Optimal measures for elliptic problems

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Abstract

We study an optimization problem for an elliptic PDE where the unknown is a measure. The state equation is a relaxed form of the classical Dirichlet problem and the functional to minimize is the energy of the system. We prove a regularity result for the optimal measure, give a description of it and present some numerical examples.

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1 Introduction

Let Ω be a connected and bounded open set in $I\!R^N$, we consider the following optimization problem

$$\inf_{\mu\in\mathcal{M}_0(\Omega)}\{J(\mu)=\int_\Omega f u_\mu dx+\alpha\mu(\Omega)\} \tag{1}$$

where $\mathcal{M}_0(\Omega)$ is a set of measures and α is a nonnegative parameter. The function u_μ is the solution of the state equation

$$\begin{cases} -\Delta u+u\mu &= f \quad \text{in} \quad \Omega,\\ \qquad\quad u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

This equation is the relaxed formulation of the classical Dirichlet problem which was considered by Buttazzo and Dal Maso in the framework of shape optimization problems [3]. The fact of introducing measures as a generalization of domains or functions allows in particular to compensate the lack of compactness and to solve several optimization problems. It is not rare that those problems do not admit a classical solution but only a solution of relaxed type (see for instance [2]). Moreover, from a physical point of view, measures allow to model punctual forces or more generally, singular data. A similar optimization problem has been studied by Henrot and Maillot in the paper [6]; instead of the set of admissible measures $\mathcal{M}_0(\Omega)$, they consider the set of all functions uniformly bounded by one. They prove that in fact the minimum is of "non-relaxed type": it is a characteristic function. By identifying the function $1_A(x)$ with the subset $A\subset\Omega$, the minimization problem (1) becomes a shape optimization problem for a functional $J(A)$ defined on the family of all subsets of Ω. The minimum of $J(A)$ is then reached inside the class of subsets of Ω. In this paper, we consider a larger set of admissibility since we want to take into account also singular measures (for example in one dimension, a spring at a point x_0 can be modeled by a Dirac mass $c\delta_{x_0}$). We prove that the optimal measure is absolutely continuous with respect to the Lesbegue measure, and we obtain a characterization of the solution in which the only unknown is the support of the density. We choose the Lagrangian formulation in (1) in order to take into account the mass constraint on the measure μ. This formulation is equivalent to the optimization problem

$$\inf_{\mu\in\mathcal{M}_0(\Omega)}\{J(\mu)=\int_\Omega fu_\mu dx,\ \ \mu(\Omega)=C\}; \tag{3}$$

the equivalence is due to the fact that, in particular, the functional J is continuous with respect to the Lagrangian multiplier α. The constraint on $\mu(\Omega)$ in (3) or the penalization $\alpha\mu(\Omega)$ in (1) are very natural; without them the infimum of the energy $\int_\Omega fu_\mu dx$ would be equal to zero, reached by a measure μ with infinite mass.

This paper is organized as follows. Section 2 is devoted to the setting of the optimization problem. In Section 3, we prove some results concerning the existence and uniqueness of the solution and we give some necessary optimality conditions. Section 4 is devoted to the main result which is the characterization of the optimal measure. In Section 5, we present some numerical results.

2 Setting of the problem and notation

Let us now precise the notation used throughout this paper. The admissible set $\mathcal{M}_0(\Omega)$ we consider is the class of all nonnegative Borel measures on Ω such that $\mu(B)=0$ for every Borel subset B of Ω such that $\mathrm{Cap}(B)=0$, where the capacity of a subset A is defined by

$$\mathrm{Cap}(A)=\inf\{\int_{I\!R^N}|\nabla u|^2+u^2dx : u\in\mathcal{U}_A\}.$$

where $\mathcal{U}_A$ is the set of all functions u of the Sobolev space $H^1(I\!R^N)$ such that $u\geq 1$ almost everywhere in a neighborhood of A.

If a property $P(x)$ holds for all $x\in E$ except for the elements of a set $Z\subset E$ with $\mathrm{Cap}(Z)=0$, we say that $P(x)$ holds quasi-everywhere on E (shortly q.e on E). The expression almost everywhere (shortly a.e.) refers, as usual, to the Lebesgue measure.

A weak solution of the equation (2) is a function which belongs to the Hilbert space $X_\mu(\Omega)=H^1_0(\Omega)\cap L^2(\Omega,\mu)$ endowed with the norm

$$\Big(\int_\Omega|\nabla u|^2dx+\int_\Omega u^2d\mu\Big)^{1/2}.$$

and which solves the following variational problem

$$\int_\Omega\nabla u\nabla v dx+\int_\Omega uvd\mu=\int_\Omega fvdx \qquad \forall v\in X_\mu(\Omega). \tag{4}$$

It is well known that for all $\mu \in \mathcal{M}_0(\Omega)$ and all $f \in L^2(\Omega)$, there exists an unique u_μ weak solution of (2) (see for instance [3]).

We denote by $\mathcal{L}^N$ the $N-$dimensional Lebesgue measure and by $\mathcal{H}^N$ the $N-$dimensional Hausdorff measure. By ω_N we denote the volume of the unit $N-$dimensional ball. We recall that

$$\mathcal{L}^N(B(x,r)) = \omega_N r^N, \quad \text{and} \quad \mathcal{H}^{N-1}(\partial B(x,r)) = N\omega_N r^{N-1}$$

where $B(x,r)$ is the ball of center x and radius r. The average of a function u in the ball $B(x,r)$ is defined by

$$\bar{u}_r(x) = \frac{1}{w_N r^N} \int_{B(x,r)} u(y)dy \tag{5}$$

We recall here the classical result about Sobolev functions: if $u \in H^1(\Omega)$, then there exists a Borel set $E \subset \Omega$ such that $\text{Cap}(E) = 0$ and for all $x \in \Omega \setminus E$, the following limit exists

$$\tilde{u}(x) = \lim_{r \to 0^+} \bar{u}_r(x);$$

the function $\tilde{u}$ is called the quasi-continuous representative of u.

Let μ be a Borel measure; following the notation concerning the differentiation of measures (see [5]), we define for each point x in Ω

$$\underline{D}\mu(x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{\mathcal{L}^N(B(x,r))}. \tag{6}$$

The following result can be easily deduced from [5] (lemma 1 page 36).

Lemma 2.1 *Let $0 < \alpha < \infty$ be fixed. Let $\mu \in \mathcal{M}_0(\Omega)$ and let A be a Borel subset of Ω such that $\underline{D}\mu(x) \leq \alpha$ for $\mu-$a.e $x \in A$. Then*

$$\mu(A) \leq \alpha \mathcal{L}^N(A).$$

In particular, if there exists a constant α such that $\underline{D}\mu(x) \leq \alpha$ for $\mu-$a.e $x \in \Omega$, then the measure μ is absolutely continuous with respect to the Lesbegue measure. In this case, $\underline{D}\mu \in L^\infty(\Omega)$ and by the differentiation Theorem of Radon measures (see [5]), $\underline{D}\mu$ is then the density of μ with respect to the Lesbegue measure, that is $\mu = \underline{D}\mu\mathcal{L}^N$.

3 Existence and properties of the optimal measure

The formulation (1) previously stated leads to the following existence result of optimal measures in the set $\mathcal{M}_0(\Omega)$.

Theorem 3.1 *The problem (1) has at least one solution.*

Proof Since the space $\mathcal{M}_0(\Omega)$ endowed with the topology of the γ-convergence is a compact metric space, it is sufficient to prove that the total mass functional $\mu \mapsto \mu(\Omega)$ is sequentially lower semi-continuous for the same convergence. Let $(\mu_n)_{n\in I\!N}$ be a sequence in $\mathcal{M}_0(\Omega)$ such that $\mu_n \xrightarrow{\gamma} \mu$, then the sequence of functionals $G_n(u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 d\mu_n$ Γ-converges to the functional $G(u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 d\mu$ in $L^2(\Omega)$ (see [4], for the definition and properties of Γ-convergence). Let K be a compact subset of Ω and let $u \in H_0^1(\Omega)$ such that $u \leq 1$ in Ω and $u = 1$ in K. Due to the Γ-convergence of $(G_n)_{n\in I\!N}$ to G, we derive

$$\mu(K) \leq \int_\Omega u^2 d\mu \leq \liminf_{n\to\infty} \int_\Omega u^2 d\mu_n \leq \liminf_{n\to\infty} \mu_n(\Omega).$$

Now let $(K_m)_{m\in I\!N}$ be an increasing sequence of compact sets such that $\Omega = \cup_{m=1}^\infty K_m$; then

$$\mu(\Omega) = \lim_{m\to\infty} \mu(K_m) \leq \liminf_{n\to\infty} \mu_n(\Omega),$$

which concludes the proof. □

Remark 3.2 The argument used in the proof above allows more generally to prove the existence of optimal solutions for minimization problems of the form

$$\min\{J(\mu) : \mu \in \mathcal{M}_0(\Omega), \mu(\Omega) \leq C\} \quad \text{or} \quad \min\{J(\mu) + \alpha\mu(\Omega) : \mu \in \mathcal{M}_0(\Omega)\}$$

whenever $J : \mathcal{M}_0(\Omega) \longrightarrow [0, +\infty]$ is a $\gamma-$l.s.c functional.

The aim of the next lemma is to give some necessary conditions of optimality for the solution of the optimization problem (1).

Lemma 3.3 *Let μ be an optimal measure of (1) and u the weak solution of (2) associated to this measure. Then*

$$|u| \leq \sqrt{\alpha} \ \ \mathcal{L}^N - a.e., \quad and \quad |u| = \sqrt{\alpha} \ \ \mu - a.e.$$

Proof We follow a classic scheme introduced in [3]. For two different perturbations $\{\mu_\varepsilon\}_{\varepsilon>0}$ of μ, we compute

$$\frac{dJ(\mu_\varepsilon)}{d\varepsilon}|_{\varepsilon=0} = \lim_{\varepsilon\to 0^+} \frac{J(\mu_\varepsilon) - J(\mu)}{\varepsilon}$$

and deduce the optimality conditions from the fact that this derivative is nonnegative. We denote by u_ε the solution associated to μ_ε. By equation (4) we have

$$J(\mu_\varepsilon) - J(\mu) = -\int_\Omega u_\varepsilon u[d\mu_\varepsilon - d\mu] + \alpha[\mu_\varepsilon(\Omega) - \mu(\Omega)].$$

Let $\mu_\varepsilon = \mu + \varepsilon\phi$ be the first family of perturbations with ϕ a nonnegative function of $\mathcal{D}(\Omega)$. We easily see that $u_\varepsilon \longrightarrow u$ in $H_0^1(\Omega)$ and using the last equality we get

$$\frac{dJ(\mu_\varepsilon)}{d\varepsilon}|_{\varepsilon=0} = \int_\Omega [\alpha - u^2]\phi dx.$$

This implies that $|u| \leq \sqrt{\alpha}\ \mathcal{L}^N-$a.e.

Let now $\mu_\varepsilon = (1-\varepsilon)\mu$ be the second family of perturbation. For $\varepsilon < 1$, the measure μ_ε still belongs to $\mathcal{M}_0(\Omega)$. By the same procedure as above we derive

$$\int_\Omega [u^2 - \alpha] d\mu \geq 0.$$

Since $|u| \leq \sqrt{\alpha}\ \mathcal{L}^N-$a.e., the quasi-continuous representative $\tilde{u}$ of u verifies $|\tilde{u}| \leq \sqrt{\alpha}$ q.e. in Ω, hence $|u| \leq \sqrt{\alpha}\ \mu-$a.e. in Ω. Then the last inequality implies that $|u| = \sqrt{\alpha}\ \mu-$a.e. $\square$

We are now in a position to prove the uniqueness of the optimal measure.

Lemma 3.4 *Suppose that f is nonnegative. Then there exists an unique measure which satisfies the optimality conditions of Lemma 3.3. In particular, the optimal measure of problem (1) is unique.*

Proof Let μ_1 and μ_2 be two measures which satisfy the optimality conditions, and let u_1 and u_2 be respectively the associated solutions. Let $\phi \in \mathcal{D}(\Omega)$. Using the variational formulation (4) with ϕ as a test function for the couples (u_1, μ_1) and (u_2, μ_2) and subtracting the two expressions obtained, we get

$$\int_\Omega \nabla(u_1 - u_2)\nabla\phi dx = \sqrt{\alpha}\int_\Omega \phi d(\mu_2 - \mu_1), \tag{7}$$

To obtain the above expression, we used also the fact that $u_1 = \sqrt{\alpha}\ \mu_1-$a.e. and $u_2 = \sqrt{\alpha}$ μ_2-a.e. (u_1 and u_2 are nonnegative since f is nonnegative). Now taking $\phi = u_1 - u_2$ in the last equality, we derive

$$\int_\Omega |\nabla(u_1 - u_2)|^2 dx = \sqrt{\alpha}\Big[\int_\Omega (u_1 - \sqrt{\alpha})d\mu_2 - \int_\Omega (\sqrt{\alpha} - u_2)d\mu_1)\Big].$$

Using the fact that $u_1 \leq \sqrt{\alpha}$ q.e. and $u_2 \leq \sqrt{\alpha}$ q.e. in Ω, we obtain that $u_1 = u_2$. Then (7) implies that $\mu_1 = \mu_2$. $\square$

4 Characterization of the optimal measure

The main result we prove in this paper is the following.

Theorem 4.1 *For every nonnegative function $f \in L^2(\Omega)$, the optimal measure μ is absolutely continuous with respect to the Lesbegue measure and moreover there exists a set $K \subset \Omega$ such that*

$$\mu = \frac{f}{\sqrt{\alpha}}\mathcal{L}^N \llcorner K.$$

We first give two lemmas which will be useful in the proof of the main theorem.

Lemma 4.2 *For every $u \in H_0^1(\Omega)$ the average function $\bar{u}_r$ defined by (5) is continuously differentiable with respect to r on $I\!R_+^*$, and its derivative is given by*

$$\frac{d\bar{u}_r}{dr}(x) = \frac{1}{w_N r^N} \int_{\partial B(x,r)} u(s) ds - \frac{N}{w_N r^{N+1}} \int_{B(x,r)} u(y) dy.$$

Proof By the Co-area formula (see, for example [5]), we can write

$$\bar{u}_r(x) = \frac{1}{w_N r^N} \int_0^r \int_{\partial B(x,t)} u(s) d\mathcal{H}^{n-1}(s) dt$$

where $u|_{\partial B(x,t)}$ is the trace of u which belongs to $L^2(\partial B((x,t))$ for all $t > 0$. To get the result it is sufficient to prove that the function

$$G(t) = \int_{\partial B(x,t)} u(s) d\mathcal{H}^{n-1}(s)$$

is continuous on $I\!R_+^*$. Let $(\varphi_n)_{n \in I\!N}$ be a sequence of functions in $\mathcal{D}(\Omega)$ which converges to u in $H_0^1(\Omega)$; then for all $t > 0$ and $h > 0$ small enough, we have

$$|G(t+h) - G(t)| \leq C \|u - \varphi_n\|_{H_0^1(\Omega)} + |\int_{\partial B(x,t+h)} \varphi_n(s) d\mathcal{H}^{n-1}(s) - \int_{\partial B(x,t)} \varphi_n(s) d\mathcal{H}^{n-1}(s)|$$

where C is the supremal on a small interval around t of the norm of the trace operators from $H^1(B(x,t))$ into $L^2(\partial B(x,t))$. In the right side of the last inequality, the second term goes to zero as h goes to zero (by smoothness of the function φ_n) and then the function G is continuous. $\square$

Lemma 4.3 *Let $u \in H_0^1(\Omega)$ and suppose that there exists $c \in I\!R$ such that $u \leq c$ $\mathcal{L}^N-$a.e.; then q.e. in $\{\tilde{u} = c\}$*

$$\liminf_{r \to 0^+} \frac{1}{r} \frac{d\bar{u}_r}{dr}(x) \leq 0. \tag{8}$$

Proof As $u \leq c$ $\mathcal{L}^N-$a.e., for all $x \in \{\tilde{u} = c\}$ and all $r > 0$, $\bar{u}_r(x) \leq \tilde{u}(x)$. Let now $x \in \Omega$ such that (8) is false; then there exists $\delta > 0$ such that for r small enough, $\frac{d\bar{u}_r}{dr}(x) > \delta r$ and so $\bar{u}_r(x) > \tilde{u}(x)$. This implies that the subset of Ω such that (8) is false intersected with $\{\tilde{u} = c\}$ is a null capacity set. $\square$

Proof of Theorem 4.1 We will show the result in three steps. In the first one, we will assume $f \in L^\infty(\Omega)$ and prove that the optimal measure μ is absolutely continuous with respect to the Lesbegue measure and that its density g verifies the following inequality

$$g \leq \frac{2^{N+1}}{\sqrt{\alpha}} f. \tag{9}$$

In the second one, we will generalize this result for all nonnegative $f \in L^2(\Omega)$. We will conclude the proof in the last step.

• First step.
Let x be a point of Ω and let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a family of functions defined by

$$\varphi_\varepsilon(y) = \begin{cases} 1 - \frac{|y-x|^2}{\varepsilon^2} & \text{if } \ |y-x| < \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

For all $\varepsilon > 0$, the function φ_ε belongs to $H_0^1(\Omega) \cap H^2(B(x,\varepsilon))$ and

$$\nabla\varphi_\varepsilon = -\frac{2}{\varepsilon^2}(y-x), \quad \Delta\varphi_\varepsilon = -\frac{2N}{\varepsilon^2}, \ \text{ and } \ \frac{\partial\varphi_\varepsilon}{\partial \mathrm{n}}|_{|\mathrm{y}-\mathrm{x}|=\varepsilon} = -\frac{2}{\varepsilon}.$$

Let u be the solution of equation (2) associated to the optimal measure μ. By the Green formula, we have

$$\int_\Omega \nabla u \nabla \varphi_\varepsilon dx = \frac{2N}{\varepsilon^2}\int_{B(x,\varepsilon)} u(y)dy - \frac{2}{\varepsilon}\int_{\partial B(x,\varepsilon)} u(s) d\mathcal{H}^{N-1}(s).$$

Using the function $\bar{u}_\varepsilon$ defined by (5), by Lemma 4.2 we obtain that

$$\frac{1}{w_N \varepsilon^N}\int_\Omega \nabla u \nabla\varphi_\varepsilon dx = -\frac{2}{\varepsilon}\frac{d\bar{u}_\varepsilon}{d\varepsilon}(x). \tag{10}$$

On the other hand, thanks to Lemma 3.3 and to the fact that u is nonnegative because f is nonnegative, the couple (u,μ) verifies the following equation

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx - \sqrt{\alpha}\int_\Omega v d\mu \qquad \forall v \in X_\mu(\Omega).$$

By taking $v = \varphi_\varepsilon$ in the last equation and using (10), we have

$$-\frac{2}{\varepsilon}\frac{d\bar{u}_\varepsilon}{d\varepsilon}(x) = \frac{1}{w_N\varepsilon^N}\Big[\int_\Omega f\varphi_\varepsilon dx - \sqrt{\alpha}\int_\Omega \varphi_\varepsilon d\mu\Big]. \tag{11}$$

The function φ_ε verifies

$$\int_\Omega \varphi_\varepsilon d\mu \geq \frac{1}{2}\mu(B(x,\frac{\varepsilon}{2}))$$

and since $f \in L^\infty(\Omega)$,

$$\frac{1}{w_N\varepsilon^N}\int_\Omega f\varphi_\varepsilon dx \leq \|f\|_{L^\infty(\Omega)}.$$

Using the two last expressions in (11), we have

$$\frac{2}{\varepsilon}\frac{d\bar{u}_\varepsilon}{d\varepsilon}(x) \geq \frac{\sqrt{\alpha}\mu(B(x,\frac{\varepsilon}{2}))}{2w_N\varepsilon^N} - \|f\|_{L^\infty(\Omega)}.$$

Passing to the limit in the last inequality, we obtain by (6)

$$\liminf_{\varepsilon\to 0}\frac{2}{\varepsilon}\frac{d\bar{u}_\varepsilon}{d\varepsilon}(x) \geq \frac{\sqrt{\alpha}}{2^{N+1}}\underline{D}\mu(x) - \|f\|_{L^\infty(\Omega)}. \tag{12}$$

On the other hand, Lemma 4.3 implies that $\mu-$a.e. in Ω

$$\liminf_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\frac{d\bar{u}_\varepsilon}{d\varepsilon}(x)\leq 0. \tag{13}$$

Combining (12) and (13), we obtain that $\mu-$a.e. in Ω

$$\underline{D}\mu(x)\leq\frac{2^{N+1}}{\sqrt{\alpha}}\|f\|_{L^\infty(\Omega)}. \tag{14}$$

It means that the measure μ is absolutely continuous with respect to the Lebesgue measure. Let us now come back to the equality (11) in order to obtain more informations on the optimal measure. We have $\mathcal{L}^N-$a.e. in Ω

$$\lim_{\varepsilon\to 0}\frac{1}{w_N\varepsilon^N}\int_\Omega f\varphi_\varepsilon dx\leq f(x);$$

following the same procedure than before, we obtain instead of (14) that $\mathcal{L}^N-$a.e. in Ω

$$\underline{D}\mu(x)\leq\frac{2^{N+1}}{\sqrt{\alpha}}f(x).$$

Then the density g of μ satisfies

$$g\leq\frac{2^{N+1}}{\sqrt{\alpha}}f.$$

• Second step.

Let now f be a nonnegative function of $L^2(\Omega)$. The sequence $(f_n)_{n\in\mathbb{N}}$ defined by $f_n=f\wedge n$ converges to f in $L^2(\Omega)$ as n goes to $+\infty$. Let μ_n be the optimal measure associated to f_n, and by step one, let g_n be the density of μ_n. By inequality (9), up to a subsequence, there exists $h\in L^2(\Omega)$ such that $g_n\to h$ weakly in $L^2(\Omega)$. Let μ be the measure in $\mathcal{M}_0(\Omega)$ with density h, we will prove that μ is the optimal measure associated to f. Let us denote, for all $f\in L^2(\Omega)$ and all $\mu\in\mathcal{M}_0(\Omega)$

$$E(f,\mu)=\int_\Omega fu_\mu dx+\mu(\Omega).$$

By an usual way in optimal design we have

$$\lim_{n\to\infty}E(f_n,\mu_n)=E(f,\mu) \tag{15}$$

and also for all $\nu\in\mathcal{M}_0(\Omega)$,

$$\lim_{n\to\infty}E(f_n,\nu)=E(f,\nu). \tag{16}$$

Since μ_n is optimal, for all $\nu\in\mathcal{M}_0(\Omega)$,

$$E(f_n,\nu)\geq E(f_n,\mu_n).$$

The statements (15) and (16) imply that for all $\nu\in\mathcal{M}_0(\Omega)$

$$E(f,\nu)\geq E(f,\mu)$$

and so μ is the optimal measure.

• Third step.
Let u be the solution associated to μ. By the optimality conditions, u is the weak solution of

$$\begin{cases} -\Delta u + \sqrt{\alpha} g &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

The smoothness properties of the Laplacian imply that $u \in H^2_{loc}(\Omega)$. By a well-known property of Sobolev functions we have $\nabla u = 0$ a.e. in $\{u = \sqrt{\alpha}\}$ and $\Delta u = 0$ a.e. in $\{u = \sqrt{\alpha}\}$. Since by step two μ is absolutely continuous with respect to the Lebesgue measure, we have $\Delta u = 0$ $\mu-$a.e. in ω, so that by equation (2) we conclude that $g(x) = 0$ or $g(x) = \frac{f(x)}{\sqrt{\alpha}}$ and the set K is nothing else than the support of g intersected with Ω. □

Remark 4.4 The set K is a compact set for the induced topology in Ω, but it seems economical for the functional J that K is a compact set included in Ω. Without any smoothness assumption on the boundary of Ω, there exists some simple counterexamples. However, if we suppose that $\partial\Omega$ is regular in the sense of Wiener and $f \in L^p(\Omega)$ with $p > N/2$, then the solution u_μ is continuous at the boundary of Ω, and so the distance between K and $\partial\Omega$ is strictly positive (see for instance [7]).

Remark 4.5 Consider the solution u_0 of equation (2) with a measure $\mu = 0$; then for all $\mu \in \mathcal{M}_0(\Omega)$, we have $u_\mu \le u_0$. This implies that if $\sup_\Omega u_0 \le \sqrt{\alpha}$, then, by the optimality conditions, the solution of the optimization problem (1) is the null measure. On the other side, if $\sup_\Omega u_0 > \alpha$ then the optimal measure is nonzero and the solution u_μ is the weak solution of the over-determined problem

$$\begin{cases} -\Delta u &= f \quad \text{in} \quad \Omega \setminus K \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ u &= \sqrt{\alpha} \quad \text{on} \quad \partial K \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial K. \end{cases} \tag{17}$$

5 Numerical examples

5.1 The radial case

In this section Ω is the unit ball $B(0,1)$ in $I\!R^N$ and f is radial in the sense that $f(x) = f(r(x))$ where $r(x) = \|x\|$. In this case we see easily that the optimal measure is radial.

Theorem 5.1 *There exists $t \in [0,1]$ such that the solution of the optimization problem (1) is given by*

$$\mu = \frac{f}{\sqrt{\alpha}} \mathcal{L}^N \llcorner B(0,t)$$

Proof Let K be the support of the optimal measure. Since f is radial, the measure obtained by a rotation of K around 0 with an angle θ is again a solution of the optimization problem. By the uniqueness of the optimal solution (Lemma 3.4), K is radial. Let now $t = \inf\{r \in [0,1], \mathcal{L}^N(K \setminus B(0,r)) = 0\}$. It is clear that the solution u of problem (2) with the measure μ is radial and so it is equal to $\sqrt{\alpha}$ on $\partial(B(0,t))$. The function $u - \sqrt{\alpha}$ is solution of the following equation

$$\int_{B(0,t)} \nabla(u-\sqrt{\alpha})\nabla\varphi dx = \int_{B(0,t)} f\varphi dx \qquad \forall\varphi \in \mathcal{D}(B(0,t))$$

By taking $u - \sqrt{\alpha}$ as test function in the expression above, we obtain that

$$\int_{B(0,t)} |\nabla(u-\sqrt{\alpha})|^2 dx = \int_{B(0,t)} f(u-\sqrt{\alpha})dx$$

Since f is nonnegative and $u \leq \sqrt{\alpha}$, we deduce of the last equation that $u = \sqrt{\alpha}$ in $B(0,t)$. It implies in particular that $\nabla u = 0$ in $B(0,t)$ and then

$$\sqrt{\alpha}\int_{B(0,t)} \varphi d\mu = \int_{B(0,t)} f\varphi dx \qquad \forall\varphi \in \mathcal{D}(B(0,t))$$

This last equation means that $\mu \llcorner B(0,t) = \frac{f}{\sqrt{\alpha}}\mathcal{L}^N \llcorner B(0,t)$, hence $K = B(0,t)$. $\square$

In some simple cases we can give the exact solution of the optimization problem. Let us consider now $f = 1$. The solution u associated to the optimal measure μ solves the over-determined problem (17) with $K = B(0,t)$. Since u is radial we have

$$\Delta u = \Delta r \frac{du}{dr} + \operatorname{div} r \frac{d^2u}{dr^2}$$

with $\Delta r = \frac{N-1}{r}$ and $\operatorname{div} r = 1$. Then $u(r)$ is solution of

$$\left\{ \begin{array}{rcll} -u'' - \frac{N-1}{r}u' & = & 1 & \text{on} \quad (t,1) \\ u(1) & = & 0 & \\ u(t) & = & \sqrt{\alpha} & \\ u'(t) & = & 0. & \end{array} \right.$$

The solution of the previous equation is given by

$$u(r) = \left\{ \begin{array}{ll} \frac{1}{4}(1-r^2) + \frac{t^2}{2}\ln r & for \quad N=2 \\ \frac{1}{2N}(1-r^2) + \frac{t^N}{N(2-N)}(\frac{1}{r^{N-2}} - 1) & for \quad N \geq 3, \end{array} \right.$$

Then we find t as function of α by adding the condition $u(t) = \sqrt{\alpha}$.

5.2 Numerical computations

In this section, we present some numerical results. We define an algorithm based on that developed by Bendsøe and Sigmund for homogenization problems (see the book [1] or the web site *www.topopt.dtu.dk/*). For the numerical computations, we consider all the measures through their density: the nonnegative function ρ. Then the functional to minimize is given by

$$J(\rho) = \int_\Omega f u_\rho dx + \alpha \int_\Omega \rho dx, \tag{18}$$

where u_ρ is the solution of

$$\begin{cases} -\Delta u + \rho u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

The technique we use is a gradient descent method. We follow the direction given by the derivative of J with respect to ρ

$$\frac{dJ(\rho)}{d\rho} = \int_\Omega [\alpha - u_\rho^2] dx.$$

We present three different examples. We consider the square as the design region Ω. The two first examples (from Figure 1 to Figure 5) deal with the optimization problem with a fixed Lagrangian (1). The last example (from Figure 6 to Figure 8) deals with the optimization problem under a volume constraint (3).

The first example we treat, is for a function f equal to one. Figure 1 represents the density of the optimal measure. On the left of Figure 2, we give the support of the optimal measure and on the right, the corresponding solution. For a nonconstant function f (Figure 3), the optimal measure is given in Figure 4 and the support and the solution in Figure 5.

For fixed volume constraint, we calculate the Lagrangian at each iteration by a bisectioning algorithm to ensure that the measure has a mass equal to the constraint. In the last example, the function f is given in Figure 6. For a mass constraint equal to 0.05, the optimal measure is given in Figure 7. The contour of the optimal measure and the corresponding solution is given in Figure 8.

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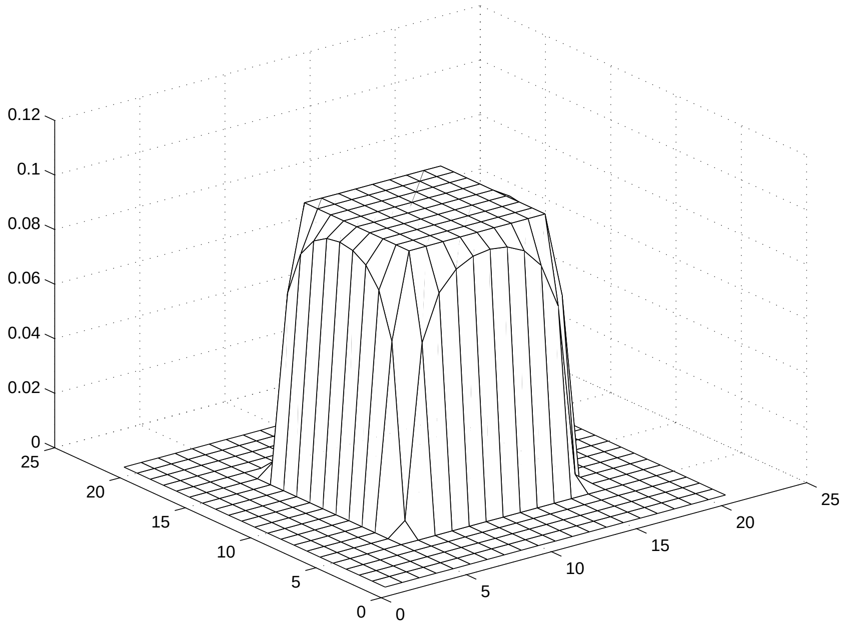

Figure 1: Optimal measure, mesh 20 by 20, Lagrangian $\alpha = 100$

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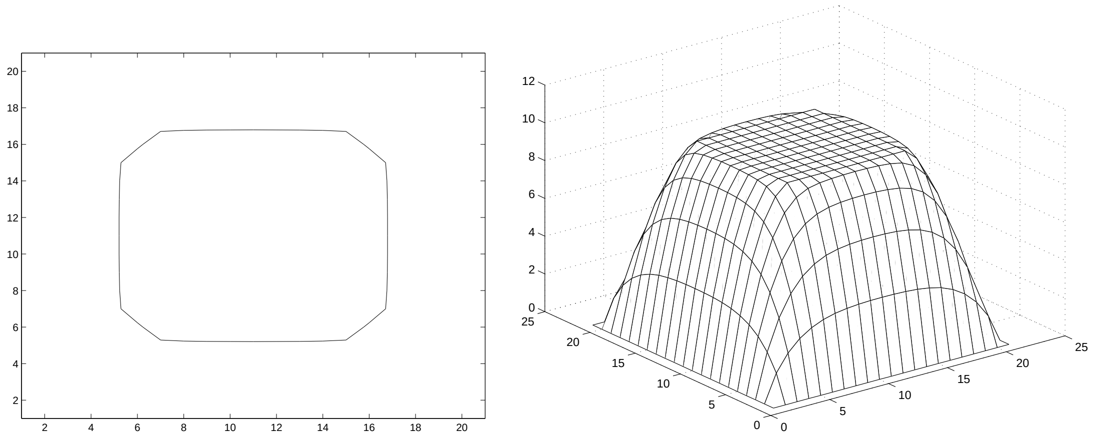

Figure 2: Support and solution.

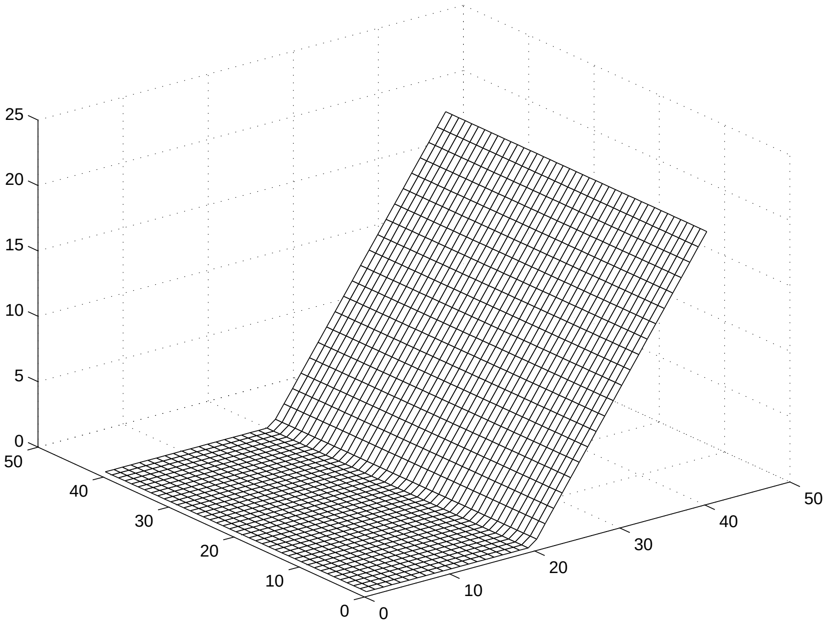

Figure 3: Function f, mesh 40 by 40.

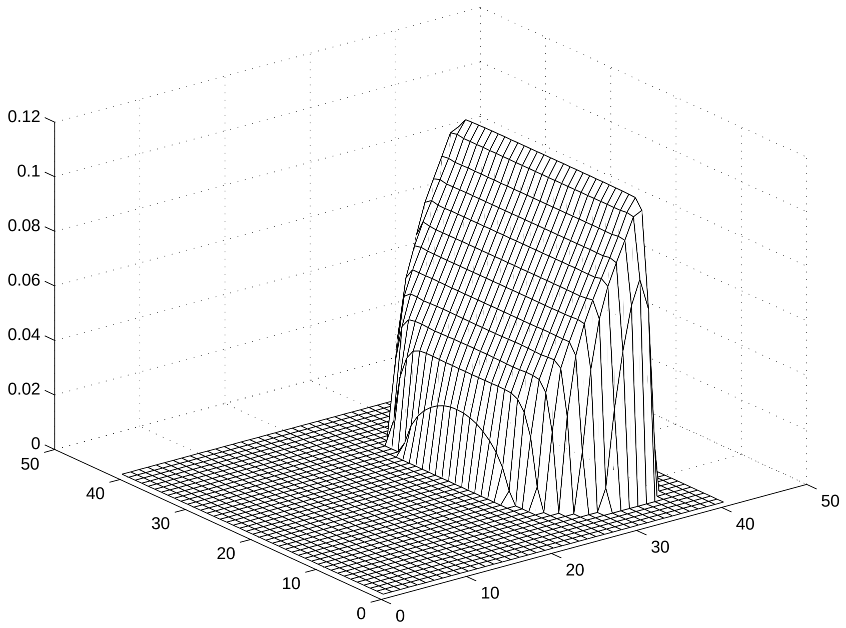

Figure 4: Optimal measure, Lagrangian $\alpha = 2e4$.

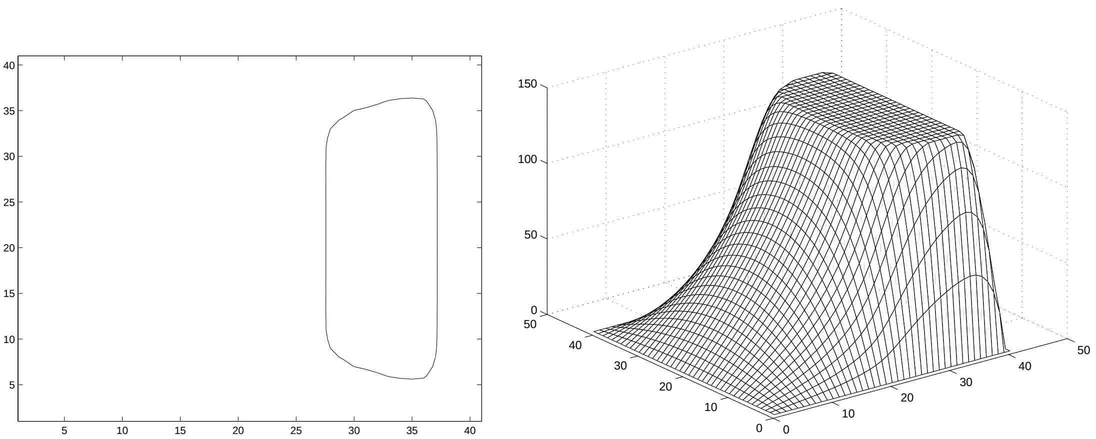

Figure 5: Support and solution.

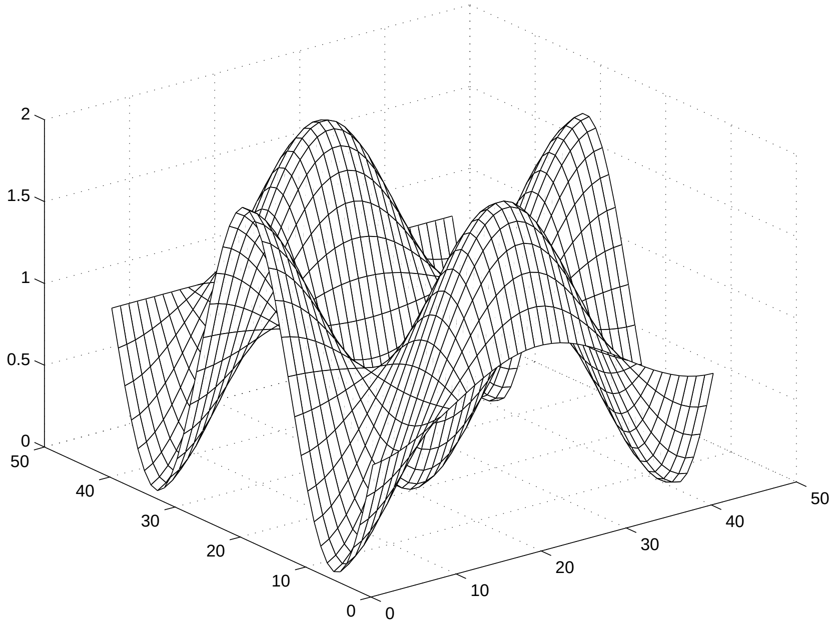

Figure 6: Function f, mesh 40 by 40.

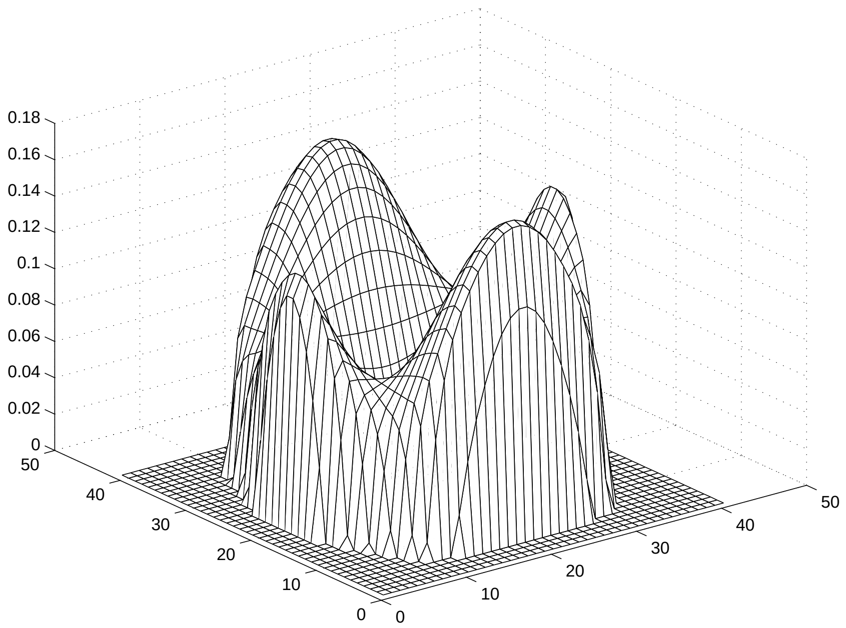

Figure 7: Optimal measure, volume 0.05

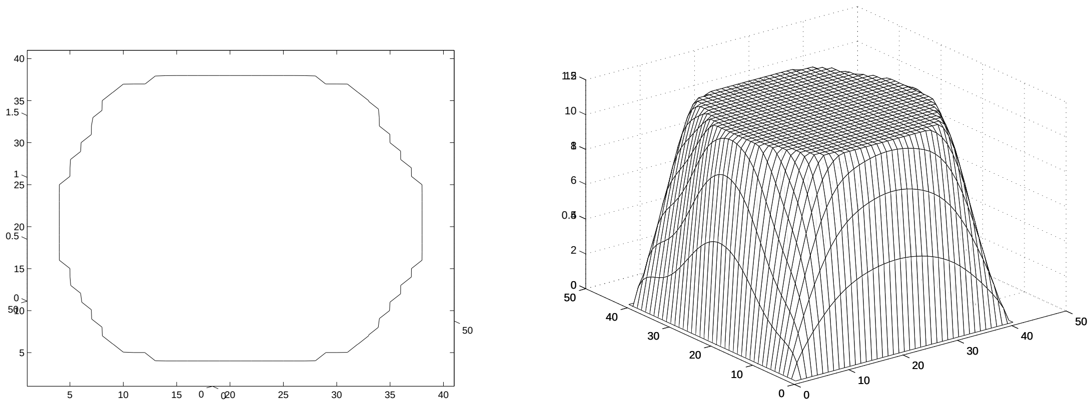

Figure 8: Support and solution, volume 0.05 .