Positivity, symmetry and uniqueness for minimizers of second order Sobolev inequalities *

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Abstract

We prove that minimizers for subcritical second order Sobolev embeddings in the unit ball are unique, positive and radially symmetric. Since the proofs of the corresponding first order results cannot be extended to the present situation we apply new and recently developed techniques.

1 Introduction

Let $\mathbf{B}$ denote the unit ball in $\mathbb{R}^n$ $(n \geq 2)$ and let $\|\cdot\|_q$ denote the $L^q(\mathbf{B})$ norm. Consider the *second order* Sobolev spaces

$$\text{either} \qquad \mathcal{H} = H^2_0(\mathbf{B}) \qquad \text{or} \qquad \mathcal{H} = H^2(\mathbf{B}) \cap H^1_0(\mathbf{B}) \; . \tag{1.1}$$

In view of the Sobolev-Rellich-Kondrachov Theorem, both spaces in (1.1) compactly embed into $L^p(\mathbf{B})$ for any $1 \leq p < 2_* = \frac{2n}{n-4}$, with the convention that $2_* = \infty$ if $n = 2, 3, 4$. These embedding properties are well explained through the inequalities

$$S_p \|u\|_p^2 \leq \|\Delta u\|_2^2 \qquad \text{for all } u \in \mathcal{H} \; , \qquad S_p \; = \min_{w \in \mathcal{H} \setminus \{0\}} \frac{\|\Delta w\|_2^2}{\|w\|_p^2} \; . \tag{1.2}$$

Since the embeddings are compact, for any $1 \leq p < 2_*$ there exists a minimizer u_p for S_p, namely there exists a nontrivial function $u_p \in \mathcal{H}$ such that $S_p \|u_p\|_p^2 = \|\Delta u_p\|_2^2$. The main result of this paper is the following.

Theorem 1. *Let $2 < p < 2_*$. Then, in both cases (1.1), the minimization problem (1.2) has, up to multiplication by a constant, a unique solution u_p which is positive, radially symmetric and radially decreasing.*

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We recall that, in case $\mathcal{H} = H_0^2(\mathbf{B})$, minimizers of (1.2) are weak solutions (see (1.7) below for a definition) of the following boundary value problem with Dirichlet boundary conditions

$$\begin{cases} \Delta^2 u = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{1.3}$$

whereas in case $\mathcal{H} = H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$, minimizers of (1.2) are weak solutions of the corresponding problem with Navier boundary conditions, i.e.,

$$\begin{cases} \Delta^2 u = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = \Delta u = 0 & \text{on } \partial\mathbf{B}. \end{cases} \tag{1.4}$$

As an interesting consequence of the positivity of minimizers of (1.2) and the Hopf boundary lemma, we infer that the best Sobolev constant S_p for $\mathcal{H} = H^2 \cap H_0^1$ in (1.2) is *strictly smaller* than the constant S_p for $\mathcal{H} = H_0^2$. In fact, this is true on an arbitrary bounded domain and not just on the ball $\mathbf{B}$, see Section 5 below. This is in striking contrast with the *critical case* $p = 2_*$ $(n \geq 5)$ for which van der Vorst [18] showed that the two embedding constants coincide.

Our second main result is concerned with the uniqueness of positive solutions of (1.3), (1.4).

Theorem 2. *Let* $2 < p < 2_*$. *Then:*

(i) Problem (1.4) has a unique positive solution which is radially symmetric and radially decreasing.

(ii) Problem (1.3) has a unique radial positive solution which is radially decreasing.

It is worth pointing out that we prove radial symmetry of minimizers of (1.2) in both cases (1.1), but we cannot prove radial symmetry of arbitrary positive solutions of (1.3). Thus we state the following

Open question: *Is every positive solution of (1.3) radially symmetric ?*

If this is true, then problem (1.3) also has a unique positive solution by Theorem 2 (ii).

The present paper is motivated by the results available for the first order Sobolev space $H_0^1(\mathbf{B})$ which compactly embeds into $L^p(\mathbf{B})$ for any $1 \leq p < 2^* = \frac{2n}{n-2}$, with the convention that $2^* = \infty$ if $n = 2$. These embeddings read

$$\Sigma_p \|v\|_p^2 \leq \|\nabla v\|_2^2 \qquad \text{for all } v \in H_0^1(\mathbf{B}) \ , \qquad \Sigma_p = \min_{w \in H_0^1 \setminus \{0\}} \frac{\|\nabla w\|_2^2}{\|w\|_p^2} \ . \tag{1.5}$$

If v_p is a minimizer for Σ_p, then also $|v_p|$ is a minimizer. And since a minimizer for Σ_p satisfies the Euler equation

$$\begin{cases} -\Delta v = |v|^{p-2}v & \text{in } \mathbf{B} \\ v = 0 & \text{on } \partial\mathbf{B} \ , \end{cases} \tag{1.6}$$

v_p may be assumed to be positive by the maximum principle. Elliptic regularity enables us to infer that v_p is smooth and therefore, for $p \geq 2$, v_p is radially symmetric and radially decreasing according to [10, Theorem 1]. Finally, from [10, Lemma 2.3] we know that there exists a unique positive solution of (1.6). By combining all these facts we conclude that, up to multiplicative constants, there exists a unique minimizer v_p for Σ_p and that it is positive, radially symmetric and radially decreasing. Let us also mention that positivity and radial symmetry (but not uniqueness!) may be proved via Schwarz symmetrization, see [16].

Summarizing, in order to obtain these properties for v_p, besides the maximum principle the following tools have been used in the literature. Firstly, the possibility of replacing $v \in H_0^1(\mathbf{B})$ with $|v|$. Secondly, the symmetry and the uniqueness results of [10]. Unfortunately, none of these tools can be used for embeddings of the second order Sobolev spaces $\mathcal{H}$ in (1.1). Indeed, if $u \in H^2(\mathbf{B})$ it is in general not true that $|u| \in H^2(\mathbf{B})$. The same implication is also false for the Schwarz symmetrization, see [7, 8] for a recent discussion of the problem. Moreover, the full extension of the symmetry result in [10] seems out of reach for the corresponding fourth order elliptic equations, see [15]. Finally, the uniqueness statement in [10, Lemma 2.3] does not hold for the corresponding fourth order ordinary differential equation, since also the "shooting concavity" $u''(0)$ represents a degree of freedom. Hence, in order to prove our results, we need to follow different methods. We obtain the positivity of minimizers u_p by using arguments inspired by [9] and [18]. Then, in the space $H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$ we may apply the symmetry result by Troy [17] to obtain its radially symmetry. In the space $H_0^2(\mathbf{B})$ the situation is more involved and we introduce a new technique based on *polarization*. This two-point rearrangement has been applied successfully in variational problems posed in first order Sobolev spaces or L^p-spaces, see e.g. [2, 3, 5, 6, 14] and the references therein. Its applicability to higher order problems is new and somewhat surprising, since in general polarized H^2-functions do not belong to H^2 anymore. Once we know that minimizers u_p are positive and radially symmetric, we combine a suitable scaling with a comparison argument recently developed by McKenna-Reichel [12] in order to prove uniqueness of radial positive solutions of (1.3) and (1.4).

The paper is organized as follows. In Section 2, we prove that, up to reflection $u \mapsto -u$, minimizers of the minimization problem (1.2) are strictly positive in $\mathbf{B}$. In Section 3 we then show that both (1.3) and (1.4) have a unique radial positive solution. Combining this with Troy's radial symmetry result [17] for positive solutions of (1.4), we obtain Theorem 2. Also, Theorem 1 follows in the case where $\mathcal{H} = H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$. In Section 4 we then show that, in case $\mathcal{H} = H_0^2(\mathbf{B})$, every minimizer of the minimization problem (1.2) is radially symmetric and radially decreasing. This completes the proof of Theorem 1 for $\mathcal{H} = H_0^2(\mathbf{B})$. In Section 5 we prove the strict inequality between the embedding constants S_p mentioned above.

Finally, we collect some definitions and notations used throughout the paper. We say that a function $u \in H_0^2(\mathbf{B})$ is a weak solution of (1.3) if

$$\int_{\mathbf{B}} \Delta u \Delta v \; dx = \int_{\mathbf{B}} |u|^{p-2} \, uv \; dx \qquad \text{for all } v \in H_0^2(\mathbf{B}). \tag{1.7}$$

Moreover, we say that a function $u \in H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$ is a weak solution of (1.4) if (1.7) holds for all $v \in H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$. By [11, Theorem 1], every weak solution of (1.3) is in fact a classical solution. Also, by [18, Lemma B.3], every weak solution of (1.4) is a classical solution. We endow both Hilbert spaces $\mathcal{H} = H_0^2(\mathbf{B})$ and $\mathcal{H} = H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$ with the scalar product

$$(u, v) = \int_{\mathbf{B}} \Delta u \Delta v \; dx \qquad \text{for } u, v \in \mathcal{H}. \tag{1.8}$$

For a subset $A \subset \mathbb{R}^n$, we denote by $\operatorname{int}(A)$, $\overline{A}$, and ∂A the interior, the closure, and the boundary of A.

2 Positivity of minimizers for (1.2)

The existence of minimizers for problems (1.2) may, in both cases (1.1), be obtained by a standard argument from the calculus of variations based on the compact embeddings $\mathcal{H} \subset L^p(\mathbf{B})$. In this section we prove that, if u is a minimizer, then u or $-u$ is strictly positive on $\mathbf{B}$. We give two proofs of this fact, both based on the following maximum principle:

Lemma 1. *Let $\mathcal{K} = \{w \in \mathcal{H};\ w \geq 0 \text{ a.e. in } \mathbf{B}\}$ and assume that $u \in \mathcal{H}$ satisfies*

$$\int_{\mathbf{B}} \Delta u \Delta v \geq 0 \qquad \textit{for all } v \in \mathcal{K}\ ;$$

then $u \in \mathcal{K}$. Moreover, one has either $u \equiv 0$ or $u > 0$ a.e. in B.

Proof. When $\mathcal{H} = H^2(\mathbf{B}) \cap H^1_0(\mathbf{B})$, the statement follows by the maximum principle for the operator $-\Delta$: take an arbitrary nonnegative $\varphi \in C^\infty_c(\mathbf{B})$ and use as test function $v_\varphi \in \mathcal{K}$ such that $-\Delta v_\varphi = \varphi$. When $\mathcal{H} = H^2_0(\mathbf{B})$, the statement is a consequence of Boggio's principle [4], see [9, Lemma 2] and [1, Lemma 16] for the details. □

In view of Lemma 1, strict positivity of the minimizer u of (1.2) follows if we show that $u \in \mathcal{K}$.

First proof. We use the decomposition method in dual cones developed in [9]. We consider the dual cone of $\mathcal{K}$, namely

$$\mathcal{K}' = \left\{ w \in \mathcal{H};\ \int_{\mathbf{B}} \Delta w \Delta v \leq 0 \text{ for all } v \in \mathcal{K} \right\}.$$

For contradiction, assume that a minimizer u of (1.2) is sign-changing. By the Proposition in [13] there exists a unique couple $(u_1, u_2) \in \mathcal{K} \times \mathcal{K}'$ such that $u = u_1 + u_2$ and $\int_{\mathbf{B}} \Delta u_1 \Delta u_2 = 0$. Consider the function $v := u_1 - u_2$. Then, since $u_1 \geq 0$ and $u_2 < 0$ (by Lemma 1) for a.e. $x \in \mathbf{B}$ we have $v(x) > |u(x)|$. Hence, $\|v\|_p > \|u\|_p$. Moreover, by orthogonality

$$\int_{\mathbf{B}} |\Delta v|^2 = \int_{\mathbf{B}} |\Delta u_1|^2 + \int_{\mathbf{B}} |\Delta u_2|^2 = \int_{\mathbf{B}} |\Delta u|^2.$$

Therefore,

$$\frac{\|\Delta u\|_2^2}{\|u\|_p^2} > \frac{\|\Delta v\|_2^2}{\|v\|_p^2},$$

which contradicts the assumption that u minimizes (1.2). □

Second proof. Let u be a minimizer for (1.2). Modulo scaling, we may assume that u is solution of (1.3) or (1.4). Suppose by contradiction that u is sign-changing in $\mathbf{B}$ and let w be a solution of the following problem

$$\begin{cases} \Delta^2 w = |u|^{p-1} & \text{in } \mathbf{B} \\ w = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\mathbf{B} \end{cases} \qquad \text{or} \qquad \begin{cases} \Delta^2 w = |u|^{p-1} & \text{in } \mathbf{B} \\ w = \Delta w = 0 & \text{on } \partial\mathbf{B}\ . \end{cases} \tag{2.1}$$

Lemma 1 implies that $w > |u|$ in $\mathbf{B}$. Hence, multiplying the equations in (2.1) by w and integrating by parts we obtain

$$\|\Delta w\|_2^2 = \int_{\mathbf{B}} w|u|^{p-1}\,dx < \int_{\mathbf{B}} w^2|u|^{p-2}\,dx \le \|w\|_p^2\|u\|_p^{p-2}$$

so that

$$\frac{\|\Delta w\|_2^2}{\|w\|_p^2} < \|u\|_p^{p-2} = \frac{\|u\|_p^p}{\|u\|_p^2} = \frac{\|\Delta u\|_2^2}{\|u\|_p^2}.$$

This contradicts the fact that u is a minimizer for (1.2). $\square$

Remark 1. A third proof which only works in the case where $\mathcal{H} = H^2(\mathbf{B}) \cap H_0^1(\mathbf{B})$ can also be obtained arguing as in [18]. It consists in showing that a minimizer u of (1.2) necessarily has Δu which does not change sign in $\mathbf{B}$, see also Lemma 7 below.

3 Uniqueness of positive radial solutions for (1.3)-(1.4)

In this section we prove that the boundary value problems (1.3) and (1.4) admit at most one positive radial solution. Although the proofs for the two problems are somewhat similar, we treat the two cases separately so that the differences become clear. In both cases, we will use the following comparison principle for radial functions due to McKenna-Reichel [12]:

Proposition 1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing. Let $u, v \in C^4([0, R])$ be positive functions such that*

$$\begin{cases} \Delta^2 v(r) - f(v(r)) \ge \Delta^2 u(r) - f(u(r)) & \text{in } [0, R) \\ v(0) \ge u(0),\ \ v'(0) = u'(0) = 0,\ \ \Delta v(0) \ge \Delta u(0),\ (\Delta v)'(0) = (\Delta u)'(0) = 0. \end{cases}$$

Then we have

(i) $v(r) \ge u(r),\ v'(r) \ge u'(r),\ \Delta v(r) \ge \Delta u(r),\ (\Delta v)'(r) \ge (\Delta u)'(r)$ *for any* $r \in [0, R]$

(ii) if there exists $\rho \in (0, R)$ *such that* $v > u$ *in* $(0, \rho)$ *then* $v(r) > u(r),\ v'(r) > u'(r),\ \Delta v(r) > \Delta u(r),\ (\Delta v)'(r) > (\Delta u)'(r)$ *for any* $r \in (0, R]$.

3.1 Uniqueness for (1.3)

We assume by contradiction that $u_1 \ne u_2$ are two positive radial solutions of (1.3). In radial coordinates $r = |x|$, the functions u_1 and u_2 solve the following initial value problem (for some A_1, A_2, B_1, B_2):

$$\begin{cases} u_i^{iv} + \dfrac{2(n-1)}{r}u_i''' + \dfrac{(n-1)(n-3)}{r^2}u_i'' - \dfrac{(n-1)(n-3)}{r^3}u_i' = u_i^{p-1} & r \in (0,1) \\ u_i(0) = A_i,\ \ u_i'(0) = 0,\ \ u_i''(0) = B_i,\ \ u_i'''(0) = 0 \end{cases} \quad i = 1, 2\ . \tag{3.1}$$

Moreover, the Dirichlet boundary conditions become

$$u_i(1) = u_i'(1) = 0 \quad \text{for } i = 1, 2. \tag{3.2}$$

Applying Proposition 1 to problem (3.1), we deduce that either $A_1 \neq A_2$ or $B_1 \neq B_2$ since otherwise we have $u_1 \equiv u_2$. If $A_1 = A_2$ then we necessarily have $B_1 \neq B_2$ and, up to switching u_1 with u_2, we may assume that $B_1 > B_2$. Since $u_1(0) = u_2(0) = A_1 = A_2$ and $u_1'(0) = u_2'(0) = 0$, there exists $\rho > 0$ such that

$$u_1(r) > u_2(r) \qquad \text{for all } r \in (0, \rho) \tag{3.3}$$

so that by Proposition 1 (ii) we infer $u_1(1) > u_2(1)$ which contradicts (3.2).

Therefore, from now on we may assume that $A_1 \neq A_2$. We define

$$v_i(r) = A_i^{-1} u_i \left(A_i^{-\frac{p-2}{4}} r \right) \qquad \text{for all } r \in \left(0, A_i^{\frac{p-2}{4}} \right), \quad i = 1, 2 \tag{3.4}$$

and

$$\widetilde{B}_i = v_i''(0), \quad R_i = A_i^{\frac{p-2}{4}}, \qquad i = 1, 2. \tag{3.5}$$

Then for $i = 1, 2$ the functions v_i solve the ordinary differential equation (3.1) on $(0, R_i)$ and satisfy $v_i(0) = 1$ and $v_i(R_i) = v_i'(R_i) = 0$. We may assume that $\widetilde{B}_1 \neq \widetilde{B}_2$ since otherwise by Proposition 1 we have $v_1 \equiv v_2$ and, in turn, $u_1 \equiv u_2$. Moreover, up to switching v_1 with v_2, we may suppose that $\widetilde{B}_1 > \widetilde{B}_2$. Applying to v_1 and v_2 the same argument employed for (3.3), by Proposition 1 (ii) it follows that

$$R_1 > R_2 \ \text{ and } \ v_1(r) > v_2(r) \qquad \text{for all } r \in (0, R_2]. \tag{3.6}$$

Denote by B_{R_1} and B_{R_2} the open balls centered at the origin whose radii are respectively R_1 and R_2. Then by (3.4)-(3.5) we have for $i = 1, 2$

$$\begin{cases} \Delta^2 v_i = v_i^{p-1} & \text{in } B_{R_i} \\ v_i = \dfrac{\partial v_i}{\partial \nu} = 0 & \text{on } \partial B_{R_i}. \end{cases} \tag{3.7}$$

In both cases $i = 1, 2$, multiplying the equation in (3.7) by v_j for $j \neq i$ and integrating by parts (over B_{R_2}) four times yields

$$\begin{aligned} \int_{B_{R_2}} v_1^{p-1} v_2 \, dx &= \int_{B_{R_2}} \Delta^2 v_1 \; v_2 \, dx \\ &= \int_{B_{R_2}} v_1 \Delta^2 v_2 \, dx - \int_{\partial B_{R_2}} \left(\frac{\partial (\Delta v_2)}{\partial \nu} v_1 - \Delta v_2 \frac{\partial v_1}{\partial \nu} \right) dS \\ &= \int_{B_{R_2}} v_2^{p-1} v_1 \, dx - \int_{\partial B_{R_2}} \left(\frac{\partial (\Delta v_2)}{\partial \nu} v_1 - \Delta v_2 \frac{\partial v_1}{\partial \nu} \right) dS, \end{aligned} \tag{3.8}$$

where we used the fact that $v_2 \in H_0^2(B_{R_2})$.

We now give a sign to the boundary integral in (3.8).

Lemma 2. *Let v_1 and v_2 be the functions defined in (3.4). Then we have*

$$\int_{\partial B_{R_2}} \left(\frac{\partial (\Delta v_2)}{\partial \nu} v_1 - \Delta v_2 \frac{\partial v_1}{\partial \nu} \right) dS > 0$$

Proof. First of all note that the initial value problem (3.1) for v_1 and v_2 reads

$$\begin{cases} \left(r^{n-1}\left(\Delta v_i\right)'\right)' = r^{n-1}v_i^{p-1} & \text{in } (0,R_i) \\ v_i\left(0\right)=1, \;\; v_i'\left(0\right)=0, \;\; \Delta v_i\left(0\right)=n\widetilde{B}_i, \;\; \left(\Delta v_i\right)'\left(0\right)=0 & \end{cases} \qquad i=1,2, \tag{3.9}$$

where the initial conditions in the second line of (3.9) follow from a straightforward application of De l'Hospital's rule. After integration in (3.9) we have

$$\left(\Delta v_2\right)'\left(R_2\right)=\frac{1}{R_2^{n-1}}\int_0^{R_2} s^{n-1}v_2^{p-1}(s)\;ds>0. \tag{3.10}$$

Moreover, since $v_2(R_2)=v_2'\left(R_2\right)=0$ and $v_2(r)>0$ in $\left(0,R_2\right)$, it follows that

$$\Delta v_2\left(R_2\right)=v_2''(R_2)\geq 0. \tag{3.11}$$

Next we turn to the function v_1. We claim that

$$\frac{\partial v_1}{\partial \nu}<0 \qquad \text{on } \partial B_{R_2}. \tag{3.12}$$

We have

$$\left(\Delta v_1\right)'\left(r\right)=\frac{1}{r^{n-1}}\int_0^{r} s^{n-1}v_1^{p-1}(s)\;ds>0 \qquad \text{for all } r\in(0,R_1) \tag{3.13}$$

which implies that $r\mapsto \Delta v_1\left(r\right)$ is strictly increasing in $(0,R_1)$. Two cases may occur; we show that in both cases (3.12) holds.

First case: $\Delta v_1\left(R_1\right)\leq 0$. In this case, we have $-\Delta(-\Delta v_1)=v_1^{p-1}\geq 0$ in B_{R_1} and $-\Delta v_1\geq 0$ on ∂B_{R_1}. The maximum principle for $-\Delta$ then implies $\Delta v_1<0$ in B_{R_1}, that is, $(r^{n-1}v_1'(r))'=r^{n-1}\Delta v_1(r)<0$ on $[0,R_1)$ and (3.12) follows since $r^{n-1}v_1'(r)|_{r=0}=0$.

Second case: $\Delta v_1\left(R_1\right)>0$. In this case, since the map $r\mapsto r^{n-1}v_1'(r)$ equals zero both at $r=0$ and at $r=R_1$, by (3.13) we know that the map $r\mapsto \Delta v_1(r)=r^{1-n}(r^{n-1}v_1'(r))'$ admits exactly one change of sign in $(0,R_1)$. Hence, $v_1'\left(r\right)<0$ for any $r\in(0,R_1)$, which proves (3.12).

Finally, by (3.6) it is clear that $v_1\left(R_2\right)>0$. This inequality, combined with (3.10)-(3.12), proves the lemma. □

By (3.8) and Lemma 2, we obtain

$$\int_{B_{R_2}} v_1v_2\left(v_1^{p-2}-v_2^{p-2}\right)\;dx<0 \tag{3.14}$$

which contradicts (3.6). This contradiction proves uniqueness of positive radial solutions of (1.3).

3.2 Uniqueness for (1.4)

We assume by contradiction that $u_1\neq u_2$ are two positive radial solutions of (1.4). In radial coordinates $r=\left|x\right|$, the functions u_1 and u_2 satisfy (3.1) and the Navier boundary conditions

$$u_i\left(1\right)=\Delta u_i\left(1\right)=0 \qquad \text{for } i=1,2.$$

As in the previous subsection, using the scaling (3.4), we introduce the functions v_1 and v_2 and we assume $\widetilde{B}_1 > \widetilde{B}_2$ so that we have again (3.6). Moreover, v_1 and v_2 solve

$$\begin{cases} \Delta^2 v_i = v_i^{p-1} & \text{in } B_{R_i} \\ v_i = \Delta v_i = 0 & \text{on } \partial B_{R_i} \end{cases} \qquad i = 1, 2.$$

After multiplication and integration by parts, we have

$$\int_{B_{R_2}} v_1^{p-1} v_2 \, dx = \int_{B_{R_2}} \Delta^2 v_1 \; v_2 \, dx$$

$$= \int_{B_{R_2}} v_1 \Delta^2 v_2 \, dx - \int_{\partial B_{R_2}} \left(\Delta v_1 \frac{\partial v_2}{\partial \nu} + \frac{\partial (\Delta v_2)}{\partial \nu} v_1 \right) \, dS$$

$$= \int_{B_{R_2}} v_1 v_2^{p-1} \, dx - \int_{\partial B_{R_2}} \left(\Delta v_1 \frac{\partial v_2}{\partial \nu} + \frac{\partial (\Delta v_2)}{\partial \nu} v_1 \right) \, dS \tag{3.15}$$

since $v_2 = \Delta v_2 = 0$ on ∂B_{R_2}. Next, we prove the following

Lemma 3. *Let v_1 and v_2 be the functions defined above. Then we have*

$$\int_{\partial B_{R_2}} \left(\Delta v_1 \frac{\partial v_2}{\partial \nu} + \frac{\partial (\Delta v_2)}{\partial \nu} v_1 \right) \, dS > 0.$$

Proof. Note that v_1 and v_2 satisfy again (3.9). Since

$$(\Delta v_1)'(r) = \frac{1}{r^{n-1}} \int_0^r s^{n-1} v_1^{p-1}(s) \, ds > 0 \qquad \text{for all } r \in (0, R_1),$$

the map $r \mapsto \Delta v_1(r)$ is strictly increasing in $(0, R_1)$ and since $\Delta v_1(R_1) = 0$ it follows that

$$\Delta v_1 < 0 \qquad \text{on } \partial B_{R_2} \tag{3.16}$$

Since $\Delta v_2(R_2) = 0$, by (3.10) we have $\Delta v_2(r) < 0$ for all $r \in (0, R_2)$. This implies

$$\left(r^{n-1} v_2'(r) \right)' = r^{n-1} \Delta v_2(r) < 0 \qquad \text{for all } r \in (0, R_2)$$

and since $v_2'(0) = 0$ we obtain

$$\frac{\partial v_2}{\partial \nu} < 0 \quad \text{on } \partial B_{R_2}. \tag{3.17}$$

The statement of the lemma follows from (3.6), (3.10) and (3.16)-(3.17). ∎

By (3.15) and Lemma 3, we obtain again (3.14) which proves uniqueness of positive radial solutions of (1.4).

4 Radial symmetry of the Sobolev minimizers under Dirichlet boundary conditions

In this section we prove the following.

Theorem 3. *If $u \in H_0^2(\mathbf{B})$ is a minimizer for (1.2), then u is* Schwarz symmetric*, i.e., it is radially symmetric and nonincreasing in the radial variable.*

Let $H \subset \mathbb{R}^n$ be an affine half-space, and let $\sigma_H : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection at the boundary ∂H of H. Let $C_0(\mathbb{R}^n)$ be the space of continuous functions on $\mathbb{R}^n$ with compact support. For $v \in C_0(\mathbb{R}^n)$, we define the *polarization* v_H of v relative to H by

$$v_H(x) = \begin{cases} \max\{v(x), v(\sigma_H(x))\}, & x \in H, \\ \min\{v(x), v(\sigma_H(x))\}, & x \in \mathbb{R}^n \setminus H. \end{cases}$$

We note that, by a straightforward argument, $\|v_H\|_p = \|v\|_p$ for all $v \in C_0(\mathbb{R}^n)$, $1 \le p \le \infty$ and all affine half-spaces $H \subset \mathbb{R}^n$. Moreover, we have the identity

$$v(x) + v(\sigma_H(x)) = v_H(x) + v_H(\sigma_H(x)) \qquad \text{for every } x \in \mathbb{R}^n. \tag{4.1}$$

Now let $\mathcal{H}_0$ denote the family of all closed affine half-spaces $H \subset \mathbb{R}^n$ such that $0 \in \operatorname{int}(H)$. Then we have the following useful characterization, which follows directly from [6, Lemma 6.4].

Proposition 2. *A function $v \in C_0(\mathbb{R}^n)$ is Schwarz symmetric (with respect to the origin) if and only if $v = v_H$ for every $H \in \mathcal{H}_0$.*

If $u \in H_0^2(\mathbf{B})$ is a minimizer for (1.2), then u solves (1.3). As already mentioned, in view of [11, Theorem 1] we then have $u \in C^\infty(\overline{\mathbf{B}})$ so that, by trivial extension, u may be seen as a function in $C_0(\mathbb{R}^n)$. And by Proposition 2, the problem of showing Schwarz symmetry of u is reduced to showing that $u = u_H$ for every $H \in \mathcal{H}_0$. To follow this approach, we first need some crucial estimates for the Green's function $G = G(x,y)$ of Δ^2 on $\mathbf{B}$ relative to the Dirichlet boundary conditions. It is convenient to introduce the quantity

$$\theta(x,y) = \begin{cases} (1-|x|^2)(1-|y|^2) & \text{if } x, y \in \mathbf{B} \\ 0 & \text{if } x \notin \mathbf{B} \text{ or } y \notin \mathbf{B}. \end{cases}$$

Then for $x, y \in \mathbf{B}$, $x \neq y$ we have the following representation due to Boggio, see [4, p.126]:

$$G(x,y) = c_n |x-y|^{4-n} \int_1^{\left(\frac{\theta(x,y)}{|x-y|^2}+1\right)^{1/2}} \frac{\tau^2-1}{\tau^{n-1}}\, d\tau = \frac{c_n}{2}|x-y|^{4-n} \int_0^{\frac{\theta(x,y)}{|x-y|^2}} \frac{z}{(z+1)^{n/2}}\, dz. \tag{4.2}$$

Here c_n is a positive constant which only depends on the dimension n. In the following, we will assume that G is extended in a trivial way to $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$, i.e., $G(x,y) = 0$ if $|x| \ge 1$ or $|y| \ge 1$. Then formula (4.2) is valid for all $x, y \in \mathbb{R}^n$, $x \neq y$. For $h \in C_0(\mathbb{R}^n)$ we consider the function $\mathcal{G}h : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathcal{G}h(x) = \int_{\mathbb{R}^n} G(x,y)h(y)\, dy.$$

Then $\mathcal{G}h \equiv 0$ on $\mathbb{R}^n \setminus \mathbf{B}$, and $\mathcal{G}h|_{\mathbf{B}}$ is the unique solution of the problem

$$\begin{cases} \Delta^2 u = h & \text{in } \mathbf{B} \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbf{B}. \end{cases}$$

To avoid lengthy notations, from now on we write $\overline{x}$ instead of $\sigma_H(x)$ for any $x \in \mathbb{R}^n$ whenever the underlying affine half-space H is understood.

Lemma 4. *Let $H \in \mathcal{H}_0$. Then for $x, y \in H, x \neq y$ we have*

$$G(x,y) \geq \max\{G(x,\overline{y}), G(\overline{x},y)\}, \tag{4.3}$$

$$G(x,y) - G(\overline{x},\overline{y}) \geq |G(x,\overline{y}) - G(\overline{x},y)|. \tag{4.4}$$

Moreover, if $x, y \in \text{int}(\mathbf{B} \cap H)$, then we have strict inequalities in (4.3) and (4.4).

Proof. We first note that, since $H \in \mathcal{H}_0$, we have $|\overline{z}| \geq |z|$ for all $z \in H$. Hence, if $x \notin \mathbf{B}$ or $y \notin \mathbf{B}$, then also $\overline{x} \notin \mathbf{B}$ or $\overline{y} \notin \mathbf{B}$, and both sides of the inequalities (4.3) and (4.4) are zero in this case. Therefore it suffices to consider $x, y \in \text{int}(\mathbf{B} \cap H)$ and to prove the strict inequality in (4.3) and (4.4). It is easy to see that

$$|x-y| = |\overline{x} - \overline{y}| < |x - \overline{y}| = |\overline{x} - y| \qquad \text{for } x, y \in \text{int}(H). \tag{4.5}$$

Moreover, since $|\overline{x}| > |x|$ and $|\overline{y}| > |y|$ for all $x, y \in \text{int}(H)$, we have

$$\theta(x,y) \;>\; \max\{\theta(x,\overline{y}), \theta(\overline{x},y)\} \;\geq\; \min\{\theta(x,\overline{y}), \theta(\overline{x},y)\} \;>\; \theta(\overline{x},\overline{y}). \tag{4.6}$$

We now write $G(x,y) = \frac{c_n}{2} H(|x-y|^2, \theta(x,y))$ with

$$H : (0,\infty) \times [0,\infty) \to \mathbb{R}, \qquad H(s,t) = s^{2-\frac{n}{2}} \int_0^{t/s} \frac{z}{(z+1)^{n/2}}\, dz.$$

We first verify the following properties of H:

$$\partial_s H(s,t) < 0, \tag{4.7}$$

$$\partial_t H(s,t) > 0, \tag{4.8}$$

$$\partial_s \partial_t H(s,t) < 0 \tag{4.9}$$

for $s, t > 0$. Indeed,

$$\partial_s H(s,t) = (2 - \frac{n}{2}) s^{1-\frac{n}{2}} \int_0^{t/s} \frac{z}{(z+1)^{n/2}}\, dz - \frac{t^2}{s(t+s)^{n/2}} \tag{4.10}$$

so that (4.7) immediately follows for $n \geq 4$. Since $3x + 2 < 2(x+1)^{3/2}$ for all $x > 0$, putting $n = 3$ in (4.10) we obtain

$$\partial_s H(s,t) = \frac{3st + 2s^2 - 2\sqrt{s}(t+s)^{3/2}}{s(t+s)^{3/2}} < 0$$

which proves (4.7) for $n = 3$. Finally, since $\frac{x}{x+1} < \log(x+1)$ for all $x > 0$, putting $n = 2$ in (4.10) we obtain

$$\partial_s H(s,t) = \frac{t}{t+s} - \log\frac{t+s}{s} < 0$$

which proves (4.7) also for $n = 2$.

Moreover,

$$\partial_t H(s,t) = \frac{t}{(t+s)^{n/2}} > 0 \qquad \text{and} \qquad \partial_s\partial_t H(s,t) = -\frac{nt}{2(t+s)^{n/2+1}} < 0$$

so that (4.8)-(4.9) are also true.

From (4.7) and (4.8) it follows that

$$H(s_1,t_1) > H(s_2,t_2) \qquad \text{if } s_1 < s_2,\ t_1 > t_2, \tag{4.11}$$

while (4.8) and (4.9) imply that

$$\begin{aligned} H(s_1,t_4) - H(s_1,t_1) &= \int_{t_1}^{t_4} \partial_t H(s_1,t)\,dt > \int_{t_1}^{t_4} \partial_t H(s_2,t)\,dt > \int_{\min\{t_2,t_3\}}^{\max\{t_2,t_3\}} \partial_t H(s_2,t)\,dt \\ &= |H(s_2,t_2) - H(s_2,t_3)| \qquad \text{if } 0 < s_1 < s_2,\ 0 < t_1 < t_2, t_3 < t_4. \end{aligned} \tag{4.12}$$

The strict inequality in (4.3) follows directly from (4.5), (4.6), and (4.11). Moreover, the strict inequality in (4.4) follows from (4.5), (4.6), and (4.12). □

Lemma 5. *Let $H \in \mathcal{H}_0$, let $f \in C_0(\mathbb{R}^n)$ be a nonnegative function with support contained in $\overline{\mathbf{B}}$, and let $u = \mathcal{G}f$, $w = \mathcal{G}f_H$. Then:*

$$w(x) \geq w(\overline{x}) \qquad \textit{for } x \in H, \tag{4.13}$$

$$w(x) \geq u_H(x) \qquad \textit{for } x \in H, \tag{4.14}$$

$$w(x) + w(\overline{x}) \geq u_H(x) + u_H(\overline{x}) \qquad \textit{for } x \in \mathbb{R}^n. \tag{4.15}$$

Moreover, if $f \not\equiv f_H$, then the inequality (4.15) is strict for every $x \in \operatorname{int}(\mathbf{B} \cap H)$.

Proof. Let $x \in H$. Then, since $f_H(y) \geq f_H(\overline{y})$ for all $y \in H$, we have

$$\begin{aligned} w(x) - w(\overline{x}) &= \int_{\mathbb{R}^n} [G(x,y) - G(\overline{x},y)] f_H(y)\,dy \\ &= \int_H \Big([G(x,y) - G(\overline{x},y)] f_H(y) + [G(x,\overline{y}) - G(\overline{x},\overline{y})] f_H(\overline{y}) \Big)\,dy \\ &\geq \int_H \Big([G(x,y) - G(\overline{x},y)] + [G(x,\overline{y}) - G(\overline{x},\overline{y})] \Big) f_H(\overline{y})\,dy. \end{aligned}$$

By Lemma 4, the integrand in the last line is nonnegative, hence (4.13) follows. Next, using (4.1) and Lemma 4, we obtain

$$\begin{aligned} w(x) - u(x) &= \int_{\mathbb{R}^n} G(x,y)(f_H(y) - f(y))\,dy \\ &= \int_H \Big(G(x,y)[f_H(y) - f(y)] + G(x,\overline{y})[f_H(\overline{y}) - f(\overline{y})] \Big)\,dy \\ &= \int_H \Big(G(x,y) - G(x,\overline{y}) \Big) [f_H(y) - f(y)]\,dy \geq 0. \end{aligned} \tag{4.16}$$

Moreover,

$$\begin{aligned} w(x) - u(\overline{x}) &= \int_{\mathbb{R}^n} [G(x,y)f_H(y) - G(\overline{x},y)f(y)]\,dy \\ &= \int_H \Big(G(x,y)f_H(y) - G(\overline{x},y)f(y) + G(x,\overline{y})f_H(\overline{y}) - G(\overline{x},\overline{y})f(\overline{y}) \Big)\,dy. \end{aligned} \tag{4.17}$$

To estimate the integrand in (4.17), we distinguish two cases. If $y \in H$ is such that $f_H(y) = f(y)$, then also $f_H(\overline{y}) = f(\overline{y})$ and by (4.3)-(4.4) we have

$$\begin{aligned} &G(x,y)f_H(y) - G(\overline{x},y)f(y) + G(x,\overline{y})f_H(\overline{y}) - G(\overline{x},\overline{y})f(\overline{y}) \\ &\qquad = [G(x,y) - G(\overline{x},y)]f_H(y) + [G(x,\overline{y}) - G(\overline{x},\overline{y})]f_H(\overline{y}) \\ &\qquad \geq \Big(G(x,y) - G(\overline{x},y) + G(x,\overline{y}) - G(\overline{x},\overline{y}) \Big) f_H(\overline{y}) \geq 0. \end{aligned}$$

On the other hand, if $y \in H$ is such that $f_H(y) = f(\overline{y})$, then $f_H(\overline{y}) = f(y)$ and by (4.3)-(4.4) we obtain

$$\begin{aligned} &G(x,y)f_H(y) - G(\overline{x},y)f(y) + G(x,\overline{y})f_H(\overline{y}) - G(\overline{x},\overline{y})f(\overline{y}) \\ &\qquad = [G(x,y) - G(\overline{x},\overline{y})]f_H(y) + [G(x,\overline{y}) - G(\overline{x},y)]f_H(\overline{y}) \\ &\qquad \geq \Big(G(x,y) - G(\overline{x},\overline{y}) + G(x,\overline{y}) - G(\overline{x},y) \Big) f_H(\overline{y}) \geq 0. \end{aligned}$$

Going back to (4.17) we find $w(x) - u(\overline{x}) \geq 0$, and together with (4.16) this yields $w(x) \geq u_H(x)$ for $x \in H$. Hence (4.14) holds.

To prove (4.15), we may assume that $x \in H$. Since

$$\begin{aligned} w(x) + w(\overline{x}) &= \int_{\mathbb{R}^n} [G(x,y) + G(\overline{x},y)]f_H(y)\,dy \\ &= \int_H \Big([G(x,y) + G(\overline{x},y)]f_H(y) + [G(x,\overline{y}) + G(\overline{x},\overline{y})]f_H(\overline{y}) \Big)\,dy \end{aligned}$$

and

$$\begin{aligned} u(x) + u(\overline{x}) &= \int_{\mathbb{R}^n} [G(x,y) + G(\overline{x},y)]f(y)\,dy \\ &= \int_H \Big([G(x,y) + G(\overline{x},y)]f(y) + [G(x,\overline{y}) + G(\overline{x},\overline{y})]f(\overline{y}) \Big)\,dy, \end{aligned}$$

we find

$$\begin{aligned} &w(x) + w(\overline{x}) - [u(x) + u(\overline{x})] \\ &= \int_H \Big([G(x,y) + G(\overline{x},y)](f_H(y) - f(y)) + [G(x,\overline{y}) + G(\overline{x},\overline{y})](f_H(\overline{y}) - f(\overline{y})) \Big)\,dy \\ &= \int_H \Big(G(x,y) + G(\overline{x},y) - [G(x,\overline{y}) + G(\overline{x},\overline{y})] \Big)(f_H(y) - f(y))\,dy \geq 0 \end{aligned} \tag{4.18}$$

again by (4.1) and Lemma 4. By (4.1) we also obtain

$$w(x) + w(\overline{x}) \geq u(x) + u(\overline{x}) = u_H(x) + u_H(\overline{x}) \qquad \text{for } x \in \mathbb{R}^n. \tag{4.19}$$

To conclude the proof, we note that $f \equiv f_H \equiv 0$ on $\mathbb{R}^n \setminus \mathbf{B}$. This follows since $H \in \mathcal{H}_0$, $f \geq 0$ in $\mathbf{B}$ and $f \equiv 0$ on $\mathbb{R}^n \setminus \mathbf{B}$. Moreover, if $f(y) \neq f_H(y)$ for some $y \in \mathrm{int}(H \cap \mathbf{B})$, then, for fixed $x \in \mathrm{int}(\mathbf{B} \cap H)$, the integrand in (4.18) is strictly positive in a neighborhood of y by the strict inequality in (4.4). Hence the inequality in (4.19) is strict if $f \not\equiv f_H$ and $x \in \mathrm{int}(\mathbf{B} \cap H)$. This completes the proof of the Lemma. □

Lemma 6. *Let $H \in \mathcal{H}_0$, let $f \in C_0(\mathbb{R}^n)$ be a nonnegative function with support contained in $\overline{\mathbf{B}}$, and let $u = \mathcal{G}f$, $w = \mathcal{G}f_H$. Then:*

$$\int_{\mathbf{B}} w(x) u_H^{p-1}(x)\, dx \leq \int_{\mathbf{B}} w^2(x) u_H^{p-2}(x)\, dx \qquad \textit{for all } p \geq 2. \tag{4.20}$$

Moreover, if equality holds in (4.20), then $f \equiv f_H$.

Proof. Without loss of generality, we assume that f is not identically zero. We use (4.13)-(4.15) to estimate

$$\begin{aligned}
\int_{\mathbf{B}} w^2(x) u_H^{p-2}(x)\, dx - \int_{\mathbf{B}} w(x) u_H^{p-1}(x)\, dx &= \int_{\mathbf{B}} w(x) u_H^{p-2}(x)[w(x) - u_H(x)]\, dx \\
&= \int_H \Big(w(x) u_H^{p-2}(x)[w(x) - u_H(x)] + w(\overline{x}) u_H^{p-2}(\overline{x})[w(\overline{x}) - u_H(\overline{x})] \Big)\, dx \\
&\geq \int_H \Big(w(x) u_H^{p-2}(x)[w(x) - u_H(x)] + w(\overline{x}) u_H^{p-2}(\overline{x})[u_H(x) - w(x)] \Big)\, dx \qquad (4.21) \\
&= \int_H \Big(w(x) u_H^{p-2}(x) - w(\overline{x}) u_H^{p-2}(\overline{x}) \Big)[w(x) - u_H(x)]\, dx \\
&\geq \int_H w(x) \Big(u_H^{p-2}(x) - u_H^{p-2}(\overline{x}) \Big)[w(x) - u_H(x)]\, dx \geq 0.
\end{aligned}$$

Hence (4.20) is true. Moreover, if equality holds in (4.20), then we also have equality in (4.21), which implies that either $w(\overline{x}) - u_H(\overline{x}) = u_H(x) - w(x)$ for some $x \in \mathrm{int}(H \cap \mathbf{B})$, or $w(\overline{x}) u_H^{p-1}(\overline{x}) = 0$ for all $x \in \mathrm{int}(H \cap \mathbf{B})$. In the first case, Lemma 5 yields $f \equiv f_H$. In the second case we conclude that $\mathbf{B} \subset H$, since w and u_H are both positive on $\mathbf{B}$. But then we also have $f \equiv f_H$, since $f \equiv 0$ on $\mathbb{R}^n \setminus H$. □

Proposition 3. *Let $u \in H_0^2(\mathbf{B})$ be a minimizer for (1.2), and let $H \in \mathcal{H}_0$. Then $u = u_H$.*

Proof. Without loss of generality, we may assume that u is a positive solution of (1.3). We set $f = u^{p-1}$. Then u coincides with the restriction of $\mathcal{G}f$ to $\mathbf{B}$. We also put $w = \mathcal{G}f_H$. Then, by Lemma 6 we have

$$\|\Delta w\|_2^2 = \int_{\mathbf{B}} w f_H\, dx = \int_{\mathbf{B}} w u_H^{p-1}\, dx \leq \int_{\mathbf{B}} w^2 u_H^{p-2}\, dx \leq \|w\|_p^2 \|u_H\|_p^{p-2} = \|w\|_p^2 \|u\|_p^{p-2} \tag{4.22}$$

so that

$$\frac{\|\Delta w\|_2^2}{\|w\|_p^2} \leq \|u\|_p^{p-2} = \frac{\|u\|_p^p}{\|u\|_p^2} = \frac{\|\Delta u\|_2^2}{\|u\|_p^2}. \tag{4.23}$$

Since u is a Sobolev minimizer, we conclude that equality holds in (4.23), so that by going back to (4.22) we find

$$\int_{\mathbf{B}} w u_H^{p-1}\, dx = \int_{\mathbf{B}} w^2 u_H^{p-2}\, dx.$$

Hence $u^{p-1} \equiv f \equiv f_H \equiv u_H^{p-1}$ by virtue of Lemma 6, which implies that $u = u_H$. $\square$

Now the proof of Theorem 3 is completed by combining Proposition 2 and Proposition 3.

5 The strict inequality between the embedding constants

In this section we prove the following strict inequality.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with $\partial\Omega \in C^{1,\alpha}$, let $2 < p < 2_*$ and let*

$$S_p^1(\Omega) = \min_{w \in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta w\|_2^2}{\|w\|_p^2}\ , \qquad S_p^2(\Omega) = \min_{w \in H^2 \cap H_0^1(\Omega)\setminus\{0\}} \frac{\|\Delta w\|_2^2}{\|w\|_p^2}\ . \tag{5.1}$$

Then, $S_p^1(\Omega) > S_p^2(\Omega)$.

The proof relies on the following lemma.

Lemma 7. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with $\partial\Omega \in C^{1,\alpha}$, let $2 < p < 2_*$ and let $u_p \in H^2 \cap H_0^1(\Omega)$ be a minimizer for $S_p^2(\Omega)$. Then, $u_p \in C^{4,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. Moreover, up to a change of sign, $u_p > 0$ in Ω and $\frac{\partial u_p}{\partial \nu} < 0$ on $\partial\Omega$.*

Proof. Up to a Lagrange multiplier, the minimizer u_p satisfies

$$\int_\Omega \Delta u_p \Delta \varphi = \int_\Omega |u_p|^{p-2} u_p \varphi \qquad \text{for all } \varphi \in H^2 \cap H_0^1(\Omega)\ .$$

Then, by elliptic regularity (see [18, Lemma B.3]) we infer that $u_p \in C^{4,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

Let u be the solution of the following problem

$$\begin{cases} -\Delta u = |\Delta u_p| & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ . \end{cases}$$

For contradiction, if u_p is not of one sign then the maximum principle implies $u > |u_p|$ in Ω. Hence, $\|u\|_p > \|u_p\|_p$ while $\|\Delta u\|_2 = \|\Delta u_p\|_2$. This contradicts the fact that u_p minimizes $S_p^2(\Omega)$. This shows that $u_p > 0$ and also that $-\Delta u_p \geq 0$ in Ω. By the boundary point lemma, we then conclude that $\frac{\partial u_p}{\partial \nu} < 0$ on $\partial\Omega$. $\square$

We can now complete the proof of Theorem 4. Since $H_0^2(\Omega) \subset H^2 \cap H_0^1(\Omega)$, we clearly have $S_p^1(\Omega) \geq S_p^2(\Omega)$. Assume for contradiction that equality holds and let $u_p \in H_0^2(\Omega)$ be a minimizer for $S_p^1(\Omega)$. Then, u_p is also a minimizer for $S_p^2(\Omega)$ which satisfies $\frac{\partial u_p}{\partial \nu} = 0$ on $\partial\Omega$. This contradicts Lemma 7 and proves Theorem 4.

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