LOWER SEMICONTINUITY RESULTS FOR FREE DISCONTINUITY ENERGIES

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Abstract. We establish new lower semicontinuity results for energy functionals containing a very general volume term of polyconvex type and a surface term depending in a discontinuous way on the spatial variable.

Keywords: Semicontinuity, Existence, Capacity, Fractures.
AMS-MSC: 49J45, 49Q20, 74R10.

1. Introduction

In this paper we present new lower semicontinuity results for free discontinuity energies with a polyconvex volume term and a jump term depending on a possibly discontinuous integrand. Energies of this type occur in fracture mechanics when one considers quasistatic evolution of stratified, heterogeneous materials. Moreover, scalar energies to which our results apply include generalized Mumford–Shah type functionals.

In the last years, many mathematicians considered variational models for the evolution of the fracture process. In [21] Francfort and Marigo presented a model for the quasistatic growth of brittle cracks in elastic materials which is described by an integral functional including very general bulk and crack energies. The crack growth admits a quasistatic evolution, i.e., at each time the equilibrium is obtained by the competition between the elastic energy of the body and the dissipation energy of the fracture process.

In other more recent papers ([13],[14],[15] and [16]) this quasistatic evolution is studied in the framework of nonlinear elasticity. In these papers a precise mathematical formulation of the problem is given in the SBV setting of special functions of bounded variation. The bulk energy of the uncracked part of the body is given by

$$\mathcal{W}(u) := \int_{\Omega\setminus\Gamma} W(x, \nabla u(x))\, dx,$$

where Γ is the crack, the function $u : \Omega \setminus \Gamma \to I\!R^m$ is the unknown deformation of the body and $W(x, \xi)$ is a quasiconvex function with respect to ξ which describes the material. The energy needed to produce the crack Γ admits the form

$$\mathcal{K}(\Gamma) := \int_{\Gamma} k(x, \nu(x))\, d\mathcal{H}^{N-1}, \tag{1.1}$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and the function k depends on the position x and on the orientation ν. This function describes the "toughness" of the material in different locations and directions, thus including the case of heterogeneous and anisotropic materials. The existence of the quasistatic evolution is obtained by a time discretization and by minimizing the total energy

$$\mathcal{F}(u, \Gamma) := \mathcal{W}(u) + \mathcal{K}(\Gamma)\,,$$

at each discretization step.
In all papers quoted above the existence of minimizers is assured by a compactness theorem due to Ambrosio (see [5] and [8]) and by standard hypotheses which guarantee the lower semicontinuity of $\mathcal{W}$ and $\mathcal{K}$ (see [6] and [7]).
In this paper we prove that this lower semicontinuity still holds for a general integrand W of polyconvex type and under weaker assumptions on the integrand k.
More precisely, we consider a volum term of the type

$$\mathcal{W}(u) := \int_{\Omega\setminus\Gamma} W(x, u(x), \nabla u(x))\, dx,$$

where the integrand $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times N} \to [0, +\infty]$ is a Carathéodory function, polyconvex in the last variable, satisfying

$$W(x, u, \xi) \geq C|\xi|^{m\wedge N} \qquad (x, s, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times N}.$$

A result of the same type in SBV framework for a polyconvex energy is obtained in [24] under different growth conditions involving all the adjoints.
Moreover we consider a surface term of the type (1.1) and we allow jumps of the function $k(\cdot, \nu)$ by requiring only a BV dependence on x. More generally, we assume a lower semicontinuity of $k(\cdot, \nu)$ with respect to the C_1 capacity, which is satisfied in particular by the lower approximate limit $k^-(\cdot, \nu)$ of the BV function $k(\cdot, \nu)$.
This setting seems to apply to the case of composite or stratified media, where the energy needed to create the crack may change from point to point in a discontinuous way. More precisely, we consider a fracture energy of the type

$$\mathcal{K}(u) := \int_{J_u} k(x, \nu_u(x))\gamma(|u^+(x) - u^-(x)|)\, d\mathcal{H}^{N-1},$$

where $|u^+ - u^-|$ is the difference of the trace of u on both sides of J_u, ν_u is the normal to the jump set J_u and the function γ depends on the material. For $k(x, \nu) = 1$ the energy was proposed by Barenblatt in [10] (see also [25]), while in [13] and [14] the authors consider the case where $\gamma(s) = 1$.
Our lower semicontinuity theorem applies also to the model functional

$$\int_\Omega |\nabla u|^p\, dx + \alpha \int_\Omega |u - g|^q\, dx + \int_{J_u} a^-(x)\, d\mathcal{H}^{N-1}(x), \tag{1.2}$$

where $m = 1$ and $a \in \mathrm{BV}(\Omega)$ is a bounded function such that $a(x) > 0$ for $\mathcal{H}^{N-1}$-a.e. x. Functional (1.2) may be viewed as a generalized Mumford–Shah functional where in the image reconstruction one emphasizes the contours contained in a given region of Ω by giving appropriate values to the weight a.
In order to prove the lower semicontinuity of the volume term we follow the proof of the analogous theorem in [24] which is based on a preliminary compactness result for the adjoints of SBV functions. More precisely, we prove that by assuming, for sake of simplicity, $m = N$ and by considering a sequence $u_h, u \in \mathrm{SBV}(\Omega, \mathbb{R}^N)$ such that $u_h \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and

$$\sup_{h\in\mathbb{N}} \Big\{ \|u_h\|_{L^\infty(\Omega,\mathbb{R}^N)} + \int_\Omega |\nabla u_h|^N\, dx + \mathcal{H}^{N-1}(J_{u_h}) \Big\} < \infty,$$

then there exists a subsequence $\{u_{h_j}\}$ such that $\det\nabla u_{h_j}$ converges in the biting sense to $\det\nabla u$. This extends to the SBV framework a well-known result due to Zhang (see [28]) on weak convergence of determinants.
In order to prove the lower semicontinuity of the surface term we use some methods introduced previously in [4] (see also [2], [3], [17] and [18]) for general integral functionals defined in BV. More precisely, using an explicit construction given in [23], we get a suitable sequence of smooth functions approximating from below the discontinuous integrand $k(x,\nu)$ in $(\Omega\setminus A)\times I\!R^N$, where A is an open set with arbitrarily small 1-capacity. The lower semicontinuity of the approximating functionals is easily obtained via the chain rule formula for BV functions, while the general case is recovered by using the capacitary quasi-potentials of the sets A.

2. Notation and preliminaries

2.1. **Notation.** Throughout the paper $N>1$, $m\geq 1$ are fixed integers and the letter c denotes a strictly positive constant, whose value may change from line to line.
Given $x_0\in I\!R^N$ and $\rho>0$, $B_\rho(x_0)$ denotes the ball in $I\!R^N$ centered at x_0 with radius ρ. For the sake of simplicity, we set $B_\rho=B_\rho(0)$.
Let Ω be a bounded open subset of $I\!R^N$ with Lipschitz boundary. We denote by $\mathcal{A}(\Omega)$ the family of all open subsets A of Ω and by $\mathcal{B}(\Omega)$ the σ-algebra of all Borel subsets B of Ω.
Let $\mathcal{L}^N$ denote the Lebesgue measure on $I\!R^N$ and $\mathcal{H}^{N-1}$ the Hausdorff measure of dimension $(N-1)$ on $I\!R^N$.

2.2. **Approximate limits and BV functions.** If $u\in L^1_{\rm loc}(\Omega;I\!R^m)$ and $x\in\Omega$, the *precise representative of u at x* is defined as the unique value $\widetilde{u}(x)\in I\!R^m$ such that

$$\lim_{\rho\to0^+}\frac{1}{\rho^N}\int_{B_\rho(x)}|u(y)-\widetilde{u}(x)|\,dx=0\,.$$

The set of points in Ω where the precise representative of x is not defined is called the *approximate singular set* of u and denoted by S_u.

Let $u\in L^1_{\rm loc}(\Omega;I\!R^m)$ and $x\in\Omega$. We say that x is an approximate jump point of u if there exist $a,b\in I\!R^m$ and $\nu\in\mathbb{S}^{N-1}$, such that $a\neq b$ and

$$\lim_{\rho\to0^+}\fint_{B^+_\rho(x,\nu)}|u(y)-a|\,dy=0\qquad\text{and}\qquad\lim_{\rho\to0^+}\fint_{B^-_\rho(x,\nu)}|u(y)-b|\,dy=0$$

where $B^\pm_\rho(x,\nu):=\{y\in B_\rho(x)\ :\ \langle y-x,\nu\rangle\gtrless0\}$. The triplet (a,b,ν) is uniquely determined by the previous formulas, up to a permutation of a,b and a change of sign of ν, and it is denoted by $(u^+(x),u^-(x),\nu_u(x))$. The Borel functions u^+ and u^- are called the *upper and lower approximate limit* of u at the point $x\in\Omega$. The set of approximate jump points of u is denoted by J_u.
We recall that the space $\mathrm{BV}(\Omega;I\!R^m)$ of *functions of bounded variation* is defined as the set of all $u\in L^1(\Omega;I\!R^m)$ whose distributional gradient Du is a bounded Radon measure on Ω with values in the space $I\!M^{m\times N}$ of $m\times N$ matrices.
We recall the usual decomposition

$$Du=\nabla u\,\mathcal{L}^N+D^cu+(u^+-u^-)\otimes\nu_u\,\mathcal{H}^{N-1}\lfloor J_u\,,$$

where ∇u is the Radon-Nikodým derivative of Du with respect to the Lebesgue measure and $D^c u$ is the *Cantor part* of Du. For the sake of simplicity, we denote by $D^s u = D^c u + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \lfloor J_u$.

We recall that the space $\mathrm{SBV}(\Omega; \mathbb{R}^m)$ of *special functions of bounded variation* is defined as the set of all $u \in \mathrm{BV}(\Omega; \mathbb{R}^m)$ such that $D^s u$ is concentrated on S_u; i.e., $|D^s u|(\Omega \setminus S_u) = 0$.
Let $p > 1$. The space $\mathrm{SBV}^p(\Omega; \mathbb{R}^m)$ is defined as the set of functions $u \in \mathrm{SBV}(\Omega; \mathbb{R}^m)$ with $\nabla u \in L^p(\Omega; \mathbb{M}^{m\times N})$ and $\mathcal{H}^{N-1}(S_u) < \infty$. We will say that a sequence $\{u_n\}$ converges to u weakly in $\mathrm{SBV}^p(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ if $u_n(x) \to u(x)$ almost everywhere in Ω, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^p(\Omega; \mathbb{M}^{m\times N})$, and $\|u_n\|_\infty$ and $\mathcal{H}^{N-1}(S_{u_n})$ are bounded uniformly with respect to n.
We define $\mathrm{GSBV}(\Omega; \mathbb{R}^m)$ the space of *generalized special functions of bounded variation* as the set of all functions $u : \Omega \to \mathbb{R}^m$ such that $\phi(u) \in \mathrm{SBV}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ for every $\phi \in \mathcal{C}^1(\mathbb{R}^m; \mathbb{R}^m)$ with $supp(\nabla\phi) \subset\subset \mathbb{R}^m$.
We define $\mathrm{GSBV}^p(\Omega; \mathbb{R}^m)$ as the set of functions $u \in \mathrm{GSBV}(\Omega; \mathbb{R}^m)$ such that $\nabla u \in L^p(\Omega; \mathbb{M}^{m\times N})$ and $\mathcal{H}^{N-1}(S_u) < \infty$. We will say that a sequence $\{u_n\}$ converges to u weakly in $\mathrm{GSBV}^p(\Omega; \mathbb{R}^m)$ if $u_n, u \in \mathrm{GSBV}^p(\Omega; \mathbb{R}^m)$, $u_n(x) \to u(x)$ almost everywhere in Ω, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^p(\Omega; \mathbb{M}^{m\times N})$ and $\mathcal{H}^{N-1}(S_{u_n})$ is bounded uniformly with respect to n.

Finally, we recall a classical compactness result due to Ambrosio (see [5], [8] and [9, Theorem 4.8]).

Theorem 2.1. *Let $\{u_n\}$ be a sequence in $\mathrm{GSBV}^p(\Omega; \mathbb{R}^m)$ satisfying*

$$\sup_{n\in\mathbb{N}} \left[\|u_n\|_1 + \int_\Omega |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty \, .$$

Then there exists a subsequence $\{u_{n_k}\} \subset \mathrm{GSBV}^p(\Omega; \mathbb{R}^m)$ weakly converging in $\mathrm{GSBV}^p(\Omega; \mathbb{R}^m)$ to $u \in \mathrm{GSBV}^p(\Omega; \mathbb{R}^m)$, i.e., $u_{n_k}(x) \to u(x)$ for almost every $x \in \Omega$, $\nabla u_{n_k} \rightharpoonup \nabla u$ weakly in $L^P(\Omega; \mathbb{R}^{m\times N})$ and $\mathcal{H}^{N-1}(J_{u_{n_k}})$ is equibounded.

For a general survey on the spaces of BV, SBV, SBV^p, GSBV and GSBV^p functions we refer for instance to [9].

2.3. **Capacity.**

Given an open set $A \subset \mathbb{R}^N$, the 1-*capacity* of A is defined by setting

$$C_1(A) := \inf \left\{ \int_{\mathbb{R}^N} |D\varphi| \, dx \ : \ \varphi \in W^{1,1}(\mathbb{R}^N), \quad \varphi \geq 1 \quad \mathcal{L}^N\text{–a.e. on } A \right\} .$$

Then, the 1-capacity of an arbitrary set $B \subset \mathbb{R}^N$ is given by

$$C_1(B) := \inf\{C_1(A) \ : \ A \supseteq B, \ A \text{ open}\} \, .$$

It is well known that capacities and Hausdorff measure are closely related. In particular, we have that for every Borel set $B \subset \mathbb{R}^N$

$$C_1(B) = 0 \qquad \Longleftrightarrow \qquad \mathcal{H}^{N-1}(B) = 0 \, .$$

Definition 2.2. Let $B \subset \mathbb{R}^N$ be a Borel set with $C_1(B) < +\infty$. Given $\varepsilon > 0$, we call *capacitary ε-quasi-potential* (or simply *capacitary quasi-potential*) of B a function $\varphi_\varepsilon \in W^{1,1}(\mathbb{R}^N)$, such that $0 \le \widetilde{\varphi}_\varepsilon \le 1$ $\mathcal{H}^{N-1}$-a.e. in $I\!R^N$, $\widetilde{\varphi}_\varepsilon = 1$ $\mathcal{H}^{N-1}$-a.e. in B and

$$\int_{I\!R^N} |D\varphi_\varepsilon|\, dx \le C_1(B) + \varepsilon\,.$$

We recall that a function $g : I\!R^N \to I\!R$ is said C_1-quasi continuous if for every $\varepsilon > 0$ there exists an open set A, with $C_1(A) < \varepsilon$, such that $g|_{A^c}$ is continuous on A^c; C_1-quasi lower semicontinuous and C_1-quasi upper semicontinuous functions are defined similarly.
It is well known that if g is a $W^{1,1}$ function, then its precise representative $\widetilde{g}$ is C_1-quasi continuous (see [19, Sections 9 and 10]). Moreover, to every BV function g, it is possible to associate a C_1-quasi lower semicontinuous and a C_1-quasi upper semicontinuous representative, as stated by the following theorem (see [11], Theorem 2.5).

Theorem 2.3. *For every function $g \in \mathrm{BV}(\Omega)$, the approximate upper limit g^+ and the approximate lower limit g^- are C_1-quasi upper semicontinuous and C_1-quasi lower semicontinuous, respectively.*

In particular, if B is a Borel subset of $I\!R^N$ with finite perimeter, then χ_B^- is C_1-quasi lower semicontinuous and χ_B^+ is C_1-quasi upper semicontinuous.

2.4. Jointly convex functions.

Definition 2.4. Let $K \subset I\!R^m$ be a compact set and $\phi : K \times K \times I\!R^N \to [0, +\infty)$. We say that ϕ is *jointly convex* if there exists a sequence of functions $g_j \in \mathcal{C}(K; I\!R^N)$ such that

$$\phi(r, t, \xi) = \sup_{j \in I\!N} \langle g_j(r) - g_j(t), \xi\rangle \qquad \text{for all } (r, t, \xi) \in K \times K \times I\!R^N.$$

Remark 2.5. We recall that a class of jointly convex functions ϕ can be obtained in the following way:

$$\phi(r, t, \xi) = \gamma(|r - t|)\varphi(\xi)$$

where γ is a lower semicontinuous, increasing and subadditive function with $\gamma(0) = 0$ and φ is convex, positively 1-homogeneous and even (see Example 5.23 in [9]).

2.5. Approximation results.

We recall here a few approximation results that will be used in the sequel.
Let $u \in L^1(B_1)$ be a nonnegative function. The *local maximal function* $M(u)$ is defined by

$$M(u)(x) = \sup\left\{ \fint_{B_\rho(x)} u(y)\, dy : 0 < \rho < 1 - |x| \right\} \qquad \text{for all } x \in B_1\,.$$

If u belongs to $L^p(B_1)$ for some $p > 1$, then (see [27, Chapter 1])

$$\int_{B_1} M^p(u)\, dx \le c \int_{B_1} u^p\, dx\,, \tag{2.1}$$

for a suitable constant c depending only on n and p.
Using maximal functions, one can get a Lusin-type approximation of SBV functions by means of Lipschitz functions as in the next theorem due to Ambrosio (see [7, Theorem 2.3]).

Theorem 2.6. *Let u be a function from* $\mathrm{SBV}(B_1;I\!R^m)\cap L^\infty(B_1;I\!R^m)$ *and* $\lambda>0$. *Set*

$$E:=\{x\in B_1: M(|Du|)(x)\le\lambda\}\,.$$

Then, for any $\rho\in(0,1)$, there exists a Lipschitz function $v:B_\rho\to I\!R^m$ such that $u(x)=v(x)$ for $\mathcal{L}^N$-a.e. $x\in E\cap B_\rho$ and

$$\mathrm{Lip}(v,B_\rho)\le c(n)m\lambda+\frac{2m\|u\|_\infty}{1-\rho}\,,$$

for some constant $c(n)$, depending only on n. Moreover, if $|\nabla u|\in L^p(B_1)$ for some $p>1$, then, for any $C\in\mathcal{B}(B_1)$,

$$\mathcal{L}^N(\{x\in C: M(|Du|)(x)>2\lambda\})\le\lambda^{-p}\int\limits_{C\cap\{M(|\nabla u|)>\lambda\}}M^p(|\nabla u|)\,dx+\frac{2c(n)\|u\|_{L^\infty(B_1)}}{\lambda}\mathcal{H}^{N-1}(J_u)\,.$$

Next result, also known as Chacon's biting lemma, allows to recover some equi-integrability from a sequence which is only bounded in L^1 (see e.g. [1, Lemma I.7] or [9, Lemma 5.32]).

Lemma 2.7. *Let $\{u_h\}$ be a bounded sequence in $L^1(\Omega;I\!R^m)$. Then, there exists a subsequence $\{u_{h_j}\}$ and a decreasing sequence $E_n\subset\mathcal{B}(\Omega)$, such that $\mathcal{L}^N(E_n)\to 0$ as $n\to\infty$ and the sequence $\{u_{h_j}\}$ is equi-integrable in $\Omega\setminus E_n$ for any $n\in I\!N$.*

In view of this lemma it is then natural to set the following definition.

Definition 2.8. Let $\{u_h\}$ be a bounded sequence in $L^1(\Omega;I\!R^m)$. We say that $\{u_h\}$ *converges in the biting sense* to $u\in L^1(\Omega;I\!R^m)$ if there exists a decreasing sequence $E_n\subset\mathcal{B}(\Omega)$, such that $\mathcal{L}^N(E_n)\to 0$ as $n\to\infty$ and $u_h\rightharpoonup u$ weakly in $L^1(\Omega\setminus E_n;I\!R^m)$ for any $n\in I\!N$.

Thus, Lemma 2.7 above simply states that given any bounded sequence in L^1, there exists always a subsequence converging to some L^1 function in the biting sense.
Let us state also the following simple lemma (see Theorem 1.2 of [5]).

Lemma 2.9. *Let $\{u_h\}\subset L^1(\Omega,I\!R^m)$ be an equi-integrable sequence, $u\in L^1(\Omega,I\!R^m)$, and let us assume that*

$$\liminf_{h\to+\infty}\int\limits_\Omega|u_h-w|\,dx\ge\int\limits_\Omega|u-w|\,dx,$$

for every $w\in L^1(\Omega,I\!R^m)$. Then u_h weakly converges to u in $L^1(\Omega,I\!R^m)$.

We recall that a function $W:I\!\!M^{m\times N}\to(-\infty,+\infty]$ is *polyconvex* if there exists a convex function $G:I\!R^\tau\to(-\infty,+\infty]$ such that

$$W(\xi)=G(\mathcal{M}(\xi))\qquad\text{for all }\xi\in I\!\!M^{m\times N}\,,$$

where $\mathcal{M}(\xi)$ is the vector whose components are all the minors of the matrix ξ and $\tau=\tau(N,m)$ is the dimension of $\mathcal{M}(\xi)$. For $k=1,\dots,N\wedge m$, we denote by $\mathrm{adj}_k\xi$ the vector whose components are the minors of the matrix ξ of order k. We denote the dimension of $\mathrm{adj}_k\xi$ by τ_k. Notice that $\tau_k=\binom{N}{k}\binom{m}{k}$.
In the next lemma, using the same argument as in the proof of Theorem 1.1 in [22], we obtain the lower semicontinuity for a functional whose integrand is the supremum of convex functions.

Lemma 2.10. *Let $h, h_j : \Omega \times I\!R^m \times I\!R^m \times I\!R^N \to [0, +\infty)$, $j \in I\!N$, be Borel functions, convex and positively* 1*-homogeneous in the last variable and such that*

$$h(x, r, t, \xi) = \sup_{j \in I\!N} h_j(x, r, t, \xi) \qquad \text{for all } (x, r, t, \xi) \in (\Omega \setminus N_0) \times I\!R^m \times I\!R^m \times I\!R^N ,$$

where $N_0 \subset \Omega$ is a Borel set with $\mathcal{H}^{N-1}(N_0) = 0$. If the functionals $\mathcal{F}_{h_j}(\cdot, \Omega)$ defined by

$$\mathcal{F}_{h_j}(u) := \int_{\Omega \cap J_u} h_j(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}$$

are weakly lower semicontinuous in $\mathrm{SBV}^p(\Omega; I\!R^m)$*, then $\mathcal{F}_h$, defined similarly, is weakly lower semicontinuous in* $\mathrm{SBV}^p(\Omega; I\!R^m)$ *too.*

3. Semicontinuity results

For every $A \in \mathcal{A}(\Omega)$ and every $u \in \mathrm{GSBV}^p(\Omega; I\!R^m)$, $p > 1$, we set

$$\begin{aligned} G(u, A) &= \int_A W(x, u, \nabla u)\, dx + \int_{A \cap J_u} h(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1} , \\ F(u, A) &= G(u, A) + \int_\Omega |u - U_0|^q\, dx , \end{aligned} \tag{3.1}$$

where $q \geq 1$, $W : \Omega \times I\!R^m \times I\!M^{m \times N} \to [0, +\infty]$ is a Carathéodory function, $h : \Omega \times I\!R^m \times I\!R^m \times I\!R^N \to [0, +\infty)$ is a Borel function and $U_0 : \Omega \to I\!R^m$ belongs to $L^\infty(\Omega; I\!R^m)$.
Our aim is to prove a lower semicontinuity theorem for this functional, along sequences $\{u_n\}$ in $\mathrm{GSBV}^p(\Omega; I\!R^m)$ such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and $\|\nabla u_n\|_p$, $\mathcal{H}^{N-1}(J_{u_n})$ are uniformly bounded with respect to $n \in I\!N$.
As the lower semicontinuity of the last term in F is trivial, the result can be obtained proving, for every $A \in \mathcal{A}(\Omega)$, the lower semicontinuity of the two functionals

$$(u, A) \mapsto \int_A W(x, u, \nabla u)\, dx \qquad \text{and} \qquad (u, A) \mapsto \int_{A \cap J_u} h(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1} \tag{3.2}$$

separately.
We shall first discuss the lower semicontinuity of the surface term. As we said in the introduction the new feature of the results presented here is that the function h may possibly be discontinuous with respect to x and in the first part of this section we shall consider different structure assumptions on h. Concerning the volume term, a result due to Ambrosio [9, Theorem 5.29] settles the quasi-convex case (see also [7] and [26]) under suitable growth conditions. However, for possible applications to fracture mechanics it is more natural to specialize to polyconvex integrands, where more general growth assumptions are allowed. This case will be discussed in the second part of this section.

3.1. **Lower semicontinuity of the surface term.** Next result is an almost straightforward generalization of Theorem 5.22 in [9]. However, since it is going to be a key ingredient for proving the more general Theorem 3.3, we present it here in details.

Proposition 3.1. *Let $a:\Omega\to[0,+\infty)$ be a locally bounded function belonging to $W^{1,1}(\Omega)$ and coinciding with its precise representative and $\phi:I\!R^m\times I\!R^m\times I\!R^N\to[0,+\infty)$ be a jointly convex function. Then, for every $\{u_n\}\subset \mathrm{SBV}^p(\Omega;I\!R^m)$ and $u\in\mathrm{SBV}^p(\Omega;I\!R^m)$ such that $u_n(x)\to u(x)$ for almost every $x\in\Omega$ and*

$$\sup_{n\in I\!N}\left[\|u_n\|_\infty+\int_\Omega|\nabla u_n|^p\,dx+\mathcal{H}^{N-1}(J_{u_n})\right]<+\infty\,,$$

we have

$$\int_{\Omega\cap J_u} a(x)\phi(u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1}\le\liminf_{n\to+\infty}\int_{\Omega\cap J_{u_n}} a(x)\phi(u_n^-,u_n^+,\nu_{u_n})\,d\mathcal{H}^{N-1}\,. \tag{3.3}$$

Proof. We argue as in the proof of Theorem 5.22 in [9] with the obvious modifications due to the explicit dependence on the spatial variable x.
Let

$$C:=\sup_{n\in I\!N}\left[\|u_n\|_\infty+\int_\Omega|\nabla u_n|^p\,dx+\mathcal{H}^{N-1}(J_{u_n})\right]$$

and $\overline{B}(0,C)\subset I\!R^m$ be the closed ball of radius C, centered at the origin. By the definition of jointly convex function and taking into account that ϕ is nonnegative, there exists a sequence of functions $g_j\in\mathcal{C}(\overline{B}(0,C);I\!R^N)$ such that

$$\phi(r,t,\xi)=\sup_{j\in I\!N}\left[\langle g_j(r)-g_j(t),\xi\rangle\vee 0\right].$$

By Lemma 2.10, it is enough to prove the lower semicontinuity for functionals of the type

$$u\to\int_{\Omega\cap J_u}\left[\langle a(x)\big(g(u^+)-g(u^-)\big),\nu_u\rangle\vee 0\right]\,d\mathcal{H}^{N-1}\,. \tag{3.4}$$

It is not restrictive to assume that $g\in\mathcal{C}_0^\infty(I\!R^m;I\!R^N)$, since the general case can be obtained by a standard approximation as in [9, proof of Theorem 5.22]. Let us now fix $\psi\in C_0^1(\Omega)$, $0\le\psi\le1$. The lower semicontinuity of the functional in (3.4) will follow if we prove the continuity of

$$u\to\int_{\Omega\cap J_u}\left[\langle a(x)\big(g(u^+)-g(u^-)\big),\nu_u\rangle\right]\psi(x)\,d\mathcal{H}^{N-1}\,.$$

Using the chain rule formula for BV functions (see [9, Theorem 3.9, Example 3.97]), we have

$$\int_{\Omega\cap J_u} a(x)\langle g(u^+)-g(u^-),\nu_u\rangle\,\psi\,d\mathcal{H}^{N-1}=-\int_\Omega a(x)\langle\nabla\psi(x),g(u(x))\rangle\,dx$$
$$-\int_\Omega\psi(x)\,\langle\nabla a(x),g(u(x))\rangle\,dx-\int_\Omega\psi(x)\,a(x)\,\mathrm{tr}\left[\nabla g(u(x))\cdot\nabla u(x)\right]dx\,.$$

Notice that

$$\int_\Omega a(x)\langle\nabla\psi(x),g(u(x))\rangle\,dx=\lim_{n\to+\infty}\int_\Omega a(x)\langle\nabla\psi(x),g(u_n(x))\rangle\,dx\,; \tag{3.5}$$

$$\int_\Omega\psi(x)\,\langle\nabla a(x),g(u(x))\rangle\,dx=\lim_{n\to+\infty}\int_\Omega\psi(x)\,\langle\nabla a(x),g(u_n(x))\rangle\,dx\,; \tag{3.6}$$

$$\int_{\Omega} \psi(x)\, a(x)\, \mathrm{tr}\,[\nabla g(u(x)) \cdot \nabla u(x)]\, dx = \lim_{n\to+\infty} \int_{\Omega} \psi(x)\, a(x)\, \mathrm{tr}\,[\nabla g(u_n(x)) \cdot \nabla u_n(x)]\, dx\,. \tag{3.7}$$

In fact, since g is continuous, $\{g(u_n)\}$ converges almost everywhere to $g(u)$ and is equibounded in $L^\infty(\Omega)$. Thus, taking into account that $a\nabla\psi$ and $\psi\nabla a$ belong to $L^1(\Omega; I\!\!R^N)$, (3.5) and (3.6) hold. In order to prove equality (3.7), we observe that $a\,\psi \in L^\infty(\Omega)$, $\nabla g(u_n) \to \nabla g(u)$ strongly in $L^{p'}(\Omega; I\!\!M^{N\times M})$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^p(\Omega; I\!\!M^{m\times N})$. This concludes the proof. □

Proposition 3.2. *Let $a : \Omega \to [0, +\infty)$ be a locally bounded function from $W^{1,1}(\Omega)$, coinciding with its precise representative, $\gamma : [0, +\infty) \to [0, +\infty)$ be a lower semicontinuous, increasing and subadditive function such that $\gamma(0) = 0$, and $\varphi : I\!\!R^N \to [0, +\infty)$ be a convex, even and positively 1-homogeneous function. Then, for every $\{u_n\} \subset \mathrm{GSBV}^p(\Omega; I\!\!R^m)$ and $u \in \mathrm{GSBV}^p(\Omega; I\!\!R^m)$ such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and*

$$\sup_{n\in I\!\!N} \left[\int_{\Omega} |\nabla u_n|^p\, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty\,, \tag{3.8}$$

we have

$$\int_{\Omega\cap J_u} a(x)\gamma(|u^+ - u^-|)\varphi(\nu_u)\, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} a(x)\gamma(|u_n^+ - u_n^-|)\varphi(\nu_{u_n})\, d\mathcal{H}^{N-1}\,. \tag{3.9}$$

Proof. Firstly we assume that

$$\sup_{n\in I\!\!N} \left[\|u_n\|_\infty + \int_{\Omega} |\nabla u_n|^p\, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty\,.$$

By Remark 2.5, the function $\phi(r, t, \nu) = \gamma(|r - t|)\varphi(\nu)$ is jointly convex; moreover, $u_n, u \in \mathrm{GSBV}^p(\Omega; I\!\!R^m) \cap L^\infty(\Omega; I\!\!R^m) \subset \mathrm{SBV}^p(\Omega; I\!\!R^m)$. Hence, by Proposition 3.1 the thesis follows. The general case $u_n, u \in \mathrm{GSBV}^p(\Omega; I\!\!R^m)$ satisfying (3.8) can be obtained as in the proof of Theorem 3.7 in [6]. □

We extend now the last result to a more general case, pointing out that this is the most significative case in the framework of quasi-static models in mechanics of fractures (see [14]).

Theorem 3.3. *Let $k : \Omega \times I\!\!R^N \to [0, +\infty)$ be a locally bounded Borel function satisfying*

(3.10) $\quad k(\cdot, \xi)$ *is C_1-quasi lower semicontinuous for every $\xi \in I\!\!R^N$;*

(3.11) $\quad k(x, \cdot)$ *is convex and positively 1-homogeneous in $I\!\!R^N$ for every $x \in \Omega$;*

(3.12) $\quad k(x, \xi) = k(x, -\xi)$ *for every $(x, \xi) \in \Omega \times I\!\!R^N$;*

(3.13) $\quad k(x, \xi) > 0$ *for every $(x, \xi) \in (\Omega \setminus N_0) \times (I\!\!R^N \setminus \{0\})$, where $\mathcal{H}^{N-1}(N_0) = 0$.*

Let $\gamma : [0, +\infty) \to [0, +\infty)$ be a locally bounded, lower semicontinuous, increasing and subadditive function such that $\gamma(0) = 0$. Then, for every $u_n \subset \mathrm{GSBV}^p(\Omega; I\!\!R^m)$ and $u \in \mathrm{GSBV}^p(\Omega; I\!\!R^m)$ such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and

$$\sup_{n\in I\!\!N} \left[\int_{\Omega} |\nabla u_n|^p\, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty\,,$$

we have

$$\int_{\Omega\cap J_u} \gamma(|u^+-u^-|)k(x,\nu_u)\,d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} \gamma(|u_n^+-u_n^-|)k(x,\nu_{u_n})\,d\mathcal{H}^{N-1}\,. \tag{3.14}$$

Proof. Step 1. We follow some ideas contained in the proof of Theorem 3.6 in [4]. Notice that since k is locally bounded in $\Omega\times I\!\!R^N$ and positively 1-homogeneous with respect to ξ, for any open set $\Omega'\subset\subset\Omega$, there exists a constant $\Lambda'=\Lambda'(\Omega')$ such that

$$0\le k(x,\xi)\le \Lambda'|\xi| \qquad \text{for all } (x,\xi)\in\Omega'\times I\!\!R^N\,. \tag{3.15}$$

Condition (3.15), together with the convexity of k with respect to ξ immediately yields that

$$|k(x,\xi_1)-k(x,\xi_2)|\le c_0\Lambda'|\xi_1-\xi_2| \qquad \text{for all } (x,\xi_1),(x,\xi_2)\in\Omega'\times I\!\!R^N\,, \tag{3.16}$$

for some constant $c_0>0$. Let us now fix $h\in I\!\!N$ and a dense sequence $\{\xi_i\}\subset I\!\!R^N$. Thanks to (3.10) for all i there exists an open set $A_{i,h}\subset\Omega'$, $A_{i,h}\supset N_0$, with $C_1(A_{i,h})<1/(h2^i)$, such that $k(\cdot,\xi_i)$ is lower semicontinuous in $\Omega\setminus A_{i,h}$. Setting $A_h=\cup_i A_{i,h}$ we obtain that A_h is open, $C_1(A_h)\le 1/h$ and we may assume that $\{A_h\}$ is a decreasing sequence. Making use of (3.16), one easily gets that k is lower semicontinuous in $(\Omega'\setminus A_h)\times I\!\!R^N$.
We claim that, given h and $x_0\in\Omega'\setminus A_h$, for all $\varepsilon>0$ there exists $\delta>0$ such that

$$k(x_0,\xi)\le(1+\varepsilon)k(x,\xi) \tag{3.17}$$

for all $(x,\xi)\in(\Omega'\setminus A_h)\times I\!\!R^N$ such that $|x-x_0|<\delta$.
To prove this, we argue by contradiction, assuming that for some $x_0\in(\Omega'\setminus A_h)$ and $\varepsilon_0>0$ there exist two sequences $\{x_i\}\subset\Omega'\setminus A_h$, with $|x_i-x_0|<1/i$, and $\{\xi_i\}\subset I\!\!R^N$ such that

$$k(x_0,\xi_i)>(1+\varepsilon_0)k(x_i,\xi_i)\,. \tag{3.18}$$

Clearly, by the positive 1-homogeneity of $k(x_i,\cdot)$, we may assume that $|\xi_i|=1$, for every $i\in I\!\!N$; hence, up to a subsequence, there exists $\xi_0\in\mathbb{S}^{N-1}$ such that $\xi_i\to\xi_0$. Then, letting $i\to+\infty$ in (3.18) and using the lower semicontinuity of k and the continuity of $k(x_0,\cdot)$, we get

$$k(x_0,\xi_0)=\lim_{i\to+\infty}k(x_0,\xi_i)\ge(1+\varepsilon_0)\liminf_{i\to+\infty}k(x_i,\xi_i)\ge(1+\varepsilon_0)k(x_0,\xi_0)\,.$$

Hence, $k(x_0,\xi_0)=0$, which is a contradiction since $x_0\in\Omega\setminus N_0$. This proves the claim; i.e., (3.17) holds.

Step 2. We use now a construction similar to the one in the proof of Lemma 8(c) in [23], to show that for any h there exist $\{a_j^h\}\subset C_0^\infty(I\!\!R^N)$ and $\psi_j^h: I\!\!R^N\to[0,\infty)$ such that, for all $j\in I\!\!N$, $0\le a_j^h\le 1$, ψ_j^h is a convex, positively 1-homogeneous and even function satisfying

$$k(x,\xi)=\sup_{j\in I\!\!N}a_j^h(x)\psi_j^h(\xi) \qquad \text{for all } (x,\xi)\in(\Omega'\setminus A_h)\times I\!\!R^N\,, \tag{3.19}$$

$$0\le\psi_j^h(\xi)\le\Lambda'|\xi| \qquad \text{for all } \xi\in I\!\!R^N\,. \tag{3.20}$$

To this aim, fix $\varepsilon>0$ and $x_0\in\Omega'\setminus A_h$ and choose $\delta>0$ such that (3.17) holds. Let $\alpha_{\varepsilon,x_0}\in C_0^\infty(B_\delta)$, with $0\le\alpha_{\varepsilon,x_0}\le 1$ and $\alpha_{\varepsilon,x_0}(x_0)=1$, and $\varrho\in C_0^\infty(B_1)$ such that $\varrho\ge0$, $\varrho(\xi)=\varrho(-\xi)$ and $\int\varrho\,d\xi=1$. Define $\varrho_\varepsilon(\xi)=\varepsilon^{-n}\varrho(\xi/\varepsilon)$ and for all $x,\xi\in I\!\!R^N$

$$k_{\varepsilon,x_0}(\xi)=\frac{1}{1+\varepsilon}\int_{I\!\!R^N}\varrho_\varepsilon(\eta)k(x_0,\xi+\eta)\,d\eta-\varepsilon c_0\Lambda', \qquad k_{n,\varepsilon,x_0}(x,\xi)=\alpha_{\varepsilon,x_0}(x)\sigma_n(k_{\varepsilon,x_0}(\xi))\,,$$

where for all $n \in I\!N$, $\sigma_n \in C^\infty(I\!R)$ is a nonnegative increasing convex function such that

$$\sigma_n(t) \uparrow t \vee 0 \qquad \text{for all } t \in I\!R\,. \tag{3.21}$$

By construction, each $k_{n,\varepsilon,x_0}(x,\xi)$ is a smooth nonnegative function, convex and even in ξ. In fact, since $\varrho_\varepsilon(\xi) = \varrho_\varepsilon(-\xi)$ and k is even in ξ we have immediately that $k_{\varepsilon,x_0}(\xi) = k_{\varepsilon,x_0}(-\xi)$. Notice also that for all $(x,\xi) \in (\Omega' \setminus A_h) \times I\!R$

$$k_{n,\varepsilon,x_0}(x,\xi) \le k(x,\xi)\,. \tag{3.22}$$

Indeed, this inequality is trivial if $|x - x_0| \ge \delta$, since $\alpha_{\varepsilon,x_0}(x) = 0$. On the other hand, if $x \in B_\delta(x_0) \cap \Omega' \setminus A_h$, from (3.17) and (3.16) we get

$$k_{\varepsilon,x_0}(\xi) \le \int_{I\!R^N} \varrho_\varepsilon(\eta) k(x,\xi+\eta)\,d\eta - \varepsilon c_0 \Lambda' \le \int_{I\!R^N} \varrho_\varepsilon(\eta)\left[k(x,\xi) + c_0\Lambda'|\eta|\right] d\eta - \varepsilon c_0\Lambda' \le k(x,\xi)\,.$$

Hence, inequality (3.22) follows from (3.21), taking into account that $k(x,\xi) \ge 0$ and $0 \le \alpha_{\varepsilon,x_0}(x) \le 1$. Finally, observe that from these inequalities, (3.21) and (3.15) we have also

$$\sigma_n(k_{\varepsilon,x_0}(\xi)) \le k(x_0,\xi) \le \Lambda'|\xi|\,. \tag{3.23}$$

Thus, using (3.22) and the fact that $\alpha_{\varepsilon,x_0}(x_0) = 1$ for all $x_0 \in \Omega' \setminus A_h$, we get that

$$k(x,\xi) = \sup_{\varepsilon>0} \sup_{x_0 \in \Omega'\setminus A_h} \sup_{n\in I\!N} k_{n,\varepsilon,x_0}(x,\xi) \qquad \text{for all } (x,\xi) \in (\Omega'\setminus A_h)\times I\!R\,.$$

From this equality, it easily follows (see for instance [20, Lemma 9.2]) that there exists a sequence $(n_j, \varepsilon_j, x_j) \in I\!N \times (0,\infty) \times (\Omega' \setminus A_h)$ such that

$$k(x,\xi) = \sup_{j\in I\!N} \alpha_{\varepsilon_j,x_j}(x)\sigma_{n_j}(k_{\varepsilon_j,x_j}(\xi))$$

for all $(x,\xi) \in (\Omega'\setminus A_h)\times I\!R$. Finally, we set, for $x \in \Omega'\setminus A_h$, $\xi \in I\!R$,

$$a_j^h(x) = \alpha_{\varepsilon_j,x_j}(x), \qquad \psi_j^h(\xi) = \sup_{t>0} \frac{\sigma_{n_j}(k_{\varepsilon_j,x_j}(t\xi))}{t}$$

Notice that by construction the functions ψ_j^h are convex, 1-homogeneous, even and from (3.23) it is clear that they satisfy (3.20). Moreover, since k is 1-homogeneous we have also that

$$\begin{aligned} k(x,\xi) &= \sup_{t>0} \frac{k(x,t\xi)}{t} = \sup_{t>0} \sup_{j\in I\!N} \alpha_{\varepsilon_j,x_j}(x) \frac{\sigma_{n_j}(k_{\varepsilon_j,x_j}(t\xi))}{t} \\ &= \sup_{j\in N} \sup_{t>0} a_j^h(x) \frac{\sigma_{n_j}(k_{\varepsilon_j,x_j}(t\xi))}{t} = \sup_{j\in N} a_j^h(x)\psi_j^h(\xi)\,, \end{aligned}$$

thus proving (3.19).

Step 3. Now we argue as in the proof of Theorem 3.4 in [4]. Let $\varphi_h \in W^{1,1}(I\!R^N)$ be a capacitary quasi-potential of A_h. More precisely, let us assume that there exists a Borel set $N_h \subset I\!R^N$, with $C_1(N_h) = \mathcal{H}^{N-1}(N_h) = 0$, such that $0 \le \widetilde{\varphi}_h(x) \le 1$ for every $x \in I\!R^N \setminus N_h$, $\widetilde{\varphi}_h = 1$ on $A_h \setminus N_h$ and

$$\int_{I\!R^N} |\nabla \widetilde{\varphi}_h|\,dx \le C_1(A_h) + \frac{1}{h} < \frac{2}{h}\,.$$

Set, for all $j \in I\!N$, $\widetilde{\alpha}_j^h(x) = \max\{a_j^h(x) - \widetilde{\varphi}_h(x), 0\}$ for all $x \in I\!R^N$. We have that, since $\widetilde{\varphi}_h(x) \ge 0$,

$$0 \le \widetilde{\alpha}_j^h(x) \le 1, \quad a_j^h(x) \ge \widetilde{\alpha}_j^h(x) \ge a_j^h(x) - \widetilde{\varphi}_h(x) \quad \text{for all } x \in I\!R^N\,. \tag{3.24}$$

Moreover, setting $\widetilde{N} = \cup_h N_h$, $C_1(\widetilde{N}) = \mathcal{H}^{N-1}(\widetilde{N}) = 0$, for every $h, j \in I\!N$ we have that

$$k(x,\xi) \geq \widetilde{\alpha}_j^h(x)\psi_j^h(\xi) \qquad \text{for all } (x,\xi) \in (\Omega' \setminus \widetilde{N}) \times I\!R^N\,. \tag{3.25}$$

Finally, we set for all $h, j \in I\!N$

$$g_j^h(x,\xi) = \widetilde{\alpha}_j^h(x)\psi_j^h(\xi), \qquad g^h(x,\xi) = \sup_{j\in I\!N} g_j^h(x,\xi),$$

for all $(x,\xi) \in \Omega' \times I\!R^N$. Notice that the functions $\widetilde{\alpha}_j^h, \gamma, \psi_j^h$ satisfy the assumptions of a, γ, φ in Proposition 3.2. Therefore, the functionals $\mathcal{F}_j^h$ defined by

$$\mathcal{F}_j^h(u,\Omega') := \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)\, g_j^h(x,\nu_u)\, d\mathcal{H}^{N-1}$$

satisfy the inequality (3.9), with Ω replaced by Ω'. Hence by Lemma 2.10 the same is true for the functionals $\mathcal{F}^h$, defined by

$$\mathcal{F}^h(u,\Omega') := \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)\, g^h(x,\nu_u)\, d\mathcal{H}^{N-1}\,,$$

for any $h \in I\!N$.
To prove (3.14), we fix $h \in I\!N$ and set

$$\psi_h(\xi) = \sup_{j\in I\!N} \psi_j^h(\xi) \qquad \text{for all } \xi \in I\!R^N\,.$$

From (3.25), (3.24), (3.19) and (3.20), we get that

$$\begin{aligned}
\liminf_{n\to+\infty} \int_{\Omega'\cap J_{u_n}} \gamma(|u_n^+ - u_n^-|)k(x,\nu_{u_n})\, d\mathcal{H}^{N-1} &\geq \liminf_{n\to+\infty} \int_{\Omega'\cap J_{u_n}} \gamma(|u_n^+ - u_n^-|)g^h(x,\nu_{u_n})\, d\mathcal{H}^{N-1} \\
&\geq \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)g^h(x,\nu_u)\, d\mathcal{H}^{N-1} \geq \int_{\Omega'\cap J_u\setminus A_h} \gamma(|u^+ - u^-|)g^h(x,\nu_u)\, d\mathcal{H}^{N-1} \\
&\geq \int_{\Omega'\cap J_u\setminus A_h} \gamma(|u^+ - u^-|)k(x,\nu_u)\, d\mathcal{H}^{N-1} - \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)\widetilde{\varphi}_h(x)\psi_h(\nu_u)\, d\mathcal{H}^{N-1} \\
&\geq \int_{\Omega'\cap J_u\setminus A_h} \gamma(|u^+ - u^-|)k(x,\nu_u)\, d\mathcal{H}^{N-1} - \Lambda \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)\widetilde{\varphi}_h(x)\, d\mathcal{H}^{N-1}\,.
\end{aligned} \tag{3.26}$$

Since $\widetilde{\varphi}_h \to 0$ strongly in $W^{1,1}(I\!R^N)$ as $h \to \infty$, we have that, up to a subsequence, $\widetilde{\varphi}_h(x) \to 0$ for $\mathcal{H}^{N-1}$-almost every $x \in I\!R^N$ (see Proposition 1.2 in [12]). Therefore, letting $h \to +\infty$ in (3.26), recalling that $A_{h+1} \subset A_h$ for all h and that $\mathcal{H}^{N-1}(\cap_h A_h) = 0$ and taking into account that γ is locally bounded, from the Dominated Convergence Theorem we get the thesis in Ω' for $u \in \mathrm{GSBV}^p(\Omega; I\!R^m) \cap L^\infty(\Omega; I\!R^m) \subset \mathrm{SBV}^p(\Omega; I\!R^m)$. Finally, inequality (3.14) holds, letting $\Omega' \nearrow \Omega$. The general case $u \in \mathrm{GSBV}^p(\Omega; I\!R^m)$ can be obtained as in the proof of Theorem 3.7 in [6]. □

As a consequence of previous proposition, we obtain the following result.

Corollary 3.4. *Let* $k : \Omega \times I\!R^N \to [0, +\infty)$ *be a locally bounded function satisfying all the assumptions of Theorem 3.3. Then, for every* $\{u_n\} \subset \mathrm{GSBV}^p(\Omega; I\!R^m)$ *and* $u \in \mathrm{GSBV}^p(\Omega; I\!R^m)$ *such that such that* $u_n(x) \to u(x)$ *for almost every* $x \in \Omega$ *and*

$$\sup_{n \in I\!N} \left[\int_\Omega |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty \,,$$

we have

$$\int_{\Omega \cap J_u} k(x, \nu_u) \, d\mathcal{H}^{N-1} \leq \liminf_{n \to +\infty} \int_{\Omega \cap J_{u_n}} k(x, \nu_{u_n}) \, d\mathcal{H}^{N-1} \,. \tag{3.27}$$

Proof. It is enough to consider the function $\phi(x, r, t, \nu) = \gamma(|t - r|) k(x, \nu)$, where $\gamma(0) = 0$ and $\gamma(s) = 1$ for $s > 0$. Hence the conclusion follows from Theorem 3.3. □

Corollary 3.5. *Let* $k : \Omega \times I\!R^N \to [0, +\infty)$ *be a locally bounded Borel function such that* $k(\cdot, \xi) \in BV(\Omega)$ *and coincides with its approximate lower limit for every* $\xi \in I\!R^N$. *Assume also that* k *satisfies* (3.11)–(3.13). *Then the same conclusion of Corollary 3.4 holds.*

Proof. It is a direct consequence of Theorem 2.3 and Corollary 3.4. □

3.2. **Lower semicontinuity of the volume term.** As we said before, though the result due to Ambrosio deals with a general quasi-convex integrand W, the growth assumptions one has to make on W are often too strong for possible applications. The idea is then to replace quasi-convexity with the (less general) polyconvexity, with the advantage of allowing a more general growth. A first result in this spirit is contained in the next theorem, proved in [24].

Theorem 3.6. *Let* $W : \Omega \times I\!R^m \times I\!R^{m \times N} \to [0, +\infty]$ *be a Carathéodory function, polyconvex in the last variable, satisfying*

$$W(x, u, \xi) \geq \sum_{k=1}^{m \wedge N} \beta_k |\mathrm{adj}_k \xi|^{p_k} \qquad \text{for all } (x, s, \xi) \in \Omega \times I\!R^m \times M\!\!I^{m \times N} \,,$$

where $\beta_k > 0$ *for every* $k = 1, \ldots, m \wedge N$, *and the exponents* p_k *satisfy the following inequalities*

$$p_1 \geq 2 \,, \qquad p_k \geq \frac{p_1}{p_1 - 1} \quad \text{if } k = 2, \ldots, m \wedge N - 1, \qquad p_{m \wedge N} > 1 \,. \tag{3.28}$$

Then, if $\{u_n\} \subset \mathrm{GSBV}(\Omega; I\!R^m)$ *is a sequence such that* $u_n \to u$ *strongly in* $L^1(\Omega; I\!R^m)$, *with* $u \in \mathrm{GSBV}(\Omega; I\!R^m)$ *and* $\sup_n \mathcal{H}^{N-1}(J_{u_h}) < \infty$, *we have*

$$\int_\Omega W(x, u, \nabla u) \, dx \leq \liminf_{n \to +\infty} \int_\Omega W(x, u_n, \nabla u_n) \, dx \,.$$

The proof of Theorem 3.6 is an immediate consequence of the following compactness result (see [24]) for the adjoints of a SBV function.

Theorem 3.7. *Let* $u_n, u \in \mathrm{SBV}(\Omega, I\!R^m)$ *such that* $u_n \to u$ *strongly in* $L^1(\Omega, I\!R^m)$. *Assume that*

$$\sup_{n \in I\!N} \Big\{ \|u_n\|_{L^\infty(\Omega, I\!R^m)} + \sum_{k=1}^{N \wedge m} \int_\Omega |\mathrm{adj}_k \nabla u_n|^{p_k} \, dx + \mathcal{H}^{N-1}(J_{u_n}) \Big\} < \infty \,,$$

where the exponents p_k *satisfy (3.28). Then, for all* $k = 1, \dots, N \wedge m$, $\mathrm{adj}_k \nabla u_n \rightharpoonup \mathrm{adj}_k \nabla u$ *weakly in* $L^{p_k}(\Omega, I\!R^{\tau_k})$.

Our next task is to extend this result in order to get a compactness result in a somewhat limit situation and then to deduce a new semicontinuity theorem for the polyconvex case.

Theorem 3.8. *Let* $u_n, u \in \mathrm{SBV}(\Omega, I\!R^m)$ *such that* $u_n \to u$ *strongly in* $L^1(\Omega, I\!R^m)$*, where* $N, m \geq 2$*. Assume that*

$$\sup_{n \in I\!N} \Big\{ \|u_n\|_{L^\infty(\Omega, I\!R^m)} + \int_\Omega |\nabla u_n|^{N \wedge m} \, dx + \mathcal{H}^{N-1}(J_{u_n}) \Big\} < \infty \,. \tag{3.29}$$

Then, for all $k = 1, \dots, N \wedge m - 1$, $\mathrm{adj}_k \nabla u_n \rightharpoonup \mathrm{adj}_k \nabla u$ *weakly in* $L^{(N \wedge m)/k}(\Omega, I\!R^{\tau_k})$*. Moreover, there exists a subsequence* $\{u_{n_j}\}$ *such that* $\mathrm{adj}_{N \wedge m} \nabla u_{n_j}$ *converges in the biting sense to* $\mathrm{adj}_{N \wedge m} \nabla u$.

Proof. Notice that, if $k = 1, \dots, N \wedge m - 1$, from assumption (3.29) it follows that $\{\mathrm{adj}_k \nabla u_n\}$ is a sequence bounded in $L^{(N \wedge m)/k}(\Omega, I\!R^{\tau_k})$, hence the assertion follows from Theorem 3.7, choosing $p_1 = N \wedge m$ and $p_k = (N \wedge m)/k$. Therefore, we have only to prove the assertion in the limit case $k = N \wedge m$, which is also the most difficult since now the sequence $\{\mathrm{adj}_{N \wedge m} \nabla u_n\}$ is only bounded in L^1. To this aim we follow the argument introduced in [24] to prove Theorem 3.7. However, instead of referring the reader to the proof contained in that paper and listing the changes needed in our case we give all the details. Also, to simplify the notation and make our argument clearer we shall assume $m = N$. The general case is treated in the same way.

Step1. As in [24], we start by assuming that $u_n \to u = \xi x + b$ strongly in $L^1(B_1)$, where $\xi \in I\!R^{N \times N}$ and $b \in I\!R^N$ are given and that $\mathcal{H}^{N-1}(J_{u_n}) \to 0$ as $n \to \infty$. Then, given any nonnegative function $a \in L^\infty(B_1)$ and any function $w \in L^1(B_1)$ we claim that

$$\int_{B_1} a(x) |\mathrm{det}\xi - w(x)| \, dx \leq \liminf_{n \to \infty} \int_{B_1} a(x) |\mathrm{det} \nabla u_n - w(x)| \, dx \,. \tag{3.30}$$

To prove this inequality we use an induction argument on the number of components of u_n belonging to $W^{1,N}(B_1)$. In fact, if all components of each function u_n belong to $W^{1,N}(B_1)$, (3.30) follows from a well known semicontinuity property of quasi-convex integrals (see e.g. [1, Theorem II.4]).

Assume now that (3.30) holds true whenever the last $N - j$ components of each u_n belong to $W^{1,N}(B_1)$, for some $j = 0, \dots, N-1$. We are now going to prove that (3.30) still holds if only the last $N - j - 1$ components are in $W^{1,N}(B_1)$, i.e. $u_n^1, \dots, u_n^{j+1} \in \mathrm{SBV}(B_1)$, while $u_n^{j+2}, \dots, u_n^N \in W^{1,N}(B_1)$. To this aim, let us fix w and let us assume, without loss of generality, that the $\liminf$ on the right-hand side of (3.30) is indeed a limit and that $u_n(x) \to u(x)$ for $\mathcal{L}^N$-a.e. $x \in B_1$.

From the assumption (3.29) and from the estimate (2.1) it follows that $\{M^N(|\nabla u_n^{j+1}|)\}$ is bounded in $L^1(B_1)$, hence Lemma 2.7 applies. Therefore, passing possibly to a subsequence, we may assume that for any $\varepsilon > 0$ there exist a Borel set $C_\varepsilon \subset B_1$ and a positive number $\delta < \mathcal{L}^N(B_1)$ such that for any Borel set $C \subset B_1 \setminus C_\varepsilon$, with $\mathcal{L}^N(C) < \delta$, we have for all n

$$\int_C M^N(|\nabla u_n^{j+1}|) \, dx < \varepsilon \,. \tag{3.31}$$

Let us fix $\varepsilon > 0$ and $\rho \in (0,1)$ such that $\mathcal{L}^N(B_\rho) > \delta$. For any n and any $\lambda > 0$, let us denote by $u^{j+1}_{n,\lambda}$ the Lipschitz approximation of u^{j+1}_n in B_ρ provided by Theorem 2.6. Hence $u^{j+1}_{n,\lambda}(x) = u^{j+1}_n(x)$ $\mathcal{L}^N$-a.e in $B_\rho \setminus E_{n,\lambda}$, where

$$E_{n,\lambda} = \{x \in B_1 : M(|Du^{j+1}_n|) > \lambda\}\,.$$

Moreover,

$$\mathrm{Lip}(u^{j+1}_{n,\lambda}, B_\rho) \leq c\Big(\lambda + \frac{1}{1-\rho}\Big) \tag{3.32}$$

and, for every $C \in \mathcal{B}(B_1)$,

$$\mathcal{L}^N(E_{n,\lambda} \cap C) \leq \left(\frac{2}{\lambda}\right)^N \int\limits_{C\cap\{M(|\nabla u^{j+1}_n|)>\lambda/2\}} M^N(|\nabla u^{j+1}_n|)\,dx + \frac{c}{\lambda}\mathcal{H}^{N-1}(J_{u^{j+1}_n})\,, \tag{3.33}$$

where c is a constant depending only on N and $\sup_n \|u_n\|_{L^\infty(B_1,I\!R^m)}$. Notice that from (3.33) and (2.1) it follows that there exists λ_ε such that $\mathcal{L}^N(E_{n,\lambda}) < \delta$, for all $\lambda > \lambda_\varepsilon$ and every n. Therefore, from (3.31) we get in particular that

$$\int\limits_{E_{n,\lambda}\setminus C_\varepsilon} M^N(|\nabla u^{j+1}_n|)\,dx < \varepsilon \qquad \text{for all } \lambda > \lambda_\varepsilon \text{ and every } n \in I\!N\,. \tag{3.34}$$

Let us fix also $\lambda > \max\{\lambda_\varepsilon, 1\}$. From (3.32) and from the fact that $|u^{j+1}_{n,\lambda}(x)| \leq C$ in $B_\rho \setminus E_{n,\lambda}$ (which is not empty, since $\mathcal{L}^N(B_\rho) > \delta > \mathcal{L}^N(E_{n,\lambda})$), it follows that $\{u^{j+1}_{n,\lambda}\}$ is bounded in $L^\infty(B_\rho)$. Therefore, passing possibly to another (and again not relabelled) subsequence, we may assume that $\{u^{j+1}_{n,\lambda}\}$ converges weakly* in $W^{1,\infty}(B_\rho)$ to a Lipschitz function u^{j+1}_λ. Moreover, since for any $n \in I\!N$

$$\mathcal{L}^N(\{x \in B_\rho : u^{j+1}_{n,\lambda} \neq u^{j+1}_n\}) \leq \mathcal{L}^N(E_{n,\lambda})\,,$$

by the lower semicontinuity of the functional $v \to \mathcal{L}^N(\{x \in B_\rho : v(x) \neq 0\})$ with respect to the $\mathcal{L}^N$-a.e. convergence, the a.e. convergence of u_n to u and (3.33), we get that

$$\mathcal{L}^N(\{x \in B_\rho : u^{j+1}_\lambda \neq u^{j+1}\}) \leq \frac{c}{\lambda}\,, \tag{3.35}$$

where c is a positive constant independent of n, λ and ε. We can now estimate, for any $n \in I\!N$,

$$\begin{aligned}\int\limits_{B_1} a(x)|\mathrm{det}_k \nabla u_n - w(x)|dx &\geq \int\limits_{(B_\rho\setminus E_{n,\lambda})\setminus C_\varepsilon} a(x)|\det(\nabla u^1_n, \dots, \nabla u^j_n, \nabla u^{j+1}_{n,\lambda}, \dots, \nabla u^N_n) - w(x)|dx \\ &= \int\limits_{B_\rho} a(x)\chi_{B_\rho\setminus C_\varepsilon}(x)|\det(\nabla u^1_n, \dots, \nabla u^j_n, \nabla u^{j+1}_{n,\lambda}, \dots, \nabla u^N_n) - w(x)|\,dx \\ &\quad - \int\limits_{E_{n,\lambda}\setminus C_\epsilon} a(x)|\det(\nabla u^1_n, \dots, \nabla u^j_n, \nabla u^{j+1}_{n,\lambda}, \dots, \nabla u^N_n) - w(x)|\,dx\,.\end{aligned}$$

Letting $n \to \infty$ on both sides of this inequality and using the fact that (3.30) (with $a(x)$ replaced by $a(x)\chi_{B_\rho\setminus C_\varepsilon}(x)$) holds true if the last $N-j$ components of each u_n are in $W^{1,N}(B_\rho)$, we get that

$$\liminf_{n\to+\infty} \int\limits_{B_1} a(x)|\det\nabla u_n - w(x)|\,dx \tag{3.36}$$

$$\geq \int_{B_\rho \setminus C_\varepsilon} a(x)|\det(\nabla u^1, \ldots, \nabla u^j, \nabla u_\lambda^{j+1}, \ldots, \nabla u^N) - w(x)|\, dx - \limsup_{n\to+\infty} I^\varepsilon_{n,\lambda}\,,$$

where

$$I^\varepsilon_{n,\lambda} = \int_{E_{n,\lambda}\setminus C_\epsilon} a(x)|\det(\nabla u_n^1, \ldots, \nabla u_n^j, \nabla u_{n,\lambda}^{j+1}, \ldots, \nabla u_n^N) - w(x)|\, dx\,.$$

In order to estimate $I^\varepsilon_{n,\lambda}$, we recall (3.32), (3.33), and (3.34), thus getting

$$\begin{aligned} I^\varepsilon_{n,\lambda} &\leq c\Big(\lambda + \frac{1}{1-\rho}\Big) \int_{E_{n,\lambda}\setminus C_\varepsilon} |\mathrm{adj}_{N-1}(\nabla u_n^1, \ldots, \nabla u_n^j, \nabla u_n^{j+2}, \ldots \nabla u_n^N)|\, dx + c\int_{E_{n,\lambda}} |w|\, dx \\ &\leq c\Big(\lambda + \frac{1}{1-\rho}\Big)\Big(\int_{B_1} |\nabla u_n|^N dx\Big)^{\frac{N-1}{N}} \Big(L^N(E_{n,\lambda}\setminus C_\varepsilon)\Big)^{\frac{1}{N}} + \omega(\lambda) \\ &\leq c\Big(\lambda + \frac{1}{1-\rho}\Big)\Big(\frac{1}{\lambda^N} \int_{(E_{n,\lambda}\setminus C_\varepsilon)\cap\{M^N(|\nabla u_n^{j+1}|)>\lambda/2\}} M^{p_1}(|\nabla u_n^{j+1}|)dx + \frac{1}{\lambda}\mathcal{H}^{N-1}(J_{u_n^{j+1}})\Big)^{\frac{1}{N}} + \omega(\lambda) \\ &\leq c\Big(\lambda + \frac{1}{1-\rho}\Big)\Big(\frac{\varepsilon}{\lambda^N} + \frac{1}{\lambda}\mathcal{H}^{N-1}(J_{u_n^{j+1}})\Big)^{\frac{1}{N}} + \omega(\lambda)\,, \end{aligned}$$

where $\omega(\lambda)$ is a quantity, independent of ε and h, converging to 0 as λ goes to infinity, and c is a constant depending only on N and $\sup_n \|u_n\|_{L^\infty(B_1,\mathbb{R}^m)}$. Letting n go to infinity in the previous estimate and recalling that $\mathcal{H}^{N-1}(J_{u_n}) \to 0$ as $n \to \infty$, we get

$$\limsup_{n\to\infty} I^\varepsilon_{n,\lambda} \leq c\frac{1}{\lambda}\Big(\lambda + \frac{1}{1-\rho}\Big)\varepsilon^{\frac{1}{N}}\,.$$

In conclusion, recalling (3.36), we have that if $\lambda > \max\{\lambda_\varepsilon, 1\}$, then

$$\liminf_{n\to\infty} \int_{B_1} a(x)|\det\nabla u_n - w(x)|dx \geq \int_{B_\rho\setminus C_\varepsilon} a(x)|\det(\nabla u^1, \ldots, \nabla u^j, \nabla u_\lambda^{j+1}, \ldots, \nabla u^N) - w(x)|dx - \frac{c\varepsilon^{\frac{1}{N}}}{1-\rho} - \omega(\lambda)\,.$$

Therefore, we have in particular that

$$\liminf_{n\to\infty} \int_{B_1} a(x)|\det\nabla u_n - w(x)|dx \geq \int_{(B_\rho\setminus C_\varepsilon)\cap\{u_\lambda^{j+1}=u^{j+1}\}} a(x)|\det\xi - w(x)|dx - \frac{c\varepsilon^{\frac{1}{N}}}{1-\rho} - \omega(\lambda)\,.$$

Recalling (3.35), we let first λ go to ∞, then ε to zero, and $\rho \to 1$, thus obtaining (3.30) when the last $N-j-1$ components of u_n belong to $W^{1,N}(B_1)$.

Step 2. We now turn to a general sequence satisfying the assumption (3.29) for $m = N$. Fix $a \in L^\infty(\Omega)$ nonnegative and $w \in L^1(\Omega)$. We want to show that

$$\int_\Omega a(x)|\det\nabla u - w(x)|\, dx \leq \liminf_{n\to\infty} \int_\Omega a(x)|\det\nabla u_n - w(x)|\, dx\,. \tag{3.37}$$

To this aim we may assume, without loss of generality, that the lim inf on the right-hand side of (3.37) is a limit. Passing possibly to a (not relabelled) subsequence and observing that the sequence $|\det\nabla u_n - w|$ is bounded in $L^1(\Omega)$, we may assume that there exist two nonnegative Radon measure in Ω, say μ and λ, such that

$$\mathcal{H}^{N-1}\lfloor J_{u_n} \rightharpoonup \mu, \quad |\det\nabla u_n - w| \rightharpoonup \lambda \qquad \text{weakly* in } \Omega\,. \tag{3.38}$$

Moreover, from the assumption (3.29) we may also assume that there exists a nonnegative Radon measure ν such that

$$|\nabla u_n|^N \lfloor \mathcal{L}^N \rightharpoonup \nu \qquad \text{weakly* in } \Omega\,. \tag{3.39}$$

Clearly, (3.37) is proved if we show that

$$\frac{d\lambda}{d\mathcal{L}^N}(x) \geq a(x)|\det\nabla u(x) - w(x)| \qquad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega\,. \tag{3.40}$$

To this aim, let us consider the points $x \in \Omega$ such that

$$\lim_{\rho\to 0} \frac{\mu(B_\rho(x))}{\rho^{N-1}} = 0\,, \tag{3.41}$$

$$\frac{d\lambda}{d\mathcal{L}^N}(x) < \infty, \qquad \lim_{\rho\to 0} \frac{\nu(B_\rho(x))}{\rho^N} < \infty\,, \tag{3.42}$$

and set

$$G = \{x \in \Omega : x \text{ is a Lebesgue point for } u,\, a \text{ and } w,\, \nabla u(x) \text{ exists and (3.41), (3.42) hold}\}\,.$$

Notice that since the set of points in Ω where (3.41) does not hold is a Borel set of σ-finite $\mathcal{H}^{N-1}$-measure (see e.g. [9, Theorem 2.56]) and the set where (3.42) does not hold has $\mathcal{L}^N$ measure zero, we have that $\mathcal{L}^N(\Omega \setminus G) = 0$.
Let us fix $x_0 \in G$ and choose an infinitesimal sequence ρ_j such that $\mu(\partial B_{\rho_j}(x_0)) = \lambda(\partial B_{\rho_j}(x_0)) = 0$ for all j. From the strong convergence of u_n to u and from (3.38), (3.39), it follows that there exists a strictly increasing sequence of integers n_j, such that

$$\begin{cases} \dfrac{1}{\rho_j^{N+1}} \displaystyle\int_{B_{\rho_j}(x_0)} |u_{n_j}(x) - u(x)|\,dx < \dfrac{1}{j}\,, \\[2ex] \left| \displaystyle\fint_{B_{\rho_j}(x_0)} a(x)|\det\nabla u_{n_j} - w(x)|\,dx - \dfrac{d\lambda}{d\mathcal{L}^N}(x_0)\right| < \dfrac{1}{j}\,, \\[2ex] \displaystyle\sup_{j\in\mathbb{N}} \dfrac{1}{\rho_j^N} \int_{B_{\rho_j}(x_0)} |\nabla u_{n_j}|^N dx < \infty\,, \\[2ex] \dfrac{1}{\rho_j^{N-1}} \left| \mathcal{H}^{N-1}(J_{u_{n_j}} \cap B_{\rho_j}(x_0)) - \mu(B_{\rho_j}(x_0))\right| < \dfrac{1}{j}\,, \end{cases} \tag{3.43}$$

where the last inequality follows from the fact that since $\mu(\partial B_{\rho_j}(x_0)) = 0$ for all j, then we have that $\mathcal{H}^{N-1}(J_{u_n} \cap B_{\rho_j}(x_0)) \to \mu(B_{\rho_j}(x_0)$ as $n \to \infty$ and the second inequality follows similarly from the fact that $\lambda(\partial B_{\rho_j}(x_0)) = 0$. Let us set

$$v_j(y) = \frac{u_{n_j}(x_0 + \rho_j y) - u(x_0)}{\rho_j} \qquad \text{for all } j \in \mathbb{N} \text{ and } y \in B_1\,.$$

Notice that from $(3.43)_1$ we have

$$\int_{B_1} |v_j(y) - \nabla u(x_0)y|dy \leq \int_{B_1} \frac{|u_{h_j}(x_0+\rho_j y) - u(x_0+\rho_j y)|}{\rho_j} dy + \int_{B_1} \left| \frac{u(x_0+\rho_j y) - u(x_0)}{\rho_j} - \nabla u(x_0)y \right| dy$$

$$\leq \frac{1}{\rho_j^{N+1}} \int\limits_{B_{\rho_j}(x_0)} |u_{n_j}(x) - u(x)|\, dx + \frac{1}{\rho_j^{N+1}} \int\limits_{B_{\rho_j}(x_0)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|\, dx$$

$$\leq \frac{1}{j} + \frac{1}{\rho_j^{N+1}} \int\limits_{B_{\rho_j}(x_0)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|\, dx\,,$$

hence $v_j(y) \to \nabla u(x_0)y$ in $L^1(B_1, I\!R^m)$. Moreover, $(3.43)_4$ yields that

$$\mathcal{H}^{N-1}(J_{v_j} \cap B_1) = \frac{1}{\rho_j^{N-1}}\Big[\mathcal{H}^{N-1}(J_{u_{n_j}} \cap B_{\rho_j}(x_0)) - \mu(B_{\rho_j}(x_0))\Big] + \frac{\mu(B_{\rho_j}(x_0))}{\rho_j^{N-1}} < \frac{1}{j} + \frac{\mu(B_{\rho_j}(x_0))}{\rho_j^{N-1}}$$

and thus we have also that $\mathcal{H}^{N-1}(J_{v_j} \cap B_1) \to 0$ as $j \to \infty$. Therefore, since from $(3.43)_3$ we have in particular

$$\sup_{j \in I\!N} \int\limits_{B_1} |\nabla v_j|^N dy < \infty\,,$$

we may apply what we have proved in Step 1 to the sequence $\{v_j\}$. To this aim, notice that since x_0 is a Lebesgue point for a, the functions $a(x_0 + \rho_j y)$ converge in $L^1(B_1)$ and, up to a subsequence, also a.e. to $a(x_0)$. Therefore, if $\varepsilon > 0$ there exists $C_\varepsilon \subset B_1$, $\mathcal{L}^N(C_\varepsilon) < \varepsilon$, such that $a(x_0 + \rho_j y) \to a(x_0)$ uniformly to $a(x_0)$ in $B_1 \setminus C_\varepsilon$. Thus, recalling $(3.43)_2$, we have

$$\mathcal{L}^N(B_1 \backslash C_\varepsilon) a(x_0) |\det \nabla u(x_0) - w(x_0)| \leq \liminf_{j\to\infty} \int\limits_{B_1} \chi_{B_1 \backslash C_\varepsilon}(y) a(x_0) |\det \nabla v_j - w(x_0)|\, dy$$

$$= \liminf_{j\to\infty} \int\limits_{B_1} \chi_{B_1 \backslash C_\varepsilon}(y) a(x_0 + \rho_j y) |\det \nabla v_j - w(x_0)|\, dy$$

$$\leq \liminf_{j\to\infty} \int\limits_{B_1} a(x_0 + \rho_j y) |\det \nabla v_j - w(x_0)|\, dy = \liminf_{j\to\infty} \frac{1}{\rho_j^N} \int\limits_{B_{\rho_j}(x_0)} a(x) |\det \nabla u_{n_j} - w(x_0)| dx$$

$$\leq \lim_{j\to\infty} \frac{1}{\rho_j^N} \int\limits_{B_{\rho_j}(x_0)} a(x) |\det \nabla u_{n_j} - w(x)| dx + \lim_{j\to\infty} \frac{1}{\rho_j^N} \int\limits_{B_{\rho_j}(x_0)} |w(x) - w(x_0)| dx = \mathcal{L}^N(B_1) \frac{d\lambda}{d\mathcal{L}^N}(x_0).$$

Letting $\varepsilon \to 0^+$, (3.40) follows, thus completing the proof of (3.37).

Step 3. To conclude the proof, we use Lemma 2.7 thus getting a subsequence $\{\det \nabla u_{n_j}\}$ converging in the biting sense to some function $d \in L^1(\Omega)$. We claim that $d = \det \nabla u$. To show this, let us consider a decreasing sequence of sets E_i such that $\mathcal{L}^N(E_i) \to 0$ as $i \to \infty$ $\det \nabla u_{n_j} \rightharpoonup d$ weakly in $L^1(\Omega \setminus E_i)$ for all i. Fix i and apply (3.37) with $a = \chi_{\Omega \backslash E_i}$. We then get that for any $w \in L^1(\Omega)$

$$\int\limits_{\Omega \backslash E_i} |\det \nabla u - w(x)|\, dx \leq \liminf_{j\to\infty} \int\limits_{\Omega \backslash E_i} |\det \nabla u_{n_j} - w(x)|\, dx$$

and from this inequality we have also that for any $w \in L^1(\Omega)$

$$\int\limits_{\Omega} |\chi_{\Omega \backslash E_i}(x) \det \nabla u - w(x)|\, dx \leq \liminf_{j\to\infty} \int\limits_{\Omega} |\chi_{\Omega \backslash E_i}(x) \det \nabla u_{n_j} - w(x)|\, dx\,.$$

Since the sequence $\{\det \nabla u_{n_j}\}$ is equi-integrable in $\Omega \setminus E_i$, clearly the sequence $\{\chi_{\Omega \backslash E_i} \det \nabla u_{n_j}\}$ is equi-integrable in Ω. Therefore, from Lemma 2.9 we may conclude that $\chi_{\Omega \backslash E_i} \det \nabla u_{n_j} \rightharpoonup$

$\chi_{\Omega\setminus E_i}\det\nabla u$ weakly in $L^1(\Omega)$, hence $\det\nabla u_{n_j} \rightharpoonup \det\nabla u$ weakly in $L^1(\Omega \setminus E_i)$ for all i, thus proving the assertion. □

From the result just proved we obtain immediately the following lower semicontinuity result for polyconvex functionals.

Theorem 3.9. *Let $W : \Omega \times I\!R^m \times I\!R^{m\times N} \to [0, +\infty]$ be a Carathéodory function, polyconvex in the last variable. Then, if $u_n, u \in \mathrm{GSBV}(\Omega; I\!R^m)$ are such that $u_n \to u$ strongly in $L^1(\Omega; I\!R^m)$ and*

$$\sup_{n\in I\!N}\Big\{\int_\Omega |\nabla u_n|^{N\wedge m}\,dx + \mathcal{H}^{N-1}(J_{u_n})\Big\} < \infty\,,$$

then we have

$$\int_\Omega W(x, u, \nabla u)\,dx \le \liminf_{n\to+\infty}\int_\Omega W(x, u_n, \nabla u_n)\,dx\,. \tag{3.44}$$

Proof. Let us first assume that $u_n, u \in \mathrm{SBV}(\Omega; I\!R^m)$ satisfy the assumption (3.29). Since W is a convex function of the minors, (3.44) follows immediately from the strong convergence of u_n to u and the weak convergence of $\mathrm{adj}_k\nabla u_n$ to $\mathrm{adj}_k\nabla u$, for $k < N \wedge m$ and the biting convergence of $\mathrm{adj}_{N\wedge m}\nabla u_n$ to $\mathrm{adj}_{N\wedge m}\nabla u$. The general case $u \in \mathrm{GSBV}(\Omega; I\!R^m)$ can be obtained as in the proof of Theorem 3.7 in [6].

□

Aknowlegment. The work by N.Fusco was partially supported by the European Research Council under FP7, Advanced Grant n. 226234 *Analytic Techniques for Geometric and Functional Inequalities.*

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