ISOPERIMETRIC SETS ON CARNOT GROUPS

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ABSTRACT. We prove the existence of isoperimetric sets in any Carnot group, that is, sets minimizing the intrinsic perimeter among all measurable sets with prescribed Lebesgue measure. We also show that, up to a null set, these isoperimetric sets are open, bounded, their boundary is Ahlfors-regular and they satisfy the condition B. Furthermore, in the particular case of the Heisenberg group, we prove that any reduced isoperimetric set is a domain of isoperimetry. All these properties are satisfied with implicit constants that depend only on the dimension of the group and on the prescribed Lebesgue measure.

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An isoperimetric inequality in the Heisenberg group equipped with its Carnot-Carathéodory metric was first proved by P. Pansu ([Pan82]) and then, in different forms and in the general framework of Carnot groups, by several authors. More specifically, given a Carnot group $\mathbb{G}$ with homogeneous dimension Q, it is proved in [GN96] that there exists a dimensional constant $C > 0$ such that, for any set F with finite $\mathbb{G}$-perimeter $P_{\mathbb{G}}(F, \mathbb{G})$ and finite Lebesgue measure $|F|$, the following inequality holds,

$$C\,|F|^{(Q-1)/Q} \leq P_{\mathbb{G}}(F, \mathbb{G}).$$

The first result we prove here is the existence of isoperimetric sets (Theorem 2.5). This means that if we set

$$C_I = \inf(P_{\mathbb{G}}(F, \mathbb{G});\ |F| = 1),$$

then, for any given $v > 0$, there exists a set $E \subset \mathbb{G}$ such that $|E| = v$ and

$$P_{\mathbb{G}}(E, \mathbb{G}) = C_I\,|E|^{(Q-1)/Q}.$$

In other words, the set E minimizes the $\mathbb{G}$-perimeter among all sets with prescribed Lebesgue measure.

We also prove some intrinsic properties of isoperimetric sets. Namely, if E is as above, there exists a unique set E_1, which will be called a *reduced* isoperimetric set, equivalent to E, which is open, bounded, whose boundary is Ahlfors-regular of dimension $Q-1$ and which satisfies the so-called condition B (Theorem 2.11 and Theorem 4.3). Moreover, in the particular case of the Heisenberg group $\mathbb{H}^n$, we prove that E_1 is also a domain of isoperimetry, that is, a relative isoperimetric inequality on E_1 holds (Theorem 2.16). As a consequence we get that any reduced isoperimetric set in $\mathbb{H}^n$ is connected. Let us stress that our proof of the fact that reduced isoperimetric sets are domains of isoperimetry relies on fine properties

2000 *Mathematics Subject Classification.* 28A75, 22E25, 49Q20

of sets with finite perimeter in $\mathbb{H}^n$ proved in [FSSCa] and for which an extension to the case of Carnot groups of step 2 has been recently announced by the same authors. This should allow us to extend our proof to this more general case.

In the Euclidean context such results are well known and one knows even more. Up to a null set, sets minimizing the classical perimeter among all sets with prescribed Lebesgue measure are Euclidean balls. We stress that the geometry of Carnot groups is, in general, far from being Euclidean, even locally. For instance, it is well known that, except for the trivial case, the Hausdorff dimension of the group is strictly larger than the topological one. Going back to our isoperimetric problem, it has been proved in [Mon00] that, in the case of the Heisenberg group, intrinsic balls are not optimal. Then the fact that isoperimetric sets do exist gives one more difference with the Euclidean setting. On the other hand Carnot groups are equipped with intrinsic translations and dilations that still give rise to a rich underlying structure and it turns out that such groups share with Euclidean spaces some nice features. The existence of an isoperimetric inequality is one of them. One can also try to extend to this context, and more generally to the context of metric measure spaces, some aspects of Geometric Measure Theory. This has been investigated by many authors in the last few years. We will more specifically refer here to [Amb01] and [FSSCa] where a general discussion and many references on this subject can be found (see also [FSSC99]). Then, using counterparts of results that are classical in the Euclidean context, one can extend to Carnot groups some techniques used in the study of Euclidean variational problems and especially here those involving a perimeter term under a volume constraint.

The rest of this paper is organized as follows. In Section 1 we recall the definition of a Carnot group. We also recall the notion of sets with finite perimeter in this context together with some of their properties that will be used after. In Section 2 we state our main results and give a rough description of their proofs. Section 3 is devoted to the proof of the existence of isoperimetric sets. In Section 4 we prove that, up to a null set, isoperimetric sets are reduced isoperimetric sets, that is, they are open, their boundary is Ahlfors-regular and they satisfy the condition B. Finally, in Section 5 we turn our attention to the Heisenberg group $\mathbb{H}^n$ and prove that any reduced isoperimetric set in $\mathbb{H}^n$ is a domain of isoperimetry.

1. Preliminaries

In this section we recall the definition of a Carnot group, the notion of sets with finite perimeter in this context and recall for further reference some of their properties.

Let us begin with the definition of a Carnot group. Let $\mathbb{G}$ be a connected, simply connected and nilpotent Lie group whose Lie algebra $\mathcal{G}$ admits a stratification, that is,

$$\mathcal{G} = \oplus_{j=1}^{k} V_j, \quad [V_1, V_j] = V_{j+1}, \quad V_k \neq \{0\}, \quad V_j = \{0\} \text{ if } j > k,$$

where $V_1, \dots, V_k$ are linear subspaces of $\mathcal{G}$ and $[V_1, V_j]$ is the subspace of $\mathcal{G}$ generated by the elements of the form $[Y, Z]$ with $Y \in V_1$ and $Z \in V_j$. One can define a natural family δ_λ, $\lambda > 0$, of dilations on $\mathcal{G}$ that are given by

$$\delta_\lambda \left(\sum_{j=1}^{k} Y_j\right) = \sum_{j=1}^{k} \lambda^{\alpha_j} Y_j$$

whenever $Y_j \in V_j$, $j = 1, \dots, k$, and where $1 = \alpha_1 < \alpha_2 \leq \dots \leq \alpha_k$ are integers (see [FS82, Chapter 1]). Let $m = \dim V_1$ and $n = \sum_{j=1}^{k} \dim V_j$. It is well known that, identifying $\mathcal{G}$ with $\mathbb{G}$ via the exponential map, one can define global coordinates on $\mathbb{G} \simeq \mathbb{R}^n$ such that the group law can be written in the form

$$x.\, y = x + y + P(x, y) \tag{1.1}$$

for any x, $y \in \mathbb{R}^n$, where $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ has polynomial components, $P_1 = \dots = P_m = 0$, and P_j, $j > m$, depends only on x_i and y_i for $i < j$, and also such that the dilations induced on $\mathbb{G}$ by the family δ_λ, and still denoted by δ_λ, are of the form

$$\delta_\lambda (x_1, \dots, x_n) = (\lambda\, x_1, \dots, \lambda\, x_m, \lambda^{\beta_{m+1}}\, x_{m+1}, \dots, \lambda^{\beta_n}\, x_n)$$

where $\beta_i = \alpha_j$ for $j \geq 2$ and $1 + \sum_{l<j} \dim V_l \leq i \leq \sum_{l \leq j} \dim V_l$ (see [FS82, Chapter 1]). We shall systematically use this identification in the sequel.

Let $X_1, \dots, X_m$ be a family of left invariant vector fields that is an orthonormal basis of $V_1 \simeq \mathbb{R}^m$ at the origin, that is, $X_1(0) = \partial_{x_1}, \dots, X_m(0) = \partial_{x_m}$. Then, identifying as usual vector fields and associated first order differential operators, we have that X_j is of the form

$$X_j(x) = \partial_{x_j} + \sum_{i=m+1}^{n} a_{ij}(x)\, \partial_{x_i}, \tag{1.2}$$

for $j = 1, \dots, m$, where a_{ij} are polynomials such that $a_{ij}(\delta_\lambda(x)) = \lambda^{\beta_i - 1}\, x$ and $a_{ij}(x) = a_{ij}(x_1, \dots, x_{i-1})$ (see [FS82, Proposition 1.26]). From now on, we shall refer to $X_1, \dots, X_m$ as the family of canonical generating vector fields. We say that an absolutely continuous curve $\gamma : [0, T] \to \mathbb{G}$ is a sub-unit curve if there exist measurable functions $h_1, \dots, h_m : [0, T] \to \mathbb{R}$ such that $\sum_j h_j^2(s) \leq 1$ and

$$\dot{\gamma}(s) = \sum_{j=1}^{m} h_j(s) X_j(\gamma(s))$$

for a.e. $s \in [0, T]$. The family $X_1, \dots, X_m$ generates $\mathcal{G}$ as a Lie algebra, hence it satisfies the Chow-Hörmander's condition and it follows from Chow's theorem that any two points x and y in $\mathbb{G}$ can be joined by a sub-unit curve. Then the Carnot-Carathéodory distance between x and y is defined as

$$d_c(x,y) = \inf\{T > 0\,;\text{ there exists a sub-unit curve } \gamma : [0,T] \to \mathbb{G} \\ \text{such that } \gamma(0) = x,\ \gamma(T) = y\}.$$

Note that one can show that given x and y in $\mathbb{G}$ there exists a sub-unit curve joining x and y whose length is exactly $d_c(x,y)$. For more details about Carnot-Carathéodory spaces, see e.g. [Bel96] or [Gro96].

Definition 1.3. *The structure given by $(\mathbb{G}, . , \delta_\lambda, d_c)$ as defined above is said to be a Carnot group.*

Example 1.4. The space $\mathbb{R}^n$ equipped with the usual addition, the dilations $\delta_\lambda(x) = \lambda x$ for any $x \in \mathbb{R}^n$, and its Euclidean structure gives the first example of Carnot group. In this case the stratification is trivial, $V_1 = \mathbb{R}^n$, and $X_j = \partial_{x_j}$ for $j = 1, \dots, n$.

Example 1.5. The simplest non trivial example of Carnot group is given by the Heisenberg group $\mathbb{H}^n$. As a set, $\mathbb{H}^n$ coincides with $\mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$. The group law is

$$x.x' = [\zeta, t].[\zeta', t'] = [\zeta + \zeta', t + t' + 2\,\mathrm{Im} \sum_{j=1}^{n} \zeta_j \bar{\zeta}'_j],$$

where we denote points in $\mathbb{H}^n$ by $x = [\zeta, t] = [\xi + i\eta, t]$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$. The dilations are given by $\delta_\lambda([\zeta, t]) = [\lambda\zeta, \lambda^2 t]$ and the family of canonical generating vector fields is given by

$$\begin{aligned} X_j &= \partial_{\xi_j} + 2\eta_j \partial_t \qquad j = 1, \dots, n \\ X_{n+j} &= \partial_{\eta_j} - 2\xi_j \partial_t \qquad j = 1, \dots, n. \end{aligned}$$

Let us now collect for further reference a list of properties of Carnot groups. We fix for the rest of this section a Carnot group $(\mathbb{G}, . , \delta_\lambda, d_c)$ equipped with the structure defined above. We denote by

$$B(x,r) = \{y \in \mathbb{G};\ d_c(x,y) < r\},$$
$$\overline{B}(x,r) = \{y \in \mathbb{G};\ d_c(x,y) \le r\},$$

the respectively open and closed ball with center $x \in \mathbb{G}$ and radius $r > 0$. We shall write B_r and $\overline{B}_r$ when $x = 0$.

First it follows easily from the choice of the vector fields X_j and the definition of the Carnot-Carathéodory distance that d_c behaves well with respect to left translations and dilations, that is,

$$d_c(\tau_x(y), \tau_x(z)) = d_c(y, z),$$
$$d_c(\delta_\lambda(y), \delta_\lambda(z)) = \lambda\, d_c(y, z),$$

for any three points x, y and z in $\mathbb{G}$, $\lambda > 0$, and where $\tau_x : \mathbb{G} \to \mathbb{G}$ denotes the left translation,

$$\tau_x(y) = x.\, y,$$

for any x, $y \in \mathbb{G}$.

Next the n-dimensional Lebesgue measure dx on $\mathbb{G} \simeq \mathbb{R}^n$ is the Haar measure of the group, that is, if $|F|$ denotes the Lebesgue measure of any measurable set F, we have

$$|\tau_x(F)| = |F|. \tag{1.6}$$

To see this one can for instance compute explicitely the Jacobian of the corresponding change of variables using our identification and the explicit form of the group law given in (1.1). Similarly, it is also clear that

$$|\delta_\lambda(F)| = \lambda^Q\, |F|, \tag{1.7}$$

where

$$Q = \sum_{j=1}^{k} \alpha_j \dim V_j \tag{1.8}$$

is the homogeneous dimension of the group.

In particular, it follows that

$$|B(x, r)| = \omega_Q\, r^Q \tag{1.9}$$

for any $x \in \mathbb{G}$, $r > 0$ and with $\omega_Q = |B_1|$. Then it turns out that the Hausdorff dimension of $(\mathbb{G}, d_c)$ is Q. Thus, eventhough it is well known that the topology induced on $\mathbb{G}$ by d_c coincides with the Euclidean topology on $\mathbb{R}^n$, this space is in general far from being Euclidean from a geometrical point of view because $Q > n$ as soon as $\dim V_1 < n$.

Finally, we will denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measures obtained from d_c in the usual way.

We now turn to the definition of sets with finite $\mathbb{G}$-perimeter. Recall that we always identify $\mathbb{G}$ with $\mathbb{R}^n$ via the coordinates given in (1.1) and that $X_1, \dots, X_m$ denotes the family of canonical generating vector fields given by (1.2).

If Ω is an open subset of $\mathbb{G}$ and $F \subset \mathbb{G}$ is measurable, we set

$$P_{\mathbb{G}}(F, \Omega) = \sup \left(\int_{\Omega \cap F} \operatorname{div}_{\mathbb{G}} \phi\, dx\, ;\ \phi \in C_0^1(\Omega, \mathbb{R}^m),\ \|\phi\|_\infty \le 1 \right),$$

where

$$\operatorname{div}_{\mathbb{G}} \phi = \sum_{j=1}^{m} X_j \phi_j$$

for any $\phi = (\phi_1, \ldots, \phi_m) \in C_0^1(\Omega, \mathbb{R}^m)$. Here $C_0^1(\Omega, \mathbb{R}^m)$ denotes the space of $\mathbb{R}^m$-valued functions of class C^1 (in the usual Euclidean sense) with compact support.

Definition 1.10. *We say that a measurable set F has finite $\mathbb{G}$-perimeter in Ω if $P_{\mathbb{G}}(F, \Omega) < +\infty$. When $\Omega = \mathbb{G}$ we simply say that F has finite $\mathbb{G}$-perimeter.*

Equivalently it follows from Riesz' representation theorem that F has finite $\mathbb{G}$-perimeter if and only if the distributional $\mathbb{G}$-gradient $\nabla_{\mathbb{G}} \mathbf{1}_F = (X_1 \mathbf{1}_F, \ldots, X_m \mathbf{1}_F)$ of its characteristic function $\mathbf{1}_F$ can be represented by a vector-valued measure. Moreover the set function $A \mapsto P_{\mathbb{G}}(F, A)$ is the restriction to the open subsets of $\mathbb{G}$ of a finite Borel measure which coincides with the total variation $|\nabla_{\mathbb{G}} \mathbf{1}_F|$ of the measure $\nabla_{\mathbb{G}} \mathbf{1}_F$. In particular there exists a $|\nabla_{\mathbb{G}} \mathbf{1}_F|$-measurable function $\nu_F : \Omega \to \mathbb{R}^m$ such that $|\nu_F(x)| = 1$ for $|\nabla_{\mathbb{G}} \mathbf{1}_F|$-a.e. $x \in \mathbb{G}$ and the following equality between measures holds,

$$\nabla_{\mathbb{G}} \mathbf{1}_F = \nu_F \, |\nabla_{\mathbb{G}} \mathbf{1}_F|. \tag{1.11}$$

Sets with finite perimeter have already been widely studied in the Euclidean setting. Their study and more generally that of BV spaces associated to locally Lipschitz vector fields has also already been quite widely investigated (see e.g. [FSSC96], [GN96] and the references therein). Using a slightly different approach, which turns out to be equivalent to the previous one in our present setting (see [Amb01, Example 3.2]), functional properties of BV spaces in Ahlfors-regular metric measure spaces that support a weak (1,1)-Poincaré inequality have been studied in [Mir] and fine properties of sets with finite perimeter in [Amb01]. Let us also mention that in the case of the Heisenberg group a structure theorem generalizing that of De Giorgi in the Euclidean context is proved in [FSSCa] (see also [FSSCb] where some results are extended to more general Carnot groups).

Let us now state properties of sets with finite $\mathbb{G}$-perimeter that will be relevant for our purposes. The following ones can be easily recovered from the definition of the $\mathbb{G}$-perimeter (see also the relaxation procedure used in [Amb01]). Let $\Omega \subset \mathbb{G}$ be open and F and F' be two measurable subsets of $\mathbb{G}$, then we have

$$P_{\mathbb{G}}(F, \Omega) = P_{\mathbb{G}}(F^c, \Omega), \tag{1.12}$$

$$P_{\mathbb{G}}(F, \Omega) = P_{\mathbb{G}}(F', \Omega) \quad \text{whenever} \quad |(F \triangle F') \cap \Omega| = 0, \tag{1.13}$$

(1.14)

$F \mapsto P_{\mathbb{G}}(F, \Omega)$ is lower semicontinuous with respect to the $L^1_{loc}(\Omega)$-topology,

$$P_{\mathbb{G}}(F \cap F', \Omega) + P_{\mathbb{G}}(F \cup F', \Omega) \leq P_{\mathbb{G}}(F, \Omega) + P_{\mathbb{G}}(F', \Omega). \tag{1.15}$$

In (1.12) and in the following F^c denotes the complement of F and in (1.13) $F \triangle F' = (F \setminus F') \cup (F' \setminus F)$ denotes the symmetric difference between F and F'. We shall say that F and F' are equivalent if $|F \triangle F'| = 0$. If F has finite $\mathbb{G}$-perimeter we get in particular that the two measures $P_{\mathbb{G}}(F, .)$ and $P_{\mathbb{G}}(F^c, .)$ coincide and, if F' is equivalent to F, then F' has finite $\mathbb{G}$-perimeter and $P_{\mathbb{G}}(F, .) = P_{\mathbb{G}}(F', .)$ as an equality between measures. This is very standard and will be of constant use.

The following compactness result is proved in [GN96].

Theorem 1.16. *Let $(F_k)_{k\geq 0}$ be a sequence of measurable sets such that*

$$\sup_{k\geq 0} P_{\mathbb{G}}(F_k, \mathbb{G}) < +\infty.$$

Then there exists a subsequence converging in $L^1_{loc}(\mathbb{G})$ to some measurable set F with finite $\mathbb{G}$-perimeter.

Here convergence means convergence of the corresponding characteristic functions. Next recalling that the vector fields X_j are invariant under left translations, homogeneous of degree one with respect to the dilations and recalling also the behaviour of the Lebesgue measure under left translations and dilations (see (1.6) and (1.7)), one can easily see that for any measurable set F we have

$$P_{\mathbb{G}}(\tau_x(F), \mathbb{G}) = P_{\mathbb{G}}(F, \mathbb{G}) \quad \text{for any } x \in \mathbb{G}, \tag{1.17}$$

$$P_{\mathbb{G}}(\delta_\lambda(F), \mathbb{G}) = \lambda^{Q-1} P_{\mathbb{G}}(F, \mathbb{G}) \quad \text{for any } \lambda > 0. \tag{1.18}$$

The following lemma, whose proof can be found in [Amb01, Lemma 3.5], gives a canonical relation between the $\mathbb{G}$-perimeter of a set F on the boundary of a ball and the derivative of its volume inside this ball.

Lemma 1.19. *Let F be a set with finite $\mathbb{G}$-perimeter and $x \in \mathbb{G}$ be fixed. Then, for a.e. $r > 0$, we have*

$$P_{\mathbb{G}}(F \cap B(x,r), \partial B(x,r)) \leq m'(r),$$
$$P_{\mathbb{G}}(F \setminus B(x,r), \partial B(x,r)) \leq m'(r),$$

where $m(r) = |F \cap B(x,r)|$.

We now wish to end this list of general properties of sets with finite $\mathbb{G}$-perimeter with a representation formula of the $\mathbb{G}$-perimeter in terms of the Hausdorff measure $\mathcal{H}^{Q-1}$ restricted to the so-called essential boundary. Recall that Q is the homogeneous dimension of the group (see (1.8)). For any measurable set F and any $t \in [0,1]$, we define $F(t)$ as the set of points where the volume density of F equals t,

$$F(t) = \left\{ x \in \mathbb{G};\ \lim_{r\to 0} \frac{|F \cap B(x,r)|}{|B(x,r)|} = t \right\}.$$

Next we define the essential boundary $\partial^* F$ of F as the set of points where the volume density of F is neither 0 nor 1, $\partial^* F = \mathbb{G} \setminus (F(1) \cup F(0))$. The following representation formula is proved in [Amb01].

Theorem 1.20. *Let F be a set with finite $\mathbb{G}$-perimeter. Then*

$$P_{\mathbb{G}}(F, B) = \int_{B \cap \partial^* F} \theta_F \, d\mathcal{H}^{Q-1}$$

for any Borel set B and for some Borel function $\theta_F : \mathbb{G} \to [c', +\infty)$ where c' is a positive dimensional constant.

It follows from this and the definition of ν_F (see (1.11)) that if F has finite $\mathbb{G}$-perimeter then

$$|\nu_F(x)| = 1 \quad \text{for } \mathcal{H}^{Q-1}\text{-a.e. } x \in \partial^* F, \tag{1.21}$$

$$\nabla_{\mathbb{G}} \mathbf{1}_F = \nu_F \, \theta_F \, \mathbf{1}_{\partial^* F} \, d\mathcal{H}^{Q-1}, \tag{1.22}$$

where the last equality is an equality between measures.

In fact we will need this representation formula only in section 5, where we will study more specifically isoperimetric sets in the Heisenberg group $\mathbb{H}^n$ (see the example 1.5 for the definition of $\mathbb{H}^n$). As already mentioned, refined properties of sets with finite perimeter in $\mathbb{H}^n$ have been proved in [FSSCa]. Let us simply state what we will need here and which is a direct consequence of that paper.

Proposition 1.23. *Let F be a set with finite $\mathbb{H}$-perimeter. Then*

$$\mathcal{H}^{Q-1}(\partial^* F \setminus F(1/2)) = 0.$$

2. Isoperimetric sets: definition and results

In this section we define precisely the problem considered in this paper and state our main results. We fix a Carnot group $(\mathbb{G}, \,.\,, \delta_\lambda, d_c)$ and set

$$q = (Q-1)/Q, \tag{2.1}$$

where Q is the homogeneous dimension of the group (see (1.8)). First we recall the following isoperimetric inequality proved in [GN96].

Theorem 2.2 (Isoperimetric inequality)**.** *There exists a constant $C > 0$ such that if F is any set with finite $\mathbb{G}$-perimeter then either F or F^c has finite Lebesgue measure and we have*

$$C \min(|F|, |F^c|)^q \le P_{\mathbb{G}}(F, \mathbb{G}).$$

In other words, if we set

$$C_I = \inf(P_{\mathbb{G}}(F,\mathbb{G});\ |F| = 1), \tag{2.3}$$

that is, C_I is the best constant for the previous isoperimetric inequality, we have $C_I > 0$ and, thanks to the scaling properties of the Lebesgue measure and of the $\mathbb{G}$-perimeter (see (1.7) and (1.18)),

$$C_I\,|F|^q \le P_{\mathbb{G}}(F,\mathbb{G}) \tag{2.4}$$

for any set F with finite $\mathbb{G}$-perimeter such that $|F| < +\infty$.

Our first result gives the existence of isoperimetric sets, that is, sets for which the isoperimetric inequality (2.4) is actually an equality.

Theorem 2.5. *For any $v > 0$, there exists a set E with finite $\mathbb{G}$-perimeter such that $|E| = v$ and*

$$P_{\mathbb{G}}(E,\mathbb{G}) = C_I\,|E|^q. \tag{2.6}$$

This will be proved in Section 3. Any set E satisfying (2.6) will be called an isoperimetric set. Alternatively, an isoperimetric set E is characterized by the following minimality property,

$$P_{\mathbb{G}}(E,\mathbb{G}) \le P_{\mathbb{G}}(F,\mathbb{G}) \tag{2.7}$$

for any measurable set F such that $|E| \le |F| < +\infty$. Indeed if E is isoperimetric and if F is any measurable set such that $|E| \le |F| < +\infty$, we have by (2.4)

$$P_{\mathbb{G}}(E,\mathbb{G}) = C_I\,|E|^q \le C_I\,|F|^q \le P_{\mathbb{G}}(F,\mathbb{G}).$$

On the other hand if E satisfies (2.7) for all F with $|E| \le |F| < +\infty$, it automatically satisfies the same minimality condition among all sets F such that $|F| = |E|$, that is, E is isoperimetric.

In the Euclidean context it is well known that isoperimetric sets are, up to a null set, Euclidean balls. We will not be able here to give such a precise description of isoperimetric sets in $\mathbb{G}$. However we prove some intrinsic properties of them. Before stating the results we need some definitions.

Definition 2.8 (Ahlfors-regular sets)**.** *Let $S \subset \mathbb{G}$ be closed. We say that S is Ahlfors-regular (of codimension 1) if there exists a Borel measure μ supported in S and a constant $C > 1$ such that*

$$C^{-1}r^{Q-1} \le \mu(B(x,r)) \le Cr^{Q-1} \tag{2.9}$$

for all $x \in S$ and $r \le 1$.

The Ahlfors-regularity is a uniform and scale invariant way to say that the Hausdorff dimension of S is $Q-1$. Note that if S is Ahlfors-regular and μ is a measure that satisfies (2.9) then μ is equivalent to the measure $\mathcal{H}^{Q-1}$ restricted to S and the implicit constant in this equivalence depends only on Q and on the constant that shows up in (2.9). The point is that for all $x \in S$, $r > 0$ and $R > r$, one can cover $S \cap B(x, R)$ with at most $C(R/r)^{Q-1}$ balls with radius r, where C is a constant which depends only on Q and the Ahlfors-regularity constant for S.

Definition 2.10 (Condition B)**.** *Let $F \subset \mathbb{G}$ be open. We say that F satisfies the condition B if there exists a constant $C > 0$ such that for any ball B centered on ∂F with radius $r \leq 1$ there exist two balls B_1 and B_2 with radius Cr such that $B_1 \subset F \cap B$ and $B_2 \subset B \setminus \overline{F}$.*

This condition is a quantitative, uniform and scale invariant way of saying that the boundary of F separates well F from its complement. In Euclidean spaces it is known that sets satisfying the condition B and whose boundary is Ahlfors-regular have strong rectifiability properties (see e.g. [DJ90] or [DS93]). In the context of Carnot groups we do not know whether the Ahlfors-regularity and the condition B imply more regularity. However it might be useful to know that isoperimetric sets satisfy these properties.

Note that if E is isoperimetric and E' is equivalent to E then E' is still an isoperimetric set. Thus before stating properties that hold everywhere on the boundary, we will have first to clean up our isoperimetric sets.

Theorem 2.11. *Let E be an isoperimetric set. There exists a unique set E_1 equivalent to E such that E_1 is open, ∂E_1 is Ahlfors-regular and E_1 satisfies the condition B. Moreover the implicit constants in these properties can be chosen depending only on Q and $|E|$.*

This will be proved in Section 4. We also refer to Theorem 4.3 for a precise description of E_1. In other words, Theorem 2.11 tells us that, up to a null set, any isoperimetric set has an Ahlfors-regular boundary and satisfies the condition B. For simplicity, let us fix the terminology with the following definition.

Definition 2.12. *We say that a set E is a reduced isoperimetric set if E is an open isoperimetric set whose boundary is Ahlfors-regular and that satisfies the condition B.*

Let us point out that we will also have as a consequence of Theorem 4.3 that the topological boundary and the essential boundary of any reduced isoperimetric set coincide (see Proposition 4.4). Moreover any reduced isoperimetric set E is bounded and its diameter can be bounded by a constant which depends only on Q and $|E|$. This will follow either from the proof of Theorem 2.5 or from the Ahlfors-regularity of its boundary (see Proposition 4.5).

In the final section of this paper we will study more precisely isoperimetric sets in the Heisenberg group $\mathbb{H}^n$ (see the example 1.5 for the definition of $\mathbb{H}^n$). Thanks to the refined properties of sets with finite $\mathbb{H}$-perimeter recalled at the end of Section 1 we will be able to prove that any reduced isoperimetric set E in $\mathbb{H}^n$ is a domain of isoperimetry. This means that a relative isoperimetric inequality on

E holds. Before stating the result, let us recall for further reference that Carnot-Carathéodory balls in general Carnot groups are domains of isoperimetry.

Definition 2.13 (Domain of isoperimetry)**.** *We say that an open set $F \subset \mathbb{G}$ is a domain of isoperimetry if there exists a constant $C > 0$ such that for any measurable set $A \subset F$, we have*

$$C \min(|A|, |F \setminus A|)^q \leq P_{\mathbb{G}}(A, F). \tag{2.14}$$

A constant such that (2.14) holds will be called an isoperimetric constant for F. The following theorem is proved in [GN96].

Theorem 2.15. *Balls are domains of isoperimetry with an isoperimetric constant that can be chosen depending only on Q.*

Let us now return to our isoperimetric sets in $\mathbb{H}^n$. We will prove in Section 5 the following property.

Theorem 2.16. *Let $E \subset \mathbb{H}^n$ be a reduced isoperimetric set. Then E is a domain of isoperimetry with an isoperimetric constant that can be chosen depending only on Q.*

Note that any domain of isoperimetry is connected so that reduced isoperimetric sets in $\mathbb{H}^n$ are connected.

Let us now give an outline of the proof of these results. To get the existence of isoperimetric sets we take a minimizing sequence $(E_k)_{k\geq 1}$ with $|E_k| = 1$ and we want to show that in the limit we get the desired isoperimetric set. By compactness one can always assume that the minimizing sequence converges in L^1_{loc} to some set E. By lower semicontinuity of the perimeter we have $P_{\mathbb{G}}(E, \mathbb{G}) \leq C_I$ where C_I is the isoperimetric constant defined in (2.3). And E will be isoperimetric provided we show that $|E| = 1$. For an arbitrary minimizing sequence this is not true in general because some mass could disappear at infinity in the limit. To construct an appropriate minimizing sequence we use a concentration-compactness type argument. First we find for each E_k a ball of radius 1 where the amount of E_k is at least some positive constant that do not depend on k (see Lemma 3.1). Next we translate the sets E_k in such a way that they all concentrate inside the unit ball centered at 0. Then one cannot find a substantial part of E_k far away from B_1 because E_k almost minimizes the perimeter among all sets with Lebesgue measure 1. This fits the intuition that isoperimetric sets must be bounded and connected and thus cannot spread out too much as soon as they concentrate somewhere. Therefore we get that no mass disappear in the limit, that is, $|E| = 1$, hence E is isoperimetric.

To prove Theorem 2.11, we follow a standard scheme that has already been used in the Euclidean context to get the same kind of properties for hypersurfaces subject to various quasiminimality conditions (see for instance [DS98]). We will prove the Ahlfors-regularity with $\mu = P_{\mathbb{G}}(E,.)$ for any isoperimetric set E (see Definition 2.8). The upper estimate in (2.9) follows from a simple comparison argument. On the other hand, the lower one and the condition B are strongly related to the behaviour of the proportion of E and of its complement inside balls. In fact it is sufficient to prove a good decay for this proportion provided it is small enough (see Lemma 4.2). This will be done through appropriate volume-preserving modifications of E and use of the minimality condition (2.7). Then the construction of the required reduced isoperimetric set E_1 follows from standard arguments that do not use anymore the fact that E is isoperimetric.

To prove that any reduced isoperimetric set E in the Heisenberg group $\mathbb{H}^n$ is a domain of isoperimetry, we simply consider a subset A of E, apply the global isoperimetric inequality to A and $E \setminus A$ and add up these inequalities to conclude. We stress that, to make this rigourous, one need to check some technicalities, in particular Lemma 5.3. The proof of this last lemma relies strongly on fine properties of sets with finite $\mathbb{H}$-perimeter, namely Proposition 1.23, and would work in any Carnot group where the analogue of this proposition holds. As already mentioned, B. Franchi, R. Serapioni and F. Serra Cassano recently announced a generalization of this result to Carnot groups of step 2. However the question to know whether an analogue of Proposition 1.23 as stated here holds in the general case remains up to now open.

Finally, let us state a last convention. In what follows, the letter C will always denote a positive constant whose value can change at each occurrence.

3. Existence

In this section we prove Theorem 2.5 which gives the existence of isoperimetric sets in any Carnot group. We let $(\mathbb{G}, \,.\,, \delta_\lambda, d_c)$ be as before a fixed Carnot group. First of all note that it is sufficient to prove the existence of an isoperimetric set E with Lebesgue measure 1. Indeed if v is any fixed positive constant and $\lambda = v^{1/Q}$ then, thanks to (1.7) and (1.18), $|\delta_\lambda(E)| = v$ and $\delta_\lambda(E)$ is isoperimetric.

As mentioned at the end of Section 2, to get the existence of isoperimetric sets, we consider a minimizing sequence for the $\mathbb{G}$-perimeter, each element of the sequence having Lebesgue measure 1, and want to show that, up to a subsequence and up to translations, this sequence converges to the required isoperimetric set. The main issue here is to prove that the limiting set still has Lebesgue measure 1. In fact we will prove a little bit more, namely that the limiting set is also essentially bounded (see Lemma 3.8). We stress that one can show with the same arguments as those of the proof of Lemma 3.8 that any isoperimetric set is essentially bounded. This will also easily follow from Theorem 2.11 and will be proved in Proposition 4.5.

Going back to the proof of the existence we will use a concentration - compactness type argument and the following lemma gives the first step in this direction.

Indeed, for each element E_k of the minimizing sequence, it will give us a ball of radius 1 with at least a fixed amount of mass of E_k inside.

Lemma 3.1. *Let F be a set with positive and finite $\mathbb{G}$-perimeter and Lebesgue measure. Assume that $m \in (0, \omega_Q/2)$ is such that $|F \cap B(x,1)| < m$ for all $x \in \mathbb{G}$. Then we have*

$$C|F|^Q P_{\mathbb{G}}(F, \mathbb{G})^{-Q} \leq m$$

for some constant $C > 0$ that depends only on Q.

Proof. Let F and m be as in the statement. Let $\mathcal{A}$ be a maximal family of points in $\mathbb{G}$ such that $d_c(x, x') \geq 1/2$ for all $x, x' \in \mathcal{A}$ with $x \neq x'$ and $|F \cap B(x, 1/2)| > 0$ for all $x \in \mathcal{A}$. Then $\cup_{x\in\mathcal{A}} B(x, 1)$ covers almost all of F, that is,

$$|F \setminus \bigcup_{x\in\mathcal{A}} B(x,1)| = 0.$$

Otherwise there would exist a point $y \in \mathbb{G}$ such that

$$|(F \setminus \bigcup_{x\in\mathcal{A}} B(x,1)) \cap B(y,1/2)| > 0. \tag{3.2}$$

By maximality of $\mathcal{A}$ we would have $y \in B(x, 1/2)$ for some $x \in \mathcal{A}$ and then $B(y, 1/2) \subset B(x, 1)$ which contradicts (3.2). Hence we have

$$\begin{aligned} |F| &\leq \sum_{x\in\mathcal{A}} |F \cap B(x,1)| \\ &\leq m^{1/Q} \sum_{x\in\mathcal{A}} |F \cap B(x,1)|^q \\ &\leq C m^{1/Q} \sum_{x\in\mathcal{A}} P_{\mathbb{G}}(F, B(x,1)), \end{aligned} \tag{3.3}$$

where q is defined in (2.1). The last inequality follows from the relative isoperimetric inequality for intrinsic balls (see Theorem 2.15). This uses the assumption $m \leq \omega_Q/2$ which implies that $\min(|F \cap B(x,1)|, |B(x,1) \setminus F|) = |F \cap B(x,1)|$. Now the balls $B(x,1)$, $x \in \mathcal{A}$, have bounded overlap because the balls $B(x, 1/4)$, $x \in \mathcal{A}$, are pairwise disjoint. Indeed if we set for each $z \in \mathbb{G}$,

$$\mathcal{A}(z) = \{x \in \mathcal{A};\ z \in B(x,1)\},$$

we have

$$\bigcup_{x\in\mathcal{A}(z)} B(x,1/4) \subset B(z,2),$$

and using (1.9) we get

$$4^{-Q}\omega_Q \,\mathrm{card}(\mathcal{A}(z)) = |\bigcup_{x\in\mathcal{A}(z)} B(x,1/4)| \leq |B(z,2)| = 2^Q \omega_Q$$

and the claim follows. Thus we have

$$\sum_{x\in\mathcal{A}} P_{\mathbb{G}}(F, B(x,1)) \le C P_{\mathbb{G}}(F, \mathbb{G}).$$

Combining this inequality with (3.3) we get

$$|F| \le C m^{1/Q} P_{\mathbb{G}}(F, \mathbb{G}),$$

which is exactly what we want. □

Now we consider a minimizing sequence $(E_k)_{k\ge 1}$ of sets with finite $\mathbb{G}$ - perimeter such that

$$|E_k| = 1 \quad \text{and} \quad P_{\mathbb{G}}(E_k, \mathbb{G}) \le C_I \left(1 + \frac{1}{k}\right) \tag{3.4}$$

for all $k \ge 1$. Then, by Lemma 3.1, one can find an absolute constant $m_0 > 0$ that does not depend on k and, for each $k \ge 1$, a point $x_k \in \mathbb{G}$ such that $|E_k \cap B(x_k, 1)| \ge m_0$. Indeed this is satisfied for any $m_0 < C\,(2C_I)^{-Q}$ where C is given by Lemma 3.1. For technical reasons we shall also assume that $m_0 < 1/2$. Thus, considering the sequence $(\tau_{-x_k}(E_k))_{k\ge 1}$, which we still denote by $(E_k)_{k\ge 1}$, we have

$$|E_k \cap B_1| \ge m_0 \tag{3.5}$$

and (3.4) still holds for all $k \ge 1$ by left invariance of the Lebesgue measure and of the $\mathbb{G}$-perimeter (see (1.6) and (1.17)). Now, by Theorem 1.16, one can extract a subsequence, still denoted by $(E_k)_{k\ge 1}$, such that the characteristic functions $\mathbf{1}_{E_k}$ converge in $L^1_{loc}(\mathbb{G})$ to the characteristic function of some set E with finite $\mathbb{G}$-perimeter. By lower semicontinuity of the $\mathbb{G}$-perimeter (see (1.14)), we have

$$P_{\mathbb{G}}(E, \mathbb{G}) \le \liminf_{k\to+\infty} P_{\mathbb{G}}(E_k, \mathbb{G}) \le C_I, \tag{3.6}$$

and E will be isoperimetric provided we show that $|E| = 1$. Note that we have

$$m_0 \le \lim_{k\to+\infty} |E_k \cap B_1| = |E \cap B_1| \quad \text{and} \quad |E| \le \liminf_{k\to+\infty} |E_k| \le 1. \tag{3.7}$$

Lemma 3.8. *There exists a radius $R_Q > 0$, depending only on Q, such that $|E \cap B_{R_Q}| = 1$.*

Proof. Let $r \ge 2$ be fixed. Assume that $|E \cap B_r| < 1$ and let us show that $r \le R_Q$ for some positive dimensional constant R_Q. For all $\rho > 0$ and $k \ge 1$ we set $m_k(\rho) = |E_k \cap B_\rho|$ and $m(\rho) = |E \cap B_\rho|$. Applying the isoperimetric inequality (2.4) to $E_k \cap B_\rho$ and $E_k \setminus B_\rho$, and using Lemma 1.19, we get

$$\begin{aligned} C_I\, m_k(\rho)^q &\le P_{\mathbb{G}}(E_k \cap B_\rho, \mathbb{G}) \\ &= P_{\mathbb{G}}(E_k, B_\rho) + P_{\mathbb{G}}(E_k \cap B_\rho, \partial B_\rho) \\ &\le P_{\mathbb{G}}(E_k, B_\rho) + m_k'(\rho) \end{aligned}$$

and

$$\begin{aligned} C_I\,(1-m_k(\rho))^q &\leq P_{\mathbb{G}}(E_k \setminus B_\rho, \mathbb{G}) \\ &= P_{\mathbb{G}}(E_k, \mathbb{G}\setminus \overline{B}_\rho) + P_{\mathbb{G}}(E_k \setminus B_\rho, \partial B_\rho) \\ &\leq P_{\mathbb{G}}(E_k, \mathbb{G}\setminus \overline{B}_\rho) + m_k'(\rho), \end{aligned}$$

for a.e. $\rho > 0$. Adding up these inequalities and using (3.4), we get

$$\begin{aligned} C_I\,(m_k(\rho)^q + (1-m_k(\rho))^q) &\leq P_{\mathbb{G}}(E_k, \mathbb{G}) + 2\, m_k'(\rho) \\ &\leq C_I \left(1+\frac{1}{k}\right) + 2\, m_k'(\rho), \end{aligned}$$

that is,

$$f(m_k(\rho)) - \frac{1}{k} \leq C\, m_k'(\rho) \tag{3.9}$$

for a.e. $\rho > 0$ where $C > 0$ is an absolute constant and f is defined for all $t \in [0,1]$ by

$$f(t) = t^q + (1-t)^q - 1. \tag{3.10}$$

Note that $f(t) = f(1-t)$ for all $t \in [0,1]$, that $f(0) = 0$ and that f is increasing on $[0,1/2]$. Since $m(r) < 1$ and $\lim_{k\to+\infty} m_k(r) = m(r)$, one can find $\eta(r) \in (0, m_0)$ and $k(r) \geq 1$ large enough such that, for all $k \geq k(r)$,

$$m_k(r) \leq 1 - \eta(r) \quad \text{and} \quad \frac{2}{k} \leq f(\eta(r)).$$

Then we have

$$0 < \frac{f(\eta(r))}{2} \leq f(\eta(r)) - \frac{1}{k} \leq f(t) - \frac{1}{k} \tag{3.11}$$

for all $k \geq k(r)$ and $t \in [\eta(r), 1-\eta(r)]$ (recall that $m_0 < 1/2$ so that $\eta(r) < 1/2 < 1-\eta(r)$). By (3.5) and by choice of $\eta(r)$, we have

$$\eta(r) \leq m_0 \leq m_k(\rho) \leq m_k(r) \leq 1-\eta(r) \tag{3.12}$$

for all $\rho \in [1, r]$. Thus (3.11) holds in particular with $t = m_k(\rho)$ and it follows from (3.9) that

$$1 \leq C \frac{m_k'(\rho)}{f(m_k(\rho)) - \frac{1}{k}}$$

for a.e. $\rho \in [1,r]$. Integrating over $[r/2, r]$ and using (3.12), we get

$$\begin{aligned} r &\leq C \int_{\frac{r}{2}}^{r} \frac{m_k'(\rho)}{f(m_k(\rho)) - \frac{1}{k}}\, d\rho \\ &= C \int_{m_k(\frac{r}{2})}^{m_k(r)} \frac{dt}{f(t)-\frac{1}{k}} \leq C \int_{\eta(r)}^{1-\eta(r)} \frac{dt}{f(t)-\frac{1}{k}}. \end{aligned}$$

Then, thanks to (3.11), one can pass to the limit when k goes to infinity and one gets

$$r \leq C \int_{\eta(r)}^{1-\eta(r)} \frac{dt}{f(t)} \leq C \int_0^1 \frac{dt}{f(t)}$$

for some absolute constant $C > 0$. Hence, if R_Q denotes the right-hand side of this inequality, R_Q is finite, depends only on Q and we have just proved that $r \leq R_Q$ whenever $|E \cap B_r| < 1$. It follows that $|E \cap B_{R_Q}| \geq 1$ and, since $|E| \leq 1$ (see (3.7)), we get the required conclusion. □

To conclude, we now have $|E| = 1$. By (3.6) and by definition of C_I (see (2.3)) we get that $P_{\mathbb{G}}(E, \mathbb{G}) = C_I$, thus E is isoperimetric and Theorem 2.5 is proved.

4. Ahlfors-regularity and condition B

This section is devoted to the proof of Theorem 2.11. This theorem gives for any isoperimetric set E an equivalent set E_1 whose boundary is Ahlfors-regular (here with $\mu = P_{\mathbb{G}}(E, .)$, see Definition 2.8), and which satisfies the condition B (Definition 2.10), with implicit constants that can be chosen depending only on Q and $|E|$. We refer to Theorem 4.3 for a more precise version of this result. Throughout this section $(\mathbb{G}, \,.\,, \delta_\lambda, d_c)$ denotes as usual a Carnot group and E a fixed isoperimetric set in $\mathbb{G}$.

First we deal with the upper estimate in the Ahlfors-regularity.

Lemma 4.1. *There exists a constant $C > 0$ that depends only on Q and $|E|$ such that*

$$P_{\mathbb{G}}(E, B(x, r)) \leq C\, r^{Q-1}$$

for all $x \in \mathbb{G}$ and $r > 0$.

Proof. We fix $x \in \mathbb{G}$, $r > 0$ and we set $F = E \cup B(x, r)$. We have

$$\begin{aligned} P_{\mathbb{G}}(F, \mathbb{G}) &= P_{\mathbb{G}}(E, \mathbb{G} \setminus \overline{B}(x, r)) + P_{\mathbb{G}}(E \cup B(x, r), \partial B(x, r)) \\ &\leq P_{\mathbb{G}}(E, \mathbb{G}) - P_{\mathbb{G}}(E, B(x, r)) + P_{\mathbb{G}}(E \cup B(x, r), \partial B(x, r)). \end{aligned}$$

On the other hand, by (1.12), Lemma 1.19 and (1.9), we also have

$$\begin{aligned} P_{\mathbb{G}}(E \cup B(x, r), \partial B(x, r)) &= P_{\mathbb{G}}(E^c \setminus B(x, r), \partial B(x, r)) \\ &\leq (Q\, \omega_Q\, r^{Q-1} - m'(r)) \\ &\leq C\, r^{Q-1} \end{aligned}$$

for a.e. $r > 0$ and where $m(.)$ denotes the nondecreasing function $r \mapsto |E \cap B(x, r)|$. Hence it follows that, for a.e. $r > 0$,

$$P_{\mathbb{G}}(F, \mathbb{G}) \leq P_{\mathbb{G}}(E, \mathbb{G}) - P_{\mathbb{G}}(E, B(x, r)) + C\, r^{Q-1}.$$

On the other hand, we have $|E| \le |F|$ and it follows from the minimality property (2.7) of E that

$$P_{\mathbb{G}}(E, \mathbb{G}) \le P_{\mathbb{G}}(F, \mathbb{G}).$$

Then we simply combine these inequalities to get the required conclusion for a.e. $r > 0$ and the conclusion follows easily for any $r > 0$. □

To prove the lower estimate in the Ahlfors-regularity and the condition B we introduce the function h defined on $\mathbb{G} \times \mathbb{R}^+$ by

$$h(x, r) = r^{-Q} \min(|E \cap B(x, r)|, |B(x, r) \setminus E|).$$

This function measures the proportion of E and of its complement inside any ball $B(x, r)$. The key point in the proof is the following lemma about the behaviour of h whenever it is small enough.

Lemma 4.2. *There exists $\varepsilon > 0$, depending only on Q and $|E|$, such that for all $x \in \mathbb{G}$ and $r \le 1$, if $h(x, r) \le \varepsilon$, then*

$$|B(x, r/2) \setminus E| = 0 \quad \text{or} \quad |E \cap B(x, r/2)| = 0.$$

Proof. Let $\varepsilon > 0$ to be fixed small later and assume for simplicity that $|E| = 1$. The general case follows by a simple scaling argument. Let $x \in \mathbb{G}$ and $r \le 1$ be such that $h(x, r) \le \varepsilon$. We distinguish two cases.

Case 1. If $h(x, r) = r^{-Q}|B(x, r) \setminus E|$, the proportion of the complement of E inside $B(x, r)$ is very small and we set $E_t = E \cup B(x, t)$ for any $t > 0$. First let us estimate the $\mathbb{G}$-perimeter of this new competitor. Set $m(t) = |B(x, t) \setminus E|$. Arguing as in Lemma 4.1, we have

$$\begin{aligned} P_{\mathbb{G}}(E_t, \mathbb{G}) &= P_{\mathbb{G}}(E, \mathbb{G} \setminus \overline{B}(x, t)) + P_{\mathbb{G}}(E \cup B(x, t), \partial B(x, t)) \\ &\le P_{\mathbb{G}}(E, \mathbb{G}) - P_{\mathbb{G}}(E, B(x, t)) + P_{\mathbb{G}}(E^c \setminus B(x, t), \partial B(x, t)) \\ &\le P_{\mathbb{G}}(E, \mathbb{G}) - P_{\mathbb{G}}(E, B(x, t)) + m'(t) \end{aligned}$$

for a.e. $t > 0$, where the first inequality follows from (1.12) and the last one from Lemma 1.19. On the other hand, we have for a.e. $t > 0$,

$$\begin{aligned} P_{\mathbb{G}}(B(x, t) \setminus E, \mathbb{G}) &= P_{\mathbb{G}}(E^c, B(x, t)) + P_{\mathbb{G}}(B(x, t) \setminus E, \partial B(x, t)) \\ &\le P_{\mathbb{G}}(E, B(x, t)) + m'(t), \end{aligned}$$

where the last inequality follows once again from (1.12) and Lemma 1.19. Thus

$$P_{\mathbb{G}}(E_t, \mathbb{G}) \le P_{\mathbb{G}}(E, \mathbb{G}) - P_{\mathbb{G}}(B(x, t) \setminus E, \mathbb{G}) + 2\, m'(t).$$

Now we have $|E_t| \ge |E|$, hence (2.7) implies

$$P_{\mathbb{G}}(E, \mathbb{G}) \le P_{\mathbb{G}}(E_t, \mathbb{G}).$$

It follows that

$$P_{\mathbb{G}}(B(x, t) \setminus E, \mathbb{G}) \le 2\, m'(t),$$

and the isoperimetric inequality (2.4) gives

$$C_I\, m(t)^q \le 2\, m'(t)$$

for a.e. $t > 0$. Suppose now that $m(t) > 0$ for all $t \in [r/2, r]$, otherwise there is nothing to prove. Then one can rewrite the previous inequality as

$$C \le m'(t)/m(t)^q$$

for a.e. $t \in [r/2, r]$ and some constant $C > 0$ which depends only on Q. By integration between $r/2$ and r, we get

$$r \le C\,(m(r)^{1/Q} - m(r/2)^{1/Q}) \le C\, m(r)^{1/Q} \le C\, \varepsilon^{1/Q}\, r$$

for a suitable constant $C > 0$ depending only on Q. This is clearly impossible if ε is small enough and gives the contradiction. Therefore $|B(x, r/2) \setminus E| = 0$ as wanted.

Case 2. We suppose now that $h(x, r) = r^{-Q}|E \cap B(x, r)|$ and proceed in a way similar to the previous one. The percentage of E inside $B(x, r)$ is now quite small, hence natural competitors are built by removing $B(x, t)$ from E. However, this competitor must be modified (dilated) since its Lebesgue measure, in the non-trivial case, is strictly less than that of E.

For $t > 0$ set $m(t) = |E \cap B(x, t)|$. When $t \le r$, we have $m(t) \le |E \cap B(x, r)| \le \varepsilon$ because $r \le 1$, hence $m(t) < 1$ whenever ε is small enough and one can define

$$\lambda_t = (1 - m(t))^{-1/Q}$$

for $t \le r$. Let us set $E^t = \delta_{\lambda_t}(E \setminus B(x, t))$. Then $|E^t| = |E|$ and E^t is an admissible candidate for E. Hence (2.7) gives

$$P_{\mathbb{G}}(E, \mathbb{G}) \le P_{\mathbb{G}}(E^t, \mathbb{G}).$$

On the other hand, using (1.18) and arguing as before, we have

$$\begin{aligned}
P_{\mathbb{G}}(E^t, \mathbb{G}) &= \lambda_t^{Q-1} P_{\mathbb{G}}(E \setminus B(x, t), \mathbb{G}) \\
&\le \lambda_t^{Q-1}(P_{\mathbb{G}}(E, \mathbb{G}) - P_{\mathbb{G}}(E, B(x, t)) + m'(t)) \\
&\le \lambda_t^{Q-1}(P_{\mathbb{G}}(E, \mathbb{G}) - P_{\mathbb{G}}(E \cap B(x, t), \mathbb{G}) + 2\, m'(t)) \\
&\le \lambda_t^{Q-1}(P_{\mathbb{G}}(E, \mathbb{G}) - C_I\, m(t)^q + 2\, m'(t))
\end{aligned}$$

for a.e. $t \le r$. Then, combining these inequalities and recalling that $P_{\mathbb{G}}(E, \mathbb{G}) = C_I$, we get

$$\lambda_t^{1-Q} - 1 + m(t)^q \le C\, m'(t),$$

that is,

$$f(m(t)) \le C\, m'(t)$$

for a.e. $t \le r$, some constant $C > 0$ which depends only on Q and with

$$f(v) = v^q + (1 - v)^q - 1.$$

We have $\lim_{v\to 0} v^{-q} f(v) = 1$ and $m(t) \leq \varepsilon$ for $t \leq r$. Hence, if ε is small enough, $f(m(t)) \geq m(t)^q/2$ whenever $t \leq r$. Suppose now that $m(t) > 0$ for all $t \in [r/2, r]$, otherwise there is nothing to prove. Then we have

$$C \leq m'(t)/m(t)^q$$

for a.e. $t \in [r/2, r]$ and some constant $C > 0$ which depends only on Q. Now we conclude as in the previous case, integrating over $[r/2, r]$ to get a contradiction. □

Now Theorem 2.11, and the refined version given by Theorem 4.3 below, follow by quite standard arguments that do not use anymore that E is an isoperimetric set (see e.g. [DS98] where the same kind of arguments are used in a different context). However for the convenience of the reader we detail these arguments in the sequel.

Theorem 4.3. *We set*

$$E_1 = \{x \in \mathbb{G};\ \textit{there exists } r > 0 \textit{ such that } |B(x,r) \setminus E| = 0\},$$
$$E_0 = \{x \in \mathbb{G};\ \textit{there exists } r > 0 \textit{ such that } |E \cap B(x,r)| = 0\},$$
$$S = \{x \in \mathbb{G};\ h(x,r) > \varepsilon \textit{ for all } r \leq 1\},$$

where ε is given by Lemma 4.2. Then the following conclusions hold:

- (*i*) *E_1, E_0 and S form a partition of $\mathbb{G}$,*
- (*ii*) *E_0 and E_1 are open,*
- (*iii*) *$E_1 = E(1)$ and $E_0 = E(0)$, hence E and E_1, resp. E^c and E_0, are equivalent,*
- (*iv*) *$S = \partial E_1 = \partial E_0$ and S is Ahlfors-regular,*
- (*v*) *E_1 and E_0 satisfy the condition B.*

Moreover the implicit constants that show up in (iv) and (v) can be chosen depending only on Q and $|E|$.

Proof. (*i*) This is an immediate consequence of Lemma 4.2.

(*ii*) This is clear from the definition of E_1 and E_0.

(*iii*) Let us show that E_1 coincides with the set of Lebesgue points $E(1)$ of E. Indeed if x is a Lebesgue point of E then $h(x,r) = r^{-Q}|B(x,r) \setminus E|$ if r is small enough and $h(x,r)$ tends to zero when r goes to zero. Thus $x \in E_1$ by Lemma 4.2. The other inclusion follows from the definition of E_1. Hence E and E_1 are equivalent. Similar arguments with E replaced by its complement show that E_0 is the set of Lebesgue points of E^c and then E^c and E_0 are equivalent.

(*iv*) Note that we clearly have $S \subset \partial E$. Since E_1 and E_0 are open and disjoint and because E_1, E_0 and S form a partition of $\mathbb{G}$, we have $\partial E_1 \cup \partial E_0 \subset S$. On the other hand, by definition of S and since E and E_1, respectively E_0 and E^c, are equivalent, we also have $S \subset \partial E_1 \cap \partial E_0$. Hence $S = \partial E_1 = \partial E_0$.

Next we prove that S is Ahlfors-regular with $\mu = P_{\mathbb{G}}(E,.) = P_{\mathbb{G}}(E_1,.)$ in (2.9). We have $\mathrm{spt}(\mu) \subset \partial E_1 = S$. The upper estimate follows from Lemma 4.1. To prove the lower estimate we take $x \in S$ and $r \le 1$. By definition of S and by the relative isoperimetric inequality for intrinsic balls (Theorem 2.15) we have

$$C\,\varepsilon^{(Q-1)/Q} r^{Q-1} \le C(r^Q h(x,r))^{(Q-1)/Q} \le P_{\mathbb{G}}(E, B(x,r)),$$

which is exactly what we want.

(v) We prove now that E_1 and E_0 satisfy the condition B. Let $x \in S$ and $r \le 1/2$ be fixed. Set

$$Z = \{z \in B(x,r/2)\,;\; d_c(z,S) \le sr/2\}$$

where $0 < s < 1$ will be fixed small later. We have $|Z| \le C\,s\,r^Q$. To see this we take a maximal family $\mathcal{A}$ of points in $S \cap B(x,r)$ at mutual distance $\ge sr/2$. The balls $B(y, sr/4)$, $y \in \mathcal{A}$, are pairwise disjoint, S is Ahlfors-regular and $\cup_{y\in\mathcal{A}} B(y,sr/4) \subset B(x,2r)$. Thus we have

$$\begin{aligned} \mathrm{card}(\mathcal{A}) &\le C\,(sr)^{-Q+1} \sum_{y\in\mathcal{A}} P_{\mathbb{G}}(E, B(y,sr/4)) \\ &\le C\,(sr)^{-Q+1} P_{\mathbb{G}}(E, B(x,2r)) \\ &\le C\,s^{-Q+1}. \end{aligned}$$

Going back to Z, we have

$$Z \subset \bigcup_{y\in\mathcal{A}} B(y,sr)$$

and then $|Z| \le C\ \mathrm{card}(\mathcal{A})\,(sr)^Q \le C\,s\,r^Q$ as required. Since $x \in S$, we also have

$$|E_1 \cap B(x,r)| \ge \varepsilon\, r^Q \quad \text{and} \quad |E_0 \cap B(x,r)| \ge \varepsilon\, r^Q.$$

Then, if s is small enough, depending on Q and ε, one can find $z_1 \in (E_1 \cap B(x,r)) \setminus Z$ and $z_0 \in (E_0 \cap B(x,r)) \setminus Z$, and then

$$\begin{aligned} B(z_1, sr/2) &\subset E_1 \cap B(x,r) = B(x,r) \setminus \overline{E}_0, \\ B(z_0, sr/2) &\subset E_0 \cap B(x,r) = B(x,r) \setminus \overline{E}_1. \end{aligned}$$

This gives the required conclusion for any $x \in S = \partial E_1 = \partial E_0$ and $r \le 1/2$. And the conclusion follows easily for any $r \le 1$ (with a slightly different implicit constant in the definition of the condition B). □

To complete the proof of Theorem 2.11 it only remains to prove the uniqueness of sets equivalent to E that satisfy the required conclusions and this follows from the fact that two equivalent open sets that both satisfy the condition B coincide.

Then, with Theorem 2.11 in hand, one can always assume that isoperimetric sets are, up to a null set, reduced isoperimetric sets according to Definition 2.12. On the other hand we explicitely stress here that because of uniqueness in Theorem 2.11 and thanks to Theorem 4.3 we have a precise description of any reduced isoperimetric set. Indeed, if E is a reduced isoperimetric set, then E coincides with the set $E_1 = E(1)$ given by the previous construction and moreover we have $\partial E = S = \mathbb{G} \setminus (E(1) \cup E(0)) = \partial^* E$. Note also that $\mathbb{G} \setminus \overline{E} = E_0 = E(0)$.

Proposition 4.4. *Let E be a reduced isoperimetric set. Then the topological boundary of E and its essential boundary coincide.*

Note also that the previous arguments show that the implicit constants in the Ahlfors-regularity and in the condition B can always be chosen depending only on Q and $|E|$ for any reduced isoperimetric set E.

Finally it follows from the properties of reduced isoperimetric sets that they are bounded. Recall that one could have shown the following proposition using the same arguments as those of the proof of Lemma 3.8.

Proposition 4.5. *Let E be a reduced isoperimetric set. Then E is bounded and its diameter is bounded by a constant that depends only on Q and $|E|$.*

Proof. Let E be a reduced isoperimetric set. The lemma follows from the Ahlfors-regularity of ∂E. Let $\mathcal{A}$ be a maximal family of points in ∂E at mutual distance ≥ 2. We have

$$\mathrm{card}(\mathcal{A}) \leq C \sum_{x \in \mathcal{A}} P_{\mathbb{G}}(E, B(x,1)) \leq C\, P_{\mathbb{G}}(E, \mathbb{G}),$$

because ∂E is Ahlfors-regular and the balls $B(x,1)$, $x \in \mathcal{A}$, are pairwise disjoint. Since $\partial E \subset \cup_{x \in \mathcal{A}} B(x,2)$ and $|E| < +\infty$, we get

$$\mathrm{diam}\, E = \mathrm{diam}\, \partial E \leq C$$

where C depends only on Q and $|E|$. □

5. Domain of Isoperimetry

In this section we focus on isoperimetric sets in the Heisenberg group and prove Theorem 2.16. Recall that the Heisenberg group $\mathbb{H}^n$ has been defined in Example 1.5 and that Theorem 2.16 tells us that any reduced isoperimetric set in $\mathbb{H}^n$ is a domain of isoperimetry. Namely if $E \subset \mathbb{H}^n$ is a fixed reduced isoperimetric set (Definition 2.12) and A is any subset of E, we want to prove that there exists a constant $C > 0$, depending only on Q, such that

$$C \min(|A|, |E \setminus A|)^q \leq P_{\mathbb{H}}(A, E).$$

The underlying idea to get this inequality is quite simple. We apply the global isoperimetric inequality to A and $E \setminus A$ and add these inequalities to conclude. To make this argument rigourous one needs however to check some (nontrivial) technicalities which rely on fine properties of sets with finite $\mathbb{H}$-perimeter. As already pointed out, the crucial point, proved in Lemma 5.3, holds actually in any Carnot group for which the analogue of Proposition 1.23 holds. This has been recently announced to be satisfied in Carnot groups of step 2 by B. Franchi, R. Serapioni and F. Serra Cassano, the general case remaining however unanswered up to now.

Let us postpone all the technicalities at the end of the section and first give the proof of the desired relative isoperimetric inequality on reduced isoperimetric sets. Let E and A be as above. One can assume that $P_{\mathbb{H}}(A,E) = P_{\mathbb{H}}(E \setminus A, E) < +\infty$ (recall that E is open) otherwise there is nothing to prove. We also assume that $|A| \leq |E|/2$, otherwise we consider $E \setminus A$ instead of A, and we set $v = |A|/|E|$. Then A and $E \setminus A$ have finite $\mathbb{H}$-perimeter in $\mathbb{H}^n$. This follows from Lemma 5.1, proved a few lines below, recalling that, since E is a reduced isoperimetric set, $\partial E = \partial^* E$ is Ahlfors-regular and that, by Theorem 1.20, $\mathcal{H}^{Q-1}(\partial E) \leq C\, P_{\mathbb{H}}(E, \mathbb{H}^n) < +\infty$. Then the isoperimetric inequality (2.4) applied to A and $E \setminus A$ gives

$$\begin{aligned} C_I\, |E|^q v^q &\leq P_{\mathbb{H}}(A, \mathbb{H}^n), \\ C_I\, |E|^q (1-v)^q &\leq P_{\mathbb{H}}(E \setminus A, \mathbb{H}^n). \end{aligned}$$

We now wish to add these inequalities and would like to recover in the right-hand side the $\mathbb{H}$-perimeter of E plus two times the $\mathbb{H}$-perimeter of A inside E as anyone would expect. This will follow from Lemma 5.3 because E is open, $\partial E = \partial^* E$ and A has finite $\mathbb{H}$-perimeter in $\mathbb{H}^n$. Hence we get

$$\begin{aligned} C_I\, |E|^q (v^q + (1-v)^q) &\leq P_{\mathbb{H}}(A, \mathbb{H}^n) + P_{\mathbb{H}}(E \setminus A, \mathbb{H}^n) \\ &= P_{\mathbb{H}}(E, \mathbb{H}^n) + 2\, P_{\mathbb{H}}(A, E). \end{aligned}$$

Thus, since E is isoperimetric, we have

$$C_I\, |E|^q (v^q + (1-v)^q - 1) \leq 2\, P_{\mathbb{H}}(A, E).$$

And Lemma 5.9, proved at the end of this section, with $\alpha = q$ gives a constant $C > 0$ that depends only on Q such that

$$C|A|^q = C|E|^q v^q \leq P_{\mathbb{H}}(A, E)$$

which is exactly what we want. This concludes the proof of Theorem 2.16.

Let us now end this section with the various lemmas to which we have referred in this proof. The first one holds on general Carnot groups. Thus we return to the general case to state and prove it and $(\mathbb{G}, \,.\,, \delta_\lambda, d_c)$ denotes in the following lemma a Carnot group as in Definition 1.3.

Lemma 5.1. *Let F be an open subset of $\mathbb{G}$ with Ahlfors-regular boundary and such that $\mathcal{H}^{Q-1}(\partial F) < +\infty$. If $A \subset F$ has finite $\mathbb{G}$-perimeter in F, then A has finite $\mathbb{G}$-perimeter in $\mathbb{G}$.*

Let us stress that the assumption on $\mathcal{H}^{Q-1}(\partial F)$ to be finite is essential, one can easily find counterexamples where F is a set with finite $\mathbb{G}$-perimeter but

$\mathcal{H}^{Q-1}(\partial F) = +\infty$. On the other hand the Ahlfors-regularity of ∂F is here mostly for technical convenience.

Proof. Let $\delta \in (0,1)$ be fixed. Let $\mathcal{A}$ be a maximal family of points in ∂F at mutual distance $\geq \delta$. The balls $B(x, \delta/2)$, $x \in \mathcal{A}$, are pairwise disjoint. Thus, since ∂F is Ahlfors-regular (see Definition 2.8 and what follows), we have

$$\begin{aligned} (\operatorname{card} \mathcal{A})\, \delta^{Q-1} &\leq C \sum_{x \in \mathcal{A}} \mathcal{H}^{Q-1}(\partial F \cap B(x, \delta/2)) \\ &\leq C \mathcal{H}^{Q-1}(\partial F) < +\infty. \end{aligned} \tag{5.2}$$

In particular the family $\mathcal{A}$ is finite and one can easily construct two open sets $F'' \subset\subset F' \subset\subset F$ such that

$$F \setminus F'' \subset \cup_{x \in \mathcal{A}} B(x, \delta).$$

Let $A \subset F$ be such that $P_{\mathbb{G}}(A, F) < +\infty$. Set

$$A_\delta = A \cup (\cup_{x \in \mathcal{A}} B(x, \delta)).$$

We have $P_{\mathbb{G}}(A, F') \leq P_{\mathbb{G}}(A, F) < +\infty$. Next let us explicitely note that

$$P_{\mathbb{G}}(B(x, \delta), \mathbb{G}) = C\, \delta^{Q-1}$$

for any $x \in \mathbb{G}$ and for some constant $C > 0$ which depends only on Q. To see this we first apply Lemma 1.19 to the set $\mathbb{G}$ and use (1.9) to get that $P_{\mathbb{G}}(B(x, r), \mathbb{G}) \leq C\, r^{Q-1}$ for a.e. $r > 0$. In particular $P_{\mathbb{G}}(B(x, r), \mathbb{G})$ is finite for a.e. $r > 0$. Then we use the invariance under left translations and the homogeneity of the $\mathbb{G}$-perimeter (see (1.17) and (1.18)) to get that $P_{\mathbb{G}}(B(x, r), \mathbb{G}) = P_{\mathbb{G}}(B_1, \mathbb{G})\, r^{Q-1}$ for any $r > 0$. Hence it follows that $P_{\mathbb{G}}(B(x, r), \mathbb{G})$ is finite for any $r > 0$ and the required formula holds. Then we get

$$\begin{aligned} P_{\mathbb{G}}(\cup_{x \in \mathcal{A}} B(x, \delta), F') &\leq \sum_{x \in \mathcal{A}} P_{\mathbb{G}}(B(x, \delta), F') \\ &\leq C(\operatorname{card} \mathcal{A})\, \delta^{Q-1} \leq C \mathcal{H}^{Q-1}(\partial F) < +\infty, \end{aligned}$$

by (1.15) and (5.2). Hence A_δ has finite $\mathbb{G}$-perimeter in F' and

$$\begin{aligned} P_{\mathbb{G}}(A_\delta, F') &\leq P_{\mathbb{G}}(A, F') + P_{\mathbb{G}}(\cup_{x \in \mathcal{A}} B(x, \delta), F') \\ &\leq P_{\mathbb{G}}(A, F) + C \mathcal{H}^{Q-1}(\partial F). \end{aligned}$$

We have $A_\delta \setminus \overline{F}'' = (\cup_{x \in \mathcal{A}} B(x, \delta)) \setminus \overline{F}''$, then, arguing in a similar way, we get

$$\begin{aligned} P_{\mathbb{G}}(A_\delta, \mathbb{G} \setminus \overline{F}'') &= P_{\mathbb{G}}(\cup_{x \in \mathcal{A}} B(x, \delta), \mathbb{G} \setminus \overline{F}'') \\ &\leq \sum_{x \in \mathcal{A}} P_{\mathbb{G}}(B(x, \delta), \mathbb{G} \setminus \overline{F}'') \leq C \mathcal{H}^{Q-1}(\partial F) < +\infty, \end{aligned}$$

and A_δ has finite $\mathbb{G}$-perimeter in $\mathbb{G} \setminus \overline{F}''$. Hence A_δ has finite $\mathbb{G}$-perimeter and

$$\begin{aligned} P_{\mathbb{G}}(A_\delta, \mathbb{G}) &\leq P_{\mathbb{G}}(A_\delta, F') + P_{\mathbb{G}}(A_\delta, \mathbb{G} \setminus \overline{F}'') \\ &\leq P_{\mathbb{G}}(A, F) + C\mathcal{H}^{Q-1}(\partial F). \end{aligned}$$

Moreover $(A_\delta \setminus A) \subset \cup_{x\in\mathcal{A}} B(x,\delta)$, thus

$$|A_\delta \setminus A| \leq C(\operatorname{card} \mathcal{A})\, \delta^Q \leq C\, \delta \mathcal{H}^{Q-1}(\partial F)$$

and $\mathbf{1}_{A_\delta}$ converges to $\mathbf{1}_A$ in L^1 when δ goes to zero. By lower semicontinuity of the $\mathbb{G}$-perimeter, we get

$$\begin{aligned} P_{\mathbb{G}}(A, \mathbb{G}) &\leq \liminf_{\delta\to 0} P_{\mathbb{G}}(A_\delta, \mathbb{G}) \\ &\leq P_{\mathbb{G}}(A, F) + C\mathcal{H}^{Q-1}(\partial F) < +\infty \end{aligned}$$

and A has finite $\mathbb{G}$-perimeter in $\mathbb{G}$. □

Next we give the crucial lemma that makes our proof of Theorem 2.16 work. As already mentioned, it strongly relies on fine properties of sets with finite $\mathbb{H}$-perimeter in $\mathbb{H}^n$.

Lemma 5.3. *Let F be an open set in $\mathbb{H}^n$ with finite $\mathbb{H}$-perimeter such that $\mathcal{H}^{Q-1}(\partial F \setminus \partial^* F) = 0$. Then, for any set $A \subset F$ with finite $\mathbb{H}$-perimeter in $\mathbb{H}^n$, we have*

$$P_{\mathbb{H}}(A, \mathbb{H}^n) + P_{\mathbb{H}}(F \setminus A, \mathbb{H}^n) = P_{\mathbb{H}}(F, \mathbb{H}^n) + 2\, P_{\mathbb{H}}(A, F).$$

Proof. Let F and A be as in the statement. We have $\mathbf{1}_F = \mathbf{1}_A + \mathbf{1}_{F\setminus A}$ hence

$$\nabla_{\mathbb{H}}\, \mathbf{1}_F = \nabla_{\mathbb{H}}\, \mathbf{1}_A + \nabla_{\mathbb{H}}\, \mathbf{1}_{F\setminus A},$$

that is, thanks to (1.22),

$$\nu_F\, \theta_F\, \mathbf{1}_{\partial^* F}\, d\mathcal{H}^{Q-1} = \nu_A\, \theta_A\, \mathbf{1}_{\partial^* A}\, d\mathcal{H}^{Q-1} + \nu_{F\setminus A}\, \theta_{F\setminus A}\, \mathbf{1}_{\partial^*(F\setminus A)}\, d\mathcal{H}^{Q-1}.$$

In particular, we have

$$\nu_F\, \theta_F\, \mathbf{1}_{\partial^* F} = \nu_A\, \theta_A\, \mathbf{1}_{\partial^* A} + \nu_{F\setminus A}\, \theta_{F\setminus A}\, \mathbf{1}_{\partial^*(F\setminus A)} \tag{5.4}$$

$\mathcal{H}^{Q-1}$-a.e.. Since $|\nu_F(x)\, \theta_F(x)| \geq c'$ for $\mathcal{H}^{Q-1}$-a.e. $x \in \partial^* F$ (see Theorem 1.20 and (1.21)), it follows that

$$\mathcal{H}^{Q-1}(\partial^* F \setminus (\partial^* A \cup \partial^*(F \setminus A))) = 0. \tag{5.5}$$

Next we want to prove that

$$\mathcal{H}^{Q-1}(\partial^* F \cap \partial^* A \cap \partial^*(F \setminus A)) = 0. \tag{5.6}$$

This follows from Proposition 1.23. Indeed it tells us that $\mathcal{H}^{Q-1}$-a.e. point in $\partial^* A \cap \partial^*(F\setminus A)$ is a point of volume density $1/2$ for A and $F\setminus A$ and thus a point of volume density 1 for F. But it also tells us that $\mathcal{H}^{Q-1}$-a.e. point in $\partial^* F$ is a point of volume density $1/2$ for F. Therefore (5.6) follows. Furthermore, using (5.4) and (5.6), we get that

$$\nu_F\,\theta_F = \nu_A\,\theta_A \qquad \mathcal{H}^{Q-1} - \text{a.e. on } \partial^* F \cap \partial^* A,$$
$$\nu_F\,\theta_F = \nu_{F\setminus A}\,\theta_{F\setminus A} \qquad \mathcal{H}^{Q-1} - \text{a.e. on } \partial^* F \cap \partial^* (F\setminus A),$$

hence,

$$\theta_F = \theta_A \qquad \mathcal{H}^{Q-1} - \text{a.e. on } \partial^* F \cap \partial^* A, \tag{5.7}$$
$$\theta_F = \theta_{F\setminus A} \qquad \mathcal{H}^{Q-1} - \text{a.e. on } \partial^* F \cap \partial^* (F\setminus A), \tag{5.8}$$

because ν_F, ν_A and $\nu_{F\setminus A}$ are unit vectors and θ_F, θ_A and $\theta_{F\setminus A}$ are positive functions.

Let us now compute separately $P_{\mathbb{H}}(A,\mathbb{H}^n)$, $P_{\mathbb{H}}(F\setminus A,\mathbb{H}^n)$ and $P_{\mathbb{H}}(F,\mathbb{H}^n)$. Since F is open and $A\subset F$ and since $\mathcal{H}^{Q-1}(\partial F\setminus\partial^* F)=0$, we have

$$\begin{aligned} P_{\mathbb{H}}(A,\mathbb{H}^n) &= P_{\mathbb{H}}(A,F) + \int_{\partial^* A\cap\partial F} \theta_A\, d\mathcal{H}^{Q-1} \\ &= P_{\mathbb{H}}(A,F) + \int_{\partial^* A\cap\partial^* F} \theta_A\, d\mathcal{H}^{Q-1}, \end{aligned}$$

and it follows from (5.7) that

$$P_{\mathbb{H}}(A,\mathbb{H}^n) = P_{\mathbb{H}}(A,F) + \int_{\partial^* A\cap\partial^* F} \theta_F\, d\mathcal{H}^{Q-1}.$$

Similarly,

$$\begin{aligned} P_{\mathbb{H}}(F\setminus A,\mathbb{H}^n) &= P_{\mathbb{H}}(F\setminus A,F) + \int_{\partial^* (F\setminus A)\cap\partial F} \theta_{F\setminus A}\, d\mathcal{H}^{Q-1} \\ &= P_{\mathbb{H}}(A,F) + \int_{\partial^* (F\setminus A)\cap\partial^* F} \theta_{F\setminus A}\, d\mathcal{H}^{Q-1} \\ &= P_{\mathbb{H}}(A,F) + \int_{\partial^* (F\setminus A)\cap\partial^* F} \theta_F\, d\mathcal{H}^{Q-1}. \end{aligned}$$

Hence we get

$$\begin{aligned} &P_{\mathbb{H}}(A,\mathbb{H}^n) + P_{\mathbb{H}}(F\setminus A,\mathbb{H}^n) \\ &\qquad = 2\,P_{\mathbb{H}}(A,F) + \int_{\partial^* A\cap\partial^* F} \theta_F\, d\mathcal{H}^{Q-1} + \int_{\partial^* (F\setminus A)\cap\partial^* F} \theta_F\, d\mathcal{H}^{Q-1}. \end{aligned}$$

On the other hand, thanks to (5.5) and (5.6), we also have

$$\begin{aligned} P_{\mathbb{H}}(F,\mathbb{H}^n) &= \int_{\partial^* F} \theta_F\, d\mathcal{H}^{Q-1} \\ &= \int_{\partial^* F\cap\partial^* A} \theta_F\, d\mathcal{H}^{Q-1} + \int_{\partial^* F\cap\partial^* (F\setminus A)} \theta_F\, d\mathcal{H}^{Q-1}, \end{aligned}$$

and the claim follows. □

Let us conclude with a last technical lemma that was used to get the final conclusion in the proof of Theorem 2.16.

Lemma 5.9. *For any $\alpha \in (0,1)$, there exists a constant $\gamma > 0$ depending only on α such that*

$$v^\alpha + (1-v)^\alpha - 1 \geq \gamma\, v^\alpha$$

for any $v \in [0,1/2]$.

Proof. Let $\alpha \in (0,1)$ be fixed. We fix $\beta \in (2^{1-\alpha}\alpha, 2^{1-\alpha})$ and set

$$g(v) = (1-v)^\alpha - 1 + \beta v.$$

We have $g(0) = 0$ and

$$g'(v) = -\alpha(1-v)^{\alpha-1} + \beta \geq 0$$

whenever $v \in [0,1/2]$. Hence we have $g(v) \geq 0$ for all $v \in [0,1/2]$ and thus

$$\begin{aligned} v^\alpha + (1-v)^\alpha - 1 &\geq v^\alpha - \beta v \\ &\geq (1-\beta v^{1-\alpha})v^\alpha \geq (1-2^{\alpha-1}\beta)v^\alpha, \end{aligned}$$

and the lemma follows because $\gamma = 1 - 2^{\alpha-1}\beta > 0$. □

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