Continuum limits of discrete systems without convexity hypotheses

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1 Introduction

The use of non-convex discrete energies to derive a continuum theory from lattice interactions by using a variational approach has been recently considered by Truskinovsky [19] and by Braides, Dal Maso and Garroni [9]. In both papers the authors consider an array of material points linked by nearest-neighbour interactions and an energy of the form

$$E_\varepsilon(u) = \sum_i \varepsilon\psi_\varepsilon\Big(\frac{u_i - u_{i-1}}{\varepsilon}\Big), \tag{1.1}$$

where ε is the distance between neighbouring points in the reference configuration, u_i denotes the position of the i-th material point ($i = 1, \dots, L/\varepsilon$) in the deformed configuration and ψ_ε is a Lennard Jones type potential characterized by being convex until a threshold T_ε and then concave. In both cases, though with some technical differences, the authors highlight that the effect of the convex part is to give rise to a bulk (elastic) energy, while the concave part contributes to a penalization of fracture. Minimizers, local minimizers and stationary points under suitable boundary conditions and body forces are characterized as having only a finite number of interactions lying in the concave region of ψ_ε which give rise to fracture in the limit description, while all the other interactions contribute to the bulk. The limit continuum energy can be explicitly described by using the terminology of Γ-convergence (which is a variational convergence that ensures the convergence of minimum problems, see [13], [8] or [10]), and the theory of free-discontinuity functionals, of the form

$$\mathcal{E}(u) = \int_{(0,L)\setminus S(u)} f(\dot u)\, dt + \sum_{t\in S(u)} g([u]), \tag{1.2}$$

defined on piecewise-Sobolev functions on the segment $(0, L)$ ($S(u)$ denotes the set of discontinuities of u and $[u]$ stands for the jump of u), and can be carried to dimension higher than one by using the framework of special functions of bounded variation (see [5]). In the case treated in [9] the limit energy densities f and g are defined as the limit of the convex part of ψ_ε and of a suitable scaling of its concave part, respectively.

In this paper we deal with the case when the energies ψ_ε do not have a concave-convex behaviour. One of the motivations of this study lies in the fact that interatomic potential may indeed show a wiggly profile, and that, even when starting from Lennard Jones potentials, superposition of interactions at different lattice spacings may produce non convex-concave potentials (see e.g. [18], [16]). Another application of our results is the determination of discrete systems giving rise to continuum theories with bistable surface energies, which may justify formation of microcracks (see [15]). Our description of the passage from a discrete to a continuum theory shows that a 'homogenization' process may take place. As a consequence, periodic microstructure may appear in minimizers, underlined by a convexification of $\psi_\varepsilon(z)$ for 'small' values of z, as well as a fragmentation of fracture, mathematically translated into a 'subadditive envelope' on $\psi_\varepsilon(z)$ for 'large' values of z. This 'regularization' process produces convex functions f_ε defined by considering the values of ψ_ε in a suitable interval $I_\varepsilon = [T_\varepsilon^-, T_\varepsilon^+]$ and subadditive functions g_ε obtained by considering the values of ψ_ε outside I_ε, whose asymptotic behaviour as $\varepsilon \to 0$ describes the limit energy densities f and g in (1.2), respectively. The precise statement of this convergence result can be found in Theorems 3.1, 3.5 and 3.7.

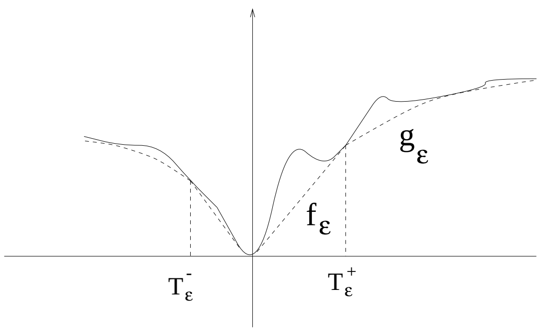

Figure 1: 'regularization of ψ_ε'.

We underline that, contrary to the case of a convex-concave behaviour, a sequence of minimizers for E_ε may not converge in $L^1(0, L)$ to minimizers of the continuum limit, but only in measure, due to the possible fragmentation of fracture. Moreover, the limit g is not described anymore by simply taking the pointwise limit of g_ε, but by a more complex limit procedure. We refer to [11] for some extensions of our results to the N-dimensional setting.

For technical reasons we divide our analysis in three different cases, according to the growth hypotheses satisfied by the potentials ψ_ε. In Section 3.2 we consider potentials with a locally superlinear growth, such as scaled Lennard Jones potentials. In this case strong compactness properties ensure that the limit is completely described by formula (1.2). In particular, we may obtain as limits Griffith type functionals. Subsequently, Section 3.2 is devoted to potentials of linear growth, in which case an additional 'relaxation' argument must be applied to the functionals in (1.2). The complete form of the Γ-limit is then obtained on the space of functions of bounded variation (Theorem 3.4), with an interaction between bulk and surface energy densities characteristic of Barenblatt type energies. Finally, we treat the case of potentials satisfying growth conditions of mixed type, obtaining a convergence result with very weak hypotheses that comprise Lennard Jones type potentials and the cases treated in [19] and [9]. It is worth explaining that in the cases considered in Sections 3.1 and 3.2 the particle interactions is not getting weaker with the separation, and may even get stronger. However, such results, besides their technical importance in the proof of those in Section 3.3, may be used to obtain finite-difference approximation of energies in fracture mechanics (see [11], [1]).

2 Notation and preliminaries

For a set A of $\mathbb{R}$ we denote $\operatorname{int} A$ the interior of A. We write $\operatorname{sgn} t$ and $[t]$ to denote the sign of t and the integer part of t, respectively, and $t \wedge s = \min\{s,t\}$. We write $\mathcal{L}_1(A)$ or $|A|$ to denote the Lebesgue measure of $A \subset \mathbb{R}$, $\#$ for the counting measure and δ_t for the Dirac mass at t. We use standard notation for Sobolev and Lebesgue spaces. If ϕ is a measurable function then $\fint_B \phi\, dx$ is its mean value on the set B. If μ is a (signed) Borel measure then μ^+ and μ^- denote its positive and negative parts, respectively, and $|\mu|$ its total variation; if B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) := \mu(A \cap B)$. The letter c will denote a strictly positive constant whose value may vary from line to line.

2.1 Functions of bounded variation

We recall that the space $BV(a,b)$ of *functions of bounded variation* on (a,b) is defined as the space of functions $u \in L^1(a,b)$ whose *distributional derivative* Du is a signed Borel measure. For each such u there exists $f \in L^1(a,b)$, a (at most countable) set $S(u) \subset (a,b)$, a sequence of real numbers $(a_t)_{t\in S(u)}$ with $\sum_t |a_t| < +\infty$ and a non-atomic measure $D_c u$ singular with respect to the Lebesgue measure such that the equality of measures $Du = f\,\mathcal{L}_1 + \sum_{t\in S(u)} a_t\delta_t + D_c u$ holds. It can be easily seen that for such functions the left hand-side and right hand-side *approximate limits* $u^-(t)$, $u^+(t)$ exist at every point, and that $S(u) = \{t \in \mathbb{R} : u^-(t) \neq u^+(t)\}$ and $a_t = u^+(t) - u^-(t) =: [u](t)$. We will write $\dot u = f$, which is an approximate gradient of u. $D_c u$ is called the *Cantor part* of Du. A sequence u_j converges weakly to u in $BV(a,b)$ if $u_j \to u$ in $L^1(a,b)$ and $\sup_j |Du_j|(a,b) < +\infty$.

The space $SBV(a,b)$ of *special functions of bounded variation* is defined as the space of functions $u \in BV(a,b)$ such that $D_c u = 0$; i.e., whose distributional derivative Du can be written as $Du = \dot{u}\,\mathcal{L}_1 + \sum_{t \in S(u)}(u^+(t) - u^-(t))\delta_t$. This notation describes a particular case of a SBV-functions space as introduced by De Giorgi and Ambrosio [14]. We will mainly deal with functionals whose natural domain is that of piecewise-$W^{1,p}$ functions, which is a particular sub-class of $SBV(a,b)$ corresponding to the conditions $\dot{u} \in L^p(a,b)$ and $\#(S(u)) < +\infty$, but we nevertheless use the more general SBV notation for future reference and for further generalization to higher dimensions (see [4]). For an introduction to BV and SBV functions we refer to the book by Ambrosio, Fusco and Pallara [5], while approximation methods for free-discontinuity problems are discussed by Braides [7].

A class of energies on $SBV(a,b)$ are those of the form (1.2) with $f, g : \mathbb{R} \to [0,+\infty]$. Lower semicontinuity conditions on $\mathcal{E}$ are equivalent to requiring that f is lower semicontinuous and convex and g is lower semicontinuous and subadditive; i.e., $g(x+y) \le g(x) + g(y)$. The latter can be interpreted as a condition penalizing fracture fragmentation, whereas convexity penalizes oscillations. If φ is not lower semicontinuous and convex (respectively, subadditive) then we may consider its *lower semicontinuous and convex* (respectively, *subadditive*) *envelope*; i.e., the greatest lower semicontinuous and convex (respectively, subadditive) function not greater than φ, that we denote by φ^{**} (respectively, $\mathrm{sub}^-\varphi$). We recall that if φ is continuous then a simple formula characterizes $\mathrm{sub}^-\varphi$:

$$\mathrm{sub}^-\varphi(z) = \inf\Big\{\sum_{j=1}^N \varphi(z_i) : N \in \mathbb{N}, \sum_{j=1}^N z_i = z\Big\}.$$

For a discussion on the role of these conditions for the lower semicontinuity of $\mathcal{E}$ we refer to [7] Section 2.2 or [8]. Energies in BV must satisfy further compatibility conditions between f and g (see e.g. Theorem 2.3 below and the subsequent remark)

2.2 Γ-convergence

We recall the definition of De Giorgi's Γ-*convergence* in a metric space space (X,d): given a family of functionals $F_n : X \to [0,+\infty]$, $n \in \mathbb{N}$, for $u \in X$ we define

$$F'(u) = \Gamma(d)\text{-}\liminf_n F_n(u) := \inf\Big\{\liminf_n F_n(u_n) \ : \ \lim_n d(u_n,u) = 0\Big\}, \qquad (2.1)$$

and

$$F''(u) = \Gamma(d)\text{-}\limsup_n F_n(u) := \inf\Big\{\limsup_n F_n(u_n) \ : \ \lim_n d(u_n,u) = 0\Big\}. \qquad (2.2)$$

Note that the functions F' and F'' are lower semicontinuous. If these two quantities coincide then their common value is called the Γ-*limit* of the sequence (F_n) at

u, and is denoted by $\Gamma\text{-}\lim_n F_n(u)$ or $\Gamma(d)\text{-}\lim_n F_n(u)$. Equivalently, $F(u) = \Gamma\text{-}\lim_n F_n(u)$ if and only if the two following conditions are satisfied:

(i) (*lower semicontinuity inequality*) for all sequences (u_n) converging to u in X we have $F(u) \leq \liminf_n F_n(u_n)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_n) converging to u in X such that $F(u) \geq \limsup_n F_n(u_n)$.

We will use as d the L^1-metric or a metric giving convergence in measure.

For a comprehensive study of Γ-convergence we refer to the book of Dal Maso [13] (see also [8] and [10] Part II). The reason for the introduction of this notion is explained by the following fundamental theorem.

Theorem 2.1 *Let $F = \Gamma\text{-}\lim_n F_n$, and let a compact set $K \subset X$ exist such that $\inf_X F_n = \inf_K F_n$ for all n. Then*

$$\exists \min_X F = \lim_n \inf_X F_n.$$

Moreover, if (u_n) is a converging sequence such that $\lim_n F_n(u_n) = \lim_n \inf_X F_n$ then its limit is a minimum point for F.

We recall a simple 1-dimensional Γ-convergence result.

Proposition 2.2 *For any $n \in \mathbb{N}$ let $h_n : \mathbb{R} \to [0, +\infty]$ be convex functions, and let $h : \mathbb{R} \to [0, +\infty]$ satisfy $h(x) \leq \liminf_n h_n(x_n)$ for all $x \in \mathbb{R}$ and $x_n \to x$ (i.e., $h \leq \Gamma\text{-}\liminf_n h_n$). Then for all $\phi_n, \phi \in L^p(a, b)$ such that $\phi_n \rightharpoonup \phi$ weakly in $L^p(a, b)$, we have*

$$\liminf_n \int_a^b h_n(\phi_n)\, dt \geq \int_a^b h(\phi)\, dt.$$

Proof. Let $(B_i)_{i \in I}$ be a Borel partition of (a, b). Then, for ϕ_n, ϕ as above, by convexity we have

$$\int_a^b h_n(\phi_n)\, dt \geq \sum_I |B_i| h_n \left(⨍_{B_i} \phi_n\, dt \right).$$

Hence,

$$\liminf_n \int_a^b h_n(\phi_n)\, dt \geq \sum_I |B_i| \liminf_n h_n \left(⨍_{B_i} \phi_n\, dt \right) = \sum_I |B_i| h \left(⨍_{B_i} \phi\, dt \right).$$

Let $w_{\mathcal{B}}$ be defined as $w_{\mathcal{B}}(x) = \sum_i \big(⨍_{B_i} u\, dy \big) \chi_{B_i}(x)$. It suffices to consider a sequence of Borel partitions $(\mathcal{B}_n)$ such that $w_{\mathcal{B}_n} \to u$ pointwise a.e. and use Fatou's Lemma. □

The following theorem is an easy corollary of [2] Theorem 6.3 and will be widely used in the next section.

Theorem 2.3 *For all $n \in \mathbb{N}$ let $f_n, g_n : \mathbb{R} \to [0, +\infty]$ be lower semicontinuous functions. Let $\alpha > 0$ exists such that*

1) f_n is convex and

$$\alpha(|z| - 1) \leq f_n(z) \qquad \text{for every } z \in \mathbb{R},$$

2) g_n is subadditive and

$$\alpha(|z| - 1) \leq g_n(z) \qquad \text{for every } z \in \mathbb{R}.$$

and suppose that $f, g : \mathbb{R} \to [0, +\infty]$ exist such that $\Gamma\text{-}\lim_n f_n = f$ on $\mathbb{R}$ and $\Gamma\text{-}\lim_n g_n = g$ on $\mathbb{R} \setminus \{0\}$. For notation's convenience we set $g(0) = 0$. Let $\mathcal{H}_n : BV(a, b) \to [0, +\infty]$ be defined as

$$\mathcal{H}_n(u) := \begin{cases} \displaystyle\int_a^b f_n(\dot{u})\,dx + \sum_{S(u)} g_n([u]) & \text{if } u \in SBV(a, b) \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}_n$ Γ-converge with respect to the weak topology of $BV(a, b)$ to the functional $\mathcal{H} : BV(a, b) \to [0, +\infty)$ defined by

$$\mathcal{H}(u) := \int_0^l \overline{f}(\dot{u})\,dx + \sigma^+ D_c u^+(a, b) + \sigma^- D_c u^-(a, b) + \sum_{S(u)} \overline{g}([u])$$

(recall that $D_c u^\pm$ denote the positive/negative part of the Cantor measure $D_c u$), where

$$\overline{f}(z) := \inf\{f(z_1) + g^0(z_2) : z = z_1 + z_2\}, \ \overline{g}(z) := \inf\{f^\infty(z_1) + g(z_2) : z = z_1 + z_2\},$$

$$f^\infty(z) = \lim_{t \to +\infty} \frac{f(tz)}{t}, \quad g^0(z) = \lim_{t \to +\infty} tg\Big(\frac{z}{t}\Big), \quad \text{and} \quad \sigma^\pm = \lim_{t \to +\infty} \frac{\overline{f}(\pm t)}{t}$$

for all $z \in \mathbb{R}$.

Remark 2.4 Note that if we take $g_n = g$ and $f_n = f$ we recover the well-known compatibility hypothesis $f^\infty = g^0$ for weakly lower semicontinuous functionals on $BV(a, b)$.

If $f(0) = 0$ then it can be easily seen that $\overline{f} = (f \wedge g^0)^{**}$ and $\overline{g} = \mathrm{sub}^-(f^\infty \wedge g)$.

3 The main results

For future reference, we state and prove the convergence results allowing for a more general dependence on the underlying lattice than in the Introduction, at the expense of a slightly more complex notation.

We begin by identifying the functions defined on a lattice with a subset of measurable functions. Consider an open interval (a,b) of $\mathbb{R}$ and two sequences (λ_n), (a_n) of positive real numbers with $a_n \in [a, a+\lambda_n)$ and $\lambda_n \to 0$. For $n \in \mathbb{N}$ let $a \leq x_n^1 < \ldots < x_n^{N_n} < b$ be the partition of (a,b) induced by the intersection of (a,b) with the set $a_n + \lambda_n\mathbb{Z}$. We define $\mathcal{A}_n(a,b)$ the set of the restrictions to (a,b) of functions constant on each $[a+k\lambda_n, a+(k+1)\lambda_n)$, $k \in \mathbb{Z}$. A function $u \in \mathcal{A}_n(a,b)$ will be identified by $N_n + 1$ real numbers $c_n^0, \ldots, c_n^{N_n}$ such that

$$u(x) = \begin{cases} c_n^i & \text{if } x \in [x_n^i, x_n^{i+1}),\ i = 1, \ldots, N_n - 1 \\ c_n^0 & \text{if } x \in (a, x_n^1) \\ c_n^{N_n} & \text{if } x \in [x_n^{N_n}, b). \end{cases} \tag{3.1}$$

For $n \in \mathbb{N}$ let $\psi_n : \mathbb{R} \to [0, +\infty]$ be a given Borel function and define $E_n : L^1(a,b) \to [0,+\infty]$ as

$$E_n(u) = \begin{cases} \displaystyle\sum_{i=1}^{N_n - 1} \lambda_n \psi_n\left(\frac{u(x_n^{i+1}) - u(x_n^i)}{\lambda_n}\right) & x \in \mathcal{A}_n(a,b) \\ +\infty & \text{otherwise in } L^1(a,b). \end{cases} \tag{3.2}$$

The following sections contain the description of the asymptotic behaviour of E_n as $n \to +\infty$.

3.1 Potentials with local superlinear growth

We first treat the case when the potentials ψ satisfy locally a growth condition of order $p > 1$. This is the case of non-convex potentials introduced by Blake and Zisserman in some models in Computer Vision [6] and of the scaled Lennard Jones potentials which justify Griffith theory as a first-order effect [19]. The main feature of the limit energy is that its surface energy density G satisfies $G(z) \geq c > 0$.

Theorem 3.1 *For all $n \in \mathbb{N}$ let $T_n^\pm \in \mathbb{R}$ exist with*

$$\lim_n T_n^\pm = \pm\infty, \qquad \lim_n \lambda_n T_n^\pm = 0, \tag{3.3}$$

and such that, if we define $F_n, G_n : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ as

$$F_n(z) \quad = \quad \begin{cases} \psi_n(z) & T_n^- \leq z \leq T_n^+ \\ +\infty & z \in \mathbb{R} \setminus [T_n^-, T_n^+] \end{cases} \tag{3.4}$$

$$G_n(z) \quad = \quad \begin{cases} \lambda_n \psi_n\left(\dfrac{z}{\lambda_n}\right) & z \in \mathbb{R} \setminus [\lambda_n T_n^-, \lambda_n T_n^+] \\ +\infty & \textit{otherwise} \end{cases} \tag{3.5}$$

the following conditions are satisfied: there exists $p > 1$ such that

$$F_n(z) \geq c(|z|^p - 1) \qquad \forall z \in \mathbb{R} \tag{3.6}$$

$$G_n(z) \geq c > 0 \qquad \forall z \neq 0 \tag{3.7}$$

and, moreover, there exist $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $G : \mathbb{R} \to [0, +\infty]$, *such that*

$$\Gamma\text{-}\lim_n F_n^{**} = F \ \ on\ \mathbb{R}, \tag{3.8}$$

$$\Gamma\text{-}\lim_n \mathrm{sub}^- G_n = G \ \ on\ \mathbb{R}. \tag{3.9}$$

Then, $(E_n)_n$ Γ*-converges to* E *with respect to the convergence in measure on* $L^1(a, b)$*, where*

$$E(u) = \begin{cases} \displaystyle\int_a^b F(\dot{u})\, dt \ + \sum_{t \in S(u)} G([u](t)) & u \in SBV(a, b) \\ +\infty & otherwise\ in\ \ L^1(a, b). \end{cases}$$

Remark 3.2 Note that hypotheses (3.8) and (3.9) are not restrictive upon passing to a subsequence by a compactness argument. This remark also holds for Theorems 3.5 and 3.7. Moreover, if F is finite everywhere then Γ-convergence in (3.8) can be replaced by pointwise convergence.

Proof. For simplicity of notation we deal with the case $T_n^+ = -T_n^- =: T_n$, the general case following by simple modifications. Without loss of generality we may assume

$$\sup_n \inf_{z \in \mathbb{R}} F_n(z) < +\infty; \tag{3.10}$$

otherwise we trivially have $F \equiv +\infty$ and consequently $E \equiv +\infty$.

With fixed $u \in L^1(a, b)$ and a sequence $(u_n) \subseteq \mathcal{A}_n(a, b)$ such that $u_n \to u$ in measure and $\sup_n E_n(u_n) < +\infty$. Up to a subsequence, we can suppose in addition that u_n converges to u pointwise a.e. We now construct for each $n \in \mathbb{N}$ a function $v_n \in SBV(a, b)$ and a free-discontinuity energy such that v_n still converges to u and we can use that energy to give a lower estimate for $E_n(u_n)$. Set

$$I_n := \left\{ i \in \{1, \ldots, N_n - 1\} : \left| \frac{u_n(x_n^{i+1}) - u_n(x_n^i)}{\lambda_n} \right| > T_n \right\} \tag{3.11}$$

and

$$v_n(x) := \begin{cases} u_n(x_n^1) & \text{if } x \in (a, x_n^1) \\ c_n^i + \dfrac{(c_n^{i+1} - c_n^i)}{\lambda_n}(x - x_n^i) & x \in [x_n^i, x_n^{i+1}),\ i \notin I_n \\ u_n(x) & x \text{ elsewhere in } (a, b). \end{cases} \tag{3.12}$$

We have that, for $\varepsilon > 0$ fixed,

$$\begin{aligned} & \{x : |v_n(x) - u_n(x)| > \varepsilon\} \\ \subseteq \quad & \{x \in [x_n^i, x_n^{i+1}),\ i \notin I_n,\ |u_n(x_n^{i+1}) - u_n(x_n^i)| > \varepsilon\} \cup (a, x_n^1). \end{aligned} \tag{3.13}$$

Since, for $i \notin I_n$ we have $|u_n(x_n^{i+1}) - u_n(x_n^i)| \leq \lambda_n T_n$, then $\{x : |v_n(x) - u_n(x)| > \varepsilon\}$ consists at most of the interval (a, x_n^1) if n is large enough. Hence, the sequence $(v_n)_n$ converges to u in measure and pointwise a.e. Moreover, by (3.7)

$$c\#I_n \leq E_n(u_n) \leq M, \tag{3.14}$$

with $M = \sup_n E_n(u_n)$. By the equiboundedness of $\#I_n$, we can suppose that $S(v_n) = \{x_n^{i+1}\}_{i \in I_n}$ tends to a finite set. For the local nature of the arguments in the following reasoning, we can also assume that S consists of only one point $x_0 \in (a, b)$.

Now, consider the sequence $(w_n)_n$ defined by

$$w_n(x) = \begin{cases} v_n(a) + \displaystyle\int_{(a,x)} \dot{v}_n(t)\, dt & \text{if } x < x_0 \\ v_n(a) + \displaystyle\int_{(a,x)} \dot{v}_n(t)\, dt + \sum_{t \in S(v_n)} [v_n](t) & \text{if } x \geq x_0. \end{cases} \tag{3.15}$$

Note that $w_n(a) = v_n(a)$, $\dot{w}_n = \dot{v}_n$, $S(w_n) = \{x_0\}$ and $[w_n](x_0) = \sum_{t \in S(v_n)} [v_n](t)$. Such a sequence still converges to u a.e. Indeed, since x_0 is the limit point of the sets $S(v_n)$, for any $\eta > 0$ fixed we can find $n_0(\eta) \in \mathbb{N}$ such that for any $n \geq n_0(\eta)$ and for any $i \in I_n$ $|x_0 - x_n^{i+1}| < \eta$. Hence, by construction, for any $n \geq n_0(\eta)$ and for any $x \in (a, b) \setminus [x_0 - \eta, x_0 + \eta]$, $w_n(x) = v_n(x)$, that is, the two sequences (v_n) and (w_n) have the same pointwise limit. Since $\dot{w}_n = \dot{v}_n$ on (a, b), by (3.6) we have that $\|\dot{w}_n\|_{L^p(a,b)} \leq M$. Then, using Poincaré's inequality on each interval, it can be easily seen that $(w_n)_n$ is equibounded in $W^{1,p}((a, b) \setminus \{x_0\})$. Since it also converges to u pointwise a.e., by using a compactness argument, we get that $u \in W^{1,p}((a, b) \setminus \{x_0\})$ and, up to subsequences,

$$\dot{w}_n \rightharpoonup \dot{u} \quad \text{weakly in } L^p(a, b).$$

Moreover, since for any two points $a < x_1 < x_0 < x_2 < b$ we have

$$w_n(x_2) = w_n(x_1) + \int_{x_1}^{x_2} \dot{w}_n\, dt + [w_n](x_0)$$

$$u(x_2) = u(x_1) + \int_{x_1}^{x_2} \dot{u}\, dt + [u](x_0),$$

taking points x_1, x_2 in which w_n converges to u and passing to the limit as $n \to +\infty$, we have

$$[w_n](x_0) \to [u](x_0). \tag{3.16}$$

We can now rewrite our functionals in terms of v_n:

$$\begin{aligned} E_n(u_n) &= \sum_{i \notin I_n} \lambda_n \psi_n(\dot{v}_n) + \sum_{i \in I_n} G_n([v_n](x_n^{i+1})) \\ &= \int_a^b F_n(\dot{v}_n)\, dt + \sum_{t \in S(v_n)} G_n([v_n](t)). \end{aligned}$$

¿From (3.14) we also have

$$
\begin{aligned}
E_n(u_n) &\geq \int_a^b F_n(\dot{v}_n)\,dt + \mathrm{sub}^- G_n\Big(\sum_{t\in S(v_n)}[v_n](t)\Big)\\
&\geq \int_a^b F_n^{**}(\dot{w}_n)\,dt + \mathrm{sub}^- G_n([w_n](x_0))
\end{aligned}
$$

as $n \to +\infty$. Passing to the liminf as $n \to +\infty$, using (3.16) and Proposition 2.2, we have

$$
\begin{aligned}
\liminf_n E_n(u_n) &\geq \liminf_n \int_a^b F_n^{**}(\dot{w}_n)\,dt + \liminf_n \mathrm{sub}^- G_n([w_n](x_0))\\
&\geq \int_a^b F(\dot{u})\,dt + G([u](x_0))
\end{aligned}
$$

as desired.

We now turn our attention to the construction of recovery sequences for the Γ-limsup. We may assume in what follows that $\inf_{z\in\mathbb{R}} F_n(z) = F_n(0)$.

Step 1 We first prove the limsup inequality for u affine on (a,b). Set $\xi = \dot{u}$; we can assume, upon a slight translation argument, that $F(\xi) = \lim_n F_n^{**}(\xi)$. Then, for each n in $\mathbb{N}$ we can find $\xi_n^1, \xi_n^2 \in \mathbb{R}$, $t_n \in [0,1]$ such that

$$
\begin{aligned}
|t_n\xi_n^1 + (1-t_n)\xi_n^2 - \xi| &\leq \frac{\sqrt{\lambda_n}}{2(b-a)}\\
t_nF_n(\xi_n^1) + (1-t_n)F_n(\xi_n^2) &\leq F_n^{**}(\xi) + o(1)\\
|\xi_n^i| &\leq c = c(\xi).
\end{aligned}
\tag{3.17}
$$

Note that in the last inequality the choice of the constant c can be chosen independent of n thanks to (3.6) and (3.10). It can be easily seen that it is not restrictive to make the following assumptions on ξ_n^i:

$$
\xi_n^1 > \xi, \quad F_n(\xi_n^1) \leq F_n(\xi_n^2), \quad (|\xi_n^1| + |\xi_n^2|)\sqrt{\lambda_n} \leq 1. \tag{3.18}
$$

We define a piecewise-affine function $v_n \in L^1(a,b)$ with the following properties:

$$
\begin{aligned}
&v_n(x) = u(x) \text{ on } (a, x_n^1],\\
&\dot{v}_{n\,|\,[x_n^i, x_n^{i+1})} := v_n^i \in \{\xi_n^1, \xi_n^2\},
\end{aligned}
$$

and v_n^i is defined recursively by

$$
\begin{cases}
v_n^1 = \xi_n^1\\
v_n^{i+1} = \begin{cases} v_n^i & \text{if } \frac{\sqrt{\lambda_n}}{2} \leq v_n(a_n) + \sum_{j=1}^{i} v_n^j\lambda_n + v_n^i\lambda_n - u(x_n^{i+1}) \leq \sqrt{\lambda_n}\\[2mm] \xi_n^1 + \xi_n^2 - v_n^i & \text{otherwise.}\end{cases}
\end{cases}
\tag{3.19}
$$

Since $0 \le v_n - u \le \sqrt{\lambda_n}$ by definition, $(v_n)_n$ converges to u uniformly, and hence in measure and, moreover,

$$\beta_n^1 := \# \left\{ i \in \{0, \dots, N_n\} : \ v_n^i = \xi_n^1 \right\} \ge t_n N_n. \tag{3.20}$$

Indeed, from (3.17), (3.18) and (3.19) we deduce

$$\begin{aligned} \lambda_n N_n \left(t_n(\xi_n^1 - \xi) + (1 - t_n)(\xi_n^2 - \xi) \right) &\le \frac{\sqrt{\lambda_n}}{2} \le v_n(x_n^{N_n}) - u(x_n^{N_n}) \\ &= \beta_n^1(\xi_n^1 - \xi)\lambda_n + (N_n - \beta_n^1)(\xi_n^2 - \xi)\lambda_n, \end{aligned}$$

so that

$$(\beta_n^1 - t_n N_n)(\xi_n^1 - \xi_n^2) \ge 0.$$

Now, consider the sequence $(u_n) \subseteq A_n(a, b)$ defined by

$$\begin{aligned} u_n(x_n^i) = v_n(x_n^i) & \qquad \text{for } i = 1, \dots, N_n, \\ u_n(a) = v_n(a) \quad \text{and} & \quad u_n(b) = v_n(b). \end{aligned}$$

Since (3.13) still holds with u_n, v_n as above, it can be easily checked that $(u_n)_n$ converges to u in measure. Hence, recalling (3.17), (3.18) and (3.20),

$$\begin{aligned} E_n(u_n) &= \lambda_n F_n(\xi_n^1)\beta_n^1 + (N_n - \beta_n^1)\lambda_n F_n(\xi_n^2) \\ &\le t_n \lambda_n N_n F_n(\xi_n^1) + (1 - t_n) N_n \lambda_n F_n(\xi_n^2) \\ &\le N_n \lambda_n \left(F_n^{**}(\xi) + o(1) \right) \le (b - a) F_n^{**}(\xi) + o(1). \end{aligned}$$

Taking the limsup as $n \to +\infty$ we get

$$\limsup_n E_n(u_n) \le F(\xi)(b - a) = E(u).$$

The same construction as above works also in the case of a piecewise-affine function: let $[a, b] = \bigcup [a_j, b_j]$ with $a_1 = a$, $b_j = a_{j+1}$ and $\dot{u}$ constant on each (a_j, b_j), then it suffices to repeat the procedure above on each (a_j, b_j) to provide functions v_n^j in $A_n(a_j, b_j)$ such that

$$\begin{aligned} & v_n^j \to u_{|\,(a_j, b_j)} \qquad \text{in measure} \\ & \limsup_n \sum_{\{i:\ x_n^i \in (a_j, b_j)\}} \lambda_n \psi_n \left(\frac{v_n^j(x_n^{i+1}) - v_n^j(x_n^i)}{\lambda_n} \right) \le \int_{a_j}^{b_j} F(\dot{u}) \, dx. \end{aligned}$$

With j fixed define $y_n^j := \max\{x_n^i \in (a_j, b_j)\}$. Then, the recovery sequence u_n is defined in (a_j, b_j) as

$$u_n(x) = v_n^j(x) - \sum_{\ell < j} (v_n^{\ell+1}(y_n^\ell + \lambda_n) - v_n^\ell(y_n^\ell)).$$

Since $u(x) + \frac{\sqrt{\lambda_n}}{2} \leq v_n^j(x) \leq u(x) + \sqrt{\lambda_n}$ by construction, and $|u(y_n^\ell + \lambda_n) - u(y_n^\ell)| \leq c\lambda_n$, we have that $u_n \to u$ in measure and

$$E_n(u_n) = \sum_j \sum_{\{i:\ x_n^i \in (a_j,b_j)\}} \lambda_n \psi_n \left(\frac{v_n^j(x_n^{i+1}) - v_n^j(x_n^i)}{\lambda_n} \right) + cF_n(0)\lambda_n.$$

By a density argument we can extend the result to functions in $W^{1,p}(a,b)$.

Step 2 Let u be of the form $z\chi_{(x_0,b)}$ with $G(z) < +\infty$ and let z_n be a recovery sequence for $G(z) = \Gamma\text{-}\lim_n \mathrm{sub}^- G_n(u)$. The sequence $\mathrm{sub}^- G_n(z_n)$ is bounded, hence, by (3.7), upon possibly considering a suitable subsequence, there exists an integer N not depending on n such that

$$\mathrm{sub}^- G_n(z_n) = \sup_\varepsilon \inf \left\{ \sum_{i=1}^N G_n(z^i) : \left| \sum_{i=1}^N z^i - z_n \right| < \varepsilon \right\}.$$

Hence, for all n we can find N points $\{z_n^1, \dots, z^N\}$ such that

$$\lim_n \sum_{i=1}^N z_n^i = z \quad \text{and} \quad \lim_n \sum_{i=1}^N G_n(z_n^i) = G(z). \tag{3.21}$$

Let $i_n \in \{1, \dots, N_n\}$ be the index such that $x_0 \in [x_n^{i_n}, x_n^{i_n+1})$ and, for n large, define w_n as in (3.1) with

$$c_n^i = \begin{cases} 0 & \text{if } i \leq i_n \\ \sum_{j \leq (i-i_n)} \left(z_n^j\right) & \text{if } i_n < i \leq i_n + N \\ \sum_{j=1}^N \left(z_n^j\right) & \text{if } i > i_n + N. \end{cases} \tag{3.22}$$

Clearly $(w_n)_n \to u$ in measure and

$$E_n(w_n) = \sum_{i=1}^N \lambda_n \psi_n \left(\frac{z_n^i}{\lambda_n} \right) = \sum_{i=1}^N G_n(z_n^i) + (b-a)F_n(0)\,;$$

the estimate follows from (3.21) by passing to the limit as $n \to +\infty$.

Step 3 Let $u \in SBV(a,b)$ be such that $E(u) < +\infty$, then

$$u = v + w \text{ with } v(x) = \int_a^x \dot{u}\, dt + c \text{ and } w(x) = \sum_{j=1}^m z_j \chi_{[x_j,b)}.$$

For $j = 1, \dots m$ let w_n^j be defined as in Step 2 with jumps in $\bigcup_j \{x_n^{i+i_{n,j}}\}_{i=1}^{N_j}$ and let v_n be a recovery sequence for v such that it is constant on each $[x_n^{i_{n,j}}, x_n^{i_{n,j}} + \lambda_n N_j)$.

The sequence $u_n = v_n + \sum_{j=1}^{m} w_n^j$ converges in measure to u and

$$\limsup_n E_n(u_n) = \limsup_n \Big(E_n(v_n) + \sum_{j+1}^{m} E_n(w_n^j) \Big) \le E(v) + E(w) = E(u),$$

as desired. $\square$

Corollary 3.3 *Let $\psi_n : \mathbb{R} \to [0, +\infty]$ satisfy the hypotheses of Theorem 3.1. Assume that, in addition, for all $n \in \mathbb{N}$, $F_n = F_n^{**}$ on $[T_n^-, T_n^+]$ and $G_n = \mathrm{sub}^- G_n$ on $\mathbb{R} \setminus [\lambda_n T_n^-, \lambda_n T_n^+]$. Then, for any $u \in L^1(a,b)$, $E_n(u)$ Γ-converges to $E(u)$ with respect to the strong topology of $L^1(a,b)$.*

Proof. It suffices to produce a recovery sequence converging strongly in $L^1(a,b)$. Note that in Step 1, by the convexity of F_n, we can choose $\xi_n^1 = \xi_n^2 = \xi_n$ in (3.17). Then $v_n = u$ and u_n turns out to be the piecewise-constant interpolation of u at points $\{x_n^i\}$. It is easy to check that $u_n \to u$ strongly in $L^1(a,b)$. It remains to show that also for functions of the form $z\chi_{[x_0,b)}$ it is possible to exhibit a sequence that converges strongly in $L^1(a,b)$. To this end it suffices to note that in Step 2, since $G_n = \mathrm{sub}^- G_n$ locally on $\mathbb{R} \setminus \{0\}$, we can find a sequence (z_n) such that (3.21) is replaced by $\lim_n z_n = z$ and $\lim_n G_n(z_n) = G(z)$. Hence, the sequence w_n defined by (3.22) converges to u strongly in $L^1(a,b)$ and it is a recovery sequence. $\square$

Remark 3.4 *Note that the hypotheses of the previous corollary are satisfied if ψ_n is convex and lower semicontinuous on $[T_n^-, T_n^+]$ and concave and lower semicontinuous on $(-\infty, T_n^-]$ and $[T_n^+, +\infty)$*

3.2 Potentials with linear growth

In this section we will consider energy potentials ψ_n such that

$$\psi_n(z) \ge \alpha(|z| - 1) \qquad \forall z \in \mathbb{R} \tag{3.23}$$

for some $\alpha > 0$. For this kind of energies we can still prove a convergence result to a free-discontinuity energy, whose volume and surface densities are obtained by a suitable interaction of the limit functions F, G of the two 'regularized' scalings of ψ_n. Note that in the following statement the sequences $T_n^\pm$ are arbitrary. The weaker growth condition (3.23) allows the limit surface energy density $G(w)$ to have a linear growth for small values of the opening w, which was not possible with the hypotheses of Theorem 3.1. However, (3.23) forces G to have a globally-linear growth.

Theorem 3.5 *Let $\psi_n : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ satisfy* (3.23)*. For all $n \in \mathbb{N}$ let $T_n^{\pm} \in \mathbb{R}$ satisfy properties* (3.3) *and let $F_n : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $G_n : \mathbb{R} \to [0, +\infty]$ be defined as in* (3.4)*. Assume that $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $G : \mathbb{R} \to [0, +\infty]$ exist such that*

$$\Gamma\text{-}\lim_n F_n^{**} = F \ \text{ on } \mathbb{R}, \tag{3.24}$$

$$\Gamma\text{-}\lim_n \operatorname{sub}^- G_n = G \ \text{ on } \mathbb{R} \setminus \{0\}. \tag{3.25}$$

For notation's convenience we set $G(0) = 0$. Then, $(E_n)_n$ Γ-converges to E with respect to the convergence in $L^1(a, b)$ and the convergence in measure, where

$$E(u) = \begin{cases} \displaystyle\int_a^b \overline{F}(\dot{u})\, dx + \sum_{S(u)} \overline{G}([u]) + c_1 Du_c^+(a, b) + c_{-1} Du_c^-(a, b) & \\ & \text{if } u \in BV(a, b) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\overline{F}(z) := \inf\{F(z_1) + G^0(z_2) : z_1 + z_2 = z\},$$
$$\overline{G}(z) := \inf\{F^\infty(z_1) + G(z_2) : z_1 + z_2 = z\},$$

$c_1 := \overline{F}^\infty(1)$ *and* $c_{-1} := \overline{F}^\infty(-1)$.

Remark 3.6 Thanks to (3.23) the theorem can be restated also with respect to the weak convergence in $BV(a, b)$. Indeed, sequences converging in measure along which the functionals E_n are equibounded are weakly compact in BV.

Proof. Again we deal with the case $T_n^+ = -T_n^- =: T_n$, the general case being achieved by slight modifications. Let $u_n, u \in L^1(a, b)$ be such that $u_n \to u$ in measure and $E_n(u_n) \leq c$. Analogously to the proof of Theorem 3.1, we will estimate $E_n(u_n)$ by a free-discontinuity energy computed on a sequence v_n converging to u weakly in $BV(a, b)$. Let I_n and v_n be defined as in (3.11) and (3.12), respectively. Note that $v_n \to u$ in measure and that v_n has equibounded total variation on (a, b). Indeed, by hypothesis (3.23) we have

$$|Dv_n|(a, b) = \sum_{i=1}^{N_n - 1} |u_n(x_n^{i+1}) - u_n(x_n^i)| \leq \frac{1}{\alpha} E_n(u_n) + c.$$

¿From this inequality we easily get that $u \in BV(a, b)$, in particular the Γ-liminf is finite only on $BV(a, b)$.

Up to passing to a subsequence we may assume that v_n converges to u weakly in $BV(a, b)$; moreover, by construction we have

$$E_n(u_n) \geq \int_a^b F_n^{**}(\dot{v}_n)\, dt + \sum_{S(v_n)} \operatorname{sub}^- G_n([v_n]).$$

Hence, it suffices to apply Theorem 2.3 to the functionals on the left hand side to get the Γ-liminf inequality.

To obtain the converse inequality it suffices to provide a sequence v_n converging to u in $L^1(a,b)$ such that

$$\limsup_n E_n(v_n) \le E_1(u) := \int_a^b F(\dot u)\,dt + \sum_{S(u)} G([u])$$

when $u \in SBV(a,b)$. The general estimate will be then obtained by relaxation (i.e. by taking $f_n = F$ and $g_n = G$ in Theorem 2.3). By a standard approximation argument it is sufficient to prove this inequality in the simpler cases of u linear and of u with a single jump. Let $u(t) = \xi t$; we may assume that $F(\xi) < +\infty$. Moreover, we may assume in what follows that $\inf_{z\in\mathbb{R}} F_n(z) = F_n(0)$. Then we can find ξ_n^1, ξ_n^2 such that the analog of (3.17) holds. In this case, $|\xi_n^1|, |\xi_n^2|$ are not necessarily equibounded; nevertheless we have by definition $|\xi_n^1|, |\xi_n^2| \le T_n$ since $F(\xi) < +\infty$. Thus we can construct the functions v_n as in the proof of the Γ-limsup-inequality of Theorem 3.1, up to a slight modification. Indeed, if we replace $\sqrt{\lambda_n}$ with $\lambda_n T_n$ in (3.17) and (3.19), all those inequalities still hold. In particular we have that $|u(x) - v_n(x)| \le \lambda_n T_n$ in (a,b). Thus v_n is a recovery sequence converging to u in $L^\infty(a,b)$.

As for the case of $u = z\chi_{(x_0,b)}$ with $G(z) < +\infty$, let $z_n = \sum_{i=1}^{M_n} z_n^i$ be such that $G(z) = \lim_n \sum_{i=1}^{M_n} G_n(z_n^i)$. Note that since $\lim_n \sum_{i=1}^{M_n} z_n^i = z$, by taking (3.23) into account, we may assume that $\sum_{i=1}^{M_n} (z_n^i)^+ \le c|z|$ and the same for the negative terms. We may assume also that $|z_n^i| \ge \lambda_n T_n$ for any i. Hence, by arguing separately on the positive and negative part of (z_n^i), we easily get that

$$M_n \le \frac{c|z|}{\lambda_n T_n}. \tag{3.26}$$

Finally we can construct a sequence of functions w_n defined as in (3.22) where we replace N with M_n. By taking (3.26) into account we easily get

$$|\{x : w_n(x) \ne u(x)\}| \le \lambda_n M_n \le \frac{c|z|}{T_n}.$$

Then $w_n \to u$ in $L^\infty(a,b)$ and, by construction,

$$\limsup_n E_n(w_n) = G(z) = E_1(u).$$

The desired upper estimate follows then by standard arguments. □

3.3 Potentials of Lennard Jones type

In this section we treat the case of potentials with non-symmetric growth conditions, which still ensure weak-BV compactness of sequences with equibounded energies. These conditions are satisfied for example by Lennard Jones potentials. The limit jump energy density G is finite only on positive jumps, where it may be trivially 0 (i.e., jumps are not penalized at all), or may have a linear growth at 0, but be globally bounded on $(0,+\infty)$; e.g.,

$$G(z)=\min\{c_1 z, c_2\},\qquad \text{or}\qquad G(z)=\frac{c_1 z}{c_2+z}$$

for $z>0$.

Theorem 3.7 *Let $\psi_n:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ satisfy*

$$\psi_n(z)\geq(|z|^p-1)\qquad \text{for all } z<0. \tag{3.27}$$

for some $p>1$. For all $n\in\mathbb{N}$ let $T_n\in\mathbb{R}$ satisfy

$$\lim_n T_n=+\infty,\qquad \lim_n \lambda_n T_n=0, \tag{3.28}$$

and let $F_n:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$, $G_n:\mathbb{R}\to[0,+\infty]$ be defined by

$$F_n(z) = \begin{cases}\psi_n(z) & z\leq T_n\\ +\infty & z>T_n\end{cases} \tag{3.29}$$

$$G_n(z) = \begin{cases}\lambda_n\psi_n\left(\dfrac{z}{\lambda_n}\right) & \text{if } z>\lambda_n T_n\\ +\infty & \text{otherwise.}\end{cases} \tag{3.30}$$

Assume that there exist $F:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$, $G:\mathbb{R}\to[0,+\infty]$ such that

$$\Gamma\text{-}\lim_n F_n^{**}=F \text{ on } \mathbb{R}, \tag{3.31}$$

$$\Gamma\text{-}\lim_n \mathrm{sub}^- G_n=G \text{ on } \mathbb{R}\setminus\{0\}. \tag{3.32}$$

For notation's convenience we set $G(0)=0$. Then, $(E_n)_n$ Γ-converges to E with respect to the convergence in $L^1_{\rm loc}(a,b)$ and the convergence in measure, where

$$E(u)=\begin{cases}\displaystyle\int_a^b \overline{F}(\dot u)\,dx+\sum_{S(u)}\overline{G}([u])+\sigma Du_c^+(a,b) & \text{if } u\in BV_{\rm loc}(a,b)\ D_c u^-=0\ ,\\ & \text{and } [u]>0 \text{ on } S(u),\\ +\infty & \text{otherwise in } L^1(a,b).\end{cases}$$

where $\overline{F}$ and $\overline{G}$ are defined as in Theorem 3.5 and $\sigma:=\overline{F}^\infty(1)$.

Proof. Let $u_n \to u$ in measure and be such that $E_n(u_n) \le c$, and assume that $u_n \to u$ also pointwise. Set

$$I_n = \{i \in \{1, \dots, N_n\} : \; u_n(x_n^{i+1}) - u_n(x_n^i) > \lambda_n T_n\},$$

and let v_n be the sequence of functions defined as in (3.12) with this choice of I_n. Note that $v_n \to u$ in measure. By taking hypothesis (3.27) into account we have the following estimate on the negative part of the (classical) derivative of v_n

$$\int_a^b |(\dot v_n)^-|^p \, dt \le \sum_{i \notin I_n} \lambda_n \Big(\frac{(u_n(x_n^{i+1}) - u_n(x_n^i))^-}{\lambda_n} \Big)^p \le \frac{E_n(u_n)}{\alpha} + c.$$

Hence, with fixed $\delta > 0$ and with fixed x_1, x_2 points in $(a, a+\delta)$, $(b-\delta, b)$, respectively, in which v_n converges pointwise to u, we get

$$\Big| \int_{a+\delta}^{b-\delta} (\dot v_n)^+ \, dt + \sum_{S(v_n)\cap(a+\delta, b-\delta)} [v_n] \Big| \le |v_n(x_2) - v_n(x_1)| + \int_a^b (\dot v_n)^- \, dt.$$

It follows that v_n is bounded in $BV_{\rm loc}(a,b)$. Since $v_n \to u$ in measure we get that v_n converges in $BV_{\rm loc}(a,b)$ to u and hence $u \in BV_{\rm loc}(a,b)$. With fixed $\eta > 0$, consider

$$F_n^\eta(z) = (F_n)^{**}(z) + \eta|z|, \qquad G_n^\eta(z) = \mathrm{sub}^- G_n(z) + \eta|z|.$$

For every $\delta \in (0, (b-a)/2)$ we have

$$E_n(u_n) + \eta c(\delta) \ge \int_{a+\delta}^{b-\delta} F_n^\eta(\dot v_n) \, dt + \sum_{S(v_n)\cap(a+\delta,b-\delta)} G_n^\eta([v_n]),$$

where $c(\delta) = \sup_n |Dv_n|(a+\delta, b-\delta)$. We can apply Theorem 2.3 and obtain for every η and δ

$$\begin{aligned} &\liminf_n E_n(u_n) + \eta c(\delta) \\ \ge\; &\int_{a+\delta}^{b-\delta} \overline{F}(\dot u) \, dt + \overline{F}^\infty(1)|D_c u^+|(a+\delta, b-\delta) + \sum_{S(u)\cap(a+\delta,b-\delta)} \overline{G}([u]). \end{aligned}$$

By letting $\eta \to 0$ and subsequently $\delta \to 0$ we obtain the desired inequality.

The construction of a recovery sequence for the Γ-limsup follows the same procedure as in the proof of Theorem 3.5. □

4 Examples

Example 4.1 (i) The typical example of a sequence of functions which satisfy the hypotheses of Theorem 3.1 (and indeed of Corollary 3.3) is given (fixed (λ_n)

converging to 0 and $C > 0$) by

$$\psi_n(z) = \frac{1}{\lambda_n}\big((\lambda_n z^2) \wedge C\big),$$

with $p = 2$, $T_n = \sqrt{C/\lambda_n}$,

$$F_n(z) = \begin{cases} z^2 & |z| \le \sqrt{C/\lambda_n} \\ +\infty & \text{otherwise,} \end{cases} \qquad G_n(z) = \begin{cases} C & |z| > \sqrt{C/\lambda_n} \\ +\infty & |z| \le \sqrt{C/\lambda_n}, \end{cases}$$

so that

$$E(u) = \int_a^b |\dot u|^2\, dt + C\#(S(u))$$

on $SBV(a,b)$. Discrete energies of this form have been proposed by Blake and Zisserman [6]. The continuum limit was first obtained by Chambolle [12].

(ii) Theorem 3.1 allows also to treat asymmetric cases. As an example, let

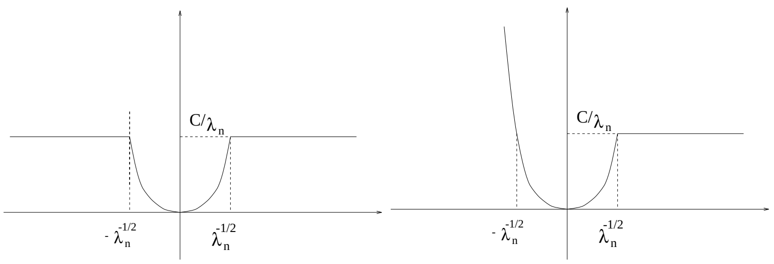

Figure 2: the potentials ψ_n in Example 4.1 (i) and (ii).

$$\psi_n(z) = \begin{cases} \frac{1}{\lambda_n}\big((\lambda_n z^2) \wedge C\big) & \text{if } z > 0 \\ z^2 & \text{if } z \le 0. \end{cases}$$

In this case the Γ-limit (with respect to both the convergence in measure and L^1 convergence) is given by

$$E(u) = \begin{cases} \displaystyle\int_a^b |\dot u|^2\, dt + C\#(S(u)) & \text{if } u \in SBV(a,b) \text{ and } u^+ > u^- \text{ on } S(u) \\ +\infty & \text{otherwise.} \end{cases}$$

Note that

$$G(z) = \begin{cases} C & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ +\infty & \text{if } z < 0 \end{cases}$$

forbids negative jumps.

(iii) (Lennard Jones type potentials) Let $\psi : \mathbb{R} \to [0, +\infty]$ be a lower semi-continuous function and satisfy $\psi(z) = 0$ if and only if $z = 0$, $\psi(z) \geq \alpha(|z|^p - 1)$ for $z < 0$ and $\lim_{z\to+\infty} \psi(z) = C$. Let $\psi_n = \psi$ for all n. Then we can apply Theorem 3.7 and obtain $F = \psi^{**}$ and

$$G(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ +\infty & \text{if } z < 0. \end{cases}$$

Note that $F(z) = 0$ if $z \geq 0$.

(iv) (scaled Lennard Jones type potentials) Let ψ be as in the previous example, and choose

$$\psi_n(z) = \frac{1}{\lambda_n}\psi(z).$$

Then we can apply Theorem 3.1 with

$$F(z) = \begin{cases} 0 & \text{if } z = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

and G as in Example (ii) above. In this case the limit energy E is finite only on piecewise-constant functions with a finite number of positive jumps. On such functions $E(u) = C\#(S(u))$.

Examples 4.1(iii) and (iv) are taken from [19], where the notation of Γ-convergence is not explicitly used. Other examples of discrete functionals related to softening phenomena can be found in [9].

We now give an example which illustrates the effect of the operation of the subadditive envelope.

Example 4.2 If we take

$$\psi_n(z) = z^2 \wedge \Big(\frac{1}{\lambda_n} + (|z|\sqrt{\lambda_n} - 1)^2\Big)$$

with λ_n converging to 0, then we obtain $F(z) = z^2$ and

$$G(z) = \mathrm{sub}^-(1 + z^2) = \min\Big\{k + \frac{z^2}{k} \ : \ k = 1, 2, \ldots\Big\}.$$

We now give an example which shows that the Γ-convergence result stated in Theorem 3.1 is sharp, in the sense that it cannot be improved to a result also with respect to the strong convergence in $L^1(a, b)$, unless by adding further hypotheses as in Corollary 3.3.

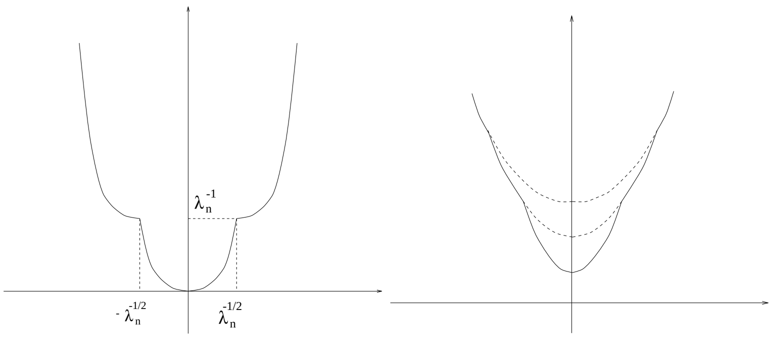

Figure 3: the functions ψ_n and G in Example 4.2.

Example 4.3 Consider a lattice of step size $\lambda_n = \frac{1}{n^2}$ and the functions $\psi_n : \mathbb{R} \to [0, +\infty)$ defined as

$$\psi_n(x) := \begin{cases} x^2 & \text{if } |x| \leq n \\ 3n^2 & \text{if } n < |x| < n + n^2 r_n \\ n^2 & \text{if } |x| \geq n + n^2 r_n \end{cases}$$

where $(r_n)_n$ is a fixed sequence of positive real numbers. The functions ψ_n satisfy the hypotheses of Proposition 3.1 with $T_n = n$, $F(x) = x^2$ and $G \equiv 2$ on $\mathbb{R} \setminus \{0\}$, independently of (r_n). Denote

$$E(u) := \begin{cases} \displaystyle\int_a^b \dot{u}^2\, dt + 2\#(S(u)) & \text{if } u \in SBV(a,b) \\ +\infty & \text{otherwise in } L^1(a,b), \end{cases}$$

$$E^1(u) := \begin{cases} \displaystyle\int_a^b \dot{u}^2\, dt + 3\#(S(u)) & \text{if } u \in SBV(a,b) \\ +\infty & \text{otherwise in } L^1(a,b), \end{cases}$$

and with $E^{'}(u)$, $E^{''}(u)$ the Γ-liminf and the Γ-limsup of E_n with respect to the strong topology of $L^1(a,b)$ defined as in (2.1) and (2.2), respectively. We will see that the sequence E_n has different Γ-limits in $L^1(a,b)$ depending on the choice of r_n.

Case 1 If $r_n = n$, then $E^{'}(u) = E^{''}(u) = E(u)$. In particular the Γ-limit in L^1 coincides with the Γ-limit in measure.

Proof. Since the strong convergence in $L^1(a,b)$ implies the convergence in measure, we easily get $E^{'}(u) \geq E(u)$. We now prove that $E(u) \geq E^{''}(u)$ for a $u \in SBV(a,b)$ with $\dot{u} \in L^\infty(a,b)$, $u(a+) = 0$ and only a positive jump z in

x_0. If i_n is the index such that $x_0 \in [x_n^{i_n}, x_n^{i_n+1})$, we define $u_n \in \mathcal{A}_n(a,b)$ as

$$u_n(x_n^i) = \begin{cases} \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt & \text{if } i < i_n \\ \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt + n + 1 + z & \text{if } i = i_n \\ \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt + z & \text{if } i > i_n. \end{cases} \tag{4.1}$$

We have

$$\begin{aligned} & \lim_n \int_a^b |u_n - u|\,dt \\ = \; & \lim_n \sum_i \int_{x_n^i}^{x_n^{i+1}} \int_{x_n^i}^{x} |\dot{u}|\,ds\,dt + (n+1)(x_n^{i_n+1} - x_0) + (n+1+z)(x_0 - x_n^{i_n}) \\ \leq \; & \lim_n c\frac{1}{n} = 0. \end{aligned}$$

Hence, $u_n \to u$ strongly in $L^1(a,b)$ and, moreover,

$$E_n(u_n) \leq \sum_{i \neq i_n} \int_{x_n^i}^{x_n^{i+1}} \dot{u}^2\,dt + G_n(n + z + c_n) + G_n(-n - c_n) \leq E(u),$$

where c_n is definitively positive. □

Case 2 If $r_n = n^2$, then $E^{'}(u) = E^{''}(u) = E^1(u)$. In particular the Γ-limits in measure and in the L^1 convergence are different.

Proof. Note that in this case the sequence u_n defined in (4.1) is a recovery sequence for $E^1(u)$. Indeed, definitively, $c_n + z < n^2$, hence $E(u) < E_n(u_n) = E^1(u) + o(1)$. Note also that the sequence v_n defined by

$$v_n(x_n^i) = \begin{cases} \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt & \text{if } i < i_n \\ \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt + z + 1 + n^2 & \text{if } i = i_n \\ \displaystyle\int_a^{x_n^{i+1}} \dot{u}\,dt + z & \text{if } i > i_n. \end{cases}$$

is such that $E_n(v_n) = E(u) + o(1)$ but it converges to u only in measure, that is, $(v_n)_n$ is a recovery sequence for the convergence in measure but not for the strong convergence in $L^1(a,b)$.

It remains only to prove the liminf inequality. With fixed a function $u \in L^1(a,b)$ and a sequence $(u_n)_n \subset \mathcal{A}_n(a,b)$ such that $u_n \to u$ strongly in $L^1(a,b)$

and $\sup_n E_n(u_n) < +\infty$. Proceeding as in the proof of Proposition 3.1 we get that $u \in SBV(a,b)$ with a finite set of jumps that we assume to be non-empty (otherwise there is nothing to prove). For any $t \in S(u)$, let $I_n(t)$ be the set of points of I_n whose limit point is t (this set is definitively non-empty). Taking notations (3.1) and (3.11) into account, we claim that there exists at least an index $i \in I_n(t)$ such that $|c_n^{i+1} - c_n^i| < n^2$. Indeed, for any $x \in [x_n^i, x_n^{i+1})$, we have

$$|c_n^{i+1} - c_n^i| \leq |c_n^{i+1} - u(x+\lambda_n)| + |u(x+\lambda_n) - u(x)| + |u(x) - c_n^i|.$$

Integrating on $[x_n^i, x_n^{i+1})$ and summing on $i \in I_n(t)$, we get

$$\frac{1}{n^2} \sum_{i \in I_n(t)} |c_n^{i+1} - c_n^i| \leq 2\int_a^b |u_n(x) - u(x)|\, dx + \int_a^{b-\lambda_n} |u(x+\lambda_n) - u(x)|\, dx.$$

If $|c_n^{i+1} - c_n^i| \geq n^2$ for each $i \in I_n(t)$, the left hand side term remains bounded from below. Indeed,

$$1 \leq \#(I_n(t)) \leq \frac{1}{n^2} \sum_{i \in I_n(t)} |c_n^{i+1} - c_n^i|.$$

Since the right term goes to 0 as n goes to $+\infty$, we get a contradiction. Hence, there exists at least one point $i(t) \in I_n(t)$ such that $n < i(t) < n + n^4$. Now, it can be easily checked that $E_n(u_n) \geq E(u)$. □

Case 3 If $r_n = n$ for n even and $r_n = n^2$ for n odd, then $E^{'}(u) = E^1(u)$, $E^{''}(u) = E(u)$. Hence, the Γ-limit with respect to the L^1-convergence may not exist.

5 A remark on second-neighbour interactions

In [11] the effect of long-range interaction in discrete systems has been investigated. In particular, [11] Theorem 5.3 can be applied to second-neighbour interactions, by considering functionals of the form

$$E_n(u) = \sum_i \lambda_n \psi_n^1\Big(\frac{u(x_n^{i+1}) - u(x_n^i)}{\lambda_n}\Big) + \sum_i 2\lambda_n \psi_n^2\Big(\frac{u(x_n^{i+2}) - u(x_n^i)}{2\lambda_n}\Big). \quad (5.1)$$

If both sequences of functions $(\psi_n^i)_n$ satisfy conditions of Corollary 3.3 and some additional growth conditions from above, then the conclusions of Theorem 3.1 hold with

$$F(z) = \lim_n \Big(\psi_n^1(z) + 2\psi_n^2(z)\Big),$$

and

$$G(z) = \lim_n \lambda_n \Big(\psi_n^1\Big(\frac{z}{\lambda_n}\Big) + 4\psi_n^2\Big(\frac{z}{2\lambda_n}\Big)\Big).$$

This means that E_n can be decomposed as the sum of three 'nearest-neighbour type' functionals, with underlying lattices $\lambda_n\mathbb{Z}, 2\lambda_n\mathbb{Z}$ and $\lambda_n(2\mathbb{Z}+1)$, respectively, whose Γ-convergence can be studied separately. We now show that a similar conclusion does not hold if we remove the convexity/concavity hypothesis on ψ_n^i.

Example 5.1 Let (λ_n) be a sequence of positive numbers converging to 0, and let $M > 2$ be fixed. Let E_n be given by (5.1) with

$$\psi_n^k(z) = \begin{cases} z^2 & \text{if } |z| \leq 1/\sqrt{k\lambda_n} \\ \frac{1}{k\lambda_n} G^k\Big(k\lambda_n z\Big) & \text{if } |z| > 1/\sqrt{k\lambda_n} \end{cases}$$

$(k = 1, 2)$, where

$$G^1(z) = \begin{cases} M & \text{if } |z| < 8 \\ 1 & \text{if } |z| \geq 8 \end{cases} \qquad G^2(z) = \begin{cases} 1 & \text{if } |z| \leq 1 \\ M & \text{if } |z| > 1. \end{cases}$$

Neither G^i is subadditive and we have

$$\mathrm{sub}^- G^1(z) = \begin{cases} 2 & \text{if } |z| < 8 \\ 1 & \text{if } |z| \geq 8 \end{cases} \qquad \mathrm{sub}^- G^2(z) = \begin{cases} 1 & \text{if } |z| \leq 1 \\ 2 & \text{if } |z| > 1. \end{cases}$$

We can view E_n as the sum of a first-neighbour interaction functional and two second-neighbour interaction functionals, to whom we can apply separately Theorem 3.1, obtaining the limit functionals

$$E^1(u) = \int_a^b |\dot{u}|^2\,dt + \sum_{S(u)} \mathrm{sub}^- G^1([u])$$

for the first, and

$$E^2(u) = \int_a^b |\dot{u}|^2\,dt + \sum_{S(u)} \mathrm{sub}^- G^2([u])$$

for each of the second ones. We will show that the Γ-limit of E_n is strictly greater than $E^1(u) + 2E^2(u)$ at some $u \in SBV(a,b)$.

Let u be given simply by $u = \chi_{(t_0,b)}$ with $t_0 \in (a,b)$. In this case $E^1(u) + 2E^2(u) = 4$. Suppose that there exist $u_n \in \mathcal{A}_n(a,b)$ converging to u and such that $\limsup_n E_n(u_n) \leq 4$. In this case it can be easily seen that for n large enough there must exist i_n such that

$$u_n(x^{i_n}) - u_n(x^{i_n-1}) > 4, \qquad u_n(x^{i_n+1}) - u_n(x^{i_n}) < -4,$$

but

$$|u_n(x^{i_n-1}) - u_n(x^{i_n-2})| < 1, \qquad |u_n(x^{i_n+2}) - u_n(x^{i_n+1})| < 1.$$

This implies that

$$u_n(x^{i_n}) - u_n(x^{i_n-2}) > 3, \qquad u_n(x^{i_n+2}) - u_n(x^{i_n}) < -3,$$

so that $\limsup_n E_n(u_n) \geq 2M$, which gives a contradiction.

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