OPTIMAL TRANSPORTATION NETWORKS AS FLAT CHAINS

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Abstract. We provide a model of optimization of transportation networks (e.g. urban traffic lines, subway or railway networks) in a geografical area (e.g. a city) with given density of population and that of services and/or workplaces, the latter being the destinations of everyday movements of the former. The model is formulated in terms of Federer-Fleming theory of currents, and allows to get both the position and the necessary capacity of the optimal network. Existence and some qualitative properties of solutions to the respective optimization problem are studied. Also, in an important particular case it is shown that the model proposed is equivalent to another known model of optimization of optimal transportation network, the latter not using the language of currents.

1. Introduction

Let φ^+, φ^- stand for finite Borel measures with compact support in $\mathbb{R}^n$ and of equal total mass $\varphi^+(\mathbb{R}^n) = \varphi^-(\mathbb{R}^n)$, the former representing the density of population, the latter the density of workplaces or services in some geographical area (e.g. a city). The scope of this paper is to provide a reasonable model of choosing a "transportation network" (e.g the set of subway, or, generally speaking, urban traffic lines) in a city characterised by the distributions $\varphi^\pm$. The network to be chosen has to facilitate the transportation of the population to the services. The model we consider is based primarily on the Monge-Kantorovich theory of optimal mass transport, but is expressed in terms of Federer-Fleming theory of currents. Apart from the fact that a language of currents, as we will show later, is extremely natural for such urban planning problems, it also allows to formulate the models which take into consideration the degree to which the pattern of behaviour of the population is "individualistic". Such models allow as well to find naturally not only the position of the network to be constructed, but also the network capacity which is intrinsic in the model. Below we discuss in a more detailed way the formulations of the models studied in this paper.

1.1. Transport problems.

The classical Monge-Kantorovich optimal transportation problems consists in finding the the "optimal" way of transporting φ^+ to φ^-. One of the many equivalent formulations of the latter reads as follows [8, 1]: *find a finite Borel measure μ_{opt} (called transport density) and a Borel measurable unit vectorfield ν_{opt} in $\mathbb{R}^n$ (called field of transportation directions) which minimizes the total mass $\mu(\mathbb{R}^n)$ among all the couples (μ, ν) as above satisfying the Monge-Kantorovich transport equation*

$$div\, \mu\nu = \varphi^+ - \varphi^-$$

in the sense of distributions.

One might easily reformulate this problem using the language of Federer-Fleming theory of currents. In this case we identify $\varphi^\pm$ with zero-dimensional flat chains.

The work of the second author was partially supported by the Italian government program "Incentivazione alla mobilità di studiosi stranieri e italiani residenti all'estero" .

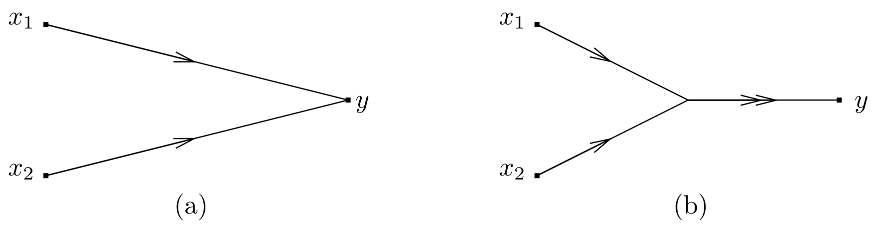

FIGURE 1. Solutions to Monge-Kantorovich transport problem with $\varphi^+ := \delta_{x_1} + \delta_{x_2}$ and $\varphi^- := 2\delta_y$ in the case *(a)* $\alpha = 1$ and *(b)* $\alpha < 1$.

The respective formulation would read: *find a flat chain T_{opt} minimizing the total mass $\mathbb{M}(T)$ among all the one-dimensional real flat chains T satisfying*

$$\partial T = \varphi^+ - \varphi^-. \tag{1}$$

Clearly, once one finds $T_{opt} = \tau_{T_{opt}} \wedge \mu_{T_{opt}}$, i.e. $\tau_{T_{opt}}$ is the orientation of T_{opt} while $\mu_{T_{opt}}$ is the underlying measure, one gets then $\mu_{opt} = \mu_{T_{opt}}$ and $\nu_{opt} = \tau_{T_{opt}}$, and, vice versa, if one knows (μ_{opt}, ν_{opt}), then one gets $T_{opt} = \nu_{opt} \wedge \mu_{opt}$.

The above Monge-Kantorvich problem formulated in terms of currents admits the further far reaching generalization which is obtained by minimizing a generic α-mass $\mathbb{M}^\alpha$ (for some given $\alpha \in [0, 1]$) of the current T instead of $\mathbb{M}$. Namely, such a general formulation reads as follows: *find a flat chain T_{opt} minimizing the α-mass $\mathbb{M}^\alpha(T)$ among all the one-dimensional real flat chains T of finite mass satisfying* (1). The background idea of this generalization is given by the following example. Let $\varphi^+ := \delta_{x_1} + \delta_{x_2}$ be the sum of two Dirac masses and $\varphi^- := 2\delta_y$ be just one Dirac mass where the points x_1, x_2 and y are positioned as in Figure 1. Then the solution of the classical problem (i.e. with $\alpha = 1$) is given by the one-dimensional real polyhedral chain provided in Figure 1 (a), i.e. the transportation occurs along the segments connecting the source points x_1 and x_2 with the destination y. However, the solution for the problem with $\alpha < 1$ looks like that given in Figure 1 (b). In other words, the role of parameter α is that of making it more convenient for the people living in the source points to make together part of their trip to the destination instead of moving "individually". If one interprets the solution T_{opt} as an optimal transportation network which provides the movement of φ^+ to φ^-, then clearly it contains the information both on the directions of the movement and on the capacity of the network in each point.

In discrete settings such models were introduced and studied for communication networks in [14], for pipelines [6] and drainage networks in [15]. They are quite natural in fluid mechanics (and therefore also in the traffic flow models based on the latter) when modeling the flow of liquids in tubes subject to Poiseuille's law which implies the increase of resistence as the tube becomes thinner [11, 16, 5, 4]. In continuous settings such models were introduced in [23] and in a different though equivalent formulation in [16].

1.2. **Optimal transportation networks.** We now propose a more general model for choosing the optimal transportation network. In fact, suppose that one has to provide a set of fast transportation routes (i.e. a subway and/or a set of urban transportation lines) in a given city. We further call such routes the transportation network to be constructed. The flow of people moving along these routes will be modelled by a one-dimensional real flat chain S, while the flow of people moving without the use of such routes will be modelled by a one-dimensional real flat chain T. It is reasonable to suppose that the cost of using transportation network (for example the time spent for the travel) be proportional to $\mathbb{M}^\beta(S)$ with coefficient

$B \geq 0$ and with some given $\beta \in [0,1]$, while the cost of movement of the population without using the network be proportional to $\mathbb{M}^\alpha(T)$ with coefficient $A \geq 0$ and with some given $\alpha \in [0,1]$. Then the number

$$W(T,S) := A\mathbb{M}^\alpha(T) + B\mathbb{M}^\beta(S)$$

represents the overall cost of everyday movement of the population. Clearly, the parameters α and β model the degree to which the behaviour of the population is "individualistic" (i.e. when both are equal to one one may assume that such a behaviour is completely individualistic, while the less they are, the more it is convenient for the people to make part of their itinerary together). It is further reasonable to assume that the cost of construction of the transportation network depend only on $\mathbb{M}^\delta(S)$ (in simplest applications one would have even $\delta = 0$, i.e. the cost of construction depends only on the length of the network) according to some given monotone nondecreasing function $H\colon \mathbb{R}^+ \to \mathbb{R}^+$. Therefore, the number

$$\mathfrak{F}(T,S) := W(T,S) + H(\mathbb{M}^\delta(S)) \tag{2}$$

represents the total expenses of everyday movement of the population together with the cost of building the transportation network. One assumes also that the total flow of the population $T + S$ transports φ^+ to φ^-, that is, the relationship

$$\partial(T+S) = \varphi^+ - \varphi^- \tag{3}$$

holds. The following quite natural minimization problem describes the optimal choice of the transportation network.

Problem 1. *Find a couple of one-dimensional real flat chains (T_{opt}, S_{opt}) minimizing $\mathfrak{F}$ among all the couples of real one-dimensional flat chains (T,S) of finite mass, satisfying* (3).

Note that in the particular case $B \geq A$ Problem 1 reduces to the version of the Monge-Kantorovich problem studied in [23] and mentioned in paragraph 1.1. Some of the qualitative properties of some particular solutions to such a transportation problem (namely those which can be obtained as limits of solutions to appropriate approximating discrete problems) have been studied in [25, 24].

In this paper we study the existence (Theorem 8.1) as well as qualitative properties of solutions of Problem 1 like acyclicity (Theorem 10.1), rectifiability (Theorem 10.2) and properties of the support (Theorem 10.4). In particular, we provide the conditions under which the respective solutions are rectifiable and have some subtler regularity properties, namely, when the current S_{opt} representing the transportation network to be constructed may be represented as a rectifiable current concentrated on a closed set (which gives the position of the optimal network) and with an u.s.c. density (representing network capacity) strictly greater than some nonnegative threshold (Theorem 10.7).

As an illustration of the results obtained, we summarize in the theorem below the assertions regarding Problem 1 under particular, though rather general conditions on problem data, which we consider to be most interesting for applications.

Theorem 1.1. *Let $A > 0$, $\alpha \in [0,1)$, $B \geq 0$, $H\colon [0,+\infty) \to [0,+\infty)$ strictly increasing, strictly concave and unbounded (i.e. $H(l) \to +\infty$ when $l \to +\infty$), $\delta \in [0,\alpha)$. Let also $\beta \in [0,\alpha]$ with $A > B$ if $\beta = \alpha$. Finally, suppose $\varphi^\pm$ be finite positive Borel measures with compact support in $\mathbb{R}^n$ and such that $\varphi^+(\mathbb{R}^n) = \varphi^-(\mathbb{R}^n)$.*

Then Problem 1 admits a solution, i.e. there exists a pair of real one-dimensional flat chains (T_{opt}, S_{opt}) which minimizes the functional

$$\mathfrak{F}(T,S) = A\mathbb{M}^\alpha(T) + B\mathbb{M}^\beta(S) + H(\mathbb{M}^\delta(S))$$

among all pairs (T,S) of flat chains with finite mass and such that $\partial(T+S) = \varphi^+ - \varphi^-$. Moreover every such optimal pair (T_{opt}, S_{opt}) enjoys the following properties.

(i) *S_{opt} is a rectifiable current representable as $S_{opt} = \theta_{S_{opt}}[\![\Sigma_{opt}]\!]$, where Σ_{opt} is a compact and countably $(\mathcal{H}^1, 1)$-rectifiable set, and the density $\theta_{S_{opt}}(x)$ is u.s.c. and satisfies $\inf_{x\in\Sigma_{opt}} \theta_{S_{opt}}(x) = \theta_0 > 0$.*
(ii) *T_{opt} is a rectifiable current disjoint from S_{opt} in the sense that the measures $\mu_{T_{opt}}$ and $\mu_{S_{opt}}$ are mutually singular. One has also that $\sup_{x\in\mathbb{R}^n} \theta_{T_{opt}}(x) \leq \theta_0$, where $\theta_{T_{opt}}$ is the density of T_{opt}.*
(iii) *$T_{opt} + S_{opt}$ is acyclic.*

One should note that some of the above results are quite natural. In fact, consider for simplicity the case when $H(l) = Cl$ for some constant $C > 0$. Then it is easy to note that the Problem 1 reduces then to the version of the Monge-Kantorovich problem from paragraph 1.1, but with the mass $\mathbb{M}^g$ instead of the mass $\mathbb{M}^\alpha$, where

$$g(t) := At^\alpha \wedge (Bt^\beta + Ct^\delta).$$

In fact, if R_{opt} is a flat chain solving this problem, then it is rectifiable due to the general rectifiability theorem from [22], hence $R_{opt} = \theta[\![\Sigma]\!]$ for some countably $(\mathcal{H}^1, 1)$-rectifiable set Σ, and therefore

$$\begin{aligned}\mathbb{M}^g(R_{opt}) = A\int_{\{x\in\Sigma\,:\,\theta(x)<d\}} \theta^\alpha\, d\mathcal{H}^1 + B\int_{\{x\in\Sigma\,:\,\theta(x)\geq d\}} \theta^\beta\, d\mathcal{H}^1 + \\ C\int_{\{x\in\Sigma\,:\,\theta(x)\geq d\}} \theta^\delta\, d\mathcal{H}^1 \\ = A\mathbb{M}^\alpha(T) + B\mathbb{M}^\beta(S) + H(\mathbb{M}^\delta(S)) = \mathfrak{F}(T,S),\end{aligned} \tag{4}$$

where $d > 0$ is the unique number such that $Ad^\alpha = Bd^\beta + Cd^\delta$ and

$$T := R_{opt}\llcorner\{x \in \Sigma \,:\, \theta(x) < d\}, \qquad S := R_{opt}\llcorner\{x \in \Sigma \,:\, \theta(x) \geq d\}.$$

Vice versa, if a pair (T, S) "almost" solves Problem 1 (in the sense that $\mathfrak{F}(T, S)$ is close to the infimum of $\mathfrak{F}$ on the class of admissible pairs of flat chains), then one can show that up to decreasing even more the functional $\mathfrak{F}$ one may assume both T and S rectifiable, while denoting $R := T + S$, one has $R = \theta[\![\Sigma]\!]$ for some countably $(\mathcal{H}^1, 1)$-rectifiable set Σ, and

$$T = R\llcorner\{x \in \Sigma \,:\, \theta(x) < d\}, \qquad S = R\llcorner\{x \in \Sigma \,:\, \theta(x) \geq d\}$$

(of course, technical details omitted for the moment), and hence

$$\mathbb{M}^g(R) = \mathfrak{F}(T, S).$$

Thus once the existence of a minimizer R_{opt} to $\mathbb{M}^g$ on the class of real one-dimensional flat chains R of finite mass satisfying $\partial R = \varphi^+ - \varphi^-$ is established (which can be done, for instance, using the machinery developed in [23] for $\mathbb{M}^\alpha$ instead of $\mathbb{M}^g$), we get the existence of a solution (T, S) to Problem 1 given by (4). The threshold θ_0 from Theorem 1.1 is then given by $\theta_0 := d$. Of course, the qualitative properties of the solution (e.g. that S may be assumed to be concentrated on a closed set and have an u.s.c. density) are slightly more delicate even in this simple case.

We also show that in a particular case when $\alpha = \beta = 1$ and $\delta = 0$ the Problem 1 is naturally equivalent to a particular case of problem studied in [9] of optimizing a transportation network under condition that the prices per unit distance of travelling with and without the help of the transportation network are constant.

The background idea we use in most of the results is the representation of normal one-dimensional currents through measures over the appropriately metrized set of Lipschitz-continuous paths in $\mathbb{R}^n$ (called further *transports*). The idea of using such a representation when dealing with one-dimensional currents goes back to [19], although in a different context such measures were already used in [7]. In context of transportation and urban problems such measures were employed in [10, 9] and, implicitly, also in [16, 5]. The description of mass transportation through transports happens to be in fact more precise with respect to that using currents.

2. Notation and preliminaries

2.1. **Measures.** Unless otherwise explicitly stated, all the measures we will be dealing with in the sequel are nonnegative Borel measures over $\mathbb{R}^n$. We denote by $\phi \wedge \psi$ the maximum nonnegative Borel measure μ satisfying $\mu(e) \leq \psi(e) \wedge \phi(e)$ for all Borel $e \subset \mathbb{R}^n$. If φ is a signed measure, we denote by $\varphi^{\pm}$ its positive and negative parts respectively.

We will say that a sequence of signed Radon measures ϕ_ν converges in *narrow* sense to a signed Radon measure ϕ, if $\int_{\mathbb{R}^n} f\, d\phi_\nu \to \int_{\mathbb{R}^n} f\, d\phi$ as $\nu \to \infty$ for every bounded and continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$.

Let $\Theta^*_k(\mu, x)$ and $\Theta_{k*}(\mu, x)$ stand for upper and lower k-dimensional density of the measure μ in $x \in \mathbb{R}^n$. Namely,

$$\Theta^*_k(\mu, x) := \limsup_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\omega_k \rho^k}, \qquad \Theta_{k*}(\mu, x) := \liminf_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\omega_k \rho^k}.$$

2.2. **Currents.** For basic notation on currents we refer to [17, 18]. Here we recall rather briefly some principal facts we will use in the sequel. We will always deal with real currents, i.e. currents with real coefficients. If T is a current, for every open $U \subset \mathbb{R}^n$ we set

$$\mu_T(U) := \sup\{T(\omega) \,:\, \operatorname{supp} \omega \subset U, \|\omega\|_{L^\infty} \leq 1\}.$$

We set also $\mathbb{M}(T) := \mu_T(\mathbb{R}^n)$ and call this quantity the *mass* of T. It is well known that when $\mathbb{M}(T) < +\infty$, then μ_T defines a finite Radon measure and T is representable as $T = \tau_T \wedge \mu_T$ for some unit simple k-vector field τ_T, in the sense that

$$T(\omega) = \int_{\mathbb{R}^n} \langle \tau_T(x), \omega(x) \rangle \, d\mu_T(x)$$

for every regular differential k-form ω. In this case we set for every $\theta \in L^1(\mathbb{R}^n; \mu_T)$

$$T \wedge \theta(\omega) := \int_{\mathbb{R}^n} \theta(x) \langle \tau_T(x), \omega(x) \rangle \, d\mu_T(x),$$

and $T \llcorner A := T \wedge 1_A$, where 1_A stands for the characteristic function of a Borel set $A \subset \mathbb{R}^n$. We say that a sequence of currents T_ν converges to a current T in the flat norm (written $T_\nu \overset{\mathcal{F}}{\rightharpoonup} T$) whenever $\mathbb{F}(T_\nu - T) \to 0$, where

$$\mathbb{F}(T) := \inf\{\mathbb{M}(A) + \mathbb{M}(B) \,:\, T = A + \partial B\}.$$

Clearly, the topology induced by the flat norm is stronger than the weak topology of currents. We say that T is a *normal current*, if $\mathbb{M}(T) < +\infty$ and $\mathbb{M}(\partial T) < +\infty$, and, finally, that T is a *flat chain*, if there is a sequence of normal currents $\{T_\nu\}$ such that $T_\nu \overset{\mathcal{F}}{\rightharpoonup} T$.

We call T a *rectifiable current*, if there exists a countably $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset \mathbb{R}^n$, a function $\theta \in L^1(\mathcal{H}^k \llcorner \Sigma)$ (called *multiplicity* or *density* of T) such that $T = \tau_T \wedge \theta \mathcal{H}^k \llcorner \Sigma$, while the unit simple k-vector $\tau_T\colon \Sigma \to \mathbb{R}^n$ is an *orientation* of Σ, in the sense that for $\mathcal{H}^k$-a.e. $x \in \Sigma$ the vector $\tau_T(x)$ defines the approximate tangent plane to Σ at x. In this case we write also $T = \theta[\![\Sigma]\!]$ when an orientation on Σ is prescribed. One can show that if T is a flat chain with finite mass $\mathbb{M}(T) < +\infty$, then T is a rectifiable current, if and only if for some countably $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset \mathbb{R}^n$ one has $T = T \llcorner \Sigma$, or, in other words, $\mu_T = \mu_T \llcorner \Sigma$ (see [3, Theorem 4.5]).

A k-dimensional *simplicial current* is a rectifiable current $[\![\Sigma]\!]$, where $\Sigma \subset \mathbb{R}^n$ is a k-dimensional simplex (i.e. a convex envelope of $k+1$ points, in particular, a segment, if $k = 1$). Finally, we say that a current T is a *polyhedral chain*, if it can be written as a finite linear combination of simplicial currents supported on simplices with mutually disjoint interiors. Polyhedral chains (and hence rectifiable currents) are a dense subset of flat chains with respect to the flat norm.

Given a rectifiable current $T = \theta[\![\Sigma]\!]$ and a concave monotone nondecreasing function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $g(0) = 0$, we define the g-mass of T by the formula

$$\mathbb{M}^g(T) := \int_\Sigma g(\theta(x))\, d\mathcal{H}^k(x).$$

In particular, if $g(t) := t^\alpha$ for the given $\alpha \in [0,1]$, then the above expression defines the α-mass of T, namely,

$$\mathbb{M}^\alpha(T) := \int_\Sigma \theta^\alpha(x)\, d\mathcal{H}^k(x).$$

The functional $\mathbb{M}^g$ (in particular, $\mathbb{M}^\alpha$) is lower semicontinuous on rectifiable currents with respect to the flat norm convergence (this fact can be proven by a technique used in the proof of lemma 3.2.14 from [13]). Hence it can be extended to a lower semicontinuous functional defined on all flat chains. In the sequel we will write $\mathbb{M}$ instead of $\mathbb{M}^1$ (and call it simply *mass*) as one is accustomed to.

The following easy consequence of the rectifiability theorem due to White [22, theorem 8.1] used in the sequel is due to the fact that there exist no nonconstant continuous curve $\theta\colon [0,1] \to \mathbb{R}$ having finite α-length defined by the formula

$$\theta \mapsto \int_0^1 \left(|\theta'(t)| + |\theta'(t)|^\alpha\right)\, dt$$

when $\alpha \in [0,1)$.

Theorem 2.1 (White)**.** *Let T be a current such that $\mathbb{M}(T) < +\infty$ and $\mathbb{M}^\alpha(T) < +\infty$ for some $\alpha \in [0,1)$. Then T is rectifiable.*

3. Subcurrents of flat chains

In the sequel we will be frequently using the notion of a subcurrent of a given current as introduced in the definition below.

Definition 3.1. *We say that S is a* subcurrent *of T, and write $S \leq T$, where T and S are k-dimensional currents, whenever*

$$\mathbb{M}(T - S) + \mathbb{M}(S) \leq \mathbb{M}(T).$$

We now provide a series of remarks concerning the above definition.

Remark 3.2. Since the inequality

$$\mathbb{M}(T - S) + \mathbb{M}(S) \geq \mathbb{M}(T)$$

always holds true, then S is a subcurrent of T, if and only if the equality actually holds.

Remark 3.3. If $S \leq T$ and $R \leq S$, then $R \leq T$. In fact,

$$\begin{aligned}\mathbb{M}(T) &\geq \mathbb{M}(S) + \mathbb{M}(T-S) \geq \mathbb{M}(R) + \mathbb{M}(S-R) + \mathbb{M}(T-S)\\ &\geq \mathbb{M}(R) + \mathbb{M}(T-R),\end{aligned}$$

because of the triangle inequality $\mathbb{M}(T - R) \leq \mathbb{M}(T - S) + \mathbb{M}(S - R)$.

Remark 3.4. Let T be a current with finite mass $\mathbb{M}(T) < +\infty$ and let $e \subset \mathbb{R}^n$ be a Borel set. Then $T \llcorner e \leq T$. In fact,

$$\mathbb{M}(T) = \mu_T(\mathbb{R}^n) = \mu_T(e) + \mu_T(\mathbb{R}^n \setminus e) = \mathbb{M}(T \llcorner e) + \mathbb{M}(T - T \llcorner e).$$

Remark 3.5. Notice that $S \le T$ in general *does not* imply

$$\mathbb{M}^\alpha(T) = \mathbb{M}^\alpha(T - S) + \mathbb{M}^\alpha(S)$$

when $\alpha \in [0,1)$ (take for example $T \neq 0$, $S = T/2$). However, if $S = T\llcorner e$ for some Borel set $e \subset \mathbb{R}^n$, then the above relationship holds whenever $\mathbb{M}^\alpha(T) < +\infty$ and $\mathbb{M}(T) < +\infty$. In fact, in this case, T turns out, in view of Theorem 2.1, to be a rectifiable current $T = \theta[\![\Sigma]\!]$ for some $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset \mathbb{R}^n$ and some $\theta \in L^1(\mathcal{H}^1\llcorner\Sigma)$. Then

$$\mathbb{M}^\alpha(T) = \int_\Sigma |\theta|^\alpha \, d\mathcal{H}^k = \int_{\Sigma\setminus e} |\theta|^\alpha \, d\mathcal{H}^k + \int_{\Sigma\cap e} |\theta|^\alpha \, d\mathcal{H}^k = \mathbb{M}^\alpha(T - S) + \mathbb{M}^\alpha(S).$$

Remark 3.6. If T is a current with finite mass $\mathbb{M}(T) < +\infty$, $S \le T$, then for every Borel set $e \subset \mathbb{R}^n$ one has $S\llcorner e \le T\llcorner e$. In fact, by the triangle inequality

$$\mathbb{M}(T\llcorner e) \le \mathbb{M}((T - S)\llcorner e) + \mathbb{M}(S\llcorner e)$$
$$\mathbb{M}(T\llcorner \mathbb{R}^n \setminus e) \le \mathbb{M}((T - S)\llcorner \mathbb{R}^n \setminus e) + \mathbb{M}(S\llcorner \mathbb{R}^n \setminus e)$$

for every Borel $e \subset \mathbb{R}^n$, while if we sum the above inequalities, then as a result we get an equality since $S \le T$. Hence the above inequalities are in fact equalities for all Borel $e \subset \mathbb{R}^n$. In particular, this also implies

$$\mu_T = \mu_{T-S} + \mu_S, \tag{5}$$

and hence $\mu_S \le \mu_T$. On the other hand, if (5) holds, then $S \le T$ since

$$\mathbb{M}(S) + \mathbb{M}(T - S) = \mu_S(\mathbb{R}^n) + \mu_{T-S}(\mathbb{R}^n) = \mu_T(\mathbb{R}^n) = \mathbb{M}(T).$$

The following lemma gives an easy characterization of subcurrents of flat chains of finite mass.

Lemma 3.7. *Let a current T have finite mass $\mathbb{M}(T) < +\infty$, and assume $S \le T$. Then, the representation $T = \tau_T \wedge \mu_T$ implies $S = \tau_T \wedge \sigma\mu_T$, $\mu_S = \sigma\mu_T$ for some Borel function $\sigma\colon \mathbb{R}^n \to \mathbb{R}$ satisfying $0 \le \sigma \le 1$ (in other words, $S = T \wedge \sigma$). In particular, if T is a rectifiable flat chain, then so is S.*

Further, if T is a flat chain, then $\mathbb{M}^\alpha(S) \le \mathbb{M}^\alpha(T)$ for all $\alpha \in [0,1]$. If, moreover, $\mathbb{M}(T - S) \neq 0$ and $\mathbb{M}^\alpha(T) < +\infty$, then one has also $\mathbb{M}^\alpha(S) < \mathbb{M}^\alpha(T)$ for all $\alpha \in (0,1]$.

Proof. By Remark 3.6 one has $\mu_S \le \mu_T$ and hence $\mu_S = \sigma\mu_T$ for some Borel function σ satisfying $0 \le \sigma \le 1$. Since according to the same remark, $\mu_{T-S} = \mu_T - \mu_S$, we get also $\mu_{T-S} = (1-\sigma)\mu_T$. Representing then $S = \tau_S \wedge \mu_S$ and $T - S = \tau_{T-S} \wedge \mu_{T-S}$, we get

$$\tau_T \wedge \mu_T = T = \tau_S \wedge \mu_S + \tau_{T-S} \wedge \mu_{T-S} = (\sigma\tau_S + (1-\sigma)\tau_{T-S}) \wedge \mu_T.$$

Hence, $\tau_T = \sigma\tau_S + (1-\sigma)\tau_{T-S}$, and, minding that both τ_T, τ_S and τ_{T-S} are unit vectors, we observe that whenever $\sigma(x) > 0$ one has $\tau_S(x) = \tau_T(x)$. In particular $\sigma\tau_S = \sigma\tau_T$ and hence $\tau_S \wedge \mu_S = \tau_S \wedge \sigma\mu_T = \tau_T \wedge \sigma\mu_T$. This concludes the proof of the first claim.

If T is a rectifiable flat chain, then $\mu_T = \sigma\mathcal{H}^k\llcorner\Sigma$ for some countably $(\mathcal{H}^k, k)$-rectifiable $\Sigma \subset \mathbb{R}^n$ and some $\theta \in L^1(\Sigma; \mathcal{H}^k)$, while $\tau_T(x)$ orients the approximate tangent plane to Σ at x for $\mathcal{H}^k$-a.e. $x \in \Sigma$. One has then $S = \tau_T \wedge \sigma\theta\mathcal{H}^k\llcorner\Sigma$ which means that S is still rectifiable.

Finally, let T be a flat chain of finite mass $\mathbb{M}(T) < +\infty$. Suppose $\alpha < 1$ (otherwise the conclusion follows trivially from the definition of a subcurrent) and $\mathbb{M}^\alpha(T) < +\infty$ (otherwise there is nothing to prove). Then by Theorem 2.1, we know that T is rectifiable, i.e. $T = \tau_T \wedge \theta\mathcal{H}^k\llcorner\Sigma$ with Σ, θ and τ_T as above. Then

$$\mathbb{M}^\alpha(T) = \int_\Sigma |\theta|^\alpha \, d\mathcal{H}^k.$$

But then $S = \tau_T \wedge \sigma\theta \mathcal{H}^k \llcorner \Sigma$ and hence

$$\mathbb{M}^\alpha(S) = \int_\Sigma |\sigma\theta|^\alpha \, d\mathcal{H}^k \leq \mathbb{M}^\alpha(T)$$

since $|\sigma| \leq 1$. Moreover, if $\mathbb{M}^\alpha(T) < +\infty$, then $\mathbb{M}^\alpha(S) = \mathbb{M}^\alpha(T)$ only when $\sigma = 1$ $\mathcal{H}^k$-a.e. over Σ, which means $T = S$ and hence $\mathbb{M}(T - S) = 0$. □

Remark 3.8. If $S \leq T$ then $S \leq T+S$. In fact, by Lemma 3.7, one has $S = \tau_T \wedge \sigma\mu_T$ and hence $T + S = \tau_T \wedge (1 + \sigma)\mu_T$ so that $\mu_{T+S} = (1 + \sigma)\mu_T$ which means that $T \leq T + S$ and hence $S \leq T + S$.

Lemma 3.9. *Let A and B be subcurrents of T. Then $A \leq A + B$ and consequently $B \leq A + B$. If, moreover, $\mu_A \wedge \mu_B = 0$, then $A + B \leq T$.*

Proof. By Lemma 3.7 we have $A = \tau_T \wedge \sigma_A \mu_T$ and $B = \tau_T \wedge \sigma_B \mu_T$ with $0 \leq \sigma_A \leq 1$, $0 \leq \sigma_B \leq 1$. Then $A + B = \tau_T \wedge (\sigma_A + \sigma_B)\mu_T$ and hence $\mu_{A+B} = \mu_A + \mu_B$, which means $A \leq A + B$.

If we also suppose $\mu_A \wedge \mu_B = 0$, we will have then that $\sigma_A + \sigma_B \leq 1$. Hence $\mu_{T-A-B} = (1 - \sigma_A - \sigma_B)\mu_T$. Therefore, $\mu_{A+B} + \mu_{T-A-B} = \mu_T$, which means $A + B \leq T$. □

Lemma 3.10. *Let T_ν be a sequence of currents, $S_\nu \leq T_\nu$, and suppose that both $S_\nu \rightharpoonup S$ and $T_\nu \rightharpoonup T$ weakly as currents as $\nu \to \infty$, while $\mathbb{M}(T_\nu) \to \mathbb{M}(T)$. Then, $\mathbb{M}(T) < +\infty$ implies that $S \leq T$ and $\mathbb{M}(S_\nu) \to \mathbb{M}(S)$.*

Proof. Consider the sequence $\{T_\nu - S_\nu\}$ which converges to $T - S$ in the weak sense of currents. By the lower semicontinuity of $\mathbb{M}$ we know that

$$\begin{aligned} \mathbb{M}(S) + \mathbb{M}(T - S) &\leq \liminf_{k\to\infty} \mathbb{M}(S_\nu) + \liminf_{k\to\infty} \mathbb{M}(T_\nu - S_\nu) \\ &\leq \liminf_{k\to\infty} [\mathbb{M}(S_\nu) + \mathbb{M}(T_\nu - S_\nu)] \\ &\leq \liminf_{k\to\infty} \mathbb{M}(T_\nu) = \mathbb{M}(T), \end{aligned} \tag{6}$$

i.e. $S \leq T$. Since we also have $\mathbb{M}(T) \leq \mathbb{M}(S) + \mathbb{M}(T - S)$, the inequalities in (6) actually are equalities. Also, since $\mathbb{M}(T - S) \leq \liminf_\nu \mathbb{M}(T_\nu - S_\nu)$ we obtain $\mathbb{M}(S) = \liminf_\nu \mathbb{M}(S_\nu)$. This is also true for every subsequence of S_ν, hence we have full convergence of the sequence $\mathbb{M}(S_\nu)$ to $\mathbb{M}(S)$ as $\nu \to \infty$. □

We give now the following definition which will be crucial in the sequel.

Definition 3.11. *Let T be a current with $\mathbb{M}(T) < +\infty$. We say that C is a* cycle *of a current T, if $C \leq T$ and $\partial C = 0$. We say that T is* acyclic*, if $C = 0$ is the only cycle of T.*

We are able to prove now the existence of a "maximum cycle" of every current T with finite mass, i.e. such a cycle that $T - C$ is acyclic.

Proposition 3.12. *Every current T with finite mass $\mathbb{M}(T) < +\infty$ contains such a cycle C that $T - C$ is acyclic.*

Proof. Let

$$\xi = \sup\{\mathbb{M}(C)\colon C \text{ is a cycle of } T\}.$$

Clearly $\xi < +\infty$ since $\mathbb{M}(C) \leq \mathbb{M}(T)$ for every cycle C of T. Also $\xi \geq 0$ since $C = 0$ is always a cycle of T.

Step 1. We claim that there exists a cycle C of T such that $\mathbb{M}(C) = \xi$. In fact, by definition of ξ, there exists a sequence $\{C_\nu\}$ of cycles of T such that

$$\mathbb{M}(C_\nu) \geq \xi - \frac{1}{\nu}.$$

Clearly $\mathbb{M}(C_\nu) \le \mathbb{M}(T)$ and $\mathbb{M}(\partial C_\nu) = 0$ so, up to a subsequence (not relabeled), the currents C_ν converge to a limit C with $\partial C = 0$. By Lemma 3.10 (applied with $T_\nu := T$) the current C itself is a cycle of T and $\mathbb{M}(C) = \lim_\nu \mathbb{M}(C_\nu) = \xi$.

Step 2. We only have to prove that $T - C$ is acyclic. Let D be any cycle of $T - C$. Since $D \le T - C$ and $C \le T$ we also have $T - C - D \le T - C$ and $T - C \le T$ so we get (Remark 3.3) $T - C - D \le T$ and $C + D \le T$. Hence we actually have

$$\mathbb{M}(T) - \mathbb{M}(C) = \mathbb{M}(T - C) = \mathbb{M}(D) + \mathbb{M}(T - C - D) = \mathbb{M}(D) + \mathbb{M}(T) - \mathbb{M}(C + D)$$

i.e. $\mathbb{M}(C) + \mathbb{M}(D) = \mathbb{M}(C + D)$ which reads $C \le C + D$. Since $C + D$ is a cycle of T we have $\mathbb{M}(C + D) \le \xi$ and minding that $\mathbb{M}(C) = \xi$ we have $\mathbb{M}(D) = 0$ i.e. $D = 0$. Since this is true for every cycle D of $T - C$, we conclude that $T - C$ is acyclic. □

Finally, the following easy assertion will be used in the sequel.

Lemma 3.13. *Let T be a polyhedral k-dimensional chain and let $S \le T$ be its subcurrent such that $\partial S \le \partial T$. Then S is itself polyhedral.*

Proof. One has $T = \sum_{i=1}^k \theta_i [\![\Sigma_i]\!]$, where $\Sigma_i \in \mathbb{R}^n$ are pairwise disjoint k-simplices, while $\theta_i \in \mathbb{R}$. Since $S \le T$, then by Lemma 3.7 one has $S = T \wedge \sigma$ for some Borel function σ satisfying $0 \le \sigma \le 1$. For each $i = 1, \dots, k$ one has that σ is constant over the each simplex Σ_i, since otherwise one would not have $\partial S \le \partial T$. Hence, $S = \sum_{i=1}^k \theta_i \sigma_i [\![\Sigma_i]\!]$, where σ_i is the value of σ over Σ_i, or, in other words, S is still polyhedral. □

4. Concave functionals on flat chains

We give the following definition which will be crucial in the sequel.

Definition 4.1. *We say that the functional $T \mapsto F(T) \in [0, +\infty]$ defined on k-dimensional real flat chains with finite mass is*

(i) concave *(resp.* strictly concave*), if the function $f\colon [-1, +\infty) \to \mathbb{R}$ defined by*

$$f(t) := F(T + tS)$$

is concave (resp. strictly concave) whenever $F(T) < +\infty$, $S \le T$, $S \ne 0$.

(ii) nondecreasing*, if*

$$F(S) \le F(T)$$

whenever $S \le T$, $S \ne T$. We say that F is strictly increasing*, if under the same hypotheses we get a strict inequality.*

Notice that the above definition of concavity of the functional F can be viewed as the usual concavity of F restricted in the directions given by subcurrents. As an example notice that $\mathbb{M}^\alpha$ will be proven to be concave in this sense, but not in the usual sense, in fact one clearly has

$$\mathbb{M}^\alpha\left(\frac{1}{2}T + \frac{1}{2}(-T)\right) = \mathbb{M}^\alpha(0) = 0 < \frac{1}{2}\mathbb{M}^\alpha(T) + \frac{1}{2}\mathbb{M}^\alpha(-T)$$

for every $T \ne 0$.

Remark 4.2. Suppose F is a concave (resp. strictly concave) functional defined on real flat chains of finite mass. If $H\colon [0, +\infty] \to [0, +\infty]$ is a concave (resp. strictly concave) function, then the functional $T \mapsto H(F(T))$ is concave (resp. strictly concave). In fact, assume $F(T) < +\infty$, $S \le T$. If the function $f\colon [-1, +\infty) \to \mathbb{R}$ defined by the formula $f(t) := F(T + tS)$ is concave (resp. strictly concave) and H is itself concave (resp. strictly concave) then so is $H \circ f$.

Remark 4.3. Clearly, a sum of concave functionals is still concave, and is strictly concave once at least one of the summands is concave.

The following result shows that the functional $\mathbb{M}^\alpha$ is concave for $\alpha \in [0,1]$ and strictly concave for $\alpha \in (0,1)$.

Lemma 4.4. *Let $\alpha \in [0,1]$, let T be a k-dimensional real flat chain satisfying $\mathbb{M}(T) < +\infty$, $\mathbb{M}^\alpha(T) < +\infty$ and assume $S \le T$. Consider the function $f\colon [-1,+\infty) \to \mathbb{R}$ defined by*

$$f(t) := \mathbb{M}^\alpha(T + tS).$$

The following properties hold:

(i) *f is concave;*
(ii) *if $S \neq 0$ and $\alpha \in (0,1)$, then f is strictly concave;*
(iii) *if $\alpha = 0$, then f is constant on $(-1,+\infty)$;*
(iv) *if $\alpha = 1$, then f is affine;*
(v) *if $S = 0$, then f is constant.*

Proof. By Lemma 3.7 one has $S = \sigma T$ for some Borel function σ satisfying $0 \le \sigma \le 1$. For $t \ge -1$ one has $1 + t\sigma \ge 0$ and $\mu_{T+tS} = (1+t\sigma)\mu_T$, so that

$$\mathbb{M}(T+tS) = \mathbb{M}((1+t\sigma)T) = \int_{\mathbb{R}^n} |1+t\sigma|\, d\mu_T = \int_{\mathbb{R}^n} (1+t\sigma)\, d\mu_T = \mathbb{M}(T) + t\mathbb{M}(S),$$

which proves the claim for $\alpha = 1$. If further, $\alpha \in [0,1)$ and $\mathbb{M}^\alpha(T) < +\infty$, then by Theorem 2.1, $T = \tau_T \wedge \theta \mathcal{H}^k \llcorner \Sigma$ for some countably $(\mathcal{H}^k, k)$-rectifiable $\Sigma \subset \mathbb{R}^n$ and some $\theta \in L^1(\mathcal{H}^k \llcorner \Sigma)$, while $\tau_T(x)$ orients the approximate tangent plane to Σ at x for $\mathcal{H}^k$-a.e. $x \in \Sigma$. Then, for $t \ge -1$, we get

$$\mathbb{M}^\alpha(T+tS) = \int_\Sigma |(1+t\sigma)\theta|^\alpha\, d\mathcal{H}^k = \int_\Sigma (1+t\sigma)^\alpha |\theta|^\alpha\, d\mathcal{H}^k,$$

which is concave in t for all $\alpha \in (0,1)$, and is strictly concave in t, if $S \neq 0$. Finally, for the case $\alpha = 0$ we have

$$f(t) = \mathbb{M}^0(T+tS) = \int_\Sigma \phi((1+t\sigma)\theta)\, d\mathcal{H}^k = \int_\Sigma \phi(1+t\sigma)\phi(\theta)\, d\mathcal{H}^k,$$

where

$$\phi(s) := \begin{cases} 0, & s = 0,\\ 1, & s > 0,\end{cases}$$

and hence

$$f(t) = \begin{cases} \mathbb{M}^0(T-S), & t = -1,\\ \mathbb{M}^0(T), & t > -1,\end{cases}$$

which is constant for $t > -1$ and concave for $t \ge -1$. □

In the following lemmata, it is convenient to represent the functional $\mathfrak{F}$ in Problem 1 as $\mathfrak{F}(T,S) := F(T) + G(S)$, where

$$F(T) := A\mathbb{M}^\alpha(T), \qquad G(S) := B\mathbb{M}^\beta(S) + H(\mathbb{M}^\delta(S)).$$

We formulate first a very easy auxiliary result regarding concavity of functionals F and G defined above.

Lemma 4.5. *The functional F is concave and nondecreasing. It is strictly concave when $A \neq 0$ and $\alpha \in (0,1)$. The functional G is concave and nondecreasing (resp. strictly concave) whenever either H is concave or $\delta = 0$ (resp. either $B \neq 0$ and $\beta \in (0,1)$, or $\delta \in (0,1)$ and H is strictly increasing and concave).*

Proof. The assertion regarding F follows from Lemma 4.4 (i),(ii). By the same Lemma the functional $S \mapsto B\mathbb{M}^\beta(S)$ is concave (resp. strictly concave when $B \neq 0$ and $\beta \in (0,1)$). Further, if H is concave (resp. strictly concave and strictly increasing, while $\delta \in (0,1)$) we get from Lemma 4.4 (i),(ii) and Remark 4.2 (minding that H is assumed monotone nondecreasing) that the functional $S \mapsto H(\mathbb{M}^\delta(S))$ is concave (resp. strictly concave). Finally, if $\delta = 0$, then by Lemma 4.4 (iii) the latter functional is constant. Putting all these facts together, one proves the assertion on G. □

Lemma 4.6. *Let F and G be two concave, nondecreasing functionals on currents, and let $\mathfrak{F}(T,S) := F(T)+G(S)$. Let T and S be real one-dimensional flat chains of finite mass such that $\mathfrak{F}(T,S) < +\infty$ and either T or S is rectifiable. Then there are two real one-dimensional flat chains T' and S' of finite mass such that $T'+S' = T+S$, $\mu_{T'} \wedge \mu_{S'} = 0$, $\operatorname{supp}(T'+S') \subset \operatorname{supp} T \cup \operatorname{supp} S$ and $\mathfrak{F}(T',S') \le \mathfrak{F}(T,S)$.*

If, moreover, T and S are not disjoint (in the sense that $\mu_T \wedge \mu_S \neq 0$) and either F or G is strictly concave and strictly increasing, then one has the strict inequality $\mathfrak{F}(T',S') < \mathfrak{F}(T,S)$.

Proof. Suppose first that T is rectifiable, i.e. $T = \tau_T \wedge \theta_T \mathcal{H}^1 \llcorner \Sigma_T$ (mind that $\Sigma_T \subset \mathbb{R}^n$ is countably $(\mathcal{H}^1,1)$-rectifiable, while $\tau_T(x)$ orients the approximate tangent plane to Σ at x for $\mathcal{H}^1$-a.e. $x \in \Sigma$). Let $\sigma := \mu_T \wedge \mu_S$. If $\sigma \neq 0$ (otherwise one may just take $T' := T$, $S' := S$), then there is a Borel set $\Sigma \subset \Sigma_T$ (hence Σ is also countably $(\mathcal{H}^1,1)$-rectifiable) on which σ is concentrated. Observe that $\sigma(\Sigma \setminus \Sigma_S) = 0$, because

$$\mu_S(\Sigma \setminus \Sigma_S) \le \mu_S(\Sigma_T \setminus \Sigma_S) = 0,$$

the latter equality being valid in view of the fact that $\mu_S(E \setminus \Sigma_S) = 0$ for every countably $(\mathcal{H}^1,1)$-rectifiable set $E \subset \mathbb{R}^n$. Hence, we may assume without loss of generality $\Sigma \subset \Sigma_S$. We have also $\sigma = \theta \mathcal{H}^1 \llcorner \Sigma$, where $\theta = \theta_T \wedge \theta_S$.

Set now

$$\Sigma^\pm := \{x \in \Sigma \colon \tau_S(x) = \pm \tau_T(x)\}.$$

Since $S \llcorner \Sigma_S$ is rectifiable, then so is $S \llcorner \Sigma$, which implies $\mathcal{H}^1(\Sigma \setminus (\Sigma^+ \cup \Sigma^-)) = 0$. Hence, minding $\sigma \ll \mathcal{H}^1 \llcorner \Sigma$, we get

$$\sigma(\Sigma \setminus (\Sigma^+ \cup \Sigma^-)) = 0.$$

We first focus our attention on Σ^- and show that one may assume without loss of generality that $\sigma(\Sigma^-) = 0$. In fact, if $\sigma(\Sigma^-) > 0$, the setting $R := \tau_T \wedge \theta \mathcal{H}^1 \llcorner \Sigma^-$, one gets $R \neq 0$, while, clearly, $R \le T$ and $-R \le S$. Set now $\tilde{T} := T - R$, $\tilde{S} := S + R$, and note that $\tilde{T} + \tilde{S} = T + S$. Further, since $\tilde{T} \le T$, we have $F(\tilde{T}) \le F(T)$, and since $\tilde{S} \le S$, we have $G(\tilde{S}) \le G(S)$ while at least one of the above inequalities is strict, if either F or G is strictly increasing. Thus we get $\mathfrak{F}(\tilde{T},\tilde{S}) \le \mathfrak{F}(T,S)$ (with strict inequality if either F or G is strictly increasing). Hence, if one substitutes $\tilde{T}$ for T and $\tilde{S}$ for S, one will find that, by construction, $\sigma(\Sigma^-) = 0$. Therefore, from now on we assume without loss of generality that σ is concentrated on Σ^+, and that $\Sigma = \Sigma^+$.

For each $t \in [0,1]$ define

$$T_t := T + S \llcorner \Sigma - t(T \llcorner \Sigma + S \llcorner \Sigma), \qquad S_t := S + T \llcorner \Sigma - (1-t)(T \llcorner \Sigma + S \llcorner \Sigma)$$

and notice that $T_t + S_t = T + S$. Also notice that $T \llcorner \Sigma + S \llcorner \Sigma$ is a subcurrent of both $T + S \llcorner \Sigma$ and $S + T \llcorner \Sigma$. Applying Lemma 4.7 (with $T + S \llcorner \Sigma$ instead of T, $S + T \llcorner \Sigma$ instead of S and $T \llcorner \Sigma + S \llcorner \Sigma$ instead of R), we get that $t \mapsto \mathfrak{F}(T_t, S_t)$ is concave (resp. strictly concave if either F or G is so). It follows that $t \in [0,1] \mapsto \mathfrak{F}(T_t,S_t)$ attains its minimum (resp. strict minimum) in either $\bar{t} = 0$ or $\bar{t} = 1$. Let $T' = T_{\bar{t}}$, $S' = S_{\bar{t}}$. Then we have $\mathfrak{F}(T',S') \le \mathfrak{F}(T,S)$ (resp. $\mathfrak{F}(T',S') < \mathfrak{F}(T,S)$ under either of the conditions (i)–(v) of Lemma 4.7 and when $T \llcorner \Sigma + S \llcorner \Sigma \neq 0$, the latter being true when $\mu_T \wedge \mu_S > 0$). To conclude the proof of the statement for rectifiable T, we only have to check that $\mu_{T'} \wedge \mu_{S'} = 0$. This is true by construction: in fact, if $\bar{t} = 0$, then

$$T' = T + S \llcorner \Sigma, \qquad S' = S - S \llcorner \Sigma$$

which means that $\mu_{S'}$ is concentrated on $\mathbb{R}^n\setminus\Sigma$, while $T'\llcorner(\mathbb{R}^n\setminus\Sigma) = T$, $S'\llcorner(\mathbb{R}^n\setminus\Sigma) = S$ and hence

$$\mu_{T'} \wedge \mu_{S'} \leq (\mu_{T'}\llcorner(\mathbb{R}^n \setminus \Sigma)) \wedge (\mu_{S'}\llcorner(\mathbb{R}^n \setminus \Sigma)) = (\mu_{T}\llcorner(\mathbb{R}^n \setminus \Sigma)) \wedge (\mu_{S}\llcorner(\mathbb{R}^n \setminus \Sigma)) = 0.$$

The case $\bar{t} = 1$ is completely analogous, since then

$$T' = T - T\llcorner\Sigma, \qquad S' = S + T\llcorner\Sigma,$$

and hence $\mu_{T'}$ is concentrated outside of Σ, while $T'\llcorner(\mathbb{R}^n\setminus\Sigma) = T$, $S'\llcorner(\mathbb{R}^n\setminus\Sigma) = S$.

The case when S is rectifiable, while T may be arbitrary, is considered in a completely symmetric way. □

Lemma 4.7. *Let F and G be two concave functionals defined on real flat chains with finite mass and let $\mathfrak{F}(T,S) := F(T) + G(S)$. Suppose that T, S, R be given real flat chains of finite mass such that $R \leq T$ and $R \leq S$. Then the function*

$$[0,1] \ni t \mapsto \mathfrak{F}(T - tR, S - (1-t)R)$$

is concave. Moreover the same function is strictly concave, if $R \neq 0$ and either F or G is strictly concave. In particular, if $\mathfrak{F}$ is defined as in Problem 1, then the respective function is concave, if either H is concave, or $\delta = 0$. In this case the same function is strictly concave, if $R \neq 0$ and either of the following conditions hold:

(i) *$\delta = 0$, $\alpha \in (0,1)$ and $A \neq 0$, or*
(ii) *$\delta = 0$, $\beta \in (0,1)$ and $B \neq 0$, or*
(iii) *H is concave, $\alpha \in (0,1)$ and $A \neq 0$, or*
(iv) *H is concave, $\beta \in (0,1)$ and $B \neq 0$, or*
(v) *H is strictly increasing and concave, while $\delta \in (0,1)$.*

Proof. From Definition 4.1 we know that both the functions

$$t \mapsto F(T - tR), \qquad \text{and} \qquad t \mapsto G(S - (1-t)R)$$

are concave for $t \in [0,1]$. It suffices to refer now to Remark 4.3. The case when $\mathfrak{F}$ is as in Problem 1 follows then from Lemma 4.5. □

5. Auxiliary lemmata

We will need the following auxiliary assertions on convergence of measures and currents.

Lemma 5.1. *Let ϕ be a signed finite Borel measure with compact support in $\mathbb{R}^n$, $\phi(\mathbb{R}^n) = 0$. Then there exists a sequence of finite weighted sums of Dirac measures ϕ_ν such that*

$$\phi_\nu^\pm \rightharpoonup \phi^\pm, \qquad \phi_\nu(\mathbb{R}^n) = 0.$$

Proof. Consider two sequences of of finite weighted sums of Dirac measures $\psi_\nu^+ \rightharpoonup \phi^+$ and $\psi_\nu^- \rightharpoonup \phi^-$ in the $*$-weak sense of measures when $\nu \to \infty$ (note that here ψ_ν^+ and ψ_ν^- *do not* denote positive and negative parts of some signed measure ψ_ν, but just some positive measures; in fact, it may happen that $\psi_\nu^+ \wedge \psi_\nu^- \neq 0$).

Consider the quantity $\lambda_\nu := \psi_\nu^-(\mathbb{R}^n) - \psi_\nu^+(\mathbb{R}^n)$ and set

$$\begin{aligned} \tilde{\psi}_\nu^+ &:= \psi_\nu^+ + \lambda_\nu\delta_0, & \tilde{\psi}_\nu^- &:= \psi_\nu^-, & &\text{if } \lambda_\nu \geq 0, \\ \tilde{\psi}_\nu^+ &:= \psi_\nu^+, & \tilde{\psi}_\nu^- &:= \psi_\nu^- - \lambda_\nu\delta_0, & &\text{otherwise.} \end{aligned}$$

In this way we have $\tilde{\psi}_\nu^+(\mathbb{R}^n) = \tilde{\psi}_\nu^-(\mathbb{R}^n)$, while both measures $\tilde{\psi}_\nu^\pm$ are still non-negative. Moreover we notice that $\lambda_\nu \to 0$ because $\psi_\nu^\pm(\mathbb{R}^n) \to \phi^\pm(\mathbb{R}^n)$ as $\nu \to \infty$ and $\phi^-(\mathbb{R}^n) - \phi^+(\mathbb{R}^n) = \phi(\mathbb{R}^n) = 0$. In particular, $\lambda_\nu\delta_0 \rightharpoonup 0$ and hence $\tilde{\psi}_\nu^\pm \rightharpoonup \phi^\pm$ $*$-weakly in the sense of measures as $\nu \to \infty$.

We modify now $\tilde\psi_\nu^\pm$ into $\phi_\nu^\pm$ so that $\phi_\nu^+ \wedge \phi_\nu^- = 0$. To achieve this result we define $\mu_\nu := \tilde\psi_\nu^+ \wedge \tilde\psi_\nu^-$ and

$$\phi_\nu^\pm := \tilde\psi_\nu^\pm - \mu_\nu.$$

Given any $*$-weakly convergent (in the sense of measures) subsequence of $\{\mu_\nu\}$, for its limit one has $\mu \le \phi^\pm$ because $\mu_\nu \le \tilde\psi_\nu^\pm$ and $\tilde\psi_\nu^\pm \rightharpoonup \phi^\pm$ $*$-weakly in the sense of measures as $\nu \to \infty$. Therefore, minding that $\phi^+ \wedge \phi^- = 0$, one has $\mu = 0$. Hence, $\mu_\nu \rightharpoonup 0$, which implies that $\phi_\nu^\pm \rightharpoonup \phi^\pm$ $*$-weakly in the sense of measures as $\nu \to \infty$. On the other hand, by construction, $\phi_\nu^+ \wedge \phi_\nu^- = 0$ and hence the measure $\phi_\nu := \phi_\nu^+ - \phi_\nu^-$ has $\phi_\nu^\pm$ as positive and negative parts. Moreover we easily find that

$$\phi_\nu(\mathbb{R}^n) = \phi_\nu^+(\mathbb{R}^n) - \phi_\nu^-(\mathbb{R}^n) = \tilde\psi_\nu^+(\mathbb{R}^n) - \tilde\psi_\nu^-(\mathbb{R}^n) = 0,$$

concluding the proof. □

Lemma 5.2. *Let ψ_ν be signed measures on $\mathbb{R}^n$ such that $\psi_\nu \rightharpoonup 0$ $*$-weakly in the sense of measures as $\nu \to \infty$, $\operatorname{supp}\psi_\nu \subset K \Subset \mathbb{R}^n$, $\psi_\nu(\mathbb{R}^n) = 0$ and $\psi_\nu^\pm(\mathbb{R}^n) < +\infty$. Then there exists a real flat chain R_ν such that $\partial R_\nu = \psi_\nu$ and $\mathbb{M}(R_\nu) \to 0$ as $\nu \to \infty$. Moreover, if ψ_ν is a finite sum of signed Dirac masses, then one may choose R_ν polyhedral.*

Proof. Let R_ν provide the minimum of $T \mapsto \mathbb{M}(T)$ among all the flat chains T satisfying $\partial T = \psi_\nu$. In other words, R_ν solves the classical Monge-Kantorovich optimal transportation problem of transporting ψ_ν^+ to ψ_ν^- as announced in the subsection 1.1. Then $\mathbb{M}(R_\nu)$ is a Wasserstein distance between ψ_ν^+ to ψ_ν^- which metrizes the $*$-weak topology of measures over the set of finite nonnegative Borel measures over the compact $K \subset \mathbb{R}^n$. Hence, $\mathbb{M}(R_\nu) \to 0$ whenever $\psi_\nu \rightharpoonup 0$ $*$-weakly in the sense of measures as $\nu \to \infty$. It is also well-known that if ψ_ν is a finite sum of signed Dirac masses, then R_ν is polyhedral. □

Lemma 5.3. *Let T be a one-dimensional real normal current. Then there is a sequence of one-dimensional real polyhedral chains T_ν which converges in the flat norm to T, $\mathbb{M}(T_\nu) \to \mathbb{M}(T)$ and also $(\partial T_\nu)^\pm \rightharpoonup (\partial T)^\pm$ $*$-weakly in the sense of measures as $\nu \to \infty$. Moreover, if T is acyclic, then one can choose T_ν to be acyclic too.*

Proof. Let $\{\phi_\nu\}$ be a sequence of finite Borel measures constructed by Lemma 5.1 applied with $\phi := \partial T$. Let also S_ν be one-dimensional real polyhedral chains satisfying $S_\nu \overset{\mathcal{F}}{\rightharpoonup} T$, $\mathbb{M}(S_\nu) \to \mathbb{M}(T)$ as $\nu \to \infty$. By Lemma 5.2 applied with $\psi_\nu := \phi_\nu - \partial S_\nu$ there is a sequence of one-dimensional real polyhedral chains R_ν with $\partial R_\nu = \psi_\nu$ and $\mathbb{M}(R_\nu) \to 0$. Then the current $T_\nu := S_\nu + R_\nu$ satisfy the first part of the lemma being proven. In fact, $T_\nu \overset{\mathcal{F}}{\rightharpoonup} T$ as $\nu \to \infty$ and $\mathbb{M}(T_\nu) \le \mathbb{M}(S_\nu) + \mathbb{M}(R_\nu)$. Passing to the limit we obtain

$$\mathbb{M}(T) \le \liminf_\nu \mathbb{M}(T_\nu) \le \lim_\nu \mathbb{M}(S_\nu) = \mathbb{M}(T),$$

and hence, $\mathbb{M}(T_\nu) \to \mathbb{M}(T)$ as $\nu \to \infty$. Also $(\partial T_\nu)^\pm = \phi^\pm \rightharpoonup (\partial T)^\pm$ as $\nu \to \infty$ by construction.

If T is acyclic, we modify T_ν in the following way. Let C_ν be the cycle of T_ν given by Proposition 3.12 such that $T'_\nu := T_\nu - C_\nu$ is acyclic. Up to a subsequence (not relabeled), $C_\nu \overset{\mathcal{F}}{\rightharpoonup} C$ as $\nu \to \infty$. Hence, by Lemma 3.10, $\mathbb{M}(C_\nu) \to \mathbb{M}(C)$ as $\nu \to \infty$ and C is a cycle of T. Since the only cycle of T is 0 we conclude that $\mathbb{M}(C_\nu) \to 0$, which means that $T'_\nu \overset{\mathcal{F}}{\rightharpoonup} T$ and $\mathbb{M}(T'_\nu) \to \mathbb{M}(T)$ as $\nu \to \infty$. It remains to observe that $\partial T'_\nu = \partial T_\nu = \phi_\nu$, while by Lemma 3.13 T'_ν is still polyhedral. □

6. Currents versus transports

We call two Lipschitz-continuous curves $\hat\theta_1$, $\hat\theta_2$: $[0,1]\to\mathbb{R}^n$ equivalent, if there is a continuous surjective nondecreasing function (called usually "reparameterization") ϕ: $[0,1]\to[0,1]$ such that $\hat\theta_1(t)=\hat\theta_2(\phi(t))$ for all $t\in[0,1]$. Let then Θ stand for the set of equivalence classes of Lipschitz-continuous paths. In this way each $\theta\in\Theta$ can be clearly identified with some directed rectifiable curve. In the sequel we will frequently slightly abuse the language, identifying the elements of Θ (i.e. directed rectifiable curves) with their parameterizations (i.e. Lipschitz-continuous paths parameterizing such curves), when it cannot lead to a confusion. We consider the set Θ to be equipped with the distance

$$d_\Theta(\theta_1,\theta_2):=\inf\left\{\max_{t\in[0,1]}|\hat\theta_1(t)-\hat\theta_2(t)|\,:\,\hat\theta_i \text{ parameterization of } \theta_i, i=1,2\right\}, \tag{7}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. It is easy to see that $\theta_\nu\to\theta$ in Θ implies the Hausdorff convergence of the respective traces, though the converse is clearly not true. It is further important to mention that clearly every subset of Θ made by all paths with uniformly bounded length is clearly compact with respect to the introduced topology. This implies that the whole metric space Θ is σ-compact (i.e. a countable union of compact sets).

In the sequel we will also use the following notions. We say that $\sigma\in\Theta$ is contained in a given $\theta\in\Theta$, if for some parameterizations of σ and θ and for some affine nondecreasing ϕ: $[0,1]\to[0,1]$ one has $\theta(\phi(t))=\sigma(t)$ for all $t\in[0,1]$, which means that σ represents a "piece" of θ. Finally, we call $\theta\in\Theta$ an arc, if it is injective.

To each $\theta\in\Theta$ we associate the integral one-dimensional current $[\![\theta]\!]$ defined by the formula

$$[\![\theta]\!](\omega):=\int_0^1\langle\dot\theta(t),\omega(\theta(t))\rangle\,dt$$

(note that the latter integral does not depend on the parameterization of θ so it is well defined on equivalence classes $\theta\in\Theta$). We also define the *parametric length* of θ as

$$\ell(\theta):=\int_0^1|\dot\theta(t)|\,dt.$$

Clearly, one has

$$\mathbb{M}([\![\theta]\!])=\sup\{[\![\theta]\!](\omega)\,:\,\|\omega\|_\infty\le 1\}\le\ell(\theta).$$

The following rather simple assertion is valid.

Lemma 6.1. *The map $\theta\in\Theta\mapsto[\![\theta]\!]$ is a continuous embedding of each subset of curves from Θ with uniformly bounded lengths into the space of integral one-dimensional currents endowed with weak topology of currents.*

Proof. Let $\theta_\nu\in\Theta$ be curves with uniformly bounded length, i.e. $\ell(\theta_\nu)\le C<+\infty$ for all $\nu\in\mathbb{N}$. One has to prove that $\theta_\nu\to\theta\in\Theta$ as $\nu\to\infty$ implies $[\![\theta_\nu]\!](\omega)\to[\![\theta]\!](\omega)$ for every $\mathcal{C}^\infty$ 1-form ω. Consider the parameterizations of θ_ν with $|\dot\theta_\nu|\le C$ for all $t\in[0,1]$. Since $\theta_\nu(t)$ for all $t\in[0,1]$ and for all $\nu\in\mathbb{N}$ are all contained in some neighborhood of θ, one has that the sequence θ_ν is weakly compact in $W^{1,2}([0,1];\mathbb{R}^n)$. Hence, up to a subsequence (not relabeled) $\theta_\nu\rightharpoonup\sigma$ weakly in $W^{1,2}([0,1];\mathbb{R}^n)$ as $\nu\to\infty$ for some $\sigma\in W^{1,2}([0,1];\mathbb{R}^n)$, which in particular means that $\sigma=\theta$, and hence $\dot\theta_\nu\rightharpoonup\dot\theta$ weakly in $L^2([0,1];\mathbb{R}^n)$ as $\nu\to\infty$. Hence,

$$[\![\theta_\nu]\!](\omega)=\int_0^1\langle\dot\theta_\nu(t),\omega(\theta_\nu(t))\rangle\,dt\to\int_0^1\langle\dot\theta(t),\omega(\theta(t))\rangle\,dt=[\![\theta]\!](\omega)$$

as $\nu\to\infty$. □

Given a transport η on Θ we define a functional T_η on 1-forms as follows

$$T_\eta(\omega) := \int_\Theta [\![\theta]\!](\omega)\, d\eta(\theta). \tag{8}$$

The following theorem shows that T_η is a normal current under natural assumptions on η.

Theorem 6.2. *Let η be a finite Borel measure on Θ satisfying*

$$\int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta) < +\infty.$$

Then (8) defines a normal one-dimensional current $T = T_\eta$ on $\mathbb{R}^n$ with

$$\partial T = \eta(1) - \eta(0), \ \textit{where } \eta(i) := (t_i)_{\#}\eta, \quad t_i(\theta) := \theta(i), \qquad i = 0, 1.$$

In particular, if $\eta(1) \wedge \eta(0) = 0$, then

$$(\partial T)^+ = \eta(1), \qquad (\partial T)^- = \eta(0).$$

Furthermore, for all Borel sets $e \subset \mathbb{R}^n$ one has

$$\mu_T(e) \le \int_\Theta \mathbb{M}([\![\theta]\!] \llcorner e)\, d\eta(\theta). \tag{9}$$

Proof. We have to prove that $T = T_\eta$ is continuous on $\mathcal{C}^\infty$ 1-forms, has finite mass and finite boundary mass. According to the definition of mass

$$\mathbb{M}(T) := \sup\{T(\omega) \colon |\omega(x)| \le 1 \text{ for all } x \in \mathbb{R}^n\}$$

and hence

$$\mathbb{M}(T) \le \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta) < +\infty.$$

Analogously, the relationships

$$\mu_T(U) = \sup\{T(\omega) \colon |\omega(x)| \le 1, \operatorname{supp}\omega \subset U \text{ for all } x \in \mathbb{R}^n\},$$
$$\mathbb{M}([\![\theta]\!] \llcorner U) = \sup\{([\![\theta]\!](\omega) \colon |\omega(x)| \le 1, \operatorname{supp}\omega \subset U \text{ for all } x \in \mathbb{R}^n\},$$

for every open set $U \subset \mathbb{R}^n$ imply

$$\mu_T(e) \le \int_\Theta \mathbb{M}([\![\theta]\!] \llcorner e)\, d\eta(\theta)$$

for every open set $e \subset \mathbb{R}^n$, and hence, for every Borel set $e \subset \mathbb{R}^n$. Finally, the computation of the boundary

$$\begin{aligned}
\partial T(f) = T(df) &= \int_\Theta \left(\int_0^1 \langle \nabla f(\theta(t)), \dot\theta(t)\rangle\, dt \right) d\eta(\theta) \\
&= \int_\Theta \left(\int_0^1 \frac{d}{dt} f \circ \theta\, dt \right) d\eta(\theta) \\
&= \int_\Theta [f(\theta(1)) - f(\theta(0))]\, d\eta(\theta) \\
&= \int_\Theta f(t_1(\theta))\, d\eta(\theta) - \int_\Theta f(t_0(\theta))\, d\eta(\theta) \\
&= \int_{\mathbb{R}^n} f(x)\, d(\eta(1) - \eta(0)),
\end{aligned}$$

concludes the proof. □

It is worth mentioning that the inequality in (9) may be strict, as the following example shows.

Example 1. *Let e_i, $i = 1,2$ stand for the unit vectors along axis x_i in $\mathbb{R}^2$, and let $\Theta_1 \subset \Theta$ be a set of paths θ in $Q := [0,1]\times[0,1]$ admitting a parameterization $\theta(t) = (t, x_2)$, $t \in [0,1]$, for some $x_2 \in [0,1]$. Define η_1 by the formula*

$$\eta_1(e) := \mathcal{H}^1(t_0(e \cap \Theta_1))$$

for all Borel $e \subset \Theta$, where $t_0(\theta) := \theta(0)$. Clearly, $T_{\eta_1} = e_1 \wedge \mathcal{L}^2 \llcorner Q$. Analogously, letting $\Theta_2 \subset \Theta$ be a set of paths θ admitting a parameterization $\theta(t) = (x_1, t)$, $t \in [0,1]$, for some $x_1 \in [0,1]$, and defining η_2 by the formula

$$\eta_2(e) := \mathcal{H}^1(t_0(e \cap \Theta_2))$$

for all Borel $e \subset \Theta$, we get $T_{\eta_2} = e_2 \wedge \mathcal{L}^2 \llcorner Q$. Now, setting $\eta := \eta_1 + \eta_2$, one has $T_\eta = T_{\eta_1} + T_{\eta_2} = (e_1 + e_2) \wedge \mathcal{L}^2 \llcorner Q$, and hence, $\mathbb{M}(T_\eta) = \sqrt{2}$, while

$$\int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta = \int_{\Theta_1} \mathbb{M}([\![\theta]\!])\, d\eta_1 + \int_{\Theta_2} \mathbb{M}([\![\theta]\!])\, d\eta_2 = 2 > \mathbb{M}(T_\eta).$$

We now prove a converse statement, i.e. that given a normal real one-dimensional current T, there is a transport η satisfying $T = T_\eta$.

Theorem 6.3. *Given a one-dimensional acyclic real normal current T with compact support, there exists a Borel measure η over Θ such that $T = T_\eta$ as defined by* (8) *and*

$$\mathbb{M}(T) = \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta). \tag{10}$$

Moreover, one can choose an η so that

$$\eta(1) = (\partial T)^+, \quad \eta(0) = (\partial T)^-, \tag{11}$$

where $(\partial T)^\pm$ are the positive and the negative part of the measure ∂T respectively, while η-a.e. $\theta \in \Theta$ is an arc.

Remark 6.4. In view of Theorem 6.2, the claim (10) is equivalent to a formally weaker one

$$\mathbb{M}(T) \geq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta).$$

Apart from the claim (11), which is indeed used in the sequel, the above theorem is in fact contained (though in quite different terminology) in theorem C from [19]. Since the relationship between one-dimensional flat chains of finite mass and transports is of utmost importance in the sequel, we provide here the complete and independent proof of the result.

Before proving the above Theorem 6.3 in the general case, we need to prove a similar assertion valid only for one-dimensional real *polyhedral* chains, as given by the following lemma.

Lemma 6.5. *Let T be a one-dimensional real polyhedral chain. Then there exists a Borel measure η over Θ such that $T = T_\eta$ and*

$$\mathbb{M}(T) = \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta) = \int_\Theta \ell(\theta)\, d\eta(\theta) \tag{12}$$

and η-a.e. θ is supported on $\operatorname{supp} T$. Further, if T is also acyclic, then one can choose η so as to satisfy

$$\eta(1) = (\partial T)^+, \qquad \eta(0) = (\partial T)^-. \tag{13}$$

If one does not require (13)*, one can choose η so as to have $\mathcal{H}^1(\theta) \leq \operatorname{diam} \operatorname{supp} T$ for η-a.e. $\theta \in \Theta$.*

Proof. Every one-dimensional real polyhedral chain T can be written as a finite sum

$$T = \sum_\nu \theta_\nu T_\nu,$$

where $\theta_\nu > 0$ are real multiplicities, and T_ν are currents associated to oriented segments $T_\nu = [\![a_\nu, b_\nu]\!]$ (called further *edges* of T) with non overlapping interior.

Step 1. If T is a generic one-dimensional real polyhedral chain, consider the Lipschitz curves σ_ν defined by $\sigma_\nu(t) := (1-t)a_\nu + tb_\nu$ for all $t \in [0,1]$, and set

$$\eta := \sum_\nu \theta_\nu \delta_{\sigma_\nu},$$

where δ_{σ_ν} is the Dirac measure concentrated on $\sigma_\nu \in \Theta$. Clearly, we have

$$T(\omega) = \sum_\nu \int_{\sigma_\nu} \theta_\nu\, \omega \cdot (b_\nu - a_\nu) = \sum_\nu \theta_\nu [\![\sigma_\nu]\!](\omega) = \int_\Theta [\![\sigma]\!](\omega)\, d\eta(\sigma), \tag{14}$$

i.e. $T = T_\eta$. By construction one also has $\mathbb{M}([\![\sigma]\!]) = \ell(\sigma)$ for η-a.e. $\sigma \in \Theta$, and hence

$$\mathbb{M}(T) = \sum_\nu |\theta_\nu| \cdot |b_\nu - a_\nu| = \int_\Theta \ell(\sigma)\, d\eta(\sigma), \tag{15}$$

while η-a.e. $\theta \subset \Theta$ is a segment, $\theta \subset \operatorname{supp} T$, and hence $\mathcal{H}^1(\theta) \le \operatorname{diam} \operatorname{supp} T$.

Step 2. To consider the case of an acyclic T, we introduce some extra notation. We say that an ordered finite collection of edges $(T_{\nu_1}, \dots, T_{\nu_N})$, where $T_{\nu_i} := [\![a_{\nu_i}, b_{\nu_i}]\!]$, $i = 1, \dots, N$, is a *path* in T, if $b_{\nu_i} = a_{\nu_{i+1}}$ for $i = 1, \dots, N-1$. We say that such a path is *closed*, if also $b_{\nu_N} = a_{\nu_1}$. Choosing $\theta_0 > 0$ to be the minimum of θ_ν over all ν, we notice that the current

$$\sum_{i=1}^N \theta_0 T_{\nu_i}$$

is a subcurrent of T. An acyclic T contains therefore no closed paths. Finally, given a path in T, we can extend it *forward*, if there exists an edge T_ν of T such that $a_\nu = b_{\nu_N}$, and *backward*, if there exists and edge T_ν such that $b_\nu = a_{\nu_1}$.

Let T be acyclic. We consider a path with a single edge $T_{\bar\nu}$ such that $\theta_{\bar\nu} = \theta_0$. We extend this path as much as possible forward and backward. At each extension step the path is not closed, hence the path is composed by different edges. Since there is only a finite number of edges in T, this extension process must finish in a finite number of steps. We obtain in this way a *maximal path* containing $T_{\bar\nu}$. Let $(T_{\nu_1}, \dots, T_{\nu_N})$ be this maximal path and consider the corresponding current

$$P_0 := \sum_{i=1}^N \theta_0 T_{\nu_i}.$$

Clearly, P_0 is a subcurrent of T and $\partial P_0 = [\![b_{\nu_N}]\!] - [\![a_{\nu_1}]\!]$. Since this path is maximal, there is no edge T_ν with endpoint $b_\nu = a_{\nu_1}$, and thus $[\![a_{\nu_1}]\!]$ is a subcurrent of $(\partial T)^-$. Analogously $[\![b_{\nu_N}]\!]$ is a subcurrent of $(\partial T)^+$. One has

$$P_0(\omega) = \int_\Theta [\![\sigma]\!](\omega)\, d\eta_0(\sigma),$$

where $\eta_0 := \theta_0 \delta_{\sigma_0}$ is the Dirac measure with mass θ_0 concentrated on the curve $\sigma_0 \in \Theta$, the latter curve representing the polygonal line $[a_1, b_1] \circ \ldots \circ [a_N, b_N]$ (starting at a_1 and ending at b_N). Hence $\eta_0(1) = (\partial P_0)^+$ and $\eta_0(0) = (\partial P_0)^-$.

The current $T' = T - P_0$ is itself a polyhedral acyclic current with strictly less edges than T, because the edge $T_{\bar\nu}$ is not included in T'. Repeating the previous construction with T' in place of T we find a subcurrent P_1 representing a path in T' and such that

$$P_1(\omega) = \int_\Theta [\![\sigma]\!](\omega)\, d\eta_1(\sigma)$$

with $\eta_1(1) = (\partial P_1)^+$ and $\eta_1(0) = (\partial P_1)^-$. A finite number of such steps will clearly exhaust T and yield a decomposition $T = \sum_{i=0}^k P_i$ such that the corresponding measure $\eta := \sum_{i=0}^k P_i$ has the required properties. □

We are now able to prove the general Theorem 6.3.

Proof. (of Theorem 6.3) We divide the proof in two steps.

Step 1. Given an arbitrary one-dimensional real flat chain T, we consider a sequence T_ν of one-dimensional polyhedral chains which converges to T in the flat norm and $\mathbb{M}(T_\nu) \to \mathbb{M}(T)$ as $\nu \to \infty$, hence in particular $\mathbb{M}(T_\nu) \leq \mathbb{M}(T) + 1$ for all sufficiently large $\nu \in \mathbb{N}$. By Lemma 6.5, for each T_ν we find a transport η_ν satisfying

$$
\begin{aligned}
T_\nu(\omega) &= \int_\Theta [\![\theta]\!](\omega)\, d\eta_\nu(\theta),\\
\mathbb{M}(T) &= \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta_\nu(\theta) = \int_\Theta \ell(\theta)\, d\eta_\nu(\theta)
\end{aligned} \tag{16}
$$

for all $\mathcal{C}^\infty$ 1-forms ω. Since T is acyclic, we choose T_ν according to Lemma 5.3, i.e. so that in addition $(\partial T_\nu)^\pm \rightharpoonup (\partial T)^\pm$ in the $*$-weak sense of measures when $\nu \to \infty$. In this case by Lemma 6.5 one can choose a transport η_ν satisfying additionally

$$
\eta_\nu(1) = (\partial T_\nu)^+, \qquad \eta_\nu(0) = (\partial T_\nu)^-. \tag{17}
$$

In particular, the total masses $\eta_\nu(\Theta)$ are uniformly bounded.

In view of (16) we have the estimate

$$
\int_\Theta \ell(\theta)\, d\eta_\nu = \mathbb{M}(T_\nu) \leq \mathbb{M}(T) + 1.
$$

Further, without loss of generality we may assume that the traces of η_ν-a.e. $\theta \in \Theta$ are supported on some compact $\Omega \subset \mathbb{R}^n$. We may invoke therefore Lemma 6.7 below, obtaining that up to a subsequence (not relabeled) $\eta_\nu \rightharpoonup \eta$ in the narrow sense of measures for some finite Borel measure η, and, moreover, that one may pass to the limit as $\nu \to \infty$ in both sides of the first relationship of (16) obtaining therefore $T(\omega) = T_\eta(\omega)$ for each $\mathcal{C}^\infty$ 1-form ω, and hence $T = T_\eta$. One shows in addition that (11) is valid by passing to the limit as $\nu \to \infty$ in both sides of (17).

Furthermore, note that

$$
\mathbb{M}(T_\nu) = \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta_\nu(\theta) \tag{18}
$$

by the second relationship of (16). Hence, minding that the functional $\theta \in \Theta \to \mathbb{M}([\![\theta]\!])$ is l.s.c., and hence, the integral in the right-hand side of the above relationship is l.s.c. with respect to narrow convergence of η_ν, by passing to a limit in both sides of (18) as $\nu \to \infty$, we deduce

$$
\mathbb{M}(T) = \lim_\nu \mathbb{M}(T_\nu) = \lim_\nu \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta_\nu(\theta) \geq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta),
$$

which provides (10) once one recalls Remark 6.4.

We also consider for further use the functional $M' \colon \Theta \to \mathbb{R}^+$ defined by

$$
M'(\eta) := \int_\Theta \ell(\theta)\, d\eta. \tag{19}
$$

It is l.s.c. with respect to the narrow convergence of measures (because the parametric length $\ell(\cdot)$ is l.s.c. in Θ). Hence, minding that for each η_ν one has by construction $M'(\eta_\nu) = \mathbb{M}(T_\nu)$, we get $M'(\eta) \leq \mathbb{M}(T) < +\infty$.

Step 2. Finally, for T acyclic, we consider an η minimizing M' over the set E of all the transports η' satisfying $T = T_{\eta'}$, as well as (10) and (11). To prove the existence of such an η recall that the latter set of transports is nonempty in view of Step 1. Consider now a minimizing sequence $\{\eta_\nu\} \subset E$ for M'. By the final remark of Step 1 one has $M'(\eta_\nu) \leq C < +\infty$ for some $C > 0$. Further, without loss of generality we may assume that the traces of η_ν-a.e. $\theta \in \Theta$ are supported on some compact $\Omega \subset \mathbb{R}^n$. Hence by Lemma 6.7, the sequence $\{\eta_\nu\}$ admits a subsequence (further, as usual, not relabeled) converging to some transport η in the narrow sense of measures, while $T = T_{\eta_\nu} \to T_\eta$ in the weak sense of currents as $\nu \to \infty$, and thus $T = T_\eta$. Since by the same lemma $\eta_\nu(i) \rightharpoonup \eta(i)$, $i = 0, 1$, in the narrow sense of measures, then also (11) holds for η, while acting as in Step 1, we get the validity of (10) for η. Summing up, we get $\eta \in E$. Minding that M' is l.s.c. with respect to the narrow convergence of measures, we get that η is a minimizer of M' over E.

Let $f\colon \Theta \to \Theta$ and $g\colon \Theta \to \Theta$ be given by Lemma 6.6. Then $T_{f_\# \eta}$ is a cycle of $T = T_\eta$. Hence, $T_{f_\#\eta} = 0$. This means $[\![f(\theta)]\!] = 0$ for η-a.e. $\theta \in \Theta$. We have thus $[\![g(\theta)]\!] = [\![\theta]\!]$ for η-a.e. $\theta \in \Theta$. Hence, $T_{g_\#\eta} = T_\eta = T$, and $g_\#\eta \in E$, so

$$M'(g_\#\eta) = \int_\Theta \ell(g(\theta))\, d\eta(\theta) \le \int_\Theta \ell(\theta)\, d\eta(\theta).$$

Therefore, by the minimality of η for M', we get $\ell(g(\theta)) = \ell(\theta)$, hence η-a.e. $\theta \in \Theta$ is an arc, which concludes the proof. □

The following technical assertions have been used in the proof of Theorem 6.3.

Lemma 6.6. *The following assertions are valid.*

(i) *There is a map $f\colon \Theta \to \Theta$ measurable with respect to all transports such that $f(\theta)$ is a loop (i.e. a simple closed curve) contained in $\theta \in \Theta$, such that*

$$\ell(f(\theta)) \ge 1/2 \sup\{\ell(\sigma) \,:\, \sigma \text{ is a loop contained in } \theta\}.$$

(ii) *There is a map $g\colon \Theta \to \Theta$ measurable with respect to all transports such that for all $\theta \in \Theta$ one has that $\theta = g(\theta) \cup f(\theta)$ (as traces), $[\![\theta]\!] = [\![g(\theta)]\!] + [\![f(\theta)]\!]$, while*

$$\ell(g(\theta)) < \ell(\theta),$$

unless θ is an arc, and, finally, $g(\theta) = \theta$, if and only if θ is an arc.

Proof. We construct a map $f\colon \Theta \to \Theta$ satisfying claim (i) as follows. For every $\theta \in \Theta$ and $x \in \theta$ we let $C(\theta, x)$ stand for the set of curves contained in θ starting and ending at x in the sense that

$$C(\theta, x) = \Big\{\tilde\theta \in \Theta\colon \tilde\theta(t) = \theta((1-t)s_1 + ts_2) \\ \text{for some } 0 \le s_1 \le s_2 \le 1,\ \theta(s_1) = \theta(s_2) = x\Big\}.$$

In case $x \notin \theta$ we define $C(\theta, x)$ to be a set consisting just of a single curve θ_x defined by $\theta_x(t) := x$ for all $t \in [0, 1]$, i.e. of a "constant" curve the trace of which reduces to just one point x. Note that $\theta_x \in C(\theta, x)$ for all $x \in \mathbb{R}^n$. Defined in this way, the multivalued map

$$(\theta, x) \in \Theta \times \mathbb{R}^n \mapsto C(\theta, x) \subset \Theta$$

is u.s.c. (as a multivalued map), and hence Borel measurable. Therefore, recalling that $\ell\colon \Theta \to \mathbb{R}$ is l.s.c. one gets the Borel measurability of the single-valued map

$$\lambda\,:\, \theta \in \Theta \mapsto \sup_{x \in \mathbb{R}^n} \sup\{\ell(\sigma) \,:\, \sigma \in C(\theta, x)\} \in \mathbb{R}.$$

Clearly, $\lambda(\theta)$ gives is the supremum of the length of the loops contained in θ. Finally, we define

$$F\,:\, \theta \in \Theta \mapsto \left\{\sigma \in \bigcup_{x \in \theta} C(\theta, x) \,:\, \ell(\sigma) \ge \lambda(\theta)/2\right\} \subset \Theta.$$

By the von Neumann-Aumann measurable selection theorem (theorem III.22 and III.23 from [12], or, equivalently, corollary 5.5.8 from [20]) one can find a selection $f\colon \theta \to \Theta$ of the multivalued map F which is measurable with respect to all transports η. Clearly, $f(\theta)$ is as announced in the statement being proven.

Define now $g\colon \Theta \to \Theta$ as a union of two curvilinear segments, by setting

$$g(\theta) := [\theta(0), f(\theta)(0)] \circ [f(\theta)(1), \theta(1)].$$

Clearly, $g(\theta)$ is obtained by "cancelling" the loop $f(\theta)$ from θ. The properties of g announced in claim (ii) follow immediately since $\ell(g(\theta)) \leq \ell(\theta) - \lambda(\theta)/2$, while $g(\theta) = \theta$, if and only if $f(\theta) = \theta_x$ for some $x \in \theta$, i.e. when θ is an arc. □

Lemma 6.7. *Let $\{\eta_\nu\}$ be a sequence of nonnegative finite Borel measures over Θ with uniformly bounded total masses, and denote $T_\nu := T_{\eta_\nu}$. Assume that for some one-dimensional real flat chain T with $\mathbb{M}(T) < +\infty$ one has $T_\nu \rightharpoonup T$ weakly in the sense of currents, $\mathbb{M}(T_\nu) \to \mathbb{M}(T)$ as $\nu \to \infty$, and*

$$M'(\eta_\nu) := \int_\Theta \ell(\theta)\, d\eta_\nu \leq C < +\infty$$

for all $\nu \in \mathbb{N}$, while there is such a compact $\Omega \subset \mathbb{R}^n$ that for each $\nu \in \mathbb{N}$, the traces of η_ν-a.e. $\theta \in \Theta$ are supported in Ω. Then there exists a transport η such that up to a subsequence (not relabeled), $\eta_\nu \rightharpoonup \eta$ (and in particular, $\eta_\nu(i) \rightharpoonup \eta(i)$, $i = 0, 1$) in the narrow sense of measures. Further, one has $T = T_\eta$, if either of the following two conditions hold:

(i) *all η_ν are concentrated on some compact subset of Θ (independent of ν), or*
(ii) *T is acyclic and*

$$\mathbb{M}(T_\nu) = \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta_\nu(\theta)$$

for all $\nu \in \mathbb{N}$.

Proof. Since for every $c > 0$ one has

$$M'(\eta_\nu) = \int_\Theta \ell(\theta)\, d\eta_\nu \geq c\eta_\nu(\{\ell(\theta) > c\}),$$

we conclude

$$\eta_\nu(\{\ell(\theta) > c\}) \leq C/c.$$

Recalling now that the set $\{\theta \in \Theta\colon \ell(\theta) \leq c, \theta \subset \Omega\}$ is a compact subset of Θ, we see that the sequence η_ν is tight in the sense of measures. Hence, up to a subsequence (not relabeled), $\eta_\nu \rightharpoonup \eta$ as $\nu \to \infty$ in the narrow sense of measures for some finite Borel measure η over Θ. The convergence $\eta_\nu(i) \rightharpoonup \eta(i)$, $i = 0, 1$, as $\nu \to \infty$ follows from the fact that a push-forward operator by means of a continuous function is continuous with respect to narrow convergence of measures.

In the case when (i) holds, i.e. all η_ν are concentrated on some compact subset of Θ, one immediately gets

$$T_\nu(\omega) = \int_\Theta [\![\theta]\!](\omega)\, d\eta_\nu(\theta) \to \int_\Theta [\![\theta]\!](\omega)\, d\eta(\theta) = T_\eta(\omega)$$

as $\nu \to \infty$, and hence $T = T_\eta$.

Consider now the case when (ii) holds, and in particular, η_ν are non necessarily concentrated on some (unique) compact subset of Θ. We show first that

$$\phi(k) := \limsup_\nu \int_{\{\ell(\theta) > k\}} \ell(\theta)\, d\eta_\nu(\theta) \to 0 \text{ when } k \to +\infty. \tag{20}$$

In fact, otherwise there exists a $c > 0$ such that for a subsequence of η_ν (not relabeled) one has

$$\int_{\{\ell(\theta) > \nu\}} \ell(\theta)\, d\eta_\nu(\theta) \geq c.$$

Consider then $\eta'_\nu := \eta_\nu \llcorner \{\ell(\theta) > \nu\}$, and $S_\nu := T_{\eta'_\nu}$. By Remark 6.9, each S_ν is a subcurrent of T_ν, and hence by Lemma 3.10 one gets that up to a subsequence (again not relabeled) $S_\nu \rightharpoonup S$ weakly in the sense of currents as $\nu \to \infty$, while S is a subcurrent of T and $\mathbb{M}(S) \geq c$. On the other hand, since $\eta'_\nu \rightharpoonup 0$,

$$\partial S_\nu = \eta'_\nu(1) - \eta'_\nu(0) \rightharpoonup 0$$

weakly in the sense of measures as $\nu \to \infty$, hence $\partial S = 0$ and, by acyclicity of T, one gets $S = 0$, giving a contradiction. Hence, the claim (20) is proven.

Fix now an arbitrary regular 1-form ω, and for each $\theta \in \Theta$, $k \in \mathbb{N}$ set

$$f_k(\theta) := \begin{cases} [\![\theta]\!](\omega), & \ell(\theta) \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

One gets

$$\left| \int_\Theta [\![\theta]\!](\omega)\, d\eta_\nu(\theta) - \int_\Theta f_k(\theta)\, d\eta_\nu(\theta) \right| = \left| \int_{\{\ell(\theta)>k\}} [\![\theta]\!](\omega)\, d\eta_\nu(\theta) \right|$$
$$\leq \|\omega\|_\infty \int_{\{\ell(\theta)>k\}} \mathbb{M}([\![\theta]\!])\, d\eta_\nu(\theta) \leq \|\omega\|_\infty \int_{\{\ell(\theta)>k\}} \ell(\theta)\, d\eta_\nu(\theta) = \|\omega\|_\infty \phi(k).$$

Minding that for each k fixed, by Lemma 6.1 the map f_k in Θ, one has

$$\int_\Theta f_k(\theta)\, d\eta_\nu(\theta) \to \int_\Theta f_k(\theta)\, d\eta(\theta),$$

as $\nu \to \infty$, and we arrive at the estimate

$$\int_\Theta f_k(\theta)\, d\eta(\theta) - \|\omega\|_\infty \phi(k) \leq \liminf_\nu \int_\Theta [\![\theta]\!](\omega)\, d\eta_\nu(\theta) \leq \limsup_\nu \int_\Theta [\![\theta]\!](\omega)\, d\eta_\nu(\theta)$$
$$\leq \int_\Theta f_k(\theta)\, d\eta(\theta) + \|\omega\|_\infty \phi(k).$$

Letting $k \to +\infty$ in the above estimate and taking into account (20), we get

$$T_\nu(\omega) \to \sup_k \int_\Theta f_k(\theta)\, d\eta(\theta) = \int_\Theta [\![\theta]\!](\omega)\, d\eta(\theta) = T_\eta(\omega)$$

as $\nu \to \infty$, which allows to conclude that $T = T_\eta$. □

It is worth remarking that the requirement of acyclicity of the "limit current" T in (ii) of the above Lemma 6.7 is essential as shown in the example below.

Example 2. *Consider the sequence of curves in $\mathbb{R}^2$ admitting the parameterization $\theta_\nu(t) := (1 + t/\nu)(\cos(2\pi\nu t), \sin(2\pi\nu t))$, $t \in [0,1]$, and define $\eta_\nu := \frac{1}{\nu}\delta_{\theta_\nu}$ be the transport concentrated on $\theta_\nu \in \Theta$ and having total mass $1/\nu$. Define also $\bar\theta(t) := (\cos(2\pi t), \sin(2\pi t))$ and let $\eta := \delta_{\bar\theta}$ be the transport concentrated on $\bar\theta$ with unit total mass. Clearly $\eta_\nu \rightharpoonup 0$ in the narrow sense of measures as $\nu \to \infty$ (in fact, $\eta_\nu(\Theta) = 1/\nu$). On the other hand, $T_{\eta_\nu} \overset{\mathcal{F}}{\rightharpoonup} T_\eta \neq 0$ as $\nu \to \infty$. However, this is not in contradiction with the above Lemma 6.7 because clearly $\partial T_\eta = 0$, i.e. T_η is a cycle.*

We concentrate now our attention on the restriction to a given Borel set of the currents of the form $T = T_\eta$.

Proposition 6.8. *Let T be a normal one-dimensional current and η be such a transport that $T = T_\eta$ and*

$$\mathbb{M}(T) = \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta). \tag{21}$$

Then $\mu_T = \mu_{[\![\theta]\!]} \otimes \eta$, i.e.

$$\mu_T(e) = \int_\Theta \mathbb{M}([\![\theta]\!] \llcorner e)\, d\eta(\theta), \tag{22}$$

and, moreover,

$$T\llcorner e(\omega) = \int_\Theta [\![\theta]\!]\llcorner e(\omega)\, d\eta(\theta), \tag{23}$$

for every Borel set $e \subset \mathbb{R}^n$, where

$$[\![\theta]\!]\llcorner e(\omega) = \int_{\theta^{-1}(e)} \langle \dot\theta(t), \omega(\theta(t))\rangle\, dt$$

(note that the latter integral is independent on the choice of parameterization of θ).

Remark 6.9. The relationship (21) implies also that for every Borel $e \subset \Theta$ the current $S := T_{\eta\llcorner e}$ is a subcurrent of T. In fact, in this case $T - S = T_{\eta\llcorner e^c}$, where $e^c := \Theta \setminus e$, and thus, by Theorem 6.2

$$\begin{aligned} \mathbb{M}(S) &\leq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta\llcorner e(\theta), \\ \mathbb{M}(T-S) &\leq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta\llcorner e^c(\theta). \end{aligned}$$

Hence, summing the above inequalities, one gets

$$\mathbb{M}(S) + \mathbb{M}(T-S) \leq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta) = \mathbb{M}(T).$$

Proof. The claim (22) follows immediately since by Theorem 6.2 one has

$$\mu_T(e) \leq \int_\Theta \mathbb{M}([\![\theta]\!]\llcorner e)\, d\eta(\theta)$$

for every Borel set $e \subset \mathbb{R}^n$, while according to (21) the latter estimate becomes an equality for $e := \mathbb{R}^n$.

Since $\mu_T = \mu_{[\![\theta]\!]} \otimes \eta$, then $f_\nu \to g$ in $L^1(\mu_T)$ as $\nu \to \infty$, implies $f_\nu \to g$ in $L^1(\mu_{[\![\theta]\!]})$ for η-a.e. $\theta \in \Theta$. We use this observation to prove the last claim (23). For this purpose let $\{f_\nu\}$ be a sequence of smooth functions which converge to 1_e in $L^1(\mu_T)$ as $\nu \to \infty$. Since μ_T has finite total mass, $T\llcorner e(\omega) := \lim_\nu T(f_\nu \wedge \omega)$. But then

$$\begin{aligned} T(f_\nu \wedge \omega) &= \int_\Theta [\![\theta]\!](f_\nu \wedge \omega)\, d\eta(\theta) \\ &= \int_\Theta \left(\int_\theta f_\nu(\xi) \langle \omega(\xi), \tau_\theta(\xi)\rangle\, d\mu_{[\![\theta]\!]}(\xi) \right) d\eta(\theta). \end{aligned}$$

As we just observed, for η-a.e. θ one has then

$$\int_\theta f_\nu(\xi)\langle \omega(\xi), \tau_\theta(\xi)\rangle d\mu_{[\![\theta]\!]}(\xi) \to \int_{\theta\cap e} \langle \omega(\xi), \tau_\theta(\xi)\rangle\, d\mu_{[\![\theta]\!]}(\xi) =: [\![\theta]\!]\llcorner e(\omega).$$

as $\nu \to \infty$. Moreover,

$$\left| \int_\theta f_\nu(\xi)\langle \omega(\xi), \tau_\theta(\xi)\rangle\, d\mu_{[\![\theta]\!]}(\xi) \right| \leq \|\omega\|_\infty \left| \int_\theta f_\nu\, d\mu_{[\![\theta]\!]} \right|$$

and for η-a.e. θ one has

$$\int_\theta f_\nu\, d\mu_{[\![\theta]\!]} \to \mu_{[\![\theta]\!]}(e)$$

as $\nu \to \infty$. Hence,

$$\left| \int_\theta f_\nu(\xi)\langle \omega(\xi), \tau_\theta(\xi)\rangle\, d\mu_{[\![\theta]\!]} \right| \leq 2\|\omega\|_\infty \mu_{[\![\theta]\!]}(e).$$

Since

$$\int_\Theta \mu_{[\![\theta]\!]}(e)\, d\eta(\theta) \leq \int_\Theta \mathbb{M}([\![\theta]\!])\, d\eta(\theta) \leq \mathbb{M}(T),$$

one also has that the functions

$$\theta \mapsto \left| \int_\theta f_\nu \langle \omega, \tau_\theta \rangle \, d\mu_{[\![\theta]\!]} \right|$$

is bounded by a function in $L^1(\eta)$. Hence by the Lebesgue convergence theorem, we obtain the desired result (23). □

7. Mass estimates

We first announce the following technical lemma which is practically contained in the proof of the rectifiability theorem for currents.

Lemma 7.1. *Let T be a k-dimensional real flat chain with finite mass $\mathbb{M}(T) < \infty$, and set*

$$\theta_T(x) := \Theta_k^*(\mu_T, x), \qquad \Sigma_T := \{x \in \mathbb{R}^n \colon 0 < \theta_T(x) < +\infty\}.$$

Then Σ_T is countably $(\mathcal{H}^k, k)$-rectifiable, and for $\mathcal{H}^k$-a.e. $x \in \Sigma_T$ one has

$$\Theta_k^*(\mu_T, x) = \Theta_{k*}(\mu_T, x),$$

while

$$\mu_T \llcorner \Sigma = \theta_T \mathcal{H}^k \llcorner (\Sigma_T \cap \Sigma)$$

for every countably $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset \mathbb{R}^n$.

Proof. We first claim

$$\mathcal{H}^k \llcorner \Sigma_T \ll \varphi := \mu_T \llcorner \Sigma_T.$$

In fact, if $e \subset \Sigma_T$, then

$$e = \bigcup_{j=1}^\infty e^j, \text{ where } e^j := \left\{ x \in e \colon \Theta_k^*(\mu_T, x) \geq \frac{1}{j} \right\},$$

and thus $\mu_T(e) = 0$ implies by [2, Theorem 2.56] the estimate

$$\mathcal{H}^k(e^j) \leq j\mu_T(e^j) \leq j\mu_T(e) = 0,$$

hence, $\mathcal{H}^k(e) = 0$ proving the announced claim.

Assume now that $e \subset \Sigma_T$ is purely $(\mathcal{H}^k, k)$-unrectifiable. Then $\mu_T(e) = 0$ by theorem 3.1 from [21], and hence $\mathcal{H}^k(e) = 0$, which proves the countable $(\mathcal{H}^k, k)$-rectifiability of Σ_T.

Observe now that $\varphi = 1_{\Sigma_T} \mu_T$, and hence

$$\frac{\varphi(B_r(x))}{\mu_T(B_r(x))} \to 1$$

as $r \to 0^+$ for μ_T-a.e. $x \in \Sigma_T$, and hence also for $\mathcal{H}^k$-a.e. $x \in \Sigma_T$. Minding that

$$\frac{\varphi(B_r(x))}{\omega_k r^k} = \frac{\varphi(B_r(x))}{\mu_T(B_r(x))} \cdot \frac{\mu_T(B_r(x))}{\omega_k r^k},$$

we get

$$\Theta_k^*(\varphi, x) = \Theta_k^*(\mu_T, x) \text{ and } \Theta_{k*}(\varphi, x) = \Theta_{k*}(\mu_T, x) \tag{24}$$

for $\mathcal{H}^k$-a.e. $x \in \Sigma_T$.

We now claim

$$\varphi \ll \mathcal{H}^k \llcorner \Sigma_T. \tag{25}$$

In fact, if $e \subset \Sigma_T$, then

$$e = \bigcup_{j=1}^\infty e_j, \text{ where } e_j := \{x \in e \colon \Theta_k^*(\mu_T, x) \leq j\}.$$

Hence, $\mathcal{H}^k(e) = 0$ implies by [2, Theorem 2.56] the estimate

$$\mu_T(e_j) \leq 2^k j \mathcal{H}^k(e_j) \leq 2^k j \mathcal{H}^k(e) = 0,$$

and therefore, $\mu_T(e) = 0$ proving $\varphi \ll \mathcal{H}^k \llcorner \Sigma_T$.

Minding now that (25) implies $\Theta_k^*(\varphi, x) = \Theta_{k*}(\varphi, x)$ for $\mathcal{H}^k$-a.e. $x \in \Sigma_T$, we get from (24) that

$$\Theta_k^*(\mu_T, x) = \Theta_{k*}(\mu_T, x)$$

for $\mathcal{H}^k$-a.e. $x \in \Sigma_T$.

Finally, to show the last claim of the statement being proven, it is enough to prove it for an arbitrary countably $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset \mathbb{R}^n$ satisfying $\mathcal{H}^k(\Sigma) < +\infty$. Clearly,

$$\mu_T(\Sigma \cap \Sigma_T) = \varphi(\Sigma) = \int_{\Sigma_T} \theta_T \, d\mathcal{H}^k. \tag{26}$$

We write now

$$\mu_T(\Sigma \setminus \Sigma_T) = \mu_T(\Sigma \cap \{\Theta_k^*(\mu_T, x) = +\infty\}) + \mu_T(\Sigma \cap \{\Theta_k^*(\mu_T, x) = 0\}).$$

But

$$\begin{aligned} \mathcal{H}^k\big(\{\Theta_k^*(\mu_T, x) = +\infty\}\big) &= \mathcal{H}^k\left(\bigcap_{j=1}^{\infty} \{\Theta_k^*(\mu_T, x) \geq j\}\right) \\ &= \inf_j \mathcal{H}^k\left(\{\Theta_k^*(\mu_T, x) \geq j\}\right) \leq \inf_j \frac{1}{j} \mu_T\left(\{\Theta_k^*(\mu_T, x) \geq j\}\right) \\ &\leq \inf_j \mu_T(\mathbb{R}^n)/j = 0, \end{aligned}$$

hence, $\mu_T\{\Theta_k^*(\mu_T, x) = +\infty\})) = 0$ by (25). On the other hand,

$$\begin{aligned} \mu_T(\Sigma \cap \{\Theta_k^*(\mu_T, x) = 0\}) &= \mu_T\left(\bigcap_{j=1}^{\infty} \{x \in \Sigma \colon \Theta_k^*(\mu_T, x) \leq 1/j\}\right) \\ &= \inf_j \mu_T\left(\{x \in \Sigma \colon \Theta_k^*(\mu_T, x) \leq 1/j\}\right) \\ &\leq \inf_j \frac{2^k}{j} \mathcal{H}^k(\Sigma) = 0. \end{aligned}$$

Putting the above estimates together, we get $\mu_T(\Sigma \cap \Sigma_T) = 0$, which together with (26) concludes the proof of the last claim. □

Given a transport η, we define the *transiting mass* function $a_\eta \colon \mathbb{R}^n \to \mathbb{R}$ by setting

$$a_\eta(x) := \eta(\{\theta \in \Theta : x \in \Theta\}).$$

In other words, $a_\eta(x)$ measures the number of people passing through the point $x \in \mathbb{R}^n$. We may now announce the following result.

Lemma 7.2. *Let T be a one-dimensional real flat chain with compact support and finite mass $\mathbb{M}(T) < +\infty$. Let η be given by Theorem 6.3 and let θ_T be defined as in Lemma 7.1. Then*

(i) *a_η is u.s.c.;*
(ii) *if T is acyclic, then $\theta_T(x) = a_\eta(x)$ for $\mathcal{H}^1$-a.e. $x \in \mathbb{R}^n$.*

Proof. For each $x \in \mathbb{R}^n$ define the function $1_x \colon \Theta \to \mathbb{R}$ by the formula

$$1_x(\sigma) := \begin{cases} 1, & \text{if } x \in \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, 1_x is u.s.c. To prove (i), it is enough therefore to observe that

$$a_\eta(x) = \int_\Theta 1_x(\sigma) \, d\eta(\sigma)$$

and to apply Fatou's lemma.

To prove (ii), it is enough to show that for an acyclic T one has

(A) for each countably $(\mathcal{H}^1, 1)$-rectifiable set $\Sigma \subset \mathbb{R}^n$ one has $\theta_T(x) = a_\eta(x)$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$;
(B) $\theta_T(x) \geq a_\eta(x)/2$ for $\mathcal{H}^1$-a.e. $x \in \mathbb{R}^n$.

In fact, the set Σ_T is countably $(\mathcal{H}^1, 1)$-rectifiable by Lemma 7.1, and hence $\theta_T(x) = a_\eta(x)$ for $\mathcal{H}^1$-a.e. $x \in \Sigma_T$ by (A). On the other hand, for $\mathcal{H}^1$-a.e. $x \notin \Sigma_T$ one has $\theta_T(x) = 0$, and hence $a_\eta(x) = 0$ by (B), which shows (ii).

We thus prove now (A) and (B). To show the validity of (A), recall that due to Theorem 6.3, minding that η-a.e. $\sigma' \in \Theta$ is an arc, one has for all $\sigma \in \Theta$ the following equalities

$$\begin{aligned}
\mu_T(\sigma) &= \int_\Theta \mathbb{M}([\![\sigma']\!] \llcorner \sigma)\, d\eta(\sigma') = \int_\Theta \mathcal{H}^1(\sigma' \cap \sigma)\, d\eta(\sigma') \\
&= \int_\Theta \left(\int_\sigma 1_x(\sigma')\, d\mathcal{H}^1(x) \right) d\eta(\sigma') = \int_\sigma \left(\int_\Theta 1_x(\sigma')\, d\eta(\sigma') \right) d\mathcal{H}^1(x) \\
&= \int_\sigma a_\eta\, d\mathcal{H}^1.
\end{aligned}$$

On the other hand, by Lemma 7.1,

$$\mu_T(\sigma) = \int_\sigma \theta_T\, d\mathcal{H}^1,$$

which proves

$$\int_\sigma \theta_T\, d\mathcal{H}^1 = \int_\sigma a_\eta\, d\mathcal{H}^1$$

for every Lipschitz curve σ. The latter clearly implies (A).

To prove (B) let a point $x \in \mathbb{R}^n$ be fixed and let $\varepsilon > 0$. Consider the sets

$$\begin{aligned}
A(x) &:= \{\sigma \in \Theta \colon x \in \sigma\}, \\
A_\rho(x) &:= \{\sigma \in A(x) \colon \mathbb{M}([\![\sigma]\!] \llcorner B_\rho(x)) \geq \rho\}, \\
A'_\rho(x) &:= A(x) \setminus A_\rho(x), \\
C^+_\rho(x) &:= \{\sigma \in \Theta \colon \sigma(0) \in B_\rho(x)\}, \\
C^-_\rho(x) &:= \{\sigma \in \Theta \colon \sigma(1) \in B_\rho(x)\}, \\
C_\rho(x) &:= C^+_\rho(x) \cap C^-_\rho(x).
\end{aligned}$$

One has $\eta(C^\pm_\rho(x)) = (t_{0,1})_\# \eta(B_\rho(x)) = (\partial T)^\pm(B_\rho(x))$ and hence $\eta(C_\rho(x)) \leq (\partial T)^+(B_\rho(x)) \wedge (\partial T)^-(B_\rho(x)) \to 0$ as $\rho \to 0^+$. In particular there exists $\delta > 0$ such that for every $\rho < \delta$ one has $\eta(C_\rho(x)) < \varepsilon$.

Notice that in view of Theorem 6.3, η-a.e. $\sigma \in \Theta$ is an arc. Hence for η-a.e. $\sigma \in A(x)$ if either $\sigma(0) \notin B_\rho(x)$ or $\sigma(1) \notin B_\rho(x)$ one has $\mathbb{M}([\![\sigma]\!] \llcorner B_\rho(x)) \geq \rho$. This proves that $\eta(A'_\rho(x) \setminus C_\rho(x)) = 0$. Hence $\eta(A'_\rho(x)) \leq \varepsilon$ and, consequently, $\eta(A_\rho(x)) = \eta(A(x)) - \eta(A'_\rho(x)) \geq a_\eta(x) - \varepsilon$.

To conclude, note that

$$\begin{aligned}
\frac{\mathbb{M}(T \llcorner B_\rho(x))}{2\rho} &= \frac{1}{2\rho} \int_\Theta \mathbb{M}([\![\sigma]\!] \llcorner B_\rho(x))\, d\eta(\sigma) \\
&\geq \frac{1}{2\rho} \int_{A_\rho(x)} \mathbb{M}([\![\sigma]\!] \llcorner B_\rho(x))\, d\eta(\sigma) \\
&\geq \frac{\rho(a_\eta(x) - \varepsilon)}{2\rho} = (a_\eta(x) - \varepsilon)/2,
\end{aligned}$$

so that for $\mathcal{H}^1$-a.e. $x \in \mathbb{R}^n$ one has $\theta_T(x) \geq (a_\eta(x) - \varepsilon)/2$ and since this is true for every $\varepsilon > 0$, the conclusion (B) follows. □

Theorem 7.3. *If T is an acyclic one-dimensional normal current, then*

$$\theta_T(x) \leq \frac{1}{2}\mathbb{M}(\partial T)$$

for $\mathcal{H}^1$-a.e. $x \in \mathbb{R}^n$.

Proof. If T is an acyclic normal current, then from Theorem 6.3 one has $T = T_\eta$ for some transport η such that $\eta(0) = (\partial T)^+$. Then, minding that Σ_T is countably $(\mathcal{H}^1, 1)$-rectifiable by Lemma 7.1, we get from Lemma 7.2(ii) that $\theta_T(x) = a_\eta(x)$ for $\mathcal{H}^1$-a.e. $x \in \Sigma_T$. But

$$a_\eta(x) \leq \eta(\Theta) \leq \eta(0)(\mathbb{R}^n) = \frac{1}{2}\mathbb{M}(\partial T),$$

and thus $\theta_T(x) \leq \mathbb{M}(\partial T)/2$ for $\mathcal{H}^1$-a.e. $x \in \Sigma_T$. On the other hand, $\mathcal{H}^1$-a.e. on $\mathbb{R}^n \setminus \Sigma_T$ one has $\theta_T = 0$ (since it has been shown in the proof of Lemma 7.1 that $\mathcal{H}^1(\{\theta_T = +\infty\}) = 0$), which concludes the proof. □

The assertion below may be regarded as a version of the Sobolev-Poincaré inequality for one-dimensional real flat chains.

Theorem 7.4. *Let T be an acyclic one-dimensional real flat chain of finite mass $\mathbb{M}(T) < +\infty$ and assume that $S \leq T$. Then*

$$\mathbb{M}^\beta(S) \leq \frac{1}{2^{\beta-\alpha}}\mathbb{M}^\alpha(S)\mathbb{M}(\partial T)^{\beta-\alpha}$$

for all $\alpha \in [0,1]$, $\beta \in [\alpha, 1]$. In particular,

$$\mathbb{M}(S) \leq \frac{1}{2^{1-\alpha}}\mathbb{M}^\alpha(S)\mathbb{M}(\partial T)^{1-\alpha}$$

for every $\alpha \in [0,1]$.

Proof. By Theorem 7.3 the claim is easily proven when S is *rectifiable* and T is normal. In fact, in this case one may consider $\Sigma_S \subset \Sigma_T$, $\theta_S \leq \theta_T$. Therefore

$$\begin{aligned}
\mathbb{M}^\beta(S) &= \int_{\Sigma_S} \theta_S^\beta \, d\mathcal{H}^1 = (\mathbb{M}(\partial T)/2)^\beta \int_{\Sigma_S} \left(\frac{\theta_S}{\mathbb{M}(\partial T)/2}\right)^\beta d\mathcal{H}^1 \\
&\leq (\mathbb{M}(\partial T)/2)^\beta \int_{\Sigma_S} \left(\frac{\theta_S}{\mathbb{M}(\partial T)/2}\right)^\alpha d\mathcal{H}^1 \\
&= \frac{1}{2^{\beta-\alpha}}\mathbb{M}^\alpha(S)\mathbb{M}(\partial T)^{\beta-\alpha}.
\end{aligned}$$

To prove the claim in the general case, it is enough to note that we may assume $\alpha < 1$ (otherwise there is nothing to prove), and then the hypothesis $\mathbb{M}(T) < +\infty$ provides $\mathbb{M}(S) < +\infty$, and hence, by Theorem 2.1, S is rectifiable. One may suppose also $\mathbb{M}(\partial T) < +\infty$ (otherwise there is still nothing to prove), which guarantees that T is normal. Hence the validity of the inequality being proven follows. □

8. Existence of solutions

To illustrate the developed technique we prove the existence of solutions to the announced Problem 1 in an important particular case when the function $H(\cdot)$ is concave.

Theorem 8.1. *Let $\varphi^\pm$ be finite nonnegative Borel measures with compact support in $\mathbb{R}^n$, satisfying $\varphi^+(\mathbb{R}^n) = \varphi^-(\mathbb{R}^n)$. Assume also the function H to be concave, while $H(l) \to +\infty$ as $l \to +\infty$, $A > 0$, $\alpha < 1$, and either $\alpha > \beta \vee \delta$, or $\alpha = \beta > \delta$, but $A > B$. Then the functional $\mathfrak{F}$ attains its minimum value on the set of pairs of one-dimensional real flat chains of finite mass (T, S), which satisfy* (3)*. In other words, Problem 1 in this case admits solutions.*

Proof. Assume the existence of such a pair of one-dimensional real flat chains of finite mass (T_0, S_0), that $\partial(T_0 + S_0) = \varphi^+ - \varphi^-$ and

$$\mathfrak{F}(T_0, S_0) < +\infty$$

(otherwise $\mathfrak{F} \equiv +\infty$, and hence, there is nothing to prove). Here and below for the sake of brevity we denote $\phi := \varphi^+ - \varphi^-$.

We may also assume that $H \not\equiv +\infty$ over $(0, +\infty)$. In fact, in the opposite case Problem 1 admits a trivial solution $(T, 0)$, where T is a real flat chain minimizing $\mathbb{M}^\alpha$ among all one-dimensional real flat chains of finite mass satisfying $\partial T = \phi$.

We divide the proof in several steps.

Step 1. We first show the existence of a minimizing sequence $\{(T_\nu, S_\nu)\}$ for the functional $\mathfrak{F}$ of pairs of real rectifiable currents, which satisfy condition (1), have uniformly bounded masses and also satisfy condition $\mu_{T_\nu} \wedge \mu_{S_\nu} = 0$.

Let $\{(T'_\nu, S'_\nu)\}$ be an arbitrary minimizing sequence for the functional $\mathfrak{F}$, which satisfies (1). Then $\mathfrak{F}(T'_\nu, S'_\nu) < +\infty$. In view of the assumption on H we have $\mathbb{M}^\delta(S'_\nu) < +\infty$, and hence, one may apply Theorem 10.2, which gives rectifiability of S'_ν. Since $\mathbb{M}^\alpha(T'_\nu) < +\infty$, then according to the same theorem also T'_ν is rectifiable. In view of Lemma 4.6 we may assume without loss of generality that $\mu_{T'_\nu} \wedge \mu_{S'_\nu} = 0$. In other words, for every $\nu \in \mathbb{N}$ there is such a Borel set $E_\nu \subset \mathbb{R}^n$, that

$$T'_\nu = T'_\nu \llcorner E_\nu \qquad\qquad S'_\nu = S'_\nu \llcorner (\mathbb{R}^n \setminus E_\nu).$$

According to Proposition 3.12 there is such a cycle $C_\nu \leq T'_\nu + S'_\nu$, that the current $T'_\nu + S'_\nu - C_\nu$ is acyclic. Setting

$$T_\nu := T'_\nu - C_\nu \llcorner E_\nu, \qquad\qquad S_\nu := S'_\nu - C_\nu \llcorner (\mathbb{R}^n \setminus E_\nu),$$

we get $T_\nu + S_\nu = T'_\nu + S'_\nu - C_\nu$, and hence,

$$\partial(T_\nu + S_\nu) = \partial(T'_\nu + S'_\nu) = \phi.$$

On the other hand, from $C_\nu \leq T'_\nu + S'_\nu$ one gets

$$C_\nu \llcorner E_\nu \leq (T'_\nu + S'_\nu) \llcorner E_\nu = T'_\nu.$$

Due to Remark 3.6, one has $T_\nu \leq T'_\nu$. Analogously, $S_\nu \leq S'_\nu$, and hence, applying Lemma 3.7, we get $\mathfrak{F}(T_\nu, S_\nu) \leq \mathfrak{F}(T'_\nu, S'_\nu)$, i.e. $\{(T_\nu, S_\nu)\}$ is still a minimizing sequence for the Problem 1. Let $R_\nu := T_\nu + S_\nu$. In view of acyclicity of R_ν we may apply Theorem 7.4 to get

$$\mathbb{M}(T_\nu) \leq \frac{1}{2^{1-\alpha}} \mathbb{M}^\alpha(T_\nu) \mathbb{M}(\partial R_\nu)^{1-\alpha} \leq \frac{1}{2^{1-\alpha}} \mathbb{M}^\alpha(T'_\nu) |\phi|(\mathbb{R}^n)^{1-\alpha}, \tag{27}$$

minding that $\mathbb{M}^\alpha(T_\nu) \leq \mathbb{M}^\alpha(T'_\nu)$ by Lemma 3.7, since $T_\nu \leq T'_\nu$, and that $\mathbb{M}(\partial R_\nu) = |\phi|(\mathbb{R}^n)$. In the same way we get the estimate

$$\mathbb{M}(S_\nu) \leq \frac{1}{2^{1-\delta}} \mathbb{M}^\delta(S'_\nu) |\phi|(\mathbb{R}^n)^{1-\delta}. \tag{28}$$

On the other hand, since $\mathfrak{F}(T_\nu, S_\nu) \leq \mathfrak{F}(T_0, S_0)$, then for some $C' > 0$ and for all $\nu \in \mathbb{N}$ the estimates

$$\mathbb{M}^\alpha(T'_\nu) \leq C', \qquad \mathbb{M}^\delta(S'_\nu) \leq C'$$

hol, because $A > 0$ and the functon H is unbounded. Combining the above estimates with (27) and (28), we conclude that both T_ν and S_ν, and hence also R_ν hav uniformly bounded masses $\mathbb{M}$.

Step 2. For every $d \geq 0$ and every real flat chain R we let

$$S_d(R) := R \llcorner (\Sigma_R \cap \{\theta_R \geq d\}), \qquad\qquad T_d(R) := R - S_d(R).$$

If $C \in [H'_+(\mathbb{M}^\delta(S)), H'_-(\mathbb{M}^\delta(S))]$, where $H'_\pm$ are the left and the right derivatives of the function H respectively, and $l \geq 0$, we denote also

$$F[l, C](R) := \mathbb{M}^\alpha(T_d(R)) + B\mathbb{M}^\beta(S_d(R)) + H(l) + C(\mathbb{M}^\delta(S_d(R)) - l).$$

Let $d \geq 0$ be such that $At^\alpha > Bt^\beta + Ct^\delta$ when $t \in (d, +\infty)$, and $At^\alpha < Bt^\beta + Ct^\delta$ when $t \in (0, d)$ (here and below we assume $(0, d) := \emptyset$, if $d = 0$). Then clearly

$$\mathfrak{F}(T_d(R), S_d(R)) = F[\mathbb{M}^\delta(S_d(R)), C](R), \tag{29}$$

while

$$\mathfrak{F}(T_d(R), S_d(R)) \leq F[l, C](R) \tag{30}$$

for all $l \geq 0$.

Consider a minimizing sequence for the functional $\mathfrak{F}$ constructed on Step 1. Note that since ϕ is concentrated on some ball of $\mathbb{R}^n$ then one may consider without loss of generality all T_ν and S_ν (and hence also R_ν) to be concentrated on the same ball (otherwise, projecting the latter curents to this ball will not change the boundary of their sum while not increasing any of the masses $\mathbb{M}^\lambda$, $\lambda \in [0, 1]$, and hence, not increasing the value of $\mathfrak{F}$). Since the masses $\mathbb{M}(R_\nu)$ are uniformly bounded, while $\partial R_\nu = \phi$, then up to a subsequence (not relabeled) $R_\nu \overset{\mathcal{F}}{\rightharpoonup} R$ as $\nu \to \infty$ in flat norm to some real one-dimensional flat chain R satisfying $\partial R = \phi$.

Set $l_\nu := \mathbb{M}^\delta(S_\nu)$, and choose an arbitrary $C_\nu \in [H'_+(l_\nu), H'_-(l_\nu)]$. Observe that $C_\nu \neq 0$ due to the assumption on H. Without loss of generality we may assume that up to a subsequence (again not relabeled) $l_\nu \to l$ and $C_\nu \to C$ to some $l \in [0, +\infty]$ and $C \in [0, +\infty]$ as $\nu \to \infty$. We consider separately two possible situations.

Case $l > 0$. Then $C < +\infty$.

Note that the numbers $l_\nu := \mathbb{M}^\delta(S_\nu)$ are uniformly bounded, since otherwise up to a subsequence (not relabeled), $H(l_\nu) \to +\infty$, and hence, $\mathfrak{F}(T_\nu, S_\nu) \to +\infty$ as $\nu \to \infty$, contrary to the estimate $\mathfrak{F}(T_\nu, S_\nu) \leq \mathfrak{F}(T_0, S_0) < +\infty$ for all sufficiently large ν (because the sequence $\{(T_\nu, S_\nu)\}$ is minimizing). Therefore, $l < +\infty$, which also implies that $C > 0$ (otherwise $H'_+(l) = 0$, which would mean, in view of concavity of H, that $H(t) = H(l)$ for all $t \geq l$, contrary to the assumption on unboundedness of H). We remark finally that $C \in [H'_+(l), H'_-(l)]$, since the functions $H'_\pm$, are lower and upper semicontinuous respectively.

In view of Lemma 10.5 we may assume without loss of generality that

$$\begin{aligned} \theta_{S_\nu}(x) = \theta_{R_\nu}(x) \geq d_\nu \qquad & \text{for } \mathcal{H}^1 - \text{a.e. } x \in \Sigma_{S_\nu}, \\ \theta_{T_\nu}(x) = \theta_{R_\nu}(x) < d_\nu \qquad & \text{for } \mathcal{H}^1 - \text{a.e. } x \in \Sigma_{T_\nu}, \end{aligned}$$

where the numbers $d_\nu > 0$ deend only on α, β, δ, A, B and C_ν and satisfy the relationships

$$\begin{aligned} At^\alpha < Bt^\beta + C_\nu t^\delta, \qquad & t \in (0, d_\nu), \\ At^\alpha > Bt^\beta + C_\nu t^\delta, \qquad & t \in (d_\nu, +\infty). \end{aligned} \tag{31}$$

In other words, in view of rectifiability of S_ν one has $S_\nu = S_{d_\nu}(R_\nu)$, and hence, $T_\nu = T_{d_\nu}(R_\nu)$. Thus, by (29), the equality

$$\liminf_\nu \mathfrak{F}(T_\nu, S_\nu) = \liminf_\nu \mathfrak{F}(T_{d_\nu}(R_\nu), S_{d_\nu}(R_\nu)) = \liminf_\nu F[l_\nu, C_\nu](R_\nu) \tag{32}$$

holds.

In view of rectifiability of T_ν, S_ν and R_ν, and because $\mu_{T_\nu} \wedge \mu_{S_\nu} = 0$, one can write

$$\begin{aligned} F[l_\nu, C_\nu](R_\nu) = & \int_{\Sigma_{R_\nu} \cap \{\theta_{R_\nu}(x) < d_\nu\}} A\theta^\alpha_{R_\nu}(x)\, d\mathcal{H}^1(x) + \\ & \int_{\Sigma_{R_\nu} \cap \{\theta_{R_\nu}(x) \geq d_\nu\}} \left(B\theta^\beta_{R_\nu}(x) + C_\nu \theta^\delta_{R_\nu}(x) \right) d\mathcal{H}^1(x) + \\ & H(l_\nu) - C_\nu l_\nu \\ = & \int_{\Sigma_{R_\nu}} g_\nu(\theta_{R_\nu}(x))\, d\mathcal{H}^1(x) + H(l_\nu) - C_\nu l_\nu, \end{aligned} \tag{33}$$

where

$$g_\nu(t) := (At^\alpha) \wedge (Bt^\beta + C_\nu t^\delta),$$

since $f_\nu(t) := At^\alpha - Bt^\beta - C_\nu t^\delta > 0$ when $t > d_\nu$, and $f_\nu(t) < 0$ when $t \in (0, d_\nu)$.

For all $\varepsilon > 0$ and for all sufficiently large $\nu \in \mathbb{N}$ one has

$$g_\nu(t) \geq g^\varepsilon(t) := (At^\alpha) \wedge (Bt^\beta + (C-\varepsilon)t^\delta),$$

for all $t \geq 0$. Therefore,

$$\liminf_\nu F[l_\nu, C_\nu](R_\nu) \geq \liminf_\nu \int_{\Sigma_{R_\nu}} g^\varepsilon(\theta_{R_\nu}(x))\, d\mathcal{H}^1(x) + H(l) - Cl, \tag{34}$$

Since for $\varepsilon \in [0, C)$ the function $g^\varepsilon\colon \mathbb{R}^+ \to \mathbb{R}^+$ is monotone nondecreasing and concave, and $g^\varepsilon(0) = 0$, then the functional

$$R \mapsto \int_{\Sigma_R} g^\varepsilon(\theta_R(x))\, d\mathcal{H}^1(x),$$

defined on rectifiable currents, defines an l.s.c. in the flat norm topology functional $\mathbb{M}^{g^\varepsilon}$ on the set of real flat chains according to the formula

$$\mathbb{M}^{g^\varepsilon}(R) := \inf\{\liminf_\nu \int_{\Sigma_{R_\nu}} g^\varepsilon(\theta_{R_\nu}(x))\, d\mathcal{H}^1(x)\},$$

where the infimum is taken over all the sequences of real polyhedral chains $\{R_\nu\}$ converging to R in the flat norm. Then

$$\mathbb{M}^{g^\varepsilon}(R) = \int_{\Sigma_R} g^\varepsilon(\theta_R(x))\, d\mathcal{H}^1(x),$$

if R is rectifiable [22]. By definition of $\mathbb{M}^{g^\varepsilon}$, one has therefore the inequality

$$\mathbb{M}^{g^\varepsilon}(R) \leq \liminf_\nu \int_{\Sigma_{R_\nu}} g^\varepsilon(\theta_{R_\nu}(x))\, d\mathcal{H}^1(x). \tag{35}$$

Minding (34), we get

$$\liminf_\nu F[l_\nu, C_\nu](R_\nu) \geq \mathbb{M}^{g^\varepsilon}(R) + H(l) - Cl. \tag{36}$$

From (36) and (32) one gets the inequality

$$\liminf_\nu \mathfrak{F}(T_\nu, S_\nu) \geq \mathbb{M}^{g^\varepsilon}(R) + H(l) - Cl, \tag{37}$$

which, in particular, implies $\mathbb{M}^{g^\varepsilon}(R) < +\infty$. Also,

$$\mathbb{M}(R) \leq \liminf_\nu \mathbb{M}(R_\nu) = \liminf_\nu (\mathbb{M}(T_\nu) + \mathbb{M}(S_\nu)) < +\infty,$$

since all T_ν and S_ν have uniformly bounded masses. Therefore, from the general theorem on rectifiability of flat chains [22], minding the definition of g^ε and that $\alpha < 1$, we get the rectifiability of R. Thus, $\mathbb{M}^{g^\varepsilon}(R) = \int_{\Sigma_R} g^\varepsilon(\theta_R(x))\, d\mathcal{H}^1(x)$, and hence, the inequality (37) can be rewritten as

$$\liminf_\nu \mathfrak{F}(T_\nu, S_\nu) \geq \int_{\Sigma_R} g^\varepsilon(\theta_R(x))\, d\mathcal{H}^1(x) + H(l) - Cl. \tag{38}$$

Observe that the limit $d := \lim_\nu d_\nu$ exists and satisfies

$$\begin{aligned} At^\alpha &< Bt^\beta + Ct^\delta, \qquad t \in (0, d),\\ At^\alpha &> Bt^\beta + Ct^\delta, \qquad t \in (d, +\infty). \end{aligned} \tag{39}$$

In fact, denoting by s a limit of an arbitrary subsequence d_ν (not relabeled), and passing to a limit in (39), we get

$$\begin{aligned} At^\alpha &< Bt^\beta + Ct^\delta, \qquad t \in (0, s),\\ At^\alpha &> Bt^\beta + Ct^\delta, \qquad t \in (s, +\infty), \end{aligned}$$

and hence $s = d$. Note also that $d > 0$ in view of Lemma 10.6.

Denote by d_ε such a number that

$$\begin{aligned} At^\alpha &< Bt^\beta + (C-\varepsilon)t^\delta, \qquad t\in(0,d_\varepsilon),\\ At^\alpha &> Bt^\beta + (C-\varepsilon)t^\delta, \qquad t\in(d_\varepsilon,+\infty). \end{aligned}$$

Clearly, $d_\varepsilon \le d$. Besides, $d_\varepsilon > 0$, if $\varepsilon\in[0,C)$, due to Lemma 10.6, while, as it has been just proven above, $d_\varepsilon \to d$ as $\varepsilon\to 0^+$. With the above notation

$$\begin{aligned} &\int_{\Sigma_R} g^\varepsilon(\theta_R(x))\, d\mathcal{H}^1(x) = \\ &\qquad A\mathbb{M}^\alpha(T_{d_\varepsilon}(R)) + B\mathbb{M}^\beta(S_{d_\varepsilon}(R)) + (C-\varepsilon)\mathbb{M}^\delta(S_{d_\varepsilon}(R)), \end{aligned} \tag{40}$$

and hence,

$$\begin{aligned} &\int_{\Sigma_R} g^\varepsilon(\theta_R(x))\, d\mathcal{H}^1(x) = \\ &\qquad A\mathbb{M}^\alpha(T_d(R)) + B\mathbb{M}^\beta(S_d(R)) + (C-\varepsilon)\mathbb{M}^\delta(S_d(R)) - \\ &\qquad A\mathbb{M}^\alpha(R\llcorner\Delta_\varepsilon) + B\mathbb{M}^\beta(R\llcorner\Delta_\varepsilon) + (C-\varepsilon)\mathbb{M}^\delta(R\llcorner\Delta_\varepsilon), \end{aligned} \tag{41}$$

where $\Delta_\varepsilon := \{x\in\mathbb{R}^n : d_\varepsilon \le \theta_R(x) < d\}$. Using (38), we get

$$\begin{aligned} &\liminf_\nu \mathfrak{F}(T_\nu,S_\nu) \ge \\ &\qquad F[l,C](R) - A\mathbb{M}^\alpha(R\llcorner\Delta_\varepsilon) + B\mathbb{M}^\beta(R\llcorner\Delta_\varepsilon) + \\ &\qquad (C-\varepsilon)\mathbb{M}^\delta(R\llcorner\Delta_\varepsilon) - \varepsilon\mathbb{M}^\delta(S_d(R)), \end{aligned} \tag{42}$$

because

$$F[l,C](R) = A\mathbb{M}^\alpha(T_d(R)) + B\mathbb{M}^\beta(S_d(R)) + C\mathbb{M}^\delta(S_d(R)) + H(l) - Cl.$$

At last, minding (30), we get from (42) the inequality

$$\begin{aligned} &\liminf_\nu \mathfrak{F}(T_\nu,S_\nu) \ge \\ &\quad \mathfrak{F}(T_d(R),S_d(R)) - A\mathbb{M}^\alpha(R\llcorner\Delta_\varepsilon) + B\mathbb{M}^\beta(R\llcorner\Delta_\varepsilon) + \\ &\quad (C-\varepsilon)\mathbb{M}^\delta(R\llcorner\Delta_\varepsilon) - \varepsilon\mathbb{M}^\delta(S_d(R)) = \\ &\quad \mathfrak{F}(T_d(R),S_d(R)) - \int_{\Delta_\varepsilon} (A\theta_R^\alpha(x) - B\theta_R^\beta(x) - C\theta_R^\delta(x))\, d\mathcal{H}^1(x) + \\ &\quad \varepsilon\int_{\{\theta_R(x)\ge d_\varepsilon\}} \theta_R^\delta(x)\, d\mathcal{H}^1(x) = \\ &\quad \mathfrak{F}(T_d(R),S_d(R)) - \int_{\Delta_\varepsilon} (A\theta_R^\alpha(x) - B\theta_R^\beta(x) - C\theta_R^\delta(x))\, d\mathcal{H}^1(x) + \\ &\quad \varepsilon\mathbb{M}^\delta(S_{d_\varepsilon}(R)). \end{aligned} \tag{43}$$

The estimates $\sup_{\varepsilon\in(0,C)} \mathbb{M}^{g^\varepsilon}(R) < +\infty$ and (40) imply

$$A\mathbb{M}^\alpha(T_d(R)) = \sup_{\varepsilon\in(0,C)} A\mathbb{M}^\alpha(T_{d_\varepsilon}(R)) < +\infty,$$

as well as $B\mathbb{M}^\beta(S_{d_\varepsilon}(R)) < +\infty$ and, since $C>0$, then also $\mathbb{M}^\delta(S_{d_\varepsilon}(R)) < +\infty$ for all $\varepsilon\in(0,C)$. Therefore, one may pass to a limit in $\varepsilon\to 0^+$ in (43), arriving at the inequality

$$\liminf_\nu \mathfrak{F}(T_\nu,S_\nu) \ge \mathfrak{F}(T_d(R),S_d(R)),$$

which shows that the pair $(T_d(R),S_d(R))$ is a minimizer of the functional $\mathfrak{F}$.

CASE $l=0$.

In other words, $l_\nu := \mathbb{M}^\delta(S_\nu) \to 0$ as $\nu\to\infty$. Since $S_\nu \le R_\nu$, $\partial R_\nu = \phi$, and all the currents R_ν are acyclic by construction, while $\delta\le\alpha$ according to the assumptions, then due to Theorem 7.4 $\mathbb{M}^\alpha(S_\nu)\to 0$ as $\nu\to\infty$. Thus

$$\begin{aligned} \liminf_\nu \mathfrak{F}(T_\nu,S_\nu) &= \liminf_\nu (A\mathbb{M}^\alpha(T_\nu) + B\mathbb{M}^\beta(S_\nu) + H(\mathbb{M}^\delta(S_\nu))) \\ &\ge \liminf_\nu (A\mathbb{M}^\alpha(T_\nu) + H(\mathbb{M}^\delta(S_\nu))) \\ &= \liminf_\nu (A\mathbb{M}^\alpha(R_\nu) + H(\mathbb{M}^\delta(S_\nu)) - A\mathbb{M}^\alpha(S_\nu)), \end{aligned}$$

and taking into account that

$$H(\mathbb{M}^\delta(S_\nu)) - A\mathbb{M}^\alpha(S_\nu) = H(l_\nu) - A\mathbb{M}^\alpha(S_\nu) \to 0$$

as $\nu \to \infty$, we get

$$\begin{array}{rcl}\liminf_\nu \mathfrak{F}(T_\nu, S_\nu) & \geq & \liminf_\nu A\mathbb{M}^\alpha(R_\nu) \geq A\mathbb{M}^\alpha(R) \\ & = & \mathfrak{F}(R, 0).\end{array}$$

Therefore, in this case the pair $(R, 0)$ is a minimizer of the functional $\mathfrak{F}$, which concludes the proof. □

9. Reduction to known problem formulations

In this section we consider a particular case of Problem 1 with $\alpha = \beta = 1$ and $\delta = 0$ and show that such a problem is equivalent to the classical problem of finding an optimal transportation network formulated without using the language of Federer-Fleming currents (such a formulation is studied in [9]).

Under the assumptions $\alpha = \beta = 1$ and $\delta = 0$ the Problem 1 can be stated in the following way.

Problem 2. *Find a couple of one-dimensional real flat chains* (T_{opt}, S_{opt}) *minimizing the functional* $\mathfrak{F}$ *defined by the formula*

$$\mathfrak{F}(T, S) = A\mathbb{M}(T) + B\mathbb{M}(S) + H(\mathbb{M}^0(S)),$$

among all the couples of real one-dimensional flat chains (T, S) *of finite mass, satisfying* (3).

We now define a new functional G over couples (η, Σ), where η is a transport (i.e. a nonnegative finite Borel measure over Θ) and $\Sigma \subset \mathbb{R}^n$ is a Borel set. Namely, we set

$$G(\eta, \Sigma) := \int_\Theta \left(A\mathcal{H}^1(\theta \setminus \Sigma) + B\mathcal{H}^1(\theta \cap \Sigma)\right) d\eta(\theta). \tag{44}$$

The meaning of $G(\eta, \Sigma)$ may be explained as follows. Suppose that a single citizen chooses a path $\theta \in \Theta$ in his everyday movement. Assume Σ to stand for the transportation network, so that for a citizen choosing the route θ the cost of using this network would be proportional to $\mathcal{H}^1(\theta \cap \Sigma)$ (i.e. to the length of the part of the route made with the help of the network) with coefficient $B \geq 0$. For the same citizen, moving without the use of the network by distance t, is assumed to cost At for a given $A \geq 0$. Therefore the integrand in (44) gives the individual cost of moving along the route θ. If the transport η describes the collective behaviour of the population, so that, heuristically, $\eta(\theta)$ gives the number of people choosing the route θ in their everyday movements, then $G(\eta, \Sigma)$ gives the total cost of transportation of the population to the services or workplaces. Clearly, for η to describe the pattern of behaviour of the population in the above sense, one has to require

$$\eta(0) = \varphi^+, \qquad \eta(1) = \varphi^-. \tag{45}$$

Clearly, the population as a whole chooses the way of transportation (i.e. the transport η) so as to minimize $G(\cdot, \Sigma)$ among all the transports satisfying (45) (called further *admissible transports*. In other words, the number

$$MK(\varphi^+, \varphi^-, \Sigma) := \inf \{G(\eta, \Sigma) \,:\, \eta \text{ transport satisfying (45)}\}$$

gives the cost of everyday movement of the population from their places of residence to workplaces and/or services.

We describe now another way of obtaining the same cost $MK(\varphi^+, \varphi^-, \Sigma)$ which is more used in the theory of optimal transportation. Namely, rather than using transports, it is a custom to describe the behaviour of population by so-called *transport plans*, i.e. by finite positive Borel measures γ over $\mathbb{R}^n \times \mathbb{R}^n$, so that, heuristically, $\gamma(x, y)$ gives the number of people moving from x to y. Mind that in this sense a transport plan γ gives much less information on the movement of the population than the transport η, namely, it says nothing about the routes people are choosing, but just describes source and destination points of the movement. Clearly, a transport plan γ has to satisfy the requirement on marginals

$$\pi^{\pm}_{\#}\gamma = \varphi^{\pm}, \tag{46}$$

where $\pi^{\pm}(x^+, x^-) := x^{\pm}$ (such transport plans will be further called admissible).

Under the assumptions on the cost of movement made above, it is quite reasonable to suppose that each single citizen moving from x to y would choose the route $\theta \in \Theta$ minimizing the total cost of movement, and therefore would spend

$$d_\Sigma(x, y) := \inf \left\{ A\mathcal{H}^1(\theta \setminus \Sigma) + B\mathcal{H}^1(\theta \cap \Sigma) \colon \theta \in \Theta,\ \theta(0) = x,\ \theta(1) = y \right\}.$$

If the behaviour of the population is described by a transport plan γ satisfying (46), then the total cost of transportation of the population is given by

$$\hat{G}(\gamma, \Sigma) := \int_{\mathbb{R}^n \times \mathbb{R}^n} d_\Sigma(x, y)\, d\gamma(x, y). \tag{47}$$

In [10] it has been shown that the problem of minimizing the cost $G(\cdot, \Sigma)$ among all admissible transport plan is in fact equivalent to that of minimizing the cost $\hat{G}(\cdot, \Sigma)$ among admissible transport plans. The precise meaning of this assertion is given by the statement below.

Proposition 9.1. *For each Borel set $\Sigma \subset \mathbb{R}^n$ one has*

$$MK(\varphi^+, \varphi^-, \Sigma) = \inf \left\{ \hat{G}(\gamma, \Sigma) \;:\; \gamma \text{ transport plan satisfying } (46) \right\}.$$

Further, there is an admissible transport $\eta' = \eta'(\Sigma)$ and a transport plan $\gamma' = \gamma'(\Sigma)$ (both depending on Σ) such that

$$MK(\varphi^+, \varphi^-, \Sigma) = G(\eta', \Sigma) = \hat{G}(\gamma', \Sigma).$$

One has that η'-a.e. $\theta \in \Theta$ is a simple arc. Finally, if η' is such an admissible transport that $MK(\varphi^+, \varphi^-, \Sigma) = G(\eta', \Sigma)$, then one can take $\gamma' := (p_0 \times p_1)_{\#}\eta'$. Vice versa, there is a Borel measurable function $q \colon \mathbb{R}^n \times \mathbb{R}^n \to \Theta$ such that if γ' is such an admissible transport plan that $MK(\varphi^+, \varphi^-, \Sigma) = \hat{G}(\gamma', \Sigma)$, then one can take $\eta' := q_{\#}\gamma'$.

It is important to mention that since d_Σ is easily verified to satisfy the triangle inequality, then it is well-known that

$$MK(\varphi^+, \varphi^-, \Sigma) = MK(\tilde{\varphi}^+, \tilde{\varphi}^-, \Sigma) \text{ whenever } \varphi^+ - \varphi^- = \tilde{\varphi}^+ - \tilde{\varphi}^-.$$

Supposing now that the total cost which determines the transportation network is given by the cost of everyday movement of the population $MK(\varphi^+, \varphi^-, \Sigma)$ and of the cost of constructing the network given by $H(\mathcal{H}^1(\Sigma))$ (i.e. depending only on the length $\mathcal{H}^1(\Sigma)$ of the network), we get the following natural minimization problem to find the optimal transportation network Σ.

Problem 3. *Find a Borel set $\Sigma_{opt} \subset \mathbb{R}^n$ minimizing the functional*

$$\Sigma \mapsto MK(\varphi^+, \varphi^-, \Sigma) + H(\mathcal{H}^1(\Sigma))$$

among all Borel sets $\Sigma \subset \mathbb{R}^n$.

Minding the definition of $MK(\varphi^+, \varphi^-, \Sigma)$ and the Proposition 9.1, we see that each solution $\Sigma_{opt} \subset \mathbb{R}^n$ to Problem 3 together with the respective optimal transport $\eta_{opt} := \eta'(\Sigma_{opt})$ (resp. the optimal transport plan $\gamma_{opt} := \gamma'(\Sigma_{opt})$) has to solve the following problem.

Problem 4. *Find a couple $(\eta_{opt}, \Sigma_{opt})$ (resp. $(\gamma_{opt}, \Sigma_{opt})$) minimizing the functional F (resp. $\hat{F}$) defined by*

$$F(\eta, \Sigma) := G(\eta, \Sigma) + H(\mathcal{H}^1(\Sigma)) \qquad (\text{resp. } \hat{F}(\gamma, \Sigma) := \hat{G}(\gamma, \Sigma) + H(\mathcal{H}^1(\Sigma))),$$

among all couples (η, Σ) (resp. (γ, Σ)), where η is an admissible transport (resp. γ is an admissible transport plan) and $\Sigma \subset \mathbb{R}^n$ is a Borel set.

Vice versa, if a couple $(\eta_{opt}, \Sigma_{opt})$ (resp. $(\gamma_{opt}, \Sigma_{opt})$) solves the above Problem 4, then Σ_{opt} solves Problem 3. Clearly, a solution $(\eta_{opt}, \Sigma_{opt})$ (resp. $(\gamma_{opt}, \Sigma_{opt})$) to Problem 4 gives both the optimal transportation network Σ_{opt} and the optimal pattern of behaviour of the population η_{opt} (resp. γ_{opt}). Note also that once one knows η_{opt} (resp. γ_{opt}), one can find γ_{opt} (resp. η_{opt}) as indicated in Proposition 9.1.

We now show that the Problem 4 with linear functions A and B is in fact equivalent to Problem 2 in the sense specified by the statement below. For the sake of brevity we will limit ourselves to the case $A \geq B$. The case $A < B$ is quite analogous once one observes that under this condition

$$G(\eta, \Sigma) \geq A \int_\Theta \mathcal{H}^1(\theta)\, d\eta = G(\eta, \emptyset),$$

while every optimal pair (T_{opt}, S_{opt}) solving Problem 2 has $S_{opt} = 0$, since

$$\mathfrak{F}(T, S) = A\mathbb{M}(T) + B\mathbb{M}(S) + H(\mathbb{M}^0(S)) > A\mathbb{M}(T + S) = \mathfrak{F}(T + S, 0)$$

whenever $S \neq 0$.

Theorem 9.2. *Let $A \geq B$. Then the following assertions hold:*

(i) *Suppose (T_{opt}, S_{opt}) solves Problem 2, while $\mu_{T_{opt}} \wedge \mu_{S_{opt}} = 0$ (the existence of such an optimal pair is ensured by Proposition 10.3). Let $\eta := \eta_{T_{opt}+S_{opt}}$ as defined by Theorem 6.3 and let $\Sigma := \Sigma_{S_{opt}}$. Then Σ solves Problem 3. Further, the couple (η, Σ) solves Problem 4 with $\varphi^\pm - \varphi^+ \wedge \varphi^-$ instead of $\varphi^\pm$. In particular, if φ^+ and φ^- are mutually singular, then (η, Σ) solves Problem 4.*

(ii) *Vice versa, let $(\eta_{opt}, \Sigma_{opt})$ solve Problem 4. Let $R := T_{\eta_{opt}}$ as defined by the relationship* (8)*, and let $S := R \llcorner \Sigma_{opt}$, $T := R - S$. Then (T, S) solves Problem 2.*

(iii) *Finally, $F(\eta_{opt}, \Sigma_{opt}) = \mathfrak{F}(T_{opt}, S_{opt})$.*

Remark 9.3. It is worth mentioning that once the existence of solutions to Problem 1 (hence in particular to Problem 2) is proven by Theorem 8.1, then the above Theorem 9.2 would give immediately the existence of solutions to Problem 3.

Proof. Suppose that η is an admissible transport such that η-a.e. $\theta \in \Theta$ is a simple arc (note that by Proposition 9.1 this is a case whenever (η, Σ) solves Problem 4) and $\Sigma \subset \mathbb{R}^n$ is a Borel set. Then, letting $R := T_\eta$ as defined by the relationship 8, $S := R \llcorner \Sigma$ and $T := R - S$, we have that the couple of flat chains (T, S) satisfies (3) in view of Theorem 6.2. Further, by the same theorem,

$$\mathbb{M}(S) = \mu_R(\Sigma) \leq \int_\Theta \mathbb{M}([\![\theta]\!] \llcorner \Sigma)\, d\eta(\theta) = \int_\Theta \mathcal{H}^1(\theta \cap \Sigma)\, d\eta(\theta),$$

$$\mathbb{M}(T) = \mu_R(\mathbb{R}^n \setminus \Sigma) \leq \int_\Theta \mathbb{M}([\![\theta]\!] \llcorner (\mathbb{R}^n \setminus \Sigma))\, d\eta(\theta) = \int_\Theta \mathcal{H}^1(\theta \setminus \Sigma)\, d\eta(\theta),$$

and hence, $A\mathbb{M}(T) + B\mathbb{M}(S) \leq G(\eta, \Sigma)$. On the other hand, $\mathbb{M}^0(S) \leq \mathcal{H}^1(\Sigma)$, and hence, $\mathfrak{F}(T, S) \leq F(\eta, \Sigma)$.

Suppose now that the couple of flat chains of finite mass (T, S) satisfies (3), and, further, that S is rectifiable and that both T, S and $T + S$ are acyclic, while μ_T and μ_S are mutually singular (mind that according to Theorems 10.1 and 10.2, the solutions to Problem 2 belong exactly to such class of couples of flat chains). Then $\eta := \eta_{T+S}$ as defined by Theorem 6.3 satisfies

$$\eta(0) = (\partial(T + S))^+ = \varphi^+ - \varphi^+ \wedge \varphi^-,$$
$$\eta(1) = (\partial(T + S))^- = \varphi^- - \varphi^+ \wedge \varphi^-.$$

Let $\Sigma := \Sigma_S$ as defined by Lemma 7.1. Then μ_S is concentrated on Σ. Hence, setting $R := T + S$, one gets $S = R \llcorner \Sigma$, $T = R \llcorner (\mathbb{R}^n \setminus \Sigma)$. Letting η be such that $R = T_\eta$ as defined by Theorem 6.3, we get from Proposition 6.8 that

$$\mathbb{M}(S) = \mu_R(\Sigma) = \int_\Theta \mathcal{H}^1(\theta \cap \Sigma)\, d\eta(\theta),$$
$$\mathbb{M}(T) = \mu_R(\mathbb{R}^n \setminus \Sigma) = \int_\Theta \mathcal{H}^1(\theta \setminus \Sigma)\, d\eta(\theta),$$

and hence, $G(\eta, \Sigma) = A\mathbb{M}(T) + B\mathbb{M}(S)$. Further, from definition of Σ one has $\mathbb{M}^0(S) = \mathcal{H}^1(\Sigma)$, and hence $F(\eta, \Sigma) = \mathfrak{F}(T, S)$, which concludes the proof. □

10. Qualitative properties of optimal currents

Here and below we always suppose the existence of a couple of real one-dimensional flat chains (T, S) of finite mass satisfying (3) such that $\mathfrak{F}(T, S) < +\infty$ (which means that the minimization Problem 1 is nontrivial). We further also suppose that either the penalization function H is concave, or $\delta = 0$.

10.1. **Acyclicity.** One may announce now the following easy result.

Theorem 10.1. *Let (T, S) be a pair of one-dimensional real flat chains of finite mass. Then there is such a pair of acyclic currents (T', S') that $T' \leq T$, $S' \leq S$, $\mathfrak{F}(T', S') \leq \mathfrak{F}(T, S)$ and $\partial(T' + S') = \partial(T + S)$. Moreover,*

- (i) *if $A > 0$ and T is not acylcic, then $\mathfrak{F}(T', S') < \mathfrak{F}(T, S)$;*
- (ii) *if either $B > 0$, or H is strictly increasing, and S is not acylcic, then $\mathfrak{F}(T', S') < \mathfrak{F}(T, S)$;*
- (iii) *If $A > 0$, and either $B > 0$, or H is strictly increasing, while $T + S$ is not acyclic, then $T' + S'$ acyclic, and $\mathfrak{F}(T', S') < \mathfrak{F}(T, S)$.*

In particular, if the pair (T, S) solves Problem 1, then $A > 0$ implies acyclicity if T, and either $B > 0$ or strict monotonicity of H imply acyclicity of S, while if both $A > 0$, and either $B > 0$, or H is strictly increasing, then $T + S$ is acyclic.

Proof. If T (resp. S) is acyclic, it is enough to set $T' := T$ (resp. $S' := S$). Otherwise in view of Proposition 3.12 it contains such a cycle $C \neq 0$ that the current $T' := T - C$ (resp. $S' := S - C$) is acyclic. By Lemma 3.7 one has $\mathbb{M}^\lambda(T') < \mathbb{M}^\lambda(T)$ (resp. $\mathbb{M}^\lambda(S') < \mathbb{M}^\lambda(S)$) for all $\lambda \in (0, 1]$. Therefore, miniding that $\partial T' = \partial T$ (resp. $\partial S' = \partial S$), we get $\partial(T' + S') = \partial(T + S)$, while $\mathfrak{F}(T', S') \leq \mathfrak{F}(T, S)$ (with strict inequality in cases (i) and (ii)).

We prove now (iii). According to Lemma 4.6 one may assume without loss of generality that $\mu_T \wedge \mu_S = 0$, i.e. there is such a Borel set $E \subset \mathbb{R}^n$, that

$$T = T \llcorner E \qquad S = S \llcorner (\mathbb{R}^n \setminus E).$$

If $T + S$ is not acyclic, then by Proposition 3.12 there is such a cycle $C \neq 0$, $C \leq T + S$, that the current $T + S - C$ is acyclic. Denoting

$$T' := T - C \llcorner E, \qquad S' := S - C \llcorner (\mathbb{R}^n \setminus E),$$

we get $T' + S' = T + S - C$, and hence, $\partial(T' + S') = \partial(T + S)$. On the other hand, $C \leq T + S$ implies

$$C \llcorner E \leq (T + S) \llcorner E = T,$$

By Remark 3.6 one has $T' \leq T$. In the same way, $S' \leq S$, and hence, in view of Lemma 3.7, $\mathfrak{F}(T', S') < \mathfrak{F}(T, S)$. □

It is worth remarking that with the help of Theorem 7.4 one can easily find the estimates on masses of solutions to Problem 1.

10.2. **Rectifiablity.** The following result is an easy consequence of Theorem 2.1.

Theorem 10.2. *Let (T, S) be a pair of real one-dimensional flat chains of finite mass satisfying $\mathfrak{F}(T, S) < +\infty$. Then the following assertions hold:*

(i) *if $A \neq 0$ and $\alpha < 1$, then T is rectifiable;*
(ii) *if either $B \neq 0$ and $\beta < 1$ or H is unbounded and $\delta < 1$, then S is rectifiable.*

In particular, the above assertions are valid for every optimal pair solving Problem 1.

Proof. We have $\mathfrak{F}(T, S) < +\infty$. When $A \neq 0$ one has therefore $\mathbb{M}^\alpha(T) < +\infty$, and hence (i) follows from Theorem 2.1. Analogously, if $B \neq 0$, then one has therefore $\mathbb{M}^\beta(S) < +\infty$, so that rectifiability of S follows from Theorem 2.1 when $\beta < 1$. Finally, if H is unbounded, then $\mathbb{M}^\delta(S) < +\infty$, and hence rectifiability of S follows again from Theorem 2.1 when $\delta < 1$. □

10.3. **Properties of the support.** We first show the existence of minimizing couples (T, S) solving Problem 1 such that T and S are concentrated on disjoint sets. Here and below solvability of Problem 1 will be always tacitly assumed.

Proposition 10.3. *There is a minimizing couple (T, S) solving Problem 1 such $\mu_T \wedge \mu_S = 0$. In particular, $\mathcal{H}^1(\Sigma_T \cap \Sigma_S) = 0$. Further, if either of the conditions (i)–(v) of Lemma 4.7 holds, then the above property is true for every minimizing couple (T, S) solving Problem 1.*

Proof. Follows immediately from Lemma 4.6. □

We are able to prove now the following assertion which says that whenever the penalization H is concave, then there is an optimal pair of flat chains (T, S) solving Problem 1 as follows: there is a threshold $d \geq 0$ such that T is concentrated on the set $\{x : \theta_{T+S} < d\}$, while S is concentrated on the set $\{x : \theta_{T+S} \geq d\}$, and in certain cases every optimal pair satisfies such a condition. In other words, recalling that S stands for the flow of people using the transportation network, while T stands for that of people moving by own means, it means that the transportation network has to be constructed in the set of points where the density θ_{T+S} of the total flow of people is greater than the threshold d.

Theorem 10.4 (Bathtub principle)**.** *Assume that H is concave and unbounded, and let (T', S') be any solution to Problem 1 having $\mu_{T'} \wedge \mu_{S'} = 0$ (once Problem 1 is solvable, the existence of such a pair is guaranteed by Proposition 10.3). Then there exists such an optimal pair (T, S) of flat chains solving Problem 1 that $\mu_T \wedge \mu_S = 0$ and under either of the conditions (i)–(iv) of Lemma 10.6 (with $C := H'_-(\mathbb{M}^\delta(S'))$, where H'_- stands for the left derivative of H)*

$$\theta_S(x) = \theta_{T+S}(x) \geq d \qquad \text{for } \mathcal{H}^1 - \text{a.e. } x \in \Sigma_{T+S},$$
$$\theta_T(x) = \theta_{T+S}(x) < d \qquad \text{for } \mathcal{H}^1 - \text{a.e. } x \in \Sigma_{T+S},$$

for some constant $d \geq 0$ (moreover, $d > 0$ under either of the conditions (i')–(iii') of Lemma 10.6). In particular, if S is rectifiable, then S is concentrated on the set $\{x : \theta_{T+S}(x) \geq d\}$ and T is concentrated on the set $\{x : \theta_{T+S}(x) < d\}$. Moreover, the above properties hold for all optimal pairs (T, S) satisfying $\mu_T \wedge \mu_S = 0$, if in addition one assumes that H is strictly concave.

Proof. We first note that one may restrict ourselves to consider the case $S' \neq 0$, since otherwise it is enough to choose $T = T'$, $S = S'$ and $d := \mathbb{M}(\partial T)/2$ in view of Theorem 7.3. Now the existence of the optimal pair of flat chains solving Problem 1 follows immediately from Lemma 10.5 (for the assertion in the case of rectifiable S one has just to mind that in this case one may consider S concentrated on $\Sigma_S \subset \Sigma_{T+S}$). □

The following assertions have been used in the above proof.

Lemma 10.5. *Let (T,S), $S\neq 0$, be a pair of flat chains of finite mass satisfying $\mu_T\wedge\mu_S=0$, and the function H be concave. Under either of the conditions (i)–(iv) (resp. (i')–(iii')) of Lemma 10.6 there is such a constant $d\geq 0$ (resp. $d>0$), depending only on α, β, δ, A, B and $C\in[H'_+(\mathbb{M}^\delta(S)),H'_-(\mathbb{M}^\delta(S))]$, where $H'_\pm$ stand stand for left and right derivatives of H respectively, and such a pair of flat chains (T',S'), that $T'+S'=T+S$, $\mu_{T'}\wedge\mu_{S'}=0$, $\mathfrak{F}(T',S')\leq\mathfrak{F}(T,S)$, and*

$$\begin{aligned}&\theta_{S'}(x)=\theta_{T'+S'}(x)\geq d \qquad \text{for } \mathcal{H}^1-a.e.\ x\in\Sigma_{T'+S'},\\ &\theta_{T'}(x)=\theta_{T'+S'}(x)< d \qquad \text{for } \mathcal{H}^1-a.e.\ x\in\Sigma_{T'+S'}.\end{aligned}\tag{48}$$

Moreover, if T (resp. S) is rectifiable, then so is T' (resp. S').

Furthermore, if the function H is stricly concave, and either of the properties (48) *does not hold for T, S in place of T', S', then one can find a pair (T',S') as above with $\mathfrak{F}(T',S')<\mathfrak{F}(T,S)$.*

Proof. In view of assumption on C we have

$$H(\mathbb{M}^\delta(S)+t)\leq H(\mathbb{M}^\delta(S))+Ct \qquad \text{for all } t\geq -\mathbb{M}^\delta(S),$$

with strict inequality when $t\neq 0$ and H is strictly concave (mind in the latter case $C\neq 0$ in view of the strict concavity of H). Consider the function $f\colon\mathbb{R}^+\to\mathbb{R}$ defined by the formula $f(t):=At^\alpha-Bt^\beta-Ct^\delta$. By Lemma 10.6 there is such a $d\geq 0$ that $f(t)>0$ when $t>d$, and $f(t)<0$ when $t\in(0,d)$. Moreover, $d>0$ if either of conditions (i')–(iii') of Lemma 10.6 hold. Consider the sets

$$\Sigma^+:=\{x\in\Sigma_{T+S}\colon\theta_T(x)\geq d\}, \qquad \Sigma^-:=\{x\in\Sigma_{T+S}\colon 0<\theta_S(x)<d\}$$

(with $\Sigma^-:=\emptyset$, if $d=0$). Since the densities θ_T and θ_S are Borel functions then $\Sigma^\pm$ are Borel sets so that the currents

$$\begin{aligned}R^+&=T\llcorner\Sigma^+, & R^-&=S\llcorner\Sigma^-\\ T'&=T-R^++R^-, & S'&=S-R^-+R^+,\end{aligned}$$

are well defined, while $T'+S'=T+S$. Observe that the rectifiablity of T' (resp. S') follows from that of T (resp. S) since both R^+ and R^- are rectifiable in view of Lemma 7.1. We show now that $\mathfrak{F}(T',S')\leq\mathfrak{F}(T,S)$ with strict inequality, if H is stricly concave, and either of the properties (48) does not hold for T, S in place of T', S' (all the other announced properties of the pair (T',S') are immediate). Since $R^+\leq T$, $R^-\leq S$, while $\mu_T\wedge\mu_S=0$, then

$$\begin{aligned}\mathbb{M}^\alpha(T')&=\mathbb{M}^\alpha(T-R^++R^-)=\mathbb{M}^\alpha(T)-\mathbb{M}^\alpha(R^+)+\mathbb{M}^\alpha(R^-)\\ \mathbb{M}^\beta(S')&=\mathbb{M}^\beta(S-R^-+R^+)=\mathbb{M}^\beta(S)-\mathbb{M}^\beta(R^-)+\mathbb{M}^\beta(R^+)\\ \mathbb{M}^\delta(S')&=\mathbb{M}^\delta(S-R^-+R^+)=\mathbb{M}^\delta(S)-\mathbb{M}^\delta(R^-)+\mathbb{M}^\delta(R^+),\end{aligned}$$

and therefore,

$$\begin{aligned}\mathfrak{F}(T',S')\leq{}&\mathfrak{F}(T,S)-A\mathbb{M}^\alpha(R^+)+A\mathbb{M}^\alpha(R^-)\\ &-B\mathbb{M}^\beta(R^-)+B\mathbb{M}^\beta(R^+)-C\mathbb{M}^\delta(R^-)+C\mathbb{M}^\delta(R^+).\end{aligned}\tag{49}$$

Recalling that by Lemma 7.1 the sets $\Sigma^\pm$ are countably $(\mathcal{H}^1,1)$-rectifiable, we get

$$\begin{aligned}\mathfrak{F}(T',S')&\leq\mathfrak{F}(T,S)-\int_{\Sigma^+}\Big(A\theta_T^\alpha(x)-B\theta_T^\beta(x)-C\theta_T^\delta(x)\Big)\,d\mathcal{H}^1(x)+\\ &\qquad\int_{\Sigma^-}\Big(A\theta_S^\alpha(x)-B\theta_S^\beta(x)-C\theta_S^\delta(x)\Big)\,d\mathcal{H}^1(x)\\ &=\mathfrak{F}(T,S)-\int_{\Sigma^+}f(\theta_T(x))\,d\mathcal{H}^1(x)+\int_{\Sigma^-}f(\theta_S(x))\,d\mathcal{H}^1(x)\\ &\leq\mathfrak{F}(T,S).\end{aligned}\tag{50}$$

Moreover, if the first property of (48) does not hold for T, S in place of T', S', then $\int_{\Sigma^-} f(\theta_S(x))\, d\mathcal{H}^1(x) < 0$, and hence the second inequality of (50) is strict. Further, if the second property of (48) does not hold for T, S in place of T', S', then either

$$\mathcal{H}^1(\{x \in \Sigma_{T+S} \colon \theta_T(x) > d\}) > 0,$$

or else

$$\mathcal{H}^1(\{x \in \Sigma_{T+S} \colon \theta_T(x) = d\}) > 0.$$

In the former case $\int_{\Sigma^+} f(\theta_T(x))\, d\mathcal{H}^1(x) > 0$, and hence again the second inequality of (50) is strict. In the latter case one still has $R^+ \neq 0$, and we may consider without loss of generality $\mathcal{H}^1(\Sigma^-) = 0$ (since otherwise the strict inequality $\mathfrak{F}(T', S') < \mathfrak{F}(T, S)$ has already been proven), so that $R^- = 0$. Therefore the inequality in (49) becomes strict when H is strictly concave (because $\mathbb{M}^\delta(R^+) \neq 0$), and hence the first inequality of (50) is strict, which concludes the proof. □

Lemma 10.6. *Let $f\colon [0, +\infty) \to \mathbb{R}$ be defined by*

$$f(t) := At^\alpha - Bt^\beta - Ct^\delta$$

with $\alpha, \beta, \delta \in [0, 1]$ and $A, B, C \geq 0$. Suppose also that either of the following conditions hold:

(i) *$\alpha > \beta \vee \delta$;*
(ii) *$\alpha = \beta > \delta$ and $A > B$;*
(iii) *$\alpha = \delta > \beta$ and $A > C$;*
(iv) *$\alpha = \beta = \delta$ and $A > B + C$.*

Then there is a $d \geq 0$ such that

$$f(t) \geq 0, \quad \text{if and only if} \quad t \geq d \text{ or } t = 0.$$

Moreover, $d > 0$ under either of the following conditions:

(i') *(i) holds and either $B \neq 0$ or $C \neq 0$;*
(ii') *(ii) holds and $C \neq 0$;*
(iii') *(iii) holds and $B > 0$.*

Proof. Case (i). Suppose $\beta \geq \delta$ (the other case being symmetric). Dividing by t^δ, we get

$$f(t) \geq 0, \quad \text{if and only if} \quad g(s) := As^\sigma - Bs - C \geq 0,$$

where $s := t^{\beta-\delta}$, $\sigma := \frac{\alpha-\delta}{\beta-\delta}$. Noticing that $\sigma > 1$ in the case we are considering, we get that the derivative $g'(s) = A\sigma s^{\sigma-1} - B$ is monotone nondecreasing (resp. strictly increasing, if $A > 0$). Hence, g is convex (resp. strictly convex), and minding that $g(0) = -C \leq 0$ and $g'(0) = -B \leq 0$, we get the existence of some $\bar{s} \geq 0$ such that $g(s) \geq 0$, if and only if $s \geq \bar{s}$, while $\bar{s} > 0$ if either $C \geq 0$ or $B \geq 0$. It is enough then to set $d := \bar{s}^{1/(\beta-\delta)}$.

Case (ii). Dividing by t^δ, we get that $f(t) \geq 0$, if and only if

$$(A - B)t^{\alpha/\delta} - C \geq 0,$$

which means that one may take $d := C^{\delta/\alpha}/(A - B)$.

Case (iii) is completely analogous to case (ii).

Case (iv) Dividing by t^δ, we get that $f(t) \geq 0$, if and only if

$$(A - B - C)t^\alpha \geq 0,$$

which means $d = 0$. □

Now we are able to prove that under natural conditions on problem data, the the optimal current S can be chosen to be concentrated on a closed set.

Theorem 10.7. *Suppose that H is concave and unbounded and condition (ii) of Theorem 10.2 holds. Under either of the conditions (i) or (ii) of Lemma 10.6 there exists an optimal pair (T, S) solving Problem 1 such that S is a rectifiable current representable as $S = \theta[\![\Sigma]\!]$, where $\Sigma \subset \mathbb{R}^n$ is a closed countably $(\mathcal{H}^1, 1)$-rectifiable set and $\theta \in L^1(\mathcal{H}^1 \llcorner \Sigma)$ is u.s.c., $\theta(x) \geq d$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$ and for some $d > 0$. Further, if either of the conditions (i)–(v) of Lemma 4.7 holds, while H is strictly concave, then the above assertions are true for all optimal pairs (T, S) solving Problem 1.*

Proof. Let the pair (T', S') solve the Problem 1, while $\mu_{T'} \wedge \mu_{S'} = 0$ (once Problem 1 is solvable, the existence of such a pair is guaranteed by Proposition 10.3). We may assume $S' \neq 0$ (otherwise there is nothing to prove, since we may take, for instance, $S' := S$, $\theta \equiv 1$ and $\Sigma := \emptyset$). Note now that since the assumptions (i) or (ii) of Lemma 10.6 hold, while $C := H'_-(\mathbb{M}^\delta(S)) > 0$ in view of the assumptions on H, then by Theorem 10.4 combined with Theorem 10.2 we know that there exists a $d > 0$ and an optimal pair (T, S) solving Problem 1 and satisfying $\mu_T \wedge \mu_S = 0$, such that S is rectifiable and concentrated over the set $\Sigma_S = \{x : \theta_{T+S}(x) \geq d\}$.

Let η be given by Theorem 6.3 so that $T + S = T_\eta$. By Lemma 7.2(ii), minding the acyclicity of $T + S$ one has that

$$\mathcal{H}^1(\{x : \theta_{T+S}(x) \neq a_\eta(x)\}) = 0.$$

Hence $\mathcal{H}^1(\Sigma_S \triangle \Sigma) = 0$ where $\Sigma := \{x \in \mathbb{R}^n : a_\eta(x) \geq d\}$, while Σ is closed since a_η is u.s.c. by Lemma 7.2(i). We have therefore $S = a_\eta[\![\Sigma]\!]$, which is as announced in the statement being proven. To conclude the the proof, it remains to observe that under either of the conditions (i)–(v) of Lemma 4.7 every optimal pair (T, S) solving Problem 1 satisfies $\mu_T \wedge \mu_S = 0$, while if H is strictly concave, then every such pair will have S concentrated over $\{x : \theta_{T+S}(x) \geq d\}$, and hence will satisfy conditions of the statement being proven. □

It is worth mentioning that Theorem 10.7 is only valid under concavity assumptions on H. In fact, in [9] it has been shown that even when $A = 1$, $B = 0$, but

$$H(t) := \begin{cases} 0, & t \leq l, \\ +\infty, & \text{otherwise}, \end{cases}$$

then for some measures φ^+, φ^- one could have that Problem 3 admits no solutions Σ which are closed sets. In view of Theorem 9.2 this means that no solution (T, S) to Problem 1 with $\alpha = \beta = 1$, $\delta = 0$ and A, B and H as above has the property announced in Theorem 10.7, i.e. that $S = \theta[\![\Sigma]\!]$, where $\Sigma \subset \mathbb{R}^n$ is a closed countably $(\mathcal{H}^1, 1)$-rectifiable set, and $\theta(x) > 0$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$.

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