

Homogenization of Some Low-Cost Control Problems

Rajesh Mahadevan and T. Muthukumar
Facultad de Ciencias Físicas Y Matemáticas,
Universidad de Concepción,
Concepción, Chile
e-mail: [mthirumalai,rmahadevan]@udec.cl

October 14, 2009

Abstract

The aim of this article is to study the asymptotic behaviour of some low-cost control problems. These problems motivate the study of H -convergence with weakly converging data. An improved lower bound for the limit of energy functionals corresponding to weak data is established, in the periodic case. This fact is used to prove the Γ -convergence of a low-cost problem with Dirichlet-type integral. Finally, we study the asymptotic behaviour of a low-cost problem with controls converging to measures.

Keywords: homogenization, optimal control, Γ -convergence, two-scale convergence, measure data

MSC (2000): 35B27, 49J20

1 Introduction

Let $n \geq 1$ and let Ω be a bounded open subset of \mathbb{R}^n . Let $0 < \alpha \leq \beta$ be two given positive real constants. We denote by $M(\alpha, \beta, \Omega)$ the class of all $n \times n$ matrices A with entries in $L^\infty(\Omega)$, such that,

$$\alpha|\xi|^2 \leq A(x)\xi \cdot \xi \leq \beta|\xi|^2 \quad \text{a.e. in } x, \quad \forall \xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n.$$

Let $A^t(x)$ denote the transpose of $A(x)$.

Given $A \in M(\alpha, \beta, \Omega)$, $f \in L^2(\Omega)$, $N > 0$ (a constant) and U a closed convex subset of $L^2(\Omega)$, we consider the following basic optimal control problem: find $\theta^* \in U$ such that,

$$J(\theta^*) = \min_{\theta \in U} J(\theta), \tag{1.1}$$

where the cost functional, $J : U \rightarrow \mathbb{R}$, is defined by

$$J(\theta) = I(u, \theta) + \frac{N}{2} \|\theta\|_2^2 \quad (1.2)$$

and the state $u = u(\theta)$ is the weak solution in $H_0^1(\Omega)$ of the boundary value problem

$$\begin{cases} -\operatorname{div}(A\nabla u) &= f + \theta & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

We consider the following kinds of $I(u, \theta)$:

(a) For $B \in M(c, d, \Omega)$ and symmetric, we consider Dirichlet-type integrals:

$$I(u, \theta) = \frac{1}{2} \int_{\Omega} B(x) \nabla u(\theta) \cdot \nabla u(\theta) \, dx$$

(b) For a fixed $r \in [1, \frac{n}{n-2})$,

$$I(u, \theta) = \|u(\theta)\|_r^r.$$

Then, it can be shown that the J defined above are lower semicontinuous, coercive and strictly convex, and, therefore, there exists a unique optimal control, $\theta^* \in U$ minimizing J over U (cf. [1]).

The main aim of this paper is to study the asymptotic behaviour of the optimal control problems (1.1) depending on a small parameter $\varepsilon > 0$ which represents the scale of heterogeneity of the material.

This problem was first studied in [2, 3] for varying coefficients $\{A_\varepsilon\} \subset M(\alpha, \beta, \Omega)$ in (1.3), $\{B_\varepsilon\} \subset M(c, d, \Omega)$ in the cost (1.2) corresponding to the case (a) and $N > 0$ fixed. The approach used therein consists in passing to the limit in the corresponding system of optimality conditions involving an adjoint state. A complete characterization of the asymptotic behaviour of the control problems was obtained.

In the above problem, if N is allowed to vary and degenerate by taking $N = \varepsilon$, then it is called *low-cost control* problem. The low-cost control problems were first introduced by J. L. Lions in [4] (also see [5]) and extensively studied in [6, 7, 8, 9]. The corresponding sequence of functionals J_ε is not equicoercive over $L^2(\Omega)$ and thus the sequence of optimal controls $\{\theta_\varepsilon^*\}$ is not bounded, *a priori*, in $L^2(\Omega)$. But θ_ε^* is weakly compact in $H^{-1}(\Omega)$ and thus converges to some $\theta^* \in H^{-1}(\Omega)$. This weak convergence is, in general, not enough for studying the asymptotic behaviour of the system which constitutes the optimality conditions. This case was studied in [8, 9] and a partial homogenization of the optimality system was proved when the control set is the positive cone, with or without periodicity assumptions on the coefficient matrices. In this article (cf. §4), we obtain, for the first time, a complete homogenization result for low-cost control problems by taking U to be an arbitrary closed convex set in $L^2(\Omega)$ while assuming the coefficients to be periodic. Here we prove the variational convergence of the optimal control problem in the framework of Γ -convergence.

Subsequently, in §6, we study the asymptotic behaviour of a low cost problem whose limit optimal control will be in the space of measures. Given a constant $k > 0$, let

$$U = \{\theta \in L^2(\Omega) \mid \|\theta\|_1 \leq k\}$$

be the set of all admissible controls. We consider the optimal control problem with cost functional as in case (b) governed by (1.3) with varying coefficients $A_\varepsilon \in M(\alpha, \beta, \Omega)$. The main difficulty with this problem, in contrast to the problem with costs as in case (a), is that there is no weak compactness of the optimal controls θ_ε^* , even in $H^{-1}(\Omega)$. Thus, one is unable to homogenize the control problem for a general admissible set U . However, this problem was homogenized when U is the positive cone and $r = 2$, in [8, 9], and the limit problem was obtained on the positive cone of $H^{-1}(\Omega)$. We know, by Riesz representation theorem, that any non-negative distribution in $H^{-1}(\Omega)$ is a non-negative Radon measure on Ω . Thus, we wish to consider the controls as measures. Therefore, in §6, we consider the control set U to be the class of all functions in $L^2(\Omega)$ that are bounded in $L^1(\Omega)$ and homogenize with respect to weak-* convergence of measures.

The paper is organised as follows: In §2 we recall some basic facts and tools required for the results proved in §3 and §4. In §3, we conjecture on the best lower bound of ‘generalised’ energy functionals for weakly converging data and prove the same under periodicity assumptions on the coefficients. In §4, we homogenize the periodic low-cost control problems with cost as in case (a). In §5, we present the notion of solution for measure data introduced by G. Stampacchia. We also give a G -convergence result with respect to varying measures. Finally, in §6, we study the asymptotic behaviour of the low-cost control problems with cost as in case (b).

2 Preliminaries

In this section, we introduce some basic tools and facts that will be used in this article.

2.1 G -convergence and H -convergence

For all the results in this section we refer to [10, 11, 12]. We say a sequence $\{A_\varepsilon\} \subset M(\alpha, \beta, \Omega)$ **G -converges** to A_0 (denoted as $A_\varepsilon \xrightarrow{H} A_0$) iff for any $g \in H^{-1}(\Omega)$, the solution v_ε of

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) = g & \text{in } \Omega \\ v_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \quad (2.1)$$

is such that

$$v_\varepsilon \rightharpoonup v_0 \text{ weakly in } H_0^1(\Omega) \quad (2.2)$$

where v_0 is the unique solution of

$$\begin{cases} -\operatorname{div}(A_0 \nabla v_0) = g & \text{in } \Omega \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

The matrix A_0 is called the G -limit of the sequence $\{A_\varepsilon\}$. We say a sequence of matrices H -converges to A_0 , if it G -converges and, in addition, for any $g \in H^{-1}(\Omega)$ we have

$$A_\varepsilon \nabla v_\varepsilon \rightharpoonup A_0 \nabla v_0 \text{ weakly in } (L^2(\Omega))^n \quad (2.4)$$

where v_ε and v_0 are as in (2.1) and (2.3), respectively. For symmetric matrices both the notion coincide.

Given a sequence $\{A_\varepsilon\} \subset M(\alpha, \beta, \Omega)$ which H -converges to A_0 , the sequence of corrector matrices P_ε is that which satisfies the following properties:

- (a) $P_\varepsilon \rightharpoonup I$ weakly in $(L^2(\Omega))^{n \times n}$.
- (b) $A_\varepsilon P_\varepsilon \rightharpoonup A_0$ weakly in $(L^2(\Omega))^{n \times n}$.
- (c) ${}^t P_\varepsilon A_\varepsilon P_\varepsilon \rightharpoonup A_0$ weak* in $[\mathcal{D}'(\Omega)]^{n \times n}$, the space of distributions.

One procedure to obtain the corrector matrix is by considering $\chi_\varepsilon^i \in H^1(\Omega)$, for $1 \leq i \leq n$, which are solutions of

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \chi_\varepsilon^i) = -\operatorname{div}(A_0 e_i) & \text{in } \Omega \\ \chi_\varepsilon^i = x_i & \text{on } \partial\Omega \end{cases} \quad (2.5)$$

and then by defining $P_\varepsilon e_i = \nabla \chi_\varepsilon^i$ for $1 \leq i \leq n$.

2.2 Γ -convergence

For all the results in this section we refer to [1, 13]. Let X be a topological space and let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. Let $\{F_m\}$ be a sequence of functions from X in to $\overline{\mathbb{R}}$.

For any function F on X , let $S(F)$ be the set of all lower semicontinuous functions G on X such that $G \leq F$. We define the *lower semicontinuous envelope* of F , \overline{F} , as

$$\overline{F}(x) = \sup_{G \in S(F)} G(x), \quad \forall x \in X.$$

Observe that every lower semicontinuous function is its own envelope.

We now define the *sequential Γ -lower and upper limit*. The Γ -upper limit of F_m is given by,

$$F^+(x) := \inf \left\{ \limsup_{m \rightarrow \infty} F_m(x_m) : x_m \rightarrow x \right\}.$$

Similarly, the Γ -lower limit of F_m is given by,

$$F^-(x) := \inf \left\{ \liminf_{m \rightarrow \infty} F_m(x_m) : x_m \rightarrow x \right\}.$$

We say a function F is the Γ -limit of F_m if $F = F^+ = F^-$. A characterization of the sequential Γ -limit F w.r.t the topology of X is given by the following two conditions:

(i) For every $x \in X$ and for every sequence $\{x_m\}$ converging to x in X , we have

$$\liminf_{m \rightarrow \infty} F_m(x_m) \geq F(x).$$

(ii) For every $x \in X$, there exists a sequence $\{x_m\}$ converging to x in X such that

$$\limsup_{m \rightarrow \infty} F_m(x_m) \leq F(x).$$

We now recall a result of Γ -convergence theory.

Theorem 2.1. *Let F_m Γ -converge to F in X and let x_m be a minimizer of F_m in X . If $\{x_m\}$ converges to x in X , then x is a minimizer of F in X and the minima converge,*

$$F(x) = \lim_{m \rightarrow \infty} F_m(x_m).$$

□

2.3 Two-scale convergence

For all the results in this section we refer to [14, 15, 16]. Let $Y = (0, 1)^n$ be the unit cell of \mathbb{R}^n . We say that a sequence of functions $\{v_\varepsilon\}$ in $L^2(\Omega)$ *weakly two-scale converges* to a limit $v \in L^2(\Omega \times Y)$ (denoted as $v_\varepsilon \xrightarrow{2s} v$) if

$$\int_{\Omega} v_\varepsilon(x) \phi\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Omega} \int_Y v(x, y) \phi(x, y) dy dx, \quad \forall \phi \in L^2[\Omega; C_{\text{per}}(Y)].$$

It is possible to replace $L^2[\Omega; C_{\text{per}}(Y)]$ by $\mathcal{D}[\Omega, C_{\text{per}}^\infty(Y)]$ in the above definition of weak two-scale convergence provided we add the assumption that $\{v_\varepsilon\}$ is bounded in $L^2(\Omega)$.

We say that a sequence of functions $\{v_\varepsilon\}$ in $L^2(\Omega)$ *strongly two-scale converges* to $v \in L^2(\Omega \times Y)$ (denoted as $v_\varepsilon \xrightarrow{2s} v$), if $\{v_\varepsilon\}$ weakly two-scale converges to v and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |v_\varepsilon|^2 dx = \int_{\Omega} \int_Y |v(x, y)|^2 dy dx.$$

We say that a function $\phi(x, y)$, Y -periodic in y , is an *admissible* function if the sequence $\phi_\varepsilon = \phi\left(x, \frac{x}{\varepsilon}\right)$ strongly two-scale converges to $\phi(x, y)$. The spaces $C[\Omega; L^2_{\text{per}}(Y)]$, $L^\infty[\Omega; C_{\text{per}}(Y)]$ and $L^\infty_{\text{per}}[Y; C(\bar{\Omega})]$ are some examples of classes of admissible functions.

We now state some of the main results of two-scale convergence theory that will be used in this article.

Theorem 2.2. *For any bounded sequence $\{v_\varepsilon\} \subset L^2(\Omega)$, there exists a $v \in L^2(\Omega \times Y)$ such that, v_ε weakly two-scale converges to v , for a subsequence. Also, if v_ε is bounded in $H^1(\Omega)$, then v is independent of y and is in $H^1(\Omega)$, and there exists a $v_1 \in L^2[\Omega; H^1_{\text{per}}(Y)]$ such that, up to a subsequence, ∇v_ε weakly two-scale converges to $\nabla v + \nabla_y v_1$.*

Theorem 2.3. Let $u_\varepsilon \xrightarrow{2s} u$. Then, given any sequence $v_\varepsilon \xrightarrow{2s} v$, we have that

$$\int_{\Omega} u_\varepsilon(x)v_\varepsilon(x)\tau(x) dx \rightarrow \int_{\Omega} \int_Y u(x,y)v(x,y)\tau(x) dy dx$$

for every $\tau \in C_0^\infty(\Omega)$. The $\tau \in C_0^\infty(\Omega)$ can be replaced with $\tau\left(x, \frac{x}{\varepsilon}\right)$ where $\tau \in \mathcal{D}(\Omega, C_{\text{per}}^\infty(Y))$.

We now recall a property of convex periodic functionals with respect to two-scale convergence.

Proposition 2.4 ([17, Proposition 2.5]). Let $j := j(y, \xi)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$, Y -periodic in y , convex in ξ and satisfies for some constants $a, b > 0$,

$$a|\xi|^2 \leq j(y, \xi) \leq b(1 + |\xi|^2), \quad (y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Also, let $\{v_\varepsilon\} \subset L^2(\Omega)$ be such that $v_\varepsilon \xrightarrow{2s} v \in L^2(\Omega \times Y)$. Then

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} j\left(\frac{x}{\varepsilon}, v_\varepsilon(x)\right) dx \geq \int_{\Omega \times Y} j(y, v(x, y)) dx dy.$$

□

2.4 Convex Analysis

For any function h on \mathbb{R}^n , one defines its convex conjugate h' on \mathbb{R}^n in the following way:

$$h'(x') = \sup_{x \in \mathbb{R}^n} \{\langle x, x' \rangle - h(x)\}.$$

The following inequality is called the *Fenchel's inequality* for any proper convex function h and its conjugate h' :

$$\langle x, x' \rangle \leq h(x) + h'(x'), \quad \forall x, x' \in \mathbb{R}^n.$$

If h is a quadratic convex function, say of the form

$$h(x) = \frac{1}{2} \langle x, Qx \rangle$$

where Q is a symmetric, positive definite $n \times n$ matrix, then

$$h'(x') = \frac{1}{2} \langle x', Q^{-1}x' \rangle. \tag{2.6}$$

For above results, we refer to [18]. We now comment on a classical property of commutativity of infimum and the integral. The first results along this direction was proved by Rockafellar in [19, 20]. Another version of the same was proved in [21, Theorem 1]. However we shall now state the version as given in [22, Lemma 4.3].

If $\{\Delta_k\}$ is a family of measurable set functions from Ω into \mathbb{R}^n , then there exists a measurable set function (cf. [23, Proposition 14]) Δ from Ω into \mathbb{R}^n with the following properties:

- (i) For every k , we have $\Delta_k(x) \subseteq \Delta(x)$ for a.e. $x \in \Omega$.
- (ii) If Π is a set function on Ω such that for every k , $\Delta_k(x) \subseteq \Pi(x)$ for a.e. $x \in \Omega$, then $\Delta(x) \subseteq \Pi(x)$ for a.e. $x \in \Omega$.

The set function Δ is denoted as $\Delta(x) = \text{ess-sup}_k \Delta_k(x)$.

Let E be a set of measurable functions from Ω to \mathbb{R}^n . We say E is C^1 -stable if for every finite family $\{\omega_k\}_k \subset E$ and for every non-negative family of functions $\{\psi_k\}_k \subset C^1(\overline{\Omega})$ such that $\sum_k \psi_k = 1$ in Ω , we have that $\sum_k \psi_k \omega_k \in E$. Observe that C^1 -stability implies convexity.

Lemma 2.5 ([22, Lemma 4.3]). *Let E be a C^1 -stable set and let j be Borel measurable on $\Omega \times \mathbb{R}^n$ such that $j(x, \cdot)$ is convex on \mathbb{R}^n for a.e. $x \in \Omega$. Suppose that $j(\cdot, \omega(\cdot)) \in L^1(\Omega)$, for every $\omega \in E$, and let $\Delta(x) = \text{ess-sup}_{\omega \in E} \{\omega(x)\}$ then*

$$\inf_{\omega \in E} \int_{\Omega} j(x, \omega(x)) dx = \int_{\Omega} \inf_{\zeta \in \Delta(x)} j(x, \zeta) dx.$$

□

3 Energy bounds

Given $A_\varepsilon \subset M(\alpha, \beta, \Omega)$ which H -converges to A_0 , a standard result of H -convergence is that the energies converge, *i.e.*,

$$\int_{\Omega} A_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} A_0 \nabla v_0 \cdot \nabla v_0 dx, \quad (3.1)$$

where v_ε and v_0 are the solution of (2.1) and (2.3), respectively. Moreover, one has from the theory of Γ -convergence, the following basic result.

Lemma 3.1 (cf. [1, Chapter 13]). *Given a sequence of symmetric matrices $A_\varepsilon \subset M(\alpha, \beta, \Omega)$ which G -converges to A_0 and given any sequence w_ε weakly converging to w_0 in $H_0^1(\Omega)$, we have*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon \nabla w_\varepsilon \cdot \nabla w_\varepsilon dx \geq \int_{\Omega} A_0 \nabla w_0 \cdot \nabla w_0 dx. \quad (3.2)$$

Given $\{B_\varepsilon\} \subset M(c, d, \Omega)$, a sequences of symmetric matrices and v_ε solutions of (2.1), we have from Lemma 3.1 that,

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx \geq \int_{\Omega} B_0 \nabla v_0 \cdot \nabla v_0 dx, \quad (3.3)$$

where v_0 is solution of (2.3) and B_0 is the H -limit of $\{B_\varepsilon\}$. Moreover, (3.3) remains valid when v_ε are solutions of (3.4), where g_ε converges strongly to g in $H^{-1}(\Omega)$. A question of interest is to obtain the best lower bound for the limit on the left hand side of (3.3) when v_ε is the solution of (3.4) and A_ε H -converges to A_0 . In order to state a result improving (3.3) we recall (from §2.1) that $\{P_\varepsilon\}$ is the sequence of corrector matrices associated with $\{A_\varepsilon\}$. Let B^\sharp be the weak-* limit of $\{P_\varepsilon^t B_\varepsilon P_\varepsilon\}$ in $(L^\infty(\Omega))^{n \times n}$.

Theorem 3.2 (cf. [24]). Let $g_\varepsilon \rightarrow g$ strongly in $H^{-1}(\Omega)$ and let $v_\varepsilon \in H_0^1(\Omega)$ be the weak solution of

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) = g_\varepsilon & \text{in } \Omega \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.4)$$

then

$$\int_{\Omega} B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} B^\sharp \nabla v_0 \cdot \nabla v_0 \, dx \quad (3.5)$$

where $v_0 \in H_0^1(\Omega)$ is the unique solution of (2.3). \square

Comparing (3.5) with (3.3) we observe that $B^\sharp \geq B_0$. If $B_\varepsilon = A_\varepsilon$, then we have $B^\sharp = B_0 = A_0$. For the properties of B^\sharp we refer to [24].

Remark 3.3. For g_ε weakly converging to g in $H^{-1}(\Omega)$, although v_ε converges weakly in $H_0^1(\Omega)$ (up to a subsequence) one can still not conclude that (3.3) holds. In fact, even the convergence of the energies as in (3.1) is not valid. \square

The above remark motivates us to make the following conjecture:

Conjecture 1. Let $g_\varepsilon \rightharpoonup g$ weakly in $H^{-1}(\Omega)$ and let $v_\varepsilon \in H_0^1(\Omega)$, the weak solution of (3.4), be such that $v_\varepsilon \rightharpoonup v_0$ weakly in $H_0^1(\Omega)$, where $v_0 \in H_0^1(\Omega)$ is the unique solution of (2.3). Then

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \geq \int_{\Omega} B^\sharp \nabla v_0 \cdot \nabla v_0 \, dx. \quad (3.6)$$

\square

The conjecture is open, in general. However, in this section we prove the conjecture for the periodic case with additional hypothesis on the data g_ε .

To do so, we recall the periodic set up. Let $Y = (0, 1)^n$ be the reference cell in \mathbb{R}^n . Let $A = A(x, y) \in M(\alpha, \beta, \Omega \times Y)$, Y -periodic in y . We assume that $A(x, y) = (a_{ij}(x, y))$ is in the class of admissible functions. The corrector functions χ_i , for $1 \leq i \leq n$, is defined as the solution of the cell problem

$$\begin{cases} -\operatorname{div}_y (A(x, y) [\nabla_y \chi_i(x, y) + e_i]) = 0 & \text{in } Y \\ y \mapsto \chi_i(x, y) & \text{is } Y\text{-periodic,} \end{cases} \quad (3.7)$$

where $\{e_1, \dots, e_n\}$ is the standard basis of \mathbb{R}^n . Let us define the corrector matrix as follows; $P(x, y)e_i = \nabla_y \chi_i(x, y)$. It has been shown that A_ε H -converges to A_0 which is given by,

$$(A_0)_{ij} = \int_Y A(x, y) (P(x, y)e_i + e_i) \cdot (P(x, y)e_j + e_j) \, dy. \quad (3.8)$$

Let $B = B(x, y) \in M(c, d, \Omega \times Y)$ belong to the admissible class of functions. Assume B is symmetric. Let us now define B^\sharp to be the matrix whose ij^{th} entry is given by,

$$(B^\sharp)_{ij} = \int_Y B(x, y) (P(x, y)e_i + e_i) \cdot (P(x, y)e_j + e_j) \, dy \quad (3.9)$$

We shall now recall a H -convergence result for weak data proved in [9] using two-scale convergence.

Theorem 3.4 ([9, Theorem 2.1]). *Let $\gamma < 1$ be a fixed real number. Let $v_\varepsilon \in H_0^1(\Omega)$ be the weak solution of*

$$\begin{cases} -\operatorname{div}(A(x, \frac{x}{\varepsilon}) \nabla v_\varepsilon) = g_\varepsilon & \text{in } \Omega \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g_\varepsilon \in L^2(\Omega)$ is such that $g_\varepsilon \rightharpoonup g$ weakly in $H^{-1}(\Omega)$ and $\varepsilon^\gamma g_\varepsilon$ is bounded in $L^2(\Omega)$. Then,

$$\left. \begin{aligned} v_\varepsilon &\rightharpoonup v_0 \text{ weakly in } H_0^1(\Omega) \\ A_\varepsilon \nabla v_\varepsilon &\rightharpoonup A_0 \nabla v_0 \text{ weakly in } (L^2(\Omega))^n. \end{aligned} \right\}$$

is satisfied, where $v_0 \in H_0^1(\Omega)$ is the unique solution of (2.3) and A_0 is as given in (3.8). \square

We now prove a version of Conjecture 1 for the periodic case.

Proposition 3.5. *If g_ε, g and v_ε satisfy the hypothesis as in Theorem 3.4, then the inequality, as given in (3.6), holds for B^\sharp as given in (3.9).*

Proof. Since, g_ε weakly converges to g in $H^{-1}(\Omega)$, we have v_ε bounded in $H_0^1(\Omega)$. Thus, by the compactness of two-scale convergence, there exists $v_0 \in H_0^1(\Omega)$ and $v_1 \in L^2[\Omega; H_{\text{per}}^1(Y)]$ such that

$$\nabla v_\varepsilon \xrightarrow{2s} \nabla v_0 + \nabla_y v_1(x, y).$$

Moreover, by Proposition 2.4, we have

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx \geq \int_{\Omega \times Y} B(x, y) [\nabla v_0 + \nabla_y v_1] \cdot [\nabla v_0 + \nabla_y v_1] dy dx. \quad (3.10)$$

It was shown in the proof of Theorem 3.4 that,

$$v_1(x, y) = \sum_{i=1}^n \chi_i(x, y) \frac{\partial v_0}{\partial x_i}(x)$$

and therefore, $\nabla_y v_1(x, y) = P(x, y) \nabla v_0$. Thus,

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx \geq \int_{\Omega \times Y} B(x, y) (P(x, y) + I) \nabla v_0 \cdot (P(x, y) + I) \nabla v_0 dy dx$$

and by using (3.9), we have

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx \geq \int_{\Omega} B^\sharp \nabla v_0 \cdot \nabla v_0 dx$$

Thus, we have shown (3.6) in the periodic case with B^\sharp as defined in (3.9). \square

To keep the proof of above proposition self-contained, to an extent, we derive in the following lemma the inequality (3.10) using some convex analysis arguments. We follow the line of argument as given in the proof of [17, Proposition 2.5].

Lemma 3.6. *Let $\nabla v_\varepsilon \xrightarrow{2s} \nabla v_0 + \nabla_y v_1(x, y)$ where $v_0 \in H_0^1(\Omega)$ and $v_1 \in L^2[\Omega; H_{\text{per}}^1(Y)]$ and let $B(x, y) \in M(c, d, \Omega \times Y)$, Y -periodic in y , be a symmetric matrix, then (3.10) is valid.*

Proof. Let $\Phi \in (\mathcal{D}[\Omega; C_{\text{per}}^\infty(Y)])^n$. Then, by Fenchel's inequality (for quadratic forms),

$$I_\varepsilon := \frac{1}{2} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx \geq \int_{\Omega} \nabla v_\varepsilon \cdot \Phi\left(x, \frac{x}{\varepsilon}\right) dx - \frac{1}{2} \int_{\Omega} B^{-1}\left(x, \frac{x}{\varepsilon}\right) \Phi\left(x, \frac{x}{\varepsilon}\right) \cdot \Phi\left(x, \frac{x}{\varepsilon}\right) dx$$

where B^{-1} denotes the inverse of B (which exists). We take \liminf on both sides of the inequality. By two-scale convergence of v_ε the first term on right hand side becomes,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla v_\varepsilon \cdot \Phi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega \times Y} [\nabla v_0 + \nabla_y v_1(x, y)] \cdot \Phi(x, y) dy dx.$$

Now, since B and Φ are in the admissible class of functions, so is $B^{-1}\Phi$. Therefore, using Theorem 2.3, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} B^{-1}\left(x, \frac{x}{\varepsilon}\right) \Phi\left(x, \frac{x}{\varepsilon}\right) \cdot \Phi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega \times Y} B^{-1}(x, y) \Phi(x, y) \cdot \Phi(x, y) dy dx.$$

Thus, we conclude

$$\liminf_{\varepsilon \rightarrow 0} I_\varepsilon \geq \int_{\Omega \times Y} [\nabla v_0 + \nabla_y v_1(x, y)] \cdot \Phi(x, y) dy dx - \frac{1}{2} \int_{\Omega \times Y} B^{-1}(x, y) \Phi(x, y) \cdot \Phi(x, y) dy dx.$$

Taking supremum over all $\Phi \in (\mathcal{D}(\Omega \times Y))^n$ on the right hand side and using Lemma 2.5, we obtain,

$$\liminf_{\varepsilon \rightarrow 0} I_\varepsilon \geq \int_{\Omega \times Y} \sup_{\xi \in \mathbb{R}^n} \left\{ [\nabla v_0 + \nabla_y v_1(x, y)] \cdot \xi - \frac{1}{2} B^{-1}(x, y) \xi \cdot \xi \right\} dy dx.$$

Thus, by (2.6), we get

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx \geq \int_{\Omega \times Y} B(x, y) [\nabla v_0 + \nabla_y v_1] \cdot [\nabla v_0 + \nabla_y v_1] dy dx.$$

Thus, we have shown (3.10). □

4 Dirichlet-type cost functional — Periodic Case

The purpose of this section is to announce the complete solution of a problem considered in [9, §3]. More precisely, we improve [9, Theorem 3.7] in its full generality with no assumptions

on the control set. We shall restrict ourselves to the non-perforated case, for simplicity. However, the results remain valid in perforated case with necessary modifications.

The matrices $A(x, y)$ and $B(x, y)$ are periodic in Y and is as given in the previous section. Also recall that B is symmetric. The corrector matrix P is as defined in the line following (3.7). Let U be a closed convex subset of $L^2(\Omega)$ and $f \in L^2(\Omega)$. Given $\theta \in U$, the cost functional J_ε is defined as,

$$J_\varepsilon(\theta) = \frac{1}{2} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla u_\varepsilon dx + \frac{\varepsilon}{2} \|\theta\|_2^2 \quad (4.1)$$

where $u_\varepsilon \in H_0^1(\Omega)$ is the unique solution of

$$\begin{cases} -\operatorname{div}(A(x, \frac{x}{\varepsilon}) \nabla u_\varepsilon) = f + \theta & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

Let θ_ε^* be the optimal controls and let u_ε^* be the state corresponding to θ_ε^* . Using arguments similar to that of [9, Lemma 3.2], we observe that θ_ε^* is bounded in $H^{-1}(\Omega)$ and there exists a $\theta^* \in H^{-1}(\Omega)$ such that, for a subsequence,

$$\theta_\varepsilon^* \rightharpoonup \theta^* \text{ weakly in } H^{-1}(\Omega)$$

and

$$\varepsilon^{1/2} \theta_\varepsilon^* \rightharpoonup 0 \text{ weakly in } L^2(\Omega). \quad (4.3)$$

Therefore, by Theorem 3.4, u_ε^* converges weakly in $H_0^1(\Omega)$ to $u^* \in H_0^1(\Omega)$ solving,

$$\begin{cases} -\operatorname{div}(A_0 \nabla u^*) = f + \theta^* & \text{in } \Omega \\ u^* = 0 & \text{on } \partial\Omega. \end{cases}$$

Let V be the weak closure of U in $H^{-1}(\Omega)$. By the convexity of U , V is also the strong closure in $H^{-1}(\Omega)$. We set

$$F_\varepsilon(\theta) = \begin{cases} J_\varepsilon(\theta) & \text{if } \theta \in U \\ +\infty & \text{if } \theta \in H^{-1}(\Omega) \setminus U; \end{cases}$$

where J_ε is as given in (4.1). Let the matrix B^\sharp be as defined in (3.9) and let $u = u(\theta) \in H_0^1(\Omega)$ be the weak solution of

$$\begin{cases} -\operatorname{div}(A_0 \nabla u) = f + \theta & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We set

$$F(\theta) = \begin{cases} \frac{1}{2} \int_{\Omega} B^\sharp \nabla u(\theta) \cdot \nabla u(\theta) dx & \text{if } \theta \in V \\ +\infty & \text{if } \theta \in H^{-1}(\Omega) \setminus V. \end{cases}$$

The functional F is coercive in $H^{-1}(\Omega)$ and thus, there is exists a unique minimizer in V . We now show that this minimizer is none other than θ^* .

Theorem 4.1. F_ε Γ -converges to F in the weak topology of $H^{-1}(\Omega)$. Furthermore, θ^* is the minimizer of F and $F_\varepsilon(\theta_\varepsilon^*) \rightarrow F(\theta^*)$.

Proof. Let θ_ε be a sequence weakly converging to θ in $H^{-1}(\Omega)$. Observe that it is enough to consider the case when $\theta \in V$ and $\theta_\varepsilon \in U$, for infinitely many ε , as the other cases correspond to the trivial situation ($\liminf F_\varepsilon(\theta_\varepsilon)$ is infinite). If $\theta \in V$ and $\theta_\varepsilon \in U$, we have,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(\theta_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} \left[\frac{1}{2} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2} \|\theta_\varepsilon\|_2^2 \right] \\ &\geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx. \end{aligned}$$

Observe that $\{\varepsilon^{\frac{1}{2}}\theta_\varepsilon\}$ is bounded in $L^2(\Omega)$ and converges weakly to 0. Using the inequality (3.6) as proved in Proposition 3.5, we have

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(\theta_\varepsilon) \geq \frac{1}{2} \int_{\Omega} B^\sharp \nabla u(\theta) \cdot \nabla u(\theta) \, dx = F(\theta).$$

It now remains to prove the lim sup-inequality. It is enough to consider the case $\theta \in V$ (the finite situation). By the density of U in V and convexity of U , there exists a sequence θ_ε strongly converging to θ in $H^{-1}(\Omega)$. Therefore, by Theorem 3.2, we have,

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(\theta_\varepsilon) = F(\theta).$$

Thus, we have shown the Γ -convergence of F_ε to F . Therefore, by Theorem 2.1, θ^* is the minimizer of F and $F_\varepsilon(\theta_\varepsilon^*) \rightarrow F(\theta^*)$. \square

It follows from Proposition 3.5 that for the weakly converging optimal controls, $\theta_\varepsilon^* \rightharpoonup \theta^*$ in $H^{-1}(\Omega)$, (3.6) holds for the corresponding optimal states u_ε^* . However, in the following proposition we obtain the equality in (3.6).

Lemma 4.2. For the optimal states u_ε^* and u^* , we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* \, dx = \int_{\Omega} B^\sharp \nabla u^* \cdot \nabla u^* \, dx$$

Proof. A consequence of Theorem 4.1 is that θ^* is a minimizer of F and

$$J_\varepsilon(\theta_\varepsilon^*) \rightarrow \frac{1}{2} \int_{\Omega} B^\sharp \nabla u^* \cdot \nabla u^* \, dx.$$

Observe that,

$$\frac{1}{2} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* \, dx \leq J_\varepsilon(\theta_\varepsilon^*).$$

Now, taking lim sup both sides we have,

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* \, dx \leq \int_{\Omega} B^\sharp \nabla u^* \cdot \nabla u^* \, dx$$

and comparing with (3.6), gives the equality in (3.6) for optimal states. \square

Remark 4.3. We now observe that one can, in fact, improve the convergence in (4.3) to strong convergence. Note that, as a consequence of Theorem 4.1 and Lemma 4.2,

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{2} \|\theta_\varepsilon^*\|_2^2 = \lim_{\varepsilon \rightarrow 0} \left(F_\varepsilon(\theta_\varepsilon^*) - \frac{1}{2} \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* dx \right) = 0.$$

□

In general, there is no corrector result available for weakly converging data. However, we now prove a corrector result for the optimal states.

Theorem 4.4. *Let the corrector matrix $P(x, y)$ be as defined in (3.7). We also assume that both A and B are in $C[\Omega; L_{\text{per}}^\infty(Y)]^{n \times n}$. Then,*

$$\nabla u_\varepsilon^* - \left[P\left(x, \frac{x}{\varepsilon}\right) + I \right] \nabla u^* \rightarrow 0 \text{ strongly in } L^2(\Omega).$$

Proof. Let

$$I_\varepsilon = \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \left[\nabla u_\varepsilon^* - \left[P\left(x, \frac{x}{\varepsilon}\right) + I \right] \nabla u^* \right] \cdot \left[\nabla u_\varepsilon^* - \left[P\left(x, \frac{x}{\varepsilon}\right) + I \right] \nabla u^* \right] dx.$$

Then (also using the symmetry of B),

$$\begin{aligned} I_\varepsilon &= \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* dx - 2 \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \left[P\left(x, \frac{x}{\varepsilon}\right) + I \right] \nabla u^* \cdot \nabla u_\varepsilon^* dx \\ &\quad + \int_{\Omega} B\left(x, \frac{x}{\varepsilon}\right) \left[P\left(x, \frac{x}{\varepsilon}\right) + I \right] \nabla u^* \cdot \left[P\left(x, \frac{x}{\varepsilon}\right) + I \right] \nabla u^* dx. \end{aligned}$$

To simplify notation, we rewrite above equation as,

$$I_\varepsilon = I_\varepsilon^1 - 2I_\varepsilon^2 + I_\varepsilon^3.$$

Using Lemma 4.2, we see that

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon^1 = \int_{\Omega} B^\# \nabla u^* \cdot \nabla u^* dx.$$

Since $A \in C[\Omega; L_{\text{per}}^\infty(Y)]^{n \times n}$ by the continuous dependence on data for elliptic equation, we have $P \in C[\Omega; L_{\text{per}}^2(Y)]^{n \times n}$ and hence BP in the class of admissible functions and thus, strongly two-scale converges. By Theorem 2.3

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_\varepsilon^2 &= \int_{\Omega \times Y} B(x, y) (P(x, y) + I) \nabla u^* \cdot (P(x, y) + I) \nabla u^* dy dx \\ &= \int_{\Omega} B^\# \nabla u^* \cdot \nabla u^* dx \end{aligned}$$

Similarly, we compute the last term,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_\varepsilon^3 &= \int_{\Omega \times Y} B(x, y)(P(x, y) + I)\nabla u^* \cdot (P(x, y) + I)\nabla u^* dy dx \\ &= \int_{\Omega} B^\sharp \nabla u^* \cdot \nabla u^* dx \end{aligned}$$

Therefore, by the coercivity of B ,

$$c \left\| \nabla u_\varepsilon^* - \left[P \left(x, \frac{x}{\varepsilon} \right) + I \right] \nabla u^* \right\|_2^2 \leq I_\varepsilon.$$

Now taking lim sup both sides, we have our desired result. \square

Recall from Remark 3.3 that the energy convergence (3.1) is not always true for weakly converging data. Besides Lemma 4.2, we have the following corollary.

Corollary 4.5. *For the optimal states u_ε^* and u^* , we have*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* dx = \int_{\Omega} A_0 \nabla u^* \cdot \nabla u^* dx$$

Proof. The sequence of corrector matrices P_ε introduced in §2.1 can be taken to be $P_\varepsilon = P \left(x, \frac{x}{\varepsilon} \right) + I$. Therefore, by Theorem 4.4, we have

$$\begin{aligned} \int_{\Omega} A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* dx &= \int_{\Omega} A_\varepsilon P_\varepsilon \nabla u^* \cdot P_\varepsilon \nabla u^* dx + o(1) \\ &= \int_{\Omega} P_\varepsilon^t A_\varepsilon P_\varepsilon \nabla u^* \cdot \nabla u^* dx + o(1) \\ &= \int_{\Omega} A_0 \nabla u^* \cdot \nabla u^* dx + o(1) \end{aligned}$$

where the last equality is due to the properties of corrector matrices. \square

Remark 4.6. More generally, given A_ε H -converges to A_0 , $g_\varepsilon \in H^{-1}(\Omega)$, let us assume that v_ε , the solution of (3.4), satisfies the convergences in (2.2) and (2.4). Then, the energy convergence

$$\int_{\Omega} A_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} A_0 \nabla v_0 \cdot \nabla v_0 dx$$

is, in fact, equivalent to the convergence

$$\nabla v_\varepsilon - P_\varepsilon \nabla v_0 \rightarrow 0 \text{ strongly in } L^2(\Omega)$$

under certain hypotheses, for example, A_ε symmetric.

5 Measure data

A *Borel measure* on Ω is a countably additive set function defined on the Borel subsets of Ω with values in $[-\infty, +\infty]$. The total variation of a measure λ is denoted by $|\lambda|$. Let $\mathcal{B}(\Omega)$ denote the set of all Borel measures λ with finite variation, i.e., $|\lambda|(\Omega) < +\infty$. We say a subset \mathcal{E} in $\mathcal{B}(\Omega)$ is bounded if we have $\sup_{\lambda \in \mathcal{E}} |\lambda|(\Omega) < +\infty$.

Given $A \in M(\alpha, \beta, \Omega)$, we make precise the notion of solution when $\lambda \in \mathcal{B}(\Omega)$ for the second order linear elliptic equation

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = \lambda & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.1)$$

Assume $p > n$, by the classical Sobolev Imbedding $W_0^{1,p}(\Omega) \subset C_0(\Omega)$, we have $\mathcal{B}(\Omega) \subset W^{-1,q}(\Omega)$, where $q = \frac{p}{p-1}$ is the conjugate exponent of p . In this case, u is defined to be the usual variational solution for elliptic equations. For instance, when $p = 2$ and $n < 2$, $\lambda \in H^{-1}(\Omega)$ and we have the solution $u \in H_0^1(\Omega)$ given as

$$\int_{\Omega} A(x)\nabla u \cdot \nabla w \, dx = \langle \lambda, w \rangle, \quad \forall w \in H_0^1(\Omega).$$

However, for $n > 2$, λ is not necessarily in $H^{-1}(\Omega)$ and the solution to (5.1) cannot be considered in the above sense. For this situation G. Stampacchia introduced a notion of solution in [25, Definition 9.1] using duality. We shall set $n \geq 3$ (to avoid the trivial case) for this section and the next section. Let $1^* = \frac{n}{n-1}$ denote the Sobolev conjugate of the exponent 1.

Definition 5.1. *Given $\lambda \in \mathcal{B}(\Omega)$, we say $u \in L^1(\Omega)$ is a Stampacchia solution of*

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = \lambda & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (5.2)$$

whenever

$$\int_{\Omega} u g \, dx = \int_{\Omega} v \, d\lambda, \quad \forall g \in L^\infty(\Omega), \quad (5.3)$$

and v solves

$$\begin{cases} -\operatorname{div}(A^t(x)\nabla v) = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.4)$$

□

The existence of v is well known, by Lax-Milgram theorem, and

$$\|v\|_{H_0^1(\Omega)} \leq C_0 \|g\|_2$$

where the constant C_0 depends on n, α and Ω . Henceforth C_0 will denote a generic constant depending on n, α and Ω . Assuming sufficient regularity of $\partial\Omega$, by a classical regularity

result (cf. [26]), given $s > n$ there exists ν and a constant C_0 such that v satisfies the Hölder estimate,

$$\|v\|_{C^{0,\nu}(\Omega)} \leq C_0 \|g\|_s. \quad (5.5)$$

The existence, uniqueness and regularity of Stampacchia solution u was shown in [25, Theorem 9.1]. One has the following regularity for u ,

$$\|u\|_{W_0^{1,q}(\Omega)} \leq C_0 |\lambda|, \quad \forall q \in [1, 1^*) \quad (5.6)$$

Observe that the variational solution corresponding to a data $h \in L^2(\Omega)$ is a Stampacchia solution corresponding to the measure $\lambda = h \, dx$, induced by h .

We now show the asymptotic behaviour of the Stampacchia solution under weak-* convergence in $\mathcal{B}(\Omega)$ of the data. The asymptotic behaviour of Stampacchia solution was observed in [27] for a fixed $\lambda \in \mathcal{B}(\Omega)$. The argument, however, quite simply extends to measures converging weak-* in $\mathcal{B}(\Omega)$, which we give below for completeness sake. Let $\varepsilon > 0$ be a given parameter which tends to zero and let $A_\varepsilon \in M(\alpha, \beta, \Omega)$ be a family of matrices.

Theorem 5.2 (Asymptotic Behaviour). *Let A_ε G -converge to A_0 and let λ_ε weak-* converge to λ in $\mathcal{B}(\Omega)$. If $u_\varepsilon \in L^1(\Omega)$ is the Stampacchia solution of*

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = \lambda_\varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.7)$$

then $u_\varepsilon \rightharpoonup u_0$ weakly in $W_0^{1,q}(\Omega)$, for all $q \in [1, 1^*)$ where u_0 is the Stampacchia solution of

$$\begin{cases} -\operatorname{div}(A_0(x) \nabla u_0) = \lambda & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.8)$$

Proof. We have λ_ε bounded in $\mathcal{B}(\Omega)$, since λ_ε converges weak-* in $\mathcal{B}(\Omega)$. Thus, by (5.6), there is a constant C_0 (depending on n, α and Ω , and independent of $\{A_\varepsilon\}$) such that,

$$\|u_\varepsilon\|_{W_0^{1,q}(\Omega)} \leq C_0, \quad \forall q \in [1, 1^*].$$

Consequently, there exists a subsequence (still denoted by ε), such that $u_\varepsilon \rightharpoonup u_0$ weakly in $W_0^{1,q}(\Omega)$ for $q \in [1, 1^*)$. For each ε , we have by the definition of Stampacchia solution,

$$\int_\Omega u_\varepsilon g \, dx = \int_\Omega v_\varepsilon \, d\lambda_\varepsilon, \quad \forall g \in L^\infty(\Omega), \quad (5.9)$$

where v_ε solves

$$\begin{cases} -\operatorname{div}(A_\varepsilon^t(x) \nabla v_\varepsilon) = g & \text{in } \Omega \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.10)$$

By the equi-continuity of the sequence v_ε guaranteed by the uniform Hölder estimate (5.5), v_ε converges uniformly in Ω to some v_0 (cf. [27, section 1]) and by the theory of G -convergence v_0 is the unique solution of

$$\begin{cases} -\operatorname{div}(A_0^t(x) \nabla v_0) = g & \text{in } \Omega \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.11)$$

One can pass to the limit in the right side of (5.9),

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v_{\varepsilon} d\lambda_{\varepsilon} = \int_{\Omega} v_0 d\lambda.$$

Thus,

$$\int_{\Omega} u_0 g dx = \int_{\Omega} v_0 d\lambda, \quad \forall g \in L^{\infty}(\Omega).$$

Therefore, by Definition 5.1, u_0 is the Stampacchia solution of (5.8). \square

6 Optimal Measures

Let us fix a $r \in [1, \frac{n}{n-2})$. The motivation for the choice of range for r is due to the fact that there exists a $q \in [1, 1^*)$ such that $W_0^{1,q}(\Omega)$ is compactly imbedded in $L^r(\Omega)$, for $r \in [1, \frac{n}{n-2})$. For any given constant $k > 0$, let U be the set of all $L^2(\Omega)$ functions such that their L^1 -norms are bounded, i.e.,

$$U = \{\theta \in L^2(\Omega) \mid \|\theta\|_1 \leq k\}.$$

Observe that U is closed and convex in $L^2(\Omega)$. Given $\theta \in U$ and $\{A_{\varepsilon}\} \subset M(\alpha, \beta, \Omega)$, we are interested in the asymptotic behaviour of the cost functional J_{ε} defined as,

$$J_{\varepsilon}(\theta) = \|u_{\varepsilon}(\theta)\|_r^r + \varepsilon \|\theta\|_2^2, \quad (6.1)$$

where $u_{\varepsilon} \in H_0^1(\Omega)$ is the unique solution of

$$\begin{cases} -\operatorname{div}(A_{\varepsilon} \nabla u_{\varepsilon}) &= f + \theta & \text{in } \Omega \\ u_{\varepsilon} &= 0 & \text{on } \partial\Omega. \end{cases} \quad (6.2)$$

For a fixed ε , $u_{\varepsilon} \in L^r(\Omega)$ and $\|u_{\varepsilon}(\theta)\|_r^r$, for all $r \in [1, \frac{n}{n-2})$, is continuous as function of θ for the weak topology in $L^2(\Omega)$. Thus, J_{ε} is weakly lower semicontinuous and is strictly convex. Therefore, there exists a unique $\theta_{\varepsilon}^* \in U$ which minimizes J_{ε} in U . Given any $\theta \in L^2(\Omega)$, we associate with it a measure μ , defined as follows:

$$\mu(\omega) = \int_{\omega} \theta dx, \quad \forall \text{ Borel set } \omega \text{ in } \Omega \quad (6.3)$$

and the state $u_{\varepsilon} \in H_0^1(\Omega)$ is the Stampacchia solution of

$$\begin{cases} -\operatorname{div}(A_{\varepsilon} \nabla u_{\varepsilon}) &= f + \mu & \text{in } \Omega \\ u_{\varepsilon} &= 0 & \text{on } \partial\Omega. \end{cases} \quad (6.4)$$

Let μ_{ε}^* denote the measure corresponding to θ_{ε}^* . Since U is bounded in $L^1(\Omega)$, by Banach-Alaoglu theorem, there exists a measure $\mu^* \in \mathcal{B}(\Omega)$ such that, for a subsequence, μ_{ε}^* weak-* converges to μ^* in $\mathcal{B}(\Omega)$. Let V be the weak-* closure of U in $\mathcal{B}(\Omega)$. For any $\lambda \in V$, we define the functional $J : V \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, as

$$J(\lambda) = \|u_0\|_r^r, \quad (6.5)$$

where the state u_0 is the Stampacchia solution of

$$\begin{cases} -\operatorname{div}(A_0 \nabla u_0) = f + \lambda & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.6)$$

We now extend our functionals J_ε and J to the space of Borel measures with finite variations, $\mathcal{B}(\Omega)$, in the following way:

$$F_\varepsilon(\mu) = \begin{cases} J_\varepsilon(\theta) & \text{if } \mu = \theta \, dx, \theta \in U \\ +\infty & \text{otherwise} \end{cases}$$

and

$$F(\lambda) = \begin{cases} J(\lambda) & \text{if } \lambda \in V \\ +\infty & \text{if } \lambda \in \mathcal{B}(\Omega) \setminus V. \end{cases}$$

Theorem 6.1. F_ε Γ -converges to F in the weak- $*$ topology of $\mathcal{B}(\Omega)$. Furthermore, μ^* is the minimizer of F and $F_\varepsilon(\theta_\varepsilon^*) \rightarrow F(\mu^*)$.

Proof. Let μ_ε be a sequence weakly- $*$ converging to μ in $\mathcal{B}(\Omega)$. It is enough to consider the case when $\mu \in V$ and $\mu_\varepsilon \in U$, for infinitely many ε , as the other cases correspond to the trivial situation ($\liminf F_\varepsilon(\mu_\varepsilon)$ is infinite). Let θ_ε be the function associated with μ_ε as given in (6.3). We have,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(\mu_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} [\|u_\varepsilon\|_r^r + \varepsilon \|\theta_\varepsilon\|_2^2] \\ &\geq \liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_r^r. \end{aligned}$$

For every $q \in [1, 1^*)$, $\{u_\varepsilon\}$ is bounded in $W_0^{1,q}(\Omega)$. Therefore, one can always choose a $s \in [1, 1^*)$ such that $r < s^*$. Thus, by Sobolev Imbedding, $W_0^{1,s}(\Omega)$ is compactly imbedded in $L^r(\Omega)$. Hence, there exists a u_0 such that, for a subsequence, u_ε strongly converges to u_0 in $L^r(\Omega)$. Therefore, we have,

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(\mu_\varepsilon) \geq \|u_0\|_r^r.$$

Also, by Theorem 5.2, we have that u_0 is a Stampacchia solution of (6.6) corresponding to μ . Hence,

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(\mu_\varepsilon) \geq F(\mu).$$

It now only remains to prove the limsup-inequality. To this aim, we first prove the limsup-inequality in U and extend the result to all of V using the density of U in V .

Let $F^+ := \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon$ in the weak- $*$ topology of $\mathcal{B}(\Omega)$. Given $\theta \in U$, we choose the constant sequence $\theta_\varepsilon = \theta$ in U . Thus,

$$\begin{aligned} F^+(\theta) &\leq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(\theta_\varepsilon) = \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(\theta) = \limsup_{\varepsilon \rightarrow 0} [\|u_\varepsilon\|_r^r + \varepsilon \|\theta\|_2^2] \\ &= \|u_0\|_r^r. \end{aligned}$$

Therefore,

$$F^+(\theta) \leq F(\theta), \quad \forall \theta \in U. \quad (6.7)$$

For any $\lambda \in V \setminus U$, there exists a sequence of $\{\lambda_m\} \subset U$ such that λ_m weak- * converges to λ in $\mathcal{B}(\Omega)$, as $m \rightarrow \infty$. to Consider,

$$\begin{aligned} F^+(\lambda) &\leq \liminf_{m \rightarrow \infty} F^+(\lambda_m) && \text{(by the l.s.c of } F^+) \\ &\leq \lim_{m \rightarrow \infty} F(\lambda_m) && \text{(by (6.7))} \\ &\leq F(\lambda) && \text{(by the continuity of } F) \end{aligned}$$

Thus, we have (6.7) for all $\lambda \in V$ and hence, the lim sup-inequality is proved. Therefore, F_ε Γ -converges to F . Moreover, by Theorem 2.1, μ^* is the minimizer of F and $F_\varepsilon(\theta_\varepsilon^*) \rightarrow F(\mu^*)$. \square

Remark 6.2. A consequence of Theorem 6.1 is that,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \|\theta_\varepsilon^*\|_2^2 = \lim_{\varepsilon \rightarrow 0} (F_\varepsilon(\theta_\varepsilon^*) - \|u_\varepsilon^*\|_r) = 0.$$

Thus, $\varepsilon^{\frac{1}{2}} \theta_\varepsilon^* \rightarrow 0$ strongly in $L^2(\Omega)$. \square

Conclusion

A question of interest is the regularity of the optimal controls θ^* and μ^* . To be precise, it would be of interest to obtain conditions on U or on the coefficients under which θ^* or μ^* are in $L^2(\Omega)$.

The asymptotic behaviour of the optimal control problems given by (4.1)–(4.2) has been proved for an arbitrary control set in $L^2(\Omega)$, for the periodic case. A question of further interest is whether these results can be generalized to the non-periodic case too.

Acknowledgement

The authors wish to thank Guy Bouchitté for some fruitful discussions. The first author was supported by FONDECYT Project No.3090052.

References

- [1] G. Dal Maso, An Introduction to Γ -Convergence, Progress in Nonlinear Differential Equations and Their Applications Vol. 8 (Birkhäuser Boston, 1993).
- [2] S. Kesavan and M. Vanninathan, C. R. Acad. Sci. Paris, Série A 285 (1977) 441.
- [3] S. Kesavan and J. Saint Jean Paulin, SIAM J. Control Optim. 35 (1997) 1557.
- [4] J.L. Lions, Remarks on “Cheap” Control, Topics in Numerical Analysis, edited by J. J. H. Miller, pp. 203–209, Academic Press Inc., 1973, ISBN 0-12-496950-X.

- [5] J.L. Lions, *Some Methods in the Mathematical Analysis of Systems and their Control* (Gordon and Breach, Science Publishers, Inc., 1981), ISBN 0-677-60200-6.
- [6] S. Kesavan and J. Saint Jean Paulin, *Trends in Industrial and Applied Mathematics* (2002) 251.
- [7] T. Muthukumar, *Asymptotic Behaviour of some Optimal Control Problems*, PhD thesis, University of Madras, Chennai, Institute of Mathematical Sciences, Chennai, India, 2006.
- [8] S. Kesavan and T. Muthukumar, *Proc. Indian Acad. Sci. (Math. Sci.)* 118 (2008) 133.
- [9] T. Muthukumar and A.K. Nandakumaran, *Journal of Mathematical Analysis and Applications* 351 (2009) 29.
- [10] F. Murat and L. Tartar, *H-convergence*, *Topics in the Mathematical modelling of Composite materials*, edited by A. Cherkaev and R. Kohn, , *Progress in Nonlinear Differential Equations and Their Applications* Vol. 31, pp. 21–43, Birkhäuser, Boston, 1997, English translation of [12, 11].
- [11] F. Murat, *H-convergence*, *Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger*, 1978, mimeographed notes. English translation in [10].
- [12] L. Tartar, *Cours Peccot au Collège de France*, partially written in [11]. English translation in [10], 1977.
- [13] A. Braides, *Γ -convergence for beginners*, *Oxford lecture series in Mathematics and its applications* Vol. 22 (Oxford University Press, 2002).
- [14] G. Nguetseng, *SIAM J. Math. Anal.* 20 (1989) 608.
- [15] G. Allaire, *SIAM J. Math. Anal.* 23 (1992) 1482.
- [16] D. Lukkassen, G. Nguetseng and P. Wall, *Int. J. of Pure and Appl. Math.* 2 (2002) 35.
- [17] G. Bouchitté and I. Fragalá, *SIAM J. Math. Anal.* 32 (2001) 1198.
- [18] R. Tyrrell Rockafellar, *Convex Analysis*, *Princeton Mathematical Series* Vol. 28, Second ed. (Princeton University Press, 1972).
- [19] R. Tyrrell Rockafellar, *Pacific. J. Math.* 24 (1968) 525.
- [20] R. Tyrrell Rockafellar, *Pacific. J. Math.* 39 (1971) 439.
- [21] G. Bouchitté and M. Valadier, *Journal of Functional Analysis* 80 (1988) 398.
- [22] G. Bouchitté and G. Dal Maso, *Ann. del. Scuola Norm. Sup. di Pisa* 20 (1993) 483.
- [23] M. Valadier, *J. Math. Pures Appl.* 50 (1971) 265.

- [24] S. Kesavan and M. Rajesh, Proc. Indian Acad. Sci. (Math. Sci.) 112 (2002) 337.
- [25] G. Stampacchia, Annales de l'institut Fourier 15 (1965) 189.
- [26] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of second order-
Classics in Mathematics, Reprint of the 1998 edition ed. (Springer, 2001).
- [27] A. Malusa and L. Orsina, Calculus of Variations and Partial Differential Equations 27
(2006) 179.