An approximation of Mumford-Shah energy by a family of discrete edge-preserving functionals

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Abstract

We show the Γ-convergence of a family of discrete functionals to the Mumford and Shah image segmentation functional. The functionals of the family are constructed by modifying the elliptic approximating functionals proposed by Ambrosio and Tortorelli. The quadratic term of the energy related to the edges of the segmentation is replaced by a nonconvex functional.

Keywords: Γ-convergence, finite elements, image segmentation

1 Introduction

Segmentation is an important task in image processing. The goal is to decompose an observed image into several homogeneous regions. Such a segmentation can be achieved by computing the regions or the edges limiting the regions. The most well-known segmentation functional in image processing is the one proposed by Mumford-Shah, which we write in its weak form [15, 2, 6]:

$$J^{MS}(f) = \int_\Omega (f-p)^2 dx + \int_\Omega |\nabla f|^2 dx + \mathcal{H}^1(S_f), \quad f \in SBV(\Omega), \tag{1}$$

where p is the observation, f is the unknown segmented image which should be close to p in the L^2-norm sense, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, S_f is the set of jumps of f, and SBV denotes the space of special functions of bounded variation [2].
This functional is difficult to minimize because of the lack of convexity and regularity mainly due to the term $\mathcal{H}^1(S_f)$. This kind of free-discontinuity problems can be approximated, in the sense of Γ-convergence, by sequence of more tractable functionals (see the book of Braides [8]). One of the most popular is the sequence of functionals $J_\varepsilon^{AT}(f,b)$ proposed by Ambrosio and Tortorelli [3, 4] where the new variable b controls the set of jumps S_f:

$$\begin{aligned} J_\varepsilon^{AT}(f,b) &= \int_\Omega (b^2+\kappa_\varepsilon)|\nabla f|^2 dx + \int_\Omega \left[\varepsilon|\nabla b|^2 + \frac{1}{4\varepsilon}(1-b)^2\right] dx \\ &+ \int_\Omega (f-p)^2 dx, \qquad f \in W^{1,2}(\Omega), \quad b \in W^{1,2}(\Omega;[0,1]), \end{aligned}$$

where Γ-convergence takes place as $\varepsilon \to 0^+$, and κ_ε is a positive infinitesimal faster than ε.

In practice, in the discrete setting, due to the L^2-norm of the gradient of b, the output of the Ambrosio-Tortorelli approximation is too smoothed, resulting in blurred edges in the solution. In a previous paper [16], we have proposed to use an edge-preserving φ-function for the term involving the gradient of b:

$$\begin{aligned} J_\varepsilon(f,b) &= \int_\Omega (b^2+\kappa_\varepsilon)|\nabla f|^2 dx + \int_\Omega \left[\varepsilon\varphi(|\nabla b|) + \frac{1}{4\varepsilon}(1-b)^2\right] dx \\ &+ \int_\Omega (f-p)^2 dx, \qquad f \in W^{1,2}(\Omega), \quad b \in W^{1,2}(\Omega;[0,1]), \end{aligned}$$

where

$$\varphi(t) = \frac{t^2}{1+\mu t^2}, \qquad \mu > 0.$$

This approximation gives good experimental results with sharper edges [16]. However, we can easily show [5] that the family $\{J_\varepsilon\}_\varepsilon$ does not Γ-converge to the Mumford and Shah functional, mainly due to the fact that we have no control on the gradient of the functions b_ε in the neighbourhood of edges. In order to obtain the Γ-convergence with the φ-function, we consider the subspace of $W^{1,2}(\Omega)$ of finite elements as in Chambolle and Dal Maso [9], and we introduce a sequence of discrete energies. The sharpness of the transitions at edges will be limited by the mesh size.

The paper is organized as follows. In Section 2 we define the sequence of functionals and we state the Γ-convergence result. In Section 3 we prove the compactness of the minimizers for the sequence of functionals. The proof of Γ-convergence is given in Sections 4 and 5.

2 Mathematical preliminaries and statement of the result

Let $\Omega \subset \mathbf{R}^2$ be a bounded open set. We denote by $\mathcal{B}(\Omega)$ the σ-algebra of all the Borel subsets of Ω; for any $A \in \mathcal{B}(\Omega)$ we denote by $|A|$ the two-dimensional Lebesgue measure of A, and by $\mathcal{H}^1(A)$ the one-dimensional Hausdorff measure of A.

We will use standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$. We say that $f \in L^1(\Omega)$ is a function of bounded variation in Ω, and we write $f \in BV(\Omega)$, if the distributional derivative Df of f is a vector-valued Radon measure with finite total variation in Ω. We denote by $|Df|$ the total variation of Df, and by ∇f the density of the absolutely continuous part of Df with respect to the Lebesgue measure. It can be proved [2] that ∇f coincides almost everywhere with the approximate differential of f. In the one-dimensional case we shall use the notation f' in place of ∇f.

We denote by $f^-(x)$, $f^+(x)$ the approximate lower and upper limit of f at the point x, and we denote by S_f the discontinuity set of f in an approximate sense, defined as

$$S_f = \{x \in \Omega : f^-(x) < f^+(x)\}.$$

The set S_f is negligible with respect to Lebesgue measure and it is countably

$(\mathcal{H}^1, 1)$ rectifiable, i.e., representable as a disjoint union $\cup_{i=1}^{\infty} K_i \cup N$, where $\mathcal{H}^1(N) = 0$ and K_i are compact sets, each contained in a C^1 curve $\Gamma_i \subset \mathbf{R}^2$.

Let $E \subset \mathcal{B}(\Omega)$; we define

$$P_\Omega(E) = \sup \left\{ \int_E div\ \phi\, dx : \phi \in C_0^1(\Omega; \mathbf{R}^2), |\phi| \leq 1 \right\}.$$

We say that E is a set of finite perimeter in Ω if $P_\Omega(E) < +\infty$. By Riesz's theorem (see [14]), E is a set of finite perimeter if and only if $1_E \in BV(\Omega)$, and $P_\Omega(E) = |D1_E|(\Omega)$.

The following Fleming-Rishel coarea formula (see [14]) establishes an important connection between BV functions and sets of finite perimeter:

$$|Df|(\Omega) = \int_{-\infty}^{+\infty} P_\Omega(\{x \in \Omega : f(x) > t\})\, dt. \tag{2}$$

We say that $f \in BV(\Omega)$ belongs to the space of special functions of bounded variation $SBV(\Omega)$ if [2]

$$|Df|(\Omega) = \int_\Omega |\nabla f|\, dx + \int_{S_f} |f^+ - f^-|\, d\mathcal{H}^1 .$$

We recall the definition and some properties of Γ-convergence (see [12]). Let X be a metric space, and let $F_\varepsilon : X \to [0, +\infty]$ be a family of functions indexed by $\varepsilon > 0$. We say that F_ε Γ-converge as $\varepsilon \to 0^+$ to $F : X \to [0, +\infty]$ if the following two conditions

$$\forall x_\varepsilon \to x \quad \liminf_{\varepsilon \to 0^+} F_\varepsilon(x_\varepsilon) \geq F(x), \tag{3}$$

and

$$\exists x_\varepsilon \to x \quad \limsup_{\varepsilon \to 0^+} F_\varepsilon(x_\varepsilon) \leq F(x), \tag{4}$$

are fulfilled for every $x \in X$. The Γ-limit, if it exists, is unique and lower semicontinuous. The Γ-convergence is stable under continuous perturbations, that is, $(F_\varepsilon + v)$ Γ-converge to $(F + v)$ if F_ε Γ-converge to F and v is continuous. The most important property of Γ-convergence is the following: if $\{x_\varepsilon\}_\varepsilon$ is asymptotically minimizing, i.e.

$$\lim_{\varepsilon \to 0^+} \left(F_\varepsilon(x_\varepsilon) - \inf_X F_\varepsilon \right) = 0, \tag{5}$$

and if $\{x_{\varepsilon_h}\}_h$ converge to x for some sequence $\varepsilon_h \to 0$, then x minimizes F.

In the following $\Omega \subset \mathbf{R}^2$ will denote an open polygonal domain. Let θ_0 be an angle such that $0 < \theta_0 \leq \pi/3$, and let $\nu(h)$ be a function such that $\nu(h) \geq h$ for any $h > 0$ and $\nu(h) = O(h)$ as $h \to 0^+$. Let us denote by $\{\mathbf{T}_h\}_h$ a family of triangulations of Ω made of triangles whose edges, for any $h > 0$, have length between h and $\nu(h)$, and whose angles are all greater than or equal to θ_0.

We denote by $V_h(\Omega) \subset W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ the linear finite element space

$$V_h(\Omega) = \{u : \Omega \to \mathbf{R} \ : \ u \text{ continuous}, u|_T \in P_1(T) \ \forall T \in \mathbf{T}_h\},$$

where T denotes a triangle of $\mathbf{T}_h$, $u|_T$ denotes the restriction of u to T, and $P_1(T)$ denotes the space of polynomials of degree 1 on T. We denote by $\pi_h : C^0(\overline{\Omega}) \to V_h(\Omega)$ the Lagrange interpolation operator.

Let $p \in L^\infty(\Omega)$ and $f \in SBV(\Omega)$; the weak form of the Mumford-Shah functional is defined by

$$J^{MS}(f) = \int_\Omega (f-p)^2 dx + \int_\Omega |\nabla f|^2 dx + \mathcal{H}^1(S_f).$$

We set

$$X(\Omega) = L^\infty(\Omega) \times L^\infty(\Omega; [0,1]).$$

We add a formal extra variable b to J^{MS} and we define the functional J^{MS} : $X(\Omega) \to [0, +\infty]$ by setting

$$J^{MS}(f,b) = \begin{cases} \displaystyle\int_\Omega (f-p)^2 dx + \int_\Omega |\nabla f|^2 dx + \mathcal{H}^1(S_f) & \text{if } f \in SBV(\Omega),\ b \equiv 1 \\ +\infty & \text{elsewhere on } X(\Omega). \end{cases}$$

Let $\varepsilon > 0$ and $h > 0$. We denote by $\{\kappa_\varepsilon\}_\varepsilon$ a sequence of positive numbers converging to 0 as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon \to 0} \frac{\kappa_\varepsilon}{\varepsilon} = 0.$$

We denote by p_ε a regular approximation of p as $\varepsilon \to 0$ satisfying [7]

$$p_\varepsilon \to p \text{ in } L^2(\Omega), \quad \|p_\varepsilon\|_{L^\infty(\Omega)} \leq \|p\|_{L^\infty(\Omega)}, \quad \|\nabla p_\varepsilon\|_{L^\infty(\Omega)} \leq C/\varepsilon.$$

We denote by $\varphi : [0,+\infty[\to [0,+\infty[$ the function defined by

$$\varphi(t) = \frac{t^2}{1+\mu_{\varepsilon,h}t^2},$$

where $\mu_{\varepsilon,h} > 0$.

In order to obtain the Γ-convergence result to the Mumford-Shah functional, we then consider the discrete family of functionals $J_{\varepsilon,h}(f,b) : X(\Omega) \to [0,+\infty]$ defined for any $\varepsilon, h > 0$ by:

$$\begin{aligned} J_{\varepsilon,h}(f,b) &= \int_\Omega (\pi_h(b^2)+\kappa_\varepsilon)|\nabla f|^2 dx + \int_\Omega \left[\varepsilon\varphi(|\nabla b|) + \frac{1}{4\varepsilon}\pi_h\left((1-b)^2\right)\right] dx \\ &+ \int_\Omega \pi_h\left((f-p_\varepsilon)^2\right) dx \qquad \text{if } (f,b) \in V_h(\Omega)\times V_h(\Omega;[0,1]), \end{aligned}$$

and $J_{\varepsilon,h}(f,b) = +\infty$ elsewhere in $X(\Omega)$. The integrals in $J_{\varepsilon,h}$ can be evaluated via the vertex quadrature rule, which is exact for piecewise linear functions.

The family of functionals $\{J_{\varepsilon,h}\}_{\varepsilon,h}$ has some similarities with the Bellettini-Coscia functionals $J^{BC}_{\varepsilon,h}$ [7]:

$$\begin{aligned} J^{BC}_{\varepsilon,h}(f,b) &= \int_\Omega (b+\kappa_\varepsilon)|\nabla f|^2 dx + \int_\Omega \left[\varepsilon|\nabla b|^2 + \frac{1}{4\varepsilon}\pi_h\left(1-b^2\right)\right] dx \\ &+ \int_\Omega \pi_h\left((f-p_\varepsilon)^2\right) dx \qquad \text{if } (f,b) \in V_h(\Omega)\times V_h(\Omega;[0,1]), \end{aligned}$$

and $J^{BC}_{\varepsilon,h}(f,b) = +\infty$ elsewhere in $X(\Omega)$. However, there are two main differences with our functional: the most important one is the use of the φ-function on $|\nabla b|$ and the second one is that the term $\int_\Omega \pi_h(b^2)|\nabla f|^2 + \frac{1}{4\varepsilon}\int_\Omega \pi_h\left((1-b)^2\right)$ has been replaced by $\int_\Omega b|\nabla f|^2 + \frac{1}{4\varepsilon}\int_\Omega \pi_h\left(1-b^2\right)$.

We can now state the main result of the paper.

Theorem 2.1 *Assume that $h = o(\kappa_\varepsilon)$ and that $\mu_{\varepsilon,h} = o(\varepsilon h)$. Then the family $\{J_{\varepsilon,h}\}_\varepsilon$ Γ-converges to the functional J^{MS} in the $[L^2(\Omega)]^2$ topology as $\varepsilon \to 0^+$.*

Moreover, any family $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_\varepsilon$ of absolute minimizers of $J_{\varepsilon,h}$ is relatively compact in $[L^2(\Omega)]^2$, and each of its limit points minimizes the functional J^{MS}.

3 Equicoercivity

The goal of this section is to prove the equicoercivity of the family of functionals $\{J_{\varepsilon,h}\}_\varepsilon$.

Theorem 3.1 *Assume that $h = o(\kappa_\varepsilon)$ and that $\mu_{\varepsilon,h} = o(\varepsilon h)$. Let $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_\varepsilon \subset V_h(\Omega) \times V_h(\Omega; [0,1])$ be such that*

$$\sup_{\varepsilon>0} J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) < +\infty.$$

Then the family $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_\varepsilon$ is relatively compact in the $[L^2(\Omega)]^2$ topology as $\varepsilon \to 0^+$ and any limit point is of the form $(f, 1)$ with $f \in SBV(\Omega) \cap L^\infty(\Omega)$.

For the proof of theorem 3.1, we will need the following elementary lemma about Lagrange interpolation.

Lemma 3.2 *Let b be any function of $V_h(\Omega)$. Then $b^2(x,y) \leq \pi_h(b^2)(x,y)$ for any $(x,y) \in \Omega$.*

Proof. Since the function b is in V_h, it is continuous and piecewise affine and it suffices to prove Lemma 3.2 on each triangle T of $\mathbf{T}_h$. On T, $b(x,y)$ is of the form

$$b(x,y) = Ax + By + C. \tag{6}$$

Now, the projection $\pi_h(b^2)$ of b^2 on V_h is also on triangle T of the form

$$\pi_h(b^2(x,y)) = Dx + Ey + F. \tag{7}$$

The coefficients $D, E,$ and F are determinated by the values of b^2 at the nodes $M_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, $i = 1,2,3$ of triangle T:

$$\pi_h(b^2(x_i,y_i)) = b^2(x_i,y_i), \tag{8}$$

that is to say, thanks to (6),

$$Dx_i + Ey_i + F = (Ax_i + By_i + C)^2, \quad i = 1,2,3. \tag{9}$$

Let us take the following notations:

$$V = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \quad W = \begin{pmatrix} D \\ E \\ F \end{pmatrix}, \quad Z = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \quad Z_i = \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}.$$

Then (9) is equivalent to

$$\langle W, Z_i \rangle = (\langle V, Z_i \rangle)^2, \quad i = 1, 2, 3. \tag{10}$$

where $\langle , \rangle$ denotes the scalar product in $\mathbf{R}^3$. Moreover if $M = \begin{pmatrix} x \\ y \end{pmatrix}$ belongs to T, by using the barycentric coordinates $\lambda_i(x,y)$, we have $Z = \sum_{i=1}^3 \lambda_i(x,y) Z_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^3 \lambda_i(x,y) = 1$. Therefore

$$b^2(x,y) = (\langle V, Z \rangle)^2 = (\langle V, \sum_{i=1}^3 \lambda_i(x,y) Z_i \rangle)^2 = (\sum_{i=1}^3 \lambda_i(x,y) \langle V, Z_i \rangle)^2,$$

which implies by convexity

$$\begin{aligned} b^2(x,y) \leq \sum_{i=1}^3 \lambda_i(x,y)(\langle V, Z_i \rangle)^2 &= \sum_{i=1}^3 \lambda_i(x,y) \langle W, Z_i \rangle \\ &= \langle W, Z \rangle = \pi_h(b^2)(x,y). \end{aligned}$$

Thus $b^2(x,y) \leq \pi_h(b^2)(x,y)$ and Lemma 3.2 is proved. ■

Let us now go back to the proof of Theorem 3.1.

Proof of Theorem 3.1.

We first recall the expression of our functional without interpolation:

$$\begin{aligned} J_\varepsilon(f,b) &= \int_\Omega (b^2 + \kappa_\varepsilon)|\nabla f|^2 dx + \int_\Omega \left[\varepsilon \varphi(|\nabla b|) + \frac{1}{4\varepsilon}(1-b)^2 \right] dx \\ &+ \int_\Omega (f-p)^2 dx. \end{aligned}$$

Then we have

$$J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) \leq c \tag{11}$$

for some constant c. In the sequel, we denote by c any universal positive constant which can vary from line to line. We divide the proof in two steps.

Step 1 The family $\{J_\varepsilon(f_{\varepsilon,h}, b_{\varepsilon,h})\}_{\varepsilon,h}$ is uniformly bounded.

We have $J_\varepsilon(f_{\varepsilon,h}, b_{\varepsilon,h}) \geq 0$, and

$$\begin{aligned} J_\varepsilon(f_{\varepsilon,h}, b_{\varepsilon,h}) &= J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) + \int_\Omega \left(b_{\varepsilon,h}^2 - \pi_h(b_{\varepsilon,h}^2)\right) |\nabla f_{\varepsilon,h}|^2 dx \\ &+ \frac{1}{4\varepsilon} \int_\Omega \left((1-b_{\varepsilon,h})^2 - \pi_h((1-b_{\varepsilon,h})^2)\right) dx \\ &+ \int_\Omega \left((f_{\varepsilon,h} - p)^2 - \pi_h((f_{\varepsilon,h} - p_\varepsilon)^2)\right) dx. \end{aligned} \tag{12}$$

We shortly write (12) as

$$J_\varepsilon(f_{\varepsilon,h}, b_{\varepsilon,h}) = J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) + I_{\varepsilon,h}^1 + I_{\varepsilon,h}^2 + I_{\varepsilon,h}^3. \tag{13}$$

We know with (11) that $J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) \leq c$. It remains to show that the integrals $I_{\varepsilon,h}^j, j = 1, 2, 3$ are bounded from above.
Using Lemma 3.2, it is obvious that

$$I_{\varepsilon,h}^1 = \int_\Omega \left(b_{\varepsilon,h}^2 - \pi_h(b_{\varepsilon,h}^2)\right) |\nabla f_{\varepsilon,h}|^2 dx \leq 0. \tag{14}$$

Let us consider $I_{\varepsilon,h}^2$:

$$I_{\varepsilon,h}^2 = \frac{1}{4\varepsilon} \int_\Omega \left((1-b_{\varepsilon,h})^2 - \pi_h((1-b_{\varepsilon,h})^2)\right) dx.$$

Since $b_{\varepsilon,h} \in V_h(\Omega)$ and π_h is a linear operator, we clearly have

$$I_{\varepsilon,h}^2 = \frac{1}{4\varepsilon} \int_\Omega (b_{\varepsilon,h}^2 - \pi_h(b_{\varepsilon,h}^2)) dx.$$

Again, from lemma 3.2, $I_{\varepsilon,h}^2$ is non positive, but we can in fact prove more:

$$\lim_{\varepsilon \to 0} I_{\varepsilon,h}^2 = 0. \tag{15}$$

For any $\delta_h \in (0, 1)$ we define

$$\mathbf{T}_h^1 = \{T \in \mathbf{T}_h : (1-\delta_h)|\nabla b_{\varepsilon,h}|^2 > \frac{\delta_h}{\mu_{\varepsilon,h}}\}. \tag{16}$$

Using (11) we get

$$\varepsilon \int_\Omega \frac{|\nabla b_{\varepsilon,h}|^2}{1 + \mu_{\varepsilon,h}|\nabla b_{\varepsilon,h}|^2} dx \leq c. \tag{17}$$

Thanks to the definition of $\mathbf{T}_h^1$ and the inequality

$$\frac{t^2}{1+\mu_{\varepsilon,h}t^2} \geq \min\{(1-\delta_h)t^2, \frac{\delta_h}{\mu_{\varepsilon,h}}\}, \tag{18}$$

we deduce from (17)

$$\frac{\varepsilon\delta_h}{\mu_{\varepsilon,h}}\int_{\mathbf{T}_h^1} dx + \varepsilon(1-\delta_h)\int_{\mathbf{T}_h\setminus\mathbf{T}_h^1} |\nabla b_{\varepsilon,h}|^2 dx \leq c. \tag{19}$$

Now we have

$$4\varepsilon I_{\varepsilon,h}^2 = \int_{\mathbf{T}_h^1}(b_{\varepsilon,h}^2 - \pi_h(b_{\varepsilon,h}^2))dx + \int_{\mathbf{T}_h\setminus\mathbf{T}_h^1}(b_{\varepsilon,h}^2 - \pi_h(b_{\varepsilon,h}^2))dx,$$

and, since $0 \leq b_{\varepsilon,h} \leq 1$, we get

$$4\varepsilon|I_{\varepsilon,h}^2| \leq 2\int_{\mathbf{T}_h^1} dx + \int_{\mathbf{T}_h\setminus\mathbf{T}_h^1}|b_{\varepsilon,h}^2 - \pi_h(b_{\varepsilon,h}^2)|dx.$$

On the other hand, according to classical results on interpolation theory (see Ciarlet [10]), we have the following estimate:

$$\int_{\mathbf{T}_h\setminus\mathbf{T}_h^1}|b_{\varepsilon,h}^2 - \pi_h(b_{\varepsilon,h}^2)|dx \leq ch^2 \sum_{T\in\mathbf{T}_h\setminus\mathbf{T}_h^1} \|\nabla^2(b_{\varepsilon,h}^2)\|_{L^\infty(T)},$$

but

$$\nabla^2(b_{\varepsilon,h}^2) = 2\nabla(b_{\varepsilon,h}\nabla b_{\varepsilon,h}) = 2b_{\varepsilon,h}\nabla^2 b_{\varepsilon,h} + 2\nabla b_{\varepsilon,h}\otimes\nabla b_{\varepsilon,h}.$$

Since $b_{\varepsilon,h}$ is affine on each T, and $\|\nabla b_{\varepsilon,h}\otimes\nabla b_{\varepsilon,h}\|_{L^\infty(T)} \leq |\nabla b_{\varepsilon,h}|^2$ on T, we obtain

$$\int_{\mathbf{T}_h\setminus\mathbf{T}_h^1}|b_{\varepsilon,h}^2 - \pi_h(b_{\varepsilon,h}^2)|dx \leq ch^2 \sum_{T\in\mathbf{T}_h\setminus\mathbf{T}_h^1} \|\nabla b_{\varepsilon,h}\|_{L^\infty(T)}^2 = ch^2\int_{\mathbf{T}_h\setminus\mathbf{T}_h^1}|\nabla b_{\varepsilon,h}|^2 dx.$$

Therefore

$$4\varepsilon|I_{\varepsilon,h}^2| \leq 2\int_{\mathbf{T}_h^1} dx + ch^2\int_{\mathbf{T}_h\setminus\mathbf{T}_h^1}|\nabla b_{\varepsilon,h}|^2 dx. \tag{20}$$

Now from (19) we deduce

$$\int_{\mathbf{T}_h^1} dx \leq c\frac{\mu_{\varepsilon,h}}{\varepsilon\delta_h}, \qquad \text{and} \qquad \int_{\mathbf{T}_h\setminus\mathbf{T}_h^1}|\nabla b_{\varepsilon,h}|^2 dx \leq \frac{c}{\varepsilon(1-\delta_h)}. \tag{21}$$

Using (20) combined with (21), we have

$$|I^2_{\varepsilon,h}| \le c\frac{\mu_{\varepsilon,h}}{\varepsilon^2\delta_h} + \frac{ch^2}{\varepsilon^2(1-\delta_h)},$$

and if $\delta_h \le \frac{1}{2}$,

$$|I^2_{\varepsilon,h}| \le c\frac{\mu_{\varepsilon,h}}{\varepsilon^2\delta_h} + \frac{ch^2}{\varepsilon^2}.$$

If we choose $\delta_h = \frac{\mu_{\varepsilon,h}}{\varepsilon h}$, we finally obtain

$$|I^2_{\varepsilon,h}| \le c\left(\frac{h}{\varepsilon} + \frac{h^2}{\varepsilon^2}\right),$$

from which, since $h = o(\varepsilon)$, we deduce

$$\lim_{\varepsilon\to 0} I^2_{\varepsilon,h} = 0.$$

It remains to study the asymptotic behaviour of $I^3_{\varepsilon,h}$ as $\varepsilon \to 0$:

$$I^3_{\varepsilon,h} = \int_\Omega \big((f_{\varepsilon,h} - p)^2 - \pi_h((f_{\varepsilon,h} - p_\varepsilon)^2)\big)\,dx.$$

We have $lim_{\varepsilon\to 0} I^3_{\varepsilon,h} = 0$ and this result is proved in [7] using the assumption $h = o(\kappa_\varepsilon)$.

In conclusion, going back to (12) and thanks to the previous computations we have shown that the family $\{J_\varepsilon(f_{\varepsilon,h}, b_{\varepsilon,h})\}_{\varepsilon,h}$, where J_ε is our functional without interpolation terms, is uniformly bounded.

Step 2 The family $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_{\varepsilon,h}$ is compact for the $L^2(\Omega)\times L^2(\Omega)$ strong topology.

Using step 1, we know that for some constant c

$$J_\varepsilon(f_{\varepsilon,h}, b_{\varepsilon,h}) \le c. \tag{22}$$

By the definition of J_ε and (22), we get

$$\int_\Omega (1 - b_{\varepsilon,h})^2 dx \le 4\varepsilon c,$$

hence $b_{\varepsilon,h}$ converges strongly to the constant function $b_0 \equiv 1$ in $L^2(\Omega)$. Let us examine $f_{\varepsilon,h}$.
By a truncation argument we may assume $\|f_{\varepsilon,h}\|_{L^\infty(\Omega)} \leq \|p\|_{L^\infty(\Omega)}$, hence we can extract from $\{f_{\varepsilon,h}\}$ a subsequence still denoted by $f_{\varepsilon,h}$ such that

$$f_{\varepsilon,h} \rightharpoonup g \quad \text{in} \quad L^\infty\text{-weak } *.$$

We have to show that the convergence is in $L^2(\Omega)$-strong.
Again as in step 1, we use triangles in $\mathbf{T}_h^1$ and we define

$$v_{\varepsilon,h}(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbf{T}_h^1 \\ b_{\varepsilon,h}(x) & \text{if} \quad x \in \mathbf{T}_h \setminus \mathbf{T}_h^1. \end{cases}$$

We have $v_{\varepsilon,h} \in SBV(\Omega)$ and $S_{v_{\varepsilon,h}} \subseteq \bigcup_{T \in T_h^1} \partial T$. Then, using (22), we have

$$\varepsilon \int_\Omega \varphi(|\nabla b_{\varepsilon,h}|)dx + \frac{1}{4\varepsilon}\int_\Omega (b_{\varepsilon,h}-1)^2 dx \leq c. \tag{23}$$

Using (23) and a similar reasoning to the one made in [5] or [9], we find

$$(1-\delta_h)\varepsilon \int_\Omega |\nabla v_{\varepsilon,h}|^2 dx + \frac{1}{4\varepsilon}\int_\Omega (v_{\varepsilon,h}-1)^2 dx + \frac{\delta_h \varepsilon}{\mu_{\varepsilon,h}} \sum_{T \in \mathbf{T}_h^1} |T| \leq c.$$

By the assumptions on the triangulation, the following inequality holds [9]:

$$\sum_{T \in \mathbf{T}_h^1} |T| \geq \frac{1}{6} \cdot h \sin\theta_0 \cdot \mathcal{H}^1(S_{v_{\varepsilon,h}}). \tag{24}$$

We set

$$\delta_h = \frac{6 \cdot \mu_{\varepsilon,h}}{\sin\theta_0 \cdot \varepsilon h}, \tag{25}$$

from which $\delta_h \to 0$ as $h(\varepsilon) \to 0$, since we have assumed $\mu_{\varepsilon,h} = o(\varepsilon h)$. We then obtain the following estimates:

$$(1-\delta_h)\varepsilon \int_\Omega |\nabla v_{\varepsilon,h}|^2 dx + \frac{1}{4\varepsilon}\int_\Omega (v_{\varepsilon,h}-1)^2 dx + \mathcal{H}^1(S_{v_{\varepsilon,h}}) \leq c, \tag{26}$$

for any $\delta_h \in (0,1)$, and

$$|\{x \,:\, v_{\varepsilon,h}(x) \neq b_{\varepsilon,h}(x)\}| \leq chJ_\varepsilon(f_{\varepsilon,h}, b_{\varepsilon,h}). \tag{27}$$

Let us define the function

$$\psi(y) = \int_0^y (1-\tau)d\tau.$$

We have $\psi(v_{\varepsilon,h}) \in SBV(\Omega), S_{\psi(v_{\varepsilon,h})} \subseteq S_{v_{\varepsilon,h}}$ and the approximate differential is given by $|\nabla\psi(v_{\varepsilon,h})| = (1-v_{\varepsilon,h})|\nabla v_{\varepsilon,h}|$.
Using (26) and the Young inequality, we deduce

$$(1-\delta_h)^{\frac{1}{2}} \int_\Omega |\nabla\psi(v_{\varepsilon,h})|dx + \mathcal{H}^1(S_{\psi(v_{\varepsilon,h})}) \leq c,$$

i.e. if δ_h is sufficiently small,

$$|D\psi(v_{\varepsilon,h})|(\Omega) \leq c. \tag{28}$$

Let now $\widehat{g}_{\varepsilon,h}$ be the function defined by

$$\widehat{g}_{\varepsilon,h}(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbf{T}_h^1 \\ f_{\varepsilon,h}(x) & \text{if} \quad x \in \mathbf{T}_h \setminus \mathbf{T}_h^1. \end{cases}$$

We have $\widehat{g}_{\varepsilon,h} \in SBV(\Omega)$ and $S_{\widehat{g}_{\varepsilon,h}} \subseteq \bigcup_{T\in T_h^1} \partial T$. Hence it follows $\mathcal{H}^1(S_{\widehat{g}_{\varepsilon,h}}) \leq c$, and

$$\int_\Omega v_{\varepsilon,h}^2 |\nabla\widehat{g}_{\varepsilon,h}|^2 dx \leq \int_\Omega b_{\varepsilon,h}^2 |\nabla f_{\varepsilon,h}|^2 dx \leq c. \tag{29}$$

Then we write the coarea formula (2) for the function $\psi(v_{\varepsilon,h})$ and we use (28):

$$\int_{\psi(0)}^{\psi(1)} P_\Omega\left(\{\psi(v_{\varepsilon,h}(x)) > t\}\right) dt = |D\psi(v_{\varepsilon,h})|(\Omega) \leq c. \tag{30}$$

According to Fatou's lemma the above inequality implies

$$\int_{\psi(0)}^{\psi(1)} \liminf_{\varepsilon\to 0} P_\Omega\left(\{\psi(v_{\varepsilon,h}(x)) > t\}\right) dt \leq c.$$

Therefore, possibly extracting a subsequence, there exists $t_0 \in (\psi(0), \psi(1))$ such that for all ε

$$P_\Omega\left(\{\psi(v_{\varepsilon,h}(x)) > t_0\}\right) \leq c.$$

If we set $\theta_0 = \psi^{-1}(t_0)$, we have $\theta_0 \in (0,1)$ and, since the function ψ is increasing,

$$P_\Omega\left(\{\psi(v_{\varepsilon,h}(x)) > t_0\}\right) = \mathcal{H}^1(\partial\{\psi(v_{\varepsilon,h}(x)) > t_0\}) = \mathcal{H}^1(\partial\{v_{\varepsilon,h}(x) > \theta_0\}) \leq c. \tag{31}$$

Moreover from (26) we have

$$\frac{1}{4\varepsilon}\int_\Omega (1-v_{\varepsilon,h})^2 dx \le c,$$

so that $v_{\varepsilon,h} \to 1$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Since $0 \le v_{\varepsilon,h} \le 1$, we have

$$|\{v_{\varepsilon,h}(x) \le \theta_0\}| \le \frac{1}{1-\theta_0}\int_\Omega (1-v_{\varepsilon,h})dx \to 0$$

as $\varepsilon \to 0$. Let us define

$$g_{\varepsilon,h}(x) = \begin{cases} 0 & \text{if} \quad v_{\varepsilon,h}(x) \le \theta_0 \\ \widehat{g}_{\varepsilon,h}(x) & \text{if} \quad v_{\varepsilon,h}(x) > \theta_0. \end{cases}$$

We have

$$\theta_0^2 \int_\Omega |\nabla g_{\varepsilon,h}|^2 dx \le \int_\Omega v_{\varepsilon,h}^2 |\nabla \widehat{g}_{\varepsilon,h}|^2 dx \le c. \tag{32}$$

Then $g_{\varepsilon,h} \in SBV(\Omega)$ and $S_{g_{\varepsilon,h}} \subseteq S_{\widehat{g}_{\varepsilon,h}} \cup \partial\{v_{\varepsilon,h} > \theta_0\}$. Hence, using (31), we get

$$\mathcal{H}^1(S_{g_{\varepsilon,h}}) \le \mathcal{H}^1(S_{\widehat{g}_{\varepsilon,h}}) + \mathcal{H}^1(\partial\{v_{\varepsilon,h}(x) > \theta_0\}) \le c.$$

On the other hand

$$\|g_{\varepsilon,h}\|_{L^\infty(\Omega)} \le \|\widehat{g}_{\varepsilon,h}\|_{L^\infty(\Omega)} \le \|f_{\varepsilon,h}\|_{L^\infty(\Omega)} \le \|p\|_{L^\infty(\Omega)}, \tag{33}$$

and from (29) and (32) it follows

$$\int_\Omega |\nabla g_{\varepsilon,h}|^2 dx \le \frac{1}{\theta_0^2}\int_\Omega v_{\varepsilon,h}^2 |\nabla \widehat{g}_{\varepsilon,h}|^2 dx \le \frac{1}{\theta_0^2}\int_\Omega b_{\varepsilon,h}^2 |\nabla f_{\varepsilon,h}|^2 dx \le \frac{c}{\theta_0^2}.$$

Therefore, according to the compactness Ambrosio Theorem [2], there exists $g \in SBV(\Omega) \cap L^\infty(\Omega)$ such that, up to a subsequence,

$$\begin{array}{rll} g_{\varepsilon,h} & \to g \quad \text{in} & L^2(\Omega) - \text{strong} \\ \nabla g_{\varepsilon,h} & \rightharpoonup \nabla g \quad \text{in} & L^1(\Omega) - \text{weak}. \end{array}$$

But we also have $\widehat{g}_{\varepsilon,h} \to g$ in $L^2(\Omega)$-strong since

$$\begin{aligned} \int_\Omega |\widehat{g}_{\varepsilon,h} - g|^2 dx &\le \int_\Omega |g_{\varepsilon,h} - g|^2 dx + \int_\Omega |\widehat{g}_{\varepsilon,h} - g_{\varepsilon,h}|^2 dx \\ &= \int_\Omega |g_{\varepsilon,h} - g|^2 dx + \int_{\{\widehat{g}_{\varepsilon,h} \ne g_{\varepsilon,h}\}} |\widehat{g}_{\varepsilon,h} - g_{\varepsilon,h}|^2 dx. \end{aligned} \tag{34}$$

We know that $|\{\widehat{g}_{\varepsilon,h} \neq g_{\varepsilon,h}\}| = |\{v_{\varepsilon,h} \leq \theta_0\}| \to 0$ as $\varepsilon \to 0$, so that, using (33), we obtain

$$\widehat{g}_{\varepsilon,h} \to g \quad \text{in} \quad L^2(\Omega) - \text{strong}.$$

Furthermore, $f_{\varepsilon,h}$ also tends to g as ε goes to 0:

$$\int_\Omega |f_{\varepsilon,h} - g|^2 dx \leq \int_\Omega |\widehat{g}_{\varepsilon,h} - g|^2 dx + \int_{\{f_{\varepsilon,h} \neq \widehat{g}_{\varepsilon,h}\}} |f_{\varepsilon,h} - \widehat{g}_{\varepsilon,h}|^2 dx,$$

and, using (27), we easily get that $|\{f_{\varepsilon,h} \neq \widehat{g}_{\varepsilon,h}\}| \to 0$ as $\varepsilon \to 0$. The convergence of $f_{\varepsilon,h}$ to g then follows and the proof of the theorem is completed by setting $f \equiv g$. ■

4 Lower limit

In this section we show the lower inequality of Γ-convergence for the family of functionals $J_{\varepsilon,h}$ to the Mumford and Shah functional.

First we recall some properties of the one-dimensional sections of functions $f \in SBV(\Omega)$. Let $\nu \in \mathbf{S}^1 = \{x \in \mathbf{R}^2 : |x| = 1\}$ be a fixed direction and let $E \subset \mathbf{R}^2$. We set

$$\begin{aligned} \Pi_\nu &= \{x \in \mathbf{R}^2 : \langle x, \nu \rangle = 0\}, \\ E_x &= \{t \in \mathbf{R} : x + t\nu \in E\} \quad (x \in \Pi_\nu), \\ E_\nu &= \{x \in \Pi_\nu : E_x \neq \emptyset\}. \end{aligned}$$

The sets E_x are the 1-dimensional slices of E indexed by $x \in \Pi_\nu$, and E_ν is the projection of E on Π_ν. Given $f \in SBV(\Omega)$, we define for $\mathcal{H}^1$–a.e. $x \in \Omega_\nu$ the restriction

$$f_x(t) = f(x + t\nu) \qquad \text{for a.e. } t \in \Omega_x.$$

The following slicing result can be obtained from [1], Theorem 3.3.

Lemma 4.1 *Let $f \in SBV(\Omega)$ and $\nu \in \mathbf{S}^1$, then for $\mathcal{H}^1$-a.e. $x \in \Omega_\nu$ we have:*
(a) $f_x \in SBV(\Omega_x)$;
(b) $f'_x(t) = \langle \nabla f(x + t\nu), \nu \rangle$ *for a.e.* $t \in \Omega_x$;
(c) $S_{f_x} = (S_f)_x$.

The proof of the following lemma can be found in [13], Section 3.2.22.

Lemma 4.2 *For every countably $(\mathcal{H}^1, 1)$ rectifiable set $E \subset \mathbf{R}^2$ there exists a Borel function $\nu_E : E \to \mathbf{S}^1$ such that*

$$\int_E |\langle \nu, \nu_E(x) \rangle| \, d\mathcal{H}^1(x) = \int_{E_\nu} \mathcal{H}^0(E_x) \, d\mathcal{H}^1(x) \qquad \forall \nu \in \mathbf{S}^1.$$

The function $\nu_E(x)$ is a normal unit vector to E at x in an approximate sense. We prove the following theorem.

Theorem 4.3 *Assume that $h = o(\kappa_\varepsilon)$ and that $\mu_{\varepsilon,h} = o(\varepsilon h)$. Then, for every pair $(f, b) \in X(\Omega)$ and for every sequence $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_\varepsilon \subset X(\Omega)$ converging to (f, b) in $[L^2(\Omega)]^2$ as $\varepsilon \to 0^+$, we have*

$$J^{MS}(f, b) \leq \liminf_{\varepsilon \to 0^+} J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}).$$

Proof.

Up to the extraction of a subsequence, we may assume that $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_\varepsilon \subset V_h(\Omega) \times V_h(\Omega; [0, 1])$, and

$$+\infty > L = \liminf_{\varepsilon \to 0^+} J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) = \lim_{\varepsilon \to 0^+} J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}), \tag{35}$$

otherwise the result is trivial. To simplify the notation we set $(f_\varepsilon, b_\varepsilon) = (f_{\varepsilon,h(\varepsilon)}, b_{\varepsilon,h(\varepsilon)})$, and we assume that $(f_\varepsilon, b_\varepsilon)$ converges a.e. to (f, b) as $\varepsilon \to 0^+$.

Using (12) and (13), since in the proof of Theorem 3.1 it has been shown that

$$I^1_{\varepsilon,h} \leq 0, \qquad \lim_{\varepsilon \to 0} I^2_{\varepsilon,h} = \lim_{\varepsilon \to 0} I^3_{\varepsilon,h} = 0,$$

we have

$$\liminf_{\varepsilon \to 0^+} J_{\varepsilon,h}(f_\varepsilon, b_\varepsilon) \geq \liminf_{\varepsilon \to 0^+} J_\varepsilon(f_\varepsilon, b_\varepsilon),$$

where J_ε is the functional without Lagrange interpolation. Then, it is enough to show the lower inequality for J_ε:

$$J^{MS}(f, b) \leq \liminf_{\varepsilon \to 0^+} J_\varepsilon(f_\varepsilon, b_\varepsilon). \tag{36}$$

Up to the extraction of a further subsequence, we can again assume that the liminf at the right-hand side of (36) is a finite limit.

If b were not identically equal to 1, then by the Fatou's lemma we would get

$$L \geq \liminf_{\varepsilon \to 0^+} \int_{\{b \neq 1\}} \frac{(b_\varepsilon - 1)^2}{4\epsilon} dx \geq +\infty,$$

which contradicts the assumption that $L < +\infty$. Therefore, we will assume that $b \equiv 1$. The proof follows by proving separately the following inequalities:

$$\liminf_{\varepsilon\to 0^+} \int_\Omega (b_\varepsilon^2 + \kappa_\varepsilon)|\nabla f_\varepsilon|^2 dx \geq \int_\Omega |\nabla f|^2 dx, \tag{37}$$

$$\liminf_{\varepsilon\to 0^+} \int_\Omega \left[\varepsilon\varphi(|\nabla b_\varepsilon|) + \frac{1}{4\varepsilon}(1-b_\varepsilon)^2\right] dx \geq \mathcal{H}^1(S_f). \tag{38}$$

The continuity of the term $\int |f-p|^2 dx$ with respect to the strong $L^2(\Omega)$ topology then completes the proof.

Possibly extracting a subsequence (this is allowed, since we are assuming that $J_\varepsilon(f_\varepsilon, b_\varepsilon)$ is converging) we can assume that both liminf in (37) and (38) are finite limits, denoted by L_1 and L_2 respectively. Then we divide the proof in two steps.

Step 1 Proof of the inequality (37).

Reasoning as in the proof of Theorem 3.1, by replacing in (30) $\psi(0)$ with $\psi(a)$ for some $a \in (0,1)$, we can find $\theta_0 \in (a,1)$, a function $g \in SBV(\Omega)$, and a sequence $\{g_\varepsilon\}_\varepsilon \subset SBV(\Omega)$ such that, using (29) and (32),

$$\theta_0^2 \int_\Omega |\nabla g_\varepsilon|^2 dx \leq \int_\Omega b_\varepsilon^2 |\nabla f_\varepsilon|^2 dx \leq L_1 + 1, \tag{39}$$

$g_\varepsilon \to g$ strongly in $L^2(\Omega)$, and $\nabla g_\varepsilon \rightharpoonup \nabla g$ weakly in $L^1(\Omega)$. Moreover, $f_\varepsilon \to g$ strongly in $L^2(\Omega)$, so that $f = g$ a.e.. Hence, by the properties of the approximate differential, we have $\nabla g = \nabla f$ a.e. and, using (39) and the semicontinuity theorem in SBV [2], we get

$$L_1 \geq \theta_0^2 \liminf_{\varepsilon\to 0^+} \int_\Omega |\nabla g_\varepsilon|^2 dx \geq \theta_0^2 \int_\Omega |\nabla f|^2 dx.$$

By letting $a \to 1$, also $\theta_0 \to 1$ and we obtain (37).

Step 2 Proof of the inequality (38).

We define

$$w_\varepsilon(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbf{T}_h^1 \\ b_\varepsilon(x) & \text{if} \quad x \in \mathbf{T}_h \setminus \mathbf{T}_h^1. \end{cases}$$

We have $w_\varepsilon \in SBV(\Omega)$ and $S_{w_\varepsilon} \subseteq \bigcup_{T\in T_h^1} \partial T$.

Let $A \subset \Omega$ be open; by the same method used to obtain (26) we find

$$\begin{aligned} J_\varepsilon(f_\varepsilon, b_\varepsilon) &\geq \int_A b_\varepsilon^2 |\nabla f_\varepsilon|^2 dx + (1-\delta_h)\varepsilon \int_A |\nabla w_\varepsilon|^2 dx \\ &+ \frac{1}{4\varepsilon}\int_A (b_\varepsilon - 1)^2 dx + \mathcal{H}^1(S_{w_\varepsilon} \cap A), \end{aligned}$$

where ∇w_ε denotes the approximate differential of w_ε. Let now $\nu \in \mathbf{S}^1$ be fixed; using Fubini's Theorem, Lemma 4.2, and Lemma 4.1 (b)-(c), we have

$$\begin{aligned} J_\varepsilon(f_\varepsilon, b_\varepsilon) &\geq \int_A b_\varepsilon^2 |\langle \nabla f_\varepsilon, \nu\rangle|^2 dx + (1-\delta_h)\varepsilon \int_A |\langle \nabla w_\varepsilon, \nu\rangle|^2 dx \\ &+ \frac{1}{4\varepsilon}\int_A (b_\varepsilon - 1)^2 dx + \int_{S_{w_\varepsilon}\cap A} |\langle \nu_{w_\varepsilon}, \nu\rangle| d\mathcal{H}^1 \\ &= \int_{A_\nu} d\mathcal{H}^1(x) \left\{ \int_{A_x} b_{\varepsilon x}^2 |f'_{\varepsilon x}|^2 dt + (1-\delta_h)\varepsilon \int_{A_x} |w'_{\varepsilon x}|^2 dt \right. \\ &+ \left. \frac{1}{4\varepsilon}\int_{A_x} (b_{\varepsilon x} - 1)^2 dt + \mathcal{H}^0(S_{w_{\varepsilon x}} \cap A_x) \right\}, \end{aligned} \tag{40}$$

where ν_{w_ε} is the approximate unit normal to S_{w_ε}. By using Fatou's Lemma we get

$$\begin{aligned} &\liminf_{\varepsilon \to 0^+} \left\{ \int_{A_x} b_{\varepsilon x}^2 |f'_{\varepsilon x}|^2 dt + (1-\delta_h)\varepsilon \int_{A_x} |w'_{\varepsilon x}|^2 dt \right. \\ &+ \left. \frac{1}{4\varepsilon}\int_{A_x} (b_{\varepsilon x} - 1)^2 dt + \mathcal{H}^0(S_{w_{\varepsilon x}} \cap A_x) \right\} < +\infty \end{aligned}$$

for $\mathcal{H}^1$-a.e. $x \in A_\nu$. Hence, up to the extraction of a subsequence depending on x but not on A, the quantity inside the braces is uniformly bounded with respect to ε by a positive constant K_x for $\mathcal{H}^1$-a.e. $x \in A_\nu$. Then we have

$$\int_{A_x} b_{\varepsilon x}^2 |f'_{\varepsilon x}|^2 dt + \frac{1}{4\varepsilon}\int_{A_x} (b_{\varepsilon x} - 1)^2 dt \leq K_x, \tag{41}$$

and $b_{\varepsilon x} \to 1$ a.e. on A_x for $\mathcal{H}^1$–a.e. $x \in A_\nu$ as $\varepsilon \to 0$. Moreover, using Lemma 4.1 (a) we have $f_x \in SBV(A_x)$ for $\mathcal{H}^1$–a.e. $x \in A_\nu$.

Assume now that $S_{f_x} \cap A_x \neq \emptyset$ and that $t_0 \in S_{f_x} \cap A_x$. Let $\rho_0 > 0$ be such that $(t_0 - \rho_0, t_0 + \rho_0) \subset A_x$ and let us fix $\rho \in (0, \rho_0)$. Since $b_\varepsilon \in V_h(\Omega)$, for any ε the interval $(t_0 - \rho, t_0 + \rho)$ is a finite union of intervals on which the function $b_{\varepsilon x}$ is affine. Then there are two possibilities:

(a) there exists $\delta > 0$ such that for any $\varepsilon < \delta$ we have $w_{\varepsilon x}(t) = b_{\varepsilon x}(t)$ for a.e. $t \in (t_0 - \rho, t_0 + \rho)$ (since $b_{\varepsilon x}$ is continuous, the previous equality is true for any $t \in (t_0 - \rho, t_0 + \rho)$);

(b) for any $\delta > 0$ there exist $\varepsilon < \delta$ and a set of positive measure $I_\varepsilon \subset (t_0 - \rho, t_0 + \rho)$ such that $w_{\varepsilon x}(t) \neq b_{\varepsilon x}(t)$ for a.e. $t \in I_\varepsilon$.

Case (a). Set $B_\rho(t_0) = (t_0 - \rho, t_0 + \rho)$. We have $\{w_{\varepsilon x}\}_\varepsilon \subset W^{1,2}(B_\rho(t_0))$, and

$$\begin{aligned} & (1-\delta_h)\varepsilon \int_{B_\rho(t_0)} |w'_{\varepsilon x}|^2 dt + \frac{1}{4\varepsilon} \int_{B_\rho(t_0)} (b_{\varepsilon x} - 1)^2 dt \\ \geq \; & (1-\delta_h) \left\{ \varepsilon \int_{B_\rho(t_0)} |w'_{\varepsilon x}|^2 dt + \frac{1}{4\varepsilon} \int_{B_\rho(t_0)} (w_{\varepsilon x} - 1)^2 dt \right\}. \end{aligned}$$

Since $t_0 \in S_{f_x}$, using (41) and arguing as in [4], Lemma 2.1, there exists a sequence of points $\{t_\varepsilon\}_\varepsilon \subset (t_0 - \rho, t_0 + \rho)$ such that

$$\lim_{\varepsilon \to 0^+} b_{\varepsilon x}(t_\varepsilon) = 0. \tag{42}$$

Then we have, since $w_{\varepsilon x} = b_{\varepsilon x}$ on $B_\rho(t_0)$,

$$w_{\varepsilon x} \to 1 \quad \text{a.e. on } B_\rho(t_0), \qquad \lim_{\varepsilon \to 0^+} w_{\varepsilon x}(t_\varepsilon) = 0,$$

and there exist points y_ε, $z_\varepsilon \in (t_0 - \rho, t_0 + \rho)$, with $y_\varepsilon < t_\varepsilon < z_\varepsilon$, such that

$$\lim_{\varepsilon \to 0^+} w_{\varepsilon x}(y_\varepsilon) = 1, \qquad \lim_{\varepsilon \to 0^+} w_{\varepsilon x}(z_\varepsilon) = 1.$$

Now we have

$$\begin{aligned} (1-\delta_h) \left\{ \varepsilon \int_{B_\rho(t_0)} |w'_{\varepsilon x}|^2 dt + \frac{1}{4\varepsilon} \int_{B_\rho(t_0)} (w_{\varepsilon x} - 1)^2 dt \right\} \quad & \geq \\ (1-\delta_h) \int_{B_\rho(t_0)} |w'_{\varepsilon x}||w_{\varepsilon x} - 1| dt \quad & \geq \\ (1-\delta_h) \int_{y_\varepsilon}^{z_\varepsilon} |w'_{\varepsilon x}||w_{\varepsilon x} - 1| dt. & \end{aligned}$$

It is easy to show that

$$\begin{aligned}\int_{y_\varepsilon}^{z_\varepsilon} |w'_{\varepsilon x}||w_{\varepsilon x}-1|dt &\geq \int_{y_\varepsilon}^{t_\varepsilon} -w'_{\varepsilon x}(1-w_{\varepsilon x})dt + \int_{t_\varepsilon}^{z_\varepsilon} w'_{\varepsilon x}(1-w_{\varepsilon x})dt \\ &= \frac{1}{2}[(1-w_{\varepsilon x})^2]_{y_\varepsilon}^{t_\varepsilon} - \frac{1}{2}[(1-w_{\varepsilon x})^2]_{t_\varepsilon}^{z_\varepsilon} \to 1.\end{aligned}$$

It then follows

$$\liminf_{\varepsilon\to 0^+}(1-\delta_h)\left\{\varepsilon\int_{B_\rho(t_0)}|w'_{\varepsilon x}|^2dt + \frac{1}{4\varepsilon}\int_{B_\rho(t_0)}(b_{\varepsilon x}-1)^2dt\right\} \geq 1. \tag{43}$$

Case (b). Either

$$\liminf_{\varepsilon\to 0^+}\mathcal{H}^0(S_{w_{\varepsilon x}}\cap B_\rho(t_0)) \geq 1, \tag{44}$$

or there exists a subsequence such that

$$\mathcal{H}^0(S_{w_{\varepsilon x}}\cap B_\rho(t_0)) = 0. \tag{45}$$

Notice that (45) implies that $w_{\varepsilon x}$ is continuous on $B_\rho(t_0)$ and that the set I_ε can be chosen as an open interval in $B_\rho(t_0)$. Assume that (45) holds true. Using (16) and (18) we have

$$\varepsilon\int_\Omega \varphi(|\nabla b_\varepsilon|)dx \geq \frac{\delta_h\varepsilon}{\mu_{\varepsilon,h}}\sum_{T\in\mathbf{T}_h^1}|T|,$$

from which, using (25), it follows

$$|\{x\,:\,w_\varepsilon(x)\neq b_\varepsilon(x)\}| \leq ch(L_2+1),$$

so that

$$\lim_{\varepsilon\to 0^+}|\{t\in A_x\,:\,w_{\varepsilon x}(t)\neq b_{\varepsilon x}(t)\}| = 0, \tag{46}$$

for $\mathcal{H}^1$-a.e. $x\in A_\nu$. Therefore $w_{\varepsilon x}\to 1$ a.e. on A_x for $\mathcal{H}^1$-a.e. $x\in A_\nu$. Let $x\in A_\nu$ be such that (46) holds true; if we denote by $I_\varepsilon\subset(t_0-\rho,t_0+\rho)$ the intervals such that $b_{\varepsilon x}$ is affine in I_ε and $w_{\varepsilon x}(\hat{t}_\varepsilon)\neq b_{\varepsilon x}(\hat{t}_\varepsilon)$ for some $\hat{t}_\varepsilon\in I_\varepsilon$, then we have

$$\lim_{\varepsilon\to 0^+}\sum_{I_\varepsilon\subset(t_0-\rho,t_0+\rho)}|I_\varepsilon| = 0. \tag{47}$$

Then there are two possibilities:

(i) there exists $\delta > 0$ such that for any $\varepsilon < \delta$ we have $w_{\varepsilon x}(t) = b_{\varepsilon x}(t)$ for any $t \in (t_0 - \frac{\rho}{2}, t_0 + \frac{\rho}{2})$;

(ii) for any $\delta > 0$ there exist $\varepsilon < \delta$ and an interval I_ε with the above properties such that

$$I_\varepsilon \cap (t_0 - \frac{\rho}{2}, t_0 + \frac{\rho}{2}) \neq \emptyset.$$

In the case (i), arguing as in the case (a), we again obtain the inequality (43).

Let us consider the case (ii). In this case, we cannot directly use the same proof as for (43) since we do not know the position of $\hat{t}_\varepsilon$ with respect to t_ε. Then we have to consider two subcases. As $w_{\varepsilon x} \to 1$ a.e. for $\mathcal{H}^1$-a.e. $x \in A_\nu$, we may find a sequence of points $\{z_\varepsilon\}_\varepsilon \subset (t_0 + \frac{\rho}{2}, t_0 + \rho)$ such that

$$\lim_{\varepsilon \to 0^+} w_{\varepsilon x}(z_\varepsilon) = 1. \tag{48}$$

Let us set

$$p_\varepsilon = \sup\{t \in (t_0 - \rho, t_0 + \rho) : \ t < z_\varepsilon, \ w_{\varepsilon x}(t) \neq b_{\varepsilon x}(t)\}.$$

Using (45), the function $w_{\varepsilon x}$ is continuous in $(t_0 - \rho, t_0 + \rho)$, so that we have $w_{\varepsilon x}(p_\varepsilon) = 0$ and $w_{\varepsilon x}(t) = b_{\varepsilon x}(t)$ for any $t \in [p_\varepsilon, z_\varepsilon]$ for any ε small enough. Then we have

$$\begin{aligned} & (1-\delta_h)\varepsilon \int_{B_\rho(t_0)} |w'_{\varepsilon x}|^2 dt + \frac{1}{4\varepsilon} \int_{B_\rho(t_0)} (b_{\varepsilon x} - 1)^2 dt \\ \geq \ & (1-\delta_h) \left\{ \varepsilon \int_{p_\varepsilon}^{z_\varepsilon} |w'_{\varepsilon x}|^2 dt + \frac{1}{4\varepsilon} \int_{p_\varepsilon}^{z_\varepsilon} (w_{\varepsilon x} - 1)^2 dt \right\}. \end{aligned}$$

Using (48), $w_{\varepsilon x}(p_\varepsilon) = 0$ and the fact that $w_{\varepsilon x}(t) = b_{\varepsilon x}(t)$ on $[p_\varepsilon, z_\varepsilon]$, by making the same calculus as for (43) it follows that

$$\liminf_{\varepsilon \to 0^+} (1-\delta_h) \left\{ \varepsilon \int_{p_\varepsilon}^{z_\varepsilon} |w'_{\varepsilon x}|^2 dt + \frac{1}{4\varepsilon} \int_{p_\varepsilon}^{z_\varepsilon} (w_{\varepsilon x} - 1)^2 dt \right\} \geq \frac{1}{2}. \tag{49}$$

Now we may find a sequence of points $\{y_\varepsilon\}_\varepsilon \subset (t_0 - \rho, t_0 - \rho/2)$ such that

$$\lim_{\varepsilon \to 0^+} w_{\varepsilon x}(y_\varepsilon) = 1.$$

Let us set

$$q_\varepsilon = \inf\{t \in (t_0-\rho, t_0+\rho):\ t > y_\varepsilon,\ w_{\varepsilon x}(t) \neq b_{\varepsilon x}(t)\}.$$

Arguing as before we find

$$\liminf_{\varepsilon\to 0^+}(1-\delta_h)\left\{\varepsilon\int_{y_\varepsilon}^{q_\varepsilon}|w'_{\varepsilon x}|^2dt + \frac{1}{4\varepsilon}\int_{y_\varepsilon}^{q_\varepsilon}(w_{\varepsilon x}-1)^2dt\right\} \geq \frac{1}{2}. \tag{50}$$

Since $q_\varepsilon \leq p_\varepsilon$ using (49) and (50) we then obtain

$$\liminf_{\varepsilon\to 0^+}(1-\delta_h)\left\{\varepsilon\int_{B_\rho(t_0)}|w'_{\varepsilon x}|^2dt + \frac{1}{4\varepsilon}\int_{B_\rho(t_0)}(b_{\varepsilon x}-1)^2dt\right\} \geq 1, \tag{51}$$

which concludes the study of the case (ii). Collecting the results (43), (44), (51) of all the cases considered, we get

$$\liminf_{\varepsilon\to 0^+}\left\{(1-\delta_h)\varepsilon\int_{B_\rho(t_0)}|w'_{\varepsilon x}|^2dt + \frac{1}{4\varepsilon}\int_{B_\rho(t_0)}(b_{\varepsilon x}-1)^2dt\right.$$
$$\left.+\mathcal{H}^0(S_{w_{\varepsilon x}}\cap B_\rho(t_0))\right\} \geq 1 \qquad \forall \rho \in (0,\rho_0),$$

from which it follows that the set $S_{f_x}\cap A_x$ is finite. Then, using the superadditivity of the liminf operator [4], we obtain

$$\liminf_{\varepsilon\to 0^+}\left\{(1-\delta_h)\varepsilon\int_{A_x}|w'_{\varepsilon x}|^2dt + \frac{1}{4\varepsilon}\int_{A_x}(b_{\varepsilon x}-1)^2dt + \mathcal{H}^0(S_{w_{\varepsilon x}}\cap A_x)\right\}$$
$$\geq \mathcal{H}^0(S_{f_x}\cap A_x) \qquad \text{for } \mathcal{H}^1-\text{a.e. } x\in A_\nu. \tag{52}$$

Then, using (40), Fatou's Lemma, inequality (52), Lemma 4.1 (c), and Lemma 4.2, we get

$$\begin{aligned} L_2 \ \geq\ & \liminf_{\varepsilon\to 0^+}\int_{A_\nu}d\mathcal{H}^1(x)\left\{(1-\delta_h)\varepsilon\int_{A_x}|w'_{\varepsilon x}|^2dt + \frac{1}{4\varepsilon}\int_{A_x}(b_{\varepsilon x}-1)^2dt\right. \\ +\ & \left.\mathcal{H}^0(S_{w_{\varepsilon x}}\cap A_x)\right\} \geq \int_{A_\nu}\mathcal{H}^0(S_{f_x}\cap A_x)d\mathcal{H}^1(x) = \int_{S_f\cap A}|\langle\nu_f,\nu\rangle|d\mathcal{H}^1, \end{aligned} \tag{53}$$

for any open subset $A \subset \Omega$ and any $\nu \in \mathbf{S}^1$, where ν_f is the approximate unit normal to S_f.

Now we have (see [3, 4])

$$\mathcal{H}^1(S_f) = \sup \sum_{i=1}^{k} \int_{S_f \cap A_i} |\langle \nu_f, \nu_i \rangle| d\mathcal{H}^1, \tag{54}$$

where the supremum is taken among all finite families (A_i, ν_i) with $A_i \subset \Omega$ open and pairwise disjoint, and $\nu_i \in D$, where D is a countable dense subset of $\mathbf{S}^1$. By (53) and the superadditivity of the liminf operator, any of the sums in (54) is less than L_2, from which the inequality (38) follows and the proof of the theorem is concluded. ■

5 Upper limit

In this section we show the upper inequality of Γ-convergence for the family of functionals $J_{\varepsilon,h}$ to the Mumford and Shah functional.

Let $\mathcal{W}(\Omega)$ be the set of functions $f \in SBV(\Omega)$ verifying

(i) S_f is essentially closed, i.e. $\mathcal{H}^1(\overline{S}_f \setminus S_f) = 0$;

(ii) S_f is polygonal, i.e. it is the intersection of Ω with a finite union of segments;

(iii) $f \in W^{k,\infty}(\Omega \setminus \overline{S}_f)$ for every $k \in \mathbf{N}$.

We use the following density result of the class $\mathcal{W}(\Omega)$ in $SBV(\Omega)$ proved by Cortesani and Toader [11].

Theorem 5.1 *Let $f \in SBV(\Omega) \cap L^\infty(\Omega)$ be such that*

$$\mathcal{H}^1(S_f) < +\infty, \qquad \nabla f \in L^2(\Omega; \mathbf{R}^2).$$

Then there exists a sequence $\{f^{(n)}\}_n \subset \mathcal{W}(\Omega)$ such that

(i) *$f^{(n)} \to f$ in $L^1(\Omega)$-strong;*
$\nabla f^{(n)} \to \nabla f$ in $L^2(\Omega; \mathbf{R}^2)$-strong;

(ii)

$$\limsup_{n\to+\infty} \|f^{(n)}\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)};$$

(iii)

$$\limsup_{n\to+\infty} \mathcal{H}^1(S_{f^{(n)}}) \leq \mathcal{H}^1(S_f),$$

and (iii) is an equality if $\mathcal{H}^1(\partial\Omega \cap S_f) = 0$.

Remark: in fact in (i), we have the convergence $f^{(n)} \to f$ in $L^2(\Omega)$.

To show the upper inequality we will use the functional $J^{AT}_{\varepsilon,h} : X(\Omega) \to [0, +\infty]$ which is the Ambrosio-Tortorelli functional [3] where we have incorporated a Lagrange interpolation:

$$\begin{aligned} J^{AT}_{\varepsilon,h}(f,b) &= \int_\Omega (\pi_h(b^2) + \kappa_\varepsilon)|\nabla f|^2 dx + \int_\Omega \left[\varepsilon|\nabla b|^2 + \frac{1}{4\varepsilon}\pi_h\left((1-b)^2\right)\right] dx \\ &+ \int_\Omega \pi_h\left((f-p_\varepsilon)^2\right) dx \qquad \text{if } (f,b) \in V_h(\Omega) \times V_h(\Omega;[0,1]), \end{aligned}$$

and $J^{AT}_{\varepsilon,h}(f,b) = +\infty$ elsewhere in $X(\Omega)$.

We prove the following proposition.

Proposition 5.2 *Assume that* $h = o(\kappa_\varepsilon)$. *Then, for every pair* $(f,b) \in X(\Omega)$ *there exists a sequence* $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_\varepsilon \subset X(\Omega)$ *converging to* (f,b) *in* $[L^2(\Omega)]^2$ *as* $\varepsilon \to 0^+$ *such that*

$$J^{MS}(f,b) \geq \limsup_{\varepsilon\to 0^+} J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}).$$

Proof.

Since $\varphi(t) \leq t^2$ for any $t \geq 0$, we have $J_{\varepsilon,h}(f,b) \leq J^{AT}_{\varepsilon,h}(f,b)$. Then, it is enough to show the upper inequality for $J^{AT}_{\varepsilon,h}$:

$$J^{MS}(f,b) \geq \limsup_{\varepsilon\to 0^+} J^{AT}_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}). \tag{55}$$

We can assume that the left-hand side of (55) is finite, otherwise the result is trivial. Hence, we will suppose $f \in SBV(\Omega)$, $\nabla f \in L^2(\Omega;\mathbf{R}^2)$, and $b \equiv 1$.

Let us first suppose that $f \in \mathcal{W}(\Omega)$, so that S_f is a finite union of segments. Under this assumption, Bellettini and Coscia [7] proved that there exists a sequence $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_\varepsilon \subset X(\Omega)$ converging to (f, b) in $[L^2(\Omega)]^2$ as $\varepsilon \to 0^+$ such that

$$J^{MS}(f, b) \geq \limsup_{\varepsilon\to 0^+} J^{BC}_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}). \tag{56}$$

We claim that (56) is still true for $J^{AT}_{\varepsilon,h}$.

We examine the difference $J^{AT}_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) - J^{BC}_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h})$:

$$\begin{aligned} J^{AT}_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) - J^{BC}_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) &= \int_\Omega \left[\pi_h(b^2_{\varepsilon,h}) - b_{\varepsilon,h}\right] |\nabla f_{\varepsilon,h}|^2 dx \\ &+ \frac{1}{4\varepsilon}\int_\Omega \left[\pi_h\left((1-b_{\varepsilon,h})^2\right) - \pi_h(1-b^2_{\varepsilon,h})\right] dx \\ &= I^1_{\varepsilon,h} + I^2_{\varepsilon,h}. \end{aligned}$$

Since $b^2_{\varepsilon,h} \leq b_{\varepsilon,h}$ and $b_{\varepsilon,h} \in V_h(\Omega)$, we have $\pi_h(b^2_{\varepsilon,h}) \leq \pi_h(b_{\varepsilon,h}) = b_{\varepsilon,h}$, therefore $I^1_{\varepsilon,h} \leq 0$.

By the same way, $\pi_h\left((1-b_{\varepsilon,h})^2\right) - \pi_h(1-b^2_{\varepsilon,h}) = 2\pi_h\left(b^2_{\varepsilon,h} - b_{\varepsilon,h}\right) \leq 0$, and $I^2_{\varepsilon,h} \leq 0$. Hence

$$J^{AT}_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) \leq J^{BC}_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}),$$

from which we deduce (55). Since $J_{\varepsilon,h} \leq J^{AT}_{\varepsilon,h}$, we also get

$$\limsup_{\varepsilon\to 0^+} J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) \leq J^{MS}(f, 1),$$

which yields the upper inequality under the assumption that $f \in \mathcal{W}(\Omega)$. The statement of the proposition then follows from Theorem 5.1 and a diagonal argument. ■

Finally, we can show Theorem 2.1, which summarizes all the results of this paper.

Proof of Theorem 2.1.

The Γ-convergence of the family $J_{\varepsilon,h}$ to the functional J^{MS} follows from Theorem 4.3 and Proposition 5.2.

For any $\varepsilon > 0$ the existence of a minimizer $\{(f_{\varepsilon,h}, b_{\varepsilon,h})\}_\varepsilon$ of $J_{\varepsilon,h}$ is obtained easily since in fact we search for a minimizer in a compact subset of the space $V_h(\Omega) \times V_h(\Omega; [0, 1])$ which is of finite dimension. Moreover, there exists a constant $c > 0$ such that $J_{\varepsilon,h}(f_{\varepsilon,h}, b_{\varepsilon,h}) \leq c$ for any ε.

Then, using Theorem 3.1, possibily extracting a subsequence, the family of minimizers $(f_{\varepsilon,h}, b_{\varepsilon,h})$ converges strongly in $L^2(\Omega) \times L^2(\Omega)$ to a pair $(f, 1)$ with $f \in SBV(\Omega)$. The fact that $(f, 1)$ is a minimizer of the Mumford-Shah functional follows directly from the Γ-convergence of the family $J_{\varepsilon,h}$ and the property (5) of Γ-convergence. ■

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