A stochastic selection principle in case of fattening for curvature flow

Nicolas Dirr *
Max–Planck–Institute for Mathematics in the Sciences, Leipzig

Stephan Luckhaus
Fakultät für Mathematik und Informatik, Universität Leipzig

Matteo Novaga †
Dipartimento di Matematica, Università di Pisa

Abstract

Consider two disjoint circles moving by mean curvature plus a forcing term which makes them touch with zero velocity. It is known that the generalized solution in the viscosity sense ceases to be a curve after the touching (the so-called fattening phenomenon). We show that after adding a small stochastic forcing $\epsilon \mathrm{d}W$, in the limit $\epsilon \to 0$ the measure selects *two* evolving curves, the upper and lower barrier in the sense of De Giorgi. Further we show partial results for nonzero ϵ.

1 Introduction

The evolution of a hypersurface $\Sigma(t)$ in $\mathbb{R}^n$ which flows in time with normal velocity equal to the mean curvature plus a continous forcing term has attracted a lot of attention since Brakke defined in [8] a notion of weak solution. Weak solutions are necessary for having long time existence, because the flow starting from a smooth hypersurface might create singularities, and the smooth solution might cease to exist. In addition to Brakke's varifold-based concept of weak solution which provides existence but no uniqueness, there are other ways to define the mean curvature flow beyond singularities.

One is the variational approach developed by Almgren, Taylor and Wang [1] and Luckhaus, Sturzenhecker [19], and its possible generalizations by means of the minimizing movements of De Giorgi [3].

Another way is to define the evolution of a function $u(t,x)$ by a degenerate parabolic PDE in such a way that each level set $\{x \in \mathbb{R}^n : \ u(t,x) = a\}$ evolves by mean curvature as long as it is a smooth hypersurface, see e.g. [10]. Exploiting the maximum principle for this PDE, one can define a generalized solution, called viscosity solution, which requires only continuity of u. However, the level sets of the viscosity solution may develop nonempty interior. This phenomenon is called *fattening*, and it happens precisely when the solutions of Brakke type are nonunique, see [14].

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A third approach, the so-called barrier solutions, has been introduced by E. De Giorgi [9] and developed by G. Bellettini, M. Paolini and M. Novaga [7] [5]. The idea is to consider the set bounded by the surface instead of the surface itself, and to define a unique upper and lower evolution for this set, called $M^*(t)$ and $M_*(t)$ respectively. For a brief summary of this approach see Paragraph 3.3. Fattening of the viscosity solution corresponds to $M^*(t)\setminus M_*(t)$ having nonempty interior. In the deterministic case, this approach turns out to be equivalent to the viscosity level set method.
However, fattening is thought to be a rare phenomenon in the sense that it can only happen for "few" initial surfaces. Among the levels of a given function, only a subset of measure zero can fatten at any given time t, but for general initial conditions, the meaning of "rare" is less clear.
Now consider *probabilistic* forcings, i.e the evolution

$$\mathrm{d}p(t,x)\cdot\nu(t,x) = (\kappa(t,x)+g(t))\mathrm{d}t + \epsilon \mathrm{d}W(t), \tag{1}$$

where x is a point on a fixed reference manifold, $p(t,x)$ the corresponding point on the manifold $\Sigma(t)$ at time t, κ the mean curvature and ν the outer normal at this point. W is the standard Brownian motion and ϵ a small parameter. In this way a probability measure is introduced and the conjecture that fattening is rare now takes the following form: for any regular initial surface, fattening happens with zero probability.
Let us remark that defining mean curvature flow with a stochastic forcing has turned out to be a difficult problem in itself. In [21] Yip considered the case of the noise coming from a regular random vector field, i.e. white in time but smooth in space. He used a time-step procedure and showed tightness of the resulting probability measures as well as some properties of the probability measures which are limit points of the approximating sequence. Lions and Souganidis [16], [17], [18] give a definition of viscosity solution for fully nonlinear stochastic PDEs with time dependent noise, and claim uniqueness and continuity in the initial conditions for a class of equations which covers the motion of a graph by mean curvature plus a stochastic forcing.
We avoid both approaches and show instead short-time existence for (1) in order to define upper and lower barriers in the sense of De Giorgi. We consider a particular example: two disjoint circles in $\mathbb{R}^2$ moving accordingly to (1), where g is such that they touch with zero velocity in the deterministic case (i.e. when $\epsilon = 0$). Under this assumption, fattening can occur in the deterministic case, which was first studied by Bellettini and Paolini [6]. Later Koo [15] and Gulliver and Koo [13] extended the result to the case of two touching smooth hypersurfaces of codimension 1, and gave precise upper and lower bounds on the size of the fat set.
In this paper we show that for any fixed $\epsilon > 0$ a partial nonfattening result holds (see Lemma 5.2): the fat set (if there exists one) does *not* contain a ball around the point where the circles touch, whereas in the deterministic case it contains a ball around the origin, as it was shown in [13]. Indeed, we expect that with probability 1 there is really nonfattening in a small time interval (depending on the path) after the touching time.
Moreover, we show that for $\epsilon \to 0$ the limits of the upper and lower barrier are the same, more precisely they converge in the Hausdorff distance with probability 1/2 to the deterministic upper barrier and with probability 1/2 to the deterministic lower barrier. This means the stochastic forcing selects in the limit the extremal Brakke solutions (see Theorem 3.8).
The key observation is that once the two circles touch, they have necessarily to cross, because of elementary properties of Brownian paths. But once they have crossed, the expansion coming from the mean curvature is for a short time stronger than the Brownian part, so the inner barrier contains for some time a small ball around the origin.
A result similar to ours, and obtained independently, has been announced by P. Souganidis and A. Yip.

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2 Notation

In this section we introduce some notation that we shall use in the sequel. Given two sets $A, B \subseteq \mathbb{R}^n$ we set

$$\begin{aligned} A \triangle B &:= (A \setminus B) \cup (B \setminus A), \\ \mathrm{dist}(A,B) &:= \inf_{x \in A, y \in B} |x-y|, \\ \textbf{[Hausdorff distance]} \qquad \mathrm{d}_H(A,B) &:= \sup_{x \in A} \inf_{y \in B} |x-y| + \sup_{y \in B} \inf_{x \in A} |x-y|. \end{aligned}$$

Given $E \subseteq \mathbb{R}^n$ and $\rho > 0$, we set

$$E_\rho^- := \{x \in \mathbb{R}^n : \mathrm{dist}(x, \mathbb{R}^n \setminus E) > \rho\} \qquad E_\rho^+ := \{x \in \mathbb{R}^n : \mathrm{dist}(x, E) < \rho\}.$$

For any $R > 0$, we let $B_R := \{x \in \mathbb{R}^n : |x| < R\}$.
Let $(\Omega, \sigma, \mathbb{P})$ be a probability space and $W : \Omega \to C([0,\infty))$, $\omega \to W(\cdot, \omega)$ measurable and such that $W(t)$ is a standard Brownian motion and $W(0) = 0$ almost everywhere. σ is assumed to contain sets of zero measure.
For simplicity, we shall often write ∂_t, ∂_{x_i}, ∂_{x_i,x_j} $(1 \le i,j \le n)$ instead of $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x_i}$, $\frac{\partial^2}{\partial x_i x_j}$.
Given $a, b \in \mathbb{R}$ we set $a \wedge b := \min\{a,b\}$, $a \vee b := \max\{a,b\}$.
By $f(t) = o(t)$ we mean that $\lim_{t \to 0} f(t) = 0$.

3 Setting and main results

3.1 The Existence Theorem

In order to work with the barrier solutions, we need first to show short time existence of the flow for sufficiently smooth initial surfaces.

Theorem 3.1. (Existence Theorem) *Let $\Sigma \subset \mathbb{R}^n$ be a compact embedded hypersurface of class $C^{2,\alpha}$, for some $0 < \alpha < 1$, and let $\epsilon \ge 0$. Then there exists a stopping time $T(\omega, \Sigma) > 0$, depending on the $C^{2,\alpha}$-norm of Σ, and a family of hypersurfaces $\Sigma_\omega(t)$, $t \in [0, T(\omega))$, of class $C^{2,\alpha}$ such that for any $X_0 \in \Sigma(0)$ there exists a process $X(\cdot)$ with $X(t,\omega) \in \Sigma_\omega(t)$ for $\mathbb{P}$-almost all ω which solves the following Ito equation*

$$\begin{aligned} \mathrm{d}X &= \nu(X(t,\omega),t)\kappa(X(t,\omega),t)\mathrm{d}t + \nu(X(t,\omega),t)(\epsilon \mathrm{d}W + g(t)\mathrm{d}t), \\ X(0) &= X_0, \end{aligned} \tag{2}$$

where $\kappa(t)$ and $\nu(t)$ are respectively the mean curvature and the outer normal of $\Sigma(t)$ (so we have to choose a component of $\mathbb{R}^n \setminus \Sigma$ which is "the outside"), and $g \in C^0([0,+\infty))$ is a given function.
Moreover, $\Sigma(t)$ can be approximated by hypersurfaces $\Sigma_\delta(t)$ which solve (1) with forcing term equal to $g + \frac{d}{dt} W_\delta$, where W_δ is a smooth function which converges for almost all ω to the Brownian motion in $C^{\frac{\alpha}{2}}([0,\tau])$ (e.g. the convolution with a mollifier).

We defer the proof to Section 4.

3.2 Comparison lemmas

In order to apply the barrier method, we need a maximum principle for smooth evolutions.

Lemma 3.2. *Let $\Sigma_1(t)$ and $\Sigma_2(t)$, $t \in [0,T]$, be two smooth solutions respectively of $v = \kappa + \frac{dg_1}{dt}(t)$ and $v = \kappa + \frac{dg_2}{dt}(t)$, $g_1, g_2 \in C^1([0,T])$. Assume also that*

$$\Sigma_1(0) \subseteq \Sigma_2(0), \quad \text{and} \quad (g_1(t) - g_1(0)) - (g_2(t) - g_2(0)) \geq -c(t) \quad \forall t \in [0,T] \tag{3}$$

and for some $c(t) \geq 0$, $c \in C^0([0,T])$.
For any $t \in [0,T]$, we define $D(t) := \operatorname{dist}(\Sigma_1(t), \mathbb{R}^n \setminus \Sigma_2(t))$. If $D(0) > 0$ we have

$$D(t) \geq D(0) - c(t), \qquad \forall t \in [0, T \wedge T_c), \tag{4}$$

where $T_c := \inf\{t \in [0,T] : \ c(t) > D(0)\}$. In particular, if $c = 0$ then $\Sigma_1(t) \subseteq \Sigma_2(t)$ for any $t \in [0,T]$.

Proof. Fix $t \geq 0$ such that $D(t) > 0$ and assume that $D(t) = |y_1 - y_2|$ for some $y_1 \in \Sigma_1$, $y_2 \in \Sigma_2$. We compute

$$\lim_{\delta \to 0^+} \frac{D(t+\delta) - D(t)}{\delta} = \tag{5}$$

$$\min_{\substack{y_1 \in \Sigma_1, y_2 \in \Sigma_2 \\ \text{s.t. } D(t) = |y_1 - y_2|}} \kappa_{\Sigma_1}(y_1) + \frac{d\omega_1}{dt}(t) - \kappa_{\Sigma_2}(y_2) - \frac{d\omega_2}{dt}(t) \geq \frac{d\omega_1}{dt}(t) - \frac{d\omega_2}{dt}(t),$$

since $\kappa_{\Sigma_1}(y_1) \geq \kappa_{\Sigma_2}(y_2)$ as a consequence of the fact that the function $|y - z|$, $y \in \Sigma_1$, $z \in \Sigma_2$, has a minimum for $y = y_1$, $z = y_2$. Integrating (5) we get the thesis. $\square$

If we approximate the Brownian motion with smooth functions as in Theorem 3.1, from Lemma 3.2 we obtain the following result.

Corollary 3.3. *Let $\Sigma_{\omega_1}(t)$ and $\Sigma_{\omega_2}(t)$, $t \in [0,T]$, be two solution of (2) and assume that $W(t, \omega_1) - W(t, \omega_2) > -c(t)$ for some $c(t)$. Then, if we define the function $D(\cdot)$ as in Lemma 3.2, the inequality (4) holds.*

3.3 Definition of minimal barrier

We recall the definition of barrier and minimal barrier given by De Giorgi in [9].

Definition 3.4. *Let ω be a path such that the procedure of Section 4 works for every compact $C^{2,\alpha}$ initial surface, i.e. a path which is Hölder-continous. Let $\Sigma_\omega(t)$, $t \in [a,b]$ be as in theorem 3.1, and let $\Lambda_\omega(t)$, $t \in [a,b]$ be such that $\Sigma_\omega(t) = \partial \Lambda_\omega(t)$. Let $\mathcal{F}_\omega$ be the family of all such regular set-valued functions. We say that for such a path ω of the Brownian motion a function $\phi_\omega : [t_0, +\infty) \to \mathcal{P}(\mathbb{R}^n)$, $t_0 \in \mathbb{R}$, is a* barrier *with respect to $\mathcal{F}_\omega$ if for any $\Sigma(t) \in \mathcal{F}_\omega$, $t \in [a,b] \subset \mathbb{R}$, $\Sigma(a) \subseteq \phi_\omega(a)$ implies $\Sigma(b) \subseteq \phi_\omega(b)$.*

In the following we denote by $\mathcal{B}(\mathcal{F}_\omega, t_0)$ the class of all barriers with respect to $\mathcal{F}_\omega$ starting at time t_0.

Definition 3.5. *Let $E \subseteq \mathbb{R}^n$ and $t_0 \in \mathbb{R}$. The* minimal barrier *$\mathcal{M}_\epsilon(E, t_0, \omega) : [t_0, +\infty) \to \mathcal{P}(\mathbb{R}^n)$ for the path ω starting from E at time t_0 is defined as:*

$$\mathcal{M}_\epsilon(E, t_0, \omega)(t) := \bigcap \Big\{ \phi_\omega(t) : \ \phi_\omega \in \mathcal{B}(\mathcal{F}_\omega, t_0), \ \phi_\omega(t_0) \supseteq E \Big\}.$$

We also define the upper *and* lower regularized barrier *as*

$$\mathcal{M}_{*,\epsilon}(E, t_0, \omega)(t) := \bigcup_{\rho > 0} \mathcal{M}_\epsilon(E_\rho^-, t_0, \omega)(t) \quad \mathcal{M}_\epsilon^*(E, t_0, \omega)(t) := \bigcap_{\rho > 0} \mathcal{M}_\epsilon(E_\rho^+, t_0, \omega)(t).$$

It is easy to check, see e.g. [5], that the minimal barrier $\mathcal{M}(E, t_0, \omega)$ (as well as the upper and lower regularized barriers) satisfies a semigroup property in time, i.e.

$$\mathcal{M}(E, t_0, \omega)(t) = \mathcal{M}(\mathcal{M}(E, t_0, \omega)(s), s, \omega), \qquad t_0 \leq s \leq t. \tag{6}$$

In the following, when it is clear from the context, we drop the explicit dependence of $\mathcal{M}$ (resp. $\mathcal{M}^*$, $\mathcal{M}_*$) on (t_0, ω).

Proposition 3.6. *Let $E \subseteq \mathbb{R}^n$ and let $\Sigma(t)$, $t \in [0, T]$, be a family of compact hypersurfaces of class $C^{2,\alpha}$, which evolve accordingly to (1) with $\epsilon = 0$ (i.e. deterministically). Assume also that $\Sigma(0) \subset \mathrm{int}(E)$ and let $D = \mathrm{dist}(\Sigma(0), \partial E) > 0$. Then, if we let $T_\omega > 0$ be the maximal time such that $\Sigma(t) \subseteq \mathcal{M}_{*,\epsilon}(E, \omega)(t)$ for $t \in [0, T_\omega]$, we have*

$$T_\omega \geq \inf\{t \in [0, T] : \epsilon |W(s, \omega) \wedge 0| \geq D\}. \tag{7}$$

Proof. Let $F \subset E$ be the open set such that $\Sigma(0) = \partial F$, and let $\Sigma_\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x, F) = \delta\}$, for $\delta > 0$. Then, if δ is small enough, Σ_δ is an hypersurface of class $C^{2,\alpha}$ still contained in $\mathrm{int}(E)$. Let now $\Sigma_\delta(t)$, $t \in [0, \tau_\omega]$, be the solution of (2) given by Theorem 3.1 such that $\Sigma_\delta(0) = \Sigma_\delta$. By definition we have $\mathcal{M}_{*,\epsilon}(E)(t) \supseteq \Sigma_\delta(t)$ for all $t \in [0, \tau_\omega]$.
Moreover, by Corollary 3.3 we also have $\mathrm{dist}(\Sigma(t), \Sigma_\delta(t)) \geq \delta - \epsilon \sup_{[0,t]} |W(s, \omega) \wedge 0|$. Therefore

$$\mathrm{dist}(\Sigma(t), \partial\mathcal{M}_{*,\epsilon}(E)(t)) \geq D - \epsilon \sup_{[0,t]} |W(s, \omega) \wedge 0|, \qquad t \in [0, \tau_\omega].$$

Iterating the procedure and considering $\Sigma(\tau_\omega)$ instead of $\Sigma(0)$ we get the thesis. □

The next result follows immediately from Proposition 3.6, by passing to the complementary.

Corollary 3.7. *Let E, $\Sigma(t)$, D as in Proposition 3.6 and let $F(t)$ be a family of open sets such that $\Sigma(t) = \partial F(t)$. Assume that $E \subset \mathrm{int}(F)$. Then, letting $T_\omega > 0$ be the maximal time such that $F(t) \supseteq \mathcal{M}^*_\epsilon(E, \omega)(t)$, $t \in [0, T_\omega)$, inequality (7) holds with $\sup_{[0,t]} |W(s, \omega) \vee 0|$.*

3.4 The Limit Theorem

Throughout this paragraph, we will restrict ourselves to the case of curves in $\mathbb{R}^2$, i.e. we consider the case of two disjoint circles. Fix $L > r_0 > 0$, $\epsilon \geq 0$ and let $R_\epsilon(t)$ be the process which solves

$$\mathrm{d}R_\epsilon = \left(-\frac{1}{R_\epsilon} + g(t)\right)\mathrm{d}t + \epsilon \mathrm{d}W(t) \quad t \in (0, T_\epsilon(\omega)), \qquad R_\epsilon(0) = r_0, \tag{8}$$

where $[0, T_\epsilon(\omega))$ is the maximal interval of definition. For $\epsilon = 0$, this is of course a deterministic ODE whose solution and maximal interval of definition do not depend on ω.
Let Σ_0 be the union of two two disjoint circles of radius r_0, whose centers are in $x_1 = (-L, 0)$ and $x_2 = (+L, 0)$ respectively.
Define the stopping time $t^*_\epsilon(\omega) := \inf\{t < T_\epsilon : \ R_\epsilon(t) \geq L\}$. Then for any $0 \leq t < t^*_\epsilon$ we have $\Sigma_t = B_{R_\epsilon(t)}(x^+) \cup B_{R_\epsilon(t)}(x^-)$.
Let g be of class C^1 in a neighbourhood of t^* such that for $\epsilon = 0$ the circles touch in such a way that the second derivative of $R_0(t)$ is negative for $t = t^*_0$. Let $M^*_\epsilon(t)$ and $M_{*,\epsilon}(t)$ be the upper and lower regularized barrier for the flow starting from Σ_0 (see Definition 3.5). Let also $T_0 > 0$ be the (deterministic) time at which $M_{*,0}$ shrinks to two points ($M_{*,0}$ are two circles shrinking after touching, whereas M^*_0 are two circles joining into a bean-ahaped figure and finally shrinking to a point).

Further let

$$
\begin{aligned}
S_\epsilon^+ &:= \{\omega: \ t^*_{\widetilde{\epsilon}}(\omega) < \infty \text{ for all } \widetilde{\epsilon} \le \epsilon\} \\
S_\epsilon^- &:= \{\omega: \ t^*_{\widetilde{\epsilon}}(\omega) = \infty \text{ for all } \widetilde{\epsilon} \le \epsilon\}
\end{aligned}
$$

Moreover we define

$$
\begin{aligned}
S^* := \Big\{ \lim_{\epsilon\to 0} & \Big[\mathrm{d}_H(\mathcal{M}^*_\epsilon(\Sigma_0,\omega)(t), \mathcal{M}^*_0(\Sigma_0)(t)) \\
&+ \ \mathrm{d}_H(\mathcal{M}_{*,\epsilon}(\Sigma_0,\omega)(t), \mathcal{M}^*_0(\Sigma_0)(t))\Big] = 0 \text{ for all } t \in [0, T_0)\Big\}, \\
S_* := \Big\{ \lim_{\epsilon\to 0} & \Big[\mathrm{d}_H(\mathcal{M}^*_\epsilon(\Sigma_0,\omega)(t), \mathcal{M}_{*,0}(\Sigma_0)(t)) \\
&+ \ \mathrm{d}_H(\mathcal{M}_{*,\epsilon}(\Sigma_0,\omega)(t), \mathcal{M}_{*,0}(\Sigma_0)(t))\Big] = 0 \text{ for all } t \in [0, T_0)\Big\}.
\end{aligned}
$$

Now we are able to formulate our main theorem.

Theorem 3.8. (Limit Theorem) *We have $\mathbb{P}(S^*) = \frac{1}{2}$ and $\mathbb{P}(S_*) = \frac{1}{2}$. As the two sets are disjoint, this means that almost surely the evolution converges pointwise to one of the two extremal deterministic solutions and $\lim_{\epsilon\to 0} \mathbb{P}(S^* \triangle S_\epsilon^+) = 0$, $\lim_{\epsilon\to 0} \mathbb{P}(S_* \triangle S_\epsilon^-) = 0$.*

The theorem will be the consequence of Lemma 3.9, Proposition 3.10 and Proposition 3.12 below.
First we will approximate S_ϵ^- and S_ϵ^+ for $\epsilon \to 0$ by sets which do not depend on ϵ and have probability $\frac{1}{2}$ by construction.

Lemma 3.9. *We have*

$$
\lim_{\epsilon\to 0} \mathbb{P}(S_\epsilon^-) = \lim_{\epsilon\to 0} \mathbb{P}(S_\epsilon^+) = \frac{1}{2}. \tag{9}
$$

Proof. We shall prove that there exists a centered Gaussian random variable $R_1(t,\omega)$ such that

$$
\lim_{\epsilon\to 0} \left(\mathbb{P}(S_\epsilon^- \triangle [R_1(t_0^*,\omega) > 0]) + \mathbb{P}(S_\epsilon^+ \triangle [R_1(t_0^*,\omega) < 0])\right) = 0,
$$

which gives (9).
The idea is to expand the stochastic ODE for R_ϵ in powers of ϵ, following Wentzel-Freidlin [12]. As the equation gets singular for $R_\epsilon \to 0$, we have to modify it by a smoothed version near this singularity.
Choose a $0 < R_{\min} << r_0$ and a smooth function $b(R): \mathbb{R} \to \mathbb{R}$, such that $b(r) = -\frac{1}{r}$ on $[R_{\min}, +\infty)$, and $b(r) = \mathit{const}$ on $(-\infty, R_{\min}/2)$ and replace $-\frac{1}{r}$ in (8) by $b(r)$, which gives

$$
\mathrm{d}\widetilde{R}_\epsilon = b(\widetilde{R}_\epsilon)\mathrm{d}t + g\mathrm{d}t + \epsilon\mathrm{d}W, \qquad \widetilde{R}_\epsilon(0) = r_0. \tag{10}
$$

Let R_ϵ be the solution of (8) and $\widetilde{R}_\epsilon$ the solution of (10), then R_ϵ and $\widetilde{R}_\epsilon$ coincide for $t \le \inf\{s : R_\epsilon(s) < R_{\min}\}$.
Now we expand

$$
\widetilde{R}_\epsilon(t,\omega) = R_0(t) + \epsilon R_1(t,\omega) + R_{2,\epsilon}(t,\omega), \tag{11}
$$

where R_0 solves (8) with $\epsilon = 0$, R_1 solves the linear stochastic ODE

$$
\mathrm{d}R_1 = \frac{d}{dr} b(r)|_{r=R_0(t)} R_1(t)\mathrm{d}t + \mathrm{d}W, \qquad R_1(0) = 0, \tag{12}
$$

and $R_{2,\epsilon}$ is defined as the remainder. Fix $T>0$, then by Doob's L^2-inequality

$$\mathbb{E}\left(\sup_{[0,T]}|R_1(s)|^2\right)\le 4\mathbb{E}|R_1(T)|^2\le C(R_{\min},T). \tag{13}$$

Further from expanding $b(\widetilde{R}^0_\epsilon+\epsilon R_1+R_{2,\epsilon})$ in the equation (10), and using (12), we get an equation for $R_{2,\epsilon}$, where the dW-expressions cancel. From this and Gronwall's inequality, we derive

$$|R_{2,\epsilon}(t,\omega)|\le C(T_0)\epsilon^2\sup_{[0,T_0]}|R_1(s,\omega)|^2. \tag{14}$$

¿From (13) and (14) we get $\mathbb{E}\left(\sup_{[0,T]}|R_{2,\epsilon}(t)|\right)\le C(T,R_{\min})\epsilon^2$. Note in particular that if for a fixed ω, for some $\epsilon=\epsilon_0$ and for some $0<\alpha<1$, the right hand side of (14) is smaller than $\epsilon_0^{1+\alpha}$, then this holds for all $\epsilon<\epsilon_0$.
Next observe that $R_1(t)$ is Hölder-continous for any $0<\beta<1/2$. Fix such a β and some $0<\alpha<1$ and define

$$\begin{aligned} A(\alpha,\epsilon) &:= \{\omega:\|R_{2,\epsilon}(\omega)\|_{L^\infty(0,T)}>\epsilon^{1+\alpha}\},\\ A(\beta,\epsilon) &:= \{\omega:\|R_1(\omega)\|_{C^{0,\beta}(O,T)}>\epsilon^{-\frac{\beta}{8}}\},\end{aligned}$$

then by the Markov inequality $\mathbb{P}(A(\alpha,\epsilon))\le\epsilon^{1-\alpha}$, and, e.g. by the embedding of fractional Sobolev-spaces in Hölder spaces, $\lim_{\epsilon\to0}\mathbb{P}(A(\beta,\epsilon))=0$.
Set $I(\beta,\epsilon):=[t^*-\epsilon^{\frac{4-\beta}{8}},t^*+\epsilon^{\frac{4-\beta}{8}}]$ Now note that due to the nonvanishing of the second derivative of R_0 in t_0^* we have

$$\inf_{[0,T]\setminus I(\beta,\epsilon)}(L-R_0(s))>C^{-1}\epsilon^{1-\frac{\beta}{4}}.$$

Hence either $\widetilde{R}_\epsilon(s)<L$ on $[0,T]\setminus I(\beta,\epsilon)$, or we are on $A(\epsilon):=A(\alpha,\epsilon)\cup A(\beta,\epsilon)$ for ϵ small enough.
Further observe that, unless $\omega\in A(\beta,\epsilon)$, we have

$$\sup_{s\in I(\beta,\epsilon)}|R_1(s)-R_1(t_0^*)|\le(2\epsilon^{\frac{4-\beta}{8}})^\beta\epsilon^{-\frac{\beta}{8}}=2^\beta\epsilon^{\frac{\beta(3-\beta)}{8}}.$$

So we have for small ϵ

$$\{\omega:\sup_{[0,T]}R_\epsilon(s)>L\}\quad\subseteq\quad\{\omega:R_1(t_0^*)>-\epsilon^\alpha-\epsilon^{\frac{\beta(3-\beta)}{8}}\}\cup A(\epsilon). \tag{15}$$

So $\mathbb{P}(S_\epsilon^+\setminus[R_1(t_0^*,\omega)>0])=0$ for any ϵ. (Remember that the S_ϵ^+ are increasing in ϵ.)
Now assume $R_1(t_0^*,\omega)>0$, but $\omega\notin S_\epsilon^+$. Then there is some $\widetilde{\epsilon}\le\epsilon$ such that $R_{2,\widetilde{\epsilon}}(t_0^*)<-\widetilde{\epsilon}R_1(t_0^*,\omega)$, hence $\omega\in A(\widetilde{\epsilon})\cup\{0<R_1(t_0^*)<\widetilde{\epsilon}^\alpha\}$, and the probability of the right hand side clearly tends to 0 for $\epsilon\to0$. So $[R_1(t_0^*,\omega)>0]\subseteq S_\epsilon^+\cup N_\epsilon$ where $\mathbb{P}(N_\epsilon)\to0$.
As S_ϵ^+ and S_ϵ^- are disjoint and $\mathbb{P}([R_1(t_0^*,\omega)<0]\cup[R_1(t_0^*,\omega)>0])=1$, we immediately get $\mathbb{P}(S_\epsilon^-\setminus[R_1(t_0^*,\omega)<0])\to0$.
If $\omega\in[R_1(t_0^*,\omega)<0]\setminus S_\epsilon^-$, then there is $\widetilde{\epsilon}\le\epsilon$ such that $R_{\widetilde{\epsilon}}(t)>L$ for some t. By (15), $\omega\in A(\widetilde{\epsilon})\cup[-o(\epsilon)<R_1(t_0^*)<0]$, and probability of this set tends to 0.
As R_1 has symmetric Lebesgue density, the result follows. □

Proposition 3.10. *For almost any $\omega\in\bigcup_{\epsilon>0}S_\epsilon^-$ and for any $\delta>0$ we have*

$$\lim_{\epsilon\to0}\sup_{t\in[0,T_0-\delta]}\mathrm{d}_H\left(B_{R_\epsilon(t)}(x_i),B_{R_0(t)}(x_i)\right)=0\qquad i\in\{1,2\}. \tag{16}$$

Proof. We reason as in the proof of Lemma 3.9 and we choose $R_{\min}$ such that $R_0(T_0-\delta) > R_{\min}$. Now the claim follows directly from the expansion (11) and the estimates (13) and (14). $\square$

¿From the proof of Lemma 3.9 we get the following result.

Corollary 3.11. *For almost any* $\omega \in \bigcup_{\epsilon>0} S_\epsilon^+$ *we have* $\lim_{\epsilon\to 0} t_\epsilon^* = t_0^*$.

In the sequel of the paper, we will prove the following proposition, which together with Lemma 3.9 and Proposition 3.10, gives Theorem 3.8.

Proposition 3.12. *For almost any* $\omega \in \bigcup_{\epsilon>0} S_\epsilon^+$ *we have*

$$\lim_{\epsilon\to 0}\left(\mathrm{d}_H(\mathcal{M}_\epsilon^*(\Sigma_0)(t,\omega), \mathcal{M}_0^*(\Sigma_0)(t)) + \mathrm{d}_H(\mathcal{M}_{*,\epsilon}(\Sigma_0)(t,\omega), \mathcal{M}_0^*(\Sigma_0)(t))\right) = 0. \tag{17}$$

Proposition 3.12 means that, on the set of paths where the two circles cross, the upper and lower barrier have the same limit, which coincides with the deterministic upper barrier (i.e. two circles merging into a bean-shaped curve). The proof of Proposition 3.12 is postponed to the end of Section 5.

4 Proof of the Existence Theorem

In order to define a pathwise solution of (2) for such paths that $W(t,\omega)$ is Hölder-continous, we reason as in [20, Section 8.5.4]. Without loss of generality, we shall assume $\epsilon = 1$ and $g = 0$ in (2). Indeed, for $g \in C^0$ the path $W(t,\omega)$ is Hölder-continous if and only if $\widetilde{W}(t) := \epsilon W + \int_0^t g(s)ds$ has this property.
In the following we denote by $\|\cdot\|$ the $C^{2,\alpha}$-norm. Let $D \supset \Sigma$ be a bounded open set with boundary of class $C^{2,\alpha}$ and such that $\mathrm{d}_0 \in C^{2,\alpha}(D)$, where d_0 is the signed distance function from Σ. Reasoning as in [11], [20] we can write the evolution equation (1) in the form

$$\begin{aligned}
\partial_t d &= \left(\sum_{j=1}^n \frac{\lambda_j}{1-d\lambda_j}\right)\mathrm{d}t + \mathrm{d}W = f(d, D^2 d)\mathrm{d}t + \mathrm{d}W && \text{in } D\times(0,T)\\
|\nabla d|^2 &= 1 && \text{on } \partial D\times(0,T)\\
d(0) &= \mathrm{d}_0 && \text{in } D\times\{0\},
\end{aligned}$$

where λ_j are the eigenvalues of the matrix $D^2 d$ and

$$f(u,q) = \mathrm{Tr}(q(I-uq)^{-1}),$$

which is analytic for $|u|$ and $|q|$ small enough.
Now we set $u(t,x) := d(t,x) - W(t)$, then u solves

$$\begin{aligned}
\partial_t u &= f(u+W(t), D^2 u) && \text{in } D\times(0,T)\\
|\nabla u|^2 &= 1 && \text{on } \partial D\times(0,T)\\
u(0) &= \mathrm{d}_0 && \text{in } D\times\{0\}.
\end{aligned} \tag{18}$$

As W is Hölder-continous, [20, Theorem 5.1.20], [20, Theorem 8.5.4] applies in the same way as it is used in in [20, Section 8.5.4] and gives a local $C^{1+\alpha/2,2+\alpha}$-solution. The existence time of this solution depends on the $C^{2,\alpha}$-norm of d_0 and on the $C^{\alpha/2}$-norm of $W(t,\omega)$. We briefly sketch the proof.
We define an operator $\Gamma : Y \subset C^{1+\alpha/2,2+\alpha}([0,\delta]\times D) \to Y : \Gamma(\widetilde{u}) := w$, where w solves the following problem

$$
\begin{aligned}
\partial_t w &= \mathcal{A}(u_0)w + \left[f(\tilde{u} + W(t), D^2\tilde{u}) - \mathcal{A}(u_0)\tilde{u}\right] \qquad \text{in } D \times (0,T) \\
2\nabla \mathrm{d}_0 \cdot \nabla v &= 1 - |\nabla\tilde{u}|^2 + 2\nabla \mathrm{d}_0 \cdot \nabla\tilde{u} \qquad \text{on } \partial D \times (0,T) \\
v(0) &= d_0 \qquad \text{in } D \times \{0\}.
\end{aligned}
$$

Y is chosen as a ball in $C^{1+\alpha/2,2+\alpha}$ of radius R around the initial value u_0, and $\mathcal{A}$ is the linearization around u_0, i.e. we take the derivatives of f at $(0, x, u_0, D^2u_0)$. The boundary condition is also derived by linearization.
Using the maximal regularity property of $\mathcal{A}$ in $C^{1+\alpha/2,2+\alpha}$, one can show that for δ and R small enough Γ is a contraction on Y. So by the Banach fixed point theorem there is a unique fixed point $u \in Y$ and it clearly solves (18), hence $d = u + W$ is the distance function.
Reasoning as in [20, Chapter 8], one can actually show that both $d(t,x)$ and $u(t,x)$ belongs to $C^k(D)$ for any $k \in \mathbb{N}$ and for any $t \in (0,T)$, where $T > \delta$ is the maximal interval of definition for a solution (observe that the spatially constant W disappears from the equation, when considering difference quotients).
In order to prove that the solution is a signed distance function it remains to show that $|\nabla u| = 1$ in D. Since $u(\cdot,t) \in C^{3+\alpha}$ for any $t \in (0,T)$, $\partial_{x_i}u$ solves on any D' s.t. $\overline{D}' \subset D$

$$
\partial_t \underbrace{\partial_{x_i}u}_{=\partial_{x_i}d} = \partial_{x_i}[f(d, D^2d)].
$$

Hence $w := |\nabla d|^2 - 1$ solves the same parabolic equation as in the deterministic case and we conclude as in [20] that $|\nabla u| = |\nabla d| = 1$ on D.
Consider now the stochastic ODE in the Ito sense

$$
\begin{aligned}
\mathrm{d}X(t) &= -f(d(t,X(t)), D^2d(t,X(t)))\nabla d(t,X(t))\mathrm{d}t - \nabla d(t,X(t))\mathrm{d}W, \\
X(t_0) &= x_0 \in \Sigma_{t_0}.
\end{aligned}
$$

As $D^2d = D^2u$ is Lipschitz in x, this is uniquely solvable for short times.
In order to show that $X(s)$ preserves the zero level of d, have to show that $0 = u(t,X(t)) - u(t_0,x_0) + W(t) - W(t_0)$. We can apply the Ito-formula to u, which is of class C^1 in time. Thus, using that derivatives of d and u are the same:

$$
\begin{aligned}
u(t,X(t)) &- u(t_0,x_0) = \int_{t_0}^{t} \mathrm{d}(u(s,X(s))) \\
&= \int_{t_0}^{t} \frac{1}{2}\partial_{x_i x_j}u(s,X(s))\,\partial_{x_i}u(s,X(s))\partial_{x_j}u(s,X(s))\mathrm{d}s \\
&+ \int_{t_0}^{t} \left[f(d(s,X(s)), D^2d(s,X(s)))\mathrm{d}s + \nabla u(s,X(s))\mathrm{d}X(s)\right]
\end{aligned}
$$

Using the fact $|\nabla u| = 1$ and the definition of $\mathrm{d}X$, the second integral is exactly $-(W(t) - W(t_0))$. The fact that $|\nabla u| = 1$ implies

$$
\frac{1}{2}D^2u(t,X(t)) \cdot \nabla u(t,X(t)) = \frac{1}{4}\nabla|\nabla u|^2 = 0,
$$

so the claim is shown.
In order to show convergence of evolutions forced by pathwise smooth approximations W_δ of the noise W, we need an estimate for the difference of two solutions starting from different but close initial values and (formally) forced by different Hölder- continous functions $\mathrm{d}W_1$ and $\mathrm{d}W_2$:

Lemma 4.1. *Let u_i, $i=1,2$ be two solutions of (18) with two different integrated forcings W_1 and W_2 with $W_1(0)=W_2(0)=0$, and let them start from two different initial values $u_1(0)$, $u_2(0)$, which fulfill $|\nabla u_i(0)|^2=1$ and assume further that ∇u_i is not tangent to the boundary.*
Then, there are positive constants C_1, C_2, τ, where C_1, C_2 depend only on the $C^{2,\alpha}$-norm of ∂D and $(\|u_1(0)\|_{C^{2,\alpha}}+\|u_2(0)\|_{C^{2,\alpha}})$ and τ depends also on $|W_1|+|W_2|$, such that

$$\sup_{[0,\tau]}\|u_1(t,x)-u_2(t,x)\|_{C^{2,\alpha}}\leq C_2\Big(\|u_1(t,0)-u_2(t,0)\|_{C^{2,\alpha}}+\|W_1-W_2\|_{C^{\frac{\alpha}{2}}([0,\tau])}\Big).$$

Proof. Let $v:=u_1-u_2$, then v solves

$$\begin{aligned}
\partial_t v = &\ \textstyle\sum_{ij} a_{ij}(t,x)\partial_{x_ix_j}v+c(t,x)(v+W_1-W_2) && \text{in } D\times(0,T)\\
\textstyle\sum \beta_i(t,x)\partial_{x_i}v = &\ 0 && \text{on } \partial D\times(0,T)\\
v(0) = &\ u_1(0)-u_2(0) && \text{in } D\times\{0\},
\end{aligned}$$

where

$$\begin{aligned}
a_{ij}(t,x) &= \int_0^1 \frac{\partial f}{\partial q_{ij}}\big[\sigma(u_1(t,x)+W_1(t),D^2u_1(t,x))\\
&\quad +(1-\sigma)(u_2(t,x)+W_2(t),D^2u_2(t,x))\big]\mathrm{d}\sigma\\
c(t,x) &= \int_0^1 \frac{\partial f}{\partial u}\big[\sigma(u_1(t,x)+W_1(t),D^2u_1(t,x))\\
&\quad +(1-\sigma)(u_2(t,x)+W_2(t),D^2u_2(t,x))\big]\mathrm{d}\sigma\\
\beta_i(t,x) &= \partial_{x_i}u_1(t,x)+\partial_{x_i}u_2(t,x),
\end{aligned}$$

since $(\nabla u_1+\nabla u_2)(\nabla u_1-\nabla u_2)=|\nabla u_1|^2-|\nabla u_2|^2=1-1=0$. We can find a τ as in the statement of the lemma such that on $[0,\tau]$ the boundary condition is nontangential and the $C^{(1+\alpha)/2,1+\alpha}$-norm of β_i and the $C^{\alpha/2,\alpha}$-norms of a_{ij}, c are bounded. Further the initial value fulfills the boundary condition (compatibility).
The result now follows from [20, Theorem 5.1.22] (optimal regularity for time-dependent coefficients). ☐

The following corollary is a straightforward application of the previous lemma to distance functions, provided the surfaces are so close that we can find a *common* domain D for the two distance functions.

Corollary 4.2. *Let $\Sigma_{0,i}, i=1,2$ be two $C^{2,\alpha}$-hypersurfaces where Σ_1 evolves as in (1), whereas for Σ_2 the forcing $\epsilon \mathrm{d}W$ in (1) has been replaced by a smooth forcing ϵg_2. Let also $\mathrm{d}_i(\cdot,t)$ be the signed distance function from $\Sigma_i(t)$. Then, there are positive constants C_1, C_2, τ, where C_1, C_2 depend only on $(\|\mathrm{d}_1(0)\|_{C^{2,\alpha}}+\|\mathrm{d}_2(0)\|_{C^{2,\alpha}})$ and τ depends also on $|\epsilon W+\int g|+|\int(g+\epsilon g_2)|$, such that if $\mathrm{dist}(\Sigma_{0,1},\Sigma_{0,2})\leq C_1$ then*

$$\sup_{[0,\tau]}\|\mathrm{d}_1(t,x)-\mathrm{d}_2(t,x)\|_{C^{2,\alpha}}\leq C_2\left(\|\mathrm{d}_1(t,0)-\mathrm{d}_2(t,0)\|_{C^{2,\alpha}}+\left\|\epsilon W(t)-\int_0^t\epsilon g_2(s)\mathrm{d}s\right\|_{C^{\frac{\alpha}{2}}([0,\tau])}\right).$$

Lemma 4.3. *Assume a forcing of the type $g(t)+\epsilon \mathrm{d}W$. If the initial surface Σ_0 is symmetric under rotation around the x_1-axis, then so is $\Sigma(t)$. If $\Sigma(t)$ is on a open set $(t_1,t_2)\times\Omega'$ obtained by rotating the graph of a function $h(t,x_1)$ around the x_1-axis, then h solves for all x_1 and a.a. ω*

$$\mathrm{d}h=\left(h''\,\frac{1+\frac{\epsilon^2}{2}h'^2}{1+h'^2}-\frac{n-2}{h}+\sqrt{1+h'^2}g(t)\right)\mathrm{d}t+\epsilon\sqrt{1+h'^2}\,\mathrm{d}W, \tag{19}$$

which corresponds to the Stratonovich equation

$$\partial h=\left(\frac{h''}{1+h'^2}-\frac{n-2}{h}+\sqrt{1+h'^2}g(t)\right)\mathrm{d}t+\epsilon\sqrt{1+h'^2}\partial W, \tag{20}$$

where h' denotes differentiation with respect to x_1.

Proof. The first claim follows easily from the fact that the eigenvalues of D^2u are invariant under orthogonal transformations. So the distance function $d(t,x)$ depends only on t, $r=\sqrt{x_2^2+\ldots+x_n^2}$ and on x_1.
Let $G(t,r,x_1):=d(t,r(x),x_1)-\epsilon W(t)$, then this is differentiable in time hence for any x_1 the Ito-formula can be applied to $G(t,\widetilde{h}(t,x_1),x_1)$, where the process $\widetilde{h}(t,x_1)$ solves for any fixed x_1 the following Ito-equation

$$\mathrm{d}\widetilde{h}(t,x_1)=-\left(\frac{\epsilon^2G_{rr}}{2G_r^3}+\frac{f(G+W,D^2G)}{G_r}\right)(t,\widetilde{h}(t,x_1),x_1)\mathrm{d}t-\frac{\epsilon}{G_r}(t,\widetilde{h}(t,x_1),x_1)(\mathrm{d}W+g),$$

where $u_r:=\partial_r u$. Hence we have $\mathrm{d}G=-\mathrm{d}W$. This means $d(t,\widetilde{h}(t,x),x)=0$ on the time interval $[0,T]$ which implies the relations

$$\begin{aligned}(G_r,G_{x_1}) &= \sqrt{1+(h')^2}^{\,-1}(-1,h')\\ \kappa &= G_{x_1x_1}+G_{rr}+G_r\frac{n-2}{r}\\ G_{rr} &= G_{x_1x_1}(h')^2.\end{aligned}$$

(Here we used that $|(G_r,G_{x_i})|^2(\widetilde{h}(x_1),x_1)=0$ to get a relation between G_{rx_1}, G_{rr} and $G_{x_1x_1}$.) All derivatives are taken at $(t,\widetilde{h}(t,x_1),x_1)$. This implies

$$\frac{\epsilon^2G_{rr}}{2G_r^3}=\frac{1}{2}(h'\epsilon)^2\left(\frac{\kappa}{G_r}-\frac{n-2}{r}\right),$$

so we get (19). □

5 The case of touching circles

In this Section we set $n=2$, and we consider the case when the two evolving circles collide at a time $t_\epsilon^*<+\infty$.
As first result, we shall show in Lemma 5.2 that the minimal barrier $M_{*,\epsilon}$ starting from two touching circles contains an expanding ball with center in the origin. This means that the expansive tendency of the large curvature at the origin is stronger than the driving noise, which is not able to bring the set back to the origin.

In order to show that the minimal barrier contains such a ball, it is enough to show that a sequence of barriers starting from a family of sets $\{\Lambda_n\}_n$, having smooth boundary and approximating the two touching circles, contains a ball which does not depend on n.
Let Λ_n, $n \in \mathbb{N}$, be a family of compact connected sets (beans) with smooth boundary such that:

1. $\Lambda_n \subset \Lambda_m$, if $n > m$;
2. $\bigcap_n \Lambda_n = \Lambda_{t_0^*} := B_L(x_1) \cup B_L(x_2)$, $\mathrm{dist}(\Lambda_n, B_L(x_1) \cup B_L(x_2)) = \mathrm{d}_n > 0$;
3. $(x, y) \in \Lambda_n$ iff $|y| \leq \phi_n(x)$, where ϕ_n is a positive function defined for $|x| \leq a_n(0)$, smooth for $|x| < a_n(0)$ and such that $\lim_{|x| \to a_n(0)} \phi_n(x) = 0$;
4. ϕ_n is even and has only one local minimum in $x = 0$ (except for $|x| = a_n(0)$);
5. the curvature of $\partial \Lambda_n$ has a maximum in $x = 0$, a minimum in $|x| = a_n(0)$, and no other critical point.

For any $n \in \mathbb{N}$, let $t_n^1 = t_n^1(\omega)$ be the maximal time of existence of the solution of in the sense of theorem 3.1 starting from Λ_n. Notice that, by Lemma 4.3, $\Lambda_n(t)$ is symmetric with respect to the coordinate axes, and can be written as a subgraph of a function $\phi_n(t, x)$, $|x| \leq a_n(t)$. Denote by $\Lambda_{n,\rho}$ and $\phi_{n,\rho}$ the surface and the graph starting from Λ_n but evolving with a smooth forcing term $f_\rho := g + \epsilon \frac{\mathrm{d}}{\mathrm{d}t} W_\rho$, where $W_\rho \in C^\infty$ and $W_\rho \to W$ in $C^{\alpha/2}$. The functions $\phi_{n,\rho}$ and $\partial_x \phi_{n,\rho}$ satisfy

$$\begin{aligned} \partial_t \phi_{n,\rho} &= \frac{\partial_{xx} \phi_{n,\rho}}{1 + (\partial_x \phi_{n,\rho})^2} + \sqrt{1 + (\partial_x \phi_{n,\rho})^2} f_\rho \\ \partial(\partial_x \phi_{n,\rho}) &= \frac{\partial_{xx} \partial_x \phi_{n,\rho}}{1 + (\partial_x \phi_{n,\rho})^2} - 2 \frac{\partial_x \phi_{n,\rho} (\partial_{xx} \phi_{n,\rho})^2}{(1 + (\partial_x \phi_{n,\rho})^2)^2} + \frac{\partial_x \phi_{n,\rho} \partial_{xx} \phi_{n,\rho}}{\sqrt{1 + (\partial_x \phi_{n,\rho})^2}} f_\rho \end{aligned} \tag{21}$$

Denote by $\kappa_{n,\rho} = \frac{\partial_{xx} \phi_{n,\rho}}{\left(\sqrt{1 + (\partial_x \phi_{n,\rho})^2}\right)^3}$ the curvature of $\partial \Lambda_{n,\rho}$. The function $\kappa_{n,\rho}$ satisfies the equation

$$\partial_t \kappa_{n,\rho} = \frac{\partial_{xx} \kappa_{n,\rho}}{1 + (\partial_x \phi_{n,\rho})^2} + \frac{\partial_x \phi_{n,\rho}}{\sqrt{1 + (\partial_x \phi_{n,\rho})^2)}} \partial_x \kappa_{n,\rho} f_\rho + \kappa_{n,\rho}^2 (\kappa_{n,\rho} + f_\rho).$$

Applying the Sturmian Theorem [4] to $\partial_x \phi_{n,\rho}$ and to $\partial_x \kappa_{n,\rho}$ we get that $\Lambda_{n,\rho}(t)$ has only one neck (i.e. a point where $\partial_x \phi_{n,\rho} = 0$) for $x = 0$, and the curvature has a global maximum in $x = 0$ and global minimum in $|x| = a_{n,\rho}(t)$. As $\Lambda_{n,\rho}(t) \to \Lambda_n(t)$ in $C^{2,\alpha}$ by corollary 4.2, the curvature κ_n of the limit Λ_n has the same properties.
Since $\Lambda_n(t) \supseteq B_{R_\epsilon(t)}(x_1) \cup B_{R_\epsilon(t)}(x_2)$, it follows that $\kappa_n(0, t)$ is greater than or equal to the curvature of the circle which is tangent to both $B_{R_\epsilon(t)}(x_1)$ and $B_{R_\epsilon(t)}(x_2)$, and whose center is equidistant from x_1 and x_2 (see Figure 1).
Hence $\kappa_n(0, t) \geq 2 \frac{R_\epsilon(t) - \phi_n(t)}{L^2 + \phi_n^2(t) - R_\epsilon^2(t)}$, which implies

$$\phi_n(0, t_2) - \phi_n(0, t_1) \geq \int_{t_1}^{t_2} \left(2 \frac{R_\epsilon(s) - \phi_n(s)}{L^2 + \phi_n^2(s) - R_\epsilon^2(s)} + g(s) \right) \mathrm{d}s + \epsilon \big(W(t_2) - W(t_1) \big). \tag{22}$$

After the initial smooth evolution ceases to exist, we construct another evolution starting from a new curve whose height above the origin is larger or equal to the height of the previous bean, and which in turn exists until a time $t_n^2 > t_n^1$. In this way we first obtain a (possibly finite) sequence of sets Λ_n^k and an increasing sequence of existence times t_n^k (Lemma 5.1), and later we show that there is a $\tau = \tau(\omega) > 0$ independent of n, such that $\bar{t}_n := \sup_k t_n^k \geq \tau$ (Lemma 5.2). In the following we let $D_R := B_R(x_1) \cup B_R(x_2)$, for any $R > 0$.

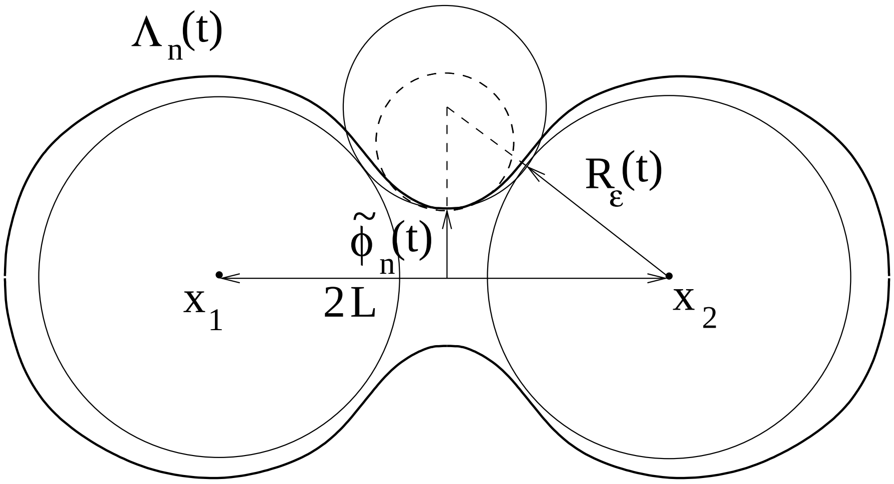

Figure 1: construction of the comparison sets Λ_n

Lemma 5.1. *For any $n \in \mathbb{N}$ and Λ_n as above, we can construct a sequence of smooth evolutions $\Lambda_n^k(t)$ for (1), defined on $[t_n^k, t_n^{k+1})$, such that*

- *$\Lambda_n^0(t) := \Lambda_n(t)$ and $D_{R_\epsilon(t)} \subseteq \Lambda_n^k(t) \subseteq \mathcal{M}_*(\Lambda_n, t_\epsilon^*)(t)$, $t \in [t_n^k, t_n^{k+1})$.*
- *Letting $\widetilde{\phi}_n^k$ be the height of Λ_n^k above the origin, the function $\widetilde{\phi}_n(s) := \widetilde{\phi}_n^k(s)$ for $s \in [t_n^k, t_n^{k+1})$ is upper semicontinous on $[t_\epsilon^*, \overline{t}_n)$, it is continuous on (t_n^k, t_n^{k+1}) and can jump only above at the times t_n^k, i.e. $\widetilde{\phi}_n(t_n^k) \geq \lim_{t \uparrow t_n^k} \widetilde{\phi}_n(t)$.*
- *When $R_\epsilon(t_n^k) \geq L$, the curvature above the origin $\kappa(t_n^k, 0)$ of $\partial\Lambda_n^k$ fulfills*

$$\frac{C(L)}{(\widetilde{\phi}_n(t_n^k))^2 \wedge \mathrm{d}_n^2} \geq \kappa(t_n^k, 0) \geq 2\frac{L - \widetilde{\phi}_n(t_n^k)}{\left(\widetilde{\phi}_n(t_n^k)\right)^2}, \tag{23}$$

  where $C(L)$ is a positive constant depending only on L and $\mathrm{d}_n := \mathrm{dist}(B_L(x_1) \cup B_L(x_2), \partial\Lambda_n) > 0$.
- *The $C^{2,\alpha}$-norm (and hence the maximal existence time) of the evolution starting from $\Lambda_n^k(t_n^k)$ depends only on the height above the origin $\widetilde{\phi}_n(t_n^k)$ and on d_n.*

Remark: As a consequence of these properties, inequality (22) holds for $\widetilde{\phi}_n$ on $[t_\epsilon^*, \overline{t})$. Indeed, between the t_n^k it is the graph of a smooth bean containing the two circles, and at t_n^k it can only jump above.

Proof. For the proof, we will use the fact that $\mathrm{dist}(B_{R_\epsilon(t)}(x_1) \cup B_{R_\epsilon(t)}(x_2), \mathbb{R}^2_\epsilon \setminus \mathcal{M}_*(\Lambda_n, t_\epsilon^*)(t))$ is nondecreasing in t, so in particular it is always greater than d_n.
Since Λ_n^1 is already constructed, we proceed inductively, assuming that Λ_n^{k-1} has already been constructed and exists until a time t_n^k. We let $\widetilde{\phi}_n^{k-1}(t_n^k) := \lim_{t \uparrow t_n^k} \widetilde{\phi}_n^{k-1}(t)$ (which always exists). If $\widetilde{\phi}_n^{k-1}(t_n^k) = 0$ or $\widetilde{\phi}_n^{k-1}(t_n^k) = L/2$ we stop the construction, if $L/2 > \widetilde{\phi}_n^{k-1}(t_n^k) > 0$ we proceed as follows. Let us distinguish two cases.
Case 1: $R(t_n^k) \leq L$.

Define $\widetilde{\Lambda}$ as the set which contains $B_{R_\epsilon(t_n^k)}(x_1) \cup B_{R_\epsilon(t_n^k)}(x_2)$ and whose boundary is of class $C^{1,1}$ and is contained in $\left(\partial B_{R_\epsilon(t_n^k)}(x_1) \cup \partial B_{R_\epsilon(t_n^k)}(x_2) \cup \partial C^+ \cup \partial C^-\right)$, where C^+ is a circle as in Figure 1, i.e a circle of radius $r := \frac{L^2 - R_\epsilon^2 + (\widetilde{\phi}_n^{k-1})^2}{2(R_\epsilon - \widetilde{\phi}_n^{k-1})}(t_n^k)$ and center in $(0, \widetilde{\phi}_n^{k-1}(t_n^k) + r)$, whereas C^- has same radius and center in $(0, -\widetilde{\phi}_n^{k-1}(t_{n,k}) - r)$. ¿From the monotonicity properties of the curvature of $\partial \Lambda_n^{k-1}(t_n^k)$, which follow from the Sturmian Theorem, it is easy to check that

$$\widetilde{\Lambda} \subseteq \Lambda_n^{k-1}(t_n^k) \subseteq \mathcal{M}_*(\Lambda_n, t_\epsilon^*)(t_n^k).$$

We now define Λ_n^k by regularizing $\widetilde{\Lambda}$ in such a way that the $C^{2,\alpha}$–norm of Λ_n^k is bounded by a constant depending only on d_n and r. This is possible since the distance between $\partial \mathcal{M}_*(\Lambda_n, t_\epsilon^*)$ and the (four) points where $\partial \widetilde{\Lambda}$ is not of class $C^{2,\alpha}$ is greater than d_n.
Case 2: $R_\epsilon(t_n^k) > L$. Here the problem is to control the curvature of the constructed curve, because the radius r defined above could be 0. We shall use the fact that two circles of radius $R_\epsilon(t_n^k) + \mathrm{d}_n/2$ are still contained in $\mathcal{M}_*(\Lambda_n, t_\epsilon^*)(t_n^k)$, because the distance of $\partial \mathcal{M}_*(\Lambda_n, t_\epsilon^*)(t_n^k)$ from $D_{R_\epsilon(t_n^k)}$ is at least d_n. Call $h := \sqrt{R_\epsilon(t_n^k)^2 - L^2}$.
Subcase 1 Assume that $\widetilde{\phi}_n^{k-1}(t_n^k) \geq \sqrt{2}h$. Proceeding as above, we get

$$r \geq \frac{(\widetilde{\phi}_n^{k-1}(t_n^k))^2}{4(L - \widetilde{\phi}_n^{k-1}(t_n^k))},$$

which gives the first inequality in (23), recalling that $\widetilde{\phi}_n^{k-1}(t_n^k) \leq L/2$. The second inequality in (23) is obvious since $R_\epsilon(t_n^k) > L$.
Subcase 2 Assume that $\widetilde{\phi}_n^{k-1}(t_n^k) < \sqrt{2}h$. In this case we define the new height as follows

$$\widetilde{\phi}_n^{k-1}(t_n^k) := \left(\lim_{t \uparrow t_n^k} \widetilde{\phi}_n^{k-1}(t)\right) \vee \left(\left(h + \frac{\mathrm{d}_n}{2}\right) \wedge 2h\right).$$

If we don't introduce any discontinuity in the function $\widetilde{\phi}_n$, the inequalities (23) follow as in *Subcase 1*. Otherwise the inequality $\widetilde{\phi}_n^{k-1}(t_n^k) \leq \sqrt{2}h$ gives the second inequality in (23), whereas the fact that $\widetilde{\phi}_n^{k-1}(t_n^k) \geq h + \frac{\mathrm{d}_n}{2}$ implies $r \geq \mathrm{d}_n^2/(8\sqrt{2}L)$, which gives the first inequality in (23). Notice also that $D_{R_\epsilon(t_n^k) + \mathrm{d}_n/2} \subseteq \widetilde{\Lambda} \subseteq \mathcal{M}_*(\Lambda_n, t_\epsilon^*)(t_n^k)$, and the distance at the four points where the smoothing takes place is larger than $\mathrm{d}_n/2$. $\square$

The following result states that the curves $\partial \Lambda_n(t)$ do not intersect the origin for a time interval τ independent of n (even if it depends on the path ω).

Lemma 5.2. *For any $1/3 < \beta' < 1/2$ and for almost any ω, there exist $\tau(\omega, \beta') > 0$ and $c(\omega) > 0$ such that $\mathcal{M}_{*,\epsilon}(\Lambda_n, t_\epsilon^*)(t_\epsilon^* + s)$ contains a ball of radius $c(\omega)s^{\beta'}$ for all $s \in [0, \tau(\omega, \beta'))$.*

Proof. Fix $\tau > 0$ and choose $1/3 < \beta < \beta' < 1/2$. We set $\tau_i := \tau/2^i$, $\delta_i := c_1(\tau)/c_2^i$ $(i \in \mathbb{N})$, where $0 < c_1(\tau) = \widehat{c}_1(\epsilon^2\tau)^{\rho_1}$ for some $0 < \rho_1 < 1$ and some $\widehat{c}_1 > 0$ and $c_2 = 2^{\rho_2}$ for some $0 < \rho_2 < 1$. The constants $\widehat{c}_1$, ρ_1 and ρ_2 can be calculated explicitly at the end of step 4. The ρ_i, $i = 1, 2$ depend on β and β'.
We divide the rest of the proof into five steps.

Step 1. We want to show that, for any $i \in \mathbb{N}$, there exists $t_i = t_i(\omega) \in (t_\epsilon^*, t_\epsilon^* + \epsilon^2\tau_i)$ such that $R_\epsilon(s) > L$ on $[t_i, t_i + (\delta_i)^{\frac{1}{\beta}}\epsilon^2\tau_i]$ with probability $1 - o(\tau)$.
Indeed, let us define t_i as follows

$$t_i := \epsilon^2\tau_i \wedge \inf\{s \geq t_\epsilon^* : R_\epsilon(s) - R_\epsilon(t_\epsilon^*) \geq \epsilon\delta_i\sqrt{\epsilon^2\tau_i}\}.$$

By applying twice the strong Markov property we get

$$
\begin{aligned}
&\mathbb{P}(t_i < \epsilon^2\tau_i, R_\epsilon > R_\epsilon(t^*_\epsilon) \text{ on } [t_i, t_i + (\delta_i)^{\frac{1}{\beta}}\epsilon^2\tau_i]) \\
&= \mathbb{P}\Big(\inf\Big\{R_\epsilon(s)\Big|s \in [t_i, t_i + (\delta_i)^{\frac{1}{\beta}}\epsilon^2\tau_i]\Big\} > L\Big|t_i < \epsilon^2\tau_i\Big)\,\mathbb{P}(t_i < \epsilon^2\tau_i) \\
&= \mathbb{P}\Big(\inf\Big\{R_\epsilon(s) - R_\epsilon(0)\Big|s \in [0, (\delta_i)^{\frac{1}{\beta}}\epsilon^2\tau_i]\Big\} > -\epsilon\delta_i\sqrt{\epsilon^2\tau_i}\Big|R_\epsilon(0) = L\Big)\cdot\mathbb{P}(t_i < \epsilon^2\tau_i) \\
&\geq \mathbb{P}(\sup_{[0,(\delta_i)^{\frac{1}{\beta}}\epsilon^2\tau_i]} \epsilon W(t) \leq c\,\epsilon\delta_i\sqrt{\epsilon^2\tau_i})\mathbb{P}(\sup_{[0,\epsilon^2\tau_i]} \epsilon W(t) \geq c\,\epsilon\delta_i\sqrt{\epsilon^2\tau_i}) \\
&\geq 1 - \tilde{c}\,\left(\delta_i^2 - \exp(-\delta_i^{2\beta-1})\right),
\end{aligned}
$$

for some positive constants c, $\tilde{c}$. We can estimate the probabilities for R_ϵ against those for the ϵ-Brownian motion ϵW, because on the set $\{\omega : \inf_{s\in[t^*_\epsilon,t^*_\epsilon+\epsilon^2\tau_i]}\{R_\epsilon(s) - R_\epsilon(t^*_\epsilon)\} > \frac{L}{2}\}$, which has probability $1-o(\epsilon^2\tau)$, the absolutely continous part of $R_\epsilon(t)$ is uniformly Lipschitz-continous. In particular, on time intervals of length $\epsilon^2\tau_i$, it is of smaller order than $\delta_i\sqrt{\epsilon^2\tau_i}$. The estimate for the Brownian motions follows directly from the fact that $\max_{[0,t]} W(s)$ has the distribution of $|W(t)|$.

Step 2. We want to show that with probability $1-o(\tau)$ the following holds: $\widetilde{\phi}_n(s) > c(s-t_i)^\beta$ for $s \in [t_i, t_i+\delta_i^{\frac{1}{\beta}}\epsilon^2\tau_i)$ and for some constant $c > 0$, whenever $[t_i, t_i+\delta_i^{\frac{1}{\beta}}\epsilon^2\tau_i) \subseteq [t^*_\epsilon, \bar{t}_n)$. Indeed, the Λ^k_n in lemma 5.1 are constructed in such a way that (23) holds on $(t^*_\epsilon, \bar{t}_n)$. Hence we get by (22) for any $t_i \leq s \leq t \leq t_i + \delta_i^{\frac{1}{\beta}}\epsilon^2\tau_i$

$$
\begin{aligned}
\widetilde{\phi}_n(t) - \widetilde{\phi}_n(s) &\geq \int_s^t \left(2\frac{L-\widetilde{\phi}_n(r)}{\widetilde{\phi}^2_n(r)} + g(r)\right)\mathrm{d}r + \epsilon(W(t) - W(s)), \qquad (24)\\
\widetilde{\phi}_n(0) &\geq 0.
\end{aligned}
$$

We have to check that the function $\theta_{\beta,c}(s) = c(s-t_i)^\beta + \epsilon(W(s) - W(t_i)) + \int_{t_i}^s g(r)\mathrm{d}r$ is a subsolution of (24) on $[t^*\epsilon, \tau \wedge \bar{t})$ with probability $1 - o(\tau)$, i.e.

$$
\theta_{\beta,c}(t) - \theta_{\beta,c}(s) \leq \int_s^t \left(2\frac{L-\theta_{\beta,c}(r)}{\theta^2_{\beta,c}(r)} + g(r)\right)\mathrm{d}r + \epsilon(W(t) - W(s)).
$$

Fix $1/2 > \alpha > \beta'$ and choose $c \leq (2L)^{\frac{1}{3}}$. We have that $|W(t) - W(t_i)| \leq c_3|t - t_i|^\alpha$ with probability $1 - o(c_3^{-1})$ for fixed α (this follows directly from the fact that $\mathbb{E}\|W(s)\|_{H^{\sigma,p}([0,t])} \leq C(\sigma,p)$ for $\sigma < \frac{1}{2}$, the embedding theorems and the Markov inequality). Hence, (notice that the Brownian terms on both sides of (24) cancel,)

$$
\theta_{\beta,c}(s)^2 \leq c(t-t_i)^\beta \text{ for } t - t_i < c_4(c, \|g \wedge 0\|_{C^0})c_3^{\frac{1}{\beta-\alpha}}. \qquad (25)
$$

Therefore, using again the Hölder-continuity of the paths of the Brownian motion, we have that $\widetilde{\phi}_n(s) > c(s-t_i)^\beta$ for all $s \in [t_i, t_i + \delta_i^{\frac{1}{\beta}}\epsilon^2\tau_i)$, whenever this interval is contained in $[t^*_\epsilon, \bar{t}_n)$, and c_3 is so small that (25) holds for the chosen τ, which happens with probability $1 - o(\tau)$. In particular, we have

$$
\widetilde{\phi}_n\big(t_i + \delta_i^{\frac{1}{\beta}}\epsilon^2\tau_i\big) \geq c\delta_i(\epsilon^2\tau_i)^\beta.
$$

Step 3. We know that the second derivative in 0 of $\phi_n(\cdot, s)$ is nonnegative for all $s \in [t^*_\epsilon, \bar{t}_n) \setminus \cup_k\{t^k_n\}$, so for $s \in [t_i + \delta_i^{\frac{1}{\beta}}\epsilon^2\tau_i, \bar{t}_n)$ we have

$$
\widetilde{\phi}_n(s) \geq \epsilon\big[W(s) - W(t_i + \delta_i^{\frac{1}{\beta}}\epsilon^2\tau_i)\big] + c\delta_i(\epsilon^2\tau_i)^\beta.
$$

Hence we have $\widetilde{\phi}_n(s) > \frac{c}{2}\delta_i(\epsilon^2\tau_i)^\beta$ for $s \in [t_i, t_i + \left(\frac{c^2}{4}\delta_i\delta_i^2(\epsilon^2\tau_i)^{2\beta-1}\right)\tau_i] \cap [t^*_\epsilon, \bar{t}_n)$ on a set Ω_i with $\mathbb{P}(\Omega_i) \geq 1 - \exp(-\delta_i^{-1})$.

We can now determine $\widehat{c}_1, \rho_1$ and ρ_2 (and thus δ_i) in such a way that the following conditions hold:

- $\left(\frac{c^2}{4}\delta_i^3\epsilon^{2(2\beta-1)}\tau_i^{2\beta-1}\right) > 2$, so $\widetilde{\phi}_n(s) > \frac{c}{2}\delta_i(\epsilon^2\tau_i)^\beta$ in $[t^*_\epsilon + \epsilon^2\tau_i, (t^*_\epsilon + \epsilon^2\tau_{i-1}) \wedge \bar{t}_n)$.
- $\delta_i(\epsilon^2\tau_i)^\beta \geq (\epsilon^2\tau_{i-1})^{\beta'}$.

Now we can argue by iteration to show that $\widetilde{\phi}_n(s) > c(s-t^*_\epsilon)^{\beta'}$ for $s \in (t^*_\epsilon, (t^*_\epsilon + \epsilon^2\tau) \wedge \bar{t}_n)$ on a set of probability of order $1 - o(\tau)$.

This implies that for almost any path ω there exists $\tau(\omega) > 0$ such that the thesis holds for the interval $[t^*_\epsilon, (t^*_\epsilon + \epsilon^2\tau) \wedge \bar{t}_n)$ instead of $[t^*_\epsilon, t^*_\epsilon + \tau)$.

Step 4. On the interval $[t^*_\epsilon + \epsilon^2\tau, (t^*_\epsilon + \tau) \wedge \bar{t}_n)$ we argue as follows. By the Hölder-continuity of the Brownian motion we know that $(L^2 - R^2_\epsilon(s)) \vee 0 \leq c_3(\omega)\epsilon(s-t^*_\epsilon)^\alpha + c_5(g, L)(s-t^*_\epsilon)$. However on $[\epsilon^2\tau, \tau)$ we have

$$\epsilon(s - t^*_\epsilon)^\alpha \leq c_6(\epsilon, \tau)(s - t^*_\epsilon)^{2\beta}, \qquad c_6(\epsilon, \tau) = \epsilon^{\rho_3}\tau^{\alpha-2\beta},$$

Hence we can estimate the denominator in (22) against $c(\tau(\omega))(s-t^*_\epsilon)^{2\beta'}$ by arguments as in step 2 for the $\tau(\omega)$ already fixed in the previous step, and we conclude that there is $c'(\tau(\omega))$ such that $\theta_{\beta',c'}$ is a subsolution for (22) on this interval.

Remark: Actually c' can be chosen *independent* of τ. This comes from the fact that we need step 4 only for $\tau \geq \epsilon^2$ and that we can assume $\alpha - 2\beta$ close to $-1/6$.

Step 5. We conclude the proof of the lemma. It remains to show that $\bar{t}_n \geq \tau(\omega)$. From *Step 2* it follows that on $[t^*_\epsilon, \bar{t}_n)$ we have $\widetilde{\phi}_n(s) \geq c(\omega)(s - t^*_\epsilon)^\beta$ hence the existence time of the smooth evolutions Λ^k_n depends only on n as long as $t^k_n \leq \tau(\omega)$ and $\widetilde{\phi}_n(t^k_n) \leq L/2$. Hence $\bar{t}_n \geq \tau(\omega) > 0$, provided that $\widetilde{\phi}_n(t) \leq L/2$ for $t \in [t^*_\epsilon, \bar{t}_n)$ (which is always satisfied for $\tau(\omega)$ small enough).

□

We are now in the position to prove Proposition 3.12.

Proof. Fix $\omega \in \bigcup_{\epsilon>0} S^+_\epsilon$ such that Corollary 3.11 holds. We set for simplicity $\mathcal{M}^*_\epsilon(t) := \mathcal{M}^*_\epsilon(\Sigma_0)(t)$, $\mathcal{M}_{*,\epsilon}(t) := \mathcal{M}_{*,\epsilon}(\Sigma_0)(t)$, $\mathcal{M}^*_0(t) := \mathcal{M}^*_0(\Sigma_0)(t)$. We divide the proof into four steps.

Step 1. Let $t < t^*_0$. Reasoning exactly as in Proposition 3.10, we get that

$$\lim_{\epsilon\to 0} \sup_{s\in[0,t]} \mathrm{d}_H\left(B_{R_\epsilon(s)}(x_i), B_{R_0(s)}(x_i)\right) = 0 \qquad i \in \{1, 2\}, \tag{26}$$

which implies (17).

Step 2. Let $t \in (t^*_0, T_0)$. We want to prove that

$$\bigcap_\lambda \bigcup_{\epsilon<\lambda} \mathcal{M}^*_\epsilon(t) = \mathcal{M}^*_0(t). \tag{27}$$

Consider the family of sets Σ_n defined at the beginning of this section. We recall that for any bounded set $A \subset \mathbb{R}^2$ and a family of sets $A_n \subset \mathbb{R}^2$ such that $A_n \downarrow A$, we have

$$\bigcap_n \mathcal{M}^*_0(A_n) = \mathcal{M}^*_0(A),$$

which implies together with (6)

$$\mathcal{M}_0^*(t) = \bigcap_n \mathcal{M}_0^*(\Sigma_n, t_0^*)(t), \qquad t > t_0^*.$$

In order to obtain (27), it is enough to prove for any $n \in \mathbb{N}$:

$$\bigcap_\lambda \bigcup_{\epsilon<\lambda} \mathcal{M}_\epsilon^*(t) \subseteq \mathcal{M}_0^*(\Sigma_n, t_0^*)(t). \tag{28}$$

Indeed, from the regularity of $\mathcal{M}_0^*(\Sigma_n, t_0^*)(t)$ (which we get for example from the gradient estimates in [2]) it follows that

$$\lim_{\tau\to t_0^*} \mathcal{M}_0^*(\Sigma_n, \tau)(t) = \mathcal{M}_0^*(\Sigma_n, t_0^*)(t) \qquad t > t_0^*.$$

Therefore, recalling that $t_\epsilon^* \to t_0^*$ for $\epsilon \to 0$ by Corollary 3.11, we obtain

$$\bigcap_\lambda \bigcup_{\epsilon<\lambda} \mathcal{M}_0^*(\Sigma_n, t_\epsilon^*)(t) = \mathcal{M}_0^*(\Sigma_n, t_0^*)(t).$$

Moreover, from Corollary 3.7 we get that for any $n \in \mathbb{N}$ there exists $\epsilon(n)$ such that

$$\mathcal{M}_\epsilon^*(t) \subseteq \mathcal{M}_0^*(\Sigma_n, t_\epsilon^*)(t)$$

for all $\epsilon < \epsilon(n)$, hence

$$\bigcap_\lambda \bigcup_{\epsilon<\lambda} \mathcal{M}_\epsilon^*(t) \subseteq \bigcap_\lambda \bigcup_{\epsilon<\lambda} \mathcal{M}_0^*(\Sigma_n, t_\epsilon^*)(t),$$

and (28) follows.

Step 3. Let $t \in (t_0^*, T_0)$. We want to show that

$$\bigcup_\lambda \bigcap_{\epsilon<\lambda} \mathcal{M}_{*,\epsilon}(t) \supseteq \mathcal{M}_0^*(t). \tag{29}$$

Given $0 \le r \le L$, denote by E_r be the union of the two tangent circles of radius r, having centers in the segment $[x_1, x_2]$ and containing the origin. Recall that, by parabolic rescaling, we get

$$\mathcal{M}_0^*(t) = \bigcup_{r\uparrow L} \mathcal{M}_0^*(E_r, t_0^*)(t), \qquad t > t_0^*. \tag{30}$$

By Lemma 5.2 and recalling (6) there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$ we can find $\tau_\delta > 0$ independent of ϵ such that

$$\mathcal{M}_{*,\epsilon}(t) \supset B_\delta(0), \qquad t \ge t_\epsilon^* + \tau_\delta.$$

Since $r < L$, we can also assume that $\mathcal{M}_{*,\epsilon}(t_\epsilon^* + \tau_\delta) \supset E_r \cup B_\delta(0)$. Applying Proposition 3.6, it follows that, for ϵ small enough (depending on δ), we have

$$\mathcal{M}_{*,\epsilon}(t) \supseteq \mathcal{M}_0^*(E_{r'}, t_\epsilon^* + \tau_\delta)(t), \qquad t \ge t_\epsilon^* + \tau_\delta,\ r' < r. \tag{31}$$

Letting $\epsilon \to 0$ and using the continuity in the variable s of $\mathcal{M}_0^*(E_{r'}, s)(t)$ for $s < t$ (which follows from the regularity of $\partial\mathcal{M}_0^*(E_{r'}, s)(t)$ [2]), inclusion (31) becomes

$$\bigcup_\lambda \bigcap_{\epsilon<\lambda} \mathcal{M}_{*,\epsilon}(t) \supseteq \mathcal{M}_0^*(E_{r'}, t_0^* + \tau_\delta)(t), \qquad t > t_0^* + \tau_\delta,\ r' < r,$$

which gives the result letting $\delta \to 0^+$ and recalling (30).

Step 4. Let us consider the case $t = t_0^*$. Since $\mathcal{M}_{*,\epsilon}(t) \supseteq B_{R_\epsilon(t)}(x_1) \cup B_{R_\epsilon(t)}(x_1)$ for any $\epsilon > 0$ and $t \in [0, T_\epsilon(\omega))$, and since $B_{R_\epsilon(t)}(x_i) \to B_{R_0(t)}(x_i)$, $i \in \{1, 2\}$, uniformly on compact subsets of $[0, T_0)$, it follows that (29) also with $t = t_0^*$.

We have to prove that also (27) holds. Indeed, assume by contradiction that there exists a point

$$z \in \left(\bigcap_\lambda \bigcup_{\epsilon<\lambda} \mathcal{M}_\epsilon^*(t_0^*) \right) \setminus \mathcal{M}_0^*(t_0^*). \tag{32}$$

By *Step 1*, for any $s \in [0, t_0^*)$ there exists $\epsilon(s)$ such that $\mathcal{M}_\epsilon^*(s) \subseteq B_L(x_1) \cup B_L(x_2)$ for any $\epsilon < \epsilon(s)$. Let now $r := \mathrm{dist}(z, B_L(x_1) \cup B_L(x_2))/2$ and let $\Sigma := \partial B_r(z)$. Let also $\Sigma_\epsilon^s(t)$, $t \in [s, s+\tau]$, $\tau > 0$, be the evolution starting from Σ at time s, which solves (1) letting ν be the unit normal pointing inside $B_r(z)$. Since $\Sigma_\epsilon^s(t)$ converges, for $\epsilon \to 0$, to the deterministic evolution starting from $B_r(z)$, we can take τ independent of s and ϵ. Moreover, it is easy to check from the definition of minimal barrier that

$$\mathcal{M}_\epsilon^*(t) \cap \Sigma_\epsilon^s(t) = \emptyset \qquad t \in [s, s+\tau]. \tag{33}$$

If we choose $s < t_0^*$ such that $t_0^* \in (s, s+\tau)$, from (33) we get

$$z \notin \left(\bigcap_\lambda \bigcup_{\epsilon<\lambda} \mathcal{M}_\epsilon^*(t_0^*) \right) \setminus \mathcal{M}_0^*(t_0^*),$$

which contradicts (32).

$\square$

References

[1] F. Almgren, J.E. Taylor, and L. Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31:387–437, 1993.

[2] S. Altshuler, S. Angenent, and Y. Giga. Mean curvature flow through singularities for surfaces of rotation. *J. Geom. Anal.*, 5(3):293–358, 1995.

[3] L. Ambrosio. Movimenti minimizzanti. *Atti Accad. Naz. delle Scienze detta dei XL*, 113, 1995.

[4] S. Angenent. On the formation of singularities in the curve shortening problem. *J. Diff. Geom.*, 33:601–633, 1991.

[5] G. Bellettini and M. Novaga. Minimal barriers for geometric evolutions. *Journal of Differential Equations*, 139:76–103, 1994.

[6] G. Bellettini and M. Paolini. Two examples of fattening for the mean curvature flow with a driving force. *Atti Acad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. 9 Math. Appl. 5*, 5:229–236, 1994.

[7] G. Bellettini and M. Paolini. Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)*, 19:43–67, 1995.

[8] K. Brakke. *The motion of a surface by its mean curvature*, volume 16. Princeton University Press, 1978.

[9] E. De Giorgi. Barriers, boundaries, motion of manifolds. *Conference held at Department of Mathematics of Pavia, March 18*, 1994.

[10] L.-C. Evans and J. Spruck. Motion of level sets by mean curvature I. *J. Differ. Geom.*, 33(3):635–681, 1991.

[11] L.-C. Evans and J. Spruck. Motion of level sets by mean curvature II. *Trans. Amer. Math. Soc.*, 330:321–332, 1992.

[12] M.I. Freidlin and A.D. Venttsel'. *Random Perturbations of Dynamical Systems*, volume 260. Springer, 1984.

[13] R. Gulliver and Y. Koo. Sharp growth rate for generalized solutions evolving by mean curvature plus a forcing term. *Preprint Max-Planck-Institute for Mathematics in the Sciences Leipzig*, 65, 1999.

[14] T. Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Memoirs. Amer. Math. Soc.*, 520, 1994.

[15] Y. Koo. A fattening principle for fronts propagating by mean curvature plus a driving force. *Comm. Part. Diff. Eq.*, 24:1035–1053, 1999.

[16] P.-L. Lions and P.E. Souganidids. Fully nonlinear stochastic partial differential equations. *C.R. Acad. Sci. Paris*, 326:1085–1092, 1998.

[17] P.-L. Lions and P.E. Souganidids. Fully nonlinear stochastic partial differential equations: nonsmooth equations and applications. *C.R. Acad. Sci. Paris*, 327:735–741, 1998.

[18] P.-L. Lions and P.E. Souganidids. Uniqueness of weak solutions of fully nonlinear stochastic partial differential equations. *Preprint*, 2000.

[19] S. Luckhaus and T. Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differ. Equ.*, 3(2):253–271, 1995.

[20] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, volume 16. Birkhäuser, 1995.

[21] N.K. Yip. Stochastic motion of mean curvature. *Arch. Ration. Mech. Anal.*, 144:313–355, 1998.

Nicolas Dirr
Max–Planck–Institute for Mathematics in the Sciences,
Augustusplatz 10/11, D–04109 Leipzig, Germany
email: ndirr@mis.mpg.de

Stephan Luckhaus
Fakultät für Mathematik und Informatik, Universität Leipzig,
Augustusplatz 10/11, D–04109 Leipzig, Germany
email: stephan.luckhaus@mis.mpg.de

Matteo Novaga
Dipartimento di Matematica, Università di Pisa,
via F. Buonarroti 2, 56127 Pisa, Italy
email: novaga@dm.unipi.it