Regularity results for a class of obstacle problems

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ABSTRACT: We prove some optimal regularity results for minimizers of the integral functional $\int f(x,u,Du)dx$ belonging to the class $K := \{u \in W^{1,p}(\Omega) : u \geq \psi\}$, where ψ is a fixed function, under standard growth conditions of p-type, i.e.

$$L^{-1}|z|^p \leq f(x,s,z) \leq L(1+|z|^p).$$

Keywords: regularity results, local minimizers, integral functionals, obstacle problems, standard growth conditions.

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1 Introduction

The aim of this paper is the study of regularity properties of local minimizers of integral functionals of the type

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x,u(x),Du(x))dx, \tag{1.1}$$

in the class $K := \{u \in W^{1,p}(\Omega,\mathbb{R}) : u \geq \psi\}$, where ψ is a fixed obstacle function, Ω is a bounded open set of $\mathbb{R}^n$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ a Carathéodory function and $u \in W^{1,1}_{\text{loc}}(\Omega,\mathbb{R})$.
The assumptions we are going to consider here are weaker than those ones usually employed in the literature in that we are not assuming that the functional considered in (1.1) admits an Euler equation, in particular we shall assume that the Lagrangian f is convex in the gradient variable in a suitably strong way, see (H2), and not necessarily twice differentiable. Such assumptions have been considered since the innovative paper of Fonseca and Fusco [11], where Lipschitz regularity results have been achieved for un-constrained local minimizers. Subsequently these results have been extended in [4], [5], [12] as far as standard functionals are considered and in [1], [8], [9], as far as the vectorial case and non-standard growth conditions are considered.
In this paper we extend the treatment of such functionals to the case of one-sided obstacle problems, providing sharp regularity results in the setting of Hölder and Morrey spaces. In particular, our results seem to be new in the standard case indeed they extend in a sharp way those obtained by Choe [2], where regularity in Morrey spaces is considered. This is possible via a more careful estimation using suitable freezing techniques. The lack of smoothness of the energy density is overcome by the use of Ekeland's variational principle, a tool that revealed to be crucial in regularity since the paper [13].
The results of this paper can be used to prove regularity theorems for obstacles problems under non standard growth conditions, see e.g. [3].

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2 Notation and statements

In the sequel Ω will denote a bounded open set in $\mathbb{R}^n$ and $B(x,R)$ the open ball $\{y \in \mathbb{R}^n : |x-y| < R\}$. As we are analysing the regularity properties of minimizers inside Ω, it is not restrictive to assume Ω smooth.
If u is an integrable function defined on $B(x,R)$, we will set

$$(u)_{x,R} = \fint_{B(x,R)} u(x)dx = \frac{1}{\omega_n R^n}\int_{B(x,R)} u(x)dx\ ,$$

where ω_n is the Lebesgue measure of $B(0,1)$. We shall also adopt the convention of writing B_R and $(u)_R$ instead of $B(x,R)$ and $(u)_{x,R}$ respectively, when the center will not be relevant or it is clear from the context; moreover, unless otherwise stated, all balls considered will have the same center. Finally the letter c will freely denote a constant, not necessarily the same in any two occurrences, while only the relevant dependences will be highlighted.
The Carathéodory function $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is supposed to satisfy a growth condition of the following type

$$L^{-1}|z|^p \le f(x,u,z) \le L(1+|z|^p)$$

for all $x\in\Omega$, $u\in\mathbb{R}$, $z\in\mathbb{R}^n$, where $p>1$ and $L\ge 1$. Next, we set

$$\varsigma(x) := |D\psi(x)|^p, \tag{2.1}$$

$$\mathcal{F}(u,\mathcal{A}) := \int_{\mathcal{A}} f(x,u(x),Du(x))dx,$$

$$\mathcal{F}_\psi(u,\mathcal{A}) := \int_{\mathcal{A}} h_\psi(x,u(x),Du(x))\,dx := \int_{\mathcal{A}} [f(x,u(x),Du(x)) + \varsigma(x)]\,dx \tag{2.2}$$

for all $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ and for all $\mathcal{A}\subset\Omega$, where $\psi\in W^{1,p}(\Omega)$ is a fixed function.
With this type of standard growth, we adopt the following notion of local minimizer and local Q-minimizer.

Definition 2.1. *We say that a function $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional $\mathcal{F}$ if $|Du(x)|^p\in L^1_{\mathrm{loc}}(\Omega)$ and*

$$\int_{\mathrm{supp}\ \phi} f(x,u(x),Du(x))dx \le \int_{\mathrm{supp}\ \phi} f(x,u(x)+\phi(x),Du(x)+D\phi(x))dx$$

for all $\phi\in W^{1,1}_0(\Omega)$ with compact support in Ω.

Definition 2.2. *We say that a function $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local Q-minimizer of the functional $\mathcal{F}$ with $Q\ge 1$ if for all $v\in W^{1,1}_{\mathrm{loc}}(\Omega)$ we have*

$$\mathcal{F}(u,H) \le Q\mathcal{F}(v,H)\ ,$$

where we set $H =: \mathrm{supp}(u-v) \subset\subset \Omega$.

We shall consider the following growth, ellipticity and continuity conditions

$$L^{-1}(\mu^2+|z|^2)^{p/2} \le f(x,u,z) \le L(\mu^2+|z|^2)^{p/2}\ , \tag{H1}$$

$$\begin{aligned}&\int_{Q_1}[f(x_0,u_0,z_0+D\phi(x)) - f(x_0,u_0,z_0)]dx\\ &\qquad\ge\ L^{-1}\int_{Q_1}(\mu^2+|z_0|^2+|D\phi(x)|^2)^{(p-2)/2}|D\phi(x)|^2dx\end{aligned} \tag{H2}$$

for some $0 \le \mu \le 1$, for all $z_0 \in \mathbb{R}^n$, $u_0 \in \mathbb{R}$, $x_0 \in \Omega$, $\phi \in \mathcal{C}_0^\infty(Q_1)$, where $Q_1 = (0,1)^n$,

$$\text{(H3)} \qquad |f(x,u,z) - f(x_0,u_0,z)| \le L\,\omega(|x-x_0| + |u-u_0|)(\mu^2 + |z|^2)^{p/2}$$

for all $z \in \mathbb{R}^n$, $u, u_0 \in \mathbb{R}$, x and $x_0 \in \Omega$, where $L \ge 1$. Here $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, nondecreasing function, vanishing at zero; we also suppose, without loss of generality, that ω is a concave, bounded and, hence, subadditive function. Let us set

$$K := \{u \in W^{1,p}(\Omega) : u \ge \psi\}.$$

Now we recall the definition of Morrey and Campanato spaces (see for example [16]).

Definition 2.3. *(Morrey spaces).*
Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $1 \le p < +\infty$ and $\lambda \ge 0$. By $L^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that, if we set $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$, we get

$$||u||_{L^{p,\lambda}(\Omega)} := \left\{ \sup_{x_0 \in \Omega,\ 0<\rho<\,\mathrm{diam}(\Omega)} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x)|^p dx \right\}^{1/p} < +\infty.$$

It is easy to see that $||u||_{L^{p,\lambda}(\Omega)}$ is a norm respect to which $L^{p,\lambda}(\Omega)$ is a Banach space.

Definition 2.4. *(Campanato spaces).*
Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $p \ge 1$ and $\lambda \ge 0$. By $\mathcal{L}^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that, if we set $\Omega(x_0,\rho) := \Omega \cap B(x_0,\rho)$, we get

$$[u]_{p,\lambda} = \left\{ \sup_{x_0 \in \Omega,\ 0<\rho<\,\mathrm{diam}(\Omega)} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x) - (u)_{x_0,\rho}|^p \, dx \right\}^{1/p} < +\infty,$$

where

$$(u)_{x_0,\rho} := \frac{1}{|\Omega(x_0,\rho)|} \int_{\Omega(x_0,\rho)} u(x)\, dx$$

is the average of u in $\Omega(x_0,\rho)$.

Also in this case it is not difficult to show that $\mathcal{L}^{p,\lambda}(\Omega)$ is a Banach space equipped with the norm

$$||u||_{\mathcal{L}^{p,\lambda}(\Omega)} = ||u||_{L^p(\Omega)} + [u]_{p,\lambda}.$$

Remark 2.5. The local variants $L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ and $\mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega)$ are defined in a standard way

$$u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega) \;\Leftrightarrow\; u \in L^{p,\lambda}(\Omega') \quad \forall\, \Omega' \subset\subset \Omega$$

$$u \in \mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega) \;\Leftrightarrow\; u \in \mathcal{L}^{p,\lambda}(\Omega') \quad \forall\, \Omega' \subset\subset \Omega.$$

The interest of Campanato's spaces lies mainly in the following result which will be used in the next sections. It can be found in [16], Sect. 2.3.

Theorem 2.6. *Let Ω be a bounded open set without internal cusps, and let $n < \lambda < n+p$. Then the space $\mathcal{L}^{p,\lambda}(\Omega)$ is isomorphic to $\mathcal{C}^{0,\alpha}(\bar{\Omega})$ with $\alpha = \frac{\lambda-n}{p}$. We also remark that, using Poincaré inequality, we have that, for a weakly differentiable function v, if $Dv \in L^{p,\lambda}(\Omega)$, then $v \in \mathcal{L}^{p,p+\lambda}(\Omega)$.*

The first result we are able to obtain is for local minimizers in K of the functional

$$\text{(2.3)} \qquad \mathcal{G}_0(w, B_R) = \int_{B_R} g(Dw(x))\, dx$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a continuous function fulfilling the following growth and ellipticity conditions

$$\text{(H4)} \qquad L^{-1}(\mu^2 + |z|^2)^{p/2} \le g(z) \le L(\mu^2 + |z|^2)^{p/2} \ ,$$

$$\text{(H5)} \quad \int_{Q_1} [g(z_0 + D\phi(x)) - g(z_0)]dx \ge L^{-1} \int_{Q_1} (\mu^2 + |z_0|^2 + |D\phi(x)|^2)^{(p-2)/2} |D\phi(x)|^2 \, dx$$

for some $0 \le \mu \le 1$, for all $z_0 \in \mathbb{R}^n, \ \phi \in \mathcal{C}_0^\infty(Q_1)$, where $Q_1 = (0,1)^n, \ L \ge 1, \ p > 1$. More precisely we have

Theorem 2.7. *Let $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (2.3) *in K, where g is a continuous function satisfying* (H4) *and* (H5) *and the function ψ fulfills the following assumption*

$$\text{(2.4)} \qquad D\psi \in L^{p,\lambda}_{\mathrm{loc}}(\Omega),$$

for some $0 < \lambda < n$. Then also $Dv \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ for the same λ.

As an immediate consequence of this result we deduce the following theorem

Theorem 2.8. *If the assumptions of Theorem* 2.7 *hold with $\lambda > n - p$, then $v \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \dfrac{n-\lambda}{p}$.*

Now if we assume that the obstacle ψ is a little more integrable, we are able to deduce the following theorem which holds for the local minimizers in K of the functional (1.1).

Theorem 2.9. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1) *in K, where f is a continuous function satisfying* (H1), (H2) *and* (H3) *and the function ψ fulfills the following assumption*

$$\text{(2.5)} \qquad D\psi \in L^{q,\lambda}_{\mathrm{loc}}(\Omega),$$

where $q = p\,r$ for some $r > 1$ and $n - p < \lambda < n$. Then $Du \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$.

Further improvements are still possible, see Remark 5.3.
Also in this case the following result can be obtained immediately from the previous one

Theorem 2.10. *Under the assumptions of Theorem* 2.9, *$u \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \dfrac{n-\lambda}{p}$.*

Finally, if the Lagrangian f is more regular and the obstacle stays in a Campanato space, we have

Theorem 2.11. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1) *in K, where f is a function of class $\mathcal{C}^2$ satisfying* (H1), (H2) *and* (H3) *and the function ψ fulfils the following assumption*

$$\text{(2.6)} \qquad D\psi \in \mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega),$$

for some $n < \lambda < n + p$. If we assume that

$$\text{(2.7)} \qquad \omega(R) \le \ L\, R^{\xi}$$

for some $0 < \xi \le 1$ and all $R \le 1$, then $Du \in \mathcal{L}^{p,\tilde\lambda}_{\mathrm{loc}}(\Omega)$ for some $\tilde\lambda \equiv \tilde\lambda(\lambda, \xi, p, n)$ such that $n < \tilde\lambda < n + p$.

In this case we have the following consequence

Theorem 2.12. *In the assumptions of Theorem* 2.11, *$u \in \mathcal{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $0 < \alpha < 1$.*

3 Preliminary results

• A classical result.

The following result is taken from [11], see also [9].

Theorem 3.1. *Let* $g : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function satisfying* (H4) *and* (H5). *Let* $w \in W^{1,p}(\Omega)$ *be a local minimizer of the functional* (2.3) *with* $B_R \subset\subset \Omega$. *Then* Dw *is locally bounded and moreover if* $0 < \rho < R/2$, *then*

$$\int_{B_\rho} (\mu^2 + |Dw(x)|^2)^{p/2} dx \leq c \left(\frac{\rho}{R}\right)^n \int_{B_R} (\mu^2 + |Dw(x)|^2)^{p/2} dx$$

with c *only depending on* p, L.

• A remark about local minimizers with obstacles.

If v is a local minimizer of the functional (2.3) in K with the Lagrangian g of class $\mathcal{C}^2$, then it is easy to see that

$$\int_\Omega A(Dv(x)) \cdot D\varphi(x)\, dx \geq 0 \qquad \forall\, \varphi \in \mathcal{C}_0^\infty(\Omega) \text{ such that } \varphi \geq 0, \tag{3.1}$$

where $A(z) := Dg(z)$ and $A(z)$ satisfies the following monotonicity and growth conditions

$$A(z) \cdot z \geq \nu\, |z|^p - c \tag{3.2}$$

for some $\nu > 0$ and

$$|A(z)| \leq L\, (1 + |z|^{p-1}). \tag{3.3}$$

This in particular yields

$$\int_\Omega A(Dv(x)) \cdot (Dv(x) - Dw(x))\, dx \leq 0 \qquad \forall\, w \in K,\ \ w - v \in W_0^{1,p}(\Omega). \tag{3.4}$$

• A higher integrability result.

If u is a local minimizer of the functional (1.1) in K, then it is possible to deduce for u a higher integrability result.

Theorem 3.2. *Let* u *be a local minimizer of the functional* (1.1) *in* K, *where the Lagrangian* f *satisfies* (H1) *and the function* ψ *fulfills* (2.5). *Then, there exist two positive constants* c, δ *depending on* p, L *such that, if* $B_R \subset\subset \Omega$, *then*

$$\left(\fint_{B_{R/2}} |Du(x)|^{p(1+\delta)} dx\right)^{1/(1+\delta)} \leq c \fint_{B_R} |Du(x)|^p\, dx + c\, R^{\frac{\lambda - n}{1+\delta}}. \tag{3.5}$$

Moreover, if the function ψ *fulfills* (2.6), *then* (3.5) *holds with* λ *replaced by* n.

Proof. Working as in [8] we can easily find the Caccioppoli inequality for local minimizers of the functional (1.1) in K, (i.e. for $Q-$minimizers of the functional (2.2))

$$\int_{B_{R/2}} |Du(x)|^p\, dx \leq c \int_{B_R} \left|\frac{u(x) - (u)_R}{R}\right|^p dx + c \int_{B_R} (|D\psi(x)|^p + 1)\, dx \tag{3.6}$$

from what we deduce

$$\fint_{B_{R/2}} |Du(x)|^p\, dx \leq c \left(\fint_{B_R} |Du(x)|^{\frac{p}{\theta}}\, dx\right)^\theta + c \fint_{B_R} (|D\psi(x)|^p + 1)\, dx$$

for some suitable $\theta \geq 1$. Now the assumption (2.5) allows us to use a classical result (see [16], Theorem 6.6) based on the Gehring's lemma and deduce that there exists $\delta \in \left(0, \dfrac{q-p}{p}\right)$ such that

$$\left(\fint_{B_{R/2}} |Du(x)|^{p(1+\delta)}\, dx \right)^{\frac{1}{1+\delta}} \leq c \fint_{B_R} |Du(x)|^p\, dx + c \left(\fint_{B_R} (|D\psi(x)|^{p(1+\delta)} + 1)\, dx \right)^{\frac{1}{1+\delta}}.$$

Using again assumption (2.5), we have

$$\begin{aligned} \left(\fint_{B_{R/2}} |Du(x)|^{p(1+\delta)}\, dx \right)^{\frac{1}{1+\delta}} &\leq c \fint_{B_R} |Du(x)|^p\, dx + c \left(\fint_{B_R} (|D\psi(x)|^{q} + 1)\, dx \right)^{\frac{1}{1+\delta}} \\ &\leq c \fint_{B_R} |Du(x)|^p\, dx + c\, R^{\frac{\lambda-n}{1+\delta}}. \end{aligned}$$

This finishes the proof. The other case is obtained in a similar way. □

• A up-to-the-boundary higher integrability result.

If u is a local minimizer of the functional (2.3) in K, then the following up-to-the-boundary higher integrability result can be rapidly deduced:

Proposition 3.3. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function fulfilling* (H4)*. Let v be a local minimizer of the functional* (2.3) *in the Dirichlet class $\{v \in u + W_0^{1,p}(B_R) : v \in K\}$, for some $u \in W^{1,p}(B_R)$, where the function ψ fulfills the assumption* (2.5)*. If moreover $u \in W^{1,\bar{q}}(B_R)$ for a certain $p < \bar{q} < q$, then there exist $p < \bar{r} < \bar{q}$ and c depending on p, L but not on u or R such that $v \in W^{1,\bar{r}}(B_{R/2})$ and*

(3.7)

$$\left(\fint_{B_{R/2}} |Dv(x)|^{\bar{r}}\, dx \right)^{p/\bar{r}} \leq c \left[\fint_{B_R} (1 + |Du(x)|^{\bar{q}})\, dx \right]^{p/\bar{q}} + c \left[\fint_{B_R} (1 + |D\psi(x)|^{\bar{q}})\, dx \right]^{p/\bar{q}}.$$

The same holds if the function ψ fulfills instead assumption (2.6).

Proof. The proof follows as in [4], the only difference is in the first step. In fact in our case the Caccioppoli inequality takes into consideration the presence of the obstacle function, so that we have, for any $B_{2\rho} \subset B_{R/2}$

$$\int_{B_\rho} |Dv(x)|^p\, dx \leq c \int_{B_{2\rho}} \left| \frac{v(x) - (v)_{2\rho}}{\rho} \right|^p dx + c \int_{B_{2\rho}} (1 + |D\psi(x)|^p)\, dx.$$

The rest of the proof follows as in the standard case. □

• A classical iteration lemma.

The following classical iteration lemma can be found for example in [16]. Here we state this lemma with a precise dependence on the constants we will need later.

Lemma 3.4. *Let $\phi(t)$ be a nonnegative and nondecreasing function. Suppose that*

$$\phi(\rho) \leq A \left[\left(\frac{\rho}{R}\right)^{\alpha} + \varepsilon \right] \phi(R) + B\, R^{\beta}$$

for all $\rho \leq R \leq R_0$, with A, B, α, β nonnegative constants, $\beta < \alpha$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$ such that if $\varepsilon < \varepsilon_0$, for all $\rho \leq R \leq R_0$ we have

$$\phi(\rho) \leq c \left[\left(\frac{\rho}{R}\right)^{\beta} \phi(R) + B\rho^{\beta} \right]$$

where c is a constant depending on α, β, A but independent of B.

• **An auxiliary decay estimate.**

This estimate, which will be useful later, is established following an idea of [10].

Proposition 3.5. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional (1.1) in K; assume that $\varsigma \in L^r(\Omega)$ where the function ς is introduced in (2.1) and $1 < r < n/p$, with $p < n$. Then for any $\varepsilon > 0$ and for any $B_\rho \subset B_R \subset \Omega$, with $R \leq 1$,*

$$\int_{B_\rho} |Du(x)|^p \, dx \leq c \left[\left(\frac{\rho}{R} \right)^{n-p+p\sigma_1} + \varepsilon \right] \int_{B_R} |Du(x)|^p \, dx + c_\varepsilon R^{n\left(1-\frac{1}{r}\right)} \left[\int_{B_R} (|\varsigma(x)|^r + 1) \, dx \right]^{\frac{1}{r}}$$

for some $0 < \sigma_1 \leq 1$, where c is a constant depending on L, n, p while c_ε depends also on ε.

Proof. Let us fix any $B_R \subset \Omega$ with $R \leq 1$, and consider the functional

$$\mathcal{F}_\psi(w, B_R) := \int_{B_R} [f(x, w(x), Dw(x)) + \varsigma(x)] \, dx$$

with $w \in V$ where $V := \{v \in u + W_0^{1,1}(B_R)\}$.
First of all we notice that

$$\inf_{w \in V} \mathcal{F}_\psi(w, B_R)$$

is finite. So let us fix any $\delta > 0$ and choose $u_\delta \in V$ such that

$$\mathcal{F}_\psi(u_\delta, B_R) \leq \inf_{w \in V} \mathcal{F}_\psi(w, B_R) + \delta R^n. \tag{3.8}$$

We want to use the minimality of u. A priori u_δ does not stay in K so we set $w_\delta := \max\{\psi, u_\delta\}$ and $\Sigma := \{x \in \mathbb{R}^n : u_\delta \geq \psi\}$. In this way $w_\delta \in K$ and, by the minimality of u, we have

$$\mathcal{F}(u, B_R) \leq \mathcal{F}(w_\delta, B_R). \tag{3.9}$$

Therefore we have

$$\begin{aligned}
\mathcal{F}_\psi(u, B_R) &= \mathcal{F}(u, B_R) + \int_{B_R} \varsigma(x) \, dx \\
&\overset{(3.9)}{\leq} \mathcal{F}(w_\delta, B_R) + \int_{B_R} \varsigma(x) \, dx \\
&= \mathcal{F}(w_\delta, B_R \cap \Sigma) + \mathcal{F}(w_\delta, B_R \setminus \Sigma) + \int_{B_R} \varsigma(x) \, dx \\
&\leq \mathcal{F}(u_\delta, B_R) + \mathcal{F}(\psi, B_R) + \int_{B_R} \varsigma(x) \, dx \\
&= \mathcal{F}_\psi(u_\delta, B_R) + \mathcal{F}(\psi, B_R) \\
&\overset{(3.8)}{\leq} \inf_{w \in V} \mathcal{F}_\psi(w, B_R) + \delta R^n + L \left(\int_{B_R} \varsigma(x) \, dx + R^n \right);
\end{aligned}$$

we set

$$H(R) := L \left(\int_{B_R} \varsigma(x) \, dx + R^n \right).$$

Then letting $\delta \to 0$ we have

$$\mathcal{F}_\psi(u, B_R) \leq \inf_{w \in V} \mathcal{F}_\psi(w, B_R) + H(R).$$

At this point, the functional $\mathcal{F}_\psi(w, B_R)$ is lower semicontinuous with respect to the topology induced on V by the distance

$$d(u_1, u_2) := (H(R))^{-1/p} R^{-n\left(1-\frac{1}{p}\right)} \int_{B_R} |Du_1(x) - Du_2(x)| \, dx;$$

then Ekeland's Lemma (see Theorem 1 in [7]) implies that there exists $v \in V$ such that

(i) $\displaystyle\int_{B_R} |Dv(x) - Du(x)|\, dx \le (H(R))^{1/p} R^{n\left(1-\frac{1}{p}\right)}$

(ii) $\mathcal{F}_\psi(v, B_R) \le \mathcal{F}_\psi(u, B_R)$

(iii) v minimizes the functional $\displaystyle\mathcal{F}_\psi(w, B_R) + \left(\frac{H(R)}{R^n}\right)^{\frac{p-1}{p}} \int_{B_R} |Dw(x) - Dv(x)|\, dx.$

Actually it is not difficult to show that $v \in u + W_0^{1,p}(B_R)$ and it is a local $Q-$minimizer (with Q depending only on L) of the functional

$$w \mapsto \int_{B_R} \left(|Dw(x)|^p + \frac{H(R)}{R^n} + 1\right) dx.$$

Then using a classical result (see for example [16], Theorem 6.7, in which the thesis (6.60) holds even without the term $|u|^{p^*}$ in both sides of the inequality) we obtain, for some $p < q_1 < n$

$$\left(⨍_{B_{R/2}} |Dv(x)|^{q_1}\, dx\right)^{1/q_1} \le c\left[\left(⨍_{B_R} |Dv(x)|^p\right)^{1/p} + \left(1 + \frac{H(R)}{R^n}\right)^{1/p}\right]; \tag{3.10}$$

we can choose $q_1 = p(1+\delta)$ where δ is the higher integrability exponent for u given by Theorem 3.2, so that the following inequality holds

(3.11)

$$\left(⨍_{B_{R/2}} |Du(x)|^{p(1+\delta)}\, dx\right)^{\frac{1}{1+\delta}} \le c ⨍_{B_R} |Du(x)|^p\, dx + c\left(⨍_{B_R} (|D\psi(x)|^{p(1+\delta)} + 1)\, dx\right)^{\frac{1}{1+\delta}}.$$

We can choose δ small enough such that $\delta < r - 1$.

At this point another classical result (see [10], Theorem 3.5, where also in this case the thesis (3.6) still holds even without the term $|u|^p$ in both sides of the inequality) entails that there exists $\sigma_1 > 0$ such that, for any $0 < \rho < R$,

$$\int_{B_\rho} |Dv(x)|^p\, dx \le c\left(\frac{\rho}{R}\right)^{n-p+p\,\sigma_1}\left[\int_{B_R} |Dv(x)|^p\, dx + H(R) + R^n\right]. \tag{3.12}$$

Now, choosing $0 < \theta < 1$ such that $\theta/q_1 + 1 - \theta = 1/p$, we obtain

$$\left(⨍_{B_{R/2}} |Du - Dv|^p\, dx\right)^{1/p} \le \left(⨍_{B_{R/2}} |Du - Dv|^{q_1}\, dx\right)^{\theta/q_1} \left(⨍_{B_{R/2}} |Du - Dv|\, dx\right)^{1-\theta}$$

and this implies, with $\varepsilon \in (0,1)$

$$\begin{aligned} &\left(⨍_{B_{R/2}} |Du(x) - Dv(x)|^p\, dx\right)^{1/p} \\ \le\,& \varepsilon\left[\left(⨍_{B_{R/2}} |Du(x)|^{q_1}\, dx\right)^{1/q_1} + \left(⨍_{B_{R/2}} |Dv(x)|^{q_1}\, dx\right)^{1/q_1}\right] \\ &+ c_\varepsilon ⨍_{B_{R/2}} |Du(x) - Dv(x)|\, dx \\ \overset{(3.11)}{\le}\,& \varepsilon\, c\left[\left(⨍_{B_R} |Du(x)|^p\, dx\right)^{1/p} + c\left(⨍_{B_R} (|\varsigma(x)|^{q_1/p} + 1)\, dx\right)^{1/q_1}\right. \\ &\left.\overset{(3.10)}{+} \left(⨍_{B_R} |Dv(x)|^p\right)^{1/p} + \left(1 + \frac{H(R)}{R^n}\right)^{1/p}\right] \overset{(i)}{+} c_\varepsilon\left(\frac{H(R)}{R^n}\right)^{1/p}. \end{aligned} \tag{3.13}$$

On the other hand,

(3.14)
$$\int_{B_R} |Dv(x)|^p\,dx \le\, L \int_{B_R} f(x,v(x),Dv(x))\,dx +\, L \int_{B_R} \varsigma(x)\,dx -\, L \int_{B_R} \varsigma(x)\,dx$$
$$= L\,\mathcal{F}_\psi(v,B_R) - L \int_{B_R} \varsigma(x)\,dx \overset{(ii)}{\le}\, L\,\mathcal{F}_\psi(u,B_R) - L \int_{B_R} \varsigma(x)\,dx$$
$$= L\,\mathcal{F}(u,B_R) \le\, c(L) \int_{B_R} (|Du(x)|^p + 1)\,dx.$$

At this point, (3.13) becomes

$$\left(\fint_{B_{R/2}} |Du(x)-Dv(x)|^p\,dx\right)^{1/p} \le\, \varepsilon\, c(L) \left[\left(\fint_{B_R} (|Du(x)|^p+1)\,dx\right)^{1/p}\right.$$
$$\left. + \left(\fint_{B_R} (|\varsigma(x)|^{q_1/p}+1)\,dx\right)^{1/q_1} + \left(1+\frac{H(R)}{R^n}\right)^{1/p}\right] +\, c_\varepsilon \left(\frac{H(R)}{R^n}\right)^{1/p}.$$

Hence, raising to the power p both sides of the previous inequality and getting rid of the averages, we get

(3.15)
$$\int_{B_{R/2}} |Du(x)-Dv(x)|^p\,dx \le \varepsilon\, c(p,L) \left[\int_{B_R} |Du(x)|^p\,dx\right.$$
$$\left. + R^n + R^{n\left(1-\frac{p}{q_1}\right)} \left(\int_{B_R} (|\varsigma(x)|^{\frac{q_1}{p}}+1)\,dx\right)^{\frac{p}{q_1}}\right] + c_\varepsilon(p)\,H(R).$$

This allows us to conclude that (here we use the fact that $\delta < r-1$)

$$\int_{B_\rho} |Du(x)|^p\,dx \le\, 2^{p-1} \int_{B_\rho} |Dv(x)|^p\,dx + 2^{p-1} \int_{B_{R/2}} |Du(x)-Dv(x)|^p\,dx$$
$$\overset{(3.12)}{\le}\, c(p,L) \left(\frac{\rho}{R}\right)^{n-p+p\sigma_1} \left[\int_{B_R} |Dv(x)|^p\,dx + H(R) + R^n\right]$$
$$\overset{(3.15)}{+}\, \varepsilon\, c(p,L) \left[\int_{B_R} |Du(x)|^p\,dx + R^n + R^{n\left(1-\frac{1}{r}\right)} \left(\int_{B_R} |\varsigma(x)|^r\,dx\right)^{1/r}\right] +\, c_\varepsilon(p)\,H(R)$$
$$\le c(p,L) \left[\left(\frac{\rho}{R}\right)^{n-p+p\sigma_1} + \varepsilon\right] \int_{B_R} |Du(x)|^p\,dx +\, c_\varepsilon\, R^{n\left(1-\frac{1}{r}\right)} \left(\int_{B_R} (|\varsigma(x)|^r+1)\,dx\right)^{1/r}$$

and this finishes the proof. □

• A Hölder regularity result

Theorem 3.6. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional* (1.1) *in K, with $p < n$; assume that $\varsigma \in L^{r,\lambda}_{\mathrm{loc}}(\Omega)$ where ς is the function introduced in* (2.1), *$1 < r < n/p$ and $0 < \lambda < n$. If moreover $n - pr < \lambda < n$, then u is locally Hölder continuous in Ω. If otherwise $p \ge n$ then u is trivially locally Hölder continuous too.*

Proof. We immediately remark that, if $p > n$ then u is trivially locally Hölder continuous in Ω due to the Sobolev embedding. On the other hand, if $p = n$ the same conclusion can be obtained using the higher integrability result (3.5) and the previous assertion. So we concentrate our discussion on the case $p < n$.

From Proposition 3.5, we have that, for any $\varepsilon > 0$ and for any $B_\rho \subset B_R \subset \Omega$, with $R \le 1$,

$$\int_{B_\rho} |Du(x)|^p\,dx \le c\left[\left(\frac{\rho}{R}\right)^{n-p+p\sigma_1} + \varepsilon\right]\int_{B_R} |Du(x)|^p\,dx + c_\varepsilon R^{n\left(1-\frac{1}{r}\right)}\left[\int_{B_R} (|\varsigma(x)|^r + 1)\,dx\right]^{\frac{1}{r}}$$

for some $0 < \sigma_1 \le 1$, where c is a constant depending on L, n, p while c_ε depends also on ε. Now, with our assumptions on the function ς, we can immediately deduce that

$$\int_{B_\rho} |Du(x)|^p\,dx \le c\left[\left(\frac{\rho}{R}\right)^{n-p+p\sigma_1} + \varepsilon\right]\int_{B_R} |Du(x)|^p\,dx + c_\varepsilon R^{n-\frac{n}{r}+\frac{\lambda}{r}}||\varsigma||_{L^{r,\lambda}(B_R)}.$$

As moreover $\lambda > n - pr$, then there exists $\sigma_2 > 0$ such that

$$\frac{\lambda}{r} - \frac{n}{r} = p\sigma_2 - p$$

and therefore

$$\int_{B_\rho} |Du(x)|^p\,dx \le c\left[\left(\frac{\rho}{R}\right)^{n-p+p\sigma_1} + \varepsilon\right]\int_{B_R} |Du(x)|^p\,dx + c_\varepsilon R^{n-p+p\sigma_2}.$$

Choosing for example $\gamma := \frac{1}{2}\min\{\sigma_1, \sigma_2\}$ and using the classical iteration Lemma 3.4, we deduce that

$$\int_{B_\rho} |Du(x)|^p\,dx \le c\left(\frac{\rho}{R}\right)^{n-p+p\gamma}\left[\int_{B_R} |Du(x)|^p\,dx + c_\varepsilon R^{n-p+p\gamma}\right].$$

At this point, Theorem 2.6 allows us to conclude that $u \in \mathcal{C}^{0,\gamma}_{\mathrm{loc}}(\Omega)$. This finishes the proof. □

From now on, since we are going to prove local regularity results, we shall assume that any local minimizer u of the functional (1.1) in K is globally Hölder continuous, that is for all $x, y \in \Omega$

$$|u(x) - u(y)| \le [u]_\gamma\,|x-y|^\gamma \le c\,|x-y|^\gamma. \tag{3.16}$$

4 Proof of Theorem 2.7

The proof of this theorem is carried on in three steps: first we establish a decay estimate for local minimizers of the functional (2.3) in K, with $g \in \mathcal{C}^2$; in a second moment we remove the smoothness of the function g by means of a standard approximation argument and finally we conclude using a classical iteration lemma.

◇ Step 1. We start by proving a first result

Proposition 4.1. *Let $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional (2.3) in K, where $g \in \mathcal{C}^2$ satisfies (H4) and (H5). If the function ψ fulfills (2.4) for some $0 < \lambda < n$, then for all $0 < \rho < R/2$ and any $\varepsilon > 0$*

$$\int_{B_\rho} |Dv(x)|^p\,dx \le c\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right]\int_{B_R} (1 + |Dv(x)|^p)\,dx + \bar{c}\,R^\lambda,$$

where $c \equiv c(p, L, \nu)$ and $\bar{c} \equiv \bar{c}(p, L, \varepsilon, \nu)$ (the coefficient ν was introduced in (3.2)).

Proof. We fix $R > 0$; then let $w \in v + W^{1,p}_0(B_R)$ be the solution of the following equation:

$$\int_{B_R} A(Dw(x)) \cdot D\varphi(x)\,dx = \int_{B_R} A(D\psi(x)) \cdot D\varphi(x)\,dx \qquad \forall\, \varphi \in W^{1,p}_0(B_R). \tag{4.1}$$

Then by the maximum principle, (for more details see for example [15]), we get that $w \geq \psi$ in B_R, since $v \geq \psi$ on ∂B_R. We also have

$$\int_{B_R} A(Dv(x)) \cdot (Dv(x) - Dw(x))\, dx \leq 0, \tag{4.2}$$

since $v - w \in W_0^{1,p}(B_R)$ and $w \geq \psi$ in B_R.
At this point let z be the solution of the following minimum problem

$$\min \left\{ \mathcal{G}_0(z, B_R) : z \in v + W_0^{1,p}(B_R) \right\}, \tag{4.3}$$

where $\mathcal{G}_0$ was introduced in (2.3). It is evident that z satisfies

$$\int_{B_R} A(Dz(x)) \cdot D\varphi(x)\, dx = 0 \qquad \forall\, \varphi \in W_0^{1,p}(B_R); \tag{4.4}$$

moreover $z = w$ on ∂B_R, so for example

$$\int_{B_R} A(Dz(x)) \cdot (Dw(x) - Dz(x))\, dx = 0. \tag{4.5}$$

First of all, from Theorem 3.1 we get for any $0 < \rho < R/2$

$$\int_{B_\rho} (\mu^2 + |Dz(x)|^2)^{p/2}\, dx \leq c \left(\frac{\rho}{R}\right)^n \int_{B_R} (\mu^2 + |Dz(x)|^2)^{p/2}\, dx$$

where the constant c only depend on p, L. Moreover using the minimality of z we get

$$\int_{B_R} |Dz(x)|^p\, dx \leq L \int_{B_R} g(Dz(x))\, dx \leq L \int_{B_R} g(Dw(x))\, dx \leq\, c(L) \int_{B_R} (1 + |Dw(x)|^p)\, dx,$$

as $w - z \in W_0^{1,p}(B_R)$.
Now, we would like to compare z and w. If $p \geq 2$, we readly have

$$\begin{aligned}
&\int_{B_R} |Dw(x) - Dz(x)|^p\, dx \\
&\leq c \int_{B_R} (1 + |Dw(x)|^2 + |Dz(x)|^2)^{(p-2)/2} |Dw(x) - Dz(x)|^2\, dx \\
&\leq c \int_{B_R} \langle A(Dw(x)) - A(Dz(x)), Dw(x) - Dz(x)\rangle\, dx \\
&\overset{(4.5)}{=} c \int_{B_R} \langle A(Dw(x)), Dw(x) - Dz(x)\rangle\, dx \\
&\overset{(4.1)}{=} c \int_{B_R} \langle A(D\psi(x)), Dw(x) - Dz(x)\rangle\, dx \\
&\overset{(3.3)}{\leq} c \int_{B_R} (|D\psi(x)|^{p-1} + 1) \cdot |Dw(x) - Dz(x)|\, dx \\
&\leq c \int_{B_R} |D\psi(x)|^{p-1} \cdot |Dw(x) - Dz(x)|\, dx + c \int_{B_R} |Dw(x) - Dz(x)|\, dx \\
&\leq c \int_{B_R} |D\psi(x)|^p\, dx + \frac{1}{4} \int_{B_R} |Dw(x) - Dz(x)|^p\, dx \\
&\quad + c\, R^{n\frac{p-1}{p}} \left(\int_{B_R} |Dw(x) - Dz(x)|^p\, dx \right)^{1/p} \\
&\leq c \int_{B_R} |D\psi(x)|^p\, dx + \frac{1}{2} \int_{B_R} |Dw(x) - Dz(x)|^p\, dx + c\, R^n,
\end{aligned}$$

where (4.1) is used with the choice $\varphi = w - z$ and where we used Young's inequality twice and the constants c depend only on p, L. So, using assumption (2.4), we get

$$\int_{B_R} |Dw(x) - Dz(x)|^p \, dx \leq c \, R^\lambda .$$

In a slightly similar way, if $1 < p < 2$, using again Young's inequality

$$\begin{aligned}
&\int_{B_R} |Dw(x) - Dz(x)|^p \, dx \\
&\leq \left(\int_{B_R} (1 + |Dw(x)|^2 + |Dz(x)|^2)^{\frac{p-2}{2}} |Dw(x) - Dz(x)|^2 \, dx \right)^{1/2} \\
&\quad \times \left(\int_{B_R} (1 + |Dw(x)|^2 + |Dz(x)|^2)^{\frac{2-p}{2}} |Dw(x) - Dz(x)|^{2p-2} \, dx \right)^{1/2} \\
&\leq c(L) \left[\int_{B_R} (|D\psi(x)|^{p-1} + 1) \cdot |Dw(x) - Dz(x)| \, dx \right]^{1/2} \left(\int_{B_R} (1 + |Dw(x)|^p) \, dx \right)^{1/2} \\
&\leq c\,(\varepsilon, L) \left[\int_{B_R} (|D\psi(x)|^{p-1} + 1) \cdot |Dw(x) - Dz(x)| \, dx \right] + \varepsilon \int_{B_R} (1 + |Dw(x)|^p) \, dx \\
&\leq c\,(\varepsilon, L, p) \int_{B_R} (|D\psi(x)|^p + 1) \, dx + \frac{1}{2} \int_{B_R} |Dw(x) - Dz(x)|^p \, dx + \varepsilon \int_{B_R} |Dw(x)|^p \, dx
\end{aligned}$$

which gives us

$$\int_{B_R} |Dw(x) - Dz(x)|^p \, dx \leq \, c(\varepsilon, L, p) \, R^\lambda + \varepsilon \int_{B_R} |Dw(x)|^p \, dx. \tag{4.6}$$

Summarizing, for any $p > 1$ we have (4.6). At this point we get

$$\begin{aligned}
\int_{B_\rho} |Dw(x)|^p dx &\leq 2^{p-1} \int_{B_\rho} |Dz(x)|^p dx + 2^{p-1} \int_{B_\rho} |Dw(x) - Dz(x)|^p dx \\
&\leq c(p) \int_{B_\rho} (\mu^2 + |Dz(x)|^2)^{p/2} dx + c(\varepsilon, L, p) \, R^\lambda + \, 2^{p-1}\varepsilon \int_{B_R} |Dw(x)|^p \, dx \\
&\leq c \, \left(\frac{\rho}{R} \right)^n \int_{B_R} (\mu^2 + |Dz(x)|^2)^{p/2} dx + \bar{c} \, R^\lambda + 2^{p-1} \, \varepsilon \int_{B_R} |Dw(x)|^p \, dx \\
&\leq c \, \left[\left(\frac{\rho}{R} \right)^n + \varepsilon \right] \int_{B_R} (1 + |Dw(x)|^p) \, dx + \bar{c} \, R^\lambda,
\end{aligned}$$

where the constant c depends only on p, L and the constant $\bar{c}$ depends on p, L, ε. Now, we would like to compare w and v. We get

$$\begin{aligned}
&\int_{B_R} |Dw(x)|^p \, dx \\
&\overset{(3.2)}{\leq} \frac{1}{\nu} \int_{B_R} \langle A(Dw(x)), Dw(x) \rangle \, dx + \, c \, R^n \\
&\overset{(4.1)}{=} \frac{1}{\nu} \int_{B_R} \langle A(Dw(x)), Dv(x) \rangle \, dx + \frac{1}{\nu} \int_{B_R} \langle A(D\psi(x)), Dw(x) - Dv(x) \rangle \, dx + \, c \, R^n \\
&\leq \frac{L}{\nu} \int_{B_R} (|Dw(x)|^{p-1} + 1) |Dv(x)| \, dx + \frac{L}{\nu} \int_{B_R} (|D\psi(x)|^{p-1} + 1) |Dw(x)| \, dx \\
&\quad + \frac{L}{\nu} \int_{B_R} (|D\psi(x)|^{p-1} + 1) |Dv(x)| \, dx + \, c \, R^n \\
&\leq \frac{1}{2} \int_{B_R} |Dw(x)|^p \, dx + c \int_{B_R} |Dv(x)|^p \, dx + c \int_{B_R} |D\psi(x)|^p \, dx + c \, R^n,
\end{aligned}$$

where (4.1) is applied with the choice $\varphi = w - v$ and where the constants c only depend on p, L, ν. So, using assumption (2.4), we have

$$\int_{B_R} |Dw(x)|^p \, dx \leq c \int_{B_R} |Dv(x)|^p \, dx + c\, R^\lambda. \tag{4.7}$$

On the other hand, working as for (4.6), it is not hard to get

$$\int_{B_R} |Dv(x) - Dw(x)|^p \, dx \leq c(\varepsilon, L, p, \nu)\, R^\lambda + \varepsilon \int_{B_R} |Dv(x)|^p \, dx$$

which is valid for all $p > 1$ and for any $\varepsilon > 0$. Indeed, when developing the analogue of (4.6) with $v(x)$ replacing $z(x)$, it is sufficient to write

$$\int_{B_R} \langle A(Dw(x)) - A(Dv(x)), Dw(x) - Dv(x) \rangle \, dx \leq \int_{B_R} \langle A(D(w(x)), Dw(x) - Dv(x) \rangle \, dx$$

which comes from (4.2). Now for any $0 < \rho < R/2$ and any $\varepsilon > 0$

$$\begin{aligned}
&\int_{B_\rho} |Dv(x)|^p dx \\
&\leq 2^{p-1} \int_{B_\rho} |Dw(x)|^p dx + 2^{p-1} \int_{B_\rho} |Dv(x) - Dw(x)|^p dx \\
&\leq c(p) \left[\left(\frac{\rho}{R} \right)^n + \varepsilon \right] \int_{B_R} (1 + |Dw(x)|^p) \, dx + c(\varepsilon, L, p) R^\lambda + \varepsilon\, 2^{p-1} \int_{B_R} |Dv(x)|^p \, dx \\
&\overset{(4.7)}{\leq} c \left[\left(\frac{\rho}{R} \right)^n + \varepsilon \right] \int_{B_R} (1 + |Dv(x)|^p) \, dx + \bar{c}\, R^\lambda
\end{aligned}$$

where c depends only on p, L, ν while $\bar{c}$ depends on ε, p, L, ν. This finishes the proof. □

◇ Step 2. We remove the $\mathcal{C}^2-$regularity of the function g.

Proposition 4.2. *Let $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (2.3) *in K, where g is a continuous function satisfying* (H4) *and* (H5). *If the function ψ fulfills* (2.4) *for some $0 < \lambda < n$, then for all $0 < \rho < R/2$ and any $\varepsilon > 0$*

$$\int_{B_\rho} |Dv(x)|^p \, dx \leq c \left[\left(\frac{\rho}{R} \right)^n + \varepsilon \right] \int_{B_R} (1 + |Dv(x)|^p) \, dx + \bar{c}\, R^\lambda, \tag{4.8}$$

where $c \equiv c(p, L, \nu)$ and $\bar{c} \equiv \bar{c}(p, L, \varepsilon, \nu)$.

Proof. The proof relays on a standard approximation argument, see [11], [4]. Here we confine ourselves only on a sketch of this proof. Let us consider $(G_m)_{m \in \mathbb{N}}$ to be a sequence of continuous functions defined by

$$G_m(z) := \int_{B(0,1)} \varphi(y)\, g\left(z + \frac{y}{m}\right) \, dy$$

where $\varphi : B(0,1) \to [0,1]$ is a positive and symmetric mollifier. Then for any $m \in \mathbb{N}$ it is not hard to prove, following [11], that G_m satisfies (H4) and (H5) with L replaced by a suitable constant c only dependent on L and p and independent of m and with μ^2 replaced by $\mu^2 + \frac{1}{m^2}$. At this point we define $v_m \in v + W^{1,p}_0(B_R)$ as the unique minimizer in K of the functional

$$\mathcal{G}_m(w, B_R) := \int_{B_R} G_m(Dw(x)) \, dx$$

in the Dirichlet class $v + W_0^{1,p}(B_R)$. Using a standard coercivity argument and the strict convexity of the functional (2.3), (see for example [1]), it turns out that, up to subsequences, v_m weakly converges to v in $W^{1,p}(B_R)$ and the estimate (4.8) follows passing to the limit the corresponding estimates valid uniformly for each v_m. □

♢ Step 3. At this point we have that the following estimate holds for all $0 < \rho < R/2$ and any $\varepsilon > 0$

$$\int_{B_\rho} |Dv(x)|^p \, dx \le \, c \left[\left(\frac{\rho}{R} \right)^n + \varepsilon \right] \int_{B_R} (1 + |Dv(x)|^p) \, dx + \bar{c} \, R^\lambda,$$

where $c \equiv c(p, L, \nu)$ and $\bar{c} \equiv \bar{c}(p, L, \varepsilon, \nu)$. Using Lemma 3.4, we can choose a radius $R_1 \equiv R_1(p, L, \nu)$ and a constant $\varepsilon_0 > 0$ such that, if $\varepsilon \le \varepsilon_0$, we may deduce

$$\int_{B_\rho} |Dv(x)|^p \, dx \le \, \bar{c} \, \rho^\lambda,$$

with $\bar{c} \equiv \bar{c}(p, L, \nu, \varepsilon)$ whenever $0 < \rho < R_1$, fact which we may assume without loss of generality. This allows us to conclude that $Dv \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$. □

5 Proof of Theorem 2.9

Also the proof of this result is divided into three parts: in the first step we establish a technical estimate which will be used in the last part of the proof. This estimate can be only obtained with the constraint of the function g being of class $\mathcal{C}^2$; therefore in the second step we must remove this further regularity assumption by means of another approximation argument similar to the one used in Proposition 4.2.

♢ Step 1. We first prove the following proposition.

Proposition 5.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ satisfying* (H4) *and* (H5) *with L replaced by $8^p \, L$ and $\mu > 0$. Let $u \in K$, $B_R \subset \Omega$ and let $v_0 \in W^{1,p}(\Omega)$ be a minimizer of the functional*

$$\mathcal{H}(w, B_R) := \int_{B_R} g(Dw(x)) \, dx + \theta_0 \int_{B_R} |Dw - Dv_0| \, dx := \mathcal{G}_0(w, B_R) + \theta_0 \int_{B_R} |Dw - Dv_0| \, dx$$

in the Dirichlet class

$$\overline{D} := \{ w \in K : w = u \ \text{ on } \partial B_R \}, \tag{5.1}$$

where $\theta_0 \ge 0$. Then, for all $\beta > 0$, for all $A_0 > 0$ and for any $\varepsilon > 0$ we have

$$\begin{aligned} \int_{B_\rho} |Dv_0(x)|^p \, dx \le & \, c \left[\left(\frac{\rho}{R} \right)^n + \varepsilon \right] \int_{B_R} (1 + |Dv_0(x)|^p) \, dx + \, \bar{c} \, R^\lambda \\ & + \, c \, \theta_0 \int_{B_R} |Du(x) - Dv_0(x)| \, dx + c \, R^n \theta_0^{\frac{p}{p-1}} \left[\frac{1}{A_0} \right]^{\frac{p\beta}{p-1}} \\ & + \, c \, [A_0]^{p\beta} \int_{B_R} (1 + |Du(x)|^p) \, dx \end{aligned}$$

for any $0 < \rho < R/2$, where the constants c depend only on L, p, ν while the constant $\bar{c}$ depends also on ε.

Proof. Let $v \in W^{1,p}(B_R)$ be a local minimizer of the functional (2.3) in the Dirichlet class (5.1), where g is the function introduced in the statement of Proposition 5.1. So by Proposition 4.1 we have, for any $0<\rho<R/2$ and any $\varepsilon>0$

$$\int_{B_\rho} |Dv(x)|^p\,dx \le c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon\right]\int_{B_R}(1+|Dv(x)|^p)\,dx+\bar c\,R^\lambda,$$

with the constant $c\equiv c(L,p,\nu)$ and the constant $\bar c\equiv \bar c(L,p,\nu,\varepsilon)$; thus comparing v and v_0 and using the minimality of v in $\overline{D}$, we obtain, for any $0<\rho<R/2$ and any $\varepsilon>0$

$$\int_{B_\rho} |Dv(x)|^p\,dx \le c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon\right]\int_{B_R}(1+|Dv_0(x)|^p)\,dx+\bar c\,R^\lambda, \tag{5.2}$$

where $c\equiv c(p,L,\nu)$ and $\bar c\equiv \bar c(p,L,\varepsilon,\nu)$. We remark that, for obtaining this first result, it is not necessary to assume $g\in\mathcal{C}^2$ as we could use directly Proposition 4.2.
Moreover, arguing in a standard way and using (5.2), it is possible to obtain the following inequality

$$\begin{aligned}\int_{B_\rho} |Dv_0(x)|^p\,dx \le\,& c\left[\left(\frac{\rho}{R}\right)^n+\varepsilon\right]\int_{B_R}(1+|Dv_0(x)|^p)\,dx\\ &+c\int_{B_R}(\mu^2+|Dv_0(x)|^2+|Dv(x)|^2)^{\frac{p-2}{2}}\,|Dv_0(x)-Dv(x)|^2\,dx+\bar c\,R^\lambda\end{aligned}$$

and the following estimate (since in our case we are assuming $\mu>0$)

$$\mathcal{G}_0(v_0)-\mathcal{G}_0(v)\ge c^{-1}\int_{B_R}(\mu^2+|Dv_0(x)|^2+|Dv(x)|^2)^{(p-2)/2}|Dv_0(x)-Dv(x)|^2dx.$$

It is here that we specifically need the further $\mathcal{C}^2-$regularity of the function g. On the other hand, using the minimality of v_0 and triangular inequality, we deduce

$$\begin{aligned}&\mathcal{G}_0(v_0)-\mathcal{G}_0(v)\\ \le\,&\mathcal{H}(v_0)-\mathcal{H}(v)+\theta_0\int_{B_R}|Dv_0(x)-Dv(x)|dx\\ &+\theta_0\int_{B_R}|Dv(x)-Du(x)|dx-\theta_0\int_{B_R}|Dv(x)-Du(x)|dx\\ \le\,&\theta_0\int_{B_R}|Du(x)-Dv_0(x)|dx+\int_{B_R}\left\{\theta_0\left[\frac{1}{A_0}\right]^\beta\right\}\left\{|Dv(x)-Du(x)|\,[A_0]^\beta\right\}dx\\ \le\,&\theta_0\int_{B_R}|Du(x)-Dv_0(x)|dx+c\,R^n\,\theta_0{}^{\frac{p}{p-1}}\left[\frac{1}{A_0}\right]^{\frac{p\beta}{p-1}}\\ &+c\,[A_0]^{p\beta}\int_{B_R}(1+|Du(x)|^p)dx\end{aligned}$$

for all $\beta>0$ and all $A_0>0$. Connecting all the estimates we have just obtained, we get the thesis. □

◇ Step 2. We now remove the assumption of smoothness of the function g.

Proposition 5.2. *Let $g:\mathbb{R}^n\to\mathbb{R}$ be a continuous function satisfying* (H4) *and* (H5). *Let $u\in K$, $B_R\subset\Omega$ and let $v_0\in W^{1,p}(\Omega)$ be a minimizer of the functional*

$$\mathcal{H}(w,B_R):=\int_{B_R}g(Dw(x))\,dx+\theta_0\int_{B_R}|Dw-Dv_0|\,dx:=\mathcal{G}_0(w,B_R)+\theta_0\int_{B_R}|Dw-Dv_0|\,dx$$

in the Dirichlet class (5.1) *where* $\theta_0 \geq 0$. *Then, for all* $\beta > 0$, *for all* $A_0 > 0$ *and for any* $\varepsilon > 0$ *we have*

$$\int_{B_\rho} |Dv_0(x)|^p \, dx \leq c \left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right] \int_{B_R} (1 + |Dv_0(x)|^p) \, dx + \bar{c}\, R^\lambda$$
$$+ \, c\,\theta_0 \int_{B_R} |Du(x) - Dv_0(x)| \, dx + c\, R^n \theta_0^{\frac{p}{p-1}} \left[\frac{1}{A_0}\right]^{\frac{p\beta}{p-1}}$$
$$+ \, c\,[A_0]^{p\beta} \int_{B_R} (1 + |Du(x)|^p) \, dx$$

for any $0 < \rho < R/2$, *where the constants* c *depend only on* L, p, ν *while the constant* $\bar{c}$ *depends also on* ε.

Proof. Also this result is based on a standard approximation argument similar to the one employed in Proposition 4.2; it easily follows from [11] and Proposition 5.1. See also [8]. □

♢ Step 3. We are ready to deal with the main part of the proof of Theorem 2.9.

• Freezing.

In the previous sections we remarked that if u is a local minimizer of (1.1) in K, then it is possible to apply Theorem 3.2 and get that $Du \in L^{p+\delta}(\Omega)$, for some $\delta \equiv \delta(p, L) > 0$.
Now let us fix any $R > 0$ and any $x_0 \in B_{4R}$, where $B_{4R} \subset\subset \Omega$. For any $z \in \mathbb{R}^n$ we set

$$h(z) := f(x_0, (u)_R, z),$$

$$\mathcal{H}_0(w, B_R) := \int_{B_R} h(Dw(x))dx = \int_{B_R} f(x_0, (u)_R, Dw(x))dx. \tag{5.3}$$

Let $\bar{v}$ be the local minimizer of the functional (5.3) in the Dirichlet class $\{v \in K : v \in u + W_0^{1,p}(B_R)\}$. We immediately notice that the function $h(z) := f(x_0, (u)_R, z)$ satisfies the assumption of Proposition 3.3, so it follows that there exist two constants $c, \bar{r}$ both depending on p, L and independent of R and u, such that $p < \bar{r} < p(1 + \delta)$ and

$$\begin{aligned}\left(\fint_{B_R} |D\bar{v}(x)|^{\bar{r}} \, dx\right)^{p/\bar{r}} &\leq c \left[\fint_{B_{2R}} (1 + |Du(x)|^{p(1+\delta)}) \, dx\right]^{1/(1+\delta)} \\ &+ c \left[\fint_{B_{2R}} (1 + |D\psi(x)|^{p(1+\delta)}) \, dx\right]^{1/(1+\delta)}.\end{aligned} \tag{5.4}$$

Since u is a local minimizer of the functional (1.1) in K, we obtain

$$\begin{aligned}\mathcal{H}_0(u) \leq \mathcal{H}_0(\bar{v}) &+ \int_{B_R} f(x, \bar{v}(x), D\bar{v}(x))dx - \int_{B_R} f(x, u(x), D\bar{v}(x))dx \\ &+ \int_{B_R} f(x, u(x), D\bar{v}(x))dx - \int_{B_R} f(x_0, u(x), D\bar{v}(x))dx \\ &+ \int_{B_R} f(x_0, u(x), D\bar{v}(x))dx - \int_{B_R} f(x_0, (u)_R, D\bar{v}(x))dx \\ &+ \int_{B_R} f(x_0, (u)_R, Du(x))dx - \int_{B_R} f(x_0, u(x), Du(x))dx \\ &+ \int_{B_R} f(x_0, u(x), Du(x))dx - \int_{B_R} f(x, u(x), Du(x))dx \\ = \mathcal{H}_0(\bar{v}) &+ I + II + III + IV + V.\end{aligned} \tag{5.5}$$

• **Bounds for the quantities** $I, II, \dots, V$.

We now estimate the quantities $I, II, ..., V$.

$$
\begin{aligned}
I &\le L \int_{B_R} \omega(|\bar{v}(x) - u(x)|)(\mu^2 + |D\bar{v}(x)|^2)^{p/2} dx \\
&\le c \left[\int_{B_R} (\mu^2 + |D\bar{v}(x)|^2)^{\frac{\bar{r}}{2}}\, dx\right]^{p/\bar{r}} \left[\int_{B_R} \omega^{\bar{r}/(\bar{r}-p)}(|\bar{v}(x) - u(x)|)\, dx\right]^{(\bar{r}-p)/\bar{r}} \\
&\le c\, R^n \left[\fint_{B_R} (1 + |D\bar{v}(x)|^{\bar{r}})\, dx\right]^{p/\bar{r}} \left[\fint_{B_R} \omega^{\bar{r}/(\bar{r}-p)}(|\bar{v}(x) - u(x)|)\, dx\right]^{(\bar{r}-p)/\bar{r}} \\
&\overset{(5.4)}{\le} c\, R^n \left[\left(\fint_{B_{2R}} (1 + |Du(x)|^{p(1+\delta)})\, dx\right)^{1/(1+\delta)} + R^{\frac{\lambda-n}{1+\delta}}\right] \left[\fint_{B_R} \omega(|\bar{v}(x) - u(x)|)\, dx\right]^{(\bar{r}-p)/\bar{r}} \\
&\overset{(3.5)}{\le} c\, R^n \left[\fint_{B_{4R}} (1 + |Du(x)|^p)\, dx + R^{\frac{\lambda-n}{1+\delta}}\right] \left[\fint_{B_R} \omega(|\bar{v}(x) - u(x)|)\, dx\right]^{(\bar{r}-p)/\bar{r}} \\
&\le c\, \omega^\sigma \left(\fint_{B_R} |\bar{v}(x) - u(x)|\, dx\right) \left[\int_{B_{4R}} |Du(x)|^p\, dx + R^{\frac{\lambda+n\delta}{1+\delta}}\right]
\end{aligned}
$$

where $\sigma := \frac{\bar{r}-p}{\bar{r}}$. We set

$$\lambda_1 := \frac{\lambda + n\delta}{1+\delta}$$

and we notice that $\lambda_1 \ge \lambda$ so that $R^{\lambda_1} \le R^\lambda$. Now, using Poincaré inequality and Caccioppoli inequality for local minimizers with obstacle (3.6), we have

$$
\begin{aligned}
&\omega^\sigma \left(\fint_{B_R} |\bar{v}(x) - u(x)|\, dx\right) \\
&\le \omega^\sigma \left(R \fint_{B_R} |D\bar{v}(x) - Du(x)|\, dx\right) \\
&\le \omega^\sigma \left[\left(R^p \fint_{B_R} |D\bar{v}(x) - Du(x)|^p\, dx\right)^{1/p}\right] \\
&\le \omega^\sigma \left[\left(R^p \fint_{B_R} (1 + |Du(x)|^p)\, dx\right)^{1/p}\right] \\
&\overset{(3.6)}{\le} \omega^\sigma \left\{\left[c\, R^p \left(\fint_{B_{2R}} \left|\frac{u(x) - (u)_{2R}}{R}\right|^p dx + \fint_{B_{2R}} (1 + |D\psi(x)|^p)\, dx\right)\right]^{1/p}\right\} \\
&\le \omega^\sigma \left\{\left[c\, R^p \fint_{B_{2R}} \left|\frac{u(x) - (u)_{2R}}{R}\right|^p dx\right]^{1/p} + c\, R \left[\fint_{B_{2R}} (1 + |D\psi(x)|)^q\, dx\right]^{1/q}\right\} \\
&\overset{(3.16)}{\le} c\, \omega^\sigma \left\{\left[R^p \left(\frac{[u]_\gamma^p\, R^{p\,\gamma}}{R^p}\right)\right]^{1/p} + R\, \left(R^{\lambda-n}\right)^{1/q}\right\} \\
&\le c(p)\, \omega^\sigma \left[[u]_\gamma\, R^\gamma + R^{\frac{q+\lambda-n}{q}}\right] \\
&\le c(p, \gamma)\, \omega^\sigma (R^{\tilde{m}}),
\end{aligned}
$$

where $\tilde{m} := \min\left\{\gamma, \frac{q+\lambda-n}{q}\right\}$; we notice that, as we choose $\lambda > n - p$ then $\tilde{m} > 0$.
So, finally

$$I \le c(p, \gamma)\, \omega^\sigma (R^{\tilde{m}}) \left[\int_{B_{4R}} |Du(x)|^p\, dx + R^{\lambda_1}\right].$$

Now, using the minimality of $\bar{v}$, we get

$$III \leq L \int_{B_R} \omega(|u(x)-(u)_R|)(\mu^2+|D\bar{v}(x)|^2)^{p/2}\,dx \leq c(L)\,\omega(R^\gamma) \int_{B_R} (1+|Du(x)|^p)\,dx.$$

In a similar way we estimate IV

$$IV \leq L \int_{B_R} \omega(|u(x)-(u)_R|)(\mu^2+|Du(x)|^2)^{p/2}\,dx \leq c(L)\,\omega(R^\gamma) \int_{B_R} (1+|Du(x)|^p)\,dx.$$

On the other hand

$$II \leq L \int_{B_R} \omega(|x-x_0|)(\mu^2+|D\bar{v}(x)|^2)^{p/2}\,dx \leq c(L)\,\omega(R) \int_{B_R} (1+|Du(x)|^p)\,dx;$$

also the estimate of V comes immediately

$$V \leq L \int_{B_R} \omega(|x-x_0|)(\mu^2+|Du(x)|^2)^{p/2}\,dx \leq c(L)\,\omega(R) \int_{B_R} (1+|Du(x)|^p)\,dx.$$

Collecting the previous bounds and summing up we get

$$I+II+III+IV+V \leq c(L)\,\omega^\sigma(R^{\tilde{m}}) \left[\int_{B_{4R}} |Du(x)|^p\,dx + R^{\lambda_1}\right]. \tag{5.6}$$

• **Applying Ekeland's variational principle.**

At this point, by the minimality of $\bar{v}$, from (5.5) and (5.6), we obtain

$$\mathcal{H}_0(u) \leq \inf_V \mathcal{H}_0 + H(R),$$

where we set

$$H(R) := c(L)\,\omega^\sigma(R^{\tilde{m}}) \left[\int_{B_{4R}} |Du(x)|^p\,dx + R^{\lambda_1}\right]$$

and

$$V := \left\{ v \in u + W_0^{1,1}(B_R) : v \in K \right\}.$$

Now we are in a position to apply Theorem 1 in [7]. Let us consider V equipped with the distance

$$d(w_1,w_2) := H(R)^{-\frac{1}{p}}\, R^{-n\frac{p-1}{p}} \int_{B_R} |Dw_1(x)-Dw_2(x)|dx\,.$$

It is easy to see that the functional $\mathcal{H}_0$ is lower semicontinuous with respect to the topology induced by the distance d. Then by Theorem 1 in [7], it follows that there exists $v_0 \in V$ such that

(i) $\displaystyle\int_{B_R} |Du(x)-Dv_0(x)|dx \leq [H(R)]^{\frac{1}{p}} R^{n\frac{p-1}{p}}$,

(ii) $\mathcal{H}_0(v_0) \leq \mathcal{H}_0(u)$,

(iii) v_0 is a local minimizer in V of the functional

$$w \mapsto \mathcal{H}(w) := \mathcal{H}_0(w) + \left[\frac{H(R)}{R^n}\right]^{\frac{p-1}{p}} \int_{B_R} |Dw - Dv_0|dx.$$

Working exactly as in [8], we get for v_0 the following estimates

$$L^{-1} \int_{B_R} |Dv_0(x)|^p\,dx \leq \mathcal{H}_0(v_0) \leq \mathcal{H}_0(u) \leq L \int_{B_R} (1+|Du(x)|^p)\,dx, \tag{5.7}$$

$$(5.8) \qquad \left(\fint_{B_{R/2}} |Dv_0(x)|^s dx \right)^{p/s} \le c \fint_{B_R} |Dv_0(x)|^p \, dx + c \left(1 + \frac{H(R)}{R^n} \right),$$

where $s \in (p, p(1+\delta))$; the second estimate is a higher integrability result for v_0 : in fact, from (i), (ii) and (iii) of the Ekeland's variational principle we can prove that v_0 is also a local quasi-minimizer of a suitable functional with standard growth.

• Comparison and conclusion.

Now we are ready to apply Proposition 5.2 to the function $h(z) := f(x_0, (u)_R, z)$ and to the functional

$$\mathcal{H}(w, B_R) := \mathcal{H}_0(w, B_R) + \left[\frac{H(R)}{R^n} \right]^{\frac{p-1}{p}} \int_{B_R} |Dw(x) - Dv_0(x)| \, dx.$$

We choose $A_0 = F(R) := \omega^\sigma(R^{\tilde{m}})$ in the Proposition 5.2, so, using the property (i) given by the Ekeland's variational principle, we have for every $\beta > 0$ and any $\varepsilon > 0$

$$\int_{B_\rho} |Dv_0(x)|^p \, dx \le c \left[\left(\frac{\rho}{R} \right)^n + \varepsilon \right] \int_{B_R} (1 + |Dv_0(x)|^p) \, dx + c \, [F(R)]^{p\beta} \int_{B_R} (1 + |Du(x)|^p) \, dx$$
$$+ \, c \, H(R) + c \, H(R) \, [F(R)]^{\frac{p\beta}{1-p}} + \bar{c} \, R^\lambda$$

where the constants c depend only on L, p, ν while the constant $\bar{c}$ depends also on ε. Now, for a suitable choice of β and using (5.7), we can say that

$$\int_{B_\rho} |Dv_0(x)|^p \, dx \le c \left[\left(\frac{\rho}{R} \right)^n + [F(R)]^{p\,\beta} + \varepsilon \right] \int_{B_{4R}} (1 + |Du(x)|^p) \, dx + \bar{c} \, R^\lambda.$$

So summing up, we have for any $\varepsilon > 0$

$$\int_{B_\rho} |Du(x)|^p \, dx \le c \int_{B_\rho} |Dv_0(x)|^p \, dx + c \int_{B_\rho} |Du(x) - Dv_0(x)|^p \, dx$$
$$\le c \left[\left(\frac{\rho}{R} \right)^n + [F(R)]^{p\beta} + \varepsilon \right] \int_{B_{4R}} |Du(x)|^p \, dx + \bar{c} \, R^\lambda + c \int_{B_{R/2}} |Du(x) - Dv_0(x)|^p \, dx,$$

where $c \equiv c(L, p, \nu)$ and $\bar{c} \equiv \bar{c}(L, p, \nu, \varepsilon)$.
In order to finish to proof we have to estimate the last term of the previous formula. We choose $\theta \in (0, 1)$ such that $\theta/s + 1 - \theta = 1/p$ where s was the higher integrability exponent of v_0. As $s \in (p, p(1+\delta))$, we get

$$\int_{B_{R/2}} |Du(x) - Dv_0(x)|^p \, dx$$
$$\le c \, R^n \left(\fint_{B_{R/2}} |Du(x) - Dv_0(x)|^s \, dx \right)^{\frac{\theta\, p}{s}} \left(\fint_{B_{R/2}} |Du(x) - Dv_0(x)| \, dx \right)^{(1-\theta)\, p}$$
$$\overset{(i)}{\le} c \, R^n \, [H(R)^{\frac{1}{p}} \, R^{-\frac{n}{p}}]^{(1-\theta)\, p} \left[\left(\fint_{B_{R/2}} |Du(x)|^s \, dx \right)^{\frac{\theta\, p}{s}} + \left(\fint_{B_{R/2}} |Dv_0(x)|^s \, dx \right)^{\frac{\theta\, p}{s}} \right]$$
$$\overset{(5.7)\ (5.8)}{\le} c \, R^{n\,\theta} [H(R)]^{(1-\theta)} \left[\left(\fint_{B_{R/2}} |Du(x)|^{p(1+\delta)} \, dx \right)^{\frac{\theta}{1+\delta}} + \left(\fint_{B_{4R}} |Du(x)|^p \, dx + R^{\lambda_1 - n} \right)^\theta \right]$$
$$\overset{(3.5)}{\le} c \, R^{n\,\theta} [H(R)]^{(1-\theta)} \left(\fint_{B_{4R}} |Du(x)|^p \, dx + R^{\lambda_1 - n} \right)^\theta$$
$$\le c \, F(R)^{(1-\theta)} \left[\int_{B_{4R}} |Du(x)|^p \, dx + R^{\lambda_1} \right].$$

At this point we can insert this estimate in the previous one and get, for any $0 < \rho < R/8$ and for any $\varepsilon > 0$

$$\int_{B_\rho} |Du(x)|^p\,dx \le c\,\left[\left(\frac{\rho}{R}\right)^n + [F(R)]^{p\beta} + [F(R)]^{(1-\theta)} + \varepsilon\right]\int_{B_{4R}} |Du(x)|^p\,dx + \bar c\,R^\lambda$$

where the constant c only depend on p, L, ν while the constant $\bar c$ depends also on ε. Using Lemma 3.4 and the fact that $\lim_{R\to 0}[F(R)]^{p\beta} + [F(R)]^{(1-\theta)} = 0$, we can choose a radius $R_1 \equiv R_1(p, L, \nu) > 0$ and a constant $\varepsilon_0 > 0$ such that $[F(R)]^{p\beta} + [F(R)]^{(1-\theta)} \le \varepsilon_0/2$ whenever $0 < R < 16R_1$; so if also $\varepsilon \le \varepsilon_0/2$, we may deduce

$$\int_{B_\rho} |Du(x)|^p\,dx \le \bar c\,\rho^\lambda$$

with $\bar c \equiv \bar c(p, L, \nu, \varepsilon)$, whenever $0 < \rho < R_1$, fact which we may assume without loss of generality. This allows us to conclude that $Du \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$. □

Remark 5.3. If one takes a certain $\tilde q$ sufficiently small such that $p < \tilde q < p(1+\delta)$, where δ is the higher integrability exponent obtained for Du in the proof of Theorem 2.9, then the following higher integrability result

$$⨍_{B_\rho} |Du(x)|^{\tilde q}\,dx \le c\left(⨍_{B_{2\rho}} |Du(x)|^p\,dx\right)^{\tilde q/p} + c\,⨍_{B_{2\rho}} (|D\psi(x)|^{\tilde q} + 1)\,dx$$

can be obtained (working as in the proof of Theorem 3.2). This inequality implies

$$\begin{aligned}
&\rho^{-\lambda}\,⨍_{B_\rho} |Du(x)|^{\tilde q}\,dx \\
&\le c\,\rho^{-\lambda}\left(⨍_{B_{2\rho}} |Du(x)|^p\,dx\right)^{\tilde q/p} + c\,\rho^{-\lambda}\,⨍_{B_{2\rho}} (|D\psi(x)|^{\tilde q} + 1)\,dx \\
&\le c\left[\rho^{-\lambda\,\frac{p}{\tilde q}}\,⨍_{B_{2\rho}} |Du(x)|^p\,dx\right]^{\tilde q/p} + c\,\rho^{-\lambda}\,⨍_{B_{2\rho}} (|D\psi(x)|^{\tilde q} + 1)\,dx \\
&\le c\left[\rho^{\lambda\left(1-\frac{p}{\tilde q}\right)}\,\rho^{-\lambda}\,⨍_{B_{2\rho}} |Du(x)|^p\,dx\right]^{\tilde q/p} + c\,\rho^{-\lambda}\,⨍_{B_{2\rho}} (|D\psi(x)|^{\tilde q} + 1)\,dx \\
&\le c\,\rho^{\lambda\left(\frac{\tilde q}{p}-1\right)}\left[\rho^{-\lambda}\,⨍_{B_{2\rho}} |Du(x)|^p\,dx\right]^{\tilde q/p} + c\,\rho^{-\lambda}\,⨍_{B_{2\rho}} |(D\psi(x)|^{\tilde q} + 1)\,dx;
\end{aligned}$$

at this point, if $D\psi$ satisfies (2.5), i.e. if $D\psi \in L^{q,\lambda}_{\mathrm{loc}}(\Omega)$, then certainly $D\psi \in L^{\tilde q,\lambda}_{\mathrm{loc}}(\Omega)$ and therefore, using the estimates just obtained, also $Du \in L^{\tilde q,\lambda}_{\mathrm{loc}}(\Omega)$ as $\frac{\tilde q}{p} > 1$.

6 Proof of Theorem 2.11

Let u be a local minimizer of the functional (1.1) in K; we fix a radius $R > 0$ and any $x_0 \in B_{4R}$; let $v \in u + W^{1,p}_0(B_R)$ be a local minimizer in K of the functional $\mathcal{H}_0$ introduced in (5.3). Moreover let z be the solution of the following minimum problem

$$\min\left\{\mathcal{H}_0(z, B_R) : z \in u + W^{1,p}_0(B_R)\right\}. \tag{6.1}$$

Then, using estimates (2.4) and (2.5) in [17], we can easily obtain for all $0 < \rho < R/2$

$$⨍_{B_\rho} |Dz(x) - (Dz)_\rho|^p\,dx = ⨍_{B_\rho}\left|⨍_{B_\rho} (Dz(x) - Dz(y))\,dy\right|^p\,dx$$

$$\leq \left[\sup_{x,y\in B_\rho} |Dz(x)-Dz(y)|\right]^p \leq \left[c\left(\frac{\rho}{R}\right)^{\bar\beta} \sup_{B_{R/2}} |Dz|\right]^p \leq c\left(\frac{\rho}{R}\right)^{\bar\beta\, p} \fint_{B_R} (1+|Dz(x)|^p)\,dx,$$

where $c>0,\ 0<\bar\beta<1$ and both c and $\bar\beta$ depend only on p, L.
Our aim is now to compare z and w. First of all using the minimality of z, we have

$$\int_{B_R} |Dz(x)|^p\,dx \leq c(L) \int_{B_R} (1+|Dw(x)|^p)\,dx.$$

On the other hand, from Theorem 2.6

$$D\psi \in \mathcal{L}^{p,\lambda}(\Omega) \Rightarrow D\psi \in \mathcal{C}^{0,\alpha}(\Omega)$$

where $\alpha = \dfrac{\lambda-n}{p}$. At this point, if $p\geq 2$ then it is not difficult to see that

$$|A(D\psi(x))-A(D\psi(y))| \leq c\,|x-y|^\alpha, \qquad \forall\, x,y\in\Omega.$$

Therefore we deduce

$$\begin{aligned}
&\int_{B_R} |Dw(x)-Dz(x)|^p\,dx \\
&\leq \int_{B_R} (1+|Dw(x)|^2+|Dz(x)|^2)^{(p-2)/2}|Dw(x)-Dz(x)|^2\,dx \\
&\leq \int_{B_R} \langle A(Dw(x))-A(Dz(x)), Dw(x)-Dz(x)\rangle\,dx \\
&\overset{(4.4)}{=} \int_{B_R} \langle A(Dw(x)), Dw(x)-Dz(x)\rangle\,dx \\
&\overset{(4.1)}{=} \int_{B_R} \langle A(D\psi(x)), Dw(x)-Dz(x)\rangle\,dx \\
&= \int_{B_R} \langle A(D\psi(x))-(A(D\psi))_R, Dw(x)-Dz(x)\rangle\,dx \\
&\leq \int_{B_R} |A(D\psi(x))-(A(D\psi))_R|\cdot|Dw(x)-Dz(x)|\,dx \\
&\leq c(L)\,R^\alpha \int_{B_R} (|Dw(x)|^p+1)\,dx.
\end{aligned}$$

On the other hand, if $1<p<2$ we can easily prove that

$$|A(D\psi(x))-A(D\psi(y))| \leq c\,|x-y|^{\alpha(p-1)}, \qquad \forall\, x,y\in\Omega, \tag{6.2}$$

where again $\alpha=\dfrac{\lambda-n}{p}$. This allows us to conclude that

$$\begin{aligned}
&\int_{B_R} |Dw(x)-Dz(x)|^p\,dx \\
&\leq \left(\int_{B_R} (1+|Dw(x)|^2+|Dz(x)|^2)^{\frac{p-2}{2}}|Dw(x)-Dz(x)|^2\,dx\right)^{1/2} \\
&\quad\times\left(\int_{B_R} (1+|Dw(x)|^2+|Dz(x)|^2)^{\frac{2-p}{2}}\,|Dw(x)-Dz(x)|^{2p-2}\,dx\right)^{1/2} \\
&\leq c(L)\left[\int_{B_R} \langle A(D\psi(x))-(A(D\psi))_R, Dw(x)-Dz(x)\rangle\,dx\right]^{\frac12}\cdot\left(\int_{B_R}(1+|Dw(x)|^p)\,dx\right)^{\frac12} \\
&\leq c(L)\,R^{\frac{\alpha(p-1)}{2}} \int_{B_R} (1+|Dw(x)|^p)\,dx.
\end{aligned}$$

So in both cases we deduce

$$\int_{B_R} |Dw(x) - Dz(x)|^p \, dx \le c(L) \, R^{\frac{\alpha(p-1)}{2}} \int_{B_R} (1 + |Dw(x)|^p) \, dx.$$

Moreover, using (3.2), (3.3) and (4.1) we are able to deduce

$$\begin{aligned}
&\int_{B_R} |Dw(x)|^p \, dx \\
&\overset{(3.2)}{\le} \frac{1}{\nu} \int_{B_R} \langle A(Dw(x)), Dw(x)\rangle \, dx + c\, R^n \\
&\overset{(4.1)}{\le} \frac{1}{\nu} \int_{B_R} \langle A(Dw(x)), Dv(x)\rangle \, dx + \frac{1}{\nu} \int_{B_R} \langle A(D\psi(x)), Dw(x) - Dv(x)\rangle \, dx + c\, R^n \\
&\overset{(3.3)}{\le} \frac{L}{\nu} \int_{B_R} (|Dw(x)|^{p-1} + 1)|Dv(x)| \, dx + \frac{1}{\nu} \int_{B_R} \langle A(D\psi(x)), Dw(x)\rangle \, dx \\
&\quad - \frac{1}{\nu} \int_{B_R} \langle A(D\psi(x)), Dv(x)\rangle \, dx + c\, R^n \\
&\le \frac{1}{4} \int_{B_R} |Dw(x)|^p \, dx + c \int_{B_R} |Dv(x)|^p \, dx + \frac{1}{\nu} \int_{B_R} \langle A(D\psi(x)) - (A(D\psi))_R, Dw(x)\rangle \, dx \\
&\quad - \frac{1}{\nu} \int_{B_R} \langle A(D\psi(x)) - (A(D\psi))_R, Dv(x)\rangle \, dx + c\, R^n \\
&\le \frac{1}{4} \int_{B_R} (|Dw(x)|^p + 1) \, dx + c \int_{B_R} (|Dv(x)|^p + 1) \, dx + c\, R^n
\end{aligned}$$

where we used (6.2) which is valid for all $p > 1$ and where the constants c may depend on L, p, α, ν. Thus we obtain

$$\int_{B_R} |Dw(x)|^p \, dx \le c(L, p, \alpha, \nu) \int_{B_R} (|Dv(x)|^p + 1) \, dx.$$

Moreover, working as in the previous case, it is not difficult to deduce for all $p > 1$

$$\int_{B_R} |Dv(x) - Dw(x)|^p \, dx \le c(L, p, \alpha, \nu) \, R^{\frac{\alpha(p-1)}{2}} \int_{B_R} (|Dv(x)|^p + 1) \, dx. \tag{6.3}$$

Now it is time to compare u and v. Working as in the proof of Theorem 2.9 and using the assumption (2.7), we are able to say that

$$\mathcal{H}_0(u) - \mathcal{H}_0(v) \le c(L) \, \omega^{\sigma}(R^{\tilde{m}}) \int_{B_{4R}} (|Du(x)|^p + 1) \, dx \le c(L) \, R^{\zeta} \int_{B_{4R}} (|Du(x)|^p + 1) \, dx$$

where $\zeta \equiv \zeta(\gamma, \sigma, \xi, \lambda, p)$ and γ, σ were introduced in the previous section. Arguing in a standard way, we immediately have

$$\int_{B_R} |Du(x) - Dv(x)|^p \, dx \le c(L) \, R^{\zeta/2} \int_{B_{4R}} (|Du(x)|^p + 1) \, dx \tag{6.4}$$

which holds for all $p > 1$. Thus summing up, setting

$$M := \min\left\{ \frac{\alpha(p-1)}{2}, \frac{\zeta}{2} \right\}$$

we have for any $0 < \rho < R/2$

$$\begin{aligned}
&\int_{B_\rho} |Du(x) - (Du)_\rho|^p \, dx \\
&\le \int_{B_\rho} |Dv(x) - (Dv)_\rho|^p \, dx + \int_{B_\rho} |Du(x) - Dv(x)|^p \, dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{B_\rho} |Dw(x) - (Dw)_\rho|^p \, dx + \int_{B_R} |Dv(x) - Dw(x)|^p \, dx + \int_{B_R} |Du(x) - Dv(x)|^p \, dx \\
&\le \int_{B_\rho} |Dz(x) - (Dz)_\rho|^p \, dx + \int_{B_\rho} |Dw(x) - Dz(x)|^p \, dx \\
&\quad \overset{(6.3)\,(6.4)}{+} \; c(L,p,\alpha,\nu) \left(R^{\frac{\alpha(p-1)}{2}} + R^{\zeta/2} \right) \int_{B_{4R}} (|Du(x)|^p + 1) \, dx \\
&\le \rho^n \fint_{B_\rho} |Dz(x) - (Dz)_\rho|^p \, dx + \, c(L,p,\alpha,\nu) \, R^M \int_{B_{4R}} (|Du(x)|^p + 1) \, dx \\
&\le c(p,L) \left(\frac{\rho}{R}\right)^{\bar\beta\, p} \left(\frac{\rho}{R}\right)^{n} \int_{B_R} (1 + |Dz(x)|^p) \, dx + \, c(L,p,\alpha,\nu) \, R^M \int_{B_{4R}} (|Du(x)|^p + 1) \, dx \\
&\le c(L,p,\alpha,\nu) \left[\left(\frac{\rho}{R}\right)^{\bar\beta\, p+n} \int_{B_R} (1 + |Dv(x)|^p) \, dx + \, R^M \int_{B_{4R}} (|Du(x)|^p + 1) \, dx \right] \\
&\le c(L,p,\alpha,\nu) \left[\left(\frac{\rho}{R}\right)^{\bar\beta\, p+n} + \, R^M \right] \int_{B_{4R}} (|Du(x)|^p + 1) \, dx .
\end{aligned}
$$

On the other hand, using Theorem 3.1 we immediately get

$$
\int_{B_\rho} |Du(x)|^p \, dx \le \, c(L,p) \, \left[\left(\frac{\rho}{R}\right)^{n} + \, R^M \right] \int_{B_{4R}} (|Du(x)|^p + 1) \, dx;
$$

then, by a standard iteration lemma, we are able to deduce the existence of a radius R_0 such that for all $R \le R_0$

$$
\int_{B_R} |Du(x)|^p \, dx \le \, c \, R^{n-\tau}
$$

for all $0 < \tau < 1$. For our purposes, we can choose any $\tau < \dfrac{p\bar\beta M}{n + p\bar\beta}$, for example $\tau := \dfrac{1}{2} \dfrac{pM\bar\beta}{n + p\bar\beta}$. At this point we choose ρ such that $\rho = \dfrac{1}{2} R^{1+\theta}$ where $\theta := \dfrac{M}{n + \bar\beta\, p}$. With such a choice of ρ, θ and τ, we have that

$$
\int_{B_\rho} |Du(x) - (Du)_\rho|^p \, dx \le \, c(L,p,\alpha,\nu) \, \rho^{\tilde\lambda} \tag{6.5}
$$

where

$$
\tilde\lambda := n + \frac{p\,\bar\beta\, M}{2(n + p\,\bar\beta + M)}.
$$

But the choice of R was arbitrary, so without loss of generality we may assume that (6.5) holds for all $0 < \rho \le R_0$. This yields the thesis. □

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