Interfacial energies for incoherent inclusions

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Abstract

We study a variational problem describing an incoherent interface between a rigid inclusion and a linearly elastic matrix. The elastic material is allowed to slip relative to the inclusion along the interface, and the resulting mismatch is penalized by an interfacial energy term that depends on the surface gradient of the relative displacement. The competition between the elastic and interfacial energies induces a threshold effect when the interfacial energy density is non-smooth: small inclusions are coherent (no mismatch); sufficiently large inclusions are incoherent. We also show that the relaxation of the energy functional can be written as the sum of the bulk elastic energy functional and the tangential quasiconvex envelope of the interfacial energy functional.

1 Introduction

In two-phase solids, for instance metal alloys, one of the phases is often segregated into inclusions distributed in the other phase's matrix.

Since the phases generally have different crystalline structure and composition, their stress-free states are often related by a transformation strain involving a dilation or a contraction. Therefore, the presence of new-phase particles generates a stress field in the matrix, and elastic energy accumulates in the system. The stored elastic energy increases with the size of the inclusions, and when these are large enough the elastic energy may be decreased by allowing the particles to break away from the matrix, by a process which essentially involves the nucleation or migration of dislocations at the interface. This process has an energetic cost, and it is the competition between this cost and the stored elastic energy which determines the threshold effect.

This phenomenon is quite common in metal alloys, and it has been demonstrated experimentally in a variety of systems, such as Cu-Co, Fe-Cu, Cu-Al, Mg-Al (see the review paper by Matthews [21]). For instance, in plastically deformed Cu-Co and Fe-Cu the critical radius has been actually determined.

A simplified mathematical description of this process is feasible, by reckoning the effect of interfacial dislocations into an interfacial energy density term which penalizes the mismatch between the phases at the interface. The concept of interfacial mismatch is better understood by viewing the phases as crystalline materials with different lattice parameters. Then the mismatch is simply the difference between the actual (i.e., deformed) lattice parameters of the two phases at the interface. When there is no mismatch, so that the crystalline structures are continuous across the interface, we say that the interface is coherent, and incoherent otherwise.

Using the notion of interfacial energy, we may formulate a simple variational model in which the total energy functional is the sum of this interfacial contribution, penalizing incoherency, and the usual elastic energy term. The competition between the elastic and interfacial energy determines the threshold effect.

In this paper we study this variational problem: we show that the existence of a critical size for coherency is directly related to the smoothness of the interfacial energy density, and give sufficient conditions for the existence of minimizers in terms of the quasiconvexity (convexity in the 2-dimensional case) of the interfacial energy function.

Precisely, we model the new-phase particle as a rigid inclusion Ω in an infinite 3-dimensional elastic matrix, within the linear approximation. The displacement $\boldsymbol{u}$ is a vector field on $\mathbb{R}^3 \setminus \Omega$, which is constrained to be tangential to the boundary of the inclusion, i.e., $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, with $\boldsymbol{n}$ the outward unit normal to $\partial\Omega$. This guarantees that cavitation does not occur at the interface.

When $\boldsymbol{u} = 0$ on $\partial\Omega$, we say that the inclusion is *coherent*, and *incoherent* otherwise. We take as a measure of incoherency the tangential gradient $D_S\boldsymbol{u}$ of the displacement on $\partial\Omega$, since

$$\boldsymbol{u} = 0 \quad \text{on}\ \partial\Omega \qquad \Leftrightarrow \qquad D_S\boldsymbol{u} = 0 \quad \text{on}\ \partial\Omega$$

For a complete description and a motivation of this choice of the measure of incoherency see Cermelli and Gurtin [6], which gives a complete model for incoherent interfaces in the nonlinear setting.

To take into account the transformation strain between the inclusion and the matrix (for crystalline solids, the difference in equilibrium lattice parameters), we assume that the elastic energy density has the form $\varphi(\boldsymbol{E} - \boldsymbol{E}_0) = \frac{1}{2}(\boldsymbol{E} - \boldsymbol{E}_0) \cdot C[\boldsymbol{E} - \boldsymbol{E}_0])$, with C the elasticity tensor, $\boldsymbol{E}$ the infinitesimal strain tensor, and $\boldsymbol{E}_0$ the transformation strain.

We assume that the *interfacial energy density* f depends on $D_S\boldsymbol{u}$ and on the orientation of the interface with respect to the matrix; since the orientation is measured by the interfacial normal $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$, with $\boldsymbol{x} \in \partial\Omega$, we may write

$$f = f(\boldsymbol{x}, D_S\boldsymbol{u}),$$

The problem we discuss in this paper is the minimization of the functional

$$F(\boldsymbol{u}) = \int_{\mathbb{R}^3\setminus\Omega} \varphi(\boldsymbol{E} - \boldsymbol{E}_0)\, dv + \int_{\partial\Omega} f(\boldsymbol{x}, D_S\boldsymbol{u})\, da,$$

on the space of functions $\boldsymbol{u} \in W^{1,2}_{\text{loc}}(\mathbb{R}^3 \setminus \Omega; \mathbb{R}^3)$ such that $\boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}_0 \in L^2(\mathbb{R}^2 \setminus \Omega; \mathbb{R}^{3\times 3})$ and $\boldsymbol{u}|_{\partial\Omega} \in W^{1,q}(\partial\Omega; \mathbb{R}^3)$, with the constraint $\boldsymbol{u}|_{\partial\Omega} \cdot \boldsymbol{n} = 0$. We treat the case $q > 1$, but it is necessary to restrict to $q \geq 2$ in the degenerate case when $\partial\Omega$ is a surface of revolution.

A central result of ours relates the smoothness of f at $D_S\boldsymbol{u} = 0$ to the threshold effect: we prove that if the interfacial energy is a smooth function of $D_S\boldsymbol{u}$, the inclusion can never be coherent, irrespective of its size. On the other hand, when the energy is non-smooth, there exist two critical values for a typical diameter ℓ of Ω, say ℓ_1 and ℓ_2, such that when the inclusion is sufficiently small, and $\ell < \ell_1$, the minimum of the functional exists and the interface is coherent, but for $\ell > \ell_2$ the interface cannot be coherent. In this connection note that the few explicit forms of the interfacial energy function resulting from microscopic calculations (van der Merwe and Fletcher et al. [24]-[13]) are indeed non-smooth and non-convex.

Our second main result is the explicit computation of the relaxed energy functional, and the discussion of existence. We prove that, granted convexity of the bulk energy, existence is determined by the convexity properties of the interfacial energy functional only, since the relaxed energy functional splits as the sum of the relaxed bulk and interface functionals. It turns out that the appropriate type of convexity needed in this context is the *tangential quasi-convexification* $\mathcal{Q}_T f$ of the interfacial energy f,

$$\mathcal{Q}_T f(\boldsymbol{x}, \boldsymbol{V}) := \inf \left\{ \int_Q f(\boldsymbol{x}, \boldsymbol{V} + \nabla_{\boldsymbol{\eta}} \boldsymbol{\varphi}(\boldsymbol{\eta}))\, d\boldsymbol{\eta} \,:\, \boldsymbol{\varphi} \in W_0^{1,\infty}(Q; T_{\boldsymbol{x}}(\partial\Omega)) \right\},$$

introduced by DACOROGNA, FONSECA, MALÝ AND TRIVISA in [9] for functionals defined on mappings from the Euclidean space to manifolds. Here $T_{\boldsymbol{x}}(\partial\Omega)$ is the tangent plane to $\partial\Omega$ at $\boldsymbol{x}$, and Q the unit square in $T_{\boldsymbol{x}}(\partial\Omega)$.

We have the following existence result: assume that Ω is not a surface of revolution and $q > 1$, or that Ω is a surface of revolution and $q \geq 2$; then, if the interfacial energy $f(\boldsymbol{x}, \cdot)$ is tangentially quasiconvex (so that $f = \mathcal{Q}_T f$), a minimizer of the functional $F(\boldsymbol{v})$ exists in an appropriate space.

In a simplified 2-dimensional setting, for which $\Omega \subset \mathbb{R}^2$, write s for an arc parameter on $\partial\Omega$, and $\boldsymbol{u}'$ for the derivative of $\boldsymbol{u}|_{\partial\Omega}$ with respect to s. The interfacial energy is now a function $f = f(s, \boldsymbol{u}')$, which reduces to a function of $\boldsymbol{u}'$ in the isotropic case, when explicit dependencies on the orientation of the interface are excluded. In this simpler case, the notion of quasiconvexity is replaced by the notion of convexity, and we have the stronger result that, if the interfacial energy is isotropic and *convex* in $\boldsymbol{u}'$, then there exists a minimizer of the energy functional.

The first theoretical treatment of the threshold effect for spherical inclusions was established in 1940 by MOTT AND NABARRO [22], who calculated the energy associated to incoherency as the energy required to nucleate dislocations at the interface. Later, Cahn and Larché treated a similar problem, penalizing incoherency by an interfacial energy depending on the misfit between the phases (cf. LARCHÉ AND CAHN [19], CAHN AND LARCHÉ [5]). The notion of an interfacial energy depending on misfit had in fact been introduced earlier by VAN DER MERWE ET AL. AND FLETCHER [12] [13] [24] [25] [26] [11].

More recently, the direct methods of the calculus of variations have been applied to study the transition from coherency to incoherency in a similar, but simpler, 2-dimensional setting, namely a deformable film on a rigid substrate. In this simplified context, the effect of the lack of convexity and non-smoothness of the interfacial energy density on the structure of the solution has first been recognized by LEO AND HU [20], who proposed a special non-convex and non-smooth interfacial energy density to describe fine incoherent patches and the threshold effect. A complete analysis of the problem for thin films is presented in CERMELLI, GURTIN AND LEONI [7], where the direct methods of the calculus of variations have been used to study a large class of interfacial energy functions, and to classify them according to the behavior of the minimizing sequences. There the simple geometry of an epitaxial film on a rigid inclusion allows one to use the full force of the calculus of variations to obtain information more detailed than in the case of a 3-dimensional inclusion. Precisely, the lack of convexity of the interfacial energy density is related to the formation of fine microstructures at the interface: depending on the growth exponent of the energy density, the minimizing sequences either present an oscillatory behavior, and determine an associated Young measure, or concentrate on small sets and are thus associated with a defect measure. On the other hand, when the interfacial energy is convex, a smooth solution does exists. These three kinds of solutions correspond physically to different fine structures of the phase interface: Young measures model an interface in which stress relaxation occurs by alternating smaller and smaller coherent and incoherent regions; defect measures correspond to finely dispersed infinitesimal jumps in the displacement, and thus to interfacial dislocations; a smooth solution may correspond to a 'glassy' interface in which the crystalline structure of the phases is completely lost at the phase boundary.

Unfortunately, such wealth of information cannot be obtained so easily in the three-dimensional setting studied here, since the mathematical features of the two cases (film on a rigid substrate and rigid inclusion in an infinite matrix) are different and, unlike the convex case, very little is known about quasiconvexity. Moreover, for an inclusion, the curvature of the interface affects the transition from coherency to incoherency as well as the fine structure of the interface.

2 Preliminaries

Let Ω, the rigid inclusion, be an open, bounded, connected subset of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The displacement field is a regular function $\boldsymbol{u} : \mathbb{R}^3 \setminus \Omega \to \mathbb{R}^3$, with strain $\boldsymbol{E} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$; when necessary, we denote the strain associated with $\boldsymbol{u}$ by $\boldsymbol{E}(\boldsymbol{u})$. Let

$$\varphi = \varphi(\boldsymbol{E} - \boldsymbol{E}_0) = \tfrac{1}{2}(\boldsymbol{E} - \boldsymbol{E}_0) \cdot C[\boldsymbol{E} - \boldsymbol{E}_0] \tag{2.1}$$

denote the elastic energy density per unit area, with C the elasticity tensor and $\boldsymbol{E}_0$ a constant fixed transformation strain. We denote by

$$\boldsymbol{T} = C[\boldsymbol{E} - \boldsymbol{E}_0]$$

the Cauchy stress, and write $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{u})$ when it becomes necessary to make explicit the dependence on $\boldsymbol{u}$. We assume that C is positive definite, i.e.,

$$\boldsymbol{E} \cdot C[\boldsymbol{E}] > 0, \tag{2.2}$$

for all symmetric $\boldsymbol{E} \neq 0$, a condition that guarantees the convexity of the elastic energy density φ (since φ is quadratic, see e.g. [8]).

Denoting by $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ the outward unit normal to $\partial\Omega$ at $\boldsymbol{x}$, with corresponding tangent plane

$$T_{\boldsymbol{x}}(\partial\Omega) = \{\boldsymbol{v} \in \mathbb{R}^3 \,:\, \boldsymbol{v} \cdot \boldsymbol{n}(\boldsymbol{x}) = 0\},$$

we assume that

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega, \tag{2.3}$$

so that $\boldsymbol{u}$ is tangential to $\partial\Omega$, i.e., $\boldsymbol{u}(\boldsymbol{x}) \in T_{\boldsymbol{x}}(\partial\Omega)$ for $\boldsymbol{x} \in \partial\Omega$.

To introduce the surface energy we need to define the *surface gradient* of the trace of $\boldsymbol{u}$ on $\partial\Omega$. Given a vector field $\boldsymbol{w}$ on $\partial\Omega$ the surface gradient of $\boldsymbol{w}$ at a point $\boldsymbol{x}_0 \in \partial\Omega$ is the linear operator $D_S\boldsymbol{w} = D_S\boldsymbol{w}(\boldsymbol{x}_0) : T_{\boldsymbol{x}_0}(\partial\Omega) \to \mathbb{R}^3$ given by

$$(D_S\boldsymbol{w})\boldsymbol{t} = \frac{d}{ds}\,\boldsymbol{w}(\boldsymbol{x}(s))\bigg|_{s=0},$$

with $\boldsymbol{x} : (-\delta, \delta) \subset \mathbb{R} \to \partial\Omega$ any regular curve with $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}(0) = \boldsymbol{t}$. If $\boldsymbol{w}$ is the trace of $\boldsymbol{u}$ at $\partial\Omega$, the surface gradient of $\boldsymbol{u}$ is just the projection on the tangent plane of the trace of $\nabla \boldsymbol{u}$:

$$D_S \boldsymbol{u} = (\nabla \boldsymbol{u})\boldsymbol{P},$$

where $\boldsymbol{P} = \mathbf{1} - \boldsymbol{n} \otimes \boldsymbol{n}$ is the projection operator onto the tangent plane. At a fixed point, the linear operator $D_S \boldsymbol{u}$ maps the tangent plane at that point into $\mathbb{R}^3$, but it is convenient to extend it to a linear operator of $\mathbb{R}^3$ in itself, such that $(D_S \boldsymbol{u})\boldsymbol{n} = 0$. With this convention, it is straightforward to define the pointwise norm of $D_S \boldsymbol{u}$ through

$$|D_S \boldsymbol{u}| = \sqrt{\operatorname{tr}\left[(D_S \boldsymbol{u})^\top D_S \boldsymbol{u}\right]}.$$

A displacement field on $\mathbb{R}^3 \setminus \Omega$ is *coherent* if it vanishes on $\partial\Omega$, or equivalently if

$$D_S \boldsymbol{u} = 0 \quad \text{on}\, \partial\Omega. \tag{2.4}$$

To prove this equivalence, note that if $D_S \boldsymbol{u} = 0 \quad$ on $\partial\Omega$ then $\boldsymbol{u}$ is constant on $\partial\Omega$, and this is incompatible with the constraint $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ unless $\boldsymbol{u} = 0$, identically.

When $\boldsymbol{u}$, or equivalently $D_S \boldsymbol{u}$, does not vanish identically, we say that the inclusion is *incoherent.*

To penalize incoherency, we introduce an interfacial energy density f, per unit area of $\partial\Omega$, which depends both on the orientation of the boundary (to account for anisotropy), and on the surface gradient of the displacement $\boldsymbol{u}$. Thus f is, a priori, a function of $(\boldsymbol{n}, D_S \boldsymbol{u})$, with domain $\{(\boldsymbol{n}, \boldsymbol{V}) \in \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \,:\, |\boldsymbol{n}| = 1 \,\text{and}\, \boldsymbol{V}\boldsymbol{n} = 0\}$. But since $\partial\Omega$ is fixed and $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ is therefore assigned, we may also write the interfacial energy density as a function

$$f = f(\boldsymbol{x}, D_S \boldsymbol{u})$$

defined on $\Sigma = \{(\boldsymbol{x}, \boldsymbol{V}) \in \partial\Omega \times \mathbb{R}^{3\times 3} \,:\, \boldsymbol{V}\boldsymbol{n}(\boldsymbol{x}) = 0\}$.

In general, we assume that f is continuous, $f(\boldsymbol{x}, \boldsymbol{V}) > 0$ if $\boldsymbol{V} \neq 0$ and $f(\boldsymbol{x}, 0) = 0$ for every $\boldsymbol{x} \in \partial\Omega$ (so that a coherent displacement is an absolute minimum of the interfacial energy density), and the growth condition

$$\frac{1}{K}(|\boldsymbol{V}|^q - 1) \leq f(\boldsymbol{x}, \boldsymbol{V}) \leq K(|\boldsymbol{V}|^q + 1), \qquad q \geq 1 \tag{2.5}$$

holds for all $(\boldsymbol{x}, \boldsymbol{V}) \in \Sigma$, with K a positive constant.

Our aim is to minimize the functional

$$F(\boldsymbol{u}) = \int_{\mathbb{R}^3\setminus\Omega} \varphi(\boldsymbol{E} - \boldsymbol{E}_0)\, dv + \int_{\partial\Omega} f(\boldsymbol{x}, D_S \boldsymbol{u})\, da, \tag{2.6}$$

on the space of sufficiently regular displacement fields $\boldsymbol{u}$ on $\mathbb{R}^3 \setminus \Omega$, such that (2.3) holds. More precisely, we minimize $F(\boldsymbol{u})$ on the space

$$\begin{aligned} W \;\; = \;\; \{ & \boldsymbol{u} \in W^{1,2}_{\text{loc}}(\mathbb{R}^3 \setminus \Omega; \mathbb{R}^3),\; \boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}_0 \in L^2(\mathbb{R}^3 \setminus \Omega; \mathbb{R}^{3\times 3}), \\ & \boldsymbol{u}|_{\partial\Omega} \in W^{1,q}(\partial\Omega; \mathbb{R}^3),\; \text{and}\; \boldsymbol{u}|_{\partial\Omega} \cdot \boldsymbol{n} = 0 \;\}. \end{aligned}$$

We also denote the space of coherent displacements by

$$W_0 = \{ \boldsymbol{u} \in W, \quad \boldsymbol{u}|_{\partial\Omega} = \mathbf{0} \;\}. \tag{2.7}$$

We conclude this section with some properties on the surface gradient which will be used in the sequel. Denoting by $\hat{D}_S \boldsymbol{u} = \boldsymbol{P} D_S \boldsymbol{u}$ the intrinsic derivative of $\boldsymbol{u}$, and by $\boldsymbol{B} = -D_S \boldsymbol{n}$ the Weingarten map, (2.3) implies that

$$D_S \boldsymbol{u} = \hat{D}_S \boldsymbol{u} + \boldsymbol{n} \otimes \boldsymbol{B}\boldsymbol{u}. \tag{2.8}$$

This decomposition shows that incoherency is due to two essentially different mechanisms: the relative deformation of the phases within the interface, as measured by the intrinsic derivative term $\hat{D}_S \boldsymbol{u}$, which does not depend on the curvature of the interface, and the mismatch due to the relative rotation of the materials at the interface, as measured by $\boldsymbol{n} \otimes \boldsymbol{B}\boldsymbol{u}$, a term which identically vanishes on flat portions of $\partial\Omega$.

Let now $(U, \boldsymbol{\chi})$ be a chart on $\partial\Omega$, with $U \subset \partial\Omega$ and $\boldsymbol{\chi} : U \to \mathbb{R}^2$, and write $R = \boldsymbol{\chi}(U) \subset \mathbb{R}^2$. We shall use, as shorthand, the notation $\boldsymbol{x}(\boldsymbol{\xi}) := \boldsymbol{\chi}(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in R$, and $\boldsymbol{\xi}(\boldsymbol{x}) := \boldsymbol{\chi}^{-1}(\boldsymbol{x})$ for $\boldsymbol{x} \in U$. Moreover, we write

$$\boldsymbol{I} = \boldsymbol{I}(\boldsymbol{x}) := D_S \boldsymbol{\chi}(\boldsymbol{x}) : T_{\boldsymbol{x}}(\partial\Omega) \to \mathbb{R}^2,$$

for the associated isomorphism between the tangent space at $\boldsymbol{x}$ and the parameter space $\mathbb{R}^2$.

Consider a function $\boldsymbol{v} : U \to \mathbb{R}^3$ such that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. Since $\boldsymbol{v}(\boldsymbol{x}) \in T_{\boldsymbol{x}}(\partial\Omega)$, the function $\boldsymbol{v}$ admits a local representation $\boldsymbol{w} : R \to \mathbb{R}^2$ defined by

$$\boldsymbol{w}(\boldsymbol{\xi}) := \boldsymbol{I}(\boldsymbol{x}(\boldsymbol{\xi}))\boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi})) \qquad \Leftrightarrow \qquad \boldsymbol{v}(\boldsymbol{x}) := \boldsymbol{I}^{-1}(\boldsymbol{x})\boldsymbol{w}(\boldsymbol{\xi}(\boldsymbol{x})), \tag{2.9}$$

for $\boldsymbol{\xi} \in R$ and $\boldsymbol{x} \in U$. Differentiating the second relation above on $\partial\Omega$ and noting that $D_S(\cdot) = \nabla_{\boldsymbol{\xi}}(\cdot)\boldsymbol{I}$, we find

$$D_S \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi})) = \boldsymbol{I}^{-1}(\boldsymbol{x}(\boldsymbol{\xi}))\nabla_{\boldsymbol{\xi}} \boldsymbol{w}(\boldsymbol{\xi})\boldsymbol{I}(\boldsymbol{x}(\boldsymbol{\xi})) + \boldsymbol{R}(\boldsymbol{\xi}, \boldsymbol{w}), \tag{2.10}$$

with $\boldsymbol{R}(\boldsymbol{\xi}, \boldsymbol{w})$ a complicated expression that is linear in $\boldsymbol{w}$ and that involves the Christoffel symbols and the Weingarten map.

3 Korn and Poincaré Inequalities

Classically, Korn's inequality states that the L^2 norm of $\boldsymbol{E}(\boldsymbol{u})$ is equivalent to the $W^{1,2}$ norm of $\boldsymbol{u}$ on a bounded domain, if $\boldsymbol{u}$ vanishes on a region of the boundary with positive area. We here extend this result to vector fields that do not necessarily vanish at the boundary, but satisfy the constraint (2.3).

More precisely, we now show that, when $\partial\Omega$ is not a surface of revolution, the constraint $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ is sufficient to ensure the validity of Korn's inequality on bounded domains containing the inclusion Ω. If $\partial\Omega$ is a surface of revolution, then a $W^{1,2}$-bound on $\boldsymbol{u}$ also requires a bound on $D_S \boldsymbol{u}$ on $\partial\Omega$.

As a preliminary result, recall that a surface of revolution S may be characterized by its invariance with respect to rotations about a given axis, i.e., there exist vectors $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$ such that if $\boldsymbol{x} \in S$ then $\boldsymbol{y} = \boldsymbol{\xi} + \boldsymbol{R}(\boldsymbol{x} - \boldsymbol{\xi}) \in S$, with $\boldsymbol{R}$ any rotation with axis $\boldsymbol{\omega}$. In other words, S contains, with any one of its points, also a circle through this point, with center on a given axis.

In what follows we denote by B_L the open ball of $\mathbb{R}^3$ centered at the origin and of radious $L > 0$.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^3$, fix $L > 0$ so that $\overline{\Omega} \subset B_L$, and consider $\boldsymbol{u} \in W^{1,2}(B_L \setminus \Omega; \mathbb{R}^3)$ such that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.*

If $\partial\Omega$ is not a surface of revolution, then there is a constant C (depending on L) such that

$$\|\boldsymbol{u}\|^2_{W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)} \leq C\|\boldsymbol{E}\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})}.$$

Alternatively, if $\partial\Omega$ is a surface of revolution, and $\boldsymbol{u}|_{\partial\Omega} \in W^{1,q}(\partial\Omega;\mathbb{R}^3)$, with $q \geq 2$, then there is a constant C (also depending on L) such that

$$\|\boldsymbol{u}\|^2_{W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)} \leq C\left(\|D_S\boldsymbol{u}\|^q_{L^q(\partial\Omega;\mathbb{R}^{3\times3})} + \|\boldsymbol{E}\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})}\right).$$

Proof. We follow the proof of Dautray and Lions [10], and expand only those steps which depart from the original proof. First recall the basic estimate (Theorem 7.3.1, page 414, Volume II [10])

$$\|\boldsymbol{u}\|^2_{W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)} \leq \text{const.}(\|\boldsymbol{E}(\boldsymbol{u})\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})} + \|\boldsymbol{u}\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^3)}). \tag{3.1}$$

Next, note that, if $\boldsymbol{E}(\boldsymbol{u}) = 0$ in $B_L \setminus \Omega$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, then either $\boldsymbol{u} = 0$ in $B_L \setminus \Omega$ or $\partial\Omega$ is a surface of revolution. In fact, if $\boldsymbol{E}(\boldsymbol{u}) = 0$, then $\boldsymbol{u}$ is a rigid motion

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{\omega} \times \boldsymbol{x}, \qquad \boldsymbol{x} \in B_L \setminus \Omega,$$

with $\boldsymbol{a}$ and $\boldsymbol{\omega}$ constant vectors. By (2.3), the restriction of $\boldsymbol{u}$ to $\partial\Omega$ is a tangential vector field, and therefore its integral lines must be contained in $\partial\Omega$. (The field $\boldsymbol{u}$ is complete since $\partial\Omega$ is compact.) But the integral lines of $\boldsymbol{u}$ with initial point a given $\boldsymbol{x}_0$ are helices with axis $\boldsymbol{\omega}$:

$$\boldsymbol{x}(s) = t\boldsymbol{a}_{||} + \boldsymbol{\xi} + \boldsymbol{R}(t)(\boldsymbol{x}_0 - \boldsymbol{\xi}), \qquad t \in \mathbb{R},$$

with $\boldsymbol{a}_{||} = |\boldsymbol{\omega}|^{-2}(\boldsymbol{a}\cdot\boldsymbol{\omega})\boldsymbol{\omega}$, $\boldsymbol{\xi} = |\boldsymbol{\omega}|^{-2}\boldsymbol{\omega} \times \boldsymbol{a}$, and $\boldsymbol{R}(s)$ a rotation about $\boldsymbol{\omega}$ of angle $|\boldsymbol{\omega}|t$. Note that, since $\partial\Omega$ is bounded, $\boldsymbol{a}_{||} = 0$.

Assume that $\boldsymbol{\omega} \neq 0$, and choose an arbitrary $\boldsymbol{x}_0 \in \partial\Omega$. Then the above formula implies that $\boldsymbol{x} = \boldsymbol{\xi} + \boldsymbol{R}(\boldsymbol{x}_0 - \boldsymbol{\xi})$ for any rotation $\boldsymbol{R}$ with axis $\boldsymbol{\omega}$, and this implies that $\partial\Omega$ must be a surface of revolution.

Hence, if Ω is not a surface of revolution, we must have $\boldsymbol{\omega} = 0$, and this, together with $\boldsymbol{a}_{||} = 0$, implies that $\boldsymbol{u} = 0$ on $B_L\setminus\Omega$. We have thus proved that $\boldsymbol{E}(\boldsymbol{u}) = 0$ implies that $\boldsymbol{u} = 0$ for any $\boldsymbol{u} \in W^{1,2}(B_L \setminus \Omega; \mathbb{R}^3)$ compatible with (2.3). We may now proceed as in [10] to show the validity of the estimate

$$\|\boldsymbol{u}\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^3)} \leq \text{const.}\|\boldsymbol{E}\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})},$$

an estimate that implies Korn's inequality, but the argument is standard and need not be given here (see [10]).

To prove Korn's inequality when $\partial\Omega$ is a surface of revolution, note first that $\boldsymbol{E}(\boldsymbol{u}) = 0$ in $B_L\setminus\Omega$ and $D_S\boldsymbol{u} = 0$ on $\partial\Omega$ imply that $\boldsymbol{u} = 0$ on $B_L\setminus\Omega$. In fact, since $D_S\boldsymbol{u} = 0$ on $\partial\Omega$ implies that $\boldsymbol{u} = 0$ on $\partial\Omega$, then $\boldsymbol{u} = 0$ everywhere, since $\boldsymbol{u}$ is a rigid displacement.

The last step of the proof is to show that, when $q \geq 2$,

$$\|\boldsymbol{u}\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^3)} \leq \text{const.}(\|D_S\boldsymbol{u}\|^q_{L^q(\partial\Omega;\mathbb{R}^{3\times3})} + \|\boldsymbol{E}\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})}),$$

from which, granted (3.1), the second part of the thesis follows. Assume, for contradiction, that the above estimate does not hold. Then there is a sequence $\{\boldsymbol{u}_n\}$ such that

$$n(\|D_S\boldsymbol{u}_n\|^q_{L^q(\partial\Omega;\mathbb{R}^{3\times3})} + \|\boldsymbol{E}(\boldsymbol{u}_n)\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})}) < \|\boldsymbol{u}_n\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^3)},$$

and thus, since $q \geq 2$,

$$n(\|D_S\boldsymbol{u}_n\|^2_{L^2(\partial\Omega;\mathbb{R}^{3\times3})} + \|\boldsymbol{E}(\boldsymbol{u}_n)\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})}) < \text{const.}\|\boldsymbol{u}_n\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^3)},$$

and dividing both sides of the above inequality by $\|\boldsymbol{u}_n\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^3)}$, this is equivalent to the existence of a sequence $\{\tilde{\boldsymbol{u}}_n\}$, with $\|\tilde{\boldsymbol{u}}_n\|_{L^2(B_L\setminus\Omega;\mathbb{R}^3)} = 1$, such that

$$\|D_S\tilde{\boldsymbol{u}}_n\|^2_{L^2(\partial\Omega;\mathbb{R}^{3\times3})} + \|\boldsymbol{E}(\tilde{\boldsymbol{u}}_n)\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})} < \text{const.}\,\frac{1}{n}.$$

Thus $\|D_S\tilde{\boldsymbol{u}}_n\|_{L^2(\partial\Omega;\mathbb{R}^{3\times3})} \to 0$ and $\|\boldsymbol{E}(\tilde{\boldsymbol{u}}_n)\|_{L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})} \to 0$. Since by (3.1) the sequence $\{\tilde{\boldsymbol{u}}_n\}$ is bounded in $W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)$, it converges weakly to a limit $\tilde{\boldsymbol{u}}$ in $W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)$. This implies that $\boldsymbol{E}(\tilde{\boldsymbol{u}}_n) \rightharpoonup \boldsymbol{E}(\tilde{\boldsymbol{u}})$ in $L^2(B_L\setminus\Omega;\mathbb{R}^{3\times3})$ and, by the lower semicontinuity of the norm, $\boldsymbol{E}(\tilde{\boldsymbol{u}}) = 0$. A similar procedure shows also that $D_S\tilde{\boldsymbol{u}} = 0$ on $\partial\Omega$. Hence, by the argument given above, we may conclude that $\tilde{\boldsymbol{u}} = 0$ on $B_L\setminus\Omega$.

On the other hand, $\tilde{\boldsymbol{u}}_n \rightharpoonup \tilde{\boldsymbol{u}}$ in $W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)$, so that $\tilde{\boldsymbol{u}}_n \to \tilde{\boldsymbol{u}}$ strongly in $L^2(B_L\setminus\Omega;\mathbb{R}^3)$. But since $\|\tilde{\boldsymbol{u}}_n\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^3)} = 1$, this implies that $\|\tilde{\boldsymbol{u}}\|^2_{L^2(B_L\setminus\Omega;\mathbb{R}^3)} = 1$, which is the desired contradiction.

□

Remark 3.2. Korn's inequality on bounded domains always holds for $\partial\Omega$ an arbitrary surface (possibly of revolution) when $\boldsymbol{u}$ is coherent.

We prove here a version of Poincaré inequality for vector fields on $\partial\Omega$. The proof follows Hebey [17], with minor modifications.

Proposition 3.3. *Let $\boldsymbol{u} \in W^{1,q}(\partial\Omega;\mathbb{R}^3)$ for $1 \leq q < 2$, and let $\Gamma \subset \partial\Omega$ be an open subset with positive area. Then there is a constant C (depending on Γ) such that*

$$\|\boldsymbol{u} - \bar{\boldsymbol{u}}_\Gamma\|_{L^q(\partial\Omega;\mathbb{R}^3)} \leq C\|D_S\boldsymbol{u}\|_{L^q(\partial\Omega;\mathbb{R}^{3\times3})}, \tag{3.2}$$

with

$$\bar{\boldsymbol{u}}_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma \boldsymbol{u}\,da.$$

Proof. Let $q \in (1,2)$. As in the proof of Korn's inequality, we must prove that

$$\inf_{\boldsymbol{u}\in H} \|D_S\boldsymbol{u}\|_{L^q(\partial\Omega;\mathbb{R}^{3\times3})} > 0,$$

with

$$H = \{\boldsymbol{u} \in W^{1,q}(\partial\Omega;\mathbb{R}^3) \,:\, \|\boldsymbol{u}\|_{L^q(\partial\Omega;\mathbb{R}^3)} = 1 \text{ and } \int_\Gamma \boldsymbol{u}\,da = 0\}.$$

Then, since $q > 1$, and by Rellich Theorem, a minimizing sequence for $\|D_S\boldsymbol{u}\|_{L^q(\partial\Omega;\mathbb{R}^3)}$ on H converges weakly in $W^{1,q}(\partial\Omega;\mathbb{R}^3)$ and strongly in $L^q(\partial\Omega;\mathbb{R}^3)$ to a function $\boldsymbol{u} \in H$. But since $\int_\Gamma \boldsymbol{u}\,da = 0$ and the area of Γ is positive, $\boldsymbol{u}$ cannot be constant, and this yields the desired conclusion.

Let $q = 1$. We prove the assertion for a scalar field $u \in W^{1,1}(\partial\Omega;\mathbb{R})$, which may be identified to one of the components of $\boldsymbol{u}$ in a fixed basis of $\mathbb{R}^3$. Note first that, since $\partial\Omega$ is compact, there exists a Green function $G(\boldsymbol{x},\boldsymbol{y})$ for the surface Laplacian Δ_S, such that

$$u(\boldsymbol{x}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u\, da + \int_{\partial\Omega} G(\boldsymbol{x},\boldsymbol{y})\Delta_S u(\boldsymbol{y})\, da_{\boldsymbol{y}}. \tag{3.3}$$

Define $h : \partial\Omega \to \mathbb{R}$ such that

$$\Delta_S h = \frac{1}{|\partial\Omega|} - \frac{1}{|\Gamma|}\chi_\Gamma,$$

with χ_Γ the characteristic function of Γ, and note that, granted (3.3), h is a smooth function on $\partial\Omega$. Moreover, the identity

$$\int_{\partial\Omega} h\Delta_S u\, da - \int_{\partial\Omega} u\Delta_S h\, da = 0,$$

implies that

$$\int_{\partial\Omega} h\Delta_S u\, da - \bar{u}_{\partial\Omega} + \bar{u}_\Gamma = 0,$$

and adding both sides of this equality to (3.3), we obtain

$$u(\boldsymbol{x}) = \bar{u}_\Gamma + \int_{\partial\Omega} H(\boldsymbol{x},\boldsymbol{y})\Delta_S u(\boldsymbol{y})\, da_{\boldsymbol{y}},$$

with $H(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{x},\boldsymbol{y}) + h(\boldsymbol{y})$. The proof now proceeds as in ([17]): the above inequality implies that

$$\int_{\partial\Omega} |u - \bar{u}_\Gamma|\, da \le \int_{\partial\Omega}\int_{\partial\Omega} |D_S H(\boldsymbol{x},\boldsymbol{y})||D_S u(\boldsymbol{y})|\, da_{\boldsymbol{x}}\, da_{\boldsymbol{y}},$$

which implies the desired inequality, granted the growth properties of the Green function and the fact that h is smooth.

□

The preceding result allows to prove a stronger version of Poincaré inequality for tangential vector fields.

Corollary 3.4. *Let $\boldsymbol{u} \in W^{1,q}(\partial\Omega;\mathbb{R}^3)$ for $1 \le q < 2$, such that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. Then there is a constant C such that*

$$\|\boldsymbol{u}\|_{L^q(\partial\Omega;\mathbb{R}^3)} \le C\|D_S\boldsymbol{u}\|_{L^q(\partial\Omega;\mathbb{R}^{3\times 3})}. \tag{3.4}$$

Proof. First note that, since $\partial\Omega$ is bounded, there is an $\boldsymbol{x}_0 \in \partial\Omega$ such that the principal curvatures $\lambda_1(\boldsymbol{x}_0)$, $\lambda_2(\boldsymbol{x}_0)$ (the eigenvalues of the Weingarten map) are both positive. Hence, there exists an open domain $\Gamma \subset \partial\Omega$, with positive Hausdorff measure, such that $\lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in \Gamma$, which implies, by the minimax property of the eigenvalues, that there is a positive constant b such that $|\boldsymbol{B}(\boldsymbol{x})\boldsymbol{s}(\boldsymbol{x})| \ge b|\boldsymbol{s}(\boldsymbol{x})|$ for any tangent vector field on Γ, uniformly in $\boldsymbol{x} \in \Gamma$. Hence

$$\int_{\partial\Omega} |D_S\boldsymbol{u}|\, da \ge \int_\Gamma |D_S\boldsymbol{u}|\, da = \int_\Gamma \sqrt{|\hat{D}_S\boldsymbol{u}|^2 + |\boldsymbol{B}\boldsymbol{u}|^2}\, da \ge b\int_\Gamma |\boldsymbol{u}|\, da.$$

But the Poincaré inequality (3.2) implies that,

$$\int_{\partial\Omega} |\boldsymbol{u}|^q \, da \leq c_1 \left(\int_{\partial\Omega} |D_S \boldsymbol{u}|^q \, da + \frac{1}{|\Gamma|} \left| \int_{\Gamma} \boldsymbol{u} \, da \right|^q \right),$$

for some constant c_1, and the thesis follows from Hölder's inequality.

□

4 Coherent displacements

In this section we prove the existence and uniqueness of a minimizer of the elastic energy functional on the space of coherent displacements, and characterize it in terms of the limit of sequences of coherent minimizers on bounded domains.

For fixed L such that $\overline{\Omega} \subset B_L$, let

$$W_0^L = \{\boldsymbol{u} \in W^{1,2}(B_L \setminus \Omega; \mathbb{R}^3) \ : \ \boldsymbol{u}|_{\partial\Omega} = 0\}, \tag{4.1}$$

and

$$F_0^L(\boldsymbol{u}) = \int_{B_L \setminus \Omega} \varphi(\boldsymbol{E} - \boldsymbol{E}_0) \, dv. \tag{4.2}$$

Since φ is strictly convex, standard arguments, together with Korn's inequality (see Remark 3.2), imply that there exists a unique minimizer $\boldsymbol{u}_{\text{co}}^L$ of F_0^L on W_0^L. Note that $\boldsymbol{u}_{\text{co}}^L$ satisfies the Euler equations

$$\begin{array}{ll} \text{Div}\, \boldsymbol{T}(\boldsymbol{u}_{\text{co}}^L) = 0 & \text{in } \ B_L \setminus \Omega, \\ \boldsymbol{T}(\boldsymbol{u}_{\text{co}}^L)\boldsymbol{n} = 0 & \text{on } \ \partial B_L. \end{array} \tag{4.3}$$

Theorem 4.1. *For any sequence $L_n \to \infty$ and any fixed L such that $\overline{\Omega} \subset B_L$,*

$$\boldsymbol{u}_{\text{co}}^{L_n} \to \boldsymbol{u}_{\text{co}} \qquad \textit{in } W_0^L,$$

where $\boldsymbol{u}_{\text{co}}$ is the unique minimizer on W_0 of the energy functional

$$F_0(\boldsymbol{u}) = \int_{\mathbb{R}^3 \setminus \Omega} \varphi(\boldsymbol{E} - \boldsymbol{E}_0) \, dv. \tag{4.4}$$

Proof. Fix any $\boldsymbol{v} \in W_0$ for which $F_0(\boldsymbol{v}) < \infty$, and consider a sequence $\{\boldsymbol{u}_{\text{co}}^{L_n}\}$ with $L_n \to +\infty$. Then, for $L_n > L$,

$$F_0^L(\boldsymbol{u}_{\text{co}}^{L_n}) \leq F_0^{L_n}(\boldsymbol{u}_{\text{co}}^{L_n}) \leq F_0^{L_n}(\boldsymbol{v}) \leq F_0(\boldsymbol{v}) < \infty. \tag{4.5}$$

Hence, by Korn's inequality on bounded domains (see Remark 3.2), the sequence $\{\boldsymbol{u}_{\text{co}}^{L_n}\}$ is bounded in W_0^L, so that (passing to a subsequence) $\boldsymbol{u}_{\text{co}}^{L_n} \rightharpoonup \boldsymbol{u}^L$ in W_0^L, for some $\boldsymbol{u}^L \in W_0^L$.

Now, for $L' > L$, apply (4.5) to the above subsequence, and extract a subsequence converging weakly to $\boldsymbol{u}^{L'}$ in $W_0^{L'}$, and thus also in W_0^L. By the uniqueness of the limit, $\boldsymbol{u}^{L'} = \boldsymbol{u}^L$ in $B_L \setminus \Omega$. Hence there exists a function $\boldsymbol{u}_{co} \in W^{1,2}_{\text{loc}}(\mathbb{R}^3 \setminus \Omega; \mathbb{R}^3)$ in $\mathbb{R}^3 \setminus \Omega$ such that, for any L, $\boldsymbol{u}_{co} = \boldsymbol{u}^L$ in $B_L \setminus \Omega$ and

$$\boldsymbol{u}_{\text{co}}^{L_n} \rightharpoonup \boldsymbol{u}_{\text{co}} \qquad \text{in } \ W^{1,2}_{loc}(B_L \setminus \Omega; \mathbb{R}^3).$$

By the lower semicontinuity of $F_0^L(\boldsymbol{u})$ and (4.5), we have

$$F_0^L(\boldsymbol{u}_{co}) \leq \liminf_{n\to+\infty} F_0^L(\boldsymbol{u}_{\rm co}^{L_n}) \leq F_0(\boldsymbol{v})$$

for any L, so that $F_0(\boldsymbol{u}_{co}) \leq F_0(\boldsymbol{v})$, and by the arbitrariness of $\boldsymbol{v} \in W_0$ it follows that

$$F_0(\boldsymbol{u}_{co}) = \min_{\boldsymbol{v}\in W_0} F_0(\boldsymbol{v}).$$

It is easy to verify, again by the strictly convexity of φ and Korn's inequality in bounded domains, that $\boldsymbol{u}_{co}$ is unique.

Next we prove regularity and strong convergence. Fix $L_0 < L$ such that $\overline{\Omega} \subset B_{L_0}$, and note that $\operatorname{Div} \boldsymbol{T}(\boldsymbol{u}_{\rm co}^{L_n}) = 0$ in $B_L \setminus \Omega$. Let ψ be a smooth function such that $\psi = 1$ in $B_{L_0} \setminus \Omega$ and $\psi = 0$ in $\mathbb{R}^3 \setminus B_L$, and let $\boldsymbol{v}^L = \psi \boldsymbol{u}_{\rm co}^L$. Then

$$\begin{array}{ll} \operatorname{Div} \boldsymbol{T}(\boldsymbol{v}^L) = \boldsymbol{f} & \text{in } \ B_L \setminus \Omega, \\ \boldsymbol{v}^L = 0 & \text{on } \ \partial(B_L \setminus \Omega), \end{array}$$

where $\boldsymbol{f} = C(\boldsymbol{E}(\boldsymbol{u}_{\rm co}^L))\nabla\psi + \operatorname{Div}[C(\boldsymbol{u}_{\rm co}^L \otimes \nabla\psi)]$.

By Theorem 2.3 in [16]

$$\|\boldsymbol{v}^L\|_{H^{k+1}(B_L\setminus\Omega;\mathbb{R}^3)} \leq \text{cost.}\,(\|\boldsymbol{f}\|_{H^{k-1}(B_L\setminus\Omega;\mathbb{R}^3)} + \|\boldsymbol{v}^L\|_{L^2(B_L\setminus\Omega;\mathbb{R}^3)}),$$

and, since $\boldsymbol{f}$ is linear in $\boldsymbol{u}_{co}^L$ and $\nabla\boldsymbol{u}_{co}^L$, this implies that

$$\|\boldsymbol{u}_{co}^L\|_{H^{k+1}(B_{L_0}\setminus\Omega;\mathbb{R}^3)} \leq \text{cost.}\,(\|\nabla\boldsymbol{u}_{co}^L\|_{H^{k-1}(B_L\setminus\Omega;\mathbb{R}^3)} + \|\boldsymbol{u}_{co}^L\|_{L^2(B_L\setminus\Omega;\mathbb{R}^3)}).$$

Taking $k = 1$ and applying this inequality to $\boldsymbol{u}_{co}^{L_n}$, and noting that the sequence $\{\boldsymbol{u}_{co}^{L_n}\}$ is bounded in $W^{1,2}(B_L \setminus \Omega; \mathbb{R}^3)$, it follows that $\{\boldsymbol{u}_{co}^{L_n}\}$ is also bounded in $H^2(B_{L_0} \setminus \Omega; \mathbb{R}^3)$. Applying this procedure iteratively, and using the arbitrariness of L_0, we find that $\boldsymbol{u}_{co}^{L_n}$ is bounded in $H^k(B_{L_0} \setminus \Omega; \mathbb{R}^3)$ for any k and any L_0. The same procedure also shows that $\boldsymbol{u}_{co} \in H^k(B_{L_0} \setminus \Omega; \mathbb{R}^3)$ for any k and L_0. Hence, by Rellich's Theorem, the sequence $\{\boldsymbol{u}_{co}^{L_n}\}$ converges strongly to $\boldsymbol{u}_{co}$ in $H^k(B_L \setminus \Omega; \mathbb{R}^3)$ for any k and L.

□

Corollary 4.2. *For any sequence $L_n \to \infty$,*

$$\boldsymbol{T}(\boldsymbol{u}_{\rm co}^{L_n})\boldsymbol{n} \to \boldsymbol{T}(\boldsymbol{u}_{\rm co})\boldsymbol{n}$$

strongly in $L^2(\partial\Omega)$.

Proof. Since the sequence $\{\boldsymbol{u}_{co}^{L_n}\}$ converges strongly to $\boldsymbol{u}_{co}$ in $H^k(B_L \setminus \Omega; \mathbb{R}^3)$ for any k and L_0, the traces of $\{\boldsymbol{u}_{\rm co}^{L_n}\}$ and $\{\nabla\boldsymbol{u}_{\rm co}^{L_n}\}$ on $\partial\Omega$ also converge strongly in $H^k(\partial\Omega)$ for any k.

□

Remark 4.3. Given $\boldsymbol{\tau} \in W^{1,q}(\partial\Omega; \mathbb{R}^3)$ such that $\boldsymbol{\tau} \cdot \boldsymbol{n} = 0$, let $\boldsymbol{u}_{\boldsymbol{\tau}}$ be the unique minimizer of $F_0(\boldsymbol{u})$ on the subspace of W such that $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{\tau}$. Then results completely analogous to Theorems 4.1 and 4.2 hold for any sequence $\{\boldsymbol{u}_{\boldsymbol{\tau}}^{L_n}\}$ of local minimizers.

5 Threshold effect

In order to study the effect of the size of the inclusion, we rescale the functional using as a parameter a characteristic length ℓ such that $|\partial\Omega| = \ell^2$; through the transformation

$$\tilde{\boldsymbol{x}} = \frac{1}{\ell}\,\boldsymbol{x},$$

the domain Ω is mapped into a new region $\tilde{\Omega}$ such that $|\partial\tilde{\Omega}| = 1$ and $|\tilde{\Omega}| = |\Omega|/\ell^3$. Thus letting

$$\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}) := \frac{1}{\ell}\boldsymbol{u}(\ell\tilde{\boldsymbol{x}}),$$

we have $\tilde{\nabla}\tilde{\boldsymbol{u}} = \nabla\boldsymbol{u}$, and the rescaled energy functional becomes

$$F(\tilde{\boldsymbol{u}}) = \ell^3\int_{\mathbb{R}^3\setminus\tilde{\Omega}}\varphi(\tilde{\boldsymbol{E}} - \boldsymbol{E}_0)\,dv + \ell^2\int_{\partial\tilde{\Omega}} f(\tilde{\boldsymbol{x}}, D_S\tilde{\boldsymbol{u}})\,da,$$

and this suggests the introduction of the functional

$$F_\ell(\boldsymbol{u}) = \int_{\mathbb{R}^3\setminus\Omega}\varphi(\boldsymbol{E} - \boldsymbol{E}_0)\,dv + \frac{1}{\ell}\int_{\partial\Omega} f(\boldsymbol{x}, D_S\boldsymbol{u})\,da, \tag{5.1}$$

where we have omitted the tilde, so that $|\partial\Omega| = 1$. Note that the functional F_ℓ coincides with F_0 on the subspace W_0 of coherent displacements.

The next result, which is central, relates the threshold effect to the smoothness of the interfacial energy density: when the interfacial energy is non smooth a small inclusion is coherent as long as its size does not exceed a threshold value.

Theorem 5.1. *Assume that*

$$f(\boldsymbol{x}, \boldsymbol{V}) \geq A|\boldsymbol{V}| \tag{5.2}$$

for all $(\boldsymbol{x}, \boldsymbol{V}) \in \Sigma$, *with* $|\boldsymbol{V}| \leq 1$, *where* A *is a positive constant. Then there is an* ℓ_0 *such that for* $\ell < \ell_0$ *the inclusion is coherent; i.e., the functional* F_ℓ *has a minimum in* W, *and the minimizer is the coherent displacement* $\boldsymbol{u}_{\rm co}$.

Proof. We split the proof in two steps:

Step 1 . We prove that

$$F_\ell(\boldsymbol{u}) - F_\ell(\boldsymbol{u}_{co}) \geq -\int_{\partial\Omega}\boldsymbol{T}(\boldsymbol{u}_{co})\boldsymbol{n}\cdot\boldsymbol{u}\,da + \frac{1}{\ell}\int_{\partial\Omega} f(\boldsymbol{x}, D_S\boldsymbol{u})\,da \tag{5.3}$$

for any $\boldsymbol{u} \in W$. For every L such that $\overline{\Omega} \subset B_L$, let

$$W^L = \{\boldsymbol{u} \in W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)\ :\ \boldsymbol{u}|_{\partial\Omega} \in W^{1,q}(\partial\Omega;\mathbb{R}^3) \text{ and } \boldsymbol{u}|_{\partial\Omega}\cdot\boldsymbol{n} = 0\}. \tag{5.4}$$

By the divergence theorem, we have, for $\boldsymbol{u} \in W^L$,

$$\begin{aligned} F_0^L(\boldsymbol{u}) - F_0^L(\boldsymbol{u}_{\rm co}^L) &= Q + \int_{\partial B_L}\boldsymbol{T}(\boldsymbol{u}_{\rm co}^L)\boldsymbol{n}\cdot(\boldsymbol{u} - \boldsymbol{u}_{\rm co}^L)\,da \\ &\quad - \int_{B_L\setminus\Omega}{\rm Div}\,\boldsymbol{T}(\boldsymbol{u}_{\rm co}^L)\cdot(\boldsymbol{u} - \boldsymbol{u}_{\rm co}^L)\,dv \\ &\quad - \int_{\partial\Omega}\boldsymbol{T}(\boldsymbol{u}_{\rm co}^L)\boldsymbol{n}\cdot(\boldsymbol{u} - \boldsymbol{u}_{\rm co}^L)\,da, \end{aligned}$$

where Q is a nonnegative quantity. Since $\boldsymbol{u}^L_{\rm co}$ satisfies the Euler equations (4.3), this implies that, for any $\boldsymbol{u} \in W$,

$$F_0^L(\boldsymbol{u}) - F_0^L(\boldsymbol{u}^L_{\rm co}) \geq -\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u}^L_{\rm co})\boldsymbol{n} \cdot \boldsymbol{u}\, da. \tag{5.5}$$

Fix $\bar{L}$ and take a sequence $L_n \to +\infty$. Then

$$F_0^{L_n}(\boldsymbol{u}) - F_0^{L_n}(\boldsymbol{u}^{L_n}_{\rm co}) \leq F_0(\boldsymbol{u}) - F_0^{\bar{L}}(\boldsymbol{u}^{L_n}_{\rm co}),$$

for $L_n > \bar{L}$, so that

$$F_0(\boldsymbol{u}) - F_0^{\bar{L}}(\boldsymbol{u}^{L_n}_{\rm co}) \geq -\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u}^{L_n}_{\rm co})\boldsymbol{n} \cdot \boldsymbol{u}\, da. \tag{5.6}$$

By Theorem 4.2,

$$\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u}^{L_n}_{\rm co})\boldsymbol{n} \cdot \boldsymbol{u}\, da \to \int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u}_{co})\boldsymbol{n} \cdot \boldsymbol{u}\, da, \tag{5.7}$$

and the last integral is independent of $\bar{L}$. Thus, taking the limit in (5.6), and using the lower-semicontinuity of $F_0^{\bar{L}}$ and Theorem 4.1, we obtain

$$F_0(\boldsymbol{u}) - F_0^{\bar{L}}(\boldsymbol{u}_{co}) \geq -\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{u}_{co})\boldsymbol{n} \cdot \boldsymbol{u}\, da,$$

and since this relation holds for any $\bar{L}$, this yields the desired result.

Step 2. Note that, by (5.3),

$$F_\ell(\boldsymbol{u}) - F_\ell(\boldsymbol{u}_{co}) \geq \int_{\partial\Omega} \left(\frac{1}{\ell} f(\boldsymbol{x}, D_S\boldsymbol{u}) - \tau_0|\boldsymbol{u}|\right) da,$$

where $\tau_0 = \sup_{\partial\Omega} |\boldsymbol{P}\boldsymbol{T}(\boldsymbol{u}_{co})\boldsymbol{n}|$, and where we have used the fact that $\boldsymbol{u}$ is tangential to $\partial\Omega$. In view of the continuity of f, the growth conditions (2.5) and (5.2), and the fact that $f(\boldsymbol{x}, \boldsymbol{V}) > 0$ for $\boldsymbol{V} \neq \boldsymbol{0}$, there is a constant $B > 0$ such that

$$f(\boldsymbol{x}, \boldsymbol{V}) \geq B|\boldsymbol{V}| \quad \text{for all } \boldsymbol{V}.$$

Consequently,

$$F_\ell(\boldsymbol{u}) - F_\ell(\boldsymbol{u}_{co}) \geq \int_{\partial\Omega} \left(\frac{B}{\ell}|D_S\boldsymbol{u}| - \tau_0|\boldsymbol{u}|\right) da;$$

but, by Poincaré inequality (3.4),

$$\|D_S\boldsymbol{u}\|_{L^1(\partial\Omega;\mathbb{R}^{3\times3})} \geq C\|\boldsymbol{u}\|_{L^1(\partial\Omega;\mathbb{R}^3)}.$$

Hence,

$$F_\ell(\boldsymbol{u}) - F_\ell(\boldsymbol{u}_{\rm co}) \geq \|\boldsymbol{u}\|_{L^1(\partial\Omega;\mathbb{R}^3)} \left(\frac{BC}{\ell} - \tau_0,\right)$$

and the right hand side of this expression is nonnegative for $\ell < \ell_0 = \frac{BC}{\tau_0}$.

□

The following result shows that: (i) coherency is lost when the inclusion is sufficiently large, and (ii) when the interfacial energy is smooth, the inclusion can never be coherent, no matter how small its size.

Theorem 5.2. *Assume that*

$$f(\boldsymbol{x}, \mathbf{V}) \le A_1 |\mathbf{V}|^p, \qquad p \ge 1, \tag{5.8}$$

for all $(\boldsymbol{x}, \mathbf{V}) \in \Sigma$, *with* $|\boldsymbol{V}| \le 1$, *where* A_1 *is a positive constant. Then there exists* $\ell_1 > 0$ *such that if* $p = 1$ *and* $\ell > \ell_1$, *or if* $p > 1$ *(and no restriction on* ℓ*), then the inclusion cannot be coherent.*

Proof. Let $\boldsymbol{u}_{\boldsymbol{\tau}}$, as in Remark 4.3, be the unique minimizer of $F_0(\boldsymbol{u})$ on the subspace of W such that $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{\tau}$, and let $\boldsymbol{u}_{\boldsymbol{\tau}}^L$ be the minimizer of $F_0^L(\boldsymbol{u})$ on the subspace of W^L such that $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{\tau}$. We prove that we may choose $\boldsymbol{\tau}$ such that $F_\ell(\boldsymbol{u}_{\boldsymbol{\tau}}) < F_\ell(\boldsymbol{u}_{\rm co})$. Again we split the proof into two steps:

Step 1 . We first prove that

$$\begin{aligned} F_0(\boldsymbol{u}_{\boldsymbol{\tau}}) - F_0(\boldsymbol{u}_{co}) \le &- \tfrac{1}{2} \textstyle\int_{\partial\Omega} C[\boldsymbol{E}(\boldsymbol{u}_{\boldsymbol{\tau}})]\boldsymbol{n} \cdot \boldsymbol{u}_{\boldsymbol{\tau}}\, da \\ &- \int_{\partial\Omega} C[\tfrac{1}{2}\boldsymbol{E}(\boldsymbol{u}_{\rm co}) - \boldsymbol{E}_0]\boldsymbol{n} \cdot \boldsymbol{u}_{\boldsymbol{\tau}}\, da. \end{aligned} \tag{5.9}$$

In fact, for any L such that $\overline{\Omega} \subset B_L$,

$$\begin{aligned} F_0^L(\boldsymbol{u}_{\boldsymbol{\tau}}^L) - F_0^L(\boldsymbol{u}_{\rm co}^L) = &\int_{B_L \setminus \Omega} \varphi(\boldsymbol{E}(\boldsymbol{u}_{\boldsymbol{\tau}}^L) - \boldsymbol{E}(\boldsymbol{u}_{\rm co}^L))\, dv \\ &+ \int_{B_L \setminus \Omega} \mathbf{T}(\boldsymbol{u}_{\rm co}^L) \cdot (\boldsymbol{E}(\boldsymbol{u}_{\boldsymbol{\tau}}^L) - \boldsymbol{E}(\boldsymbol{u}_{\rm co}^L))\, dv. \end{aligned} \tag{5.10}$$

Applying the divergence theorem, and recalling that both $\boldsymbol{u}_{\rm co}^L$ and $\boldsymbol{u}_{\boldsymbol{\tau}}^L$ are smooth minimizers and satisfy their respective Euler equations, we find

$$\begin{aligned} F_0^L(\boldsymbol{u}_{\boldsymbol{\tau}}^L) - F_0^L(\boldsymbol{u}_{\rm co}^L) = &- \tfrac{1}{2} \textstyle\int_{\partial\Omega} C[\boldsymbol{E}(\boldsymbol{u}_{\boldsymbol{\tau}}^L) - \boldsymbol{E}(\boldsymbol{u}_{\rm co}^L)]\boldsymbol{n} \cdot \boldsymbol{u}_{\boldsymbol{\tau}}^L\, da \\ &- \int_{\partial\Omega} \mathbf{T}(\boldsymbol{u}_{\rm co}^L)\boldsymbol{n} \cdot \boldsymbol{u}_{\boldsymbol{\tau}}^L\, da, \end{aligned}$$

and thus

$$\begin{aligned} F_0^L(\boldsymbol{u}_{\boldsymbol{\tau}}^L) - F_0^L(\boldsymbol{u}_{\rm co}^L) = &- \tfrac{1}{2} \textstyle\int_{\partial\Omega} C[\boldsymbol{E}(\boldsymbol{u}_{\boldsymbol{\tau}}^L)]\boldsymbol{n} \cdot \boldsymbol{u}_{\boldsymbol{\tau}}^L\, da \\ &- \int_{\partial\Omega} C[\tfrac{1}{2}\boldsymbol{E}(\boldsymbol{u}_{\rm co}^L) - \boldsymbol{E}_0]\boldsymbol{n} \cdot \boldsymbol{u}_{\boldsymbol{\tau}}^L\, da, \end{aligned} \tag{5.11}$$

Choosing a sequence $L_n \to +\infty$, we take the limit of the right-hand side of (5.11) using Theorem 4.2 and Remark 4.3; the result is the surface integrals in (5.9) (recall that $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{\tau}$ on $\partial\Omega$). Moreover,

$$F_0^{L_n}(\boldsymbol{u}_{\rm co}^{L_n}) \le F_0^{L_n}(\boldsymbol{u}_{\rm co}) \le F_0(\boldsymbol{u}_{\rm co}),$$

and for fixed L_0, the functional $F_0^{L_0}$ is lower semicontinuous; thus, using Theorem 4.1

$$F_0^{L_0}(\boldsymbol{u}_{\boldsymbol{\tau}}) \le \liminf_{n \to +\infty} F_0^{L_0}(\boldsymbol{u}_{\boldsymbol{\tau}}^{L_n}) \le \liminf_{n \to +\infty} F_0^{L_n}(\boldsymbol{u}_{\boldsymbol{\tau}}^{L_n}),$$

and taking the limit of the left-hand side of (5.11), we obtain

$$\begin{aligned} F_0^{L_0}(\boldsymbol{u}_{\boldsymbol{\tau}}) - F_0(\boldsymbol{u}_{co}) \le &- \tfrac{1}{2} \textstyle\int_{\partial\Omega} C[\boldsymbol{E}(\boldsymbol{u}_{\boldsymbol{\tau}})]\boldsymbol{n} \cdot \boldsymbol{u}_{\boldsymbol{\tau}}\, da \\ &- \int_{\partial\Omega} C[\tfrac{1}{2}\boldsymbol{E}(\boldsymbol{u}_{\rm co}) - \boldsymbol{E}_0]\boldsymbol{n} \cdot \boldsymbol{u}_{\boldsymbol{\tau}}\, da, \end{aligned}$$

which implies (5.9) since L_0 is arbitrary.

Next note that, for any $\varrho > 0$, $\boldsymbol{u}_{\varrho\boldsymbol{\tau}} = \varrho\boldsymbol{u}_{\boldsymbol{\tau}}$, where we have used the linearity of the Euler equations. Thus, applying (5.9) to $\boldsymbol{u}_{\varrho\boldsymbol{\tau}}$, we find that

$$F_0(\boldsymbol{u}_{\varrho\boldsymbol{\tau}}) - F_0(\boldsymbol{u}_{co}) \leq -\frac{\varrho^2}{2}\int_{\partial\Omega} C[\boldsymbol{E}(\boldsymbol{u}_{\boldsymbol{\tau}})]\boldsymbol{n}\cdot\boldsymbol{u}_{\boldsymbol{\tau}}\,da$$
$$-\varrho\int_{\partial\Omega} C[\tfrac{1}{2}\boldsymbol{E}(\boldsymbol{u}_{\rm co}) - \boldsymbol{E}_0]\boldsymbol{n}\cdot\boldsymbol{u}_{\boldsymbol{\tau}}\,da.$$

Choosing

$$\boldsymbol{\tau} = \boldsymbol{P}C[\tfrac{1}{2}\boldsymbol{E}(\boldsymbol{u}_{\rm co}) - \boldsymbol{E}_0]\boldsymbol{n},$$

and noting that

$$\frac{1}{\ell}\int_{\partial\Omega} f(\boldsymbol{x}, D_S\boldsymbol{u}_{\varrho\boldsymbol{\tau}})\,da \leq \frac{1}{\ell}\,B\varrho^p\int_{\partial\Omega}|D_S\boldsymbol{\tau}|^p da,$$

we find that

$$F_\ell(\boldsymbol{u}_{\varrho\boldsymbol{\tau}}) - F_\ell(\boldsymbol{u}_{\rm co}) \leq \alpha\varrho^2 - \beta\varrho + \frac{\gamma}{\ell}\,\varrho^p,$$

where α, β, γ are constants, with β and γ positive. If $p > 1$, for any fixed ℓ we may choose ϱ such that the above expression is strictly negative. If $p = 1$ take $\ell_1 = \gamma/\beta$. Then for $\ell > \ell_1$ and ϱ sufficiently small,

$$F_\ell(\boldsymbol{u}_{\varrho\boldsymbol{\tau}}) - F_\ell(\boldsymbol{u}_{\rm co}) \leq \varrho\left(\alpha\varrho - \frac{\beta}{\ell}(\ell - \ell_1)\right) < 0,$$

and this implies the desired result.

□

6 Compactness for $q > 1$

In this section we study the convergence of the minimizing sequences. We assume that $q > 1$ in the growth condition (2.5).

Theorem 6.1. *Assume that either Ω is not a surface of revolution, or that Ω is a surface of revolution and $q \geq 2$. Let $\{\boldsymbol{u}_n\} \subset W$ be a minimizing sequence for F on W, i.e.*

$$\liminf_{n\to\infty} F(\boldsymbol{u}_n) = \inf_{\boldsymbol{v}\in W} F(\boldsymbol{v}).$$

Then there is a subsequence (not relabeled) and a function $\boldsymbol{u} \in W$ such that

$$\begin{aligned} \boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \quad \text{in } W^{1,2}_{\rm loc}(\mathbb{R}^3\setminus\Omega;\mathbb{R}^3), \quad \boldsymbol{u}_n|_{\partial\Omega} \rightharpoonup \boldsymbol{u}|_{\partial\Omega} \quad \text{in } W^{1,q}(\partial\Omega;\mathbb{R}^3), \\ \boldsymbol{E}(\boldsymbol{u}_n) - \boldsymbol{E}_0 \rightharpoonup \boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}_0 \quad \text{in } L^2(\mathbb{R}^3\setminus\Omega;\mathbb{R}^{3\times3}). \end{aligned} \tag{6.1}$$

Proof. Let $\boldsymbol{u}_n$ be a minimizing sequence for F on W. By the coerciveness of the bulk and interfacial energy densities (see (2.5)), it follows that

$$\boldsymbol{E}(\boldsymbol{u}_n) - \boldsymbol{E}_0 \qquad \text{is bounded in} \quad L^2(\mathbb{R}^3\setminus\Omega;\mathbb{R}^{3\times3}) \tag{6.2}$$
$$D_S\boldsymbol{u}_n \qquad \text{is bounded in} \quad L^q(\partial\Omega;\mathbb{R}^{3\times3}). \tag{6.3}$$

The first assertion implies that a subsequence (not relabeled) of $\{\boldsymbol{E}(\boldsymbol{u}_n)-\boldsymbol{E}_0\}$ converges weakly in $L^2(\mathbb{R}^3\setminus\Omega;\mathbb{R}^{3\times 3})$ to a tensor field $\overline{\boldsymbol{E}}-\boldsymbol{E}_0\in L^2(\mathbb{R}^3\setminus\Omega;\mathbb{R}^{3\times 3})$. By Korn's inequality, the sequence $\boldsymbol{u}_n$ is bounded in $W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)$, for any ball B_L such that $B_L\supset\overline{\Omega}$. Hence the traces of $\boldsymbol{u}_n$ are uniformly bounded in $L^2(\partial\Omega;\mathbb{R}^3)$. We claim that they are actually uniformly bounded in $W^{1,q}(\partial\Omega;\mathbb{R}^3)$. Indeed by (6.3) this follows from the Poincaré inquality (3.4) when $q<2$. When $q\geq 2$, by the Sobolev Immersion Theorem (see e.g. [17]) we know that $W^{1,2}(\partial\Omega;\mathbb{R}^3)$ is contained in $L^p(\partial\Omega;\mathbb{R}^3)$ for any $p\geq 1$ and, in particular, in $L^q(\partial\Omega;\mathbb{R}^3)$. Since the traces of $\{\boldsymbol{u}_n\}$ are uniformly bounded in $W^{1,2}(\partial\Omega;\mathbb{R}^3)$ it follows that they are uniformly bounded in $L^q(\partial\Omega;\mathbb{R}^3)$ and thus, again by (6.3), in $W^{1,q}(\partial\Omega;\mathbb{R}^3)$, and the claim is proved.

A simple diagonalization argument, analogous to the procedure used in Theorem 4.1, allows to extract a subsequence (not relabeled) $\{\boldsymbol{u}_n\}$ such that $\boldsymbol{u}_n$ converges weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^3\setminus\Omega;\mathbb{R}^3)$ to some function $\boldsymbol{u}$ and the traces of $\{\boldsymbol{u}_n\}$ converge weakly in $W^{1,q}(\partial\Omega;\mathbb{R}^3)$ to the trace of $\boldsymbol{u}$. Since $\boldsymbol{E}(\boldsymbol{u}_n)-\boldsymbol{E}_0\rightharpoonup\boldsymbol{E}(\boldsymbol{u})-\boldsymbol{E}_0$ in $L^2(\mathbb{R}^3\setminus\Omega;\mathbb{R}^{3\times 3})$, it follows that $\boldsymbol{E}(\boldsymbol{u})=\overline{\boldsymbol{E}}$.

To conclude the proof it remains to show that $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. By weak convergence, for any $\psi\in C^1(\partial\Omega;\mathbb{R})$,

$$\int_{\partial\Omega}\psi\,\boldsymbol{u}_n\cdot\boldsymbol{n}\,ds\to\int_{\partial\Omega}\psi\,\boldsymbol{u}\cdot\boldsymbol{n}\,ds, \tag{6.4}$$

and since $\boldsymbol{u}_n\cdot\boldsymbol{n}=0$ on $\partial\Omega$, the proof is complete.

□

Remark. When $q=1$ in Theorem 6.1 the situation is more delicate, as a minimizing sequence converges to a limit function $\boldsymbol{u}$ which is only of bounded variation on the boundary, i.e. $\boldsymbol{u}|_{\partial\Omega}\in BV(\partial\Omega;\mathbb{R}^3)$.

7 Relaxation

For simplicity, throughout this section we assume that

$$q>\frac{4}{3}. \tag{7.1}$$

Actually, it can be shown that the integral representation formula (7.3) obtained below continues to hold when $q>1$, but when $1<q\leq\frac{4}{3}$, the proofs are rather technical and go beyond the scope of this paper. The situation is quite different when $q=1$, since, as we remarked in the previous section, a priori we only know that $\boldsymbol{u}|_{\partial\Omega}\in BV(\partial\Omega;\mathbb{R}^3)$ for the limit function $\boldsymbol{u}$ of a minimizing sequence, so that in the integral representation formula one would need to take into account extra terms due to the singular part of the measure $D_S\boldsymbol{u}$ (cf., e.g., [15]). In view of the compactness result of the previous section, for $\boldsymbol{u}\in W$ define

$$\begin{aligned}\mathcal{F}(\boldsymbol{u}):=\inf\Big\{\liminf_{n\to\infty}F(\boldsymbol{u}_n):\ \{\boldsymbol{u}_n\}\subset W,\quad &\boldsymbol{u}_n\to\boldsymbol{u}\quad\text{in }L^2_{\text{loc}}(\mathbb{R}^3\setminus\Omega;\mathbb{R}^3),\\ &\boldsymbol{u}_n|_{\partial\Omega}\rightharpoonup\boldsymbol{u}|_{\partial\Omega}\quad\text{in }W^{1,q}(\partial\Omega;\mathbb{R}^3),\\ \boldsymbol{E}(\boldsymbol{u}_n)-\boldsymbol{E}_0\rightharpoonup\boldsymbol{E}(\boldsymbol{u})-\boldsymbol{E}_0&\quad\text{in }L^2(\mathbb{R}^3\setminus\Omega;\mathbb{R}^{3\times 3})\Big\}.\end{aligned}$$

In the next theorem we show that we may separately relax the elastic and the surface energies, and characterize the relaxed surface energy functional in terms of the *tangential quasi-convexification* $\mathcal{Q}_T f$ of f, defined by

$$\mathcal{Q}_T f(\boldsymbol{x}, \boldsymbol{V}) \\ := \inf \left\{ \int_Q f(\boldsymbol{x}, \boldsymbol{V} + \nabla_{\boldsymbol{\eta}} \boldsymbol{\varphi}(\boldsymbol{\eta}))\, d\boldsymbol{\eta} \,:\, \boldsymbol{\varphi} \in W_0^{1,\infty}(Q; T_{\boldsymbol{x}}(\partial\Omega)) \right\}, \tag{7.2}$$

with $T_{\boldsymbol{x}}(\partial\Omega)$ the tangent plane to $\partial\Omega$ at $\boldsymbol{x}$, and Q the unit square in $T_{\boldsymbol{x}}(\partial\Omega)$. The definition introduced here is similar to the notion of tangential convexification introduced by Dacorogna et al. [9], for functionals defined on mappings from the Euclidean space to manifolds.

Theorem 7.1. *For $\boldsymbol{u} \in W$,*

$$\mathcal{F}(\boldsymbol{u}) = \int_{\mathbb{R}^3 \setminus \Omega} \varphi(\boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}_0)\, dv + \int_{\partial\Omega} \mathcal{Q}_T f(\boldsymbol{x}, D_S \boldsymbol{u})\, da. \tag{7.3}$$

The following Lemma uses ideas of Kristensen [18] and Fonseca and Malý [14].

Lemma 7.2. *Consider a sequence $\{u_n\} \subset W^{1,q}(\partial\Omega)$ converging weakly in $W^{1,q}(\partial\Omega)$ to $u|_{\partial\Omega}$ for some function $u \in W^{1,2}(\Omega)$. Then for any ball B_L, with $B_L \supset \overline{\Omega}$, there is a sequence $\{v_n\} \subset W^{1,2}(B_L \setminus \Omega)$ such that $\{|\nabla v_n|^2\}$ is equi-integrable, $v_n|_{\partial\Omega} = u_n$, and $v_n \rightharpoonup u$ in $W^{1,2}(B_L \setminus \Omega)$.*

Proof of Lemma 7.2. By replacing u_n with $u_n - u$, if necessary, we may assume that $u \equiv 0$. By (7.1) and the Sobolev Immersion Theorem for fractional Sobolev spaces (see e.g. Adams [2]), the following continuous inclusion holds

$$W^{1,q}(\partial\Omega) \subset W^{1-\frac{2}{3q},\frac{3}{2}q}(\partial\Omega).$$

Hence

$$u_n \in W^{1-\frac{2}{3q},\frac{3}{2}q}(\partial\Omega)$$

with

$$\|u_n\|_{W^{1-\frac{2}{3q},\frac{3}{2}q}(\partial\Omega)} \leq C\, \|u_n\|_{W^{1,q}(\partial\Omega)}\,.$$

Moreover, since the mapping

$$W^{1,\frac{3}{2}q}\,(B_L \setminus \Omega) \to W^{1-\frac{2}{3q},\frac{3}{2}q}\,(\partial\Omega)$$

is linear, continuous and onto, we may find $v_n \in W^{1,\frac{3}{2}q}(B_L \setminus \Omega)$ such that $v_n = u_n$ on $\partial\Omega$ and

$$\|v_n\|_{W^{1,\frac{3}{2}q}(B_L \setminus \Omega)} \leq C\, \|u_n\|_{W^{1-\frac{2}{3q},\frac{3}{2}q}(\partial\Omega)}\,. \tag{7.4}$$

Hence the sequence $\{v_n\}$ is uniformly bounded in $W^{1,\frac{3}{2}q}(B_L \setminus \Omega)$ and, in particular, since $q > 4/3$ it follows that $\{|\nabla v_n|^2\}$ is equi-integrable. □

Proof of Theorem 7.1. Since φ is convex, classical lower semicontinuity results give

$$\mathcal{F}(\boldsymbol{u}) \geq \int_{\mathbb{R}^3\setminus\Omega} \varphi(\boldsymbol{E}-\boldsymbol{E}_0)\,dv + \mathcal{F}_1(\boldsymbol{u}),$$

with

$$\mathcal{F}_1(\boldsymbol{u}) := \inf\left\{ \liminf_{n\to\infty} \int_{\partial\Omega} f(\boldsymbol{x}, D_S\boldsymbol{v}_n)\,da : \{\boldsymbol{v}_n\} \subset W^{1,q}(\partial\Omega;\mathbb{R}^3), \right.$$
$$\left. \boldsymbol{v}_n\cdot\boldsymbol{n}=0, \quad \boldsymbol{v}_n \rightharpoonup \boldsymbol{u}|_{\partial\Omega} \quad \text{in } W^{1,q}(\partial\Omega;\mathbb{R}^3) \right\}.$$

To prove the opposite inequality, fix $\varepsilon > 0$ and let $\{\boldsymbol{v}_n\} \subset W^{1,q}(\partial\Omega;\mathbb{R}^3)$ be such that $\boldsymbol{v}_n\cdot\boldsymbol{n} = 0$, $\boldsymbol{v}_n \rightharpoonup \boldsymbol{u}|_{\partial\Omega}$ in $W^{1,q}(\partial\Omega;\mathbb{R}^3)$ and

$$\lim_{n\to\infty} \int_{\partial\Omega} f(\boldsymbol{x}, D_S\boldsymbol{v}_n)\,da \leq \mathcal{F}_1(\boldsymbol{u}) + \varepsilon. \tag{7.5}$$

Fix B_L so that $B_L \supset \overline{\Omega}$. Since $\{\boldsymbol{v}_n\}$ is uniformly bounded in $W^{1,q}(\partial\Omega;\mathbb{R}^3)$, by the previous lemma we may find a bounded sequence $\{\tilde{\boldsymbol{v}}_n\} \subset W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)$ such that $\{|\boldsymbol{E}(\tilde{\boldsymbol{v}}_n)|^2\}$ is equi-integrable, $\tilde{\boldsymbol{v}}_n|_{\partial\Omega} = \boldsymbol{v}_n$, and $\tilde{\boldsymbol{v}}_n \rightharpoonup \boldsymbol{u}$ in $W^{1,2}(B_L\setminus\Omega;\mathbb{R}^3)$. Denoting, for every $t > 0$, $\Omega_t := \{x \in B_L\setminus\Omega |\ \mathrm{dist}(x,\partial\Omega) < t\}$, we fix some $\delta > 0$ small and define the subsets $L_\delta := \Omega_{2\delta}\setminus\overline{\Omega_\delta}$. Consider a smooth cut-off function $\psi_\delta \in C_0^\infty(\Omega_{2\delta};[0,1])$ such that $\psi_\delta = 1$ on Ω_δ. As the thickness of the strip L_δ is of order δ, we have an upper bound of the form $\|\nabla\psi_\delta\|_{L^\infty(L_\delta)} \leq C/\delta$. Define

$$\boldsymbol{w}_n := \psi_\delta\,\tilde{\boldsymbol{v}}_n + (1-\psi_\delta)\boldsymbol{u}.$$

Clearly this sequence belongs to W, converges to $\boldsymbol{u}$ in the sense (6.1), and satisfies $\boldsymbol{w}_n = \boldsymbol{v}_n$ on $\partial\Omega$. On the other hand, by the quadratic growth of φ,

$$\begin{aligned} F(\boldsymbol{w}_n) \leq & \int_{\mathbb{R}^3\setminus\Omega_\delta} \varphi(\boldsymbol{E}(\boldsymbol{u})-\boldsymbol{E}_0)\,dv + \int_{\partial\Omega} f(\boldsymbol{x}, D_S\boldsymbol{v}_n)\,da + \\ & + C\left(\int_{\Omega_{2\delta}} |\boldsymbol{E}(\tilde{\boldsymbol{v}}_n)-\boldsymbol{E}_0|^2\ dv + \int_{L_\delta} |\boldsymbol{E}(\boldsymbol{u})-\boldsymbol{E}_0|^2\ dv \right. \\ & \left. + \frac{1}{\delta^p}\int_{L_\delta} |\tilde{\boldsymbol{v}}_n - \boldsymbol{u}|^2\ dv\right). \end{aligned}$$

Thus, passing to the limit in n, we have, by (7.5),

$$\begin{aligned} \mathcal{F}(\boldsymbol{u}) \leq & \int_{\mathbb{R}^3\setminus\Omega} \varphi(\boldsymbol{E}(\boldsymbol{u})-\boldsymbol{E}_0)\,dv + \mathcal{F}_1(\boldsymbol{u}) + \varepsilon \\ & + C\left(\liminf_{n\to\infty}\int_{\Omega_{2\delta}} |\boldsymbol{E}(\tilde{\boldsymbol{v}}_n)-\boldsymbol{E}_0|^2\ dv + \int_{L_\delta} |\boldsymbol{E}(\boldsymbol{u})-\boldsymbol{E}_0|^2\ dv\right), \end{aligned}$$

where we have used the fact that $\tilde{\boldsymbol{v}}_n \to \boldsymbol{u}$ in $L^2(B_L\setminus\Omega;\mathbb{R}^3)$. Since $\{|\boldsymbol{E}(\tilde{\boldsymbol{v}}_n)|^2\}$ is equi-integrable, by letting δ go to zero we obtain

$$\mathcal{F}(\boldsymbol{u}) \leq \int_{\mathbb{R}^3\setminus\Omega} \varphi(\boldsymbol{E}(\boldsymbol{u})-\boldsymbol{E}_0)\,dv + \mathcal{F}_1(\boldsymbol{u}) + \varepsilon.$$

It now suffices to let $\varepsilon \to 0^+$.

To find an integral representation of $\mathcal{F}_1$, let $\mathcal{A}(\partial\Omega)$ be the class of all relatively open subsets of $\partial\Omega$. For any open set $A \in \mathcal{A}(\partial\Omega)$ we define

$$\mathcal{F}_1(\boldsymbol{u}; A) := \inf\Big\{ \liminf_{n\to\infty} \int_A f(\boldsymbol{x}, D_S \boldsymbol{v}_n)\, da : \{\boldsymbol{v}_n\} \subset W^{1,q}(A;\mathbb{R}^3), \\ \boldsymbol{v}_n \cdot \boldsymbol{n} = 0, \quad \boldsymbol{v}_n \rightharpoonup \boldsymbol{u}|_A \quad \text{in } W^{1,q}(A;\mathbb{R}^3) \Big\}.$$

It may be shown that $\mathcal{F}_1(u;\cdot)$ is the restriction to $\mathcal{A}(\partial\Omega)$ of a Radon measure. Without the constraint $\boldsymbol{v}_n \cdot \boldsymbol{n} = 0$ on admissible sequences, the proof of this result is standard in the Calculus of Variations, see e.g Lemma 3.1.3 and Theorem 1.0.4 of [4]. Using the fact that if $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ then $(\varphi\boldsymbol{u} + (1-\varphi)\boldsymbol{v})\cdot\boldsymbol{n} = 0$ for any scalar function φ, it is easy to see that the proofs of Lemma 3.1.3 and Theorem 1.0.4 of [4] can be carried over in our context.

Since the outward unit normal $\boldsymbol{n}$ to $\partial\Omega$ is continuous, the domain of f, namely $\Sigma = \{(\boldsymbol{x}, \boldsymbol{V}) \in \partial\Omega \times \mathbb{R}^{3\times 3} : \boldsymbol{V}\boldsymbol{n}(\boldsymbol{x}) = 0\}$, is closed in $\mathbb{R}^3 \times \mathbb{R}^{3\times3}$. Thus by Tiesze Extension Theorem we may extend f continuously to all of $\mathbb{R}^3 \times \mathbb{R}^{3\times3}$ in such a way that the growth condition (2.5) continues to hold in $\mathbb{R}^3 \times \mathbb{R}^{3\times3}$. In what follows we identify f with its extension.

Fix $\boldsymbol{x}_0 \in \partial\Omega$, let U be a neighborhood of $\boldsymbol{x}_0$ on $\partial\Omega$, with corresponding chart $\boldsymbol{\chi} : U \subset \partial\Omega \to \mathbb{R}^2$, and let $R = \boldsymbol{\chi}(U) \subset \mathbb{R}^2$. Consider a function $\boldsymbol{v} \in W^{1,q}(U;\mathbb{R}^3)$, with $\boldsymbol{v}\cdot\boldsymbol{n} = 0$, and let $\boldsymbol{w}(\boldsymbol{\xi}) := \boldsymbol{I}(\boldsymbol{x}(\boldsymbol{\xi}))\boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi}))$ denote its local representation (cf. (2.9)). We then have

$$\int_U f(\boldsymbol{x}, D_S\boldsymbol{v}(\boldsymbol{x}))\, da = \int_R f\left(\boldsymbol{x}(\boldsymbol{\xi}), D_S\boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi}))\right) J(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\ = \int_R g\left(\boldsymbol{\xi}, \boldsymbol{w}(\boldsymbol{\xi}), \nabla_{\boldsymbol{\xi}}\boldsymbol{w}(\boldsymbol{\xi})\right) d\boldsymbol{\xi},$$

with $J(\boldsymbol{\xi}) := \det \boldsymbol{I}^{-1}(\boldsymbol{x}(\boldsymbol{\xi}))$ and where the integrand $g : R \times \mathbb{R}^2 \times \mathbb{R}^{2\times2} \to [0,\infty)$ is defined by (cf. (2.10))

$$g\left(\boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{Z}\right) := f\left(\boldsymbol{x}(\boldsymbol{\xi}), \boldsymbol{I}^{-1}(\boldsymbol{x}(\boldsymbol{\xi}))\boldsymbol{Z}\boldsymbol{I}(\boldsymbol{x}(\boldsymbol{\xi})) + \boldsymbol{R}(\boldsymbol{\xi}, \boldsymbol{w})\right) J(\boldsymbol{\xi}),$$

with $\boldsymbol{R}(\boldsymbol{\xi}, \boldsymbol{w})$ defined as in (2.10). Thus

$$\mathcal{F}_1(\boldsymbol{u}; U) = \inf\Big\{ \liminf_{n\to\infty} \int_R g\left(\boldsymbol{\xi}, \boldsymbol{w}_n(\boldsymbol{\xi}), \nabla_{\boldsymbol{\xi}}\boldsymbol{w}_n(\boldsymbol{\xi})\right) d\boldsymbol{\xi} : \{\boldsymbol{w}_n\} \subset W^{1,q}(R;\mathbb{R}^2) \\ \boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \quad \text{in } W^{1,q}(R;\mathbb{R}^2) \Big\},$$

where $\boldsymbol{w} \in W^{1,q}(R;\mathbb{R}^2)$ is the function associated to $\boldsymbol{u}|_U$. Classical relaxation results (see e.g. [1] and [8]) now yield

$$\mathcal{F}_1(\boldsymbol{u}; U) = \int_R \mathcal{Q}g\left(\boldsymbol{\xi}, \boldsymbol{w}(\boldsymbol{\xi}), \nabla_{\boldsymbol{\xi}}\boldsymbol{w}(\boldsymbol{\xi})\right) d\boldsymbol{\xi},$$

where the function $\mathcal{Q}g$ is the *quasiconvex envelope* of g. It may be shown that for each fixed $(\boldsymbol{\xi}_0, \boldsymbol{w}_0, \boldsymbol{Z}_0) \in R \times \mathbb{R}^2 \times \mathbb{R}^{2\times2}$,

$$\mathcal{Q}g\left(\boldsymbol{\xi}_0, \boldsymbol{w}_0, \boldsymbol{Z}_0\right) \\ = \inf\left\{ \frac{1}{|R|} \int_R g\left(\boldsymbol{\xi}_0, \boldsymbol{w}_0, \boldsymbol{Z}_0 + \nabla_{\boldsymbol{\xi}}\boldsymbol{\varphi}(\boldsymbol{\xi})\right) d\boldsymbol{\xi} : \boldsymbol{\varphi} \in W_0^{1,\infty}(R;\mathbb{R}^2) \right\}.$$

On the other hand, this expression may also be written as

$$J(\boldsymbol{\xi}_0) \inf \left\{ \frac{1}{|R|} \int_R f\left(\boldsymbol{x}_0, \boldsymbol{I}_0^{-1} \nabla_{\boldsymbol{\xi}} \boldsymbol{\varphi}(\boldsymbol{\xi}) \boldsymbol{I}_0 + \boldsymbol{I}_0^{-1} \boldsymbol{Z}_0 \boldsymbol{I}_0 + \boldsymbol{R}(\boldsymbol{\xi}_0, \boldsymbol{w}_0)\right) d\boldsymbol{\xi} : \boldsymbol{\varphi} \in W_0^{1,\infty}(R; \mathbb{R}^2) \right\},$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(\boldsymbol{\xi}_0)$ and $\boldsymbol{I}_0 := \boldsymbol{I}(\boldsymbol{x}(\boldsymbol{\xi}_0))$. Recall that $\boldsymbol{I}_0$ induces a linear map between $T_{\boldsymbol{x}_0}(\partial\Omega)$ and $\mathbb{R}^2$ defined by $\boldsymbol{\xi} = \boldsymbol{I}_0 \boldsymbol{\eta}$, for $\boldsymbol{\eta} \in T_{\boldsymbol{x}_0}(\partial\Omega)$, and assume that $R \subset \mathbb{R}^2$ is the image through $\boldsymbol{I}_0$ of the unit square $Q \subset T_{\boldsymbol{x}_0}(\partial\Omega)$. Then any function $\boldsymbol{\varphi} \in W_0^{1,\infty}(R; \mathbb{R}^2)$ corresponds to a function $\boldsymbol{\psi} \in W_0^{1,\infty}(Q; T_{\boldsymbol{x}_0}(\partial\Omega))$ defined by

$$\boldsymbol{\psi}(\boldsymbol{\eta}) := \boldsymbol{I}_0^{-1} \boldsymbol{\varphi}(\boldsymbol{I}_0 \boldsymbol{\eta}),$$

so that

$$\boldsymbol{I}_0^{-1} \nabla_{\boldsymbol{\xi}} \boldsymbol{\varphi}(\boldsymbol{I}_0 \boldsymbol{\eta}) \boldsymbol{I}_0 = \nabla_{\boldsymbol{\eta}} \boldsymbol{\psi}(\boldsymbol{\eta}).$$

We may transform the integrals over R to integrals over Q in the expression given above for $\mathcal{Q}g$, which becomes

$$J(\boldsymbol{\xi}_0) \inf \left\{ \int_Q f\left(\boldsymbol{x}_0, \nabla_{\boldsymbol{\eta}} \boldsymbol{\psi}(\boldsymbol{\eta}) + \boldsymbol{I}_0^{-1} \boldsymbol{Z}_0 \boldsymbol{I}_0 + \boldsymbol{R}(\boldsymbol{\xi}_0, \boldsymbol{w}_0)\right) d\boldsymbol{\eta} : \boldsymbol{\psi} \in W_0^{1,\infty}(Q; T_{\boldsymbol{x}_0}(\partial\Omega)) \right\},$$

where we have used the identity $\det \boldsymbol{I}_0 = |R|$, and this implies that

$$\mathcal{Q}g\left(\boldsymbol{\xi}_0, \boldsymbol{w}_0, \boldsymbol{Z}_0\right) = J(\boldsymbol{\xi}_0) \mathcal{Q}_T f\left(\boldsymbol{x}_0, \boldsymbol{I}_0^{-1} \boldsymbol{Z}_0 \boldsymbol{I}_0 + \boldsymbol{R}(\boldsymbol{\xi}_0, \boldsymbol{w}_0)\right),$$

where we have used the definition (7.2) of tangential quasiconvexification. In particular,

$$\mathcal{Q}g\left(\boldsymbol{\xi}, \boldsymbol{w}(\boldsymbol{\xi}), \nabla_{\boldsymbol{\xi}} \boldsymbol{w}(\boldsymbol{\xi})\right) = J(\boldsymbol{\xi}) \mathcal{Q}_T f(\boldsymbol{x}(\boldsymbol{\xi}), D_S \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}))),$$

so that

$$\mathcal{F}_1(\boldsymbol{u}; U) = \int_U \mathcal{Q}_T f(\boldsymbol{x}, D_S \boldsymbol{u})\, da.$$

Since $\mathcal{F}_1(\boldsymbol{u}; \cdot)$ is the restriction to $\mathcal{A}(\partial\Omega)$ of a Radon measure, if we consider $A \in \mathcal{A}(\partial\Omega)$ with Lipschitz boundary and if for each $t > 0$, we define $A_t := \{x \in A |\ \mathrm{dist}(x, A) > t\}$, then $\mathcal{F}_1(\boldsymbol{u}; \partial A_t) = 0$ for $\mathcal{L}^1$ a.e. $t > 0$. Thus, since $\partial\Omega$ is compact, we may find a finite number of local charts $\{D_1, \ldots, D_L\}$ such that $\mathcal{F}_1(\boldsymbol{u}; \partial D_i) = 0$ and

$$\partial\Omega = \bigcup_{i=1}^{L} \overline{D_i}.$$

Hence

$$\mathcal{F}_1(\boldsymbol{u}) = \mathcal{F}_1(\boldsymbol{u}; \partial\Omega) = \sum_{i=1}^{L} \mathcal{F}_1(\boldsymbol{u}; D_i) = \sum_{i=1}^{L} \int_{D_i} \mathcal{Q}_T f(\boldsymbol{x}, D_S \boldsymbol{u}(\boldsymbol{x}))\, da = \int_{\partial\Omega} \mathcal{Q}_T f(\boldsymbol{x}, D_S \boldsymbol{u}(\boldsymbol{x}))\, da.$$

□

As an immediate application of Theorems 6.1 and 7.1 we have

Corollary 7.3 (Existence of minimizers). *Assume $\Omega \subset \mathbb{R}^3$, and that either Ω is not a surface of revolution and $q > 1$, or that Ω is a surface of revolution and $q \geq 2$. Assume also that the interfacial energy $f(\boldsymbol{x}, \cdot)$ is tangentially quasiconvex (so that $f = \mathcal{Q}_T f$). Then the minimization problem*

$$\inf_{\boldsymbol{v} \in W} F(\boldsymbol{v})$$

admits a solution $\boldsymbol{u} \in W$.

8 Two-dimensional case

The results of the preceding sections apply to plane strain, but since in that case the interface is one dimensional, quasiconvexity is replaced by convexity in the expression of the relaxed interfacial energy functional.

Let $\Omega \subset \mathbb{R}^2$, denote by $s \in [0, \ell]$ the counterclockwise arc parameter on $\partial\Omega$, and by $\boldsymbol{u}'$ the derivative of $\boldsymbol{u}(s)$ with respect to s on $\partial\Omega$. In general, we use the notation $(\cdot)' = \frac{d}{ds}$ for functions on $\partial\Omega$. Let $\boldsymbol{t}$ and $\boldsymbol{n}$ be the unit tangent and outward unit normal to $\partial\Omega$, with $\boldsymbol{t}$ pointing in the direction of increasing s. We assume that $\boldsymbol{u}$ is tangential to $\partial\Omega$, so that writing

$$\boldsymbol{u} = U\boldsymbol{t} \qquad \text{on} \quad \partial\Omega,$$

we have

$$\boldsymbol{u}' = U'\boldsymbol{t} + \kappa U \boldsymbol{n}, \tag{8.1}$$

where κ is the curvature of $\partial\Omega$. (We have used the identity $\boldsymbol{t}' = \kappa\boldsymbol{n}$.) We denote by

$$f = f(s, \boldsymbol{u}')$$

the interfacial energy density per unit length of $\partial\Omega$. We assume that f is continuous, $f(s, \boldsymbol{w}) > 0$ for $\boldsymbol{w} \neq \boldsymbol{0}$, $f(s, \boldsymbol{0}) = 0$, and the growth condition (2.5) holds.

The results in the preceding sections apply here, but we now have a somewhat simpler characterization of the relaxed interfacial energy functional.

We define the *tangential convexification* of f, denoted by $\mathcal{C}_T f = \mathcal{C}_T f(s, \boldsymbol{w})$, to be the convex envelope of the function

$$W \mapsto f(s, \boldsymbol{w} + W\boldsymbol{t}(s)),$$

computed at $W = 0$, i.e, by Carathèodory's Theorem,

$$\mathcal{C}_T f(s, \boldsymbol{w}) = \inf\{\lambda f(s, \boldsymbol{w} + W_1 \boldsymbol{t}(s)) + (1 - \lambda)\lambda f(s, \boldsymbol{w} + W_2 \boldsymbol{t}(s)) \; : \\ \lambda W_1 + (1 - \lambda) W_2 = 0\}.$$

For an alternative characterization, note that each function $f : [0, \ell] \times \mathbb{R}^2 \to \mathbb{R}$ induces a function $g : (0, \ell) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$g(s, U, V) := f\left(s, V\, \boldsymbol{t}(s) + \kappa(s)\, U \boldsymbol{n}(s)\right), \tag{8.2}$$

for every $(s, U, V) \in [0, \ell] \times \mathbb{R} \times \mathbb{R}$. For each fixed (s, U), let g^{**} be the convex envelope of $g(s, U, \cdot)$. Since g^{**} is, for each (s, U), the convex envelope of the real function

$$V \mapsto f\left(s, V\boldsymbol{t}(s) + \kappa(s)\, U\boldsymbol{n}(s)\right),$$

for each fixed (s, U), and this function (by Carathèodory's Theorem) coincides with the tangential convexification of f,

$$\mathcal{C}_T f(s, \boldsymbol{w}) = g^{**}(s, U, V),$$

with U and V defined by $\boldsymbol{w} = V\,\boldsymbol{t}(s) + \kappa(s)\, U\boldsymbol{n}(s)$.

The following theorem, analogous to Theorem 7.1, shows that the role of the tangential quasiconvexification is now played by the tangential convexification, and that, when f is isotropic, so that $f = f(\boldsymbol{u}')$ does not depend on s, the tangential convexification of f coincides with the convex envelope f^{**}.

Theorem 8.1. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then, for $\boldsymbol{u} \in W$,*

$$\mathcal{F}(\boldsymbol{u}) = \int_{\mathbb{R}^2 \setminus \Omega} \varphi(\boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}_0)\, da + \int_{\partial\Omega} \mathcal{C}_T f(s, \boldsymbol{u}')\, ds. \tag{8.3}$$

Moreover, when $f = f(\boldsymbol{w})$ is independent of s, then

$$\mathcal{C}_T f(s, \boldsymbol{w}) := f^{**}(\boldsymbol{w}).$$

Proof. To prove the first assertion, note that, granted (8.2), we may write the relaxed interfacial energy functional in Theorem 7.1 as $\mathcal{F}_1(\boldsymbol{u}) = \mathcal{G}(U)$, where

$$\mathcal{G}(U) = \inf\left\{ \liminf_{n\to\infty} \int_0^\ell g(s, U_n, U_n')\, ds : \{U_n\} \subset W^{1,q}((0,\ell);\mathbb{R}), \right. \\ \left. U_n \rightharpoonup U \quad \text{in } W^{1,q}((0,\ell);\mathbb{R}) \right\}.$$

Classical relaxation results (see e.g. [8]) give

$$\mathcal{G}(U) = \int_0^\ell g^{**}(s, U, U')\, ds, \tag{8.4}$$

where for each fixed $(s_0, U_0) \in (0, \ell) \times \mathbb{R}$ the function $g^{**}(s_0, U_0, \cdot)$ is the convex envelope of

$$g(s_0, U_0, \cdot) = f\left(s_0, \cdot\, \boldsymbol{t}(s_0) + \kappa(s_0)\, U_0 \boldsymbol{n}(s_0)\right).$$

To prove the second assertion we must show that, when $f = f(\boldsymbol{w})$ is independent of s, then for any $s \in [0, \ell]$ and $U, V \in \mathbb{R}$,

$$g^{**}(s, U, V) := f^{**}(\boldsymbol{w}),$$

with $\boldsymbol{w} = V\boldsymbol{t}(s) + \kappa(s) U\boldsymbol{n}(s)$. We first prove that f is convex if and only if $g(s, U, \cdot)$ is convex. Assume that f is convex, and choose $\lambda \in (0, 1)$. Then

$$\begin{aligned} g(s, U, \lambda V_1 + (1-\lambda) V_2) &= f(\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2) \\ &\le \lambda f(\boldsymbol{w}_1) + (1-\lambda) f(\boldsymbol{w}_2) \end{aligned}$$

$$= \lambda g(s,U,V_1) + (1-\lambda) g(s,U,V_2),$$

with $\boldsymbol{w}_1 = V_1 \boldsymbol{t}(s) + \kappa(s) U \boldsymbol{n}(s)$ and $\boldsymbol{w}_2 = V_2 \boldsymbol{t}(s) + \kappa(s) U \boldsymbol{n}(s)$. Conversely, assume that $g(s,U,\cdot)$ is convex, and choose $\boldsymbol{w}_1, \boldsymbol{w}_2$ arbitrary in $\mathbb{R}^2$. Then there is an $s_0 \in [0,\ell]$ such that

$$\boldsymbol{t}(s_0) = \frac{\boldsymbol{w}_2 - \boldsymbol{w}_1}{|\boldsymbol{w}_2 - \boldsymbol{w}_1|}, \qquad \boldsymbol{n}(s_0) = \boldsymbol{t}(s_0) \times \boldsymbol{k},$$

and note that $\boldsymbol{w}_1 \cdot \boldsymbol{n}(s_0) = \boldsymbol{w}_2 \cdot \boldsymbol{n}(s_0)$. Hence, if $\kappa(s_0) \neq 0$, we may define

$$V_1 = \boldsymbol{w}_1 \cdot \boldsymbol{t}(s_0), \qquad V_2 = \boldsymbol{w}_2 \cdot \boldsymbol{t}(s_0), \qquad U = \frac{\boldsymbol{w}_1 \cdot \boldsymbol{n}(s_0)}{\kappa(s_0)} = \frac{\boldsymbol{w}_2 \cdot \boldsymbol{n}(s_0)}{\kappa(s_0)},$$

so that $\boldsymbol{w}_1 = V_1 \boldsymbol{t}(s_0) + \kappa(s_0) U \boldsymbol{n}(s_0)$ and $\boldsymbol{w}_2 = V_2 \boldsymbol{t}(s_0) + \kappa(s_0) U \boldsymbol{n}(s_0)$. Thus

$$\begin{aligned} f(\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2) &= g(s_0, U, \lambda V_1 + (1-\lambda) V_2) \\ &\leq \lambda g(s_0,U,V_1) + (1-\lambda) g(s_0,U,V_2) \\ &= \lambda f(\boldsymbol{w}_1) + (1-\lambda) f(\boldsymbol{w}_2). \end{aligned}$$

If $\kappa(s_0) = 0$, we may choose s_0 such that there is a sequence $s_n \to s_0$ such that $\kappa(s_n) \neq 0$. Then, letting $V_{1,n} = \boldsymbol{w}_1 \cdot \boldsymbol{t}(s_n)$, $V_{2,n} = \boldsymbol{w}_2 \cdot \boldsymbol{t}(s_n)$, $U_{1,n} = \frac{\boldsymbol{w}_1 \cdot \boldsymbol{n}(s_n)}{\kappa(s_n)}$ and $U_{2,n} = \frac{\boldsymbol{w}_2 \cdot \boldsymbol{n}(s_n)}{\kappa(s_n)}$, we have

$$\begin{aligned} f(\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2) =& g(s_n, \lambda U_{1,n} + (1-\lambda) U_{2,n}, \lambda V_{1,n} + (1-\lambda) V_{2,n}) \\ \leq& \lambda g(s_n, \lambda U_{1,n} + (1-\lambda) U_{2,n}, V_{1,n}) \\ &+ (1-\lambda) g(s_n, \lambda U_{1,n} + (1-\lambda) U_{2,n}, V_{2,n}) \\ =& \lambda f(\boldsymbol{w}_{1,n}) + (1-\lambda) f(\boldsymbol{w}_{2,n}), \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{w}_{1,n} &= V_{1,n} \boldsymbol{t}(s_n) + \kappa(s_n)(\lambda U_{1,n} + (1-\lambda) U_{2,n}) \boldsymbol{n}(s_n), \\ \boldsymbol{w}_{2,n} &= V_{2,n} \boldsymbol{t}(s_n) + \kappa(s_n)(\lambda U_{1,n} + (1-\lambda) U_{2,n}) \boldsymbol{n}(s_n), \end{aligned}$$

so that

$$\boldsymbol{w}_{1,n} \to \boldsymbol{w}_1 \quad \text{and} \quad \boldsymbol{w}_{2,n} \to \boldsymbol{w}_2$$

as $n \to \infty$. The continuity of f then yields the desired result. The fact that $g^{**}(s,U,\cdot)$ coincides with f^{**} follows from the definition of convex envelope as the supremum over all the convex minorants of a given function.

□

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Note added to proof After this paper was completed, we learned of a paper of D. Kinderlehrer and G. Vergara-Caffarelli (The Relaxation of functionals with surface energies, *Asymptotic Anal.* **2** (1989) 279-298) in which they considered a functional of the form

$$\mathcal{E}(\boldsymbol{u}) := \int_{\Omega} \psi(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})\, d\boldsymbol{x} + \int_{\partial\Omega} f(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})\, da, \quad \boldsymbol{u} \in C^1\left(\overline{\Omega}; \mathbb{R}^m\right),$$

and proved that

$$\inf_{C^1\left(\overline{\Omega};\mathbb{R}^m\right)} \mathcal{E}(\boldsymbol{u}) = \inf_{C^1\left(\overline{\Omega};\mathbb{R}^m\right)} \left(\int_{\Omega} \mathcal{Q}\psi(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})\, d\boldsymbol{x} + \int_{\partial\Omega} \mathcal{Q}f(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})\, da \right),$$

where $\mathcal{Q}\psi$ and $\mathcal{Q}f$ are the quasiconvex envelopes of ψ and f. They also show an interesting property, suggested by De Giorgi, namely, that stationary points of the functional seek the minimum value of f in the normal direction for a given value of the tangential gradient.

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