathcal{E}u(x+s\xi) - \mathcal{E}u(x)|^2\, ds\, dx \\
&\quad +2|\xi|^4 \int_{\Omega^\xi_\varepsilon} \frac{1}{\varepsilon}\int_0^\varepsilon |\mathcal{E}u(x+s\xi^\perp) - \mathcal{E}u(x)|^2\, ds\, dx.
\end{aligned}
$$

Applying Fubini Theorem and then extending $\mathcal{E}u$ to 0 outside Ω yield

$$\begin{aligned} \|g - g_\varepsilon\|^2_{L^2(\Omega)} &\leq o(1) + 2|\xi|^4 \frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega |\mathcal{E}u(x+s\xi) - \mathcal{E}u(x)|^2 \, dx \, ds \\ &\quad + 2|\xi|^4 \frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega |\mathcal{E}u(x+s\xi^\perp) - \mathcal{E}u(x)|^2 \, dx \, ds, \end{aligned}$$

and so (4.9) follows by the continuity of the translation operator in $L^2(\Omega;\mathbb{R}^{2\times 2})$.

Of course, using the same argument, we can claim that the analogous inequalities of (4.8), obtained by replacing $(\xi, \xi^\perp)$ by one among the pairs $(\xi, -\xi^\perp)$, $(-\xi, \xi^\perp)$, $(-\xi, -\xi^\perp)$, hold true.

Eventually, since $f(t) \leq at$, by (4.7) and (4.8) we get

$$\limsup_{\varepsilon \to 0} \mathcal{F}^{c,\xi}_\varepsilon \left(u, \Omega^\xi_\varepsilon \setminus J^\varepsilon_u\right) \leq 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + 4a\theta|\xi|^4 \int_\Omega |\operatorname{div} u(x)|^2 \, dx$$

and the conclusion follows. □

Remark 4.3 Arguing as in the proof of Proposition 4.2 we infer that the functionals defined by

$$\mathcal{G}^\xi_\varepsilon(u) := \frac{1}{\varepsilon} \int_{\Omega^\xi_\varepsilon} g\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi} u(x)|^2\right) dx,$$

where $g(t) := (at) \wedge b$, satisfy the estimate

$$\mathcal{G}^\xi_\varepsilon(u) \leq 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + 2b \int_{J^\xi_u} |\langle \nu_u, \xi\rangle| \, d\mathcal{H}^1,$$

for any $u \in SBD(\Omega)$.

Moreover, bythe subadditivity of g and since $f(t) \leq g(t)$ by hypothesis, there holds

$$\mathcal{F}^{c,\xi}_\varepsilon(u) \leq c\left(\mathcal{G}^\xi_\varepsilon(u) + \mathcal{G}^{\xi^\perp}_\varepsilon(u)\right) \leq c\mathcal{F}^\xi(u). \tag{4.12}$$

Now we are going to prove the Γ-limsup inequality that concludes the proof of Theorem 3.1. We will obtain the recovery sequence for $u \in L^\infty(\Omega;\mathbb{R}^2)$ as suitable interpolations of the function u itself.

Proposition 4.4 *For any $u \in L^\infty(\Omega;\mathbb{R}^2)$,*

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} F^d_\varepsilon(u) \leq F^d(u).$$

Proof. It suffices to prove the inequality above for $u \in SBD^2(\Omega)$. Up to a translation argument we may assume that $0 \in \Omega$. Let $\lambda \in (0,1)$ and define $u_\lambda(x) := u(\lambda x)$ for $x \in \Omega_\lambda := \lambda^{-1}\Omega$. Notice that $\Omega \subset\subset \Omega_\lambda$ and $u_\lambda \in SBD^2(\Omega_\lambda)$. It is easy to check that $u_\lambda \to u$ in $L^1(\Omega;\mathbb{R}^2)$ as $\lambda \to 1$ and

$$\lim_{\lambda\to 1} F^d(u_\lambda, \Omega_\lambda) = F^d(u).$$

Then, by the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon\to 0} F^d_\varepsilon$, it suffices to prove that

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} F^d_\varepsilon(u_\lambda) \le F^d(u_\lambda,\Omega_\lambda),$$

for any $\lambda\in(0,1)$.
We now generalize an argument used in [22],[14]. Let $\varepsilon_j\to 0$ and consider u_λ extended to 0 outside Ω_λ. Notice that for $\alpha\in\varepsilon\mathbb{Z}^2$ and $\xi\in\mathbb{Z}^2$ we have $\varepsilon\left[\frac{\alpha}{\varepsilon}\right]_{e_1}=\alpha$ and $\varepsilon\left[\frac{\alpha+\varepsilon\xi}{\varepsilon}\right]_{e_1}=\alpha+\varepsilon\xi$, thus we get

$$\begin{aligned}\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy \;=\;& \sum_{\xi\in\mathbb{Z}^2}\rho(\xi)\int_{(0,1)^2}\sum_{\alpha\in R^\xi_{\varepsilon_j}}\varepsilon_j f\left(\frac{1}{\varepsilon_j}\Big(|D_{\varepsilon_j,\xi}u_\lambda(\varepsilon_j y+\alpha)|^2\right.\\ &\left.+\theta|\mathrm{Div}_{\varepsilon_j,\xi}u_\lambda(\varepsilon_j y+\alpha)|^2\Big)\right)\,dy,\end{aligned}$$

where T^ε_y is given by (2.5) for $\mathcal{B}=\{e_1,e_2\}$.
Then, using the change of variable $\varepsilon_j y+\alpha\to y$, we obtain

$$\begin{aligned}&\int_{(0,1)^2} F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy\\ &=\sum_{\xi\in\mathbb{Z}^2}\rho(\xi)\sum_{\alpha\in R^\xi_{\varepsilon_j}}\int_{\alpha+(0,\varepsilon_j)^2}\frac{1}{\varepsilon_j}f\left(\frac{1}{\varepsilon_j}\Big(|D_{\varepsilon_j,\xi}u_\lambda(y)|^2+\theta|\mathrm{Div}_{\varepsilon_j,\xi}u_\lambda(y)|^2\Big)\right)\,dy\\ &\le\sum_{\xi\in\mathbb{Z}^2}\rho(\xi)\mathcal{F}^{c,\xi}_{\varepsilon_j}\left(u_\lambda,\Omega_\lambda\right).\end{aligned}$$

In particular, by Proposition 4.2 and Remark 4.3, there holds

$$\begin{aligned}\limsup_j\int_{(0,1)^2} & F^d_{\varepsilon_j}(T^{\varepsilon_j}_y u_\lambda)\,dy \\ &\le\sum_{\xi\in\mathbb{Z}^2}\rho(\xi)\limsup_j\mathcal{F}^{c,\xi}_{\varepsilon_j}\left(u_\lambda,\Omega_\lambda\right)\le F^d(u_\lambda,\Omega_\lambda)<+\infty.\end{aligned}\tag{4.13}$$

Fix $\delta>0$ and set

$$C^j_\delta:=\left\{z\in(0,1)^2: F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_z u_\lambda\right)\le\int_{(0,1)^2}F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy+\delta\right\}.$$

By (4.13), we have for j large

$$|(0,1)^2\setminus C^j_\delta|\le\frac{\int_{(0,1)^2}F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy}{\int_{(0,1)^2}F^d_{\varepsilon_j}\left(T^{\varepsilon_j}_y u_\lambda\right)\,dy+\delta}\le c<1,$$

which implies

$$|C_\delta^j| > 1 - c > 0.$$

Then, by Remark 2.12, for any $j \in \mathbb{N}$ we can choose $z_j \in C_\delta^j$ such that $T_{z_j}^{\varepsilon_j} u_\lambda \to u_\lambda$ in $L^1(\Omega; \mathbb{R}^2)$ and

$$F_{\varepsilon_j}^d (T_{z_j}^{\varepsilon_j} u_\lambda) \leq \int_{(0,1)^2} F_{\varepsilon_j}^d \left(T_y^{\varepsilon_j} u_\lambda\right) \, dy + \delta. \tag{4.14}$$

Hence, by (4.13) and (4.14), there holds

$$\limsup_j F_{\varepsilon_j}^d \left(T_{z_j}^{\varepsilon_j} u_\lambda\right) \leq F^d(u_\lambda, \Omega_\lambda) + \delta,$$

from which we infer

$$\Gamma(L^1)\text{-}\limsup_j F_{\varepsilon_j}^d (u_\lambda) \leq F^d(u_\lambda, \Omega_\lambda) + \delta$$

and letting $\delta \to 0$ we get the conclusion. ☐

Remark 4.5 In the proof of the previous proposition the convexity assumption on Ω is used only to ensure that for any $\lambda > 1$ $\Omega \subset\subset \Omega_\lambda$. This condition is nedeed in order to justify the existence of some kind of extension of a SBD function outside of Ω with controlled energy. An alternative approach would involve the use of an extension theorem in SBD analogous to the one holding true in SBV (see Lemma 4.11 of [14]). So far, such a result has not been proved, yet.
Thus, Proposition 4.4 and Theorem 3.1 can be stated for open sets sharing one of the previous properties.

5 The continuous case

In this section we will prove Theorem 3.8. We "localize" the functionals $\mathcal{F}_\varepsilon^{c,\xi}$ as

$$\mathcal{F}_\varepsilon^{c,\xi}(u, A) := \frac{1}{\varepsilon} \int_{A_\varepsilon^\xi} f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi} u(x)|^2 + \theta |\mathrm{Div}_{\varepsilon,\xi} u(x)|^2\right)\right) \, dx,$$

for any $u \in L^1(\Omega; \mathbb{R}^2)$, $A \in \mathcal{A}(\Omega)$, with

$$A_\varepsilon^\xi := \{x \in \mathbb{R}^2 : [x - \varepsilon\xi, x + \varepsilon\xi] \cup [x - \varepsilon\xi^\perp, x + \varepsilon\xi^\perp] \subset A\}.$$

Proposition 5.1 *For any $u \in L^\infty(\Omega; \mathbb{R}^2) \cap L^1(\Omega; \mathbb{R}^2)$,*

$$\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon \to 0} F_\varepsilon^c(u) \geq F^c(u).$$

Proof. *Step* 1 Let us first assume $f(t)=(at)\wedge b$. Let $\varepsilon_j\to 0$, $u_j\in L^1(\Omega;\mathbb{R}^2)$, $u\in L^\infty(\Omega;\mathbb{R}^2)$ be such that $u_j\to u$ in $L^1(\Omega;\mathbb{R}^2)$ and $\liminf_j F^c_{\varepsilon_j}(u_j)=\lim_j F^c_{\varepsilon_j}(u_j)<+\infty$. In particular for a.e. $\xi\in\mathbb{R}^2$ such that $\rho(\xi)\neq 0$ $\liminf_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j)<+\infty$. Fix such a $\xi\in\mathbb{R}^2$ and $A\in\mathcal{A}(\Omega)$. Up to passing to a subsequence we may assume that $\liminf_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j,A)=\lim_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j,A)<+\infty$. We now adapt to our case a "discretization" argument used in the proof of Proposition 3.38 of [14]. In what follows, when needed, we will consider u_j and u extended to 0 outside Ω. If we define

$$g_j(x):=\begin{cases} f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u_j(x)|^2+\theta|\mathrm{Div}_{\varepsilon_j,\xi}u_j(x)|^2\right)\right) & \text{if } x\in A^\xi_{\varepsilon_j}\\ 0 & \text{otherwise in } \mathbb{R}^2,\end{cases}$$

we can write

$$\begin{aligned}\mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j,A) &= \frac{1}{\varepsilon_j}\int_{\mathbb{R}^2} g_j(x)\,dx=\frac{1}{\varepsilon_j}\sum_{\alpha\in\varepsilon_j(\mathbb{Z}\xi\oplus\mathbb{Z}\xi^\perp)}\int_{\alpha+\varepsilon_j\widetilde{Q}_\xi} g_j(x)\,dx\\ &= \frac{1}{\varepsilon_j}\sum_{\alpha\in\varepsilon_j(\mathbb{Z}\xi\oplus\mathbb{Z}\xi^\perp)}\int_{\varepsilon_j\widetilde{Q}_\xi} g_j(x+\alpha)\,dx=\int_{\widetilde{Q}_\xi}\phi_j(x)\,dx,\end{aligned}$$

where

$$\begin{aligned}\phi_j(x) &:= \sum_{\alpha\in\varepsilon_j(\mathbb{Z}\xi\oplus\mathbb{Z}\xi^\perp)}\varepsilon_j g_j(\varepsilon_j x+\alpha),\\ \widetilde{Q}_\xi &:= \{x\in\mathbb{R}^2:0\le\langle x,\xi\rangle<|\xi|^2, 0\le\langle x,\xi^\perp\rangle<|\xi^\perp|^2\}.\end{aligned}$$

Fix $\delta>0$, then, arguing as in the proof of Proposition 4.4, by using Remark 2.12, for any $j\in\mathbb{N}$ we can find $x_j\in\widetilde{Q}_\xi$ such that $T^{\varepsilon_j,\xi}_{x_j}u_j\to u$ in $L^1(\Omega;\mathbb{R}^2)$ and

$$\begin{aligned}&\mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j,A)+\delta\ge|\xi|^2\phi_j(x_j)\\ &\ge|\xi|^2\varepsilon_j\sum_{\substack{\alpha\in\varepsilon_j(\mathbb{Z}\xi\oplus\mathbb{Z}\xi^\perp)\\ \alpha\in A^\xi_{\varepsilon_j}-\varepsilon_j x_j}} f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u_j(\alpha+\varepsilon_j x_j)|^2+\theta|\mathrm{Div}_{\varepsilon_j,\xi}u_j(\alpha+\varepsilon_j x_j)|^2\right)\right)\end{aligned}$$

Now we point out that the functionals on the right hand side is of the same type of those defined in (3.4). Hence, up to slight modifications, we can proceed as in the proof of Proposition 4.1 to obtain that $u\in SBD(\Omega)$ and

$$\begin{aligned}\liminf_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j) \ \ge\ & 2a\int_\Omega|\langle\mathcal{E}u(x)\xi,\xi\rangle|^2\,dx+4a\theta|\xi|^4\int_\Omega|\mathrm{div}\,u(x)|^2\,dx\\ &+2b\Big(\int_{J^\xi_u\setminus J^{\xi^\perp}_u}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1+\int_{J^{\xi^\perp}_u\setminus J^\xi_u}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\\ &+\int_{J^\xi_u\cap J^{\xi^\perp}_u}|\langle\nu_u,\xi\rangle|\vee|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\Big).\end{aligned}$$

Finally, recalling that $\mathcal{H}^1(J_u \setminus J_u^\xi) = 0$ for a.e. $\xi \in \mathbb{R}^2$, by integrating with respect to ξ and by Fatou Lemma, we get

$$\begin{aligned}
\liminf_j F^c_{\varepsilon_j}(u_j) &\geq \int_{\mathbb{R}^2} \rho(\xi) \liminf_j \mathcal{F}^{c,\xi}_{\varepsilon_j}(u_j)\, d\xi \\
&\geq \int_{\mathbb{R}^2} \rho(\xi) \left(\int_\Omega 2a|\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 + 4a\theta|\xi|^4 |\mathrm{div}\, u(x)|^2\, dx \right) d\xi \\
&\quad + \int_{\mathbb{R}^2} 2b\rho(\xi) \left(\int_{J_u} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^1 \right) d\xi \\
&= \int_\Omega \left(\int_{\mathbb{R}^2} \rho(\xi) \left(2a|\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 + 4a\theta|\xi|^4|\mathrm{div}\, u(x)|^2\right) d\xi \right) dx \\
&\quad + \int_{J_u} \left(\int_{\mathbb{R}^2} 2b\rho(\xi) |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle|\, d\xi \right) d\mathcal{H}^1.
\end{aligned}$$

The expressions for μ, λ, γ follow after a simple computation.

Step 2 If f is any, arguing as in Step 2 of the proof of Proposition 4.1, we have that $\Gamma\left(L^1\right)$ - $\liminf_{\varepsilon\to 0} F^c_\varepsilon(u)$ is finite only if $F^c(u)$ is finite and

$$\Gamma\left(L^1\right)\text{-}\liminf_{\varepsilon\to 0} F^c_\varepsilon(u) \geq \mu_i \int_\Omega |\mathcal{E}u(x)|^2\, dx + \lambda_i \int_\Omega |\mathrm{div}\, u(x)|^2\, dx + \gamma_i \mathcal{H}^1(J_u)$$

with $\sup_i \mu_i = \mu$, $\sup_i \lambda_i = \lambda$ and $\sup_i \gamma_i = \gamma$. The thesis follows using once more Lemma 2.14. □

Let us now prove the Γ-limsup inequality which easily follows by Proposition 4.2 and Remark 4.3.

Proposition 5.2 *For any $u \in L^\infty(\Omega; \mathbb{R}^2)$,*

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} F^c_\varepsilon(u) \leq F^c(u).$$

Proof. As usual we can reduce ourselves to prove the inequality for $u \in SBD^2(\Omega)$. For such a u the recovery sequence is provided by the function itself. Indeed, by Proposition 4.2, estimate (4.12) and Fatou Lemma, we get

$$\begin{aligned}
\limsup_{\varepsilon\to 0} F^c_\varepsilon(u) &\leq \int_{\mathbb{R}^2} \rho(\xi) \limsup_{\varepsilon\to 0} \mathcal{F}^{c,\xi}_\varepsilon(u)\, d\xi \\
&\leq \int_{\mathbb{R}^2} \rho(\xi) \mathcal{F}^\xi(u)\, d\xi = F^c(u).
\end{aligned}$$

□

6 Convergence of minimum problems in the discrete case

6.1 A compactness lemma

The following lemma will be crucial to derive the convergence of the minimum problems treated in the next section.

Lemma 6.1 *Let f, ρ, F^d_ε be as in Theorem 3.1; assume in addition that Ω is a bounded Lipschitz open set. Let (u_j) be a sequence in $\mathcal{A}_{\varepsilon_j}(\Omega)$ such that*

$$\sup_j \big(F^d_{\varepsilon_j}(u_j) + \|u_j\|_{L^\infty(\Omega;\mathbb{R}^2)} \big) < +\infty. \tag{6.1}$$

Then there exists a subsequence (u_{j_k}) converging in $L^1(\Omega;\mathbb{R}^2)$ to a function $u \in SBD^2(\Omega)$.

Proof. Without loss of generality we may assume $f(t) = (at) \wedge b$. Set

$$\begin{aligned} C_j := & \sum_{\alpha \in R^{e_1}_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j} |D^{e_1}_{\varepsilon_j} u_j(\alpha)|^2 \right) + \sum_{\alpha \in R^{e_2}_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j} |D^{e_2}_{\varepsilon_j} u_j(\alpha)|^2 \right) \\ & + \sum_{\alpha \in R^{e_1+e_2}_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j} |D^{e_1+e_2}_{\varepsilon_j} u_j(\alpha)|^2 \right) + \sum_{\alpha \in R^{e_1-e_2}_{\varepsilon_j}} \varepsilon_j f\left(\frac{1}{\varepsilon_j} |D^{e_1-e_2}_{\varepsilon_j} u_j(\alpha)|^2 \right), \end{aligned}$$

then the monotonicity and subadditivity of f yield

$$\sup_j C_j \le c \, \sup_j F^d_{\varepsilon_j}(u_j) < +\infty. \tag{6.2}$$

Let

$$\begin{aligned} M_j(\alpha) := \max \big\{ & |D^{e_1}_{\varepsilon_j} u_j(\alpha)|^2;\ |D^{e_2}_{\varepsilon_j} u_j(\alpha)|^2;\ |D^{e_1}_{\varepsilon_j} u_j(\alpha + \varepsilon_j e_2)|^2; \\ & |D^{e_2}_{\varepsilon_j} u_j(\alpha + \varepsilon_j e_1)|^2;\ |D^{e_1+e_2}_{\varepsilon_j} u_j(\alpha)|^2;\ |D^{e_2-e_1}_{\varepsilon_j} u_j(\alpha + \varepsilon_j e_1)|^2 \big\} \end{aligned}$$

and

$$R_j := R^{e_1}_{\varepsilon_j} \cap R^{e_1+e_2}_{\varepsilon_j} \cap \left(R^{e_1}_{\varepsilon_j} - \varepsilon_j e_2 \right) \cap \left(R^{e_1}_{\varepsilon_j} - \varepsilon_j e_1 \right) \cap \left(R^{e_1+e_2}_{\varepsilon_j} - \varepsilon_j e_1 \right),$$

then set

$$I_j := \left\{ \alpha \in R_j :\ M_j(\alpha) \le \frac{b}{a} \varepsilon_j \right\}.$$

Consider the (piecewise affine) functions $v_j = (v_j^1, v_j^2)$ defined on $\alpha + \varepsilon_j[0,1)^2,\ \alpha \in I_j$, as

$$v_j^1(x) := \begin{cases} u_j^1(\alpha) + \frac{1}{\varepsilon_j} D_{\varepsilon_j}^{e_1} u_j(\alpha)(x_1 - \alpha_1) \\ +\frac{1}{\varepsilon_j}\left(u_j^1(\alpha + \varepsilon_j e_2) - u_j^1(\alpha)\right)(x_2 - \alpha_2) & x \in (\alpha + \varepsilon_j T^-) \cap \Omega \\ u_j^1(\alpha + \varepsilon_j(e_1 + e_2)) + \frac{1}{\varepsilon_j} D_{\varepsilon_j}^{e_1} u_j(\alpha + \varepsilon_j e_2)(x_1 - \alpha_1 - \varepsilon_j) \\ +\frac{1}{\varepsilon_j}\left((u_j^1(\alpha + \varepsilon_j(e_1 + e_2)) - u_j^1(\alpha + \varepsilon_j e_1)\right)(x_2 - \alpha_2 - \varepsilon_j) \\ & x \in (\alpha + \varepsilon_j T^+) \cap \Omega \end{cases}$$

$$v_j^2(x) := \begin{cases} u_j^2(\alpha) + \frac{1}{\varepsilon_j} D_{\varepsilon_j}^{e_2} u_j(\alpha)(x_2 - \alpha_2) \\ +\frac{1}{\varepsilon_j}\left(u_j^2(\alpha + \varepsilon_j e_1) - u_j^2(\alpha)\right)(x_1 - \alpha_1) & x \in (\alpha + \varepsilon_j T^-) \cap \Omega \\ u_j^2(\alpha + \varepsilon_j(e_1 + e_2)) + \frac{1}{\varepsilon_j} D_{\varepsilon_j}^{e_2} u_j(\alpha + \varepsilon_j e_1)(x_2 - \alpha_2 - \varepsilon_j) \\ +\frac{1}{\varepsilon_j}\left((u_j^2(\alpha + \varepsilon_j(e_1 + e_2)) - u_j^2(\alpha + \varepsilon_j e_2)\right)(x_1 - \alpha_1 - \varepsilon_j) \\ & x \in (\alpha + \varepsilon_j T^+) \cap \Omega \end{cases}$$

where $T^\pm = \{x \in (0,1)^2 :\ \pm(x_1 + x_2 - 1) \geq 0\}$ and $x_i = \langle x, e_i \rangle$, $i = 1, 2$, and $v_j = u_j$ elsewhere in Ω. Notice that on each triangle $\alpha + \varepsilon_j T^\pm$ v_j is an affine interpolation of the values of u_j on the vertices of the triangle.
By direct computation it is easily seen that for any $\alpha \in I_j$ and $x \in \alpha + \varepsilon_j(0,1)^2$ there holds

$$|\mathcal{E} v_j(x)|^2 \leq c \left(\frac{1}{\varepsilon_j}\right)^2 M_j(\alpha),$$

hence, by taking into account (6.2) and the subadditivity of f, we get

$$\sup_j \int_\Omega |\mathcal{E} v_j(x)|^2\, dx \leq c \ \sup_j C_j < +\infty. \tag{6.3}$$

Now, we provide an estimate for $\mathcal{H}^1\left(J_{v_j}\right)$. Let

$$\begin{aligned} A_j &:= \left\{\alpha \in \varepsilon_j \mathbb{Z}^2 :\ \alpha + \varepsilon_j(0,1)^2 \cap \Omega \neq \emptyset\right\}, \\ D_j &:= \left\{x \in \mathbb{R}^2 :\ d\,(x, \partial\Omega) < 2\varepsilon_j\right\}, \end{aligned}$$

and note that

$$\bigcup_{\alpha \in A_j \setminus R_j} \alpha + \varepsilon_j(0,1)^2 \subseteq D_j.$$

By the Lipschitz regularity assumption on Ω, it follows

$$\#\,(A_j \setminus R_j) \leq \frac{|D_j|}{{\varepsilon_j}^2} \leq c\ \frac{\mathcal{H}^1\,(\partial\Omega)}{\varepsilon_j},$$

and thus we get

$$
\begin{aligned}
\mathcal{H}^1\left(J_{v_j}\right) &\leq 4\varepsilon_j \#\left(A_j \setminus I_j\right) = 4\varepsilon_j \#\left(A_j \setminus R_j\right) + 4\varepsilon_j \#\left(R_j \setminus I_j\right) \\
&\leq c\left\{\mathcal{H}^1(\partial\Omega) + C_j\right\} \leq c. \qquad (6.4)
\end{aligned}
$$

Since $(v_j) \subset SBD(\Omega)$ is bounded in $L^\infty\left(\Omega;\mathbb{R}^2\right)$ and by taking into account (6.3) and (6.4), Theorem 2.8 yields the existence of a subsequence (v_{j_k}) converging in $L^1\left(\Omega;\mathbb{R}^2\right)$ to a function $u \in SBD^2(\Omega)$. The thesis follows noticing that by Proposition A.1 (u_{j_k}) is also converging to u in $L^1\left(\Omega;\mathbb{R}^2\right)$. □

Thanks to Lemma 6.1 and by taking into account Theorems 3.1 and 2.10, we have the following convergence result for obstacle problems with Neumann boundary conditions.

Theorem 6.2 *Let K be a compact subset of $\mathbb{R}^2$ and let $h \in L^1(\Omega;\mathbb{R}^2)$. Then the minimum values*

$$
\min\left\{F^d_\varepsilon(u) - \int_\Omega \langle h, u\rangle\, dx :\ u \in L^1(\Omega;\mathbb{R}^2),\ u \in K \ a.e.\right\}, \qquad (6.5)
$$

converge to the minimum value

$$
\min\left\{F^d(u) - \int_\Omega \langle h, u\rangle\, dx :\ u \in L^1(\Omega;\mathbb{R}^2),\ u \in K \ a.e.\right\}, \qquad (6.6)
$$

Moreover, for any family of minimizers (u_ε) for (6.5) and for any sequence (ε_j) of positive numbers converging to 0, there exists a subsequence (not relabeled) u_{ε_j} converging to a minimizer of (6.6).

6.2 Boundary value problems

In this section we deal with boundary value problems for discrete energies. Following [19], we separate 'interior interactions' from those 'crossing the boundary'. Let $\Omega \subset \mathbb{R}^2$ be a convex set such that $0 \in \Omega$, let $\eta > 0$ and denote by Ω_η the open set $\left\{x \in \mathbb{R}^2 :\ dist\,(x, \partial\Omega) < \eta\right\}$. Let $\varphi : \partial\Omega \to \mathbb{R}^2$ and $p_1, \ldots, p_N \in \partial\Omega$ such that φ is Lipschitz on each connected component of $\partial\Omega \setminus \{p_1, ..., p_N\}$. Then define for $\eta < dist\,(0, \partial\Omega)$ the function $\tilde\varphi : \Omega_\eta \to \mathbb{R}^2$ by

$$
\tilde\varphi(x) := \varphi\left(\tau x\right),
$$

where $\tau \geq 0$ is such that $\{\tau x\} = \{tx\}_{t\geq 0} \cap \partial\Omega$.
We remark that $\tilde\varphi \in W^{1,\infty}\left(\Omega_\eta \setminus \cup_{i=1}^N \{tp_i\}_{t\geq 0}\,;\mathbb{R}^2\right)$ and $J_{\tilde\varphi} = \cup_{i=1}^N \{tp_i\}_{t\geq 0} \cap \Omega_\eta$. The function $\tilde\varphi$ is a possible extension of φ to Ω_η. Other extensions are possible which, under regularity assumptions, yield the same result. Here we examine this 'radial' extension only, for the sake of simplicity.

With given $u \in SBD(\Omega)$, let

$$u^{\varphi}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ \\ \tilde{\varphi}(x) & \text{if } x \in \Omega_\eta \setminus \Omega, \end{cases}$$

then $u^{\varphi} \in SBD(\Omega_\eta)$ and $J_{u^{\varphi}} = J_u \cup J_{\tilde{\varphi}} \cup \{x \in \partial\Omega : \ \gamma(u)(x) \neq \varphi(x)\}$, where $\gamma(u)$ denotes the inner trace of u with respect to $\partial\Omega$. Finally, we define a suitable discretization of $\tilde{\varphi}$ by

$$\varphi_\varepsilon(\alpha) := \begin{cases} \tilde{\varphi}(\alpha) & \text{if } \alpha \in \varepsilon\mathbb{Z}^2 \cap \Omega_\eta \\ \\ 0 & \text{if } \alpha \notin \varepsilon\mathbb{Z}^2 \cap \Omega_\eta. \end{cases}$$

Let f, ρ, F^d_ε be as in Theorem 3.1 and assume in addition that $\rho(\xi) = 0$ for $|\xi| > M$, with $M \geq 2$. Let $\mathcal{B}_\varepsilon(u) := F^d_\varepsilon(u) + F^{d,\varphi}_\varepsilon(u)$, where

$$F^{d,\varphi}_\varepsilon(u) := \begin{cases} \displaystyle\sum_{|\xi|\leq M} \rho(\xi) \sum_{\alpha \in R^\xi_\varepsilon(\partial\Omega)} \varepsilon f\left(\tfrac{1}{\varepsilon}\left(|D_{\varepsilon,\xi} u^{\varphi_\varepsilon}(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon,\xi} u^{\varphi_\varepsilon}(\alpha)|^2\right)\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise} \end{cases}$$

with

$$R^\xi_\varepsilon(\partial\Omega) := \left\{\alpha \in \varepsilon\mathbb{Z}^2 \setminus R^\xi_\varepsilon(\Omega) : \ [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\xi^\perp, \alpha + \varepsilon\xi^\perp] \cap \Omega \neq \emptyset\right\}$$

and

$$u^{\varphi_\varepsilon}(\alpha) := \begin{cases} u(\alpha) & \text{if } \alpha \in \varepsilon\mathbb{Z}^2 \cap \Omega \\ \\ \varphi_\varepsilon(\alpha) & \text{if } \alpha \notin \varepsilon\mathbb{Z}^2 \cap \Omega. \end{cases}$$

$R^\xi_\varepsilon(\partial\Omega)$ represents that part of the lattice R^ξ_ε underlying interactions between pairs of points of Ω_η one inside and the other outside Ω (interactions through the boundary).

Proposition 6.3 *$\mathcal{B}_\varepsilon$ Γ-converges on $L^\infty(\Omega; \mathbb{R}^2)$ to the functional $\mathcal{B} : L^\infty(\Omega; \mathbb{R}^2) \to [0, +\infty]$ given by*

$$\mathcal{B}(u) := \begin{cases} \displaystyle F^d(u) + 2b \sum_{|\xi|\leq M} \rho(\xi) \int_{J_{u^{\varphi}} \cap \partial\Omega} \Phi^\xi\left(\gamma(u) - \varphi, \nu_{\partial\Omega}\right) d\mathcal{H}^1 & \textit{if } u \in SBD(\Omega) \\ \\ +\infty & \textit{otherwise} \end{cases}$$

with respect to both the $L^1(\Omega; \mathbb{R}^2)$-convergence and the convergence in measure, where $\nu_{\partial\Omega}$ is the inner unit normal to $\partial\Omega$ and the function $\Phi^\xi : \mathbb{R}^2 \to [0, +\infty)$ is defined by

$$\Phi^\xi(z, \nu) := \psi^\xi(z, \nu) \vee \psi^{\xi^\perp}(z, \nu),$$

with for $\eta \in \mathbb{R}^2$

$$\psi^\eta(z,\nu) := \begin{cases} |\langle \nu, \eta\rangle| & \textit{if } \langle z,\eta\rangle \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Note that if ε is sufficiently small, for every $v \in \mathcal{A}_\varepsilon(\Omega)$ we have

$$\mathcal{B}_\varepsilon(v) \geq F^d_\varepsilon\left(v^{\varphi_\varepsilon}, \Omega \cup \Omega_\eta\right) - F^d_\varepsilon\left(\varphi_\varepsilon, \Omega_\eta \setminus \overline{\Omega}\right). \tag{6.7}$$

Moreover, the regularity of φ and the assumptions on f yield

$$\limsup_{\varepsilon\to 0} F^d_\varepsilon\left(\varphi_\varepsilon, \Omega_\eta \setminus \overline{\Omega}\right) \leq c\,|\Omega_\eta \setminus \Omega| + c\,\mathcal{H}^1\left(J_{\tilde{\varphi}} \cap (\Omega_\eta \setminus \Omega)\right). \tag{6.8}$$

Let $u_\varepsilon \to u$ in measure on Ω, then $u_\varepsilon^{\varphi_\varepsilon} \to u^\varphi$ in measure on $\Omega \cup \Omega_\eta$. Thus, by Theorem 3.1 and inequalities (6.7) and (6.8), we get

$$\liminf_{\varepsilon\to 0} \mathcal{B}_\varepsilon(u_\varepsilon) \geq F^d(u^\varphi, \Omega \cup \Omega_\eta) - \omega(\eta) \geq \mathcal{B}(u) - \omega(\eta),$$

with $\lim_{\eta\to 0}\omega(\eta) = 0$. Then the Γ-$\liminf$ inequality follows by letting $\eta \to 0$.

Let $u \in L^\infty(\Omega;\mathbb{R}^2) \cap SBD^2(\Omega)$, fix $\lambda \in (0,1)$ and define $u^\varphi_\lambda \in SBD^2(\Omega_\eta)$ as

$$u^\varphi_\lambda(x) := \begin{cases} u\left(\frac{x}{\lambda}\right) & x \in \lambda\Omega \\ \tilde{\varphi}(x) & x \in \Omega_\eta \setminus \lambda\Omega, \end{cases}$$

then $u^\varphi_\lambda \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and $F^d\left(u^\varphi_\lambda, \Omega\right) \to \mathcal{B}(u)$ for $\lambda \to 1$.
Hence, to prove the Γ-$\limsup$ inequality, it suffices to show that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{B}_\varepsilon\left(u^\varphi_\lambda\right) \leq F^d\left(u^\varphi_\lambda, \Omega\right). \tag{6.9}$$

Fix $\delta > 0$, arguing as in the proof of Proposition 4.4, we can find v_ε of the form $u^\varphi_\lambda(\cdot + \tau_\varepsilon)$, with $\tau_\varepsilon \leq c\varepsilon$ and $v_\varepsilon \to u^\varphi_\lambda$ in $L^1(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0$, such that

$$\limsup_{\varepsilon\to 0} F^d_\varepsilon\left(v_\varepsilon, \Omega\right) \leq F^d\left(u^\varphi_\lambda, \Omega\right) + \delta. \tag{6.10}$$

Note that if $\beta = \alpha, \alpha \pm \varepsilon\xi, \alpha \pm \varepsilon\xi^\perp$ with $\alpha \in R^\xi_\varepsilon(\partial\Omega)$, for ε small we have

$$v^{\varphi_\varepsilon}_\varepsilon(\beta) = \begin{cases} \tilde{\varphi}(\beta + \tau_\varepsilon) & \text{if } \beta \in \varepsilon\mathbb{Z}^2 \cap \Omega \\ \tilde{\varphi}(\beta) & \text{if } \beta \notin \varepsilon\mathbb{Z}^2 \cap \Omega. \end{cases}$$

Then, by the regularity of $\tilde{\varphi}$, it can be proved that

$$F^{d,\varphi}_\varepsilon(v_\varepsilon) = O(\varepsilon),$$

hence, by (6.10),

$$\limsup_{\varepsilon\to 0} \mathcal{B}_\varepsilon(v_\varepsilon) \le F^d(u_\lambda^\varphi, \Omega) + \delta.$$

Then, inequality (6.9) follows letting $\delta \to 0$. ☐

As a consequence of Lemma 6.1 and Proposition 6.3, we get the following convergence result for boundary value problems.

Theorem 6.4 *Let K be a compact set of $\mathbb{R}^2$ and let $\mathcal{B}_\varepsilon$ be as in Proposition 6.3. Then the minimum values*

$$\min\{\mathcal{B}_\varepsilon(u) : u \in K \ a.e.\} \tag{6.11}$$

converge to the minimum value

$$\min\{\mathcal{B}(u) : u \in K \ a.e.\}. \tag{6.12}$$

Moreover, for any family of minimizers (u_ε) for (6.11) and for any sequence (ε_j) of positive numbers converging to 0, there exists a subsequence (not relabeled) u_{ε_j} converging to a minimizer of (6.12).

Proof. It easily follows from Lemma 6.1, Proposition 6.3 and Theorem 2.10. ☐

7 Generalizations

By following the approach of Section 3, different generalizations to higher dimension can be proposed. We present here one possible extension of the discrete model in $\mathbb{R}^3$ which provides as well an approximation of energies of type (1.6).

For any orthogonal pair $(\xi, \zeta) \in \mathbb{R}^3 \setminus \{0\}$ and for any $u : \mathbb{R}^3 \to \mathbb{R}^3$ define

$$\begin{aligned}
&D_\varepsilon^\xi u(x) := \langle u(x+\varepsilon\xi) - u(x), \xi\rangle, \\
&|D_{\varepsilon,\xi} u(x)|^2 := |D_\varepsilon^\xi u(x)|^2 + |D_\varepsilon^{-\xi} u(x)|^2, \\
&|D_{\varepsilon,\xi,\zeta} u(x)|^2 := |D_{\varepsilon,\xi} u(x)|^2 + |D_{\varepsilon,\zeta} u(x)|^2, \\
&|\mathrm{Div}_{\varepsilon,\xi,\zeta} u(x)|^2 := \\
&\sum_{(\sigma_1,\sigma_2,\sigma_3)\in\{1,-1\}^3} \left(\frac{1}{|\xi|^2} D_\varepsilon^{\sigma_1\xi} u(x) + \frac{1}{|\zeta|^2} D_\varepsilon^{\sigma_2\zeta} u(x) + \frac{1}{|\xi\times\zeta|^2} D_\varepsilon^{\sigma_3\xi\times\zeta} u(x) \right)^2,
\end{aligned}$$

where $\xi \times \zeta$ denotes the external product of ξ and ζ.

Let Ω be a bounded open set of $\mathbb{R}^3$ and $\mathcal{A}_\varepsilon^3(\Omega) := \{u : \Omega \to \mathbb{R}^3 \ : \ u \equiv$ *const* on $(\alpha + [0,\varepsilon)^3) \cap \Omega$ for any $\alpha \in \varepsilon\mathbb{Z}^3\}$. Then set

$$S := \{(e_1,e_2),(e_1,e_3),(e_2,e_3),(e_1+e_2,e_1-e_2),(e_1+e_3,e_1-e_3),(e_2+e_3,e_2-e_3)\}$$

and consider the sequence of functionals $F_\varepsilon^{d,3} : L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by

$$F_\varepsilon^{d,3} u := \begin{cases} \sum\limits_{(\xi,\zeta)\in S} \sum\limits_{\alpha\in R_\varepsilon^{\xi,\zeta}} \varepsilon^2 f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi,\zeta}u(\alpha)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi,\zeta}u(\alpha)|^2\right)\right) & \text{if } u \in \mathcal{A}_\varepsilon^3(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$R_\varepsilon^{\xi,\zeta} := \{\alpha \in \varepsilon\mathbb{Z}^3 : [\alpha - \varepsilon\xi, \alpha + \varepsilon\xi] \cup [\alpha - \varepsilon\zeta, \alpha + \varepsilon\zeta] \cup [\alpha - \varepsilon\xi\times\zeta, \alpha + \varepsilon\xi\times\zeta] \subset \Omega\}$$

and f, θ as in Section 3.

Theorem 7.1 *Let Ω be convex. Then $F_\varepsilon^{d,3}$ Γ-converges on $L^\infty(\Omega;\mathbb{R}^3)$ to the functional $F^{d,3} : L^\infty(\Omega;\mathbb{R}^3) \to [0,+\infty]$ given by*

$$F^d(u) = \begin{cases} 8a\int_\Omega |\mathcal{E}u(x)|^2\,dx + 4(1+2\theta)a\int_\Omega |\mathrm{div}\, u(x)|^2\,dx & \\ \qquad\qquad + 2b \sum\limits_{(\xi,\zeta)\in S} \int_{J_u} \Phi^{\xi,\zeta}(u^+ - u^-, \nu_u)\,d\mathcal{H}^2 & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

with respect to both the $L^1(\Omega;\mathbb{R}^3)$-convergence and the convergence in measure, where $\Phi^{\xi,\zeta} : \mathbb{R}^3 \to [0,+\infty)$ is defined by

$$\Phi^{\xi,\zeta}(z,\nu) := \psi^\xi(z,\nu) \vee \psi^\zeta(z,\nu) \vee \psi^{\xi\times\zeta}(z,\nu),$$

with for $\eta \in \mathbb{R}^3$

$$\psi^\eta(z,\nu) := \begin{cases} |\langle \nu,\eta\rangle| & \text{if } \langle z,\eta\rangle \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It suffices to proceed as in the proof of Theorem 3.1, by extending all the arguments to dimension 3 and taking into account Lemmas 2.11 and 2.13, that are stated in any dimension. □

A Appendix

In the previous sections in order to study the Γ-convergence of discrete energies we have identified a function u defined on a lattice with a suitable "piecewise-constant" interpolation, i.e., a function which takes on each cell of the lattice the value of u in one node of the cell itself. Then, fixed a discretization step length, we treated the convergence (in measure or L^1 strong) of discrete functions through this association.

This choice is not arbitrarily done. Indeed, the convergence of piecewise-constant interpolations ensures the convergence of any other "piecewise-affine" ones, whose values on each cell are obtained as a convex combination of the values of the discrete function in the nodes of the cell itself. Actually, the converse result also holds true, as the following proposition shows.

Proposition A.1 *Let ε be a positive parameter tending to 0 and let $\mathcal{T}_\varepsilon = (T_\varepsilon^i)_{i\in\mathbb{N}}$ be a family of n-simplices in $\mathbb{R}^n$ such that* $\mathrm{int}\,(T_\varepsilon^i)\cap\mathrm{int}\,(T_\varepsilon^j) = \emptyset$ *if* $i \neq j$, $\bigcup_i T_\varepsilon^i = \mathbb{R}^n$ *and assume also that* $\sup_i \mathrm{diam}\, T_\varepsilon^i \to 0$ *as* $\varepsilon \to 0$. *Let* $u_\varepsilon \in L^1(\mathbb{R}^n)$ *be a family of functions which are affine on the interior of each simplex* T_ε^i. *We consider the two piecewise constant functions* $\underline{u}_\varepsilon, \overline{u}_\varepsilon \in L^1(\mathbb{R}^n)$, *defined on every simplex* T_ε^i *by*

$$\underline{u}_\varepsilon := \text{ess-inf}\,_{T_\varepsilon^i} u_\varepsilon, \qquad \overline{u}_\varepsilon := \text{ess-sup}\,_{T_\varepsilon^i} u_\varepsilon.$$

Then $u_\varepsilon \to u$ *in* $L^1(\mathbb{R}^n)$ *implies* $\underline{u}_\varepsilon, \overline{u}_\varepsilon \to u$ *in* $L^1(\mathbb{R}^n)$. *The same holds if* L^1 *convergence is replaced by* L^1_{loc} *convergence or local convergence in measure.*

Proof. We begin by proving 1). With fixed ε and $i \in \mathbb{N}$ let $u_{\varepsilon,i} : T_\varepsilon^i \to \mathbb{R}$ be the unique continuous extension of $u_{\varepsilon|\mathrm{int}\,(T_\varepsilon^i)}$ to the closed simplex T_ε^i and let $y_{\varepsilon,i}^-, y_{\varepsilon,i}^+$ be two vertices of T_ε^i such that

$$u_{\varepsilon,i}(y_{\varepsilon,i}^-) = \min_{T_\varepsilon^i} u_{\varepsilon,i} \qquad u_{\varepsilon,i}(y_{\varepsilon,i}^+) = \max_{T_\varepsilon^i} u_{\varepsilon,i}.$$

If $u_{\varepsilon,i}$ is constant on T_ε^i, we suppose in addition that $y_{\varepsilon,i}^- \neq y_{\varepsilon,i}^+$ and define $\tau_\varepsilon^i := y_{\varepsilon,i}^+ - y_{\varepsilon,i}^-$. Let A_ε^i the n-simplex homothetic to T_ε^i of ratio $\frac{1}{3}$ and with homothety center in $y_{\varepsilon,i}^-$ and let $B_\varepsilon^i := A_\varepsilon^i + \frac{1}{3}\tau_\varepsilon^i$. It is easy to see that $B_\varepsilon^i \subset T_\varepsilon^i$.

We will proceed as follows: first we will construct a function v_ε on $B_\varepsilon := \cup_{i\in\mathbb{N}} B_\varepsilon^i$ which is affine on the interior of each set B_ε^i and whose distance from u in the $L^1(B_\varepsilon)$-norm tends to 0. Afterwards we will estimate two particular convex combinations of $\underline{u}_\varepsilon$ and $\overline{u}_\varepsilon$ that will allow us to estimate the oscillation $\overline{u}_\varepsilon - \underline{u}_\varepsilon$. Let v_ε be defined in $x \in \mathrm{int}\,(B_\varepsilon^i)$ as $u_\varepsilon\left(x - \frac{1}{3}\tau_\varepsilon^i\right)$. In order to prove that

$$\lim_{\varepsilon\to 0} \int_{B_\varepsilon} |v_\varepsilon - u|\, dx = 0, \tag{A.1}$$

we observe that

$$\lim_{\varepsilon\to 0} \sum_i \int_{A_\varepsilon^i} \left| u(x) - u\left(x + \frac{1}{3}\tau_\varepsilon^i\right)\right| dx = 0.$$

This would be trivial if u were continuous with compact support, and can be proved by a standard approximation argument for a general $u \in L^1(\mathbb{R}^n)$. Then we have

$$\sum_i \int_{B_\varepsilon^i} |v_\varepsilon(x) - u(x)|\, dx = \sum_i \int_{A_\varepsilon^i} \left| u_\varepsilon(x) - u\left(x + \frac{1}{3}\tau_\varepsilon^i\right)\right| dx$$
$$\leq \sum_i \int_{A_\varepsilon^i} |u_\varepsilon(x) - u(x)|\, dx + \sum_i \int_{A_\varepsilon^i} \left| u(x) - u\left(x + \frac{1}{3}\tau_\varepsilon^i\right)\right| dx \to 0,$$

which proves (A.1).

We note that $\frac{2}{3}y^-_{\varepsilon,i}+\frac{1}{3}y^+_{\varepsilon,i}$ is a maximum point of $u_{\varepsilon,i}$ on A^i_ε and a minimum point of $u_{\varepsilon,i}$ on B^i_ε. This gives

$$v_\varepsilon \leq \frac{2}{3}u_\varepsilon(y^-_{\varepsilon,i})+\frac{1}{3}u_\varepsilon(y^+_{\varepsilon,i}) \leq u_\varepsilon \qquad \text{on } B^i_\varepsilon.$$

Hence

$$\left|\frac{2}{3}u_\varepsilon(y^-_{\varepsilon,i})+\frac{1}{3}u_\varepsilon(y^+_{\varepsilon,i})-u\right| \leq \max\{|u_\varepsilon-u|,|v_\varepsilon-u|\} \qquad \text{on } B^i_\varepsilon,$$

and

$$\lim_{\varepsilon\to 0}\sum_i \int_{B^i_\varepsilon}\left|\frac{2}{3}\underline{u}_\varepsilon+\frac{1}{3}\overline{u}_\varepsilon-u\right|\,dx=0.$$

By a similar argument, we may prove also that

$$\lim_{\varepsilon\to 0}\sum_i \int_{B^i_\varepsilon}\left|\frac{1}{3}\underline{u}_\varepsilon+\frac{2}{3}\overline{u}_\varepsilon-u\right|\,dx=0.$$

Since $\overline{u}_\varepsilon$ and $\underline{u}_\varepsilon$ are constant on each simplex T^i_ε, and $|B^i_\varepsilon|=3^{-n}|T^i_\varepsilon|$, we conclude that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}|\overline{u}_\varepsilon-\underline{u}_\varepsilon|\,dx=3^n\sum_i \lim_\varepsilon \int_{B^i_\varepsilon}|\overline{u}_\varepsilon-\underline{u}_\varepsilon|\,dx=0$$

and, finally, by using the following inequalities,

$$|\underline{u}_\varepsilon-u|\leq|\underline{u}_\varepsilon-u_\varepsilon|+|u_\varepsilon-u|\leq|\underline{u}_\varepsilon-\overline{u}_\varepsilon|+|u_\varepsilon-u|,$$

we get that $\underline{u}_\varepsilon\to u$ in $L^1(\mathbb{R}^n)$ and analogously for $\overline{u}_\varepsilon$.

If $u_\varepsilon\to u$ in $L^1_{\rm loc}(\mathbb{R}^n)$ or locally in measure, it suffices to repeat the constructions and reasonings above, localizing each integral. For the local convergence in measure, one has also to replace the L^1 distance with a distance inducing the convergence in measure on a bounded set. □

Remark A.2 Note that the functions $\overline{u}_\varepsilon,\underline{u}_\varepsilon$ do not coincide with the piecewise-constant ones considered in the previous sections. Nevertheless, from the proposition above, one can easily deduce that the convergence (L^1, L^1_{loc} or locally in measure) of piecewise-affine functions considered in Lemma 6.1 implies the convergence of the piecewise-constant ones . Indeed, if we consider the piecewise-constant function w_ε defined on the cell $\alpha+[-\varepsilon/2,\varepsilon/2)^2$, $\alpha\in\varepsilon\mathbb{Z}^2$, as $u_\varepsilon(\alpha)$ for a given family of functions (u_ε) which are affine on each triangle of the form $\alpha+\varepsilon T^\pm$, then it is easy to see that $\underline{u}_\varepsilon\leq w_\varepsilon\leq\overline{u}_\varepsilon$. Thus, by the previous proposition the convergence of u_ε (L^1, L^1_{loc} or locally in measure) implies the same convergence of w_ε. Finally it suffices to note that the piecewise-constant functions considered in Lemma 6.1 can be written as $w_\varepsilon(x+\tau_\varepsilon)$ with $|\tau_\varepsilon|\leq c\varepsilon$.

Acknowledgments Our attention on this problem was drawn by Andrea Braides, after some remarks by Lev Truskinovsky. We also thank Luigi Ambrosio and Gianni Dal Maso for some useful remarks. This work is part of CNR Research Project "Equazioni Differenziali e Calcolo delle Variazioni". Roberto Alicandro and Maria Stella Gelli gratefully acknowledge the hospitality of Scuola Normale Superiore, Pisa, and Matteo Focardi that of SISSA, Trieste.

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