Ergodic problems in differential games

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Abstract

We present and study a notion of ergodicity for deterministic zero-sum differential games that extends the one in classical ergodic control theory to systems with two conflicting controllers. We show its connections with the existence of a constant and uniform long-time limit of the value function of finite horizon games, and characterize this property in terms of Hamilton-Jacobi-Isaacs equations. We also give several sufficient conditions for ergodicity and describe some extensions of the theory to stochastic differential games.

Introduction

We consider a nonlinear system in $\mathbb{R}^m$ controlled by two players

$$\dot{y}(t) = f(y(t), a(t), b(t)), \qquad y(0) = x, \tag{1}$$

and we denote with $y_x(\cdot)$ the trajectory starting at x. We are also given a bounded, uniformly continuous running cost l, and we are interested in the payoffs associated to the *long time average cost* (briefly, LTAC), namely,

$$J^\infty(x, a(\cdot), b(\cdot)) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T l(y_x(t), a(t), b(t))\, dt,$$

$$J_\infty(x, a(\cdot), b(\cdot)) := \liminf_{T\to\infty} \frac{1}{T}\int_0^T l(y_x(t), a(t), b(t))\, dt.$$

We denote with $u - \operatorname{val} J^\infty(x)$ (respectively, $l - \operatorname{val} J_\infty(x)$) the upper value of the zero-sum game with payoff J^∞ (respectively, the lower value of the game with payoff J_∞) which the 1st player $a(\cdot)$ wants to minimize while the 2nd player $b(\cdot)$ wants to maximize, and the values are in the sense of Variya-Roxin-Elliott-Kalton. We say that the LTAC game is *ergodic* if

$$u - \mathrm{val}\, J^\infty(x) = l - \mathrm{val}\, J_\infty(x) = \lambda \quad \forall\, x,$$

for some constant λ.

The terminology is motivated by the analogy with classical ergodic control theory, see, e.g., [28,10,26,13,9,22,6,7]. Note also that if the controls can take only one value the game is ergodic for all continuous l if the dynamical system $\dot{y} = f(y)$ is ergodic with a unique invariant measure (see Proposition 13 of [3] for a precise statement). Similar problems were already studied for some games, in particular by Fleming and McEneaney [18] in the context of risk-sensitive control, Carlson and Haurie [12] within the turnpike theory, and Kushner [27] for controlled nondegenerate diffusion processes. After this research was completed we also learnt of the recent paper [21]. There is a large literature on related problems for discrete-time games, see the recent survey by Sorin [34].

In order to have a compact state space we assume that the data f and l are $\mathbb{Z}^m$-periodic. First of all we show the connection between the ergodicity of the LTAC game and the existence of a constant uniform long-time limit of the lower and upper value functions of the finite horizon games with the same running cost. We call this property ergodicity of the lower (respectively, upper) game. Then we prove that the lower game is ergodic with limit λ if and only if the lower value of the discounted infinite horizon game with payoff

$$\delta \int_0^\infty l(y_x(t), a(t), b(t))\, e^{-\delta t}\, dt$$

converges uniformly to λ as the discount rate δ tends to 0. Moreover, this is also equivalent to the existence of a $\mathbb{Z}^m$-periodic viscosity χ to the Hamilton-Jacobi-Isaacs equation

$$\lambda + \min_b \max_a \{-f(y, a, b) \cdot \nabla\chi - l(y, a, b)\} = 0\,,$$

and similar statements hold for the upper value.

Next we describe two sets of conditions ensuring the previous facts and therefore the ergodicity. The first is a bounded-time controllability property of the system by one of the players, uniformly with respect to the behavior of the opponent. It is a generalization to games of a condition used for systems with a single controller by Grüne [22], Arisawa [6], and Artstein and Gaitsgory [8].

Different from the first, the second set of conditions is symmetric for the two players. We assume that some state variables y^A are asymptotically controllable by the first player, and the remaining variables y^B are asymptotically controllable by the second (see Section 2 for the precise definition).

In this case neither player can control the whole state vector $y = (y^A, y^B)$. We further assume the running cost depends only on y^A and y^B and has a saddle point, namely,

$$\min_{y^A} \max_{y^B} l(y^A, y^B) = \max_{y^B} \min_{y^A} l(y^A, y^B) =: \bar{l},$$

Then we show that the LTAC game has the value $\lambda = \bar{l}$.

In the last section we also show that for systems affected by a nondegenerate white noise the game is ergodic with no controllability assumptions on either player (see [27] for related results).

Our methods rely heavily on the Hamilton-Jacobi-Isaacs equations associated to the games, in the framework of the theory of viscosity solutions. We follow ideas of authors such as P.-L. Lions and L.C. Evans, see [28,29,15,9,7], and their developments in our papers [3,4].

Undiscounted infinite horizon control problems arise in many applications to economics and engineering, see [13,10,26] and [12,18,34] for games. Our additional motivation is that ergodicity plays a crucial role in the theory of singular perturbation problems for the dimension reduction of multiple-scale systems [25,10,26,24,20,35,36,31] and for the homogenization in oscillating media [29,15,16,1,23,30,11,5]. A general principle emerging in the papers [8,2–4] is that an appropriate form of ergodicity of the fast variables (for frozen slow variables) ensures the convergence of the singular perturbation problem, in a suitable sense.

The paper is organized as follows. Section 1 describes the connection between the ergodicity of the LTAC game and the ergodicity of the lower and upper game. Section 2 studies the ergodicity of the finite horizon games. Section 3 presents some examples. In Section 4 we give some extensions of the results of Sections 2 and 3 to diffusion processes controlled by two players, and we prove the ergodicity result for nondegenerate noise.

1 The long-time-average-cost game and ergodicity

About the system (1) and the cost we assume throughout the paper that $f : \mathbb{R}^m \times A \times B \mapsto \mathbb{R}^m$ and $l : \mathbb{R}^m \times A \times B \mapsto \mathbb{R}$ are continuous and bounded, A and B are compact metric spaces, f is Lipschitz continuous in x uniformly in a, b. In this section we do not assume the compactness of the state space.

We consider the cost funtional

$$J(T, x) = J(T, x, a(\cdot), b(\cdot)) := \frac{1}{T} \int_0^T l(y_x(t), a(t), b(t))\, dt,$$

where $y_x(\cdot)$ is the trajectory corresponding to $a(\cdot)$ and $b(\cdot)$. We denote with $\mathcal{A}$ and $\mathcal{B}$, respectively, the sets of open-loop (measurable) controls for the first and the second player, and with Γ and Δ, respectively, the sets of nonanticipating strategies for the first and the second player, see, e.g., [17,9] for the precise definition. Finally, we define the upper and lower values for the finite horizon game with average cost

$$u - \operatorname{val} J(T,x) := \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} J(T,x,a,\beta[a]),$$

$$l - \operatorname{val} J(T,x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} J(T,x,\alpha[b],b),$$

and for the LTAC game

$$u - \operatorname{val} J^\infty(x) := \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} \limsup_{T\to\infty} J(T,x,a,\beta[a]),$$

$$l - \operatorname{val} J_\infty(x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \liminf_{T\to\infty} J(T,x,\alpha[b],b).$$

We say that the *the lower game is (uniformly) ergodic* if the long time limit of the finite horizon value exists, uniformly in x, and it is constant, i.e.,

$$l - \operatorname{val} J(T,\cdot) \to \lambda \quad \text{as } T\to\infty \text{ uniformly in } \mathbb{R}^m.$$

Similarly, *the upper game is ergodic* if

$$u - \operatorname{val} J(T,\cdot) \to \Lambda \quad \text{as } T\to\infty \text{ uniformly in } \mathbb{R}^m.$$

Theorem 1.1. *If the lower game is ergodic, then*

$$l - \operatorname{val} J_\infty(x) = \lim_{T\to\infty} l - \operatorname{val} J(T,x) = \lambda \quad \forall x\in\mathbb{R}^m; \tag{2}$$

if the upper game is ergodic, then

$$u - \operatorname{val} J^\infty(x) = \lim_{T\to\infty} u - \operatorname{val} J(T,x) = \Lambda \quad \forall x\in\mathbb{R}^m. \tag{3}$$

Proof. We follow the arguments of the prooof of Theorem 8.4 in [18]. We begin wih the proof that $l - \operatorname{val} J_\infty(x) \geq \lambda$. To achieve this goal we fix $x\in\mathbb{R}^m$, $\overline{\alpha}\in\Gamma$, and $\varepsilon>0$, and construct $\overline{b}\in\mathcal{B}$ such that

$$J_\infty(x;\overline{\alpha}[\overline{b}],\overline{b}) \geq \lambda - \varepsilon. \tag{4}$$

We set $v(T,x) := l - \operatorname{val} J(T,x)$ and choose T_o such that

$$|v(T_o,y) - \lambda T_o| \leq \frac{\varepsilon T_o}{3} \quad \forall y\in\mathbb{R}^m. \tag{5}$$

By definition of v, for all $\alpha\in\Gamma$ and $z\in\mathbb{R}^m$,

$$\sup_{b\in\mathcal{B}} \int_0^{T_o} l(y_z(t),\alpha[b](t),b(t))\,dt \geq v(T_o,z), \tag{6}$$

where $y_z(\cdot) = y_z(\cdot; \alpha, b)$ solves

$$\dot{y} = f(y, \alpha[b], b), \qquad y(0) = z.$$

Then, for $z = x$ and $\alpha = \overline{\alpha}$, we can choose $\overline{b} \in \mathcal{B}$ such that

$$\int_0^{T_o} l(y_x(t), \overline{\alpha}[\overline{b}](t), \overline{b}(t))\, dt \geq v(T_o, x) - \frac{\varepsilon T_o}{3}.$$

This defines the desired control $\overline{b}$ on the interval $[0, T_o]$. Its definition on the intervals $[nT_o, (n+1)T_o)$ is obtained inductively as follows. Suppose $\overline{b}$ is defined on the interval $[0, nT_o)$. Define the strategy $\alpha_n \in \Gamma$ by $\alpha_n[b] := \overline{\alpha}[b_n]$, where

$$b_n(t) := \begin{cases} \overline{b}((t), & 0 \leq t < nT_o, \\ b(t - nT_o), & t \geq nT_o. \end{cases}$$

In (6) put $z = y_x(nT_o; \overline{\alpha}, \overline{b})$, $\alpha = \alpha_n$, and choose b_n such that

$$\int_0^{T_o} l(y_z(t), \alpha_n[b_n](t), b_n(t))\, dt \geq v(T_o, z) - \frac{\varepsilon T_o}{3}.$$

Now define $\overline{b}(t) := b_n(t + nT_o)$ for $t \in [nT_o, (n+1)T_o)$. Then

$$\int_{nT_o}^{(n+1)T_o} l(y_x(t), \overline{\alpha}[\overline{b}](t), \overline{b}(t))\, dt \geq v(T_o, z) - \frac{\varepsilon T_o}{3}.$$

By adding over n and using (5) we get

$$\int_0^{nT_o} l(y_x(t), \overline{\alpha}[\overline{b}](t), \overline{b}(t))\, dt \geq n\lambda T_o - \frac{2\varepsilon n T_o}{3}$$

and therefore

$$J(nT_o, x, \overline{\alpha}[\beta], \beta) \geq \lambda - \frac{2\varepsilon}{3}.$$

Now we write $T = nT_o + t_o$, where n is the integer part of T/T_o and $0 \leq t_o < T_o$, and observe that, for any bounded integrand $\overline{l}(t)$,

$$\frac{1}{T} \int_0^T \overline{l}(t)\, dt \geq \frac{nT_o}{nT_o + t_o} \cdot \frac{1}{nT_o} \int_0^{nT_o} \overline{l}(t)\, dt + \frac{t_o \inf \overline{l}}{nT_o + t_o}.$$

Then

$$J(T, x, \overline{\alpha}[\beta], \beta) \geq c_n J(nT_o, x, \overline{\alpha}[\beta], \beta) + c_n' \geq \lambda - \varepsilon$$

for all T large enough, because $c_n \to 1$ and $c_n' \to 0$ as $n \to \infty$. Now we let $T \to \infty$ and obtain the desired inequality (4).

In order to prove that $l - \mathrm{val}\, J_\infty(x) \le \lambda$ one fixes $\varepsilon > 0$ and constructs a strategy $\widetilde{\alpha} \in \Gamma$ such that, for all $b \in \mathcal{B}$,

$$\int_0^{nT_o} l(y_x(t), \widetilde{\alpha}[b](t), b(t))\, dt \le nT_o(\lambda + \varepsilon),$$

where T_o satisfies (5). This can be done exactly as in the last part of the proof of Theorem 8.4 in [18]. Then we divide by nT_o, send $n \to \infty$, take $\inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}}$, and finally let $\varepsilon \to 0$ to reach the conclusion.

Finally, the statement about the upper value (3) is obtained by observing that

$$-u - \mathrm{val}\, J^\infty(x) = \inf_{\beta \in \Delta} \sup_{a \in \mathcal{A}} \liminf_{T \to \infty} (-J(T, x))$$

and the left hand side is the lower value of the LTAC game with running cost $-l$ where the second player wishes to minimize and the first player to maximize. Then the conclusion follows by applying (2) to this game. □

We recall the classical Isaacs' condition, or solvability of the small game,

$$\begin{aligned} H(y,p) &:= \min_{b \in B} \max_{a \in A} \{-f(y,a,b) \cdot p - l(y,a,b)\} \\ &= \max_{a \in A} \min_{b \in B} \{-f(y,a,b) \cdot p - l(y,a,b)\}, \quad \forall y, p \in \mathbb{R}^m. \end{aligned} \tag{7}$$

It is well known that it implies the equality of the upper and the lower value of the finite horizon game, that is, the existence of the value of that game, which we denote with $\mathrm{val}\, J(T, x)$, see [17,9]. Therefore we immediately get the following consequence of Theorem 1.1.

Corollary 1.1. *Assume* (7) *and that either the lower or the upper game is ergodic. Then the LTAC game is ergodic, i.e.,*

$$l - \mathrm{val}\, J_\infty(x) = u - \mathrm{val}\, J^\infty(x) = \lim_{T \to \infty} \mathrm{val}\, J(T, x) = \lambda, \quad \forall x \in \mathbb{R}^m.$$

2 Characterizations of ergodicity

From now on we add periodicity to the standing assumptions:

$$\begin{aligned} f(y,a,b) = f(y+k,a,b), \quad l(y,a,b) = l(y+k,a,b), \\ \forall\, k \in \mathbb{Z}^m,\ y \in \mathbb{R}^m,\ a \in A,\ b \in B. \end{aligned} \tag{8}$$

This means that the state space is the m-torus $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$. The first result is a consequence of Theorem 4 in [3].

Theorem 2.1. *The following statements on the lower game are equivalent.*

(i) *The lower game is ergodic, i.e.,* $l - \mathrm{val}\, J(T,x) \to \mathrm{const}$ *uniformly in* x *as* $T \to +\infty$.

(ii) $l{-}\mathrm{val}\,\delta \int_0^\infty l(y_x(t), a(t), b(t))\, e^{-\delta t}\, dt \to \mathrm{const}$ *uniformly in* x *as* $\delta \to 0+$.

(iii) *The additive eigenvalue problem*

$$\lambda + \min_{b\in B} \max_{a\in A} \{-f(y,a,b)\cdot \nabla\chi - l(y,a,b)\} = 0 \quad \text{in } \mathbb{R}^m, \quad \chi\ \mathbb{Z}^m\text{-periodic} \tag{9}$$

has the property that

$$\sup\{\lambda \mid \text{there is a viscosity subsolution of } (9)\} = \inf\{\lambda \mid \text{there is a viscosity supersolution of } (9)\}. \tag{10}$$

If one of the above assertions is true, then the constants in (i) *and* (ii) *are equal and they coincide with the number defined by* (10)*. Moreover, the same result holds for the upper game, after replacing* $l - \mathrm{val}$ *with* $u - \mathrm{val}$ *in* (i) *and* (ii)*, and* (9) *with*

$$\lambda + \max_{a\in A} \min_{b\in B} \{-f(y,a,b)\cdot \nabla\chi - l(y,a,b)\} = 0 \quad \text{in } \mathbb{R}^m, \quad \chi\ \mathbb{Z}^m\text{-periodic.}$$

Proof. If we set $v(t,y) := l - \mathrm{val}\, J(t,y)$ it is well known [17,9] that $w(t,y) := tv(t,y)$ is the viscosity solution of the Cauchy problem for the Isaacs equation

$$w_t + H(y, D_y w) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m, \qquad w(0,y) = 0, \quad w\ \mathbb{Z}^m\text{-periodic.}$$

The equivalence of (iii) and the uniform convergence of $w(t,\cdot)/t$ to a constant as $t \to \infty$ is stated in Theorem 4 of [3], and it gives the equivalence of (i) and (iii).

Next, $w_\delta(x) := l - \mathrm{val}\, \int_0^\infty l(y_x(t), a(t), b(t))\, e^{-\delta t}\, dt$ is the viscosity solution of the Isaacs equation

$$\delta w_\delta + H(y, Dw_\delta) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta\ \mathbb{Z}^m\text{-periodic,}$$

and Theorem 4 of [3] states the equivalence of (iii) and the uniform convergence of δw_δ to a constant as $\delta \to 0+$. Therefore (ii) and (iii) are equivalent.

The equality of the three constants is also given by Theorem 4 of [3]. Finally, the proof for the upper value is the same, with the Hamiltonian $H = \min\max$ replaced by $\max\min$. □

Remark 2.1. Note that (ii) deals with a vanishing discount rate problem for infinite horizon games. The equivalence between (i) and (ii) is a differential game extension of the classical Abelian-Tauberian theorem, stating that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(t)\, dt = \lim_{\delta\to 0+} \delta \int_0^\infty \varphi(t)\ e^{-\delta t}\, dt$$

whenever one of the two limits exists. The property (iii) is a characterization of the uniform ergodicity of the lower game by a Hamilton-Jacobi-Isaacs equation. In some cases the inf and the sup in the formula (10) are attained and the number defined by (10) is the unique constant λ such that the additive eigenvalue problem (9) has a continuous viscosity solution, see Remark 3.1. In general, however, even if (iii) holds, (9) may have no continuous solution χ (see Arisawa, Lions [7]). By analogy with the theory of homogenization we call (9) the *cell problem.*

Whenever the conditions of Corollary 1.1 for the ergodicity of the LTAC game are satisfied, we have the following informations on the value of the game, namely, the constant λ.

Proposition 2.1. *Assume* (7) *and that either the lower or the upper game is ergodic. Then* $\lambda = l - \mathrm{val}\, J_\infty(x) = u - \mathrm{val}\, J^\infty(x)$ *satisfies*

$$\min_x \min_{a\in A} \max_{b\in B} l(x,a,b) \le \lambda \le \max_x \min_{a\in A} \max_{b\in B} l(x,a,b).$$

If, moreover,

$$\max_{a\in A} \min_{b\in B} \{-f(x,a,b)\cdot p - l(x,a,b)\} \ge \max_{a\in A} \min_{b\in B}\{-l(x,a,b)\} \quad \forall x, p \in \mathbb{R}^m, \tag{11}$$

(respectively, $\le$), then

$$\lambda = \min_x \min_{a\in A} \max_{b\in B} l(x,a,b), \tag{12}$$

(respectively, $\lambda = \max_x \min_a \max_b l(x,a,b)$*).*

Proof. First we use the characterization (i) of ergodicity in Theorem 2.1, and we set $v(t,y) := l - \mathrm{val}\, J(t,y)$, $w(t,y) := tv(t,y)$. It is well known [17,9] that w satisfies, in the viscosity sense,

$$w_t + H(y, D_y w) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m, \qquad w(0,y) = 0, \quad w \text{ periodic.}$$

We observe that $-t\max_y H(y,0)$ and $-t\min_y H(y,0)$ are, respectively, a sub- and a supersolution of this Cauchy problem. Therefore the comparison principle gives

$$-t\max_y H(y,0) \le w(t,y) \le -t\min_y H(y,0).$$

We divide by t and let $t \to +\infty$. Since $w(t,y)/t \to \lambda$, and $H(y,0) = \max_a \min_b\{-l(x,a,b)\}$ by (7), we get the first pair of inequalities.

To prove the second statement we assume by contradiction that $\lambda > -H(y,0)$ in a neighborhood of a minimum point of $-H(y,0) = \min_a \max_b l(x,a,b)$. Now we use the characterization (ii) of ergodicity in

Theorem 2.1, as in the proof of Theorem 2.2. With the same notations, the value function w_δ of the infinite horizon discounted game satisfies the Isaacs equation (19). By the uniform convergence of δw_δ to λ we get

$$H(y, Dw_\delta) - H(y, 0) = -\lambda - H(y, 0) + o(1) < 0 \qquad \text{as } \delta \to 0$$

in an open set. This is a contradiction with the assumption (11). □

Remark 2.2. Note that, for a running cost independent of the controls, $l = l(x)$, the condition (11) reads

$$\min_a \max_b f(x, a, b) \cdot p = \max_b \min_a f(x, a, b) \cdot p \leq 0, \tag{13}$$

in view of (7). This says that the first player has a stronger control on the vector field than the second one. The conclusion is that the LTAC value is

$$\lambda = \min_x l(x),$$

so the minimizing player can drive asymptotically the system near the minimum points of the running cost.

Next we describe some sufficient conditions for the ergodicity of the upper or the lower game. We say that the system (1) is *bounded-time controllable by the first player* if there exists $S > 0$ and for each $x, \widetilde{x} \in \mathbb{R}^m$ there exists a strategy $\widetilde{\alpha} \in \Gamma$ such that for all control functions $b \in \mathcal{B}$ there is a time $t^\# = t^\#(x, \widetilde{x}, \widetilde{\alpha}, b)$ with the properties

$$t^\# \leq S \text{ and } y_x(t^\#) - \widetilde{x} \in \mathbb{Z}^m, \tag{14}$$

where $y_x(\cdot)$ is the trajectory corresponding to the strategy $\widetilde{\alpha}$ and the control function b, i.e., it solves

$$\dot{y}(t) = f(y(t), \widetilde{\alpha}[b](t), b(t)), \qquad y(0) = x. \tag{15}$$

In other words, the first player can drive the system from any initial position x to any given state $\widetilde{x}$ on the torus $\mathbb{T}^m$ in a uniformly bounded time for all possible behaviors of the second player. Symmetrically, we say that he system (1) is *bounded-time controllable by the second player* if for some $S > 0$ and for all $x, \widetilde{x} \in \mathbb{R}^m$ there is a strategy $\widetilde{\beta} \in \Delta$ such that for all control functions $a \in \mathcal{A}$

$$\exists\ t^\# = t^\#(x, \widetilde{x}, a, \widetilde{\beta}) \leq S \text{ such that } y_x(t^\#) - \widetilde{x} \in \mathbb{Z}^m,$$

where $y_x(\cdot)$ is the trajectory corresponding to the strategy $\widetilde{\beta}$ and the control function a, i.e., it solves

$$\dot{y}(t) = f(y(t), a(t), \widetilde{\beta}[a](t)), \qquad y(0) = x.$$

For systems with a single player this notion is studied in the literature under various names such as *complete controllability* [14], *uniform exact controllability* [6], and *total controllability* [8].

Theorem 2.2. *If the system* (1) *is bounded-time controllable by the first player (respectively, by the second player), then the lower game (respectively, the upper game) is ergodic.*

Proof. The proof uses the characterization (ii) of ergodicity in Theorem 2.1, and we call w_δ the lower value function of the discounted infinite horizon problem, namely,

$$w_\delta(x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \int_0^\infty l(y_x(t), \alpha[b](t), b(t))\, e^{-\delta t}\, dt.$$

The main tool of the proof is the following Dynamic Programming Principle due to Soravia, Remark 4.2 of [33],

$$w_\delta(x) = \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \inf_{0\le t<\infty} \left\{ \int_0^t l(y(s), \alpha[b](s), b(s)) e^{-\delta s}\, ds + e^{-\delta t} w_\delta(y(t)) \right\}, \tag{16}$$

where $y(\cdot)$ is the trajectory of (15) with $\widetilde{\alpha}$ replaced by a generic α.

For fixed $x, \widetilde{x}$ we take a strategy $\widetilde{\alpha} \in \Gamma$ such that (14) holds. Then (16) and the periodicity of w_δ give

$$w_\delta(x) \le \sup_{b\in\mathcal{B}} \left\{ \int_0^{t^\#} l(y(s), \alpha[b](s), b(s)) e^{-\delta s}\, ds + e^{-\delta t^\#} w_\delta(\widetilde{x}) \right\},$$

where $y(\cdot)$ is the trajectory of (15). Since l and δw_δ are uniformly bounded, there is a constant C such that

$$\delta w_\delta(x) - \delta w_\delta(\widetilde{x}) \le C(1 - e^{-\delta S}). \tag{17}$$

Now we exchange the roles of x and $\widetilde{x}$ to get

$$\lim_{\delta\to 0+} |\delta w_\delta(x) - \delta w_\delta(\widetilde{x})| = 0 \quad \text{uniformly in } x, \widetilde{x} \in \mathbb{R}^m.$$

If for fixed $\widetilde{x}$ we choose a sequence $\delta_k \to 0$ such that $\delta_k w_{\delta_k}(\widetilde{x}) \to \mu$, we obtain the uniform convergence of $\delta_k w_{\delta_k}$ to μ.

We claim that μ is independent of the sequence δ_k. This implies the uniform convergence of the whole net δw_δ to μ, as desired. To prove the claim we recall the cell problem (9), i.e.,

$$\lambda + H(y, D\chi) = 0 \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic}, \tag{18}$$

where λ is a constant, and use the inequality

$$\lambda_1 := \sup\{\lambda \mid \exists \text{ a u.s.c. subsolution of (18)}\}$$
$$\le \lambda_2 := \inf\{\lambda \mid \exists \text{ a l.s.c. supersolution of (18)}\},$$

which follows from a standard argument based on the comparison principle for sub- and supersolutions of Hamilton-Jacobi equations (see, e.g., the proof of Theorem 1 in [6] or that of Theorem 4 in [3]). The Isaacs equation satisfied by w_δ in viscosity sense is (see, e.g., [9])

$$\delta w_\delta + H(y, Dw_\delta) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic.} \tag{19}$$

Then, for $\lambda < \mu$, w_{δ_k} is a subsolution of (18) for k large enough, so $\mu \leq \lambda_1$. The same argument gives $\lambda_2 \leq \mu$. Therefore $\mu = \lambda_1 = \lambda_2$, which proves the claim. □

An immediate consequence of this theorem and of Corollary 1.1 is the following.

Corollary 2.1. *Assume the Isaacs' condition* (7) *and that the system* (1) *is bounded-time controllable either by the first or by the second player. Then the LTAC game is ergodic, i.e.,*

$$l - \mathrm{val}\, J_\infty(x) = u - \mathrm{val}\, J^\infty(x) = \lim_{T\to\infty} \mathrm{val}\, J(T,x) = \lambda, \quad \forall x \in \mathbb{R}^m.$$

We end this section with some sufficient conditions for ergodicity that are symmetric for the two players, different from the preceding Theorem 2.2 and Proposition 2.1 where one of the two players have a much stronger hold of the system than the other. We take a system of the form

$$\begin{cases} \dot y^A(t) = f_A(y(t), a(t), b(t)), & y^A(0) = x^A \in \mathbb{R}^{m_A}, \\ \dot y^B(t) = f_B(y(t), a(t), b(t)), & y^B(0) = x^B \in \mathbb{R}^{m_B}, \\ y(t) = (y^A(t), y^B(t)), \end{cases} \tag{20}$$

and we assume that the state variables y^A are *(uniformly) asymptotically controllable* by the first player, whereas the variables y^B are asymptotically controllable by the second, in the following sense. There exists a function $\eta : [0,\infty) \to [0,\infty)$ with

$$\lim_{T\to\infty} \eta(T) = 0, \tag{21}$$

and for all $x^A, \widetilde{x}^A \in \mathbb{R}^{m_A}$, $x^B \in \mathbb{R}^{m_B}$, there is a strategy $\widetilde{\alpha} \in \Gamma$ such that, for $x = (x^A, x^B)$,

$$\frac{1}{T}\int_0^T \min_{k^A \in \mathbb{Z}^{m_A}} |y_x^A(t) - \widetilde{x}^A - k^A|\, dt \leq \eta(T), \ \forall b \in \mathcal{B}, \tag{22}$$

whereas for all $x^B, \widetilde{x}^B \in \mathbb{R}^{m_B}$, $x^A \in \mathbb{R}^{m_A}$, there is a strategy $\widetilde{\beta} \in \Delta$ such that

$$\frac{1}{T}\int_0^T \min_{k^B \in \mathbb{Z}^{m_B}} |y_x^B(t) - \widetilde{x}^B - k^B|\, dt \leq \eta(T), \ \forall a \in \mathcal{A}. \tag{23}$$

Note that the integrand in (22) is the distance between $y_x^A(t)$ and $\widetilde{x}^A$ on the m_A-dimensional torus $\mathbb{T}^{m_A} = \mathbb{R}^{m_A}/\mathbb{Z}^{m_A}$, so (22) and (21) mean that the first player can drive asymptotically y^A near $\widetilde{x}^A$, uniformly with respect to x, $\widetilde{x}^A$, and the control of the other player b. Similarly, (23) says that the second player can drive asymptotically y^B to $\widetilde{x}^B$ on the m_B-dimensional torus $\mathbb{T}^{m_B}$, uniformly with respect to x, $\widetilde{x}^B$, and a.

We will also assume that the running cost does not depend on the controls, $l = l(y^A, y^B)$, and it has a saddle point in $[0,1]^{m_A} \times [0,1]^{m_B}$, that is, it satisfies

$$\min_{y^A \in \mathbb{R}^{m_A}} \max_{y^B \in \mathbb{R}^{m_B}} l(y^A, y^B) = \max_{y^B \in \mathbb{R}^{m_B}} \min_{y^A \in \mathbb{R}^{m_A}} l(y^A, y^B) =: \bar{l}. \tag{24}$$

Proposition 2.2. *Assume the system* (1) *is of the form* (20) *with* y^A *and* y^B *asymptotically controllable, respectively, by the first and by the second player. Suppose also that* $l = l(y^A, y^B)$ *satisfies* (24) *and* (7) *holds. Then the LTAC game is ergodic and its value is the value of the static game with payoff* l*, that is,*

$$l - \mathrm{val}\, J_\infty(x^A, x^B) = u - \mathrm{val}\, J^\infty(x^A, x^B) = \bar{l}, \quad \forall (x^A, x^B) \in \mathbb{R}^m. \tag{25}$$

Proof. We first prove that, for all $x = (x^A, x^B)$,

$$\limsup_{T \to \infty} l - \mathrm{val}\, J(T, x) \le \min_{y^A \in \mathbb{R}^{m_A}} \max_{y^B \in \mathbb{R}^{m_B}} l(y^A, y^B) \quad \text{uniformly in } x \in \mathbb{R}^m. \tag{26}$$

To this goal we fix $x^A, \widetilde{x}^A, x^B$ and consider the strategy $\widetilde{\alpha}$ from the asymptotic controllability assumption. If $y_x(\cdot) = y_x(\cdot, b)$ is the corresponding trajectory, from the periodicity of l we get

$$|l(y_x^A(t), y_x^B(t)) - l(\widetilde{x}^A, y_x^B(t))| \le \omega_l(\min_{k^A \in \mathbb{Z}^{m_A}} |y_x^A(t) - \widetilde{x}^A - k^A|),$$

where ω_l is the modulus of continuity of l, i.e.,

$$|l(x) - l(y)| \le \omega_l(|x - y|), \forall x, y \in \mathbb{R}^m, \quad \lim_{r \to 0} \omega_l(r) = 0.$$

We recall that it is not restrictive to assume the concavity of ω_l, so Jensen's inequality implies

$$\frac{1}{T} \int_0^T |l(y_x^A(t), y_x^B(t)) - l(\widetilde{x}^A, y_x^B(t))|\, dt \le \omega_l(\eta(T)).$$

Then

$$\begin{aligned} l - \mathrm{val}\, J(T, x) &\le \sup_{b \in \mathcal{B}} \frac{1}{T} \int_0^T l(y_x^A(t), y_x^B(t))\, dt \\ &\le \sup_{b \in \mathcal{B}} \frac{1}{T} \int_0^T l(\widetilde{x}^A, y_x^B(t))\, dt + \omega_l(\eta(T)) \\ &\le \max_{y^B \in \mathbb{R}^{m_B}} l(\widetilde{x}^A, y^B) + \omega_l(\eta(T)). \end{aligned}$$

Now we take the $\limsup_{T\to\infty}$ of both sides and finally the min over $\widetilde{x}^A \in \mathbb{R}^{m_A}$ to get (26).

Next we observe that a symmetric proof gives

$$\liminf_{T\to\infty} u - \operatorname{val} J(T,x) \geq \max_{y^B\in\mathbb{R}^{m_B}} \min_{y^A\in\mathbb{R}^{m_A}} l(y^A,y^B) \quad \text{uniformly in } x\in\mathbb{R}^m.$$

By the Isaacs condition (7) $l - \operatorname{val} J(T,x) = u - \operatorname{val} J(T,x)$, so the assumptions (24) on l gives

$$\lim_{T\to\infty} \operatorname{val} J(T,x) = \bar{l} \quad \forall\, x\in\mathbb{R}^m.$$

The conclusion now follows from Theorem 1.1. □

Remark 2.3. If the system governing y^A is bounded-time controllable by the first player and also *stoppable*, i.e.,

$$\forall x\in\mathbb{R}^m,\ \forall b\in B,\ \exists a\in A\ :\ f_A(x,a,b)=0,$$

then the variables y^A are asymptotically controllable, because $\widetilde{x}^A$ can be reached from x^A in a time smaller than S and then the first player can keep $y^A(t)=\widetilde{x}^A$ for all later times t. In this case, if $\bar{l}=l(\widetilde{x}^A,\widetilde{x}^B)$, an optimal strategy for the first player amounts to driving the variables y^A to the saddle point $\widetilde{x}^A$ and stopping there, and the strategy of going to $\widetilde{x}^B$ and staying there forever is optimal for the second player. This kind of behavior is called a *turnpike*, see [13,12].

3 Examples

Example 1: first order controllability. Assume that for some $\nu>0$

$$B(0,\nu;m)\subset \overline{\operatorname{conv}}\{f(x,a,b)\mid a\in A\},\quad \forall x\in\mathbb{R}^m, b\in B, \tag{27}$$

where $B(0,\nu;m)$ denotes the m-dimensional open ball of radius ν centered at the origin and $\overline{\operatorname{conv}}$ the closed convex hull. From the standard theory of differential games (see, for instance, Corollary 3.7 in [32]) it is known that the system is (small-time) controllable by the first player and the time necessary to reach a point $\widetilde{x}$ from x satisfies an estimate of the form

$$t^{\#}(x,\widetilde{x},\widetilde{\alpha},b)\leq \frac{C}{\nu}|x-\widetilde{x}|.$$

Therefore the lower game is uniformly ergodic in this case. Moreover, if $l=l(x)$ it is easy to see that (13) holds, so $\lambda=\min_x l(x)$.

Example 2: higher order controllability. Consider a system of the form

$$\dot{y}(t) = \sum_{i=1}^{k-1} a_i(t) g^i(y(t)) + a_k(t) g^k(y(t), a(t), b(t)), \tag{28}$$

where the control of the first player $a = (a_1, \ldots, a_k)$ varies in a neighborhood of the origin $A \subset \mathbb{R}^k$, and all g^i with $i \leq k-1$ are C^∞ vector field in $\mathbb{R}^m$. Moreover, we suppose the full rank (Hörmander) condition on $g^1, \ldots, g^{k-1}$, that is, *the vector fields* $g^1, \ldots, g^{k-1}$ *and their commutators of any order span* $\mathbb{R}^m$ *at each point of* $\mathbb{R}^m$. By choosing $a_k \equiv 0$ we obtain a symmetric system independent of the second player. Then the classical Chow's theorem of geometric control theory says that this system is small-time locally controllable at all points of the state space. Moreover, for any small $t > 0$ the reachable set from x in time t is a neighborhood of x, and the same holds for the reachable set backward in time. From this, using the compactness of the torus $\mathbb{T}^m$, it is easy to see that the whole state space is an invariant control set in the terminology of [14]. Then the global bounded-time controllability follows from Lemma 3.2.21 in [14]. In conclusion, the full system (28) is bounded-time controllable by the first player and therefore the lower game is uniformly ergodic. As in the previous example, if λ is independent of the controls (13) holds and $\lambda = \min_x l(x)$.

Remark 3.1. If (14) holds with $t^\#(x, \widetilde{x}, \widetilde{\alpha}, b) \leq \omega(|x - \widetilde{x}|)$ for all $|x - \widetilde{x}| \leq \gamma$ and all $b \in \mathcal{B}$, for some modulus ω and $\gamma > 0$, we say that (1) is also *small-time controllable* by the first player. For such systems there exists a continuous solution χ to the additive eigenvalue problem (9), by Proposition 9.2 in [4]. The systems of the Examples 1 and 2 are indeed small-time controllable by the first player.

Example 3: separate controllability. For a system of the form (20) we can assume that the subsystem for the variables y^A either satisfies

$$B(0, \nu; m_A) \subset \overline{\text{conv}} f_A(x, A, b) \quad \forall x \in \mathbb{R}^m, b \in B,$$

or it is of the form

$$\dot{y}^A = \sum_{i=1}^{k_A - 1} a_i g_A^i(y^A) + a_{k_A} g_A^{k_A}(y, a, b),$$

where the control of the first player $a = (a_1, \ldots, a_{k_A})$ varies in a neighborhood of the origin $A \subset \mathbb{R}^{k_A}$, and the vector fields $g_A^1, \ldots, g_A^{k_A - 1}$ are of class C^∞ and satisfy the full rank condition in $\mathbb{R}^{m_A}$. Then the variables y^A are asymptotically controllable because the first player can drive them from x^A to $\widetilde{x}^A$ in bounded time and then stop there by choosing the null control. Similarly, the variables y^B are asymptotically controllable if either f_B verifies

$$B(0,\nu;m_B)\subset \overline{\text{conv}} f_B(x,a,B) \quad \forall x\in\mathbb{R}^m, a\in A,$$

or it is of the form

$$\dot{y}^B=\sum_{i=1}^{k_B-1} b_i g_B^i(y^B)+b_{k_B} g_B^{k_B}(y,a,b),$$

where the control of the second player $b=(b_1,\dots,b_{k_B})$ varies in a neighborhood of the origin $B\subset\mathbb{R}^{k_B}$, and the vector fields $g_B^1,\dots,g_B^{k_B-1}$ are of class C^∞ and satisfy the full rank condition in $\mathbb{R}^{m_B}$. Under these condition and with $l=l(y^A,y^B)$ satisfying (24) Proposition 2.2 implies the ergodicity of the LTAC game and the formula (25).

4 Ergodicity of noisy systems

In this section we study the ergodicity of the lower value for the following class of stochastic differential games. We consider the controlled diffusion process

$$dy(t)=f(y(t),a(t),b(t))dt+\sigma(y(t),a(t),b(t))dW(t), \quad y(0)=x, \qquad (29)$$

where W is an r-dimensional Brownian motion, and σ is a continuous map from $\mathbb{R}^m\times A\times B$ to the space of $m\times r$ matrices, lipschitzean in x uniformly in a,b. The finite horizon cost functional is

$$J(T,x)=J(T,x,a(\cdot),b(\cdot)):=E\left[\frac{1}{T}\int_0^T l(y_x(t),a(t),b(t))\,dt\right],$$

where E denotes the expectation. An admissible control for the first (respectively, second) player is a progressively measurable function of time taking values in A (respectively, B), and we still denote with $\mathcal{A}$ and $\mathcal{B}$ the sets of admissible controls. We also keep the notation Γ and Δ for the set of nonanticipating strategies for the first player and the second player, respectively, and we refer to [19] for the precise definitions in the stochastic setting. The lower value of the finite horizon game is

$$v(T,x):=l-\text{val}\,J(T,x):=\inf_{\alpha\in\Gamma}\sup_{b\in\mathcal{B}} J(T,x,\alpha[b],b)$$

and we say that *the lower game is ergodic* if

$$l-\text{val}\,J(T,\cdot)\to\lambda \quad \text{as } T\to\infty \text{ uniformly in } \mathbb{R}^m.$$

In addition to the periodicity in the state variable of f and l (8), we assume

$$\sigma(y,a,b) = \sigma(y+k,a,b), \quad \forall\, k \in \mathbb{Z}^m,\ y \in \mathbb{R}^m,\ a \in A,\ b \in B.$$

We are going to extend all the results of Section 2 to this setting, and we also present a theorem where the ergodicity is due to the effects of the diffusion without any controllability hypothesis. Analogous results hold for the upper game, which is defined in the obvious way, but we will not state them explicitly.

We begin with the stochastic counterpart of the Abelian-Tauberian-type Theorem 2.1, that is again a consequence of Theorem 4 in [3]. We will use the second order Hamiltonian

$$H(y,p,X) := \min_{b\in B}\max_{a\in A}\left\{-\frac{1}{2}\mathrm{trace}\left(\sigma\sigma^T(y,a,b)X\right) - f(y,a,b)\cdot p - l(y,a,b)\right\}$$

for $y, p \in \mathbb{R}^m$ and X any symmetric $m \times m$ matrix.

Theorem 4.1. *The following statements are equivalent.*
(i) *The lower game is ergodic, i.e.,* $v(T,x) \to \mathrm{const}$ *uniformly in* x *as* $T \to +\infty$.
(ii) $l - \mathrm{val}\, E[\delta \int_0^\infty l(y_x(t),a(t),b(t))\, e^{-\delta t}\, dt] \to \mathrm{const}$ *uniformly in* x *as* $\delta \to 0+$.
(iii) *The cell problem*

$$\lambda + H(y, \nabla\chi, D^2\chi) = 0 \quad \textit{in } \mathbb{R}^m, \quad \chi\ \mathbb{Z}^m\textit{-periodic} \tag{30}$$

has the property that

$$\sup\{\lambda \mid \textit{there is a viscosity subsolution of } (30)\} = \inf\{\lambda \mid \textit{there is a viscosity supersolution of } (30)\}. \tag{31}$$

If one of the above assertions is true, then the constants in (i) *and* (ii) *are equal and they coincide with the number defined by* (31).

Proof. The proof is the same as that of Theorem 2.1 after recalling that $w(t,y) := tv(t,y)$ is the viscosity solution of

$$w_t + H(y, D_y w, D^2_{yy} w) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m, \qquad w(0,y) = 0, \tag{32}$$

and $w_\delta(x) := l - \mathrm{val}\, E\left[\int_0^\infty l(y_x(t),a(t),b(t))\, e^{-\delta t}\, dt\right]$ is the viscosity solution of

$$\delta w_\delta + H(y, Dw_\delta, D^2 w_\delta) = 0 \quad \text{in } \mathbb{R}^m, \tag{33}$$

see [19]. □

Proposition 4.1. *Assume the lower game is ergodic. Then* $\lambda = \lim_{T\to\infty} v(T,x)$ *satisfies*

$$\min_x \max_{b\in B} \min_{a\in A} l(x,a,b) \le \lambda \le \max_x \max_{b\in B} \min_{a\in A} l(x,a,b).$$

If, moreover,

$$\min_{b\in B} \max_{a\in A} \left\{ -\frac{1}{2} trace \left(\sigma\sigma^T(y,a,b)X\right) - f(x,a,b)\cdot p - l(x,a,b) \right\} \ge \min_{b\in B} \max_{a\in A} \{-l(x,a,b)\} \quad \forall x,p,X,$$

(respectively, $\le$*), then*

$$\lambda = \min_x \max_{b\in B} \min_{a\in A} l(x,a,b), \tag{34}$$

(respectively, $\lambda = \max_x \max_b \min_a l(x,a,b)$*).*

Proof. The proof is the same as that of Proposition 2.1, after observing that

$$-H(y,0,0) = \max_{b\in B} \min_{a\in A} l(x,a,b).$$

□

It is well known that nondegenerate diffusion processes are ergodic (in the standard sense). The next result states the ergodicity of games involving a controlled system affected by a nondegenerate diffusion, with no controllability assumptions. It is a consequence of Theorem 7.1 in our paper [4], see also [3]. Earlier related results are due to Evans [15] and Arisawa and Lions [7].

Theorem 4.2. *Assume that the minimal eigenvalue of the matrix* $\sigma\sigma^T(y,a,b)$ *is positive for all* $y \in \mathbb{R}^m$, $a \in A$, $b \in B$. *Then the lower game is ergodic.*

Proof. The periodicity in y of σ and the compactness of A and B imply that the minimal eigenvalue of the matrix $\sigma\sigma^T$ is bounded below by a positive constant. Therefore the second order partial differential operator associated with the Hamiltonian H is uniformly elliptic. Then Theorem 7.1 in [4] shows that there exists a (unique) constant λ such that the cell problem (30) has a continuous viscosity solution χ. Then $u(t,y) := \lambda t + \chi(y)$ solves

$$u_t + H(y, D_y u, D^2_{yy} u) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^m, \qquad u(0,y) = \chi(y).$$

Since $w(t,y) := tv(t,y)$ solves (32), by the comparison principle for viscosity solutions of Cauchy problems we get

$$u(t,y) - \max\chi \le tv(t,y) \le u(t,y) + \min\chi \quad \forall t,y.$$

Therefore $v(t,\cdot) \to \lambda$ uniformly as $t \to \infty$. □

The next result is an extension to stochastic games of Theorem 2.2. We say that the system (29) is *bounded-time controllable by the first player* if for some $S > 0$ and for all $x, \widetilde{x} \in \mathbb{R}^m$ there is a strategy $\widetilde{\alpha} \in \Gamma$ such that for all admissible control functions $b \in \mathcal{B}$

$$\exists\, t^{\#} = t^{\#}(x, \widetilde{x}, \widetilde{\alpha}, b) \leq S \text{ such that } y_x(t^{\#}) - \widetilde{x} \in \mathbb{Z}^m \text{ almost surely,} \quad (35)$$

where $y_x(\cdot)$ is the solution of (29) corresponding to the controls $\widetilde{\alpha}[b]$ and b.

Theorem 4.3. *If the system* (29) *is bounded-time controllable by the first player, then the lower game is ergodic.*

Proof. We follow the argument of the proof of Theorem 2.2. We use the characterization (ii) of ergodicity in Theorem 4.1, and we call w_δ the lower value function of the discounted infinite horizon problem, namely,

$$w_\delta(x) := \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} E \left[\int_0^\infty l(y_x(t), \alpha[b](t), b(t))\, e^{-\delta t}\, dt \right].$$

The main tool of the proof is the following Dynamic Programming Principle for stochastic games due to Swiech, Corollary 2.6 (iii) of [37],

$$w_\delta(x) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \inf_{0 \leq t < \infty} E \left\{ \int_0^t l(y_x(s), \alpha[b](s), b(s)) e^{-\delta s}\, ds + e^{-\delta t} w_\delta(y_x(t)) \right\}. \quad (36)$$

For fixed $x, \widetilde{x}$ we take a strategy $\widetilde{\alpha} \in \Gamma$ such that (35) holds. Then (36) and the periodicity of w_δ give

$$w_\delta(x) \leq \sup_{b \in \mathcal{B}} E \left\{ \int_0^{t^{\#}} l(y_x(s), \widetilde{\alpha}[b](s), b(s)) e^{-\delta s}\, ds + e^{-\delta t^{\#}} w_\delta(\widetilde{x}) \right\},$$

where $y_x(\cdot)$ is the trajectory of (29) with the controls $\widetilde{\alpha}[b]$ and b. Now the proof of Theorem 2.2 shows that, along a sequence $\delta_k \to 0$, $\delta_k w_{\delta_k} \to \mu$ uniformly.

Finally, the proof that μ dos not depend on the sequence δ_k is also the same as in Theorem 2.2, with the new cell problem (30) and using the equation (33) satisfied by w_δ. □

The last result of the section is a stochastic counterpart of Proposition 2.2. We take a controlled diffusion of the form

$$\begin{cases} dy^A(t) = f_A(y(t), a(t), b(t))dt + \sigma_A(y(t), a(t), b(t))dW_A(t), \\ dy^B(t) = f_B(y(t), a(t), b(t))dt + \sigma_B(y(t), a(t), b(t))dW_B(t), \\ y(t) = (y^A(t), y^B(t)),\ y^A(0) = x^A \in \mathbb{R}^{m_A},\ y^B(0) = x^B \in \mathbb{R}^{m_B}, \end{cases} \quad (37)$$

and we assume that the state variables y^A are *asymptoticallly controllable* by the first player, and the variables y^B are asymptoticallly controllable by the second, in the following sense. There exists a function $\eta : [0,\infty) \to [0,\infty)$ with $\lim_{T\to\infty} \eta(T) = 0$, and for all $x^A, \widetilde{x}^A \in \mathbb{R}^{m_A}$, $x^B \in \mathbb{R}^{m_B}$, there is a strategy $\widetilde{\alpha} \in \Gamma$, such that, for $x = (x^A, x^B)$,

$$E\left[\frac{1}{T}\int_0^T \min_{k^A\in\mathbb{Z}^{m_A}} |y_x^A(t) - \widetilde{x}^A - k^A|\, dt\right] \le \eta(T),\ \forall b \in \mathcal{B}, \tag{38}$$

whereas for all $x^B, \widetilde{x}^B \in \mathbb{R}^{m_B}$, $x^A \in \mathbb{R}^{m_A}$, there is a strategy $\widetilde{\beta} \in \Delta$ such that

$$E\left[\frac{1}{T}\int_0^T \min_{k^B\in\mathbb{Z}^{m_B}} |y_x^B(t) - \widetilde{x}^B - k^B|\, dt\right] \le \eta(T),\ \forall a \in \mathcal{A}.$$

As in Section 2 we assume that the running cost does not depend on the controls and has a saddle point. The Isaacs condition now is

$$\min_{b\in B}\max_{a\in A}\left\{-\frac{1}{2}\text{trace}\left(\sigma\sigma^T(y,a,b)X\right) - f(y,a,b)\cdot p - l(y,a,b)\right\} = \\ \max_{a\in A}\min_{b\in B}\left\{-\frac{1}{2}\text{trace}\left(\sigma\sigma^T(y,a,b)X\right) - f(y,a,b)\cdot p - l(y,a,b)\right\}. \tag{39}$$

Proposition 4.2. *Assume the system* (29) *is of the form* (37) *with* y^A *and* y^B *asymptoticallly controllable, respectively, by the first and by the second player. Suppose also* $l = l(y^A, y^B)$ *satisfies* (24) *and* (39) *holds. Then the lower game is ergodic and its value converges to the value of the static game with payoff* l*, that is,*

$$\lim_{T\to\infty} v(T, x^A, x^B) = \bar{l}, \quad \textit{uniformly in } (x^A, x^B) \in \mathbb{R}^m. \tag{40}$$

Proof. The proof is essentially the same as that of Proposition 2.2. We begin with

$$\limsup_{T\to\infty} v(T,x) \le \min_{y^A\in\mathbb{R}^{m_A}} \max_{y^B\in\mathbb{R}^{m_B}} l(y^A, y^B) \quad \text{uniformly in } x \in \mathbb{R}^m. \tag{41}$$

We fix $x^A, \widetilde{x}^A$, x^B and consider the strategy $\widetilde{\alpha}$ from the asymptotic controllability assumption. If $y_x(\cdot)$ is the corresponding trajectory, then, by the periodicity of l, (38), and the concavity of ω_l,

$$E\left[\frac{1}{T}\int_0^T |l(y_x^A(t), y_x^B(t)) - l(\widetilde{x}^A, y_x^B(t))|\, dt\right] \le \omega_l(\eta(T)).$$

Then

$$l - \mathrm{val}\, J(T,x) \le \sup_{b\in\mathcal{B}} E\left[\frac{1}{T}\int_0^T l(y_x^A(t), y_x^B(t))\, dt\right]$$
$$\le \max_{y^B\in\mathbb{R}^{m_B}} l(\widetilde{x}^A, y^B) + \omega_l(\eta(T)).$$

Now we take the $\limsup_{T\to\infty}$ of both sides and finally the min over $\widetilde{x}^A \in \mathbb{R}^{m_A}$ to get (41). A symmetric proof gives

$$\liminf_{T\to\infty} u - \mathrm{val}\, J(T,x) \ge \max_{y^B\in\mathbb{R}^{m_B}} \min_{y^A\in\mathbb{R}^{m_A}} l(y^A, y^B) \quad \text{uniformly in } x \in \mathbb{R}^m,$$

where $u - \mathrm{val}\, J(T,x)$ denotes the upper value of the finite horizon game. By the Isaacs condition (39), the results of Fleming and Souganidis [19] imply $v(T,x) = l - \mathrm{val}\, J(T,x) = u - \mathrm{val}\, J(T,x)$ for all T, x, so the assumption (24) on l gives (40). □

Remark 4.1. The proof of Proposition 4.2 shows also that the upper game is ergodic and the upper value $u - \mathrm{val}\, J(T,x)$ converges uniformly to the saddle $\bar{l}$ as $T \to \infty$.

Remark 4.2. If the system governing y^A is bounded-time controllable by the first player and also *stoppable*, i.e.,

$$\forall x \in \mathbb{R}^m,\ \forall b \in B,\ \exists a \in A\, :\, f_A(x,a,b) = 0,\ \sigma_A(x,a,b) = 0,$$

then the variables y^A are asymptotically controllable, because $\widetilde{x}^A$ can be reached from x^A in a time smaller than S and then the first player can keep $y^A(t) = \widetilde{x}^A$ for all later times t. As in the deterministic case, an optimal strategy for each player is a turnpike: driving the system to a saddle point and stopping there.

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