One-sided Minkowski content for some classes of closed sets and applications to stochastic geometry

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Abstract

We find conditions ensuring the existence of the *one-sided Minkowski content* for d-dimensional closed sets in $\mathbb{R}^d$, in connection with regularity properties of their boundaries. Moreover, we provide a class of sets stable under finite unions for which the one-sided Minkowski content exists. It follows, in particular, that finite unions of sets with Lipschitz boundary and a type of sets with positive reach belong to this class. We find analogous conditions, stable under finite unions as well, for the existence of the mean one-sided Minkowski content of random closed sets. Finally, an application to birth-and-growth stochastic processes is briefly discussed.

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1 Introduction

This paper is concerned with the so-called *one-sided Minkowski content* $\mathcal{SM}(A)$ of a compact set $A \subset \mathbb{R}^d$. It is defined, whenever the limit exists, by

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^d(A_r \setminus A)}{r}.$$

Here we denote by A_r the closed r-neighborhood of A, and by $\mathcal{H}^k$ the Hausdorff k-dimensional measure in $\mathbb{R}^d$; to make the presentation of our results lighter, we do not give here formal definitions and we refer for this and other (typically standard) notation to the next section. If for any $x \in \partial A$ the set A is locally representable as the subgraph of a sufficiently smooth function, it is intuitive that this limit gives the surface measure of ∂A, namely $\mathcal{H}^{d-1}(\partial A)$. It is also intuitive that in many situations this coincides with the (two-sided or usual) Minkowski content $\mathcal{M}^{d-1}(\partial A)$, namely

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^d((\partial A)_r)}{2r},$$

an object on which many more results are available in the literature.

Our goals, which are motivated by problems in stochastic geometry described in more detail in the second part of this introduction (in particular from the analysis of the so-called birth-and-growth stochastic processes) are:

(a) to find general conditions ensuring the existence of the one-sided Minkowski content;

(b) to find a class of sets $\mathcal{S}$ stable under finite unions for which the one-sided Minkowski content is defined.

The first goal can be considered as a variation of the more classical theme for the Minkowski content. In this analysis, besides the topological boundary ∂A, also the smaller *essential* boundary $\partial^* A$ (i.e. the set of points where the density is neither 0 nor 1) plays an important role. We can prove for instance in Theorem 14 that, whenever the Minkowski content exists and it coincides with $\mathcal{H}^{d-1}(\partial^* A)$, then the one-sided Minkowski content exists, and has the same value. This criterion can be applied, for instance, to show that the one-sided Minkowski content exists for all sets A with a Lipschitz boundary. But, the existence of the content is only loosely related to the regularity of the boundary: for instance we characterize in Theorem 28, among all *sets with positive reach*, those for which the one-sided Minkowski content coincides with $\mathcal{H}^{d-1}(\partial^* A)$. Notice that for sets of positive reach a complete polynomial expansion of $r \mapsto \mathcal{H}^d(A_r)$ is available, the so-called Steiner formula: therefore the one-sided Minkowski content always exists, and corresponds to the first coefficient in this expansion. In connection with the most recent developments on this subject we mention the paper [12], where the authors present a generalization of the Steiner formula to closed sets; nevertheless such general formula is not polynomial in r, and so the existence of the one-sided Minkowski content cannot be obtained directly, without assuming further regularity conditions. We also mention the paper [16], where existence of the one-sided Minkowski content is proved for finite unions $\bigcup_i A_i$ of sets A_i with positive reach such that all possible finite intersections of the A_i's have positive reach as well.

The second goal is more demanding, as simple examples show that regularity properties of the boundary are not stable under unions. The same is true (see Example 2) for other typical regularity conditions considered in geometric measure theory, as positive reach, or $\mathcal{H}^{d-1}(\partial A \setminus \partial^* A) = 0$ (which implies, by the theory of sets of finite perimeter, that an approximate normal exists at $\mathcal{H}^{d-1}$-a.e. point of ∂A). Nevertheless, we are able to identify two conditions, both stable under finite unions: the first one, see (2), is a kind of quantitative non-degeneracy condition which prevents ∂A from being too sparse; simple examples (see Example 3) show that $\mathcal{SM}(A)$ can be infinite, and $\mathcal{H}^{d-1}(\partial A)$ arbitrarily small, when this condition fails. The second condition, in analogy with the above mentioned Theorem 14, is the existence of the one-sided Minkowski content and its coincidence with $\mathcal{H}^{d-1}(\partial^* A)$. The proof of stability of these two conditions, given in Theorem 19, is one of the main contributions of this paper: it requires a careful measure-theoretic analysis of the regions where the boundaries of the sets intersect, either with same or with opposite normals.

With the role of "surface measure", the one-sided Minkowski content is important in many problems arising from real applications. Moreover, it may be seen as derivative of the volume of a set with respect to its Minkowski enlargement, so that, for a time-dependent closed set, which can be taken as model for evolution problems, the one-sided Minkowski content turns to be related to evolution equations for relevant quantities associated to the model (see, e.g., [4, 14, 18].) Clearly, several real situations are studied by stochastic models (see, e.g., [19], and [13] for further applications). In a stochastic setting we deal with random closed sets and their expected volumes; so we introduce the *mean* one-sided Minkowski content of a random closed set Θ in $\mathbb{R}^d$, i.e. a measurable map from a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ to the space of closed subsets in $\mathbb{R}^d$ (see Section 6), as the limit (whenever it exists)

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_r \setminus \Theta)]}{r}. \tag{1}$$

The problem is now to find general conditions for the existence of the mean one-sided Minkowski content. Starting from the results obtained in the deterministic case, we obtain sufficient conditions on a random compact set ensuring the existence of the mean one-sided Minkowski content and its coincidence with $\mathbb{E}[\mathcal{SM}(\Theta)]$. Moreover, our results are stated in a local form, see in particular Theorem 32, and are therefore applicable to random closed sets as well. In addition, they are all stable under finite unions; this feature is particularly relevant, for instance, in connection with the so-called birth-and-growth stochastic processes. These processes, briefly described in the final part of this paper, correspond to a random union of sets in $\mathbb{R}^d$ which evolve in time according to a given growth model; the (local) mean densities of volume and of the one-sided Minkowski content are fundamental to obtain deterministic evolution equations, which are of a great interest in applications (see [7] and references therein).

2 Notation and preliminaries

In this section, after giving some basic notation, we recall some definitions and results, mainly belonging to the area of geometric measure theory, that will be used in the paper.

We work in the Euclidean space $\mathbb{R}^d$, $d \geq 2$, endowed with the usual norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. For $s \geq 0$, we denote by $\mathcal{H}^s$ the Hausdorff measure of dimension s; in particular, $\mathcal{H}^d$ is the Lebesgue measure in $\mathbb{R}^d$. $\mathcal{B}_{\mathbb{R}^d}$ denotes the Borel σ-algebra of $\mathbb{R}^d$. Given a subset A of $\mathbb{R}^d$, ∂A will be its topological boundary, A^c the complement set of A, int$A$ and cl$A$ the interior and the closure of A, respectively. For $r \geq 0$ and $x \in \mathbb{R}^d$, $B_r(x)$ is the closed ball with center x and radius r; finally, for every n we set $b_n = \mathcal{H}^n(B_1(0))$, i.e. the volume of the unit ball in $\mathbb{R}^n$.

A crucial notion in this paper is the *parallel set* of a subset of $\mathbb{R}^d$. Let $A \subset \mathbb{R}^d$ be closed and let $r \geq 0$; the parallel set of A at distance r, denoted by A_r, is defined by

$$A_r = \{x \in \mathbb{R}^d \,:\, \mathrm{dist}(x, A) \leq r\}\,.$$

A_r is also known as the Minkowski enlargement of A and can be characterized as

$$A_r = A + B_r(0) = \{x + y \,:\, x \in A\,,\, y \in B_r(0)\}\,.$$

Definition 1 (Upper and lower Minkowski content) *Let $S \subset \mathbb{R}^d$ be a closed set and let n be an integer with $0 \leq n \leq d$. The* upper *and* lower *n-dimensional Minkowski contents $\mathcal{M}^{*n}(S)$, $\mathcal{M}^n_*(S)$ are defined by*

$$\mathcal{M}^{*n}(S) := \limsup_{r \downarrow 0} \frac{\mathcal{H}^d(S_r)}{b_{d-n} r^{d-n}}, \qquad \mathcal{M}^n_*(S) := \liminf_{r \downarrow 0} \frac{\mathcal{H}^d(S_r)}{b_{d-n} r^{d-n}}$$

respectively. If $\mathcal{M}^{*n}(S) = \mathcal{M}^n_*(S) < \infty$ their common value is denoted by $\mathcal{M}^n(S)$ and S is said to admit n-dimensional Minkowski content.

$\mathcal{M}^n$ is a measure of the area of "n-dimensional sets", alternative to the n-dimensional Hausdorff measure. It poses as natural problems its existence and its comparison with $\mathcal{H}^n$. In the literature there are some general results concerning these problems, related to rectifiability properties of the involved sets; let us mention some of them. We say that a compact set $S \subset \mathbb{R}^d$ is n-rectifiable if it is representable as the image of a compact set $K \subset \mathbb{R}^n$, with $f : \mathbb{R}^n \to \mathbb{R}^d$ Lipschitz.

The following theorem is proved in [10] (p. 275).

Theorem 2 $\mathcal{M}^n(S) = \mathcal{H}^n(S)$ *for any compact n-rectifiable set $S \subset \mathbb{R}^d$.*

The following remark shows that $(d-1)$-rectifiable sets are always contained in the support of a probability measure satisfying a suitable uniform $(d-1)$-dimensional lower bound. The existence of such a probability measure is the key to provide the extension of Theorem 2 to more general classes of sets.

Remark 3 Any n-rectifiable compact set $S \subset \mathbb{R}^d$ satisfies the following property: there exist $\gamma > 0$ and a probability measure η in $\mathbb{R}^d$ satisfying

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall r \in (0,1),\ \forall x \in S. \tag{2}$$

Indeed, let us represent $S = f(K)$ and let us assume, with no loss of generality, that the Lipschitz constant L of f is strictly positive. Let $K' := K_{1/L}$ and $c := \mathcal{H}^n(K')$; the probability measure $\eta(B) := c^{-1}\mathcal{H}^n(K' \cap f^{-1}(B))$ satisfies (2) with $\gamma = b_n c^{-1} L^{-n}$ because

$$\eta(B_r(f(x))) \geq \frac{1}{c}\mathcal{H}^n(B_{r/L}(x)) = \frac{b_n}{cL^n} r^n \qquad \forall x \in K,\ \forall r \in (0,1).$$

Under the same hypothesis of density lower bound considered in the previous remark, Theorem 2 can be extended to *countably $\mathcal{H}^n$-rectifiable* compact sets. Remind that a set $S \subset \mathbb{R}^d$ is said to be countably $\mathcal{H}^n$-rectifiable if there exist countably many n-dimensional Lipschitz graphs $\Gamma_i \subset \mathbb{R}^d$ such that $S \setminus \cup_i \Gamma_i$ is $\mathcal{H}^n$-negligible. The following result is proved in [3] (p. 110).

Theorem 4 *Let $S \subset \mathbb{R}^d$ be a countably $\mathcal{H}^n$-rectifiable compact set and assume that* (2) *holds for some $\gamma > 0$ and some Radon measure η in $\mathbb{R}^d$ which is absolutely continuous with respect to $\mathcal{H}^n$. Then $\mathcal{M}^n(S) = \mathcal{H}^n(S)$.*

A simple example of set S such that $\mathcal{M}^n(S)$ does not exist for any $n \in [0, d)$ is given in [3], p. 109 (see also Example 3 in Section 5). These examples show that countable rectifiability is not sufficient for the existence of the Minkowski content.

In the sequel we will use the following proposition (see [1]), which ensures that condition (2) is sufficient to find an upper bound for $\mathcal{M}^{*n}$.

Proposition 5 *Let S be a compact subset of $\mathbb{R}^d$ such that*

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall r \in (0,1),\ x \in S$$

holds for some $\gamma > 0$ and some probability measure η in $\mathbb{R}^d$. Then

$$\frac{\mathcal{H}^d(S_r)}{b_{d-n} r^{d-n}} \leq \frac{1}{\gamma} 2^n 4^d \frac{b_d}{b_{d-n}} \qquad \forall r \in (0,2).$$

Throughout this paper we will encounter several ways to measure the boundary of a subset of $\mathbb{R}^d$; one of them is given by the notion of *perimeter*, due to De Giorgi.

Definition 6 (Perimeter) *Let E be a $\mathcal{H}^d$-measurable set of $\mathbb{R}^d$ and let $\mathcal{D} \subseteq \mathbb{R}^d$ be an open set. We say that E* has finite perimeter *in $\mathcal{D}$ if the distributional derivative of the characteristic function of E, $D\chi_E$, is an $\mathbb{R}^d$-valued finite Radon measure in $\mathcal{D}$. The* perimeter *of E* in *$\mathcal{D}$, denoted by $P(E, \mathcal{D})$, is defined as the total variation $|D\chi_E|$ in $\mathcal{D}$, namely:*

$$P(E, \mathcal{D}) := |D\chi_E|(\mathcal{D}).$$

Roughly speaking, sets of finite perimeter are those whose characteristic function has bounded variation; we refer to [3] for an exhaustive treatment of this subject. In the sequel we will write $P(E)$ instead of $P(E, \mathbb{R}^d)$. As next step we recall the notion of *essential boundary* of a set, which turns out to be closely related to sets of finite perimeter; we start from the definition of densities.

Definition 7 (d-dimensional densities) *Let E be an $\mathcal{H}^d$-measurable set in $\mathbb{R}^d$. The* upper *and* lower d-dimensional densities of E at x *are respectively defined by*

$$\Theta^*_d(E,x) := \limsup_{r\downarrow 0} \frac{\mathcal{H}^d(E\cap B_r(x))}{b_d r^d}, \qquad \Theta_{*d}(E,x) := \liminf_{r\downarrow 0} \frac{\mathcal{H}^d(E\cap B_r(x))}{b_d r^d}.$$

If $\Theta^*_d(E,x) = \Theta_{*d}(E,x)$ their common value is denoted by $\Theta_d(E,x)$.

For every $t \in [0,1]$ and every $\mathcal{H}^d$-measurable set $E \subset \mathbb{R}^d$ let $E^t := \{x \in \mathbb{R}^d : \Theta_d(E,x) = t\}$.

Definition 8 (Essential boundary) *Let E be an $\mathcal{H}^d$-measurable set in $\mathbb{R}^d$. The* essential boundary $\partial^* E$ *of E is the set*

$$\partial^* E = \mathbb{R}^d \setminus (E^0 \cup E^1)\,.$$

As a consequence of general theorems on sets with finite perimeter (see §3.5 in [3]) we have the following result, which can be used to compute the perimeter measure in terms of the $(d-1)$-dimensional Hausdorff measure. In its statement and in the sequel we adopt the notation $\mathcal{H}^n_{|B}$ for the restriction of $\mathcal{H}^n$ to B, i.e. $\mathcal{H}^n_{|B}(E) = \mathcal{H}^n(B\cap E)$ for all Borel sets $E \subset \mathbb{R}^d$.

Theorem 9 *If E has finite perimeter in an open set $\mathcal{D} \subset \mathbb{R}^d$, then the measures $|D\chi_E|$ and $\mathcal{H}^{d-1}_{|\partial^* E}$ coincide on the Borel subsets of $\mathcal{D}$. In particular $P(E,\mathcal{D}) = \mathcal{H}^{d-1}(\mathcal{D}\cap\partial^* E)$.*

We conclude this section with the following *spherical differentiation* result (see for instance [3], p. 78, for the proof).

Theorem 10 (Hausdorff and Radon measures) *Let $\mathcal{D} \subseteq \mathbb{R}^d$ be an open set, μ a positive Radon measure in $\mathcal{D}$ and $n \in \{1,\dots,d\}$. Then for any $t \in (0,\infty)$ and any Borel set $B \subseteq \mathcal{D}$ the following implications hold:*

$$\limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{b_n r^n} \ge t \quad \forall x \in B \implies \mu \ge t\mathcal{H}^n_{|B}$$

$$\liminf_{r\downarrow 0} \frac{\mu(B_r(x))}{b_n r^n} \le t \quad \forall x \in B \implies \mu \le 2^n t\mathcal{H}^n_{|B}.$$

A useful consequence of the first implication is the following: if μ and n are as above

$$B \in \mathcal{B}_{\mathcal{D}},\ \mu(B) = 0 \quad \implies \quad \lim_{r\downarrow 0} \frac{\mu(B_r(x))}{b_n r^n} = 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in B. \tag{3}$$

Putting $B = (\partial^* E)^c$ in (3) and using Theorem 9 we deduce that

$$P(E, B_r(x)) = o(r^{d-1}) \qquad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \mathbb{R}^d \setminus \partial^* E. \tag{4}$$

On the other hand, we have the density property (see Theorem 3.59 in [3])

$$\lim_{r\downarrow 0} \frac{P(E, B_r(x))}{b_{d-1} r^{d-1}} = 1 \qquad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \partial^* E. \tag{5}$$

A class of sets that will be considered are those with Lipschitz boundary. In our terminology, a compact set A has Lipschitz boundary if for every boundary point x there exists a neighborhood $\mathcal{U}$ of x such that $A\cap\mathcal{U}$ is the epigraph of a Lipschitz function. The following proposition is proved for instance in [3] (p. 159).

Proposition 11 *If $A \subset \mathbb{R}^d$ is a compact set with Lipschitz boundary then $\mathcal{H}^{d-1}(\partial A) < \infty$, A has finite perimeter in $\mathbb{R}^d$ and $P(A) = \mathcal{H}^{d-1}(\partial A)$.*

We also recall that the following inequality holds without any regularity or topological assumption on A:

$$P(A) \leq \mathcal{H}^{d-1}(\partial A). \tag{6}$$

3 The one-sided Minkowski content

In this section we introduce the notion of one-sided Minkowski content of a set and we investigate the connections of this quantity with other boundary measurements.

As mentioned in Section 1, a problem of interest in many real applications is the existence and the computation of the following limit:

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d(A_r \setminus A)}{r}, \tag{7}$$

for a given closed subset A of $\mathbb{R}^d$. Note that if A has Hausdorff dimension $\dim_{\mathcal{H}} A < d$, then $\mathcal{H}^d(A_r \setminus A) = \mathcal{H}^d(A_r)$; thus in this case conditions on A for the existence of the above limit are the same which guarantee the existence of the $(d-1)$-dimensional Minkowski content, widely studied in literature. We will consider instead the case when A is a d-dimensional set with $\mathcal{H}^d(A) > 0$. In this case the limit in (7) may be seen as the *one-sided* Minkowski content of ∂A since $A_r \setminus A$ coincides with the outer Minkowski enlargement at distance r of ∂A.

Definition 12 (Upper and lower one-sided Minkowski content) *Let $A \subset \mathbb{R}^d$ be a closed set. We define the* upper *and* lower one-sided Minkowski contents *$\mathcal{SM}^*(A)$ and $\mathcal{SM}_*(A)$, respectively, as*

$$\mathcal{SM}^*(A) := \limsup_{r \downarrow 0} \frac{\mathcal{H}^d(A_r \setminus A)}{r}, \qquad \mathcal{SM}_*(A) := \liminf_{r \downarrow 0} \frac{\mathcal{H}^d(A_r \setminus A)}{r}.$$

If $\mathcal{SM}^*(A) = \mathcal{SM}_*(A) < \infty$, we denote by $\mathcal{SM}(A)$ their common value and we say that A admits one-sided Minkowski content.

Let E be a subset of $\mathbb{R}^d$; we denote by $d_E : \mathbb{R}^d \to \mathbb{R}$ the *signed distance function* from E, defined as follows:

$$d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, E^c)$$

(where "dist" denotes the usual distance function). Note that $d_{E^c}(x) = -d_E(x)$ and

$$\{x \in \mathbb{R}^d : x \in (\partial E)_r\} = \{x \in \mathbb{R}^d : |d_E(x)| \leq r\}. \tag{8}$$

It is well known that d_E is a Lipschitz function; in particular it is almost everywhere differentiable in $\mathbb{R}^d$, with $|\nabla d_E(x)| = 1$ for any point x where it is differentiable (see for instance [2], p. 11). According to the co-area formula ((2.74) in [3]) for any Borel function $g : \mathbb{R}^d \to [0, \infty]$ we have

$$\int_E g(x)|\nabla f(x)| \mathrm{d}x = \int_{-\infty}^{+\infty} \left(\int_{E \cap \{f=t\}} g(y) \mathrm{d}\mathcal{H}^{d-1}(y) \right) \mathrm{d}t. \tag{9}$$

Choosing $g(x) \equiv 1$ and $f = d_E$ in the above equation, we have

$$\begin{aligned}\mathcal{H}^d((\partial E)_r) &= \int_{(\partial E)_r} \mathrm{d}x = \int_{(\partial E)_r} |\nabla d_E(x)| \mathrm{d}x \\ &\overset{(9)}{=} \int_{-\infty}^{+\infty} \mathcal{H}^{d-1}(\{x \in (\partial E)_r : d_E(x) = t\}) \mathrm{d}t \\ &\overset{(8)}{=} \int_{-r}^{r} \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t. \end{aligned} \tag{10}$$

Similarly we have

$$\mathcal{H}^d(E_r \setminus E) = \int_0^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t. \tag{11}$$

The next statement is an easy exercise.

Lemma 13 *Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$. If*

$$\limsup_{n\to\infty} (a_n + b_n) \le (a+b), \quad \liminf_{n\to\infty} a_n \ge a, \quad \liminf_{n\to\infty} b_n \ge b,$$

then $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$.

Now we are able to show the existence of the one-sided Minkowski content when the two-sided Minkowski content exists and coincides with the perimeter and with the Hausdorff measure of the boundary.

Theorem 14 *Let $E \subset \mathbb{R}^d$ be of finite perimeter; assume that ∂E admits $(d-1)$-dimensional Minkowski content $\mathcal{M}^{d-1}(\partial E)$ and that it coincides with $P(E)$. Then E admits one-sided Minkowski content and $\mathcal{SM}(E) = P(E)$.*

Proof. We notice first that $\mathcal{M}^{d-1}(\partial E)$ implies that ∂E is Lebesgue negligible; therefore $\{d_E < t\}$ and $\{d_E > -t\}$ converge locally in measure (see for instance [3], p. 144, for the definition) to E and E^c, respectively, as $t \downarrow 0$. Let

$$a_r := \frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t, \qquad b_r := \frac{1}{r} \int_{-r}^0 \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t.$$

Then we have

$$\begin{aligned}\limsup_{r\downarrow 0} (a_r + b_r) &= \limsup_{r\downarrow 0} \frac{1}{r} \int_{-r}^{r} \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t \\ &\overset{(10)}{=} 2 \limsup_{r\downarrow 0} \frac{\mathcal{H}^d((\partial E)_r)}{2r} = 2\mathcal{M}^{d-1}(\partial E) = 2P(E).\end{aligned}$$

Furthermore

$$\begin{aligned}\liminf_{r\downarrow 0} a_r &= \liminf_{r\downarrow 0} \frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t \\ &= \liminf_{r\downarrow 0} \int_0^1 \mathcal{H}^{d-1}(\{x : d_E(x) = tr\}) \mathrm{d}t \\ &\ge \int_0^1 \liminf_{r\downarrow 0} \mathcal{H}^{d-1}(\{x : d_E(x) = tr\}) \mathrm{d}t,\end{aligned}$$

by Fatou's Lemma. By (6) we get

$$\liminf_{r\downarrow 0} a_r \geq \int_0^1 \liminf_{r\downarrow 0} P(\{x : d_E(x) < tr\})\mathrm{d}t \geq \int_0^1 P(E)\mathrm{d}t = P(E),$$

where we have used the lower semi-continuity of the map $E \mapsto P(E, \mathcal{D})$ with respect to the local convergence in measure in $\mathcal{D}$ (see [3], p. 144). Similarly

$$\liminf_{r\downarrow 0} b_r \geq P(E^c) = P(E).$$

Using Lemma 13 we conclude $\mathcal{SM}(E) \overset{(11)}{=} \lim_{r\downarrow 0} a_r = P(E)$. □

Corollary 15 *If E is a compact subset of $\mathbb{R}^d$ with Lipschitz boundary then*

$$\mathcal{SM}(E) = P(E) = \mathcal{H}^{d-1}(\partial E) < +\infty.$$

Proof. By Proposition 11 we know that $P(E) = \mathcal{H}^{d-1}(\partial E)$, while from Theorem 4 it follows that $\mathcal{M}^{d-1}(\partial E) = \mathcal{H}^{d-1}(\partial E)$. Thus Theorem 14 applies and we have $\mathcal{SM}(E) = P(E)$. □

3.1 A class of sets stable under finite unions

Definition 16 (The class $\mathcal{S}$) *Let $\mathcal{S}$ be the class of compact subsets E of $\mathbb{R}^d$ such that:*

(i) there exist $\gamma > 0$ and a probability measure η such that

$$\eta(B_r(x)) \geq \gamma r^{d-1} \qquad \forall x \in \partial E,\ \forall r \in (0, 1); \tag{12}$$

(ii) E admits one-sided Minkowski content and $\mathcal{SM}(E) = P(E)$.

The class $\mathcal{S}$ contains, for instance, all sets E with Lipschitz boundary. Indeed, it is not hard to check that condition (i) holds, for some $\gamma > 0$, with η equal to a suitable multiple of $\mathcal{H}^{d-1}_{|\partial E}$, while (ii) is a direct consequence of Corollary 15.

Remark 17 Taking into account Theorem 10, condition (i) in Definition 16 implies that ∂E has finite $\mathcal{H}^{d-1}$ measure, and therefore that E has finite perimeter. For this reason we did not impose finiteness of perimeter in the definition.

Lemma 18 *Let $G \subset \mathbb{R}^d$ be a Borel set and assume that there exist $\gamma > 0$ and a probability measure η such that*

$$\eta(B_r(x)) \geq \gamma r^{d-1} \qquad \forall x \in \partial G,\ \forall r \in (0, 1). \tag{13}$$

Then

$$\limsup_{r\downarrow 0} \frac{\mathcal{H}^d((\partial G)_r \cap G \cap B_\rho(x))}{r} = o(\rho^{d-1}) \tag{14}$$

for $\mathcal{H}^{d-1}$-a.e. $x \in G^0 \cap \partial G$.

Proof. Let $\varepsilon > 0$ be fixed. Since for all $x \in G^0 \cap \partial G$ we have

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^d(G \cap B_r(x))}{b_d r^d} = 0\,,$$

by Egorov's Theorem we can find a compact set $K^\varepsilon \subset G^0 \cap \partial G$ such that

$$\eta(G^0 \cap \partial G \setminus K^\varepsilon) < \varepsilon \qquad \text{and} \qquad \mathcal{H}^d(G \cap B_r(x)) \leq r^d \omega(r) \quad \forall x \in K^\varepsilon, \tag{15}$$

with $\lim_{r\downarrow 0} \omega(r) = 0$. Clearly $\eta((\mathbb{R}^d \setminus K^\varepsilon) \cap K^\varepsilon) = 0$; thus by (3) we may claim that

$$\eta((\mathbb{R}^d \setminus K^\varepsilon) \cap B_\rho(x)) = o(\rho^{d-1}) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in K^\varepsilon. \tag{16}$$

Let $x \in K^\varepsilon$ such that (16) holds; consider the set $E = \{y \in \partial G \cap B_{2\rho}(x) \,:\, \mathrm{dist}(y, K^\varepsilon) > r\}$, with $r \in (0,1)$, $r < \rho$. Besicovitch covering theorem implies that there exist a set I, at most countable, and an integer number ξ depending only on d such that

$$E \subset \bigcup_{i\in I} B_r(x_i)$$

with $x_i \in E$, and every point of $\mathbb{R}^d$ belongs at most to ξ balls. Let $\mathrm{card}\,(I)$ be the cardinality of I; we have

$$\gamma r^{d-1} \mathrm{card}\,(I) = \sum_{i\in I} \gamma r^{d-1} \leq \sum_{i\in I} \eta(B_r(x_i)) \leq \xi \eta((\partial G \cap B_{2\rho}(x))_r \setminus K^\varepsilon),$$

so that

$$\mathrm{card}\,(I) \leq \xi \gamma^{-1} r^{1-d} \eta((\partial G \cap B_\rho(x))_r \setminus K^\varepsilon). \tag{17}$$

As a consequence

$$\begin{aligned} \mathcal{H}^d(E_r) &\leq \mathcal{H}^d(\bigcup_{i\in I} B_{2r}(x_i)) \overset{(17)}{\leq} \xi \gamma^{-1} r^{1-d} \eta((\partial G \cap B_\rho(x))_r \setminus K^\varepsilon) b_d (2r)^d \\ &= 2^d \xi \gamma^{-1} b_d r \eta((\partial G \cap B_{2\rho}(x))_r \setminus K^\varepsilon). \end{aligned}$$

Therefore

$$\limsup_{r\downarrow 0} \frac{\mathcal{H}^d(E_r)}{r} \leq 2^d \xi \gamma^{-1} b_d \eta(\partial G \cap B_\rho(x) \setminus K^\varepsilon) \leq 2^d \xi \gamma^{-1} b_d \eta(\mathbb{R}^d \cap B_{2\rho}(x) \setminus K^\varepsilon) \overset{(16)}{=} o(\rho^{d-1}). \tag{18}$$

Note that $(\partial G)_r \cap B_\rho(x) \subseteq E_r \cup K^\varepsilon_{2r}$. Indeed, for any $y \in (\partial G)_r \cap B_\rho(x)$ there exists $z \in \partial G$ such that $\mathrm{dist}(y,z) < r$, and then $z \in \partial G \cap B_{r+\rho}(x) \subset \partial G \cap B_{2\rho}(x)$. If $\mathrm{dist}(z, K^\varepsilon) > r$, then $z \in E$ and $y \in E_r$; if $\mathrm{dist}(z, K^\varepsilon) \leq r$, then $y \in K^\varepsilon_{2r}$. Using this inclusion, we can deduce (14) from (18) provided we show that $\mathcal{H}^d(K^\varepsilon_{2r} \cap G) = o(r)$. Let $\{B_{2r}(x_j)\}_{j\in J}$ be a covering of K^ε such that $x_j \in K^\varepsilon$ for all $j \in J$, and $\mathrm{dist}(x_i, x_j) > 2r$ for any $i \neq j$. It follows that $B_r(x_i) \cap B_r(x_j) = \emptyset$ for any $i \neq j$ and

$$\gamma r^{d-1} \mathrm{card}\,(J) = \sum_{j\in J} \gamma r^{d-1} \leq \sum_{j\in J} \eta(B_r(x_j)) \leq \eta(\mathbb{R}^d) = 1.$$

As a consequence $\mathrm{card}\,(J) \leq \gamma^{-1} r^{1-d}$, and

$$\mathcal{H}^d(K^\varepsilon_{2r} \cap G) \leq \mathcal{H}^d(\bigcup_{j\in J} (B_{4r}(x_j) \cap G)) \overset{(15)}{\leq} \gamma^{-1} r^{1-d} (4r)^d \omega(4r) = o(r).$$

In this way we proved (14) for $\mathcal{H}^{d-1}$-a.e. $x \in K^\varepsilon$. Now let $\varepsilon_i = 1/i$, $i = 1, 2, \ldots$, and consider the union B of the compact sets K^{ε_i}; we know that $\eta(G^0 \cap \partial G \setminus B) = 0$, and that (14) holds for $\mathcal{H}^{d-1}$-a.e. $x \in B$. Theorem 10 and condition (13) imply that $\eta \geq \gamma b_{d-1}^{-1} \mathcal{H}^{d-1}_{|\partial G}$; thus $\mathcal{H}^{d-1}(G^0 \cap \partial G \setminus B) = 0$ and the claim of the lemma follows. □

Theorem 19 *The class $\mathcal{S}$ is stable under finite unions.*

Proof. Let $E_1, E_2 \in \mathcal{S}$ and let E be their union. Clearly, E is compact and it is easy to check that condition (i) of Definition 16 holds with $\eta := (\eta_1 + \eta_2)/2$ and $\gamma := \min\{\gamma_1, \gamma_2\}/2$ (here η_i and γ_i, $i = 1, 2$, satisfy condition (i) relative to E_i).

The verification of the stability of condition (ii) is much less trivial. By the co-area formula, as shown in the proof of Theorem 14, we know that $\mathcal{SM}_*(F) \geq P(F)$ for any $F \subset \mathbb{R}^d$, thus condition (ii) is satisfied if $\mathcal{SM}^*(E) \leq P(E)$. In order to prove this inequality let us localize it, defining

$$\mathcal{SM}_*(F, B) := \liminf_{r \downarrow 0} \frac{|(F_r \setminus F) \cap B|}{r}, \qquad \mathcal{SM}^*(F, C) := \limsup_{r \downarrow 0} \frac{|(F_r \setminus F) \cap C|}{r},$$

for any open set B and any closed set C in $\mathbb{R}^d$. Again by the co-area formula, we have that

$$\mathcal{SM}_*(F, B) \geq P(F, B) \tag{19}$$

for any open subset B of $\mathbb{R}^d$. As a consequence, for every closed set $C \subset \mathbb{R}^d$ we have

$$\mathcal{SM}^*(F, C) \leq \mathcal{SM}^*(F) - \mathcal{SM}_*(F, C^c) \leq P(F) - P(F, C^c) = P(F, C) \qquad \forall F \in \mathcal{S}. \tag{20}$$

Further, it is easy to check that $\mathcal{SM}^*(F, C)$ is sub-additive with respect to both F and C; so by (20) we have

$$\mathcal{SM}^*(E, C) \leq \mathcal{SM}^*(E_1, C) + \mathcal{SM}^*(E_2, C) \leq P(E_1, C) + P(E_2, C) \tag{21}$$

for every closed set $C \subset \mathbb{R}^d$.

In the sequel it will also be useful a stronger version of the sub-additivity, namely

$$\mathcal{SM}^*(E, B_\rho(x)) + P(E_1 \cap E_2, \operatorname{int} B_\rho(x)) \leq \mathcal{SM}^*(E_1, B_\rho(x)) + \mathcal{SM}^*(E_2, B_\rho(x)). \tag{22}$$

Using the identity $1_{E_1 \cup E_2} + 1_{E_1 \cap E_2} = 1_{E_1} + 1_{E_2}$ and the inequality

$$1_{(E_1 \cup E_2)_r} + 1_{(E_1 \cap E_2)_r} \leq 1_{(E_1)_r} + 1_{(E_2)_r}$$

we obtain

$$\mathcal{SM}^*(E, B_\rho(x)) + \mathcal{SM}_*(E_1 \cap E_2, \operatorname{int} B_\rho(x)) \leq \mathcal{SM}^*(E_1, B_\rho(x)) + \mathcal{SM}^*(E_2, B_\rho(x)),$$

which implies (22) taking into account (19).

Note also that $\partial^* E \subset \partial^* E_1 \cup \partial^* E_2$ and that, by Proposition 2.85 in [3], for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E_1 \cap \partial^* E_2$ the generalized inner normal $\nu_{E_1}(x)$ and $\nu_{E_2}(x)$ to E_1 and E_2 respectively (see [3]), are opposite or equal. As a consequence, for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E_1 \cup \partial^* E_2$, we have that one of the following facts occurs.

1. $x \in E^1$. In this case we have that $\mathcal{SM}^*(E, B_\rho(x)) = o(\rho^{d-1})$, $\mathcal{H}^{d-1}$-a.e. Indeed, if $x \notin \partial E$ then $\mathcal{SM}^*(E, B_\rho(x)) = 0$ for ρ sufficiently small; if $x \in \partial E$ Lemma 18 applies with $G = E^c$ and observing that $\mathcal{H}^d((E_r \setminus E) \cap B_\rho(x)) \leq \mathcal{H}^d((\partial G)_r \cap G \cap B_\rho(x))$. Notice that this case includes all points $x \in \partial^* E_1 \cap \partial^* E_2$, with $\nu_{E_1}(x) = -\nu_{E_2}(x)$.

2. $x \in \partial^* E_1 \setminus \partial^* E_2$ and $x \notin E^1$. In this case $x \in (E_2)^0$ and therefore $x \in \partial^* E$. We have also

$$\mathcal{SM}^*(E, B_\rho(x)) \le b_{d-1}\rho^{d-1} + o(\rho^{d-1}) = P(E, B_\rho(x)) + o(\rho^{d-1}),$$

$\mathcal{H}^{d-1}$-a.e. This is a direct consequence of (21) and (5) (applied to E_1 and E) and (4) (applied to E_2).

3. $x \in \partial^* E_2 \setminus \partial^* E_1$ and $x \notin E^1$. As in the previous case we have that $\mathcal{SM}^*(E, B_\rho(x)) \le P(E, B_\rho(x)) + o(\rho^{d-1})$, $\mathcal{H}^{d-1}$-a.e.

4. $x \in \partial^* E_1 \cap \partial^* E_2$, with $\nu_{E_1}(x) = \nu_{E_2}(x)$. In this case E, E_1, E_2 and $E_1 \cap E_2$ have density $1/2$ at x and we can apply (22) and (5) (to E_1, E_2, E and $E_1 \cap E_2$) to obtain $\mathcal{SM}^*(E, B_\rho(x)) \le P(E, B_\rho(x)) + o(\rho^{d-1})$, $\mathcal{H}^{d-1}$-a.e.

For any $\varepsilon > 0$ let us define $\mu_\varepsilon := \varepsilon|D\chi_{E_1}| + \varepsilon|D\chi_{E_2}| + |D\chi_E|$. We have proved before that

$$\limsup_{\rho \downarrow 0} \frac{\mathcal{SM}^*(E, B_\rho(x))}{\mu_\varepsilon(B_\rho(x))} \le 1 \qquad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \partial^* E_1 \cup \partial^* E_2. \tag{23}$$

Mimicking the proof of differentiation theorems for measures, with a sub-additive set function in the numerator (instead of a measure), we are going to use (23) to show that $\mathcal{SM}^*(E) \le P(E)$ (notice that the ε term in μ_ε takes into account points which are not in $\partial^* E$, while in the union of the two essential boundaries). By Vitali-Besicovitch covering Theorem (see e.g. [3], p. 52), for any $\delta > 0$ there exists a finite covering $C_1, \dots, C_N$ of $\partial^* E_1 \cup \partial^* E_2$ where the C_i's are disjoint closed balls of $\mathbb{R}^d$, such that

$$\mu_\varepsilon((\partial^* E_1 \cup \partial^* E_2) \setminus \cup_i C_i) < \varepsilon\delta.$$

Note that $\mu_\varepsilon((\partial^* E_1 \cup \partial^* E_2) \setminus \cup_i C_i) = \mu_\varepsilon(\mathbb{R}^d \setminus \cup_i C_i)$. The balls C_i's can be chosen with centers in $\partial^* E_1 \cup \partial^* E_2$ such that the fraction in (23) does not exceed $(1+\delta)$, and further with $\mu_\varepsilon(\partial C_i) = 0$. Thus we have

$$\mathcal{SM}^*(E, C_i) \le (1+\delta)\mu_\varepsilon(C_i), \qquad i = 1, \dots, N.$$

Let $C := \mathbb{R}^d \setminus (\bigcup_{i=1}^N \mathrm{int} C_i)$; it follows from by (21) that

$$\mathcal{SM}^*(E, C) \le |D\chi_{E_1}|(C) + |D\chi_{E_2}|(C) \le \frac{\mu_\varepsilon(C)}{\varepsilon} < \delta.$$

Hence,

$$\begin{aligned}
\mathcal{SM}^*(E) &\le \mathcal{SM}^*(E, C) + \mathcal{SM}^*(E, \bigcup_{i=1}^N C_i) \le \delta + \sum_{i=1}^N \mathcal{SM}^*(E, C_i) \\
&\le \delta + (1+\delta)\sum_{i=1}^N \mu_\varepsilon(C_i) = \delta + (1+\delta)\mu_\varepsilon(\bigcup_{i=1}^N C_i) \\
&= \delta + (1+\delta)(\varepsilon P(E_1) + \varepsilon P(E_2) + P(E)).
\end{aligned} \tag{24}$$

Finally, by taking the limit in (24) as $\delta \downarrow 0$, and then as $\varepsilon \downarrow 0$, we see that $\mathcal{SM}^*(E) \le P(E)$ which proves condition (ii). □

Using the same local contents $\mathcal{SM}_*$ and $\mathcal{SM}^*$ introduced in the proof of Theorem 19, a local existence result for the one-sided Minkowski content can be proved.

Proposition 20 *Let E be a closed subset of $\mathbb{R}^d$ such that $\mathcal{SM}(E) = P(E)$. Then, for any open set A such that $P(E, \partial A) = 0$,*

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d((E_r \setminus E) \cap A)}{r} = P(E, A).$$

Proof. By the same argument in the proof of Theorem 19 we know that for any B, $C \subset \mathbb{R}^d$, open and closed respectively,

$$\mathcal{SM}_*(E,B) \geq P(E,B), \qquad \mathcal{SM}^*(E,C) \leq P(E,C),$$

and, by assumption, $P(E,\partial A) = 0$. Hence,

$$\liminf_{r\downarrow 0} \frac{\mathcal{H}^d((E_r \setminus E) \cap A)}{r} \geq P(E,A) = P(A, \mathrm{cl}A) \geq \limsup_{r\downarrow 0} \frac{\mathcal{H}^d((E_r \setminus E) \cap A)}{r}.$$

□

In particular Proposition 20 can be applied to any set in $\mathcal{S}$ and yields the following corollary.

Definition 21 (The class $\mathcal{S}_{\mathrm{loc}}$) *Let us denote by $\mathcal{S}_{\mathrm{loc}}$ the class of closed sets E such that for any $R > 0$ there exists $F \in \mathcal{S}$ with $(E\Delta F) \cap B_R(0) = \emptyset$.*

Heuristically, this class corresponds to the sets that locally coincide with sets in $\mathcal{S}$.

Corollary 22 *If $E \in \mathcal{S}_{\mathrm{loc}}$, then*

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^d((E_r \setminus E) \cap A)}{r} = P(E,A)$$

for any open set $A \subset \mathbb{R}^d$ with $P(E,\partial A) = 0$. In particular

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^d((E_r \setminus E) \cap B_R(0))}{r} = P(E, B_R(0))$$

for every $R > 0$ with at most countably many exceptions.

Proof. By the assumption of the lemma, there exists $F \in \mathcal{S}$ such that the symmetric difference $F\Delta E$ has no intersection with A_1. Then $(E_r \setminus E) \cap A = (F_r \setminus F) \cap A$ for any $r < 1$, and by Proposition 20 we have

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^d((E_r \setminus E) \cap A)}{r} = P(F,A) = P(E,A).$$

The last part of the statement follows by the fact that the set of radii $R > 0$ such that $\mathcal{H}^{d-1}(\partial E \cap \partial B_R(0)) > 0$ is at most countable. □

We close this section pointing out some simpler "regularity" properties that are stable under finite unions. A natural one, often considered in topology, is $E = \mathrm{cl}\,(\mathrm{int}\, E)$ (for instance in this way we rule out sets made by pieces of different dimensions); since most of our results are concerned with the essential boundary, we consider a weaker, and therefore more general, condition, namely $\partial E = \mathrm{cl}\,(\partial^* E)$.

Lemma 23 *Let A, B be closed subsets of $\mathbb{R}^d$ such that $\mathrm{cl}\,(\partial^* A) = \partial A$ and $\mathrm{cl}\,(\partial^* B) = \partial B$. Then $\mathrm{cl}\,(\partial^*(A \cup B)) = \partial(A \cup B)$.*

Proof. We need only to show, arguing by contradiction, that $\partial(A \cup B)$ is contained in the closure of $\partial^*(A \cup B)$. Recalling that the distributional derivative of the characteristic function is concentrated on the essential boundary, if x belongs to $\partial(A \cup B)$ but not to $\mathrm{cl}(\partial^*(A \cup B))$, then $1_{A\cup B}$ is a.e. equal to a constant c in a neighborhood $\mathcal{U}$ of x. If $c = 1$, then $A \cup B$ is dense in $\mathcal{U}$, and being closed we obtain that $A \cup B \supset \mathcal{U}$. Therefore $x \notin \partial(A \cup B)$. If $c = 0$, then both A and B have a Lebesgue negligible intersection with $\mathcal{U}$, so that x belongs neither to the closure of $\partial^* A$ nor to the closure of $\partial^* B$. It follows that $x \notin \partial A \cup \partial B$, which contains $\partial(A \cup B)$. □

4 One-sided Minkowski content of sets with positive reach

We start by recalling the definition of set with positive reach introduced by Federer in [9]. For $A \subset \mathbb{R}^d$ closed, let $\mathrm{Unp}(A)$ be the set of points having a unique projection (or foot-point) on A:

$$\mathrm{Unp}(A) := \{a \in \mathbb{R}^d \,:\, \exists!\, x \in A \text{ s.t. } \mathrm{dist}(x, A) = \|a - x\|\}.$$

The definition of $\mathrm{Unp}(A)$ implies the existence of a projection mapping $\xi_A : \mathrm{Unp}(A) \to A$ which assigns to $x \in \mathrm{Unp}(A)$ the unique point $\xi_A(x) \in A$ such that $\mathrm{dist}(x, A) = \|x - \xi_A(x)\|$. For every $a \in A$ we set:

$$\mathrm{reach}(A, a) = \sup\{r > 0 \,:\, B_r(a) \subset \mathrm{Unp}(A)\}.$$

The reach of A is then defined by:

$$\mathrm{reach}(A) = \inf_{a \in A} \mathrm{reach}(A, a),$$

and A is said to be a set with positive reach if $\mathrm{reach}(A) > 0$.

Let us briefly recall the notions of tangent and normal cone that will be used later. If A is a compact set and $x \in A$, the tangent and the normal spaces to A at x are, respectively:

$$\tan(A, x) = \{0\} \cup \left\{ u \,:\, \forall \epsilon > 0 \;\exists\, y \in A \,\text{s.t.}\, 0 < \|y - x\| < \epsilon \,,\; \left\| \frac{y - x}{\|y - x\|} - \frac{u}{\|u\|} \right\| < \epsilon \right\} ,$$

$$\mathrm{nor}(A, x) = \{v \,:\, (u, v) \le 0 \,,\, \forall u \in \tan(A, x)\} \,.$$

One of the main features of sets with positive reach is that they obey a (local and global) Steiner formula. The following statement is part of Theorem 5.6 in [9].

Theorem 24 *If $A \subset \mathbb{R}^d$ and $\mathrm{reach}(A) > 0$, then there exist unique Radon measures $\psi_0, \psi_1, \ldots, \psi_d$ over $\mathbb{R}^d$ such that, for $0 \le r < \mathrm{reach}(A)$,*

$$\mathcal{H}^d(\{x \,:\, \mathrm{dist}(x, A) \le r \text{ and } \xi_A(x) \in B\}) = \sum_{m=0}^{d} r^{d-m} b_{d-m} \psi_m(B), \tag{25}$$

whenever B is a Borel set of $\mathbb{R}^d$.

The Radon measures $\psi_0, \psi_1, \ldots, \psi_d$ are called *curvature measures associated with* A, and their supports are contained in A. Formula (25) is a local Steiner formula for A. Choosing $B = A$ and assuming that $\psi_0(A), \psi_1(A), \ldots, \psi_d(A)$ are finite (which happens for instance if A is compact) we obtain a global Steiner formula

$$\mathcal{H}^d(A_r) = \sum_{m=0}^{d} r^{d-m} b_{d-m} \Phi_m(A), \tag{26}$$

where $\Phi_i(A) = \psi_i(A)$, $i = 0, 1, \ldots, d$, are called *total curvatures* of A. As a straightforward consequence, if A is a compact set and $\mathrm{reach}(A) > 0$ then A admits one-sided Minkowski content and

$$\mathcal{SM}(A) = 2\Phi_{d-1}(A). \tag{27}$$

Sets with positive reach include the important subclass of *convex bodies*, i.e. compact convex subsets in $\mathbb{R}^d$. Steiner formulas and curvature measures for these sets are studied in [17]. In particular, if C is a convex body with non-empty interior then $\Phi_{d-1}(C) = \frac{1}{2}\mathcal{H}^{d-1}(\partial C)$ so that

$$\mathcal{SM}(C) = \mathcal{H}^{d-1}(\partial C).$$

Note also that $\mathcal{H}^{d-1}(\partial C) = P(C)$ for any d-dimensional (i.e. with non-empty interior) convex body $C \subset \mathbb{R}^d$.

Turning back to the general case of sets with positive reach, a natural problem is to compare the measurements of the boundary given by the $(d-1)$-Hausdorff measure, the perimeter and the one-sided Minkowski content. In the remaining part of this section we provide some results in this direction. We start with the following proposition.

Proposition 25 *If A is a compact subset of $\mathbb{R}^d$ with $\operatorname{reach}(A) = r > 0$, then ∂A is $(d-1)$-rectifiable.*

Proof. Let $s \in (0, r)$ be fixed. Consider the sets

$$A_s := \{x \,|\, \operatorname{dist}(x, A) \leq s\}, \quad A'_s := \{x \,|\, \operatorname{dist}(x, A) \geq s\}, \quad \Sigma_s := \{x \,|\, \operatorname{dist}(x, A) = s\}.$$

We first show that Σ_s is a $(d-1)$-rectifiable set. It is clear that Σ_s is closed, and it is compact since A is bounded. By Theorem 4.8 (5) in [9], $\operatorname{dist}(\,\cdot\,, A)$ is continuously differentiable in $\operatorname{int}(\operatorname{Unp}(A) \setminus A)$, which is an open set containing Σ_s; by (3) of the same theorem, we have that $\|\nabla\delta\| \equiv 1$ on Σ_s. By the implicit function theorem, Σ_s is locally the graph of a C^1 function of $(d-1)$-variables. Since Σ_s is compact, this local regularity property easily gives that Σ_s is $(d-1)$-rectifiable: indeed, we can write

$$\Sigma_s = \Sigma_s^1 \cup \cdots \cup \Sigma_s^N,$$

where $\Sigma_s^i = f_i(B_i)$ with B_i bounded subset of $\mathbb{R}^{d-1}$, and f_i Lipschitz function. Without loss of generality, we may suppose that the B_i's are disjoint balls. Let $B = \bigcup_{i=1}^N B_i$; by Kirszbraun extension theorem there exists a Lipschitz map f extending f_i on B_i, so that $\Sigma_s = f(B)$ and Σ_s is $(d-1)$-rectifiable.

By Theorem 4.8 (8) in [9], ξ_A is a Lipschitz function on A_s, with Lipschitz constant less than $\frac{r}{r-s}$. It is easy to see that if $x \notin A$, then $\xi_A(x) \in \partial A$. We show that the Lipschitz map

$$\xi_A : \Sigma_s \longrightarrow \partial A$$

is surjective. By Corollary 4.9 in [9], $\operatorname{reach}(A'_s) \geq s$; in particular the projection map $\xi_{A'_s} : A_s \to A'_s$ is defined in $A_s \setminus A$. Let $a \in \partial A$, and $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of points $a_i \notin A$, such that $\lim\limits_{i\to\infty} a_i = a$, with $\|a_i - a\| < s/2$ for all $i \in \mathbb{N}$. Then, by Corollary 4.9 in [9],

$$\xi_A[\xi_{A'_s}(a_i)] = \xi_A(a_i). \tag{28}$$

Observe that $\xi_{A'_s}(a_i) \in \partial A'_s$ and $\partial A'_s \subseteq \Sigma_s$. Since Σ_s is compact, there exists $y \in \Sigma_s$ such that $\xi_{A'_s}(a_i) \to y$. By the continuity of ξ_A on A_s, we have

$$\xi_A(y) = \xi_A[\lim_{i\to\infty} \xi_{A'_s}(a_i)] = \lim_{i\to\infty} \xi_A[\xi_{A'_s}(a_i)] \stackrel{(28)}{=} \lim_{i\to\infty} \xi_A(a_i) = \xi_A(a) = a,$$

i.e. ξ_A is surjective. Summarizing, we have that the map $\xi_A \circ f : B \longrightarrow \partial A$ is Lipschitz and surjective so that ∂A is $(d-1)$-rectifiable. □

As a consequence of the above proposition and Theorem 2 we have that

Corollary 26 *If A is a compact subset of $\mathbb{R}^d$ with positive reach, then $\mathcal{M}^{d-1}(\partial A) = \mathcal{H}^{d-1}(\partial A)$.*

The next result is a special case of Theorem 5.5 in [8]. We denote by $\mathbf{S}^{d-1}$ the unit sphere in $\mathbb{R}^d$.

Theorem 27 *If A is a set of positive reach in $\mathbb{R}^d$ then*

$$2\Phi_{d-1}(A) = \int_{\partial A} \mathcal{H}^0(\operatorname{Nor}(A, x) \cap \mathbf{S}^{d-1}) \mathrm{d}\mathcal{H}^{d-1}(x). \tag{29}$$

In view of (27) and taking into account that $\mathcal{H}^0(\mathrm{Nor}(A,x)\cap \mathbf{S}^{d-1}) \in \{1,2\}$ for $\mathcal{H}^{d-1}$-a.e. $x\in\partial A$ (see for instance the proof of Theorem 28 below), the previous result provides immediately the following inequalities

$$\mathcal{H}^{d-1}(\partial A) \leq \mathcal{SM}(A) \leq 2\mathcal{H}^{d-1}(\partial A)$$

for every set A with positive reach. Condition (30) in the next theorem is necessary and sufficient for having equality in the first inequality; moreover, the next theorem shows that, in the case of equality, both quantities coincide with the perimeter, and therefore the set belongs to $\mathcal{S}$.

Theorem 28 *Let $A\subset\mathbb{R}^d$ be a compact set with positive reach such that*

$$\mathcal{H}^0(\mathrm{Nor}(A,x)\cap \mathbf{S}^{d-1}) = 1 \qquad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x\in\partial A. \tag{30}$$

Then A belongs to the class $\mathcal{S}$ in Definition 16 and, in particular,

$$P(A) = \mathcal{H}^{d-1}(\partial A) = \mathcal{SM}(A). \tag{31}$$

Proof. Condition (i) in Definition 16 is fulfilled thanks to Remark 3 because we know, by Proposition 25, that ∂A is $(d-1)$-rectifiable. Let us establish the first equality in (31). By the proof of Proposition 25 we know that $\xi(x)\in\partial A$ for any $x\in A_r\setminus A$, and in particular the map $\xi_A : A_r\setminus A\to\partial A$ is surjective. This, with Theorem 4.8(12) in [9], implies that $\dim(\mathrm{Nor}(A,x))\geq 1$ for all $x\in\partial A$. By Remark 4.15(3) in [9] it follows that $\mathcal{H}^{d-1}(\{x\in\partial A : \dim(\mathrm{Nor}(A,x))\geq 2\}) = 0$. Hence we may claim that

$$\dim(\mathrm{Nor}(A,x)) = 1 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x\in\partial A. \tag{32}$$

Note that $\mathrm{Nor}(A,x))$ is a convex cone; thus by (30) and (32) we may claim that for $\mathcal{H}^{d-1}$ a.e. $x\in\partial A$

$$\exists v\in\mathbb{R}^d\,,\ |v| = 1 \ :\ \mathrm{Nor}(A,x) = \{\lambda v : \lambda\geq 0\}. \tag{33}$$

We denote by $\mathcal{B}$ the set of points of ∂A where (33) holds. We have

$$\mathcal{H}^{n-1}(\partial A\setminus\mathcal{B}) = 0.$$

Let $x\in\mathcal{B}$; as $\mathrm{reach}(A) > 0$, by Theorem 4.8(12) in [9], $\mathrm{Tan}(A,x)$ is a half-space. Again by a result proved in [9], namely Remark 4.15(2), we easily deduce that the density of A at x is greater than or equal to 1/2. On the other hand, by the definition of sets of positive reach (and the surjectivity of ξ_A onto ∂A) A admits an outer sphere at each point of its boundary. Hence the density of A at x is not greater than 1/2. Thus we have that $\mathcal{H}^{d-1}(\partial A\setminus\partial^* A)\leq\mathcal{H}^{d-1}(\partial A\setminus A^{1/2}) = 0$, and so $P(A) = \mathcal{H}^{d-1}(\partial A)$ by Theorem 9.

By Remark 17 and Corollary 26 we have that A has finite perimeter and $P(A) = \mathcal{M}^{d-1}(\partial A)$, so the second equality in (31) directly follows by Theorem 14. □

As a consequence, we have the following result on the one-sided Minkowski content for unions of sets with positive reach (which need not be sets of positive reach).

Theorem 29 *the equality $\mathcal{SM}(A) = P(A)$ holds for any union A of finitely many compact sets A_i of positive reach satisfying $\mathcal{SM}(A_i) = P(A_i)$.*

Remark 30 A local result for possibly non compact locally finite unions can be easily obtained by Proposition 20 and Corollary 22. If the union is not locally finite, the property is in general not true (see Example 1 in Section 5).

5 Examples

In this section we collect some "critical" examples of sets related to the definitions and the results that we have presented so far.

Example 1 We construct a compact set $A \subset \mathbb{R}^2$ with positive reach, such that $\mathrm{cl}(\partial^* A) = \partial A$, for which

$$P(A) < \mathcal{H}^1(A) < \mathcal{SM}(A)$$

(in particular A does not satisfy condition (30)). Let $Q = \mathbb{Q} \cap [0,1]$; then Q is a countable set $Q = \{q_1, q_2, \dots\}$. Let $\varepsilon \in (0, 1/2)$; we define

$$B_i := \left(q_i - \frac{\varepsilon}{2^i}, q_i + \frac{\varepsilon}{2^i}\right) \;\; i \in \mathbb{N}, \qquad B := \bigcup_{i=1}^{\infty} B_i, \qquad C := [0,1] \setminus B.$$

B is an open set with $\mathrm{cl}B = [0,1]$, while C is a closed subset in $[0,1]$ with $\mathcal{H}^1(C) > 0$. Let $\phi \in C^2(\mathbb{R})$ be such that $\phi(x) > 0$ in $(-1,1)$ and $\phi \equiv 0$ in $(-\infty, -1] \cup [1, \infty)$. Let us define

$$\phi_i(x) := \phi\left(\frac{2^i}{\varepsilon}(x - q_i)\right), \quad x \in \mathbb{R}, \; i \in \mathbb{N}.$$

Note that for all i, $\phi_i > 0$ in B_i and $\phi_i \equiv 0$ in $\mathbb{R} \setminus B_i$; in particular $\phi_i \equiv 0$ in C. Let $\alpha = \inf(B)$, $\beta = \sup(B)$, $I = [\alpha, \beta]$, and let $f : I \to \mathbb{R}$ be the function defined in the following way:

$$f := \sum_{i=1}^{\infty} \left(\frac{\varepsilon}{2^i}\right)^3 \phi_i.$$

It follows that $f \in C^2(I)$ and $f(x) > 0$ for all $x \in B$, $f(x) = 0$ for all $x \in C$. By the properties of f, $\mathrm{graf}(f) = \{(x,y) \in \mathbb{R}^2 : x \in I, \, y = f(x)\}$ is a set with positive reach (see [9]). Let

$$A := \{(x,y) : x \in I, \, 0 \le y \le f(x)\}.$$

A is a compact set and $A = \mathrm{cl}(\mathrm{int}A)$, since B is dense in I and $f(x) > 0$ for any $x \in B$. Moreover $\mathrm{reach}A > 0$ and

- $\mathcal{H}^0(\mathrm{Nor}(B,x) \cap \mathbf{S}^{d-1}) = 1$ for all $x \in \partial A \setminus C$,
- $\mathcal{H}^0(\mathrm{Nor}(B,x) \cap \mathbf{S}^{d-1}) = 2$ for all $x \in C$,

since $f \in C^2(I)$ and $f(x) = 0$ for all $x \in C$. In particular condition (30) is not satisfied. By (27) and (29)

$$\mathcal{SM}(A) = \mathcal{H}^1(\partial A) + \mathcal{H}^1(C).$$

Moreover, it is easy to see that $\partial^* A = \partial A \setminus C$, so that $\mathcal{SM}(A) = P(A) + 2\mathcal{H}^1(C)$. In particular, $\mathrm{cl}(\partial^* A) = \partial A$; thus A provides also an example of compact set in $\mathbb{R}^2$ with $\mathrm{cl}(\partial^* A) = \partial A$, but $\mathcal{H}^1(\partial A \setminus \partial^* A) > 0$.

Example 2 We define two sets with positive reach and with Lipschitz boundary such that their union has not positive reach (but belongs, as we know, to $\mathcal{S}$). In particular we will see that even if the perimeter of both sets equals the $(d-1)$-dimensional Hausdorff measure of the topological boundary, the perimeter of the union set, equal to the one-sided Minkowski content, is different from the $(d-1)$-dimensional measure of the topological boundary. Consider the set C and the function f in Example 1. Let E_1 and E_2 be the compact sets defined by

$$E_1 := \{(x,y) : x \in I,\, f(x) \le y \le M\}, \qquad E_2 := \{(x,y) : x \in I,\, -M \le y \le -f(x)\},$$

with $M > \max_x f(x)$. It is clear that E_i has Lipschitz boundary and $E_i \in \mathcal{S}$ with $\mathcal{SM}(E_i) = P(E_i) = \mathcal{H}^{d-1}(\partial E_i)$, for $i = 1, 2$. Notice that the set $E_1 \cup E_2$ has density 1 at each point of C; it follows that $\mathcal{SM}(E_1 \cup E_2) = P(E_1 \cup E_2) < \mathcal{H}^{d-1}(\partial(E_1 \cup E_2))$.

Example 3 Theorem 4 provides a sufficient condition for the existence of the n-dimensional Minkowski content of a countably $\mathcal{H}^n$-rectifiable compact set. In this example we present a compact set S which is the countable union of sets each admitting one-sided Minkowski content, such that $\mathcal{SM}(S)$ does not exist. In a similar way as in Example 2.103 in [3], consider the set $S := \bigcup_n S^{(n)} \cup \{0\}$, where for any $n \ge 1$ the set $S^{(n)}$ is a finite union of (sufficiently many and sufficiently small) balls in $\mathbb{R}^d$ satisfying:

(a) $S^{(n)} \subset B_{1/n}(0) \setminus B_{1/(n+1)}(0)$;

(b) $(S^{(n)})_{2^{-n-1}} \supset B_{1/n}(0) \setminus B_{1/(n+1)}(0)$;

(c) $\mathcal{H}^d(S^{(n)}) \le 2^{-1} b_d (n^{-d} - (n+1)^{-d})$.

For $\rho > 0$ let n be such that $2^{-n-1} \le \rho < 2^{-n}$, and so

$$\begin{aligned} \mathcal{H}^d(S_\rho \setminus S) \quad &\ge \quad \mathcal{H}^d(S^{(n)}_{2^{-n-1}} \setminus S) \overset{(b)}{\ge} \mathcal{H}^d(B_{1/n}(0) \setminus (B_{1/(n+1)}(0) \cup S)) \\ &\overset{(a)}{=} \quad \mathcal{H}^d(B_{1/n}(0) \setminus (B_{1/(n+1)}(0) \cup S^{(n)})) \overset{(c)}{\ge} \frac{b_d}{2}\big(n^{-d} - (n+1)^{-d}\big). \end{aligned}$$

As $n^{-d} - (n+1)^{-d} \sim n^{-d-1} \sim [\ln_2(1/\rho)]^{-d-1} \gg \rho$, we obtain that $\mathcal{SM}_*(S) = +\infty$. Since $\mathcal{H}^d(S_\rho \setminus S) \le \mathcal{H}^d((\partial S)_\rho)$, ∂S does not admit $(d-1)$-dimensional Minkowski content as well. Clearly ∂S is a countably $\mathcal{H}^{d-1}$-rectifiable compact set; this example shows the necessity, in Theorem 4, of additional conditions like (2), besides rectifiability.

6 Mean one-sided Minkowski content for random closed sets

As we mentioned in the introduction, the existence of the one-sided Minkowski content is closely related to several problems in real applications, many of them modelled in a stochastic setting. Then, considering the expected values, the problem becomes the existence of a *mean* one-sided Minkowski content.

Let Θ be a *random closed set* in $\mathbb{R}^d$, i.e. a measurable map

$$\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space, while $\mathbb{F}$ and $\sigma_{\mathbb{F}}$ denote, respectively, the class of the closed subsets in $\mathbb{R}^d$ and the σ-algebra generated by the so called *hit-or-miss topology* (see, e.g., [15]). Then it is clear that $\mathcal{H}^d(\Theta_r \setminus \Theta)/r$ is a random quantity.

Definition 31 *We say that* Θ *admits* mean *one-sided Minkowski content if*

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_r \setminus \Theta)]}{r} \tag{34}$$

exists and is finite.

To treat directly mean one-sided Minkowski contents seems to be a quite delicate problem. We assume that $\mathcal{SM}(\Theta)$ exists almost surely and we obtain the mean one-sided Minkowski content by a uniform integrability condition, which can be considered as the stochastic analogous of condition (12). In this way we may exchange the limit and the expectation in (34) and, in particular, we have that the above limit exists and coincides with $\mathbb{E}[\mathcal{SM}(\Theta)]$.

We give here a local version of the existence of the mean one-sided Minkowski content for a random closed set, so that a global version for random compact sets follows as a particular case. Note that in many stochastic processes (e.g. random lines, Boolean models, see [15]) the involved random closed set Θ is not compact, and in many real applications it is customary to look at Θ through a compact window W; thus it is clear that a local version of the mean one-sided Minkowski content is of particular interest in these cases.

Let $\mathcal{S}_{\rm loc}$ be defined in Definition 21, let $W \subset \mathbb{R}^d$ be a compact set and set, for $\Theta \in \mathcal{S}_{\rm loc}$,

$$\Gamma_W(\Theta) := \max\big\{\gamma \geq 0 \,:\, \exists \text{ a probability measure } \eta \text{ such that} \\ \eta(B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial\Theta \cap W_1,\ \forall r \in (0,1)\big\}.$$

Γ_W is strictly positive by condition (i) in Definition 16. The next theorem invokes an integrability assumption on $1/\Gamma_W(\Theta)$ and the assumption that the process is $\mathcal{S}_{\rm loc}$-valued; as in [1], to avoid the delicate problem of the measurability of $\Gamma_W(\Theta)$, we just assume the existence of an integrable random variable Y bounding $1/\Gamma_W(\Theta)$ from above (this suffices for most applications).

Theorem 32 *Let $W \subset \mathbb{R}^d$ be a compact set, assume that $\Theta \in \mathcal{S}_{\rm loc}$ almost surely, and that there exists a random variable Y with $\mathbb{E}[Y] < \infty$, such that $1/\Gamma_W(\Theta) \leq Y$ almost surely. Then, for any open set $B \subset {\rm int}W_1$ with $\mathbb{E}[P(\Theta, \partial B)] = 0$ we have*

$$\lim_{r\downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Theta_r \setminus \Theta) \cap B)]}{r} = \mathbb{E}[P(\Theta, B)]. \tag{35}$$

Proof. We proceed along the line of the proof of Theorem 17 in [1]. It is sufficient to observe that for all $r < 2$

$$\begin{aligned} \frac{\mathcal{H}^d((\Theta(\omega)_r \setminus \Theta(\omega)) \cap B)}{r} &\leq \frac{\mathcal{H}^d((\partial\Theta(\omega))_r \cap B)}{r} = \frac{\mathcal{H}^d((\partial\Theta(\omega) \cap W_1)_r \cap B)}{r} \\ &\leq \frac{\mathcal{H}^d((\partial\Theta(\omega) \cap W_1)_r)}{r} \leq Y(\omega) 2^{d-1} 4^d b_d, \end{aligned}$$

where the last inequality follows by Proposition 5. As $\mathbb{E}[Y] < \infty$, (35) holds by applying the dominated convergence theorem and Corollary 22. □

Remark 33 If $\Theta^{(1)}$ and $\Theta^{(2)}$ are two random closed sets in $\mathbb{R}^d$ satisfying the assumptions of Theorem 32 in a window W with $Y^{(1)}$ and $Y^{(2)}$, respectively, then their union $\Theta = \Theta^{(1)} \cup \Theta^{(2)}$ satisfies the same assumption (in the same window) with $Y = 2\max\{Y^{(1)}, Y^{(2)}\}$: it suffices to apply the stability of $\mathcal{S}$ (and therefore of $\mathcal{S}_{\rm loc}$) under finite unions, and the same argument (providing a pair (η, γ) for the union of two sets in $\mathcal{S}$) mentioned at the very beginning of Theorem 19.

6.1 An application to a class of birth-and-growth stochastic processes

Let us consider a random closed set in $\mathbb{R}^d$ depending upon time, which can be taken as model for the evolution of a growth process. A wide class of growth processes is given by the so called *birth-and-growth* *stochastic processes*. A birth-and-growth stochastic process is a dynamic germ-grain model (see [19, 11]) whose birth process is modelled as a *marked point process* (see e.g. [6] and reference therein). We remind that a marked point process N on $\mathbb{R}_+$ with marks in $\mathbb{R}^d$ is a point process on $\mathbb{R}_+ \times \mathbb{R}^d$ with the property that the marginal process $\{N(B \times \mathbb{R}^d) : B \in \mathcal{B}_{\mathbb{R}_+}\}$ is itself a point process. Consequently it is defined as a random measure given by

$$N = \sum_{n=1}^{\infty} \delta_{T_n, X_n} ,$$

where:

- T_n is an $\mathbb{R}_+$-valued random variable representing the time of birth of the n-th nucleus;
- X_n is an $\mathbb{R}^d$-valued random variable representing the spatial location of the nucleus born at time T_n;
- $\delta_{t,x}$ is the Dirac measure on $\mathbb{R}_+ \times \mathbb{R}^d$ concentrated at (t, x).

Hence, in particular, for any $B \in \mathcal{B}_{\mathbb{R}_+}$ and $A \in \mathcal{B}_{\mathbb{R}^d}$ bounded, we have

$$N(B \times A) = \#\{T_n \in B,\, X_n \in A\},$$

i.e. $N(B \times A)$ is the random number of germs (nuclei) born in the region A during time B, and it is finite with probability 1. Let $d \geq 2$; once born each germ generates a grain subject to surface growth with a speed $G(t, x) > 0$ which should, in general, be assumed space-time dependent. Denoted by $\Theta^t_s(y)$ the grain born at time s at point y and grown up to time t, then, for any fixed $t \in \mathbb{R}_+$, Θ^t is the random closed set given by

$$\Theta^t = \bigcup_{T_j \leq t} \Theta^t_{T_j}(X_j).$$

We assume here the *normal growth model* (see, e.g., [5]), according to which at $\mathcal{H}^{d-1}$-almost every $x \in \partial\Theta^t_{T_n}(X_n))$, growth occurs with a given strictly positive normal velocity

$$v(t, x) = G(t, x)n(t, x), \tag{36}$$

where $G(t, x)$ is a given deterministic growth field and $n(t, x)$ is the unit outer normal at point $x \in \partial\Theta^t_{T_0}(X_0)$. We assume that $0 < g_0 \leq G(t, x) \leq G_0 < \infty$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, for some $g_0, G_0 \in \mathbb{R}$ and $G(t, x)$ is sufficient regular such that the evolution problem given by (36) is well posed. It follows that Θ^t is $\mathbb{P}$-a.s a compact random set for any fixed $t \in \mathbb{R}_+$. Further, for any $x \in \mathbb{R}^d$, we may introduce a *time of capture* of point x as the random variable $T(x)$ such that $x \in \partial\Theta^{T(x)}$, being $x \in \mathrm{int}\Theta^t$ if $t > T(x)$, and $x \notin \Theta^t$ if $t < T(x)$.

The Radon-Nikodym derivatives of the expected measures $\mathbb{E}[\mathcal{H}^d(\Theta^t \cap \cdot\,)]$ and $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap \cdot\,)]$, known also as *mean densities* and denoted by $\lambda_{\Theta^t}(x)$ and by $\lambda_{\partial\Theta^t}(x)$, respectively, are relevant quantities describing the geometric process $\{\Theta^t, t \in \mathbb{R}_+\}$, associated to the growth process. The existence of a local mean one-sided Minkowski content for Θ^t plays a fundamental role in proving an evolution equation for these mean densities. Indeed, as stated in Proposition 19 in [7], if the above birth-and-growth model is such that $T(x)$ is a continuous random variable with density and Θ^t satisfies (35) with $P(\Theta^t) = \mathcal{H}^{d-1}(\partial\Theta^t)$ for any $t \in \mathbb{R}_+$, then the following evolution equation holds

$$\frac{\partial}{\partial t}\lambda_{\Theta^t}(x) = G(t, x)\lambda_{\partial\Theta^t}(x), \tag{37}$$

in a weak form.

As simple example let us consider the particular case in which each grain grows with constant rate G for any fixed $t \in \mathbb{R}_+$ and assume that the random spatial location of the nuclei is diffuse in $\mathbb{R}^d$. It is clear that Θ^t is $\mathbb{P}$-a.s. a finite union of random balls in $\mathbb{R}^d$

$$\Theta^t = \bigcup_{i:T_i \le t} B_{G(t-T_i)}(X_i),$$

$T(x)$ is a continuous random variable with density for any $x \in \mathbb{R}^d$ and Theorem 32 applies to Θ^t for any $t \in \mathbb{R}_+$ (see also [1]). Then we obtain that (37) holds with $G(t,x) = G$. Such result may be intuitively shown observing that, by the assumption $d \ge 2$, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\begin{aligned} \lim_{\Delta t \downarrow 0} \frac{\mathcal{H}^d(\Theta^{t+\Delta t}(\omega) \setminus \Theta^t(\omega) \cap B)}{\Delta t} &= \lim_{\Delta t \downarrow 0} \frac{\mathcal{H}^d(\Theta^t_{G\Delta t}(\omega) \setminus \Theta^t(\omega) \cap B)}{\Delta t} \\ &= G \lim_{r \downarrow 0} \frac{\mathcal{H}^d(\Theta^t_r(\omega) \setminus \Theta^t(\omega) \cap B)}{r} = G\mathcal{H}^{d-1}(\partial\Theta^t(\omega) \cap B) \end{aligned}$$

for any Borel set $B \subset \mathbb{R}^d$ with $\mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap \partial B)] = 0$, and then passing to the expected values.

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