ON THE PROBLEM OF OPTIMAL CUTTING

DORIN BUCUR, GIUSEPPE BUTTAZZO, AND NICOLAS VARCHON

ABSTRACT. The paper deals with the existence of an optimal cutting in a membrane satisfying the following assumptions: it has to connect two given points in order to leave the membrane the strongest possible. We prove the existence of a solution for this problem in a rather general setting and we present some open questions related to the regularity of the optimum or to possible extensions for plates in the elasticity framework.

1. INTRODUCTION

The main purpose of this paper is to raise the question of existence and regularity of an optimal "cut" in a membrane with the only constraint that the cut has to connect two or more, a priori given, points and to leave the membrane the strongest possible. More precisely, the problem, illustrated in Figure 1, can be modelized as follows.

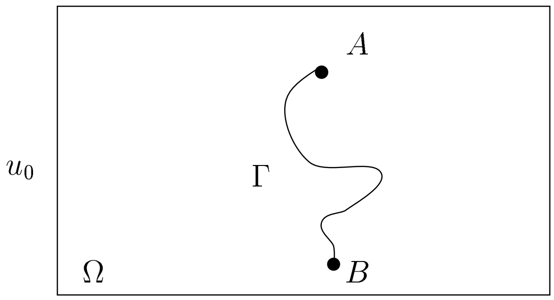

Figure 1. An admissible cut Γ in the membrane Ω.

Notice that in general admissible cuts Γ do not need to be curves; for instance if we want to connect three points we have to expect the optimal cut has a triple junction shape.

Let Ω be a two-dimensional bounded open set with a smooth boundary (say the rectangle in the figure above), $u_0 \in H^1(\Omega)$, A, B two given points in Ω. An admissible cut in Ω will simply be a compact connected subset

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Γ of $\overline{\Omega}$ containing the points A and B, and we denote by $\mathcal{U}_{ad}$ the class of admissible cuts, that is

$$\mathcal{U}_{ad} = \big\{\Gamma \subseteq \overline{\Omega} \ :\ A, B \in \Gamma,\ \Gamma \text{ is compact and connected}\big\}.$$

For every $\Gamma \in \mathcal{U}_{ad}$ the energy $\mathcal{E}(\Gamma)$ associated to Γ will be

$$\mathcal{E}(\Gamma) = \min\Big\{\int_{\Omega\setminus\Gamma} |\nabla u|^2\,dx \ :\ u \in H^1_{loc}(\Omega\setminus\Gamma),\ u = u_0 \text{ on } \partial\Omega\Big\},$$

so that the optimization problem we deal with can be written as

$$\max\big\{\mathcal{E}(\Gamma) \ :\ \Gamma \in \mathcal{U}_{ad}\big\}. \tag{1.1}$$

For fixed $\Gamma \in \mathcal{U}_{ad}$, the function $u_\Gamma \in H^1_{loc}(\Omega\setminus\Gamma)$ minimizing the energy of the membrane is the weak variational solution of the following problem

$$\begin{cases} -\Delta u_\Gamma = 0 \text{ in } \Omega\setminus\Gamma \\ \frac{\partial u_\Gamma}{\partial n} = 0 \text{ on } \partial\Gamma,\ \ u_\Gamma = u_0 \text{ on } \partial\Omega\setminus\Gamma. \end{cases} \tag{1.2}$$

A similar situation occurs in the so-called image segmentation problem where, given a function $g \in L^2(\Omega)$, the energy of a segmentation Γ is

$$\mathcal{E}(\Gamma) = \min\Big\{\int_{\Omega\setminus\Gamma} |\nabla u|^2\,dx + \int_{\Omega\setminus\Gamma} (u-g)^2\,dx \ :\ u \in H^1(\Omega\setminus\Gamma)\Big\}.$$

The optimal segmentation of the image g is then obtained by minimizing the Mumford-Shah functional, i.e. by solving the optimization problem

$$\min\big\{\mathcal{E}(\Gamma) + \mathcal{H}^1(\Gamma) \ :\ \Gamma \in \mathcal{U}_{ad}\big\} \tag{1.3}$$

where $\mathcal{U}_{ad}$ is the family of compact sets contained in $\overline{\Omega}$ and $\mathcal{H}^1(\Gamma)$ is the one dimensional Hausdorff measure of Γ. For a given Γ, the minimizer u_Γ of the energy satisfies in this case the following equation

$$\begin{cases} -\Delta u_\Gamma + u_\Gamma = g \text{ in } \Omega\setminus\Gamma \\ \frac{\partial u_\Gamma}{\partial n} = 0 \text{ on } \Gamma\cup\partial\Omega. \end{cases} \tag{1.4}$$

Let us observe that problems (1.1) and (1.3) are somehow similar, in the sense that for a given Γ the minimizer of the energy solves an elliptic equation with homogeneous Neumann boundary conditions on Γ. From this point of view, one has to study in both problems the dependence of the solution of a Neumann problem with respect to the geometric variation of Γ.

Problems (1.1) and (1.3) are nevertheless deeply different. First of all, in problem (1.1) one has to maximize the energy, while in problem (1.3) one has to minimize it. This is the main reason for which problem (1.3) can be seen as a minimum problem in the space SBV (see [1]), the "crack" Γ being seen as the "jump set" of the SBV function u.

A second difference is that the elliptic equation (1.4) has a zero order term, therefore the solution belongs to the Sobolev space $H^1(\Omega\setminus\Gamma)$. The solution of problem (1.2) belongs only to the Dirichlet space (see [9])

$$L^{1,2}(\Omega\setminus\Gamma) = \big\{u \in H^1_{loc}(\Omega\setminus\Gamma) \ :\ \nabla u \in L^2(\Omega\setminus\Gamma)\big\},$$

which coincides with $H^1(\Omega \setminus \Gamma)$ only if Γ is smooth enough, for instance Lipschitz continuous (see [10]).

Another main difference between problems (1.1) and (1.3) is the presence of the penalty term given by the Hausdorff measure. Without this term the minimizer of the Mumford-Shah functional would not exist in general, the infimum being equal to zero. On the other hand, the presence of this term in functional (1.1) is not necessary (for the existence of a solution). From an intuitive point of view, since one looks for the strongest membrane, the length of the crack should be not too big. One could add in problem (1.1) a penalty term given by the Hausdorff measure, by considering the following

(1.5)
$$\max_{\Gamma \in \mathcal{U}_{ad}} \min\{\int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx - \alpha \mathcal{H}^1(\Gamma) : u \in H^1_{loc}(\Omega \setminus \Gamma), u = u_0 \text{ on } \partial\Omega \setminus \Gamma\},$$

where $\alpha > 0$ is fixed. In this case, the existence of an optimal crack could be derived as a consequence of the result of Chambolle and Doveri [6].

An approach by duality was followed by Dal Maso and Toader in [7] where they studied a model for the quasi-static growth of brittle fracture. They dealt with an optimization problem of the type

$$\min_{\Gamma \in \mathcal{U}_{ad}} \min\{\int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx + \alpha \mathcal{H}^1(\Gamma) : u \in H^1_{loc}(\Omega \setminus \Gamma), u = u_0 \text{ on } \partial\Omega \setminus \Gamma\},$$

for which they proved the existence of an optimum.

In this paper, we prove the existence of a solution for problem (1.1) in a general setting: an anisotropic membrane subject to an external force. The main technical difficulty of this setting is that for a given Γ, the function which minimizes the energy does not have, in general, a conjugate, therefore a direct approach by duality is not possible. We refer to [4, 5] for some results in shape optimization of cracks based on duality (see also [11, 12]). Here, we use an approach based on the Mosco convergence of $L^{1,2}$-spaces along with an adjustment procedure for the traces on $\partial\Omega$ in order to prove the existence of a solution to problem (1.1).

Section 2 is devoted to the proof of the existence result, while in Section 3 we make some additional remarks and present some open questions.

2. Existence of an optimal cutting

Let Ω be a two-dimensional bounded open connected set. For simplicity, we suppose that the boundary of Ω is Lipschitz (see [8]). Consequently the number of the connected components of Ω^c is finite.

For $i = 1, \ldots, l$ let K_i be l compact sets contained in $\overline{\Omega}$ and $K \subseteq \overline{\Omega}$ be a compact set such that $\cup_{i=1}^{l} K_i \subseteq K$. Let $f \in L^2(\Omega)$ such that $\mathrm{supp\ f \cap K} = \emptyset$.

Remark 2.1. The assumption that $\mathrm{supp\ f \cap K} = \emptyset$ is made for technical reasons that will be clear in the proof of Theorem 2.2. However, we want to stress the fact that the most interesting case is when $f \equiv 0$, so that the only datum of the problem is the boundary condition u_0.

We also notice that, when the datum K is regular enough (for instance with a Lipschitz boundary), then, thanks to the equality $L^{1,2}(\Omega \setminus K) = H^1(\Omega \setminus K)$, the assumption $\mathrm{supp}\ \mathrm{f} \cap \mathrm{K} = \emptyset$ can be relaxed into the weaker one $f = 0$ a.e. on K.

Remark also that the optimization criterium (1.1) rules out the admissible Γ with $\mathcal{E}(\Gamma) = -\infty$. This automatically implies that the optimization (1.1) is performed on the class of cuts Γ such that the integral of f vanishes on every connected component of Γ which does not touch the boundary $\partial\Omega$ on a set of positive capacity.

In the sequel, we denote by $\mathcal{U}_{ad}$ the following admissible class of "cuts" which is supposed to be non-empty.

$$\mathcal{U}_{ad} = \big\{\Gamma \ : \quad \Gamma = \cup_{i=1}^{l} \Gamma^i, \\ \forall i = 1, \dots, l \quad K_i \subseteq \Gamma^i \subseteq K,\ \Gamma^i \text{ compact connected}\big\},$$

and for every $\Gamma \in \mathcal{U}_{ad}$ we consider the energy

$$\mathcal{E}(\Gamma) = \min\big\{E(u, \Gamma) \ : \ u \in H^1_{loc}(\Omega \setminus \Gamma),\ u = u_0 \text{ on } \partial\Omega\big\}. \tag{2.1}$$

where

$$E(u, \Gamma) = \frac{1}{2}\int_{\Omega\setminus\Gamma} < A\nabla u.\nabla u > \,dx - \int_{\Omega} fu\,dx.$$

Here $u_0 \in H^1(\Omega)$ is a given function and $A \in L^\infty(\Omega, I\!R^4)$ is a given symmetric matrix satisfying for some $\alpha > 0$ the ellipticity condition

$$< A\xi.\xi > \geq \alpha|\xi|^2 \quad \text{for every } \ \xi \in I\!R^2.$$

If $u \in H^1_{loc}(\Omega \setminus \Gamma)$, the trace of u on $\partial\Omega$ does not exist in general, even if $\partial\Omega$ is smooth. Nevertheless, in our case $\nabla u \in L^2(\Omega \setminus \Gamma)$, hence u belongs to the Dirichlet space $L^{1,2}(\Omega\setminus\Gamma)$ (see [9]). In that case, the trace of u on $\partial\Omega\setminus\Gamma$ is well defined, since $\partial\Omega$ is supposed to be Lipschitz. A second equivalent way to give sense to the equality $u = u_0$ on $\partial\Omega \setminus \Gamma$, is as follows. Let us fix an extension u_0^* of u_0 outside Ω, say in $\Omega^* \setminus \Omega$, where Ω^* is a Lipschitz bounded open set such that $\overline{\Omega} \subseteq \Omega^*$. The trace of u is equal to u_0 on $\partial\Omega \setminus \Gamma$ if and only if the function

$$u^* = \begin{cases} u(x) \text{ if } x \in \Omega \setminus \Gamma \\ u_0^*(x) \text{ if } x \in \Omega^* \setminus (\Omega \cup \Gamma) \end{cases} \tag{2.2}$$

belongs to $L^{1,2}(\Omega^* \setminus \Gamma)$.

For every fixed $\Gamma \in \mathcal{U}_{ad}$ problem (2.1) has a solution. This is an immediate consequence of the fact that the support of f is compactly embedded in $\overline{\Omega}\setminus K$ and that, thanks to remark 2.1, the integral of f vanishes on the connected sets of $\overline{\Omega} \setminus K$ not touching $\partial\Omega$ on a set of positive capacity. In fact, if a connected component of $\overline{\Omega} \setminus \Gamma$ contains a part of the support of f and does not touch $\partial\Omega$ on a set of positive capacity, in this region the solution is defined up to a constant, the gradient being fixed. With this remark, the solution is unique (more precisely its gradient is unique) and belongs to the Dirichlet space $L^{1,2}(\Omega \setminus \Gamma)$.

The main result of this section is contained in the following theorem.

Theorem 2.2. *The optimization problem*

$$\max\left\{\mathcal{E}(\Gamma)\ :\ \Gamma\in\mathcal{U}_{ad}\right\} \tag{2.3}$$

has at least one solution.

Proof. In order to prove the existence of a solution for problem (2.3), we follow the direct method of the calculus of variations. Let $\{\Gamma_n\}_n\subseteq\mathcal{U}_{ad}$ be a maximizing sequence for (2.3). Without loss of generality, we can suppose that for every $i=1,\dots,l$

$$\Gamma_n^i\xrightarrow{H}\Gamma^i,$$

the convergence being understood in the Hausdorff metric (see for instance [6, 13]). We denote $\Gamma=\cup_{i=1}^{l}\Gamma^i$, the Hausdorff limit of Γ_n. Our purpose is to prove that Γ is a solution for problem (2.3). Notice that for every $i=1,\dots,l$ the set Γ^i is compact, connected and $K_i\subseteq\Gamma_i\subseteq K$, hence $\Gamma\in\mathcal{U}_{ad}$.

It remains to prove that for every $u\in L^{1,2}(\Omega\setminus\Gamma)$ with $u=u_0$ on $\partial\Omega$ there exists a sequence $\{u_n\}_n$ such that $u_n\in L^{1,2}(\Omega\setminus\Gamma_n)$ with $u_n=u_0$ on $\partial\Omega\setminus\Gamma_n$ and

$$E(\Gamma,u)\geq\limsup_{n\to\infty}E(\Gamma_n,u_n). \tag{2.4}$$

The construction of the sequence $\{u_n\}_n$ is strongly related to the Mosco convergence of the spaces $L^{1,2}(\Omega\setminus\Gamma_n)$ (see [4]). We observe that if $u\notin L^{1,2}(\Omega\setminus\Gamma)$ then $E(u,\Gamma)=+\infty$ and inequality (2.4) holds trivially. For $u\in L^{1,2}(\Omega\setminus\Gamma)$ we construct a sequence $u_n\in L^{1,2}(\Omega\setminus\Gamma_n)$ with $u=u_0$ on $\partial\Omega\setminus\Gamma_n$ such that

$$\tilde\nabla u_n\to\tilde\nabla u\qquad\text{strongly in } L^2(\Omega) \tag{2.5}$$

and

$$\int_\Omega u_n f\,dx\to\int_\Omega uf\,dx.$$

In relation (2.5) we denoted by $\tilde\nabla u_n$ the extension by zero of ∇u_n on Γ_n, since ∇u_n is a priori defined only on $\Omega\setminus\Gamma_n$. Of course, the function $\tilde\nabla u_n$ is not anymore a gradient on Ω.

In order to construct the sequence $\{u_n\}_n$ we recall the following lemma, which is a consequence of [4, Theorem 4.1].

Lemma 2.3. *Let $\Gamma_n,\Gamma\in\mathcal{U}_{ad}$ be such that $\Gamma_n\xrightarrow{H}\Gamma$. Then for every $u\in L^{1,2}(\Omega\setminus\Gamma)$ there exists $u_n\in\{L^{1,2}(\Omega\setminus\Gamma_n)\}_n$ such that $\tilde\nabla u_n\longrightarrow\tilde\nabla u$ strongly in $L^2(\Omega)$.*

Notice that the sequence $\{u_n\}_n$ given by lemma 2.3 is not good to conclude the proof of theorem 2.2, since the trace of u_n on $\partial\Omega\setminus\Gamma_n$ is not equal to u_0. Nevertheless, lemma 2.3 can be used to prove the following proposition.

Proposition 2.4. *Let $\Gamma_n, \Gamma \in \mathcal{U}_{ad}$ be such that $\Gamma_n \xrightarrow{H} \Gamma$. Then for every $u \in L^{1,2}(\Omega \backslash \Gamma)$ such that $u_{|\partial\Omega\backslash\Gamma} = u_0$ there exists a sequence $u_n \in L^{1,2}(\Omega\backslash\Gamma_n)$ such that $\tilde{\nabla} u_n \longrightarrow \tilde{\nabla} u$ strongly in $L^2(\Omega)$ and $u_{n|\partial\Omega\backslash\Gamma_n} = u_0$.*

Proof Let us denote by u^* the extension of u by u_0^* on $\Omega^* \setminus \Omega$. Then we apply lemma 2.3 to Ω^* and Γ_n, Γ and we find a sequence $u_n^* \in L^{1,2}(\Omega^* \setminus \Gamma_n)$ such that $\tilde{\nabla} u_n^* \longrightarrow \tilde{\nabla} u^*$ strongly in $L^2(\Omega^*)$.

For every $n \in \mathbb{N}$, let us denote by u_n the solution of the minimization problem

$$(2.6) \quad \min\{\int_{\Omega^*\backslash\Gamma_n} |\nabla\phi - \nabla u_n^*|^2 \, dx : \phi \in L^{1,2}(\Omega^* \setminus \Gamma_n), \phi = u_0^* \text{ on } \Omega^* \setminus \Omega\}.$$

Since $u_n - u_n^* \in L^{1,2}(\Omega^* \setminus \Gamma_n)$ and since $\Omega^* \setminus \Omega$ is Lipschitz, we get that $u_n - u_n^* \in H^1(\Omega^* \setminus \Omega)$. Moreover, there exists a bounded continuous linear extension operator T from $H^1(\Omega^* \setminus \Omega)$ to $H^1(\Omega^*)$. Taking as a test function in (2.6) the function $\phi = u_n^* + T(u_n - u_n^*)$ we get

$$\min\{\int_{\Omega^*\backslash\Gamma_n} |\nabla\phi - \nabla u_n^*|^2 \, dx : \phi \in L^{1,2}(\Omega^* \setminus \Gamma_n), \phi = u_0^* \text{ on } \Omega^* \setminus \Omega\} \leq$$

$$\leq \int_{\Omega^*\backslash\Gamma_n} |\nabla T(u_n - u_n^*)|^2 dx.$$

Using the Poincaré inequality on the space $\{u \in H^1(\Omega^*) : \int_{\partial\Omega^*} u dx = 0\}$ and the boundedness of the extension operator T we get

$$\int_{\Omega^*\backslash\Gamma_n} |\nabla T((u_n - u_n^*)_{|\Omega^*\backslash\Omega})|^2 dx \leq C \int_{\Omega^*\backslash\Omega} |\nabla(u_n - u_n^*)|^2 dx =$$

$$= C \int_{\Omega^*\backslash\Omega} |\nabla(u_0^* - u_n^*)|^2 dx.$$

This last term converges to zero from lemma 2.3.

Taking the restrictions of u_n to $\Omega \setminus \Gamma_n$, all the requirements are satisfied and the proof is concluded. □

Proof of theorem 2.2 (continuation). Back to the proof of theorem 2.2, we observe that the sequence $\{u_n\}_n$ defined in proposition 2.4 satisfies relation (2.4). Indeed, the gradients extended by zero converge strongly in L^2 by construction, hence using the boundedness of A we have

$$\int_\Omega < A\tilde{\nabla} u_n, \tilde{\nabla} u_n > dx \to \int_\Omega < A\tilde{\nabla} u, \tilde{\nabla} u > dx.$$

It remains to prove that

$$\int_\Omega u_n f dx \to \int_\Omega u f dx.$$

Fix a connected component U of $\overline{\Omega} \setminus K$ containing a part of the support of f. Two possibilities may occur.

Suppose first that $\text{cap}(\mathrm{U}\cap\partial\Omega)>0$. Since $\partial\Omega$ is Lipschitz and Γ is closed, the set $\partial\Omega\setminus\Gamma$ is relatively open, hence there exists an open Lipschitz set V such that $\text{supp f}\subseteq \mathrm{V}\subseteq \mathrm{U}$ and $\text{cap}(\overline{\mathrm{V}}\cap\partial\Omega)>0$. Then, the Poincaré inequality stands true in $H^1((\Omega^*\setminus\overline{\Omega})\cup V)$, so that $u_n\longrightarrow u$ strongly in $L^2(V)$, which implies

$$\int_V u_n f dx\to\int_V uf dx.$$

Suppose now that $\text{cap}(\mathrm{U}\cap\partial\Omega)=0$. In this case there exists an open Lipschitz set V such that $\text{supp f}\subseteq \mathrm{V}\subseteq \mathrm{U}$ and $\overline{V}\cap\partial\Omega=\emptyset$. By hypothesis, we have that $\int_V f dx=0$ hence the Poincaré inequality holds in $H^1(V)/\mathbb{R}$. Consequently

$$\int_V u_n f dx\to\int_V uf dx.$$

The support of f being compactly contained in $\overline{\Omega}\setminus K$, the proof is concluded. □

If K does not touch $\partial\Omega$, one could drop the hypothesis on the regularity of Ω, by simply imposing a constraint of the type $(u-u_0)\varphi\in H^1_0(\Omega\setminus\Gamma)$, where $\varphi\in C^\infty(\mathbb{R}^2)$ is a fixed function such that $\varphi=0$ on Γ and use a partition of unity.

3. Additional remarks and open problems

The uniqueness of the optimal cut does not hold in general. Trivially, let $u_0\equiv 0$, $f\equiv 0$, $K_1=\{A,B\}$, $K=\overline{\Omega}$. Then any compact connected set containing A and B solves problem (2.3).

In some particular situations one can precise exactly at least one solution of the problem. In a symmetric setting, there exists an optimal cut which is also symmetric. Indeed, let $f\equiv 0$ and Ω be a rectangle; let d be a symmetry line of the rectangle. Suppose that $K_1=\{A,B\}$ are two points on d and that u_0 is also symmetric with respect to d. It can be easily seen that a solution of problem (2.3) (with $K=\overline{\Omega}$) is the segment AB. This follows by simply observing that the harmonic function in the rectangle which is equal to u_0 on the boundary of the rectangle has zero normal derivatives on the segment AB.

There are some other situations, when the position of the optimal cut can be determined. Let Ω be a simply connected bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary and let $f\equiv 0$. Let $K_1=\{A,B\}$, the points A and B being placed on a connected level set of the conjugate function of the harmonic function v in Ω which is equal to u_0 on $\partial\Omega$, i.e.

$$\begin{cases} -\Delta v=0 \text{ in } \Omega\\ v=u_0 \text{ on } \partial\Omega.\end{cases}\tag{3.1}$$

Since Ω is simply connected, we recall that v has an harmonic conjugate, denoted ϕ and determined by the Cauchy-Riemann relations

$$\frac{\partial \phi}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial v}{\partial x}.$$

Suppose that $\phi(A) = \phi(B)$, A, B belonging to a connected component of $\phi^{-1}(\phi(A))$. Let us set Γ be a compact connected subset of $\phi^{-1}(\phi(A))$, containing A, B. In order to simplify the following proof we assume that Γ does not touch the boundary of Ω. All results remain true if Γ touches $\partial\Omega$, but a few technical difficulties appear if $\Omega \setminus \Gamma$ is not connected.

Under the previous assumptions, we observe that

$$\min\{\int_\Omega |\nabla u|^2\, dx : u \in H^1(\Omega), u = u_0 \text{ on } \partial\Omega\} = \tag{3.2}$$

$$= \min\{\int_{\Omega\setminus\Gamma} |\nabla u|^2\, dx : u \in H^1_{loc}(\Omega \setminus \Gamma), u = u_0 \text{ on } \partial\Omega\}.$$

Indeed, on the one hand, we obviously have

$$\min\{\int_\Omega |\nabla u|^2\, dx : u \in H^1(\Omega), u = u_0 \text{ on } \partial\Omega\} \geq$$

$$\geq \min\{\int_{\Omega\setminus\Gamma} |\nabla u|^2\, dx : u \in H^1_{loc}(\Omega \setminus \Gamma), u = u_0 \text{ on } \partial\Omega\}.$$

On the other hand, let $v^* \in H^1_{loc}(\Omega \setminus \Gamma)$ be the minimizer in the right hand side of the previous relation (observe that all connected components of $\Omega \setminus \Gamma$ touch $\partial\Omega$ since v^* is harmonic).

We give the following.

Lemma 3.1. *There exists a function $\phi^* \in H^1_0(\Omega)$ and a constant $c^* \in I\!R$ such that $\nabla v^* = \operatorname{curl} \phi^*$ in $\Omega \setminus \Gamma$ and*

$$\begin{cases} -\Delta \phi^* = 0 & \text{in } \Omega \setminus \Gamma \\ \phi^* = c^* & q.e. \text{ on } \Gamma \\ \frac{\partial \phi^*}{\partial n} = \partial_t u_0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

where $\partial_t u_0$ is the weak tangential derivative in the sense of distribution on $\partial\Omega$.

Proof. The function ϕ^* is the harmonic conjugate of v^* in $\Omega \setminus \Gamma$. The proof is a consequence of the fact that $\partial\Omega$ is connected and the following equalities hold

$$\int_{\partial\Omega} \frac{\partial v^*}{\partial n} d\mathcal{H}^1 = 0,$$

$$\frac{\partial v^*}{\partial n} = 0 \text{ on } \Gamma.$$

The existence of the conjugate is a consequence of a result of [10] through an approximation procedure of the non-smooth boundary Γ. We refer to [4] where this kind of result was proved in a slightly different setting. $\square$

If ϕ is the harmonic conjugate of v in Ω, then we have

$$\begin{cases} -\Delta\phi = 0 \text{ in } \Omega \\ \frac{\partial\phi}{\partial n} = \partial_t u_0 \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

By hypothesis, we also have $\phi = c$ on Γ. Consequently, by subtraction we get

$$\begin{cases} -\Delta(\phi - \phi^*) = 0 & \text{in} \quad \Omega \setminus \Gamma \\ \phi - \phi^* = c - c^* & \text{q.e. on} \quad \Gamma \\ \frac{\partial(\phi-\phi^*)}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.5}$$

If $c - c^* \neq 0$, then $\phi - \phi^*$ attains its minimum (or maximum, or both) on $\partial\Omega$, and in that point we would have, following the Hopf maximum principle, $\frac{\partial\phi^*}{\partial n} \neq 0$ which is a contradiction with (3.5). Therefore $c - c^* = 0$, hence $\phi = \phi^*$ in $\Omega \setminus \Gamma$ and $v = v^*$ in $\Omega \setminus \Gamma$. Since the level set of a non constant harmonic function cannot have a positive Lebesgue measure, the set Γ, which is contained in the level set $\{\phi = c\}$ has measure zero. Then we get

$$\int_{\Omega\setminus\Gamma} |\nabla v^*|^2\, dx = \int_\Omega |\nabla v|^2\, dx \geq$$

$$\geq \min\{\int_\Omega |\nabla u|^2\, dx : u \in H^1(\Omega), u = u_0 \text{ on } \partial\Omega\}.$$

Consequently, equality (3.2) holds and Γ is a solution for problem (2.3).

A natural question is to see whether the optimal cut Γ touches the boundary of Ω or not. There are indeed some situations in which the optimal cut Γ necessarily touches the boundary. We give the following example. Let Ω be the rectangle

$$\Omega = \{(x,y) \in \mathbb{R}^2 \ : \ x \in [0,1],\ y \in [-1,1]\}.$$

Let $u(x,y) = 2(x+1)y$ and $u_0 \in C(\partial\Omega)$ be given by the trace of u on $\partial\Omega$, i.e. $u_0 = u_{|\partial\Omega}$. We take $f \equiv 0$ and $K_1 = \{A, B\}$ with $A = (\frac{\sqrt{5}}{2} - 1, \frac{1}{2})$, $B = (\frac{\sqrt{5}}{2} - 1, -\frac{1}{2})$, and we take $K = \overline{\Omega}$. The optimal cut solving problem (2.3) is given by the curve

$$\Gamma = \{(x,y) \in \mathbb{R}^2 \ : \ x = \sqrt{y^2+1} - 1,\ y \in [-\frac{1}{2}, \frac{1}{2}]\},$$

which obviously touches the boundary of Ω at the origin.

The proof of the optimality of Γ follows the same scheme as presented in the beginning of this section where we studied the possibility of identifying the optimal cut. In fact, one can observe that the points A and B belong to a level set of the function $v(x,y) = (x+1)^2 - y^2$ which is the harmonic conjugate of u. One can moreover observe that all optimal cuts in $\overline{\Omega}$ must touch the boundary. This is an easy consequence of the uniqueness of the minimizer of $E(u,\Gamma)$ over $u \in L^{1,2}(\Omega \setminus \Gamma)$, with$u = u_0$ on $\partial\Omega$.

Remark 3.2. When the dimension n is greater than 2, the formulation of the optimal cutting problem (1.1) becomes trivial. Indeed, since curves have zero capacity in $I\!R^n$ when $n > 2$, any curve Γ joining the points A, B would be optimal. We do not know how to formulate in a natural way the optimal cutting problem (1.1) in higher dimension, in order to obtain connected sets Γ with positive capacity.

Another possible variant to investigate could be to consider the optimal cutting problem (1.1) for the p-Laplacian:

$$\mathcal{E}_p(\Gamma) = \min\{\int_{\Omega\setminus\Gamma} |\nabla u|^p dx : u = u_0 \text{ on } \partial\Omega\}.$$

It is known (see [2]) that if $p > n-1$ curves have positive p-capacity and classes of compact connected sets Γ, like $\mathcal{U}_{ad}$, are closed for a suitable γ_p-convergence. Nevertheless, the duality argument of [4], used in theorem 2.2, does not work for p-harmonic functions, which leaves the problem open.

We finally present some open problems which seem to be of interest.

Problem 1. The first question is concerned with the regularity of optimal cuts Γ. As shown at the beginning of this section, the optimal cut is not unique. The question is to prove that among all the minimizers, there exists at least a "smooth" one. This might be of interest even in the simple case of a rectangle and two points $\{A, B\}$ with u_0 smooth and no force on the membrane.

Since the regularity of the optimal cut seems rather difficult to prove, a first question would be to identify its Hausdorff dimension, in particular to prove (or disprove) that its Lebesgue measure is zero.

The following simple example shows that in general we should not expect C^1 solutions Γ, even if $f \equiv 0$ and K reduces to two points. Indeed, take Ω the unit ball and the boundary datum $u_0(x,y) = 2xy$; according to the argument at the beginning of this section, if we consider two points A and B on a level line of the harmonic conjugate function $u + 0^*(x,y) = x^2 - y^2$, then this level line is an optimal cut for problem (1.1). Now, if we take $A = (-\frac{1}{2}, \frac{1}{2})$ and $B = (\frac{1}{2}, \frac{1}{2})$, the optimal cut Γ is the Cartesian curve $y = |x|$ with $x \in [-\frac{1}{2}, \frac{1}{2}]$ which has a singular point at the origin.

Problem 2. Instead of an optimal cut in a membrane, we may study the existence of an optimal cut in a plate. Given a smooth bounded set $\Omega \subseteq I\!R^2$ and a constant $\nu \in (0,1)$, for every compact connected set $\Gamma \subseteq \Omega$ we define

$$\mathcal{E}(\Gamma) = \min\{\int_{\Omega\setminus\Gamma} [\nu(\Delta u)^2 + (1-\nu) \sum_{1\le i,j\le 2} (\partial_i\partial_j u)^2] dx :$$

$$: u \in H^2_{loc}(\Omega \setminus \Gamma), u = u_0 \text{ on } \partial\Omega \setminus \Gamma, \frac{\partial u}{\partial n} = u_1 \text{ on } \partial\Omega \setminus \Gamma\},$$

where u_0 and u_1 are both prescribed. Given two points $A, B \in \Omega$, the question is to prove the existence of a solution for the problem

$$\max_{\Gamma \in \mathcal{U}_{ad}} \mathcal{E}(\Gamma),$$

in the same frame of theorem 2.2. The difficulty comes from the fact that a result similar to the one of proposition 2.4 should be proved in H^2-spaces, and this, to our knowledge, is an open problem.

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(Dorin Bucur) CNRS-Département de Mathématiques, Université de Franche-Comté, 16 route de Gray, 25030 Besançon, France
E-mail address, Dorin Bucur: `bucur@math.univ-fcomte.fr`

(Giuseppe Buttazzo) Dipartimento di Matematica, Università di Pisa, Via Buonarroti 2, 56127 Pisa, Italy
E-mail address, Giuseppe Buttazzo: `buttazzo@dm.unipi.it`

(Nicolas Varchon) Département de Mathématiques, Université de Franche-Comté, 16 route de Gray, 25030 Besançon, France
E-mail address, Nicolas Varchon: `varchon@math.univ-fcomte.fr`