Rescaled viscosity solutions

of a quasistatic evolution problem

in non-associative plasticity

Ph.D. Thesis

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Contents

Introduction

This thesis is devoted to the study of a generalized notion of quasistatic evolution for a problem in non-associative plasticity, namely the Cam-Clay model. By the term *quasistatic* we mean that the evolution we are interested in has a so slow time scale that the system is assumed to be in equilibrium at each instant. Our notion is based on a viscoplastic approximation and a time rescaling. For reasons that will be made clearer later in this introduction, it will be introduced for this specific problem, but we think that the underlying ideas can be adapted to more general contexts.

The choice of the model we study is motivated by its considerable interest for the engineering community. It gives the conceptual framework to analyse the inelastic behavior of fine grained soils. Its framework is small strain elasto-plasticity. The linear strain Eu is defined as the symmetric part of the gradient of the displacement u with respect to a reference configuration Ω. Moreover, the strain is additively decomposed into elastic and plastic part, namely $Eu = e + p$, where the elastic part e determines the stress σ through the linear constitutive relation $\sigma = \mathbb{C}e$. The stress satisfies the standard equilibrium condition $-\operatorname{div}\sigma = f$ in Ω, where f denotes a time dependent body force.

As it is typical in plasticity, the stress is constrained to lie in a compact convex set $K(\zeta)$ of the space $\mathbb{M}^{N\times N}_{sym}$ of symmetric $n\times n$ matrices, whose size is controlled by a scalar parameter ζ and whose boundary represents the yield surface, i.e., the plastic flow is produced only when the stress meets $\partial K(\zeta)$. The other two main ingredients are the evolution laws for the plastic strain p and for the internal variable ζ. To write them explicitly, we introduce a new internal variable z, related to ζ by the equality $\zeta = V(z)$. The function V is assumed to be strictly monotone and to satisfy the condition

$$V(z) \geq \zeta_{min} > 0 \quad \text{for every } z \in \mathbb{R}\,,$$

which implies that $\zeta \geq \zeta_{min} > 0$ and prevents the set $K(\zeta)$ from shrinking to the origin. The evolution equations are

$$\dot{p} \in N_{K(\zeta)}(\sigma)\,, \tag{0.0.1}$$

$$\dot{z} = \rho_1 \star [(\rho_2 \star \operatorname{tr}\sigma)\operatorname{tr}\dot{p}\,]\,, \tag{0.0.2}$$

where $N_{K(\zeta)}(\sigma)$ is the normal cone to $K(\zeta)$ at σ, the symbol $\star$ denotes the convolution with respect to the space variable $x \in \Omega$, ρ_1 and rho_2 are convolution kernels with compact support, and tr denotes the trace of a matrix. A remarkable fact is that the evolution law (0.0.2) does not depend on K, that is to say that the scalar parameter ζ controlling the shape of the yield surface, and thus the plastic properties of the material, is not determined by simple energetic principles but evolves according to a different equation. For this reason we speak of a nonassociative nature of the problem. Moreover, equation (0.0.2) has a nonvariational structure, unlike the other equations of the model.

In the engineering literature one assumes that z is positive and bounded away from 0, so that one can take $V(z) = z$, and identify z with ζ. Moreover, the convolution products are not present in (0.0.2). We introduce them for technical reasons, namely to recover strong compactness of z from weak compactness of p. However, it is not physically unreasonable to assume that the evolution of the internal variable at a point is affected by stresses and strains in a small neighborhood: the convolution product introduces two characteristic length-scales for this interaction through the size of the support of ρ_1 and ρ_2.

An interesting feature of this model is that, under the usual assumptions on $K(\zeta)$ (see (1.3.2)-(1.3.5)), it exhibits both hardening and softening behavior, i.e., expansion and contraction of $K(\zeta)$, depending on the loading conditions. This gives rise to one of the main technical difficulties, since allowing for a softening regime results in the lack of convexity of the problem, which is the origin of some instabilities of the system and causes discontinuity of the evolution with respect to time (see for instance [8] and [9], where another model of plasticity with softening is analysed). Actually we will show in Chapter 3 that discontinuous solutions can appear also in our case, in the softening regime (see Remark 3.4).

A general mathematical framework for the study of evolutionary problems of this kind is the energetic approach to *rate-independent problems* developed by Mielke (see [31]). By the term rate-independent we mean a system with no intrinsic time scale, which reacts to a strictly monotone time reparametrisation of the data by reparametrising its solutions exactly in the same way. Rate-independent systems occur as limit problems in the study of many physical and mechanical systems where the time scales we are interested in are much longer than the intrinsic ones in the system. This approach has been widely used in the analysis of many phenomena others than elasto-plasticity, like dry friction, fracture, or shape-memory alloys. In our setting, it would result in defining a quasistatic evolution as a map $(u(t,\cdot), e(t,\cdot), p(t,\cdot), z(t,\cdot))$ satisfying at any time t a suitable *stability condition*, a *balance* between the stored and the dissipated energy (which are the "variational part" of our model), as well as the evolution equation (0.0.2).

To be definite, in our setting the stability condition is given by the stress constraint

$$\sigma(t,x) \in K(\zeta(t,x)) \quad \text{for every } t \in [0, +\infty) \tag{0.0.3}$$

and the equilibrium equation

$$-\mathrm{div}\, \sigma(t,x) = f(t,x)\,. \tag{0.0.4}$$

We emphasize that this is a **local stability** condition, not a global one (see [33] for a general discussion), since it can be regarded as the Euler conditions for a suitable minimum problem involving the plastic dissipation (as in the perfectly plastic case, see [13, Theorem 3.6]) but equivalence with global minimality has not to be expected. Indeed it has already been shown for other models of plasticity with softening (see again [8] and [9]) that a quasistatic evolution where (0.0.3)-(0.0.4) are replaced by the global minimality of the corrsponding energy-dissipation functional may laed to a physically implausible description of the behavior of the system in the softening regime.

Following these previous examples, as well as general considerations about nonsmooth rate-independent evolutionary problems (see [32] and [33]), we introduce a viscoplastic approximation of Perzyna-type (see [37, 17, 25, 36]) of the problem. Given a viscosity parameter $\varepsilon > 0$, the stress constraint (0.0.3) is dropped, and we consider a Yosida regularization of the evolution law (0.0.1), namely

$$\dot{p}_\varepsilon(t,x) = N^\varepsilon_{K(\zeta_\varepsilon(t,x))}(\sigma_\varepsilon(t,x)) ,$$

where $N^\varepsilon_K(\sigma,\zeta) := \frac{1}{\varepsilon}\big(\sigma - \pi_{K(\zeta)}(\sigma)\big)$ and $\pi_{K(\zeta)}$ is the projection onto $K(\zeta)$. The parameter $\frac{1}{\varepsilon}$ has clearly the role of penalising stresses going too far away from the elastic domain, so that we can expect to recover (0.0.3) in the limit as $\varepsilon \to 0$. This regularization has the advantage of making the right-hand side Lipschitz continuous, thus existence of solutions $u_\varepsilon(t,x)$, $e_\varepsilon(t,x)$, $p_\varepsilon(t,x)$, $z_\varepsilon(t,x)$ for the viscoplastic problem can be obtained more easily (still, the proof remains nontrivial, see Chapter 5, Section 5.2).

The study of the limit as $\varepsilon \to 0$ of these viscoplastic approximations leads to a suitable notion of generalized solution for our problem, giving a meaning to the evolution also after the first discontinuity time. We then introduce the notion of *rescaled viscosity evolution*, expressed in terms of a rescaled time s, related to the original time by the equality $t = t^\circ(s)$, where t° is a suitable nondecreasing locally Lipschitz function, depending on the problem. The intervals where t° is constant correspond to time discontinuities in the original variable t. We will see in Chapter 5 that such an evolution may be obtained passing in the limit as $\varepsilon \to 0$ a suitable time-rescaled version of the viscoplastic approximations. Indeed, an energy estimate (Theorem 5.4) allows us to prove the existence of changes of variables $t = t^\circ_\varepsilon(s)$, uniformly Lipschitz with respect to s, such that the rescaled functions $p^\circ_\varepsilon(s,x) := p_\varepsilon(t_\varepsilon(s),x)$ are uniformly Lipschitz with respect to s, in a suitable function space. The same idea has also been used in [18, 32, 33] for rate independent dissipative problems in finite dimension.

The Ascoli-Arzelà Theorem provides the existence of a subsequence (not relabelled), such that

$$t^\circ_\varepsilon(s) \to t^\circ(s) \quad \text{and} \quad p^\circ_\varepsilon(s,\cdot) \rightharpoonup p^\circ(s,\cdot) ,$$

the latter in a weak topology. A further argument, based on the uniqueness of the solution to an auxiliary variational problem, shows that

$$e^\circ_\varepsilon(s,\cdot) \rightharpoonup e^\circ(s,\cdot) , \quad u^\circ_\varepsilon(s,\cdot) \rightharpoonup u^\circ(s,\cdot) , \quad \sigma^\circ_\varepsilon(s,\cdot) \rightharpoonup \sigma^\circ(s,\cdot) .$$

The compactness ensured by the presence of the convolutions in the evolution law for the internal variable allows us to prove that

$$z^\circ_\varepsilon(s,x) \to z^\circ(s,x) \quad \text{and} \quad \zeta^\circ_\varepsilon(s,x) \to \zeta^\circ(s,x) ,$$

uniformly with respect to x. It is then easy to see that the limit functions satisfy the consitutive relations, the equilibrium condition and the additive decomposition $Eu^\circ(s,x) = e^\circ(s,x) + p^\circ(s,x)$ (see (4.2.11)). As for (0.0.2), it holds only in a weak form since, in general, the limit $p^\circ(s,\cdot)$ is just a measure and this requires an *ad-hoc* definition for the derivative (see Chapter 1, Section 1.4 and Chapter 5, Section 5.5).

Condition (0.0.2) is satisfied in the limit for those values of s for which $t^\circ(s)$ is not locally constant. The flow rule (0.0.1) holds in a suitable measure-theoretical sense, since now for a.e. s, $\dot p^\circ(s,\cdot)$ can only be interpreted as a Radon measure on Ω with values in the space of symmetric $n\times n$ matrices. The behavior of $\dot p$ for an arbitrary value of s is described by the equation

$$\dot p \in N^{ext}_{K(\zeta)}(\sigma)\,, \tag{0.0.5}$$

interpreted in a measure-theoretical sense. Here the *extended normal cone* $N^{ext}_{K(\zeta)}$ is defined by

$$N^{ext}_{K(\zeta)}(\sigma) := \begin{cases} N_{K(\zeta)}(\sigma) & \text{if } \sigma\in K(\zeta),\\ \{\lambda(\sigma-\pi_{K(\zeta)}(\sigma)) : \lambda\geq 0\} & \text{if } \sigma\notin K(\zeta)\,. \end{cases}$$

It has to be remarked that (0.0.5) is actually formulated in terms of a suitable representative $\hat\sigma$ of the stress σ (see Definition 4.1) whose existence itself has to be proved. Therefore, although (0.0.5) has the advantage of being the rigorous counterpart of (0.0.1) in our formulation, in the proof of the existence Theorem 5.6 an energetic approach is preferable. Indeed we first show (Chapter 4) that (0.0.5) can be equivalently replaced by an energy-dissipation balance (see (4.3.1)) and a partial flow rule on the intervals where t° is constant (see (4.3.2)), so that in the proof of Theorem 5.6 we tackle these two equalities instead of (0.0.5). Equality (4.3.1) is similar to the energy-dissipation balance of perfect plasticity [13] with two main differences: first, the set K, and hence the plastic dissipation, depend now on $\zeta^\circ(s,x)$; second, there is an additional dissipative term,

$$\int_0^S\int_\Omega \big(\sigma^\circ(s,x)-\pi_{K(\zeta^\circ(s,x))}(\sigma^\circ(s,x))\big):\dot p^\circ(s,x)\,dx\,ds\,, \tag{0.0.6}$$

which accounts for viscous dissipation in those intervals where $t^\circ(s)$ is locally constant (the colon denotes the scalar product between matrices). A similar term appears in [32], where an evolution problem with nonconvex energy is studied through a viscosity approximation and time rescaling.

The possibility of computing the amount of viscous dissipation occurring at jump times is the main advantage of using a rescaled time s instead of the original time t. To consider the behavior of the evolution in terms of the original time variable, one can indeed compose the rescaled viscosity evolution with the left-continuous function

$$s^\circ_-(t) := \sup\{s\in[0,+\infty) : t^\circ(s) < t\}\,,$$

which has the property that $t^\circ(s^\circ_-(t)) = t$ for every $t\geq 0$. The composite function obtained in this way is called a *viscosity evolution*: we show in Lemma 5.9 that the (unrescaled) viscosity approximations converge to this viscosity evolution for every t, except for the countable set of the discontinuity times. In Chapter 6 we prove that every viscosity evolution satisfies an energy-dissipation balance and an evolution law for the internal variable, that can be expressed in terms of integrals depending only on the original time t (see Theorems 6.7 and 6.14). However, both these integral identities contain terms concentrated on the jump times, whose value can only be determined by looking at the rescaled formulation (see Remarks 6.8 and 6.15). Theorem 6.7 shows in addition that, in the vanishing viscosity limit, the viscous dissipation is concentrated at the discontinuity times.

Compared with other rate-independent evolution problems, the main theoretical difficulty we have to confront here is that the total variation of the plastic strain with respect to time can be controlled only in a nonreflexive Banach space, while no such a control is available for the elastic part. Rather than developing an abstract setting including our problem, we preferred to do a complete case study for a model of considerable interest for the engineering community. Nevertheless, although the technical obstacles we encountered have been solved by problem-specific techniques, we think that the main ideas can be adapted to more general rate-independent evolution problems with nonconvex energy-dissipation terms. In particular, the use of a vanishing viscosity approximation to understand the behavior of the system at jump times seems to be a tool of wide applicability.

We conclude this introduction by giving a brief overview of the content of each chapter.

Chapter 1 is devoted to the basic notation and to the presentation of some abstract mathematical tools that will be employed in our proofs. In particular, Section 1.4 introduces a notion of "weak*-derivative" for functions of bounded variation with values in the dual of a separable Banach space, that allows us to reconstruct the "primitive" function by a suitable integration process. These results, which are taken from [11], will be useful in Chapter 6 to write the precise form of the energy-dissipation balance satisfied by a viscosity evolution (see (6.2.10)). The main difficulty in the proofs is that, in the present context, we can neither assume that the space is reflexive, nor that it satisfies the Radon-Nikodym property. From the general Theorems 1.3 and 1.5 we also deduce the analogous result for absolutely continuous functions Theorem 1.8, originally proved in [13, Theorem 7.1] (actually, the part concerning the existence of a weak*-derivative in the case of Lipschitz functions had already been established in [1, Theorem 3.5]), which is useful in the rescaled formulation, since the rescaled plastic strain $p(s,\cdot)$ is 1-Lipschitz continuous.

In Section 1.5 we adapt to our setting some results of approximation of Bochner integrals with Riemann sums. This is the strategy that we will follow in the proof of (4.3.1), in Chapter 5.

Chapter 2 presents the Cam-Clay model in its classical formulation (Section 2.2) as well as the main mechanical assumptions we will make in order to prove well-posedness for our notion of weak solution. In particular, following [7], Section 2.3 introduces a notion of generalized stress-strain duality which adapts to our context the generalized duality proposed by Temam (see [52] and [53]). The new proofs that were only outlined in [7] are here developed in detail. In Section 2.4 we write down the ε-regularized equations. This chapter does not contain original results. The simple inequality (2.5) between the plastic dissipation functional $\mathcal{H}(p,\zeta)$ and the generalized duality $\langle\sigma,p\rangle$ taken from [48] is only a slight modification of the analogous result [13, Proposition 2.4] for an elastic domain K independent of ζ.

In **Chapter 3** we start our investigation of the Cam-Clay model by the study of the spatially homogeneous case. Indeed, for a Dirichlet problem with no volume forces, if the system is driven by a time-dependent affine boundary condition $w(t,x)$, with the introduction of a viscous approximation the problem reduces to determine the limit behavior of the solutions of a singularly perturbed system of ODE's in a finite dimensional Banach space.

This Chapter, which presents the results of [14], shows that we cannot expect a continuous evolution and highlights the usefulness of introducing a viscous approximation and rescaling time to understand the behavior of the system at discontinuity times. In this simplified setting we do not investigate the well-posedness of the problem, which is the object of Chapter 5, but we carry out a qualitative study of the limit behavior of the solutions as the viscosity parameter ε goes to 0 only using differential equations techniques and disregarding the variational structure of (part of) the problem. Depending on the sign of two explicit scalar indicators Φ and Ψ (see (3.1.3)-(3.1.4)), we see that the limit dynamics presents, under quite generic assumptions, the alternation of three possible regimes: the *elastic regime*, when the limit equation is just the equation of linearized elasticity; the *slow dynamics*, when the stress evolves smoothly on the yield surface and plastic flow is produced; the *fast dynamics*, which may happen only in the softening regime, when viscous solutions exhibit a jump determined by a heteroclinic orbit of the auxiliary system

$$\begin{cases} \dot\sigma(s) = -\mathbb{C}(\sigma(s) - \pi_{K(z(s))}(\sigma(s))), \\ \dot z(s) = \mathrm{tr}(\sigma(s))\,\mathrm{tr}(\sigma(s) - \pi_{K(z(s))}(\sigma(s))), \end{cases}$$

which is formally obtained by a time rescaling of the system of ODE's given by the viscoplastic approximation. It can be easily shown that, in the spatially homogeneous case, this system is equivalent to (0.0.2) and (0.0.5) with the multiplier λ appearing in the definition of extended normal cone identically equal to 1.

The main result Theorem 3.31 gives also an iterative procedure to construct a viscous solution. Its proof is based on the methods developed in [47] for another model of plasticity with softening with an associative evolution law for the plastic strain and the internal variable. We chose not to present in this thesis the results of [47], since the model it studies (see also [8]) has not the same interest from the point of view of the applications, and only allows for softening. We only underline that, from a technical point of view, the limit equations (3.3.1) and (3.4.1) are rather different from those studied in [47]. In particular, showing the existence of the heteroclinic orbit governing the jump of the system is a harder task and needs further hypotheses on the yield surface. Nevertheless, to show the convergence of the viscoplastic solutions to a limit satisfying either (3.3.1) or (3.4.1) we can use some methods developed in [47] and, actually, some technical lemmas are only suitable adaptations of the corresponding results in [47].

Chapter 4 introduces the definition of *rescaled viscosity evolution* (see Definition 4.5). The rate-independence properties of this formulation are described in Remark 4.6. As we said above, it involves a suitable representative $\hat\sigma$ of the stress which has to satisfy a delicate integration-by-parts formula (see (4.2.3)) and is very difficult to handle when proving existence for such an evolution. Therefore, as a preliminary step towards the existence proof, the chapter is devoted to showing the equivalence between this formulation and an "energetic" one, where the measure-theoretic version of the flow rule (0.0.1) is replaced by the energy-dissipation balance (4.3.1) and the partial flow rule on the intervals where t° is constant (4.3.2). This latter formulation is the one originally proposed in [10] and does not require a precise representative of σ. As we said, it proves to be more manageable for the proof of the existence Theorem 5.6. After proving the equivalence Theorem 4.7, in Section 4.4 we see that, at least under a strict convexity assumption on K, the precise representative

$\hat{\sigma}$ has an intrinsic character and can be obtained in the interior of Ω as limit of spherical averages of σ. The results of this chapter are taken from [11].

Chapter 5 tackles the proof of the existence of a rescaled viscosity evolution according to Definition (4.5), or better to the equivalent formulation given by Theorem 4.7. The main difficulty is to establish the energy-dissipation balance (4.3.1), which is the major part of the proof. Comparing to earlier attempts at modeling vanishing viscosity limits, the energy balance is a key fact, as it guarantees that all quantities remain under control in the limit. The proof is based on a delicate approximation of the integrals that appear in this equality, developed in Chapter 1, Section 1.5. The main difficulty is due to the fact that we need two different approximations on the set where the stress constraint (0.0.3) is satisifed and on its complement.

Finally, in **Chapter 6** we study the behavior of the evolution in terms of the original time t, introducing the notion of *viscosity evolution*. Relying on the technical results of Chapter 1, Section 1.4, we are able to write an energy balance (Theorem 6.7) and an evolution law for the internal variable (Theorem 6.14) for a viscosity evolution. As we have already said, both these identities describe well the behavior of a viscosity evolution at its continuity points, while a careful description of the behavior at jumps requires the use of the rescaled formulation. The energy-dissipation balance (6.2.19) shows in particular that the viscous dissipation is concentrated at the jump times.

Chapter 1

Preliminaries

1.1 Overview of the chapter

In this chapter we fix some notation and we collect some abstract results which will be useful in the sequel. The plan of the chapter is the following: after introducing some basic facts about functions and measures, we turn our attention to some properties of the integral functional which describes the plastic dissipation in our model.

Eventually we introduce some tools about functions of bounded variation with values in the dual of a separable Banach space; we will neither assume that the space is reflexive, nor that it has the Radon-Nikodym properties. From these results, as a particular case, we will also deduce the results about absolutely continuous functions contained in [13, Appendix]. They also allow us to deduce a result of discrete approximation of the total plastic dissipation on a time interval that will be useful in Chapter 5.

The rest of the chapter contains some results on the approximation of Lebesgue integrals with Riemann sums which adapt to our context the well-known result of Hahn (see [22]), and a result on continuous dependence on the data for differential equations that will be useful in the study of the finite dimensional case, in Chapter 3.

1.2 Functions and measures

The Lebesgue measure on $\mathbb{R}^N$ is denoted by $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. If $X \subset \mathbb{R}^N$ is locally compact and Ξ is a finite dimensional Hilbert space, the space of bounded Ξ-valued Radon measures on X is denoted by $M_b(X;\Xi)$. When $\Xi = \mathbb{R}$, it is omitted from the notation. The space $M_b(X;\Xi)$ is endowed with the norm $\|\mu\|_1 := |\mu|(X)$, where $|\mu| \in M_b(X)$ is the variation of the measure μ. By the Riesz Representation Theorem (see, e.g., [44, Theorem 6.19]) $M_b(X;\Xi)$ is identified with the dual of $C^0_0(X;\Xi)$, the space of continuous functions $\varphi\colon X \to \Xi$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. This defines the weak* topology in $M_b(X;\Xi)$.

The space $L^1(X;\Xi)$ of Ξ-valued $\mathcal{L}^n$-integrable functions is regarded as a subspace of $M_b(X;\Xi)$, with the induced norm. The L^p norm, $1 \leq p \leq \infty$ is denoted by $\|\cdot\|_p$. We adopt the convention

$$\|v\|_p = +\infty \quad \text{whenever } v \notin L^p\,. \tag{1.2.1}$$

The brackets $\langle \cdot\,, \cdot \rangle$ denote the duality product between conjugate L^p spaces, as well as between other pairs of spaces, according to the context.

The space of *symmetric* $n \times n$ *matrices* is denoted by $\mathbb{M}^{N\times N}_{sym}$; it is endowed with the euclidean scalar product $\xi : \eta := \sum_{ij} \xi_{ij}\eta_{ij}$ and with the corresponding euclidean norm $|\xi| := (\xi : \xi)^{1/2}$. The *symmetrized tensor product* $a \odot b$ of two vectors a, $b \in \mathbb{R}^N$ is the symmetric matrix with entries $(a_i b_j + a_j b_i)/2$.

For every $u \in L^1(U; \mathbb{R}^N)$, with U open in $\mathbb{R}^N$, let Eu be the $\mathbb{M}^{N\times N}_{sym}$-valued distribution on U whose components are defined by $E_{ij}u = \frac{1}{2}(D_j u_i + D_i u_j)$. The space $BD(U)$ of functions with *bounded deformation* is the space of all $u \in L^1(U; \mathbb{R}^N)$ such that $Eu \in M_b(U; \mathbb{M}^{N\times N}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm $\|u\|_1 + \|Eu\|_1$. It is possible to prove that $BD(U)$ is the dual of a normed space (see [29] and [53]), and this defines the weak* topology of $BD(U)$. A sequence u_k converges to u weakly* in $BD(U)$ if and only if $u_k \to u$ strongly in $L^1(U; \mathbb{R}^N)$ and $Eu_k \overset{*}{\rightharpoonup} Eu$ weakly* in $M_b(U; \mathbb{M}^{N\times N}_{sym})$. For the general properties of $BD(U)$ we refer to [52]. If U is a bounded open set with Lipschitz boundary, for every function $u \in BD(U)$ the trace of u on ∂U belongs to $L^1(\partial U; \mathbb{R}^N)$. It will always be denoted by the same symbol u. Moreover, the following result holds:

$$u \in \mathcal{D}'(U; \mathbb{R}^N) \text{ and } Eu \in L^2(U; \mathbb{M}^{N\times N}_{sym}) \quad \Longrightarrow \quad u \in H^1(U; \mathbb{R}^N)\,, \tag{1.2.2}$$

where $\mathcal{D}'(U; \mathbb{R}^N)$ is the space of $\mathbb{R}^N$-valued distributions on U. This can be obtained arguing as in the proof of [52, Chapter I, Proposition 1.1].

We will use boldface letters to denote functions defined in an interval $[a, b] \subset \mathbb{R}$ and with values in a possibly infinite dimensional Banach space Y.

Throughout the paper the *reference configuration* Ω is a *bounded connected open set* in $\mathbb{R}^N$, $n \geq 2$, with *Lipschitz boundary* $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup N$. We assume that Γ_0 and Γ_1 are relatively open, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, and $\mathcal{H}^{n-1}(N) = 0$.

We shall frequently use the following closed linear subspace of $H^1(\Omega; \mathbb{R}^N)$:

$$H^1_{\Gamma_0}(\Omega; \mathbb{R}^N) := \{u \in H^1(\Omega; \mathbb{R}^N) : u = 0 \ \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_0\}\,. \tag{1.2.3}$$

1.3 The constraint set and its support function.

Let K be a closed convex cone in $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ with nonempty interior. For every $\zeta \in [0, +\infty)$ we define the closed convex set $K(\zeta)$ by

$$K(\zeta) := \{\sigma \in \mathbb{M}^{N\times N}_{sym} : (\sigma, \zeta) \in K\}\,. \tag{1.3.1}$$

When $\zeta > 0$ the set $K(\zeta)$ has nonempty interior and

$$K(\zeta) = \zeta\, K(1)\,. \tag{1.3.2}$$

We assume that $0 \in K(1)$ and $K(1)$ is bounded, hence

$$0 \in K(\zeta) \qquad \text{for every } \zeta \in [0, +\infty)\,, \tag{1.3.3}$$

and

$$|\sigma| \le M_K \zeta \qquad \text{for every } (\sigma,\zeta) \in K \tag{1.3.4}$$

for a suitable constant $M_K < +\infty$. Since K is a convex cone, (1.3.3) implies that

$$0 \le \zeta_1 \le \zeta_2 \quad \Longrightarrow \quad K(\zeta_1) \subset K(\zeta_2)\,. \tag{1.3.5}$$

For every closed convex set $C \subset \mathbb{M}^{N\times N}_{sym}$ let $\pi_C \colon \mathbb{M}^{N\times N}_{sym} \to C$ be the minimal distance projection onto C. It follows from (1.3.2) that

$$\pi_{K(\zeta)}(\sigma) = \zeta \pi_{K(1)}\big(\frac{\sigma}{\zeta}\big) \tag{1.3.6}$$

for every $\zeta > 0$ and every $\sigma \in \mathbb{M}^{N\times N}_{sym}$.

Lemma 1.1. *The map $(\sigma,\zeta) \mapsto \pi_{K(\zeta)}(\sigma)$ from $\mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$ into $\mathbb{M}^{N\times N}_{sym}$ satisfies the Lipschitz estimate*

$$|\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_1)}(\sigma_1)| \le |\sigma_2 - \sigma_1| + 2M_K|\zeta_2 - \zeta_1| \tag{1.3.7}$$

for every (σ_1,ζ_1), $(\sigma_2,\zeta_2) \in \mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$.

Proof. It is enough to prove the estimate for (σ_1,ζ_1), $(\sigma_2,\zeta_2) \in \mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$ with $0 < \zeta_1 \le \zeta_2$. Since $\pi_{K(\zeta_2)}$ has Lipschitz constant 1 on $\mathbb{M}^{N\times N}_{sym}$, from (1.3.4) and (1.3.6) we obtain

$$\begin{aligned}
|\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_1)}(\sigma_1)| &\le |\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_2)}(\sigma_1)| + |\pi_{K(\zeta_2)}(\sigma_1) - \pi_{K(\zeta_1)}(\sigma_1)| \le \\
&\le |\sigma_2 - \sigma_1| + \big|\zeta_2\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1) - \zeta_1\pi_{K(1)}(\tfrac{1}{\zeta_1}\sigma_1)\big| \le \\
&\le |\sigma_2 - \sigma_1| + M_K|\zeta_2 - \zeta_1| + \zeta_1\big|\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1) - \pi_{K(1)}(\tfrac{1}{\zeta_1}\sigma_1)\big|\,.
\end{aligned}$$

To prove (1.3.7) it is enough to show that

$$\zeta_1\big|\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1) - \pi_{K(1)}(\tfrac{1}{\zeta_1}\sigma_1)\big| \le M_K|\zeta_2 - \zeta_1|\,. \tag{1.3.8}$$

As $0 < \zeta_1 \le \zeta_2$, we have

$$\pi_{K(1)}\Big(\tfrac{1}{\zeta_1}\sigma_1 - \tfrac{\zeta_2-\zeta_1}{\zeta_1}\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1)\Big) = \pi_{K(1)}\Big(\tfrac{1}{\zeta_2}\sigma_1 + \tfrac{\zeta_2-\zeta_1}{\zeta_1}\big(\frac{1}{\zeta_2}\sigma_1 - \pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1)\big)\Big) = \pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1)\,.$$

Since $\pi_{K(1)}$ has Lipschitz constant 1 on $\mathbb{M}^{N\times N}_{sym}$, we obtain

$$\big|\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1) - \pi_{K(1)}(\frac{1}{\zeta_1}\sigma_1)\big| \le \tfrac{\zeta_2-\zeta_1}{\zeta_1}\big|\pi_{K(1)}(\tfrac{1}{\zeta_2}\sigma_1)\big| \le M_K\tfrac{\zeta_2-\zeta_1}{\zeta_1}\,,$$

which gives (1.3.8). $\square$

Let $H \colon \mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$ be defined by

$$H(\xi,\zeta) = \sup_{\sigma\in K(\zeta)} \sigma : \xi\,, \tag{1.3.9}$$

so that $H(\cdot,\zeta)$ is the support function of $K(\zeta)$. By (1.3.2) for every $(\xi,\zeta) \in \mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$ we have

$$H(\xi,\zeta) = \zeta H(\xi,1)\,. \tag{1.3.10}$$

For every $\zeta \in [0,+\infty)$ the function $\xi \mapsto H(\xi,\zeta)$ is convex and positively one-homogeneous on $\mathbb{M}^{N\times N}_{sym}$. In particular, it satisfies the triangle inequality

$$H(\xi_1+\xi_2,\zeta) \le H(\xi_1,\zeta)+H(\xi_2,\zeta) \tag{1.3.11}$$

for every $\xi_1,\xi_2 \in \mathbb{M}^{N\times N}_{sym}$ and every $\zeta \in [0,+\infty)$. From (1.3.3), (1.3.4), and (1.3.10) it follows that

$$0 \le H(\xi,\zeta) \le M_K\zeta|\xi|\,, \tag{1.3.12}$$

$$|H(\xi_2,\zeta)-H(\xi_1,\zeta)| \le M_K\zeta|\xi_2-\xi_1|\,, \tag{1.3.13}$$

$$|H(\xi,\zeta_2)-H(\xi,\zeta_1)| \le M_K|\xi||\zeta_2-\zeta_1|\,, \tag{1.3.14}$$

for every $\xi,\xi_1,\xi_2 \in \mathbb{M}^{N\times N}_{sym}$ and every $\zeta,\zeta_1,\zeta_2 \in [0,+\infty)$.

Given $\zeta \in C^0(\overline{\Omega})^+$, we define

$$\mathcal{K}(\zeta) := \{\sigma \in L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym}) : \sigma(x) \in K(\zeta(x)) \text{ for } \mathcal{L}^n\text{-a.e. } x \in \Omega\}\,. \tag{1.3.15}$$

It is obvious that when $\sigma \in C(\overline{\Omega}) \cap \mathcal{K}(\zeta)$, then $\sigma(x) \in K(\zeta(x)$ for every $x \in \overline{\Omega}$. Given $\mu \in M_b^+(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ it will be sometimes useful to consider also the space

$$\mathcal{K}_\mu(\zeta) := \{\sigma \in L^2_\mu(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym}) : \sigma(x) \in K(\zeta(x)) \text{ for } \mu\text{-a.e. } x \in \Omega\cup\Gamma_0\}\,. \tag{1.3.16}$$

For every closed convex set $\mathcal{C} \subset L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, let $\pi_{\mathcal{C}}\colon L^2(\Omega;\mathbb{M}^{N\times N}_{sym}) \to \mathcal{C}$ be the minimal distance projection onto $\mathcal{C}$. For every $\sigma \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ we define

$$d_2(\sigma,\mathcal{C}) := \|\sigma-\pi_{\mathcal{C}}(\sigma)\|_2\,, \tag{1.3.17}$$

the L^2-distance from σ to $\mathcal{C}$. It is easy to see that, if $\sigma \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, then

$$\hat\sigma = \pi_{\mathcal{K}(\zeta)}(\sigma) \quad\Longleftrightarrow\quad \hat\sigma(x) = \pi_{K(\zeta(x))}(\sigma(x)) \text{ for } \mathcal{L}^n\text{-a.e. } x \in \Omega\,. \tag{1.3.18}$$

Using the theory of convex functions of measures developed in [21], we introduce the nonnegative Radon measure $H(p,\zeta)$ defined by

$$H(p,\zeta)(B) := \int_B H(\tfrac{dp}{d\lambda}(x),\zeta(x))\,d\lambda(x)\,, \tag{1.3.19}$$

for any Borel set $B \subset \Omega\cup\Gamma_0$. Here $\lambda \in M_b(\Omega\cup\Gamma_0)^+$ is any measure such that $p \ll \lambda$; note that the homogeneity of H with respect to ξ implies that the integral does not depend on λ. Similarly, we introduce the functional functional $\mathcal{H}\colon M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})\times C^0(\overline{\Omega})^+ \to \mathbb{R}$ defined by

$$\mathcal{H}(p,\zeta) := \int_{\Omega\cup\Gamma_0} H(\tfrac{dp}{d\lambda}(x),\zeta(x))\,d\lambda(x)\,. \tag{1.3.20}$$

In particular, if $p \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, we have

$$\mathcal{H}(p,\zeta) = \int_\Omega H(p(x),\zeta(x))\,dx\,.$$

When $\dot p$ is the rate of plastic strain and ζ is the internal variable, $\mathcal{H}(\dot p,\zeta)$ represents the rate of plastic dissipation.

For every $p \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and $\zeta \in C^0(\overline{\Omega})^+$ the symbol $\partial_p\mathcal{H}(p,\zeta)$ denotes the subdifferential in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ of $\mathcal{H}(\cdot,\zeta)$ at p. Using [40, Corollary 23.5.3] and [19, Proposition IX.2.1] it is easy to show that

$$\partial_p\mathcal{H}(0,\zeta) = \mathcal{K}(\zeta)\,. \tag{1.3.21}$$

As $\mathcal{H}(p,\zeta)$ is positively homogeneous with respect to p we have

$$\partial_p\mathcal{H}(p,\zeta) \subset \partial_p\mathcal{H}(0,\zeta) = \mathcal{K}(\zeta) \tag{1.3.22}$$

for every $p \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and every $\zeta \in C^0(\overline{\Omega})^+$.

The following theorem shows that $\mathcal{H}$ can be also regarded as the support function of the closed convex set $\mathcal{K}(\zeta)\cap C^0_0(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$; this will be the point of view of the next section.

Theorem 1.2. *Let* $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$, $\zeta \in C^0(\overline{\Omega})^+$, *let* $K(\zeta)$ *satisfy* (1.3.1)-(1.3.4), *and define* $\mathcal{H}(p,\zeta)$ *as in* (1.3.20). *Then*

$$\mathcal{H}(p,\zeta) = \sup\{\int_{\Omega\cup\Gamma_0}\tau(x)\,dp(x)\colon \tau \in C^0_0(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})\cap\mathcal{K}(\zeta)\}. \tag{1.3.23}$$

Proof. The "$\geq$" inequality is trivial. To prove the converse inequality, we assume that $\Gamma_0 = \partial\Omega$: this is not restrictive, because otherwise we can proceed by inner approximation with smooth sets Ω_k whose boundary is contained in $\Omega\cup\Gamma_0$. Observe also that, by the 1-Lipschitz continuity of the projection, the supremum in (1.3.23) remains unchanged if we replace $C^0_0(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})\cap\mathcal{K}(\zeta)$ with $\mathcal{K}_{|p|}(\zeta)$, where $\mathcal{K}_{|p|}(\zeta)$ is defined in (1.3.16).

First, we suppose that $\zeta(x)$ is constant on Ω and we denote its unique value by $\tilde{\zeta}$. In this case, the result could be deduced in an abstract framework using [21, Theorem 4] and [52, Chapter II, Lemma 5.2], but we give a direct proof for the reader's convenience. We fix $\varepsilon > 0$ and we find a continuous function $q(x)$ such that

$$\int_{\overline{\Omega}} |q(x) - \tfrac{p}{|p|}(x)|\,d|p|(x) \leq \varepsilon \tag{1.3.24}$$

and consequently

$$\mathcal{H}(p,\tilde{\zeta}) \leq \int_{\overline{\Omega}} H(q(x),\tilde{\zeta})\,d|p|(x) + M_K\tilde{\zeta}\varepsilon\,. \tag{1.3.25}$$

By the compactness of $\bar{\Omega}$ and standard properties of bounded Radon measures we can find a finite family of pairwise disjoint open sets $(Q_i)_{i=1}^{j(\varepsilon)}$ such that:

$$\bar{\Omega} \subseteq \bigcup_{i=1}^{j(\varepsilon)} \bar{Q}_i \tag{1.3.26}$$

$$|q(x) - q(y)| \leq \varepsilon \quad \text{for every } x,y \in Q_i\cap\bar{\Omega} \text{ and every } 1 \leq i \leq j(\varepsilon); \tag{1.3.27}$$

$$|p|(\partial Q_i\cap\bar{\Omega}) = 0\,. \tag{1.3.28}$$

In particular, (1.3.27) and (1.3.13) easily yield:

$$|H(q(x),\tilde{\zeta}) - H(q(y),\tilde{\zeta})| < M_K\tilde{\zeta}\varepsilon \tag{1.3.29}$$

for every $x,y \in Q_i\cap\bar{\Omega}$ and every $1 \leq i \leq j(\varepsilon)$.

We choose $x_i \in Q_i \cap \bar{\Omega}$ and we find $\xi_i \in K(\tilde{\zeta})$ such that $\xi_i : q(x_i) = H(q(x_i), \tilde{\zeta})$. Then, we define

$$\xi(x) := \begin{cases} \xi_i & \text{if } x \in Q_i \cap \bar{\Omega} \\ 0 & \text{if } x \in \bar{\Omega} \setminus \bigcup_{i=1}^{j(\varepsilon)} Q_i \end{cases},$$

which is a step function satisfying $\xi \in \mathcal{K}_{|p|}(\zeta)$. We then get, using the definition of $\xi(x)$, (1.3.24), (1.3.25), and (1.3.27)-(1.3.29), that

$$\mathcal{H}(p, \tilde{\zeta}) - M_K \tilde{\zeta}\varepsilon \le \int_{\overline{\Omega}} H(q(x), \tilde{\zeta})\, d|p|(x) =$$
$$= \sum_{i=1}^{j(\varepsilon)} \int_{Q_i \cap \bar{\Omega}} H(q(x), \tilde{\zeta})\, d|p|(x) \le \sum_{i=1}^{j(\varepsilon)} \int_{Q_i \cap \bar{\Omega}} \xi_i : q(x_i)\, d|p|(x) + M_K \tilde{\zeta}\varepsilon \le$$
$$\le \int_{\overline{\Omega}} \xi(x) : \xi(x)\, d|p|(x) + 2M_K \tilde{\zeta}\varepsilon \le \int_{\overline{\Omega}} q(x)\, dp(x) + 3M_K \tilde{\zeta}\varepsilon$$

and (1.3.23) follows immediately.

To prove the general case, we use a similar argument, taking pairwise disjoint open sets $(Q_i)_{i=1}^{j(\varepsilon)}$ satisfying (1.3.26), (1.3.29) and

$$|\zeta(x) - \zeta(y)| \le \varepsilon \quad \text{for every } x, y \in Q_i \cap \bar{\Omega} \text{ and every } 1 \le i \le j(\varepsilon), \tag{1.3.30}$$

which, together with (1.3.14), gives

$$|H(\xi, \zeta(x)) - H(\xi, \zeta(x))| < M_K \varepsilon |\xi| \tag{1.3.31}$$

for every $x, y \in Q_i \cap \bar{\Omega}$ and every $1 \le i \le j(\varepsilon)$. We choose $x_i \in Q_i \cap \bar{\Omega}$. By the previous step and (1.3.28) we can find functions $\tau_i \in C_0^0(Q_i \cap \bar{\Omega}; \mathbb{M}_D^{N \times N})$ such that:

$$\int_{Q_i \cap \bar{\Omega}} \tau_i(x)\, dp(x) \ge \int_{Q_i \cap \bar{\Omega}} H(\tfrac{p}{|p|}(x), \zeta(x_i))\, d|p|(x) - \frac{\varepsilon}{j(\varepsilon)}$$

and such that $\tau_i(x) \in K(\zeta(x_i))$ for every x in $Q_i \cap \bar{\Omega}$. Putting $\varphi_i(x) := \pi_{K(\zeta(x))}(\tau_i(x))$, where $\pi_{K(\zeta(x))}$ is the canonical projection on the closed convex set $K(\zeta(x))$, we easily get

$$\int_{Q_i \cap \bar{\Omega}} \varphi_i(x)\, dp(x) \ge \int_{Q_i \cap \overline{\Omega}} H(\tfrac{p}{|p|}(x), \zeta(x_i))\, d|p|(x) - \frac{\varepsilon}{j(\varepsilon)} - \varepsilon|\mu|(Q_i \cap \bar{\Omega}),. \tag{1.3.32}$$

We then define

$$\varphi(x) := \begin{cases} \varphi_i & \text{if } x \in Q_i \cap \bar{\Omega} \\ 0 & \text{if } x \in \bar{\Omega} \setminus \bigcup_{i=1}^{j(\varepsilon)} Q_i \end{cases},$$

which is still continuous since the Q_i's are a finite collection and the functions φ_i vanish on the interfaces, and clearly belongs to $\mathcal{K}(\zeta)$. Moreover, by (1.3.32)

$$\int_{\overline{\Omega}} \varphi(x)\, dp(x) \ge (\sum_{i=1}^{j(\varepsilon)} \int_{Q_i \cap \overline{\Omega}} H(\tfrac{p}{|p|}(x), \zeta(x_i))\, d|p|(x)) - \varepsilon - \varepsilon|\mu|(\bar{\Omega}),$$

and (1.3.31), since $\frac{p}{|p|}(x) = 1$ for $|p|$-a.e. $x \in \overline{\Omega}$, finally yields

$$\int_{\overline{\Omega}} \varphi(x)\, dp(x) \ge \int_{\overline{\Omega}} H(\tfrac{p}{|p|}(x), \zeta(x_i))\, d|p|(x) - \varepsilon - 2\varepsilon|\mu|(\bar{\Omega}),$$

as required. □

1.4 Some tools about functions of bounded variation in time

It is a well-known fact for problems in elastoplasticity like the one we are going to study in the next chapters that, since the functional $\mathcal{H}$ has linear growth, they have, in general, no solution in Sobolev spaces. This is very natural from the point of view of mechanics, due to the phenomenon of strain localization. Solutions can develop shear bands, where shear deformation concentrates. Seen from a macroscopic perspective, shear bands can be thought of as sharp discontinuities of the displacement (slip surfaces). They cannot be resolved by Sobolev functions, but they find a natural mathematical representation if plastic deformations are allowed to take values in spaces of measures (see [50]).

Therefore, in our formulation, the plastic strain $\boldsymbol{p}$ will be regarded as a function from a time interval, say $[0,T]$, to the space $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ which is neither reflexive nor enjoys the Radon-Nikodym property (for this latter notion we refer to [6, Chapter 3]). A suitable weak notion of time derivative will be then needed to understand the evolution of the system. The goal of this section is to introduce such a notion for a function $\boldsymbol{f}$ of bounded variation from a time interval to the dual of a separable Banach space and to prove the representation formula (1.4.9) for the $\mathcal{H}$-variation $\mathcal{V}$ of $\boldsymbol{f}$ defined in (1.5.1), which turns out to be closely related to the total amount of plastic dissipation on a time interval, as we will see in the following. As a particular case, we will recover the analogous results for absolutely continuous functions proved in [13, Appendix].

Throughout this section X is the dual of a separable Banach space Y, and $\mathcal{K}$ is a bounded closed convex subset of Y containing the origin. Let $\mathcal{H}\colon X\to\mathbb{R}$ be its support function, defined by

$$\mathcal{H}(x):=\sup_{y\in\mathcal{K}}\langle x,y\rangle .$$

Since $\mathcal{K}$ is bounded and contains the origin, there exist a positive constant $\beta_{\mathcal{H}}$ such that

$$0\le\mathcal{H}(x)\le\beta_{\mathcal{H}}\|x\|_X\qquad\text{for every } x\in X\,. \tag{1.4.1}$$

Thanks to (1.3.1)-(1.3.4) and Theorem 1.2, it is clear that for every fixed $\zeta\in C^0(\overline{\Omega})^+$ the functional $\mathcal{H}(\cdot,\zeta)$ introduced in (1.3.20) fulfills this description with $X:=M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$, $Y:=C^0_0(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ and $\mathcal{K}:=\mathcal{K}(\zeta)$, where this one is defined in (1.3.15).

Given $\boldsymbol{f}\colon[0,T]\to X$ and $a,b\in[0,T]$, with $a\le b$, the total variation of $\boldsymbol{f}$ on $[a,b]$ is defined by

$$\mathrm{Var}(\boldsymbol{f};a,b):=\sup\Big\{\sum_{i=1}^{N}\|\boldsymbol{f}(t_i)-\boldsymbol{f}(t_{i-1})\|_X : a=t_0\le t_1\le\dots\le t_N=b,\ N\in\mathbb{N}\Big\}\,, \tag{1.4.2}$$

while the $\mathcal{H}$-variation of $\boldsymbol{f}$ on $[a,b]$ is defined by

$$\mathcal{V}(\boldsymbol{f};a,b):=\sup\Big\{\sum_{i=1}^{N}\mathcal{H}(\boldsymbol{f}(t_i)-\boldsymbol{f}(t_{i-1})) : a=t_0\le t_1\le\dots\le t_N=b,\ N\in\mathbb{N}\Big\}\,. \tag{1.4.3}$$

For every $t\in[0,T]$, $\boldsymbol{f}(t+)$ and $\boldsymbol{f}(t-)$ denote the left and right limits of $\boldsymbol{f}$ at t. It is easily seen that the map $t\mapsto\boldsymbol{f}(t+)$ is right-continuous, as well as $t\mapsto\boldsymbol{f}(t-)$ is left-continuous. The function $\boldsymbol{f}$ is extended outside $[0,T]$ by putting $\boldsymbol{f}(t)=\boldsymbol{f}(0)$ whenever

$t \le 0$ and $\boldsymbol{f}(t) = \boldsymbol{f}(T)$ whenever $t \ge T$, so that in particular $\boldsymbol{f}(0-) = \boldsymbol{f}(0)$ and $\boldsymbol{f}(T+) = \boldsymbol{f}(T)$.

We now prove a theorem about a notion of weak* Radon-Nikodym derivative for an X-valued function of bounded variation with respect to the Stieltjes measure associated to its variation.

Theorem 1.3. *Let $\boldsymbol{f}\colon [0,T] \to X$ be a function with bounded variation, and let μ be the unique Radon measure on $[0,T]$ such that $\mu([0,t]) = \mathrm{Var}(\boldsymbol{f};0,t)$ for every $t \in [0,T]$ where $t \mapsto \mathrm{Var}(\boldsymbol{f};0,t)$ is continuous. Then there exists a unique (up to μ-equivalence) function $\boldsymbol{\nu_f}\colon [0,T] \to X$ such that for every $y \in Y$ the function $t \mapsto \langle y, \boldsymbol{\nu_f}(t)\rangle$ is μ-integrable and*

$$\langle y, \boldsymbol{f}(b) - \boldsymbol{f}(a)\rangle = \int_a^b \langle y, \boldsymbol{\nu_f}(t)\rangle \, d\mu(t) \tag{1.4.4}$$

for every $a, b \in [0,T]$ with $a \le b$, such that $\mu(\{a\}) = \mu(\{b\}) = 0$. Moreover

$$\|\boldsymbol{\nu_f}(t)\|_X \le 1 \tag{1.4.5}$$

for μ-a.e. $t \in [0,T]$.

Proof. Uniqueness is trivial, so we only prove the existence of such a function. Let F be the linear span over $\mathbb{Q}$ of a countable dense set in Y. For every $y \in F$ the function $t \mapsto \langle y, \boldsymbol{f}(t)\rangle$ has bounded variation on $[0,T]$. Let ν be the unique Radon measure on $[0,T]$ such that $\nu([0,t]) = \langle y, \boldsymbol{f}(t) - \boldsymbol{f}(0)$ for every t where $\boldsymbol{f}$ is continuous. Since ν is absolutely continuous with respect to μ, by the Besicovitch Differentiation Theorem there exists a μ-negligible set N_y such that the limit

$$D^\mu_y(t) := \lim_{h \to 0^+} \frac{\langle y, \boldsymbol{f}(t+h+) - \boldsymbol{f}(t-h-)\rangle}{\mu([t-h, t+h])}$$

exists for every $t \in [0,T] \setminus N_y$, the function $t \mapsto D^\mu_y(t)$ is μ-integrableand

$$\langle y, \boldsymbol{f}(b) - \boldsymbol{f}(a)\rangle = \int_a^b D^\mu_y(t)\, d\mu(t)$$

for every $a, b \in [0,T]$ with $a \le b$, such that $\mu(\{a\}) = \mu(\{b\}) = 0$. We also notice that by definition $\mu([t-h, t+h]) \ge \|\boldsymbol{f}(t+h+) - \boldsymbol{f}(t-h-)\|_X$. Let N be the union of the sets N_y for $y \in F$. Then, $\mu(N) = 0$, the derivative $D^\mu_y(t)$ exists for every $y \in F$ and every $t \in [0,T] \setminus N$, and

$$|D^\mu_y(t)| \le \|y\|_Y\,. \tag{1.4.6}$$

Now, for $t \in [0,T]\setminus N$ consider the $\mathbb{Q}$-linear map $y \in F \mapsto D^\mu_y(t)$. This map is continuous by (1.4.6); therefore, there exists a vector in X, which we call $\boldsymbol{\nu_f}(t)$, such that

$$D^\mu_y(t) = \langle y, \boldsymbol{\nu_f}(t)\rangle$$

for every $y \in F$. Using the density of F and (1.4.6) it is easy to show that the vector $\boldsymbol{\nu_f}(t)$ satisfies

$$\langle y, \boldsymbol{\nu_f}(t)\rangle = \lim_{h \to 0^+} \frac{\langle y, \boldsymbol{f}(t+h+) - \boldsymbol{f}(t-h-)\rangle}{\mu([t-h, t+h])} \tag{1.4.7}$$

for every $y \in Y$ and every $t \in [0,T]\setminus N$, so that (1.4.4) follows again by the Besicovitch Differentiation Theorem. Inequality (1.4.5) is an obvious consequence of (1.4.6). □

Remark 1.4. Let t be an atom of μ, that is a jump point of $\boldsymbol{f}$. It then easily follows from (1.4.7) that

$$\boldsymbol{\nu}_{\boldsymbol{f}}(t) = \frac{\boldsymbol{f}(t+) - \boldsymbol{f}(t-)}{\|\boldsymbol{f}(t+) - \boldsymbol{f}(t)\|_X + \|\boldsymbol{f}(t) - \boldsymbol{f}(t-)\|_X} \tag{1.4.8}$$

for every t such that $\mu(\{t\}) > 0$.

Theorem 1.5. *Let $\boldsymbol{f}\colon [0,T] \to X$ be a left-continuous function with bounded variation, and let μ, and $\boldsymbol{\nu}_{\boldsymbol{f}}(t)$ be as in Theorem 1.3. Then, the function $t \mapsto \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t))$ is μ-integrable and*

$$\mathcal{V}(\boldsymbol{f}; a, b) = \int_a^b \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t))\, d\mu(t) \tag{1.4.9}$$

for every $a, b \in [0,T]$ with $a \le b$, such that $\mu(\{a\}) = \mu(\{b\}) = 0$.

Proof. We note that the function $t \mapsto \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t))$ is μ-measurable, since the map $t \to \langle y, \boldsymbol{\nu}_{\boldsymbol{f}}(t)\rangle$ is μ-measurable for every $y \in Y$ and $\mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t)) = \sup_{y \in \mathcal{K}_0} \langle y, \boldsymbol{\nu}_{\boldsymbol{f}}(t)\rangle$, where $\mathcal{K}_0$ is a countable dense subset of $\mathcal{K}$.

Let us fix a and b as in the statement of the theorem. If $a = t_0 \le t_1 \le \cdots \le t_{N-1} \le t_N = b$ is a subdivision of $[a,b]$ such that $\mu(\{t_i\}) = 0$ for every i, then

$$\langle y, \boldsymbol{f}(t_i) - \boldsymbol{f}(t_{i-1})\rangle = \int_{t_{i-1}}^{t_i} \langle y, \boldsymbol{\nu}_{\boldsymbol{f}}(t)\rangle\, d\mu(t) \le \int_{t_{i-1}}^{t_i} \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t))\, d\mu(t)$$

for every $1 \le i \le N$ and every $y \in \mathcal{K}$, hence

$$\mathcal{H}(\boldsymbol{f}(t_i) - \boldsymbol{f}(t_{i-1})) \le \int_{t_{i-1}}^{t_i} \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t))\, d\mu(t)$$

for every $1 \le i \le N$. Summing over i and taking the supremum over all such subdivisions, which equals to $\mathcal{V}(\boldsymbol{f}; a, b)$ thanks to the assumption $\mu(\{a\}) = \mu(\{b\}) = 0$ and the left-continuity of $\boldsymbol{f}$, we obtain

$$\mathcal{V}(\boldsymbol{f}; a, b) \le \int_a^b \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t))\, d\mu(t)\,. \tag{1.4.10}$$

To show the converse inequality, we first observe that the function $V(t) := \mathcal{V}(\boldsymbol{f}; 0, t)$ is left-continuous and non-decreasing. Let $\mu_{\mathcal{H}}$ be the unique Radon measure on $[0,T]$ such that $\mu_{\mathcal{H}}([0,t)) = V(t)$ for every $t \in (0,T]$. This measure is absolutely continuous with respect to μ, as a consequence of (1.4.1); therefore, by the Besicovitch Differentiation Theorem there exists a μ-negligible set M such that the limit

$$\frac{d\mu_{\mathcal{H}}}{d\mu}(t) := \lim_{h \to 0^+} \frac{V(t+h+) - V(t-h)}{\mu([t-h, t+h])}$$

exists for every $t \in [0,T] \setminus M$, and

$$\int_a^b \frac{d\mu_{\mathcal{H}}}{d\mu}(t)\, d\mu(t) = \mathcal{V}(\boldsymbol{f}; a, b)\,. \tag{1.4.11}$$

Let $t \in [0,T] \setminus (N \cup M)$, where N is the set defined in the previous theorem. Since $\mathcal{H}$ is positively homogeneous of degree 1, we have

$$\mathcal{H}\Big(\frac{\boldsymbol{f}(t_0+h+) - \boldsymbol{f}(t_0-h)}{\mu([t_0-h, t_0+h])}\Big) \le \frac{V(t_0+h+) - V(t_0-h)}{\mu([t_0-h, t_0+h])}$$

for every $h > 0$. Using the weak*-lower semicontinuity of $\mathcal{H}$, by (1.4.7) and by the previous inequality we get

$$\begin{aligned}\mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t)) &\le \liminf_{h\to 0^+} \mathcal{H}\Big(\frac{\boldsymbol{f}(t+h+)-\boldsymbol{f}(t_0-h)}{\mu([t-h,t_0+h])}\Big) \le \\ &\le \limsup_{h\to 0^+} \mathcal{H}\Big(\frac{\boldsymbol{f}(t+h+)-\boldsymbol{f}(t-h)}{\mu([t-h,t+h])}\Big) \le \frac{d\mu_{\mathcal{H}}}{d\mu}(t)\end{aligned}$$

for μ-a.e. $t \in [0,T]$. We now integrate with respect to μ and we obtain (1.4.9) from (1.4.10) and (1.4.11). ☐

Remark 1.6. When $\mathcal{K}$ is the unit ball of Y, then $\mathcal{V} = \mathrm{Var}$ and $\mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t)) = \|\boldsymbol{\nu}_{\boldsymbol{f}}(t)\|_X$. It follows from (1.4.5) and (1.4.9) that

$$\|\boldsymbol{\nu}_{\boldsymbol{f}}(t)\|_X = 1 \tag{1.4.12}$$

for μ-a.e. $t \in [0,T]$.

Remark 1.7. Let μ_d the diffuse part of μ, that is to say

$$\mu_d = \mu - \sum_{\tau\in J} \mu(\{\tau\})\delta_\tau\,,$$

where $J := \{\tau \in [0,T] : \mu(\{\tau\}) > 0\}$, which is at most countable. From Theorem 1.3, Theorem 1.5, and (1.4.8) we can deduce that, if $\boldsymbol{f}$ is left-continuous and has bounded variation

$$\int_a^b \langle y, \boldsymbol{\nu}_{\boldsymbol{f}}(t)\rangle\, d\mu_d(t) = \langle y, \boldsymbol{f}(b)-\boldsymbol{f}(a)\rangle - \sum_{\tau\in J\cap[a,b)} \langle y, \boldsymbol{f}(\tau+)-\boldsymbol{f}(\tau)\rangle \tag{1.4.13}$$

for every $y \in Y$ and every $0 \le a < b \le T$, and

$$\int_a^b \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{f}}(t))\, d\mu_d(t) = \mathcal{V}(\boldsymbol{f};a,b) - \sum_{\tau\in J\cap[a,b)} \mathcal{H}(\boldsymbol{f}(\tau+)-\boldsymbol{f}(\tau)) \tag{1.4.14}$$

for every $0 \le a < b \le T$. The proof is indeed obvious when $\mu(\{a\}) = \mu(\{b\}) = 0$, otherwise it can be obtained by approximation with subintervals $[a_n, b_n]$ of $[a,b]$ such that $\mu(\{a_n\}) = \mu(\{b_n\}) = 0$ for every n.

We now turn to the case of absolutely continuous functions. We recall that a function $\boldsymbol{f}\colon [a,b] \to X$ is said to be absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|\boldsymbol{f}(t_i)-\boldsymbol{f}(s_i)\|_X < \varepsilon$, whenever $a \le s_1 < t_1 \le s_2 < t_2 \le \cdots \le s_k < t_k \le b$ and $\sum_i (t_i - s_i) < \delta$. The space of these functions is denoted by $AC([0,T];X)$. For the general properties of absolutely continuous functions with values in reflexive Banach spaces we refer to [4, Appendix]. Here instead, in our more general setting, we can deduce from Theorems 1.3 and 1.5 the following one, whose original proof can be found in [13, Theorem 7.1].

Theorem 1.8. *Let $\boldsymbol{f}\colon [0,T] \to X$ be an absolutely continuous function. Then the weak*-limit*

$$\dot{\boldsymbol{f}}(t) := w^*\text{-}\lim_{s\to t} \frac{\boldsymbol{f}(s)-\boldsymbol{f}(t)}{s-t} \tag{1.4.15}$$

exists for a.e. $t \in [0,T]$. *Moreover, the function* $t \mapsto \mathcal{H}(\dot{\boldsymbol{f}}(t))$ *is measurable and*

$$\mathcal{V}(\boldsymbol{f}; a, b) = \int_a^b \mathcal{H}(\dot{\boldsymbol{f}}(t))\, dt \tag{1.4.16}$$

for every $a, b \in [0,T]$ *with* $a \le b$.

Proof. Let μ and $\boldsymbol{\nu_f}$ as in Theorem 1.3. By the absolute continuity of $\boldsymbol{f}$, μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$ on $[0,T]$, therefore $\mu = g\,\mathcal{L}^1$ with $g \in L^1([0,T])^+$. By the Lebesgue Differentiation Theorem, for $\mathcal{L}^1$-a.e. $t \in [0,T]$ we have

$$\lim_{h \to 0^+} \frac{\mu([t-h, t+h])}{2h} = g(t)\,,$$

therefore (1.4.7) gives

$$\langle y, \boldsymbol{\nu_f}(t) g(t)\rangle = \lim_{h\to 0^+} \frac{\langle y, \boldsymbol{f}(t+h) - \boldsymbol{f}(t-h)\rangle}{2h}$$

for every $y \in Y$ and $\mathcal{L}^1$-a.e. $t \in [0,T]$, and (1.4.15) follows with $\dot{\boldsymbol{f}}(t) = \boldsymbol{\nu_f}(t) g(t)$. With this, (1.4.9) and the positive 1-homogeneity of $\mathcal{H}$, the proof of (1.4.16) is trivial. ☐

Note that in this general situation $\dot{\boldsymbol{f}}$ is only weakly* measurable, therefore it may happen that $\dot{\boldsymbol{f}}$ is not Bochner integrable. If $\varphi\colon [c,d] \to [a,b]$ is nondecreasing and absolutely continuous, then the function $\boldsymbol{g}(s) := \boldsymbol{f}(\varphi(s))$ is absolutely continuous and

$$\dot{\boldsymbol{g}}(s) = \hat{\boldsymbol{f}}(\varphi(s))\dot{\varphi}(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } \; s \in [c,d], \tag{1.4.17}$$

where $\hat{\boldsymbol{f}}(t) = \dot{\boldsymbol{f}}(t)$ if the derivative (1.4.15) exists, while $\hat{\boldsymbol{f}}(t) = 0$ otherwise (this result can be obtained for instance by adapting [3, Theorem 4.2]). It follows that

$$\int_c^d \boldsymbol{h}(\varphi(s))\dot{\varphi}(s)\, ds = \int_{\varphi(c)}^{\varphi(d)} \boldsymbol{h}(t)\, dt \tag{1.4.18}$$

for every $\boldsymbol{h} \in L^1([a,b];X)$. Indeed, the derivatives with respect to d of both sides in (1.4.18) coincide $\mathcal{L}^1$-a.e. by (1.4.17).

1.5 Discrete approximation of some integrals

In this section we establish some measure theoretic results concerning a discrete approximation of some integrals that will prove useful in Chapter 5 to get the energy-dissipation balance.

Given $\boldsymbol{p}\colon [0,T] \to M_b(\Omega \cup \Gamma_0, \mathbb{M}^{N\times N}_{sym})$ and $\zeta \in C^0(\overline{\Omega})^+$, according to (1.4.3) for every $0 \le a \le b \le T$ we define

$$\mathcal{V}(\boldsymbol{p}, \zeta; a, b) := \sup \sum_{i=1}^{k} \mathcal{H}(\boldsymbol{p}(t_i) - \boldsymbol{p}(t_{i-1}), \zeta)\,, \tag{1.5.1}$$

where the supremum is taken over all finite families $t_0, t_1, \ldots, t_k$ such that $a = t_0 \le t_1 \le \cdots \le t_k = b$. If $\boldsymbol{p}$ is absolutely continuous, the weak*-derivative $\dot{\boldsymbol{p}}$ is defined by (1.4.15).

In the following Lemma we combine some elementary properties of $\mathcal{V}(\boldsymbol{p},\zeta;0,T)$ with the representation formula (1.4.9) to get a discrete approximation of the integral

$$\int_0^T \mathcal{H}(\dot{\boldsymbol{p}}(t),\zeta)\,dt\,.$$

Lemma 1.9. *Let $T>0$, let $\boldsymbol{p}\in AC([0,T],M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym}))$, let $\zeta\in C^0(\overline{\Omega})^+$, and let $\{t^i_k\}_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,T]$ satisfying*

$$0=t^0_k\le t^1_k\le\cdots\le t^{i_k}_k=T \quad \textit{and} \quad \eta_k:=\max_{1\le i\le k}(t^i_k-t^{i-1}_k)\to 0\,. \tag{1.5.2}$$

Then

$$\lim_{k\to\infty}\sum_{i=1}^{i_k}\Big|\mathcal{H}(\boldsymbol{p}(t^i_k)-\boldsymbol{p}(t^{i-1}_k),\zeta)-\int_{t^{i-1}_k}^{t^i_k}\mathcal{H}(\dot{\boldsymbol{p}}(t),\zeta)\,dt\Big|=0\,. \tag{1.5.3}$$

Proof. We first show that, if $\boldsymbol{p}$ is only assumed to be left continuous in $[0,T]$ with respect to the norm topology in $M_b(\Omega\cup\Gamma_0,\mathbb{M}^{N\times N}_{sym})$, then

$$\mathcal{V}(\boldsymbol{p},\zeta;0,T)=\lim_{k\to\infty}\sum_{i=1}^{i_k}\mathcal{H}(\boldsymbol{p}(t^i_k)-\boldsymbol{p}(t^{i-1}_k),\zeta)\,. \tag{1.5.4}$$

To get this, by (1.5.1) it is enough to prove the inequality

$$\mathcal{V}(\boldsymbol{p},\zeta;0,T)\le\liminf_{k\to\infty}\sum_{i=1}^{i_k}\mathcal{H}(\boldsymbol{p}(t^i_k)-\boldsymbol{p}(t^{i-1}_k),\zeta)\,. \tag{1.5.5}$$

Let us fix $\lambda<\mathcal{V}(\boldsymbol{p},\zeta;0,T)$. By (1.5.1) there exist an integer h and a subdivision $0=t_0\le t_1\le\cdots\le t_h=T$ such that

$$\lambda<\sum_{j=1}^{h}\mathcal{H}(\boldsymbol{p}(t_j)-\boldsymbol{p}(t_{j-1}),\zeta)\,. \tag{1.5.6}$$

For every j and k, let $\iota(j,k)$ be the greatest integer i such that $t^i_k\le t_j$. Since $t_j-\eta_k<t^{\iota(j,k)}_k\le t_j$ and $\eta_k\to 0$, inequality (1.5.6), together with the left continuity of $\boldsymbol{p}$ and the continuity of $\mathcal{H}$, gives

$$\lambda<\sum_{j=1}^{h}\mathcal{H}(\boldsymbol{p}(t^{\iota(j,k)}_k)-\boldsymbol{p}(t^{\iota(j-1,k)}_k),\zeta)$$

for k large enough. By the triangle inequality (1.3.11), this implies

$$\lambda<\sum_{i=1}^{i_k}\mathcal{H}(\boldsymbol{p}(t^i_k)-\boldsymbol{p}(t^{i-1}_k),\zeta)$$

for k large enough. Inequality (1.5.5), and thus (1.5.4), follow from the arbitrariness of $\lambda<\mathcal{V}(\boldsymbol{p},\zeta;0,t)$.

Now, if $\boldsymbol{p}$ is absolutely continuous, by Theorem 1.8 we have

$$\mathcal{V}(\boldsymbol{p},\zeta;a,b)=\int_a^b\mathcal{H}(\dot{\boldsymbol{p}}(t),\zeta)\,dt \tag{1.5.7}$$

for every $0 \le a \le b \le T$. Therefore (1.5.4) gives

$$\int_0^T \mathcal{H}(\dot{\boldsymbol{p}}(t), \zeta)\, dt = \lim_{k\to\infty} \sum_{i=1}^{i_k} \mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \zeta)\,. \tag{1.5.8}$$

Since

$$\mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \zeta) \le \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \zeta)\, dt$$

by (1.5.7), equality (1.5.3) is equivalent to (1.5.8). □

We now turn to the case of a time-dependent function $\boldsymbol{\zeta}\colon [0,T] \to C^0(\overline{\Omega})^+$, and we get an analogous result for the integral

$$\int_0^T \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t))\, dt\,.$$

Lemma 1.10. *Let $T > 0$, let $\{t_k^i\}_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,T]$ satisfying (1.5.2), and consider $\boldsymbol{p} \in AC([0,T], M_b(\Omega\cup\Gamma_0; \mathbb{M}^{N\times N}_{sym}))$ and $\boldsymbol{\zeta} \in C^0([0,T]; C^0(\overline{\Omega})^+)$. Then*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \boldsymbol{\zeta}(t_k^{i-1})) - \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t))\, dt \Big| = 0\,, \tag{1.5.9}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \boldsymbol{\zeta}(t_k^{i})) - \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t))\, dt \Big| = 0\,. \tag{1.5.10}$$

Proof. Since $t \mapsto \boldsymbol{\zeta}(t)$ is continuous, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\|\boldsymbol{\zeta}(t') - \boldsymbol{\zeta}(t)\|_\infty < \varepsilon \quad \text{for every } t',\, t \in [0,T] \text{ with } |t'-t| < \delta(\varepsilon)\,. \tag{1.5.11}$$

Let us fix $\varepsilon > 0$ and a subdivision $0 = t_0 < t_1 < \cdots < t_h = T$ such that $t_j - t_{j-1} < \delta(\varepsilon)$ for every $j = 1, \ldots, h$. By Lemma 1.9 we have

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \boldsymbol{\zeta}(t_j)) - \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t_j))\, dt \Big| = 0 \tag{1.5.12}$$

for every $j = 1, \ldots, h$.

If $t_{j-1} < t_k^i \le t_j$, by (1.5.2) for every $t_k^{i-1} \le t \le t_k^i$ we have $|t - t_{j-1}| < \delta(\varepsilon)$ and $|t - t_j| < \delta(\varepsilon)$ for k sufficiently large. Therefore (1.3.14), (1.3.20), and (1.5.11) give

$$|\mathcal{H}(p, \boldsymbol{\zeta}(t)) - \mathcal{H}(p, \boldsymbol{\zeta}(t_{j-1}))| \le M_K \varepsilon \|p\|_1 \quad \text{and} \quad |\mathcal{H}(p, \boldsymbol{\zeta}(t)) - \mathcal{H}(p, \boldsymbol{\zeta}(t_j))| \le M_K \varepsilon \|p\|_1$$

for every $p \in M_b(\Omega\cup\Gamma_0, \mathbb{M}^{N\times N}_{sym})$ and every $t_k^{i-1} \le t \le t_k^i$. Since $\boldsymbol{p}$ is absolutely continuous, this implies, thanks to Theorem 1.8, that

$$|\mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \boldsymbol{\zeta}(t_k^{i-1})) - \mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \boldsymbol{\zeta}(t_j))| \le$$

$$\le M_K \varepsilon \|\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1})\|_1 \le M_K \varepsilon \int_{t_k^{i-1}}^{t_k^i} \|\dot{\boldsymbol{p}}(t)\|_1 dt\,,$$

$$\Big| \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t))\, dt - \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t_j))\, dt \Big| \le M_K \varepsilon \int_{t_k^{i-1}}^{t_k^i} \|\dot{\boldsymbol{p}}(t)\|_1 dt\,,$$

Therefore

$$\sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \boldsymbol{\zeta}(t_k^{i-1})) - \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t))\, dt \Big| \leq$$

$$\leq \sum_{j=1}^{h} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \boldsymbol{\zeta}(t_j)) - \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t_j))\, dt \Big| + 2M_K \varepsilon \int_0^T \|\dot{\boldsymbol{p}}(t)\|_1 dt \,,$$

so (1.5.12) gives

$$\limsup_{k\to\infty} \sum_{i=1}^{i_k} \Big| \mathcal{H}(\boldsymbol{p}(t_k^i) - \boldsymbol{p}(t_k^{i-1}), \boldsymbol{\zeta}(t_k^{i-1})) - \int_{t_k^{i-1}}^{t_k^i} \mathcal{H}(\dot{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t))\, dt \Big| \leq 2M_K \varepsilon \int_0^T \|\dot{\boldsymbol{p}}(t)\|_1 dt \,.$$

Equality (1.5.9) follows now from the arbitrariness of $\varepsilon > 0$. The proof of (1.5.10) is similar. □

We now prove two lemmas concerning the approximation of Lebesgue integrals by Riemann sums. The first one, in the weaker form (1.5.14) is well-known (see [22]). For the application we have in mind we need the stronger result (1.5.13), which is related to the Saks-Henstock lemma (see [45] and [24]) used in the theory of Henstock-Kurzweil integral (see, e.g., [30]). We present here an elementary proof in the framework of Lebesgue integration, based on Fubini's theorem, taken from [16, page 63].

Lemma 1.11. *Let $T > 0$, let X be a Banach space, and let $\boldsymbol{\psi}\colon [0,T] \to X$ be a Bochner integrable function. Then there exists a sequence $(t_k^i)_{0\leq i\leq i_k}$ of subdivisions of the interval $[0,T]$ satisfying (1.5.2) such that*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i-1})\|\, dt = 0 = \lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i})\|\, dt \,. \tag{1.5.13}$$

In particular we have

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \boldsymbol{\psi}(t_k^{i-1})(t_k^i - t_k^{i-1}) = \int_0^T \boldsymbol{\psi}(t)\, dt = \lim_{k\to\infty} \sum_{i=1}^{i_k} \boldsymbol{\psi}(t_k^{i})(t_k^i - t_k^{i-1}) \,, \tag{1.5.14}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \|\boldsymbol{\psi}(t_k^i) - \boldsymbol{\psi}(t_k^{i-1})\|(t_k^i - t_k^{i-1}) = 0 \,, \tag{1.5.15}$$

where the limits in (1.5.14) are in the strong topology of X.

Proof. We extend $\boldsymbol{\psi}$ to 0 outside $[0,T]$. Set, for every $m \geq 1$ and every $i \in \mathbb{Z}$, $\tau_m^i := \frac{i}{m}$. For every $s \in [0,1]$ we have

$$\sum_{i\in\mathbb{Z}} \int_{s+\tau_m^{i-1}}^{s+\tau_m^i} \|\boldsymbol{\psi}(s+\tau_m^i) - \boldsymbol{\psi}(t)\|\, dt =$$

$$= \sum_{i\in\mathbb{Z}} \int_0^{\frac{1}{m}} \|\boldsymbol{\psi}(s+\tau_m^i) - \boldsymbol{\psi}(s+\tau_m^i - \tau)\|\, d\tau \,.$$

Observe that there are at most $m(T+1)+2$ non-zero elements in the above sums, namely those satisfying $i \in I_m := \{i \in Z : -m \le i \le mT+1\}$. Integrating in the variable s we then get

$$\int_0^1 \Big[\sum_{i\in\mathbb{Z}} \int_{s+\tau_m^{i-1}}^{s+\tau_m^i} \|\boldsymbol{\psi}(s+\tau_m^i)-\boldsymbol{\psi}(t)\|\,dt\Big]\,ds \le$$

$$\sum_{i\in I_m} \int_0^{\frac{1}{m}} \Big[\int_{-\infty}^{+\infty} |\boldsymbol{\psi}(s+\tau_m^i)-\boldsymbol{\psi}(s+\tau_m^i-\tau)\|\,ds\Big]\,d\tau = \tag{1.5.16}$$

$$= \sum_{i\in I_m} \int_0^{\frac{1}{m}} \Big[\int_{-\infty}^{+\infty} |\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s-\tau)\|\,ds\Big]\,d\tau\,.$$

Since the translations are continuous, for every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\int_{-\infty}^{+\infty} |\boldsymbol{\psi}(s)-\boldsymbol{\psi}(s-\tau)\|\,ds < \varepsilon \tag{1.5.17}$$

for $0 < \tau < \delta$. Thus, (1.5.17) and (1.5.16) imply that

$$\lim_{m\to+\infty} \int_0^1 \Big[\sum_{i\in\mathbb{Z}} \int_{s+\tau_m^{i-1}}^{s+\tau_m^i} \|\boldsymbol{\psi}(s+\tau_m^i)-\boldsymbol{\psi}(t)\|\,dt\Big]\,ds = 0\,.$$

It follows that, along a suitable subsequence $m_k \to +\infty$, we have

$$\lim_{k\to+\infty} \Big[\sum_{i\in\mathbb{Z}} \int_{s+\tau_m^{i-1}}^{s+\tau_{m_k}^i} \|\boldsymbol{\psi}(s+\tau_{m_k}^i)-\boldsymbol{\psi}(t)\|\,dt\Big]\,ds = 0 \tag{1.5.18}$$

for $\mathcal{L}^1$-a.e. $s \in [0,1]$. Let us fix $s \in [0,1]$ such that (1.5.18) holds. Let ϱ_k be the largest integer i such that $s+\tau_{m_k}^i \le 0$, and let σ_k be the smallest integer i such that $s+\tau_{m_k}^i \ge T$, and let $i_k := \sigma_k - \varrho_k$. For $i = 1, \dots, i_k - 1$ we define $t_k^i := s + \tau_{m_k}^{\varrho_k+i}$, and we set $t_k^0 := 0$ and $t_k^{i_k} := T$. It is clear that (1.5.2) is satisfied. moreover

$$\sum_{i=1}^{i_k} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t_k^i)-\boldsymbol{\psi}(t)\|\,dt =$$

$$= \sum_{i=\varrho_k+2}^{i=\sigma_k-1} \int_{s+\tau_m^{i-1}}^{s+\tau_{m_k}^i} \|\boldsymbol{\psi}(s+\tau_{m_k}^i)-\boldsymbol{\psi}(t)\|\,dt = \tag{1.5.19}$$

$$= \int_0^{a_k} |\boldsymbol{\psi}(a_k)-\boldsymbol{\psi}(t)\|\,dt + \int_{b_k}^T |\boldsymbol{\psi}(T)-\boldsymbol{\psi}(t)\|\,dt\,,$$

where $a_k := s + \tau_{m_k}^{\varrho_k+1}$ and $b_k := s + \tau_{m_k}^{\sigma_k-1}$. Since all integers between ϱ_k+2 and σ_k-1 belong to I_{m_k}, the first term in the right-hand side of (1.5.19) tends to 0 by (1.5.18). The second term is estimated by

$$\int_0^{a_k} |\boldsymbol{\psi}(a_k)-\boldsymbol{\psi}(t)\|\,dt \le \int_{s+\tau_m^{\varrho_k}}^{s+\tau_{m_k}^{\varrho_k+1}} \|\boldsymbol{\psi}(s+\tau_{m_k}^{\varrho_k+1})-\boldsymbol{\psi}(t)\|\,dt\,,$$

which also tends to 0 by (1.5.18). As $T-b_k$ is infinitesimal by the choice of σ_k, the absolute continuity of the integral yields that also the third term goes to 0. This proves (1.5.13). Equality (1.5.15) follows from (1.5.13) by the triangle inequality. $\square$

For our purposes, we will need an ad-hoc refinement of the previous lemma. To be definite, we consider a measurable subset B of $[0,T]$. In a simplified situation, say if B is a subinterval or a finite union of subintervals, it is not difficult to prove that, in order to approximate the Bochner integral of $\boldsymbol{\psi}$ over B, we can take the sequence of Riemann sums satisfying (1.5.13) and consider only the contributions of the indexes i such that $t_k^{i-1} \in B$ and $t_k^i \in B$. The next lemma shows that is is true for an arbitrary measurable set B. The reason for this technical point will be clear in Chapter 5, Section 5.6. Roughly speaking this lemma is tailored to a situation where, when proving an inequality involving integral terms for a system whose solutions can show different types of dynamics, one may need different types of approximations on the set of times where a certain regime is followed and on its complementary set.

Lemma 1.12. *Let $T > 0$, let X be a Banach space, let $\boldsymbol{\psi}\colon [0,T] \to X$ be a Bochner integrable function, let A be a measurable set in $[0,T]$ such that $\boldsymbol{\psi} = 0$ on A, let $B := [0,T] \setminus A$, and let $(t_k^i)_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,T]$ satisfying* (1.5.2) *and* (1.5.13)*, and hence* (1.5.14)*. Let us define*

$$I_k^A := \{i : 1 \le i \le i_k,\ t_k^{i-1} \in A,\ t_k^i \in A\}\,, \tag{1.5.20}$$
$$I_k^B := \{i : 1 \le i \le i_k,\ t_k^{i-1} \in B,\ t_k^i \in B\}\,, \tag{1.5.21}$$
$$J_k^{A-} := \{i : 1 \le i \le i_k,\ t_k^{i-1} \in A,\ t_k^i \in B\}\,, \tag{1.5.22}$$
$$J_k^{A+} := \{i : 1 \le i \le i_k,\ t_k^{i-1} \in B,\ t_k^i \in A\}\,, \tag{1.5.23}$$
$$J_k^A := J_k^{A-} \cup J_k^{A+}\,. \tag{1.5.24}$$

Then

$$\lim_{k\to\infty} \sum_{i\in I_k^B} \boldsymbol{\psi}(t_k^{i-1})(t_k^i - t_k^{i-1}) = \int_0^T \boldsymbol{\psi}(t)\,dt = \lim_{k\to\infty} \sum_{i\in I_k^B} \boldsymbol{\psi}(t_k^i)(t_k^i - t_k^{i-1})\,, \tag{1.5.25}$$
$$\lim_{k\to\infty} \sum_{i\in J_k^A} (\|\boldsymbol{\psi}(t_k^{i-1})\| + \|\boldsymbol{\psi}(t_k^i)\|)(t_k^i - t_k^{i-1}) = 0\,, \tag{1.5.26}$$
$$\lim_{k\to\infty} \sum_{i\in I_k^A\cup J_k^A} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t)\|\,dt = 0\,, \tag{1.5.27}$$

where the limits in (1.5.25) *are in the strong topology of X.*

Proof. By (1.5.13) we have

$$\lim_{k\to\infty} \sum_{i\in I_k^B} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i-1})\|\,dt = 0 = \lim_{k\to\infty} \sum_{i\in I_k^B} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^i)\|\,dt\,, \tag{1.5.28}$$
$$\lim_{k\to\infty} \sum_{i\in J_k^{A+}} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i-1})\|\,dt = 0 = \lim_{k\to\infty} \sum_{i\in J_k^{A-}} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^i)\|\,dt\,, \tag{1.5.29}$$
$$\lim_{k\to\infty} \sum_{i\in I_k^A\cup J_k^{A-}} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t)\|\,dt = 0 = \lim_{k\to\infty} \sum_{i\in I_k^A\cup J_k^{A+}} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t)\|\,dt\,. \tag{1.5.30}$$

Equality (1.5.27) follows from (1.5.30). Applying the triangle inequality we obtain (1.5.26) from (1.5.27) and (1.5.29). On the other hand, taking into account (1.5.20)-(1.5.23), we have

$$\sum_{i=1}^{i_k} \boldsymbol{\psi}(t_k^{i-1})(t_k^i - t_k^{i-1}) = \sum_{i\in I_k^B} \boldsymbol{\psi}(t_k^{i-1})(t_k^i - t_k^{i-1}) + \sum_{i\in J_k^{A+}} \boldsymbol{\psi}(t_k^{i-1})(t_k^i - t_k^{i-1}) , \qquad (1.5.31)$$

and the last sum tends to 0 by (1.5.26). Therefore, the first equality in (1.5.25) follows from (1.5.14) and (1.5.31). The proof of the other equality is similar. □

Remark 1.13. Let T, A, and B be as in Lemma 1.12, and let $(t_k^i)_{0\le i\le i_k}$ be a sequence of subdivisions of $[0,T]$ satisfying (1.5.2) and

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{t_k^{i-1}}^{t_k^i} |1_B(t) - 1_B(t_k^{i-1})|\, dt = 0 \ = \lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{t_k^{i-1}}^{t_k^i} |1_B(t) - 1_B(t_k^{i})|\, dt , \qquad (1.5.32)$$

where 1_B denotes the characteristic function of B, defined by $1_B(t) = 1$ for $t\in B$ and $1_B(t) = 0$ for $t \notin B$. It follows from Lemma 1.12, applied to $X = \mathbb{R}$ and $\boldsymbol{\psi}(t) = 1_B(t)$, that

$$\lim_{k\to\infty} \sum_{i\in J_k^A} (t_k^i - t_k^{i-1}) = 0 = \lim_{k\to\infty} \sum_{i\in I_k^A\cup J_k^A} \mathcal{L}^1(B\cap[t_k^{i-1}, t_k^i]) . \qquad (1.5.33)$$

Remark 1.14. Let T, A, and B be as in Lemma 1.12. If in addition $\boldsymbol{\psi}\colon [0,T]\to X$ is bounded and A is a relatively open set in $[0,T]$, then, given $(t_k^i)_{0\le i\le i_k}$ a sequence of subdivisions of $[0,T]$ satisfying (1.5.2), (1.5.13), and (1.5.32), it is not restrictive to assume that $(t_k^{i-1}, t_k^i)\subset A$ for every $i\in J_k^A$.

Indeed, if not, we can construct another subdivision satisying (1.5.2), (1.5.13), (1.5.32), and our additional request, proceeding as follows. For every $i\in J_k^{A-}$ let $t_k^{i-\frac12}$ be the supremum of the connected component of A containing t_k^{i-1}, and for every $i\in J_k^{A+}$ let $t_k^{i-\frac12}$ be the infimum of the connected component of A containing t_k^i. If $1\le i\le i_k$ and $i\notin J_k^A$, we set $t_k^{i-\frac12} := t_k^i$. Then we consider the subdivision $(\hat t_k^i)_{0\le i\le 2i_k}$ defined by $\hat t_k^i := t_k^{i/2}$, which clearly satisfies (1.5.2). Defining $\hat J_k^A$ by (1.5.24), with $\hat t_k^i$ instead of t_k^i, by construction $(\hat t_k^{i-1}, \hat t_k^i)\subset A$ for every $i\in \hat J_k^A$.

To see that (1.5.13), and (1.5.32) are satisfied, let M be an upper bound of $\|\boldsymbol{\psi}(t)\|$ on $[0,T]$. Since $t_k^{i-\frac12} = t_k^i$ for $i\notin J_k^A$ and $\|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i-\frac12})\| \le \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i-1})\| + 2M$ for every $i\in J_k^A$ and every $s\in[0,T]$, we have

$$\sum_{i=1}^{i_k} \Big(\int_{t_k^{i-1}}^{t_k^{i-\frac12}} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i-1})\|\, dt + \int_{t_k^{i-\frac12}}^{t_k^i} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i-\frac12})\|\, dt \Big) \le$$
$$\le \sum_{i=1}^{i_k} \int_{t_k^{i-1}}^{t_k^i} \|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t_k^{i-1})\|\, dt + 2M \sum_{i\in J_k^A} (t_k^i - t_k^{i-1}) .$$

Since the right-hand side tends to 0 by (1.5.13) and (1.5.33), we obtain the first equality in (1.5.13) for $\hat t_k^i$. A similar argument proves the other equality, as well as (1.5.32).

1.6 Continuous dependence on a parameter

A particular situation will occur in Chapter 3, where, due to a suitable choice of the data, our problem will reduce to determine the limit behavior of the solutions of a singularly perturbed system of ODE's in a finite dimensional Banach space. In this perspective, the following result about continuous dependence on a parameter, whose original proof can be found in [27] (see also [26]), will be useful .

Theorem 1.15. *Let f_ε and f_0 be Carathéodory functions defined on $[a,b]\times\mathbb{R}^m$ with values in $\mathbb{R}^m$, let t_ε, $t_0 \in [a,b]$, and let x_ε, $x_0 \in \mathbb{R}^m$. Assume that there exist two constants $L > 0$ and $M > 0$ such that*

$$|f_\varepsilon(t,x_2) - f_\varepsilon(t,x_1)| \le L\,|x_2 - x_1|\,, \tag{1.6.1}$$

$$|f_\varepsilon(t,x)| \le M\,, \tag{1.6.2}$$

for every $\varepsilon > 0$, every $t \in [a,b]$, and every x, x_1, $x_2 \in \mathbb{R}^m$. Let $y_\varepsilon(t)$ and $y_0(t)$ be the solutions of the Cauchy problems

$$\begin{cases} \dot y_\varepsilon(t) = f_\varepsilon(t,y(t))\,, \\ y_\varepsilon(t_\varepsilon) = x_\varepsilon\,, \end{cases} \qquad \begin{cases} \dot y_0(t) = f_0(t,y(t))\,, \\ y_\varepsilon(t_0) = x_0\,. \end{cases} \tag{1.6.3}$$

If $t_\varepsilon \to t_0$, $x_\varepsilon \to x_0$, and for every $x \in \mathbb{R}^m$

$$\int_a^t f_\varepsilon(s,x)\,ds \to \int_a^t f_0(s,x)\,ds \qquad \textit{uniformly for } t \in [a,b]\,, \tag{1.6.4}$$

then $y_\varepsilon(t) \to y_0(t)$ uniformly for $t \in [a,b]$.

Proof. It is easy to deduce from (1.6.4) that (1.6.1) and (1.6.2) hold also for $\varepsilon = 0$, therefore $y_0(t)$ is well defined on the whole $[a,b]$. Moreover it is not restrictive to take $t_0 = a$. Let $x(s)$ be a finite linear combination of characteristic functions of subintervals of $[a,b]$. Then (1.6.4) implies that

$$\int_a^t f_\varepsilon(s,x(s))\,ds \to \int_a^t f_0(s,x(s))\,ds \qquad \text{uniformly for } t \in [a,b]\,,$$

By uniform approximation, using (1.6.1) it is not difficult to prove that

$$\int_a^t f_\varepsilon(s,y_0(s))\,ds \to \int_a^t f_0(s,y_0(s))\,ds \qquad \text{uniformly for } t \in [a,b]\,. \tag{1.6.5}$$

So, let $R_\varepsilon := \sup_{t\in[a,b]} |\int_a^t f_\varepsilon(s,y_0(s))\,ds - \int_a^t f_0(s,y_0(s))\,ds|$. For every $t \in [a,b]$, by (1.6.1) and (1.6.3) we have

$$\begin{aligned} |y_\varepsilon(t) - y_0(t)| \quad &= |x_\varepsilon - x_0 + \int_a^t f_\varepsilon(s,y_\varepsilon(s))\,ds - \int_a^t f_0(s,y_0(s))\,ds| \le \\ &\le |x_\varepsilon - x_0| + L\int_a^t |y_\varepsilon(s) - y_0(s)|\,ds + R_\varepsilon\,, \end{aligned}$$

therefore the Gronwall inequality gives

$$|y_\varepsilon(t) - y_0(t)| \le e^{L(b-a)}(|x_\varepsilon - x_0| + R_\varepsilon)$$

for every $t \in [a,b]$, and the conclusion easily follows. $\square$

In the following corollary inequalities (1.6.1) and (1.6.2) are satisfied only in the intervals $[t_\varepsilon, b]$, and the conclusion is slightly weaker.

Corollary 1.16. *Let f_ε and f_0 be Carathéodory functions defined on $[a,b]\times\mathbb{R}^m$ with values in $\mathbb{R}^m$, let $t_\varepsilon \to a$, and let x_ε, $x_0 \in \mathbb{R}^m$. Assume that there exist two constants $L > 0$ and $M > 0$ such that* (1.6.1) *and* (1.6.2) *hold for every $\varepsilon > 0$, every $t \in [t_\varepsilon, b]$, and every x, x_1, $x_2 \in \mathbb{R}^m$. Let $y_\varepsilon(t)$ and $y_0(t)$ be the solutions of the Cauchy problems* (1.6.3)*. If $x_\varepsilon \to x_0$, and for every $x \in \mathbb{R}^m$ and every $\eta > 0$*

$$\int_{a+\eta}^{t} f_\varepsilon(s,x)\,ds \to \int_{a+\eta}^{t} f(s,x)\,ds \qquad \textit{uniformly for } t \in [a+\eta, b]\,,$$

then

$$\sup_{t_\varepsilon \le t \le b} |y_\varepsilon(t) - y_0(t)| \to 0$$

Proof. Define

$$g_\varepsilon(t,x) = \begin{cases} f_\varepsilon(t,x) & \text{if } t \ge t_\varepsilon \\ f_\varepsilon(t_\varepsilon, x) & \text{otherwise} \end{cases}$$

and let $z_\varepsilon(t)$ the solutions of the Cauchy problems

$$\begin{cases} \dot z_\varepsilon(t) = g_\varepsilon(t, z(t))\,, \\ z_\varepsilon(t_\varepsilon) = x_\varepsilon\,. \end{cases}$$

It is not difficult to see that previous theorem may be applied with $g_\varepsilon(t,x)$ in place of f_ε; then $z_\varepsilon(t) \to y_0(t)$ uniformly for $t \in [a,b]$; conclusion follows as, for every $\eta > 0$, when ε sufficiently small, $z_\varepsilon(t) = y_\varepsilon(t)$ in $[a+\eta, b]$ by the uniqueness of solutions to Cauchy problems. $\square$

Chapter 2

The problem and the mechanical assumptions

2.1 Overview of the chapter

The goal of this section is to introduce the Cam-Clay model in plasticy, to fix a convenient notation for the development of the study we will carry out in the next chapters and to discuss the main mechanical assumptions that we do in order to investigate the well-posedness of the problem. In the final section, we introduce the vanishing viscosity approximation which is the first tool we will use in order to give a suitable notion of generalized solution in Chapter 5, and we write down the ε-regularized equations.

2.2 The Cam-Clay model

Cam-Clay plasticity is a well established model for the description of the mechanics of fine grained soils [41, 42, 43, 46]. The framework is small strain elasto-plasticity, where the linear strain Eu is defined as the symmetric part of the gradient of the displacement u with respect to a reference configuration Ω. Moreover, the strain is additively decomposed into elastic and plastic part, namely $Eu = e + p$, where the elastic part e determines the stress σ through the linear constitutive relation $\sigma = \mathbb{C}e$, where $\mathbb{C}$ is the isotropic elasticity tensor (see (2.3.2)). The stress satisfies the standard equilibrium condition $-\mathrm{div}\,\sigma = f$ in Ω, where div is the divergence operator with respect to the space variable x and f denotes a time dependent body force.

In its classical formulation Cam-Clay plasticity rests on three main ingredients. The first one is a set of admissible stresses $K(\zeta)$, a compact convex set in the space of symmetric $n{\times}n$ matrices, whose size depends on a scalar internal variable ζ. The boundary $\partial K(\zeta)$ identifies the yield surface, while stresses in the interior of $K(\zeta)$ cause no plastic flow. In the typical applications, $\partial K(\zeta)$ are homothetic ellipsoids passing through the origin in the space $\mathbb{M}^{N\times N}_{sym}$. The technical assumptions on $K(\zeta)$ are those in the previous chapter, namely (1.3.1)-(1.3.4). The other two main ingredients are the evolution laws for the plastic strain p

and for the internal variable ζ. To write them explicitly, we introduce a new internal variable z, related to ζ by the equality $\zeta = V(z)$, where $V\colon \mathbb{R} \to (0, +\infty)$ is a globally Lipschitz nondecreasing function such that $V(z) \geq \zeta_m$ for every $z \in \mathbb{R}$ and a suitable constant $\zeta_m > 0$ (see (2.3.37)-(2.3.38)). Denoting the normal cone to $K(\zeta)$ at σ by $N_{K(\zeta)}(\sigma)$, the equations summarising the model are

(a) constitutive equations: $\sigma(t,x) = \mathbb{C}e(t,x)$ and $\zeta(t,x) = V(z(t,x))$,

(b) additive decomposition: $Eu(t,x) = e(t,x) + p(t,x)$,

(c) equilibrium condition : $-\mathrm{div}\, \sigma(t,x) = f(t,x)$,

(d) stress constraint: $\sigma(t,x) \in K(\zeta(t,x))$,

(e) flow rule: $\dot{p}(t,x) \in N_{K(\zeta(t,x))}(\sigma(t,x))$,

(f) evolution law for the internal variable: $\dot{z}(t,x) = \rho_1 \star [(\rho_2 \star \mathrm{tr}\, \sigma(t,\cdot))\, \mathrm{tr}\, \dot{p}(t,\cdot)](x)$,

accompanied by suitable boundary conditions. Here ρ_1 and ρ_2 are smooth convolution kernel with unitary mass (see (2.3.39)). The nonassociative nature of the problem is due to the fact that the evolution law (f) does not depend on K. Due to (1.3.5), if $\dot{z}(t,x) > 0$ the set $K(\zeta(t,x))$ expands leading to a hardening response. On the contrary, if $\dot{z}(t,x) < 0$ the set $K(\zeta(t,x))$ shrinks leading to a softening response.

The above formulation contains two differences with respect to the classical one, where $V(z) = z$ and the convolution kernel is not present in the evolution law for the internal variable. The main reason for introducing the convolution is technical: it ensures that a very weak convergence of σ and $\dot{p}$ implies strong convergence of the corresponding z. From the point of view of mechanics, the convolution gives a nonlocal character to the evolution law for the internal variable: it implies that the size of the yield surface at a point x is affected by pressure and volumetric plastic strain rate in a small neighborhood of x, which is not physically implausible. However, we anticipate that in the analysis of the spatially homogeneous case (Chapter 3) one can prove that z is positive and bounded away from 0, so that one can take $V(z) = z$, and identify z with ζ, as it is typical in the engineering literature. Therefore, since the convolution kernels have unitary mass, in the case where z is indepedendent of x (Chapter 3) we recover the classical formulation.

2.3 Mechanical preliminaries

The reference configuration. Throughout the paper the *reference configuration* Ω is a *bounded connected open set* in $\mathbb{R}^N$, $n \geq 2$, with *Lipschitz boundary* $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup N$. We assume that Γ_0 and Γ_1 are relatively open, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, and $\mathcal{H}^{n-1}(N) = 0$.

On Γ_0 we will prescribe a Dirichlet boundary condition. This will be done by assigning a function $w \in H^{1/2}(\partial\Omega; \mathbb{R}^N)$, or, equivalently, a function $w \in H^1(\Omega; \mathbb{R}^N)$, whose trace on Γ_0 (also denoted by w) is the prescribed boundary value. The set Γ_1 will be the part of the boundary on which the traction is prescribed.

Stress and strain. For a given displacement $u \in BD(\Omega)$ and a boundary datum $w \in H^1(\Omega;\mathbb{R}^N)$, the *elastic* and *plastic strains* $e \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ satisfy the *weak kinematic admissibility condition*

$$\begin{aligned} Eu &= e + p \text{ in } \Omega\,,\\ p &= (w-u)\odot\nu\,\mathcal{H}^{n-1} \text{ on } \Gamma_0\,, \end{aligned} \tag{2.3.1}$$

where ν is the outer unit normal to $\partial\Omega$, and the right-hand side of the second formula in (2.3.1) denotes the measure in $M_b(\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ with density $(w-u)\odot\nu \in L^1(\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ with respect to $\mathcal{H}^{n-1}$. As usual equality between measures on a set means that they agree on every Borel subset. The stress $\sigma \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ is defined by

$$\sigma := \mathbb{C}e\,, \tag{2.3.2}$$

where $\mathbb{C}$ is the *elasticity tensor*, considered as a symmetric positive definite linear operator $\mathbb{C}\colon \mathbb{M}^{N\times N}_{sym} \to \mathbb{M}^{N\times N}_{sym}$. We assume that $\mathbb{C}$ is isotropic, so that we have $\mathbb{C}\xi = 2\mu\xi + \lambda(\mathrm{tr}\xi)I$, where λ and μ are the Lamé constants. In terms of the canonical decomposition of a symmetric matrix in its spherical and deviatoric part we can write

$$\mathbb{C}\xi = 2\mu\xi_D + \kappa(\mathrm{tr}\xi)I\,, \tag{2.3.3}$$

where the constant $\mu > 0$ is the *shear modulus*, the constant $\kappa > 0$ is called *modulus of compression*, and ξ_D denotes the projection of ξ onto the space of trace-free symmetric matrices. Let $Q\colon \mathbb{M}^{N\times N}_{sym} \to [0,+\infty)$ be the quadratic form associated with $\mathbb{C}$, defined by

$$Q(\xi) := \tfrac{1}{2}\mathbb{C}\xi : \xi$$

It turns out that there exist two constants α_Q and β_Q, with $0 < \alpha_Q \le \beta_Q < +\infty$, such that

$$\alpha_Q|\xi|^2 \le Q(\xi) \le \beta_Q|\xi|^2 \tag{2.3.4}$$

for every $\xi \in \mathbb{M}^{N\times N}_{sym}$. These inequalities imply

$$|\mathbb{C}\xi| \le 2\beta_Q|\xi|\,. \tag{2.3.5}$$

The stored elastic energy $\mathcal{Q} : L^2(\Omega;\mathbb{M}^{N\times N}_{sym}) \to \mathbb{R}$ is given by

$$\mathcal{Q}(e) = \int_\Omega Q(e(x))\,dx = \tfrac{1}{2}\langle\sigma, e\rangle\,.$$

It is well known that $\mathcal{Q}$ is lower semicontinuous on $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ with respect to weak convergence.

Stress-strain duality and plastic dissipation. If $\sigma \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^N)$, then we can define a distribution $[\sigma\nu]$ on $\partial\Omega$ by

$$\langle[\sigma\nu],\psi\rangle_{\partial\Omega} := \langle\operatorname{div}\sigma,\psi\rangle_\Omega + \langle\sigma, E\psi\rangle_\Omega \tag{2.3.6}$$

for every $\psi \in H^1(\Omega;\mathbb{R}^N)$. It turns out that $[\sigma\nu] \in H^{-1/2}(\partial\Omega;\mathbb{R}^N)$ (see, e.g., [52, Chapter I, Theorem 1.2]). If, in addition, $\sigma \in L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym})$ and $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^N)$, then (2.3.6) holds for $\psi \in W^{1,1}(\Omega;\mathbb{R}^N)$. By Gagliardo's extension result [20, Theorem 1.II], it is easy to see that in this case $[\sigma\nu] \in L^\infty(\partial\Omega;\mathbb{R}^N)$ and that

$$[\sigma_k\nu] \rightharpoonup [\sigma\nu] \quad \text{weakly}^* \text{ in } L^\infty(\partial\Omega;\mathbb{R}^N), \tag{2.3.7}$$

whenever $\sigma_k \rightharpoonup \sigma$ weakly* in $L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega;\mathbb{R}^N)$.

We shall denote with $\Sigma(\Omega)$ the space

$$\Sigma(\Omega) := \{\sigma \in L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym}) : \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^N)\}\,. \tag{2.3.8}$$

Obviously, when $\sigma \in C^0(\overline{\Omega};\mathbb{M}^{N\times N}_{sym}) \cap \Sigma(\Omega)$ we have

$$[\sigma\nu] = \sigma\nu \quad \text{on } \Gamma_0\,, \tag{2.3.9}$$

where the right-hand side is the pointwise product between the matrix $\sigma(x)$ and the normal vector $\nu(x)$ at each $x \in \Gamma_0$.

The space $\Pi_{\Gamma_0}(\Omega)$ of admissible plastic strains is defined as the set of all $p \in M_b(\Omega \cup \Gamma_0;\mathbb{M}^{N\times N}_{sym})$ for which there exist $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^N)$, and $e \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ satisfying (2.3.1).

According to [7, Section 3], given $p \in \Pi_{\Gamma_0}(\Omega)$ and $\sigma \in \Sigma(\Omega)$, we can define the distribution $[\sigma : p]$ on Ω by setting, for every $\varphi \in C^\infty_c(\Omega)$,

$$\langle [\sigma : p], \varphi\rangle := -\langle \varphi u, \operatorname{div}\sigma\rangle - \langle \sigma, u \odot \nabla\varphi\rangle - \langle \sigma, \varphi e\rangle \tag{2.3.10}$$

where $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^N)$, $e \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ are as in (2.3.1). We extend the definition of $[\sigma : p]$ by setting

$$[\sigma : p] := [\sigma\nu]\cdot(w-u)\,\mathcal{H}^{n-1} \quad \text{on } \Gamma_0\,, \tag{2.3.11}$$

so that $[\sigma : p] \llcorner \Gamma_0 \in M_b(\Gamma_0)$. Actually, we have $[\sigma : p] \in M_b(\Omega \cup \Gamma_0)$ as we discuss in the next proposition, among the other properties of the distribution $[\sigma : p]$.

Proposition 2.1. *The definition of $[\sigma : p]$ does not depend on the functions u, w, e satisfying* (2.3.1)*. Moreover $[\sigma : p]$ is a bounded Radon measure on $\Omega \cup \Gamma_0$ and, if we define the duality product*

$$\langle \sigma, p\rangle := [\sigma : p](\Omega \cup \Gamma_0)\,, \tag{2.3.12}$$

we have that

$$|\langle \sigma, p\rangle| \le \|\sigma\|_\infty \|p\|_1. \tag{2.3.13}$$

Proof. By (2.3.11) it is easy to see that $[\sigma : p] \llcorner \Gamma_0$ is independent of u, w, e satisfying (2.3.1), so that we have only to check that the right-hand side of (2.3.10) is independent of the choice of $u \in BD(\Omega)$ and $e \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ such that $Eu = e + p$ in Ω. To do that we first observe that for every $\varphi \in C^1(\overline{\Omega})$ and $u \in BD(\Omega)$ we obviously have $\varphi u \in BD(\Omega)$ and

$$E(\varphi u) = \varphi Eu + u \odot \nabla\varphi\,. \tag{2.3.14}$$

Then we take u_1, $u_2 \in BD(\Omega)$ and e_1, $e_2 \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ such that $Eu_1 - e_1 = Eu_2 - e_2 = p$ in Ω. It follows that $E(u_1 - u_2) \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, so that by (1.2.2) we get

$u_1 - u_2 \in H^1(\Omega;\mathbb{R}^N)$. Therefore, for every $\varphi \in C_c^\infty(\Omega)$ we have $\varphi(u_1-u_2) \in H_0^1(\Omega;\mathbb{R}^N)$, so that (2.3.6) and (2.3.14) give

$$\langle \varphi(u_1-u_2), \operatorname{div}\sigma\rangle + \langle \sigma, (u_1-u_2)\odot\nabla\varphi\rangle + \langle \sigma, \varphi(e_1-e_2)\rangle = 0$$

for every $\varphi \in C_c^\infty(\Omega)$, as required.

We now show that $[\sigma : p]$ is a bounded Radon measure on Ω. To do that, we observe that by a standard approximation result (see, e.g., [52, Chapter II, Theorem 3.2]) there exists a sequence $v_k \in C^\infty(\bar\Omega;\mathbb{R}^N)$ such that

$$\begin{aligned} v_k &\to u \quad \text{in } L^1(\Omega;\mathbb{R}^N),\\ Ev_k &\rightharpoonup (Eu)\llcorner\Omega \quad \text{weakly}^* \text{ in } M_b(\bar\Omega;\mathbb{R}^N),\\ \|Ev_k\|_1 &\to \|Eu\|_1, \end{aligned} \tag{2.3.15}$$

and therefore, by the Sobolev embedding Theorem in BD,

$$v_k \to u \quad \text{weakly in } L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N). \tag{2.3.16}$$

We set $p_k = Ev_k - e$, so that $p_k \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$. Moreover, by (2.3.10), (2.3.15) and (2.3.16) we get

$$\langle [\sigma : p_k], \varphi\rangle \to \langle [\sigma : p], \varphi\rangle\,, \tag{2.3.17}$$

$$\|p_k\|_1 \to \|p\|_{1,\Omega} \tag{2.3.18}$$

as k goes to $+\infty$, where $\|p\|_{1,\Omega}$ is the norm of the measure $p\llcorner\Omega$. Now (2.3.6) gives

$$\langle \sigma, E(\varphi v_k)\rangle = -\langle \varphi v_k, \operatorname{div}\sigma\rangle$$

so that by (2.3.10) and the chain rule we have that

$$\langle [\sigma : p_k], \varphi\rangle = \langle \varphi\sigma, p_k\rangle$$

in the usual sense of the L^2 duality and we get the estimate

$$|\langle [\sigma : p_k], \varphi\rangle| \le \|\sigma\|_\infty \|p_k\|_1 \|\varphi\|_\infty\,.$$

From this, (2.3.17), and (2.3.18) the claim follows as well as the estimate

$$|\langle [\sigma : p], \varphi\rangle| \le \|\sigma\|_\infty \|p\|_{1,\Omega} \|\varphi\|_\infty\,. \tag{2.3.19}$$

By (2.3.11) we then deduce that $[\sigma : p] \in M_b(\Omega\cup\Gamma_0)$.

By (2.3.1), the restriction of p to Γ_0 can be identified with an element of the space $L^1(\Gamma_0;\mathbb{R}^N)$. If we assume that $\sigma \in C(\overline{\Omega}) \cap \Sigma(\Omega)$, from (2.3.1), (2.3.9), and (2.3.11) we easily deduce

$$\|[\sigma : p]\|_{1,\Gamma_0} \le \|\sigma\|_\infty \|p\|_{1,\Gamma_0} \tag{2.3.20}$$

where $\|\cdot\|_{1,\Gamma_0}$ denotes the norm of $L^1(\Gamma_0;\mathbb{R}^N)$. We claim that (2.3.20) holds for every $\sigma \in \Sigma(\Omega)$. Indeed, we can always find $\sigma_k \in C^\infty(\overline{\Omega})$ with $\|\sigma_k\|_\infty \le \|\sigma\|_\infty$ such that $\sigma_k \rightharpoonup \sigma$ weakly* in $L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega;\mathbb{R}^N)$. This follows from [13, Lemma 2.3], which is a particular case of Lemma 2.4 that we will prove later. With this fact, the claim follows from (2.3.7) and (2.3.11). Then, (2.3.13) follows from (2.3.19) and (2.3.20). □

Other properties of the measure $[\sigma : p]$ are collected in the next remark.

Remark 2.2. We claim that

$$\langle [\sigma : p], \varphi \rangle = \langle \varphi\sigma, p \rangle \tag{2.3.21}$$

for every $\sigma \in C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}) \cap \Sigma(\Omega)$ and every $\varphi \in C^0(\overline{\Omega})$, where the duality used in the right-hand side is the standard duality between continuous functions and measures. Equivalently, for every $\sigma \in C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}) \cap \Sigma(\Omega)$ and every $p \in \Pi_{\Gamma_0}(\Omega)$ we have

$$[\sigma : p] = \sigma : p \qquad \text{on } \Omega \cup \Gamma_0\,, \tag{2.3.22}$$

where the right-hand side denotes the measure defined by

$$(\sigma : p)(B) := \int_B \sigma : dp := \sum_{ij} \int_B \sigma_{ij}\, dp_{ij} \tag{2.3.23}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$. Indeed, using (2.3.13) and an approximation argument, it suffices to prove the claim for $\sigma \in C^1(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym})$. Take u, e, and w as in (2.3.1). By (2.3.9) and (2.3.11), equality (2.3.22) is trivial on any Borel subset of Γ_0, so that it only remains to check that (2.3.21) holds for every $\varphi \in C^\infty_c(\Omega)$. Using (2.3.1) and (2.3.14) we get

$$\langle \varphi\sigma, p \rangle = \langle \sigma, E(\varphi u) \rangle - \langle \sigma, u \odot \nabla\varphi \rangle - \langle \sigma, \varphi e \rangle\,.$$

Since $\varphi u = 0$ on $\partial\Omega$ the Green's formula in $BD(\Omega)$ gives $\langle \sigma, E(\varphi u) \rangle = \langle -\operatorname{div}\sigma, \varphi u \rangle$ so that (2.3.21) follows from (2.3.10).

If $\sigma_k \rightharpoonup \sigma$ weakly* in $L^\infty(\Omega; \mathbb{M}^{N\times N}_{sym})$, $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega; \mathbb{R}^N)$, from (2.3.7), (2.3.10), (2.3.11) and (2.3.13) we immediately deduce that

$$\langle [\sigma_k : p], \varphi \rangle \to \langle [\sigma : p], \varphi \rangle \tag{2.3.24}$$

for every $\varphi \in C^0(\overline{\Omega})$.

Fix $\sigma \in \Sigma(\Omega)$ and consider a sequence $\sigma_k \in C^\infty(\overline{\Omega})$ with $\|\sigma_k\|_\infty \le \|\sigma\|_\infty$ such that $\sigma_k \rightharpoonup \sigma$ weakly* in $L^\infty(\Omega; \mathbb{M}^{N\times N}_{sym})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega; \mathbb{R}^N)$, whose existence is guaranteed by [13, Lemma 2.3]. Denoting with p_a and p_s, respectively, the absolutely continuous and the singular part of p with respect to $\mathcal{L}^n$ we get from (2.3.22) that the Lebesgue decomposition of $[\sigma_k : p]$ on $\Omega \cup \Gamma_0$ is

$$[\sigma_k : p] = \sigma_k : p_a + \sigma_k : p_s\,. \tag{2.3.25}$$

Now it is easily seen, as the functions σ_k are equibounded in $C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym})$ that, when k goes to ∞, the sequence $\sigma_k : p_s$ weakly* converge (up to a subsequence) to a measure μ which is still singular with respect to $\mathcal{L}^n$. Using (2.3.24) and taking the limit in (2.3.25), we then obtain

$$[\sigma : p] = \sigma : p_a + \mu$$

on $\Omega \cup \Gamma_0$. By the uniqueness of the Lebesgue decomposition, this entails that the absolutely continuous part $[\sigma : p]_a$ of $[\sigma : p]$ with respect to $\mathcal{L}^n$ satisfies

$$[\sigma : p]_a = \sigma : p_a \quad \text{on } \Omega\,. \tag{2.3.26}$$

We also get that μ coincides exactly with the singular part $[\sigma:p]_s$ of $[\sigma:p]$ with respect to $\mathcal{L}^n$, and we have the estimate

$$|[\sigma:p]_s| \leq \|\sigma\|_\infty |p_s| \quad \text{on } \Omega \cup \Gamma_0 \,. \tag{2.3.27}$$

The following proposition provides a useful integration-by-parts formula.

Proposition 2.3. *Let* $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^N)$, $e \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, $p \in M_b(\Omega \cup \Gamma_0;\mathbb{M}^{N\times N}_{sym})$ *satisfy* (2.3.1). *Let* $\sigma \in L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym})$, $f \in L^n(\Omega;\mathbb{R}^N)$, *and* $g \in L^\infty(\Gamma_1;\mathbb{R}^N)$. *Assume that* $-\mathrm{div}\,\sigma = f$ *in* Ω, *and that* $[\sigma\nu] = g$ *on* Γ_1. *Then*

$$\langle [\sigma:p],\varphi\rangle + \langle \varphi\sigma, e - Ew\rangle + \langle \sigma, (u-w)\odot\nabla\varphi\rangle = \langle f, \varphi(u-w)\rangle_\Omega + \langle g, \varphi(u-w)\rangle_{\Gamma_1} \tag{2.3.28}$$

for every $\varphi \in C^1(\overline{\Omega})$. *Moreover*

$$\langle \sigma, p\rangle + \langle \sigma, e - Ew\rangle = \langle f, u-w\rangle_\Omega + \langle g, u-w\rangle_{\Gamma_1}. \tag{2.3.29}$$

Proof. We can assume that $\sigma \in C^\infty(\overline{\Omega})$, otherwise we can proceed by approximation exploiting [13, Lemma 2.3], (2.3.24), and (2.3.7). Let $v := u - w \in BD(\Omega)$ and $\tilde{e} = e - Ew$. By (2.3.22) we have

$$\langle [\sigma:p],\varphi\rangle = \langle \varphi\sigma, p\rangle_\Omega + \langle \varphi\sigma, p\rangle_{\Gamma_0}$$

where the dualities in the right-hand side are the standard dualities between a continuous function and a measure. Now, using (2.3.1), (2.3.9). (2.3.14), and the Green's formula in $BD(\Omega)$ we get

$$\begin{aligned}\langle \varphi\sigma, p\rangle_\Omega &= \langle \varphi\sigma, Ev - \tilde{e}\rangle_\Omega = \langle \sigma, E(\varphi v)\rangle_\Omega - \langle \sigma, v\odot\nabla\varphi\rangle - \langle \sigma, \varphi\tilde{e}\rangle = \\ &= \langle [\sigma\nu], \varphi v\rangle_{\partial\Omega} + \langle f, \varphi v\rangle - \langle \sigma, v\odot\nabla\varphi\rangle - \langle \sigma, \varphi\tilde{e}\rangle \,.\end{aligned} \tag{2.3.30}$$

On the other hand, by (2.3.1) we have

$$\langle \varphi\sigma, p\rangle_{\Gamma_0} = -\langle [\sigma\nu], \varphi v\rangle_{\Gamma_0} \,. \tag{2.3.31}$$

Since $[\sigma\nu] = g$ on Γ_1, summing (2.3.30) and (2.3.31) we obtain (2.3.28). Taking $\varphi = 1$ in Ω, we get (2.3.29). □

The following closed linear subspace of $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ will be used in our proofs:

$$\Sigma_0(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym}) : \mathrm{div}\,\sigma = 0 \text{ in } \Omega,\ [\sigma\nu] = 0 \text{ on } \Gamma_1\}. \tag{2.3.32}$$

By the weak definition of the divergence and by the symmetry of σ, it is easy to see that $\Sigma_0(\Omega) = \{E\varphi : \varphi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^N)\}^\perp$. By taking the orthogonal complements, this implies that

$$\Sigma_0(\Omega)^\perp = \{E\varphi : \varphi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^N)\}\,, \tag{2.3.33}$$

since the latter space is closed in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ as a consequence of Poincaré's and Korn's inequalities. A different proof can be obtained by using the version of De Rham's theorem proved in [35] (see also [52, Chapter 2, Proposition 1.1]) and (1.2.2).

Here and henceforth the closed convex cone $K \subset \mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$ with nonempty interior and the closed convex set $K(\zeta) \subset \mathbb{M}^{N\times N}_{sym}$ parametrised by $\zeta > 0$ satisfy (1.3.1)-(1.3.4).

For $p \in M_b(\overline{\Omega} \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$ and $\zeta \in C^0(\overline{\Omega})^+$ the measure $H(p,\zeta)$ and the functional $\mathcal{H}(p,\zeta)$ are those defined by (1.3.19), and (1.3.20), respectively.

Following the lines of [7, Proposition 3.3] and [48, Proposition 3.2] we want to investigate the connections between the measure $H(p,\zeta)$ and the measure $[\sigma : p]$ when $\sigma \in \Sigma(\Omega)$ satisfies the stress constraint for $\mathcal{L}^n$-a.e. $x \in \Omega$. To do this, we need this preliminary lemma, which is an ad-hoc refinement of [13, Lemma 2.3]. We shall denote with $B_{\mathbb{M}^{N\times N}_{sym}}$ the closed unitary ball of the space $\mathbb{M}^{N\times N}_{sym}$.

Lemma 2.4. *Let U be a bounded open set in $\mathbb{R}^N$ with the segment property, let $\zeta \in C(\bar{U})$, and let $K(\zeta)$ be as in* (1.3.2)*. Let $\sigma \in L^r(U;\mathbb{M}^{N\times N}_{sym})$, $1 \le r < +\infty$, with $\operatorname{div}\sigma \in L^r(U;\mathbb{R}^N)$ and $\sigma(x) \in K(\zeta(x))$ for $\mathcal{L}^n$-a.e. $x \in U$. Then there exists a sequence $\sigma_k \in C^\infty(\overline{U};\mathbb{M}^{N\times N}_{sym})$ such that $\sigma_k \to \sigma$ strongly in $L^r(U;\mathbb{M}^{N\times N}_{sym})$, $\operatorname{div}\sigma_k \to \operatorname{div}\sigma$ strongly in $L^r(U;\mathbb{R}^N)$, and for every $\varepsilon > 0$ there exists k_0, only depending on ε, such that $\sigma_k(x) \in K(\zeta(x)) + \varepsilon B_{\mathbb{M}^{N\times N}_{sym}}$ for every $x \in \overline{U}$.*

Proof. Since U is bounded and has the segment property, there exists a finite open cover (U_i), $i = 1,\dots,m$, of ∂U and a corresponding sequence of nonzero vectors y_i such that, if $x \in \overline{U} \cap U_i$ for some i, then $x + ty_i \in U$ for $0 < t < 1$. We set $U_0 := U$ and $y_0 := 0$. For $i = 0,\dots,m$ and $k = 1,2,\dots$ the open set $U^i_k := \{x \in U_i : x + (1/k)y_i \in U\}$ contains $\overline{U} \cap U_i$. We define $\sigma^i_k(x) := \sigma(x + (1/k)y_i)$ for every $x \in U^i_k$. By the uniform continuity of ζ, it is clear that for every $\varepsilon > 0$, when k is sufficiently large $\sigma^i_k(x) \in K(\zeta(x)) + \frac{\varepsilon}{2} B_{\mathbb{M}^{N\times N}_{sym}}$ for every $i = 0,\dots,m$ and $\mathcal{L}^n$-a.e. $x \in U^i_k$. Let (V_i), $i = 0,\dots,m$, be an open cover of $\overline{U}$ such that $V_i \subset\subset U_i$ for every i. Since $\overline{U} \cap \overline{V}_i \subset U^i_k$, for every i and k we can find a mollifier ψ^i_k of class $C^\infty_c(\mathbb{R}^N)$ such that the convolution $\sigma^i_k \star \psi^i_k$ is well defined in a neighbourhood of $\overline{U} \cap \overline{V}_i$ and

$$\|\sigma^i_k \star \psi^i_k - \sigma^i_k\|_{r,U\cap V_i} \le \frac{1}{k} \qquad \text{and} \qquad \|\operatorname{div}\,\sigma^i_k \star \psi^i_k - \operatorname{div}\,\sigma^i_k\|_{r,U\cap V_i} \le \frac{1}{k}\,. \tag{2.3.34}$$

We can clearly assume that the mollifiers ψ^i_k are supported in a ball of center 0 and radius R_k with $R_k \to 0$ when $k \to +\infty$. By the uniform continuity of ζ, for every $\varepsilon > 0$ there exists $k_0(\varepsilon)$ independent of x such that

$$\sigma^i_k(y) \in K(\zeta(x)) + \varepsilon B_{\mathbb{M}^{N\times N}_{sym}}$$

for every x in the neighbourhood of $\overline{U} \cap \overline{V}_i$ where the convolution $\sigma^i_k \star \psi^i_k$ is well defined, and every $y \in B(x, R_k)$. As $K(\zeta(x)) + \varepsilon B_{\mathbb{M}^{N\times N}_{sym}}$ is closed and convex, for every $k \ge k_0$ we have

$$\sigma^i_k \star \psi^i_k(x) \in K(\zeta(x)) + \varepsilon B_{\mathbb{M}^{N\times N}_{sym}} \tag{2.3.35}$$

for every x in a neighbourhood of $\overline{U} \cap \overline{V}_i$.

Let (φ_i), $i = 0,\dots,m$, be a C^∞ partition of unity for $\overline{U}$ subordinate to (V_i) and let

$$\sigma_k := \sum_{i=0}^m \varphi_i(\sigma^i_k \star \psi^i_k)\,.$$

Then σ_k is of class C^∞ in a neighbourhood of $\overline{U}$. Moreover, by (2.3.35), for every $\varepsilon > 0$ and every $k \ge k_0(\varepsilon)$ we get $\sigma_k(x) \in K(\zeta(x)) + \varepsilon B_{\mathbb{M}^{N\times N}_{sym}}$ for every x in a neighbourhood of $\overline{U}$. Since $\sigma_k^i \to \sigma$ strongly in $L^r(U \cap V_i; \mathbb{M}^{N\times N}_{sym})$ and $\operatorname{div}\sigma_k^i \to \operatorname{div}\sigma$ strongly in $L^q(U \cap V_i; \mathbb{R}^N)$, from (2.3.34) and from the identity

$$\operatorname{div}\sigma := \sum_{i=0}^{m} (\varphi_i \operatorname{div}\sigma + \sigma \nabla\varphi_i)$$

we finally deduce that $\sigma_k \to \sigma$ strongly in $L^r(U; \mathbb{M}^{N\times N}_{sym})$ and $\operatorname{div}\sigma_k \to \operatorname{div}\sigma$ strongly in $L^r(U; \mathbb{R}^N)$. □

We are ready to prove the required inequality.

Proposition 2.5. *Let* $\zeta \in C^0(\bar{\Omega})^+$ *and* $p \in \Pi_{\Gamma_0}(\Omega)$. *Then*

$$H(p,\zeta) \geq [\sigma : p] \quad \text{on } \Omega \cup \Gamma_0 \tag{2.3.36}$$

for every $\sigma \in \Sigma(\Omega) \cap \mathcal{K}(\zeta)$.

Proof. We can assume that $\Gamma_0 = \partial\Omega$: this is not restrictive, because otherwise we can proceed by inner approximation with smooth sets Ω_k whose boundary is contained in $\Omega \cup \Gamma_0$. Let $\varphi \in C(\bar{\Omega})$, $\varphi \geq 0$. We fix $\varepsilon > 0$; considering the sequence σ_k defined as in the previous lemma (we omit to relabel subsequences), for every $k \in \mathbb{N}$, for every $x \in \bar{\Omega}$, we get that there exists $\zeta_{k,x} \in K(\zeta(x))$ such that $|\sigma_k(x) - \zeta_{k,x}| < \varepsilon$, and so, by the Cauchy-Schwarz inequality:

$$\sigma_k(x) : (p/|p|)(x) \leq H(p/|p|(x), \zeta(x)) + \varepsilon;$$

moreover, we can clearly assume that for every $k \in \mathbb{N}$

$$\|\sigma_k\|_\infty \leq 2M_K \|\zeta\|_\infty,$$

where M_K is given by (1.3.4). Then we get, by (2.3.23) and the previous inequalities:

$$\begin{aligned}
\langle [\sigma_k : p] | \varphi \rangle &= \int_{\bar{\Omega}} \varphi(x)\, \sigma_k(x) : \frac{p}{|p|}(x)\, d|p|(x) \leq \\
&\leq \int_{\bar{\Omega}} \varphi(x)\, H(\frac{p}{|p|}(x), \zeta(x))\, d|p|(x) + \varepsilon \int_{\bar{\Omega}} \varphi(x)\, d|p|(x) = \\
&= \langle H(p,\zeta), \varphi \rangle + \varepsilon \int_{\bar{\Omega}} \varphi(x)\, d|p|(x).
\end{aligned}$$

By (2.3.24),

$$[\sigma_k : p] \rightharpoonup [\sigma : p] \quad \text{weakly}^* \text{ in } M_b(\bar{\Omega}; \mathbb{M}^{N\times N}_{sym})$$

when k goes to $+\infty$, therefore we obtain

$$\langle [\sigma : p], \varphi \rangle \leq \langle H(p,\zeta), \varphi \rangle + \varepsilon \int_{\bar{\Omega}} \varphi(x)\, d|p|(x)$$

and we get (2.3.36) in the limit when ε goes to 0. □

The internal variables. In addition to the plastic variable p, there are two internal variables $z \in C^0(\overline{\Omega})$ and $\zeta \in C^0(\overline{\Omega})^+$. They are linked by the equality

$$\zeta := V(z), \tag{2.3.37}$$

where $V\colon \mathbb{R} \to (0,+\infty)$ is a globally Lipschitz nondecreasing function. We assume that there exists a constant $\zeta_m > 0$ such that

$$V(z) \geq \zeta_m \quad \text{for every } z \in \mathbb{R}\,. \tag{2.3.38}$$

The evolution law for the internal variable is nonlocal and involves convolutions. We fix two kernels ρ_1 and ρ_2 in $C^1_c(\mathbb{R}^N)^+$ with the property that

$$\int_{\mathbb{R}^N} \rho_i(x)\,dx = 1 \tag{2.3.39}$$

for $i = 1,2$. For $\mu \in M_b(\Omega \cup \Gamma_0)$ and $i = 1,2$, the convolution $\rho_i \star \mu$ is defined for every $x \in \overline{\Omega}$ by

$$(\rho_i \star \mu)(x) := \int_{\Omega\cup\Gamma_0} \rho_i(x-y)\,d\mu(y)\,. \tag{2.3.40}$$

It is clear that $\rho_i \star \mu \in C^1(\overline{\Omega})$ and that

$$\|\rho_i \star \mu\|_\infty \leq \|\rho_i\|_\infty \|\mu\|_1 \quad \text{and} \quad \|\nabla(\rho_i \star \mu)\|_\infty \leq \|\nabla\rho_i\|_\infty \|\mu\|_1\,, \tag{2.3.41}$$

hence the linear map $\mu \mapsto \rho_i \star \mu$ is continuous from $M_b(\Omega\cup\Gamma_0)$ to $C^1(\overline{\Omega})$.

The data of the problem. We assume that the *body force* $\boldsymbol{f}(t)$, the *surface force* $\boldsymbol{g}(t)$, and the prescribed *boundary displacement* $\boldsymbol{w}(t)$ satisfy the following assumptions:

$$\begin{aligned} &\boldsymbol{f} \in H^1_{loc}([0,+\infty); L^n(\Omega;\mathbb{R}^N))\,,\\ &\boldsymbol{g} \in H^1_{loc}([0,+\infty); L^\infty(\Gamma_1;\mathbb{R}^N))\,,\\ &\boldsymbol{w} \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^N))\,. \end{aligned} \tag{2.3.42}$$

For every $t \in [0,+\infty)$ the *total load* $\boldsymbol{L}(t) \in BD(\Omega)'$ applied at time t is defined by

$$\langle \boldsymbol{L}(t), u\rangle = \langle \boldsymbol{f}(t), u\rangle_\Omega + \langle \boldsymbol{g}(t), u\rangle_{\Gamma_1} \quad \text{for every } u \in BD(\Omega)\,. \tag{2.3.43}$$

Under our assumptions $\boldsymbol{L}$ belongs to $H^1_{loc}([0,+\infty); BD(\Omega)')$ and its time derivative is given by

$$\langle \dot{\boldsymbol{L}}(t), u\rangle = \langle \dot{\boldsymbol{f}}(t), u\rangle_\Omega + \langle \dot{\boldsymbol{g}}(t), u\rangle_{\Gamma_1} \quad \text{for every } u \in BD(\Omega)\,. \tag{2.3.44}$$

Throughout the paper we will assume also the following *uniform safe-load condition*: there exist a function $\boldsymbol{\chi} \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))$ and a constant $r_0 > 0$ such that

$$-\mathrm{div}\,\boldsymbol{\chi}(t) = \boldsymbol{f}(t) \text{ in } \Omega \text{ and } [\boldsymbol{\chi}(t)\nu] = \boldsymbol{g}(t) \text{ on } \Gamma_1 \text{ for every } t \in [0,+\infty)\,, \tag{2.3.45}$$

$$B(\chi(t,x), r_0) \subset K(\zeta_m) \quad \text{for every } t \in [0,+\infty) \text{ and } \mathcal{L}^n\text{-a.e. } x \in \Omega\,, \tag{2.3.46}$$

$$\dot{\boldsymbol{\chi}}(t) \in L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym}) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,+\infty)\,, \tag{2.3.47}$$

$$t \mapsto \|\dot{\boldsymbol{\chi}}(t)\|_\infty \quad \text{belongs to } L^1_{loc}([0,+\infty))\,, \tag{2.3.48}$$

where $\chi(t,x)$ denotes the value of $\boldsymbol{\chi}(t)$ at $x \in \Omega$, and $B(\sigma, r)$ denotes the open ball in $\mathbb{M}^{N\times N}_{sym}$ with centre σ and radius r. It is easy to see that the function $t \mapsto \|\dot{\boldsymbol{\chi}}(t)\|_\infty$ is in general lower semicontinuous, therefore assumption (2.3.48) only involves the finiteness of the integral. By (1.3.5) inclusion (2.3.46) implies

$$H(\xi,\zeta) \geq \chi(t,x) : \xi + r_0|\xi| \tag{2.3.49}$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$ and every $(\xi, \zeta) \in \mathbb{M}^{N\times N}_{sym} \times [\zeta_m, +\infty)$.

We will make use of the following infinite-dimensional generalization of (2.3.49), which improves (2.3.36) for a function $\chi \in L^\infty(\Omega; \mathbb{M}^{N\times N}_{sym})$ with $\operatorname{div}\chi \in L^n(\Omega; \mathbb{R}^N)$ satisfying in addition (2.3.46).

Proposition 2.6. *Define* $\Sigma(\Omega)$ *as in* (2.3.8). *Let* $r_0 > 0$, $\zeta \in C^0(\bar{\Omega})^+$ *and* $p \in \Pi_{\Gamma_0}(\Omega)$. *Let* $\chi \in \Sigma(\Omega)$ *be a function such that*

$$B(\chi(x), r_0) \subset K(\zeta(x)) \tag{2.3.50}$$

for $\mathcal{L}^n$*-a.e.* $x \in \Omega$. *Then*

$$\mathcal{H}(p, \zeta) - \langle \chi, p \rangle \geq r_0 \|p\|_1 , \tag{2.3.51}$$

where the duality $\langle \chi, p \rangle$ *is defined by* (2.3.12).

Proof. By standard arguments in measure theory, given $\varepsilon > 0$ we can find $\tau \in C^0_0(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym}) \cap C^\infty(\overline{\Omega})$, with $\|\tau\|_\infty \leq r_0$, such that

$$r_0 \|p\|_1 + \varepsilon \leq \int_{\Omega\cup\Gamma_0} \tau : dp \,. \tag{2.3.52}$$

By (2.3.50), $\tau + \chi \in \mathcal{K}(\zeta) \cap \Sigma(\Omega)$. Therefore, by (2.3.22), (2.3.36) and (2.3.52), we get

$$r_0 \|p\|_1 + \varepsilon \leq \int_{\Omega\cup\Gamma_0} \tau : dp = \langle \tau + \chi, p \rangle - \langle \chi, p \rangle \leq \mathcal{H}(p, \zeta) - \langle \chi, p \rangle \,,$$

which concludes the proof by the arbitrariness of ε. $\square$

About the initial data, we assume that

$$u_0 \in BD(\Omega), \quad e_0 \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym}), \quad p_0 \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym}) \quad z_0 \in C^0(\overline{\Omega}) \tag{2.3.53}$$

and we define

$$\sigma_0 := \mathbb{C} e_0 \quad \text{and} \quad \zeta_0 := V(z_0). \tag{2.3.54}$$

Moreover we suppose that the following compatibility conditions are satisfied:

Weak kinematic admissibility:

$$\begin{aligned} Eu_0 &= e_0 + p_0 \quad \text{in } \Omega \,, \\ p_0 &= (\boldsymbol{w}(0) - u_0) \odot \nu \mathcal{H}^{n-1} \quad \text{in } \Gamma_0 \,; \end{aligned} \tag{2.3.55}$$

Equilibrium condition:

$$-\operatorname{div} \sigma_0 = \boldsymbol{f}(0) \quad \text{in } \Omega \,; \qquad [\sigma_0 \nu] = \boldsymbol{g}(0) \quad \text{on } \Gamma_1. \tag{2.3.56}$$

Stress constraint:

$$\sigma_0 \in \mathcal{K}(\zeta_0). \tag{2.3.57}$$

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2.4 The vanishing viscosity approach

To deal with the instabilities of the softening regime, we introduce a viscoplastic approximation of Perzyna-type (see [37, 17, 25, 36]) of our problem. Given a viscosity parameter $\varepsilon > 0$, the corresponding viscoplastic evolution $u_\varepsilon(t,x)$, $e_\varepsilon(t,x)$, $p_\varepsilon(t,x)$, $z_\varepsilon(t,x)$, $\sigma_\varepsilon(t,x)$, $\zeta_\varepsilon(t,x)$ satisfies conditions (a), (b), (c), and (f) of Section 2.2; condition (d) is dropped, while (e) is replaced by

(e$_\varepsilon$) regularized flow rule: $\dot{p}_\varepsilon(t,x) = N^\varepsilon_{K(\zeta_\varepsilon(t,x))}(\sigma_\varepsilon(t,x))$,

where $N^\varepsilon_K(\sigma,\zeta) := \frac{1}{\varepsilon}\big(\sigma - \pi_{K(\zeta)}(\sigma)\big)$ and $\pi_{K(\zeta)}$ is the projection onto $K(\zeta)$. The well-posedness of these equations is nontrivial and will be investigated in Chapter 5, Section 5.2. The underlying idea is that, since the functional resulting from the variational formulation of our problem can have multiple wells, a quasistatic evolution driven by global minimizers could prescribe abrupt jumps from one well to another one, so that is preferable to follow a path composed of local minimizers. Among them, a good selection criterion has proved to be choosing the ones that are obtained as a limit of viscoplastic evolutions when the regularizing parameter ε tends to 0 (see [32] for a general discussion).

To prepare our treatment of the viscoplastic approximation, for every $\varepsilon > 0$ we introduce the function $H_\varepsilon\colon \mathbb{M}^{N\times N}_{sym}\times[0,+\infty) \to \mathbb{R}$ defined as

$$H_\varepsilon(\xi,\zeta) = H(\xi,\zeta) + \tfrac{\varepsilon}{2}|\xi|^2, \tag{2.4.1}$$

and the corresponding integral functional $\mathcal{H}_\varepsilon\colon L^2(\Omega;\mathbb{M}^{N\times N}_{sym})\times C^0(\overline{\Omega})^+ \to \mathbb{R}$ defined by

$$\mathcal{H}_\varepsilon(p,\zeta) := \int_\Omega H_\varepsilon(p(x),\zeta(x))\,dx\,.$$

Its subdifferential $\partial_p\mathcal{H}_\varepsilon$ with respect to p satisfies the equality

$$\partial_p\mathcal{H}_\varepsilon(p,\zeta) = \partial_p\mathcal{H}(p,\zeta) + \varepsilon p \tag{2.4.2}$$

for every $(p,\zeta) \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})\times C^0(\overline{\Omega})^+$.

The convex conjugate $H^*_\varepsilon\colon \mathbb{M}^{N\times N}_{sym}\times[0,+\infty) \to \mathbb{R}$ of H_ε with respect to ξ is defined by

$$H^*_\varepsilon(\sigma,\zeta) := \sup_{\xi\in\mathbb{M}^{N\times N}_{sym}} \{\sigma:\xi - H_\varepsilon(\xi,\zeta)\}\,.$$

Since the convex conjugate H^* of H with respect to ξ satisfies $H^*(\sigma,\zeta) = 0$ for $\sigma \in K(\zeta)$ and $H^*(\sigma,\zeta) = +\infty$ for $\sigma \notin K$ (see [40, Theorem 13.2]), using [40, Theorem 16.4] one can prove that

$$H^*_\varepsilon(\sigma,\zeta) = \tfrac{1}{2\varepsilon}|\sigma - \pi_{K(\zeta)}(\sigma)|^2\,. \tag{2.4.3}$$

This implies that H^*_ε is differentiable with respect to σ, and that its gradient is given by

$$\partial_\sigma H^*_\varepsilon(\sigma,\zeta) = N^\varepsilon_K(\sigma,\zeta) := \tfrac{1}{\varepsilon}\big(\sigma - \pi_{K(\zeta)}(\sigma)\big)\,. \tag{2.4.4}$$

Note that $N^\varepsilon_K(\sigma,\zeta)$ is Lipschitz continuous on $\mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$ by Lemma 1.1

Let $\mathcal{H}^*_\varepsilon\colon L^2(\Omega;\mathbb{M}^{N\times N}_{sym})\times C^0(\overline{\Omega})^+ \to \mathbb{R}$ be the convex conjugate of $\mathcal{H}_\varepsilon$ with respect to p, and let $\mathcal{N}^\varepsilon_{\mathcal{K}}\colon L^2(\Omega;\mathbb{M}^{N\times N}_{sym})\times C^0(\overline{\Omega})^+ \to L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ be defined by

$$\mathcal{N}^\varepsilon_{\mathcal{K}}(\sigma,\zeta) := \tfrac{1}{\varepsilon}\big(\sigma - \pi_{\mathcal{K}(\zeta)}(\sigma)\big)\,. \tag{2.4.5}$$

It follows from (1.3.18) that

$$p = \mathcal{N}^{\varepsilon}_{\mathcal{K}}(\sigma,\zeta) \quad \Longleftrightarrow \quad p(x) = N^{\varepsilon}_{K}(\sigma(x),\zeta(x)) \text{ for } \mathcal{L}^n\text{-a.e. } x \in \Omega\,, \tag{2.4.6}$$

so that $\mathcal{N}^{\varepsilon}_{\mathcal{K}}\colon L^2(\Omega;\mathbb{M}^{N\times N}_{sym})\times C^0(\overline{\Omega})^+ \to L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ is Lipschitz continuous. By a general property of integral functionals (see, e.g., [19, Proposition IX.2.1]) we have

$$\mathcal{H}^*_\varepsilon(\sigma,\zeta) = \int_\Omega H^*_\varepsilon(\sigma(x),\zeta(x))\,dx\,,$$

so that, by the Dominated Convergence Theorem and by (2.4.6), its gradient $\partial_\sigma\mathcal{H}^*_\varepsilon(\sigma,\zeta)$ with respect to σ satisfies

$$\partial_\sigma\mathcal{H}^*_\varepsilon(\sigma,\zeta) = \mathcal{N}^{\varepsilon}_{\mathcal{K}}(\sigma,\zeta)\,. \tag{2.4.7}$$

Chapter 3

The spatially homogeneous case

3.1 Overview of the chapter

Our investigation of the Cam-Clay model starts by studying the spatially homogeneous case in dimension N, with no volume forces. This simplified setting is the object of this chapter, where we do not investigate the well-posedness of the problem, which is instead carried out in Chapter 5. A similar study was done in [12] for a particular loading program and for a very special yield surface. Here we extend the results of that paper to a very general class of loading paths and yield surfaces, subject only to minor restrictions.

To be definite, we assume that the system is driven by a time-dependent affine boundary condition $w(t,x)$, whose symmetrized spatial gradient $Ew(t,x)$ is independent of the space variable x and is denoted by $\xi(t)$. In this situation, one can look for spatially homogeneous solutions, assuming that the displacement $u(t,x)$ coincides with $w(t,x)$ and the unknowns, independent of x, are the elastic part $e(t)$ and the plastic part $p(t)$ appearing in the additive decomposition of the strain $Eu(t,x) = e(t) + p(t)$, as well as the scalar internal variable $z(t)$, which describes the time evolving yield surface.

In this particular case the evolution laws for $p(t)$ and $z(t)$ result in the system

$$\begin{cases} e(t) + p(t) = \xi(t)\,, \qquad \sigma(t) = \mathbb{C}e(t) \in K(z(t))\,, \\ \dot{p}(t) \in N_{K(z(t))}(\sigma(t))\,, \\ \dot{z}(t) = \operatorname{tr}(\sigma(t))\operatorname{tr}(\dot{p}(t))\,. \end{cases} \tag{3.1.1}$$

Notice that, differently from the formulation of the problem presented in Chapter 2, there is no need of introducing a dual internal variable ζ, since we are able to prove that $z(t)$ is bounded away from zero at finite times (this follows from (3.3.4) and (3.4.9)). Throughout this chapter, we shall assume that $\operatorname{tr}(\sigma) \leq 0$ for every $\sigma \in K(z)$, which reflects the compressive conditions typical of soil mechanics. Therefore, by the second equation in (3.1.1), the hardening or softening behaviour is determined only by the sign of $\operatorname{tr}(\dot{p})$. We also premit, that due to mathematical reasons, we shall impose some additional restrictions on $K(z)$ (see (3.2.10)-(3.2.11)). The main result of the chapter in its full generality needs these assumptions, but most partial results can be proved without them. This is why in the statements we will avoid these additional restrictions whenever it is possible.

With the notation of the previous chapter, the vanishing viscosity approximation reads in this case as

$$\begin{cases} e_\varepsilon(t) + p_\varepsilon(t) = \xi(t)\,, \qquad \sigma_\varepsilon(t) = \mathbb{C}e_\varepsilon(t)\,, \\ \dot p_\varepsilon(t) = N^\varepsilon_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t))\,, \\ \dot z_\varepsilon(t) = \mathrm{tr}(\sigma_\varepsilon(t))\,\mathrm{tr}(\dot p_\varepsilon(t))\,. \end{cases} \tag{3.1.2}$$

A viscosity solution $(e(t), p(t), \sigma(t), z(t))$ to (3.1.1) is defined as a left continuous map which, for almost every time t, is the pointwise limit of a sequence $(e_\varepsilon(t), p_\varepsilon(t), \sigma_\varepsilon(t), z_\varepsilon(t))$ of solutions of (3.1.2).

We want to study in detail the limit behavior as ε goes to 0 of the solutions of (3.1.2). This is done using only differential equations techniques and disregarding the variational structure of (part of) the problem. We will see that the limit dynamics presents, for a generic choice of the initial data – some degenerate cases have indeed to be excluded – the alternation of three possible regimes:

a) **Elastic regime**. This situation occurs in a time interval $[t_1, t_2]$ when the plastic part, and thus the internal variable, do not evolve, while the stress is completely determined by the prescribed boundary displacement through the relation $\sigma(t) = \mathbb{C}(\xi(t) - \xi(t_1))$, for every $t \in [t_1, t_2]$; a necessary condition for this behavior to occur is clearly $(\mathbb{C}(\xi(t) - \xi(t_1)), z(t_1)) \in K$ for every $t \in [t_1, t_2]$.

b) **Slow dynamics**. In this situation the stress evolves smoothly on the yield surface and the limit equation (3.3.1), called the equation of the slow dynamics, takes into account the production of plastic flow. The evolution can be studied using the standard time t; during this regime both hardening and softening behavior can occur.

c) **Fast dynamics**. In the softening regime, a singular behavior can occur, which requires the use of a *fast time* $s := \frac{1}{\varepsilon}t$. The corresponding limit equation (3.4.1) is called the equation of the fast dynamics. We will see that, at a jump time t, the right limit $(\sigma(t+), z(t+))$ of the solution is given by the asymptotic value for $s \to +\infty$ of the heteroclinic solution of the equation of the fast dynamics (3.4.1) issuing from the point $(\sigma(t-), z(t-))$ at $s = -\infty$.

As in the associative case, studied in [47] and in [8, Section 7], the alternation of these three regimes is determined by the sign of two scalar indicators; the first one, depending explicitly on time and on the state of the system, will be called the *elastic-inelastic indicator*. It is given by

$$\Phi(t, \sigma, z) := \nu_{K(z)}(\sigma) \cdot \mathbb{C}\dot\xi(t) \tag{3.1.3}$$

for every $(t, \sigma, z) \in [0, +\infty] \times \partial K$. Here $\nu_{K(z)}(\sigma)$ denotes the outward unit normal to $K(z)$ at σ. The second one, only depending on the state of the system, will be called the *slow-fast indicator*; its explicit expression is given by

$$\Psi(\sigma, z) := -\nu_{K(z)}(\sigma) \cdot \mathbb{C}\nu_{K(z)}(\sigma) - \tfrac{\mathrm{tr}(\sigma)\ \mathrm{tr}(\nu_{K(z)}(\sigma))}{z}[\sigma \cdot \nu_{K(z)}(\sigma)] \tag{3.1.4}$$

for every $(\sigma, z) \in \partial K$. Roughly speaking, at times where the stress meets the yield surface, positiveness of the indicator Φ does not allow the system to evolve according to the linearized elasticity equation without breaking the stress constraint, thus plastic flow has to be produced. The choice between slow and fast dynamics depends on the sign of Ψ: if it is negative, the evolution is smooth, while if it is positive, the solution has a jump. Since the quadratic form associated to the tensor $\mathbb{C}$ is positive definite and taking into account (3.2.8), comparing (3.1.4) with (3.1.1) one sees that a necessary condition for Ψ to be positive, hence to have a jump, is that the internal variable is decreasing in a neighborhood of the jump time. From a mechanical point of view, this means that the instabilities leading to a discontinuous evolution of the system are typical of the softening regime while the hardening regime is more regular.

The main result of the chapter is Theorem 3.31. It gives an iterative procedure to construct explicitly a viscous solution, upon the verification of some nondegeneracy hypotheses at each step. If these hypotheses are satisfied, the viscous solution is also unique.

3.2 Preliminary results

We consider a closed convex cone $K \subset \mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ with nonempty interior and a family of closed convex set $K(z) \subset \mathbb{M}^{N\times N}_{sym}$, parametrised by $z > 0$ (throughout the chapter the internal variables z and ζ are indeed identified), satifying (1.3.1)-(1.3.4). We observe that, for every $z > 0$, we obviously have

$$\sigma \in \partial K(z) \Longleftrightarrow (\sigma, z) \in \partial K. \tag{3.2.1}$$

We will assume that $K(1)$ is of class C^2. For every $\sigma \in \partial K(z)$, we will denote the outward unit normal to $K(z)$ at σ by $\nu_{K(z)}(\sigma)$, while $\nu_K(\sigma, z)$ will denote the outward unit normal to K at (σ, z). We shall also assume that

$$\mathrm{tr}(\sigma) \leq 0 \text{ for every } \sigma \in K(1); \tag{3.2.2}$$

this reflects the compressive conditions typical of soil mechanics. We define, for every $(\sigma, z) \in \mathbb{M}^{N\times N}_{sym} \times (0, +\infty)$, the function

$$\varrho(\sigma, z) = |\sigma - \pi_{K(z)}(\sigma)|\,; \tag{3.2.3}$$

it is a Lipschitz function, moreover it is C^1 for every $(\sigma, z) \in [\mathbb{M}^{N\times N}_{sym} \times (0, +\infty)] \setminus K$. As an elementary consequence of (1.3.6), we have the following relation:

$$\varrho(\sigma, z) = z\, \varrho(\tfrac{\sigma}{z}, 1) \qquad \text{for every } (\sigma, z) \in \mathbb{M}^{N\times N}_{sym} \times (0, +\infty). \tag{3.2.4}$$

The next proposition collects some elementary properties which will be useful in what follows.

Proposition 3.1. *Let K be a closed convex cone in $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$, and let $K(z)$ be as in* (1.3.1)*. Assume that $K(1)$ is bounded and of class C^2 and that $0 \in \partial K(1)$. Then, for every $z > 0$ and every $\sigma \in \mathbb{M}^{N\times N}_{sym} \setminus \mathrm{int}\, K(z)$, we have*

$$\nu_{K(z)}(\pi_{K(z)}(\sigma)) = \nu_{K(1)}(\pi_{K(1)}(\tfrac{1}{z}\sigma)). \tag{3.2.5}$$

Moreover, for every $(\sigma,z)\in\partial K$

$$\nu_K(\sigma,z)=\frac{1}{\sqrt{z^2+|\sigma\cdot\nu_{K(z)}(\sigma))|^2}}(z\,\nu_{K(z)}(\sigma),-\sigma\cdot\nu_{K(z)}(\sigma)). \tag{3.2.6}$$

For every $(\sigma,z)\in[\mathbb{M}^{N\times N}_{sym}\times(0,+\infty)]\setminus K$*, we have*

$$\nabla\varrho(\sigma,z)=\tfrac{1}{z}(z\,\nu_{K(z)}(\pi_{K(z)}(\sigma)),\,-\pi_{K(z)}(\sigma)\cdot\nu_{K(z)}(\pi_{K(z)}(\sigma))). \tag{3.2.7}$$

Proof. To prove (3.2.5) it suffices to consider the case when $\sigma\notin K(z)$, which is equivalent to say that $\frac{\sigma}{z}\notin K(1)$. We then have, applying (1.3.6) and (3.2.4), that

$$\begin{aligned}\nu_{K(z)}(\sigma)&=\frac{\sigma-\pi_{K(z)}(\sigma)}{\varrho(\sigma,z)}\\&=\frac{z(\frac{\sigma}{z}-\pi_{K(1)}(\frac{\sigma}{z}))}{z\,\varrho(\frac{\sigma}{z},1)}=\nu_{K(1)}(\pi_{K(1)}(\tfrac{1}{z}\sigma)),\end{aligned}$$

which proves (3.2.5).

For what concerns (3.2.7), it is well known that, for every $(\sigma,z)\in[\mathbb{M}^{N\times N}_{sym}\times(0,+\infty)]\setminus K$, $\nabla_\sigma\,\varrho(\sigma,z)=\nu_{K(z)}(\pi_{K(z)}(\sigma))$ so only the last component of the gradient has to be calculated. Together with (3.2.4) this implies that

$$\tfrac{\partial}{\partial z}\varrho(\sigma,z)=\tfrac{\partial}{\partial z}[z\,\varrho(\tfrac{\sigma}{z},1)]=\tfrac{1}{z}(\varrho(\sigma,z)-\sigma\cdot\nu_{K(1)}(\pi_{K(1)}(\tfrac{\sigma}{z}))),$$

hence we get (3.2.7) by (3.2.5) and the equality

$$\varrho(\sigma,z)-\sigma\cdot\nu_{K(z)}(\pi_{K(z)}(\sigma))=-\pi_{K(z)}(\sigma)\cdot\nu_{K(z)}(\pi_{K(z)}(\sigma)).$$

This also implies (3.2.6); indeed, by the C^2 regularity of the boundary, for every fixed $(\bar\sigma,\bar z)\in\partial K$ we may locally define an oriented distance function r from ∂K, which is a C^1-extension of ϱ to the interior of K. Then, locally we have that $K=\{(\sigma,z)|\,r(\sigma,z)\le 0\}$. It follows that the outward unit normal to K at $(\bar\sigma,\bar z)$ must be parallel to $\nabla r(\bar\sigma,\bar z)$, which by continuity is obtained by extending the right-hand side of (3.2.7) to ∂K, and this proves (3.2.6). □

Another useful property, which will be used in what follows, comes directly from the characterization of the minimal distance projection and from the fact that $0\in K(z)$ for every z; we have indeed that, for every $(\sigma,z)\in[\mathbb{M}^{N\times N}_{sym}\times(0,+\infty)]\setminus K$

$$\pi_{K(z)}(\sigma)\cdot\nu_{K(z)}(\pi_{K(z)}(\sigma)))\ge 0. \tag{3.2.8}$$

We shall often decompose $\sigma\in\mathbb{M}^{N\times N}_{sym}$ in its spherical and deviatoric part through the relation

$$\sigma=x\tfrac{I}{\sqrt{N}}+y \tag{3.2.9}$$

where $x\in\mathbb{R}$ and $y\in\mathbb{M}^{N\times N}_{D}$ are uniquely determined; here as usual $\mathbb{M}^{N\times N}_{D}$ denotes the space of trace-free symmetric matrices of order N. Notice that $\sqrt{N}x=\mathrm{tr}(\sigma)$; in particular, for every $\sigma\in K(1)$, we shall have $x\le 0$. Similarly, $\eta(t)$ and $\gamma(t)$ will denote the spherical and the deviatoric part, respectively, of the function $\xi(t)$ mentioned in the introduction.

For mathematical reasons, we shall make some additional hypotheses on the set $K(1)$, even if most of the results we are going to prove do not need them. Precisely, we shall suppose that there exist a constant $a > 0$ and two not identically zero functions g and h, defined on a bounded convex domain D of class C^2, satifying $g = h = 0$ on ∂D and $g, h \in C^2(D) \cap C(\bar{D})$ such that, decomposing $\sigma \in \mathbb{M}^{N\times N}_{sym}$ as in (3.2.9), we have

$$K(1) = \{\sigma \in \mathbb{M}^{N\times N}_{sym} | g(y) \leq x + a \leq h(y)\} \tag{3.2.10}$$

We shall also suppose that

$$g^2, h^2 \text{ are concave }. \tag{3.2.11}$$

In terms of g and h, we can reformulate our basic assumptions on $K(1)$ as follows. Convexity of the domain $K(1)$ is easily equivalent to the fact that g is convex and h is concave; as they do not identically vanish on D and they are zero on the boundary, this implies that

$$g(y) < 0 \quad \text{and} \quad h(y) > 0 \text{ for every } y \in D\,.$$

Regularity of $\partial K(1)$ implies, that, for every $\omega \in \partial D$

$$\lim_{y\to\omega,\, y\in D} |\nabla g(y)| = \lim_{y\to\omega,\, y\in D} |\nabla h(y)| = +\infty. \tag{3.2.12}$$

Moreover, both (1.3.3) and (3.2.2) are satisfied, provided we have

$$\max_{x\in D} h = h(0) = a. \tag{3.2.13}$$

An example of set satisfying all these assumptions is, for instance, any ellipsoid of the form

$$(\tfrac{x}{a} + 1)^2 + \sum_{i=1}^{m} \tfrac{y_i^2}{b_i^2} = 1,$$

where $m = \frac{N(N+1)}{2} - 1$ and y_i are the components of y with respect to an orthonormal basis of $\mathbb{M}^{N\times N}_D$. We then have the following Proposition. We omit the simple proof, which can be found in [14, Proposition 2.3 and Remark 2.4]

Proposition 3.2. *Assume that* (1.3.1)-(1.3.4), (3.2.2), *and* (3.2.10)-(3.2.11) *are satisfied. Then, there exists a constant $F > 0$ such that, for every $\sigma \in \partial K(1)$*

$$|\mathrm{tr}(\nu_{K(1)}(\sigma))| \leq F|x + a|, \tag{3.2.14}$$

where x is defined as in (3.2.9). *Moreover*

$$\mathrm{tr}(\nu_{K(1)}(\sigma)) = 0 \Longleftrightarrow x = -a\,, \tag{3.2.15}$$

and

$$\mathrm{tr}(\nu_{K(1)}(\sigma)) > 0 \Longleftrightarrow x + a > 0. \tag{3.2.16}$$

Let us fix $\xi \in C^1([0, +\infty); \mathbb{M}^{N\times N}_{sym})$. For every $\varepsilon > 0$ system (3.1.2) is equivalent to

$$\begin{cases} \varepsilon \dot{e}_\varepsilon(t) = \varepsilon\dot{\xi}(t) - \mathbb{C}e_\varepsilon(t) + \pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t))\,, \\ \varepsilon \dot{z}_\varepsilon(t) = \mathrm{tr}(\mathbb{C}e_\varepsilon(t))\, \mathrm{tr}(\mathbb{C}e_\varepsilon(t) - \pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t)))\,. \end{cases} \tag{3.2.17}$$

Lemma 3.3. *For every $\varepsilon > 0$ and for every initial condition $e_\varepsilon(0) = e_0$ and $z_\varepsilon(0) = z_0 > 0$ system* (3.2.17) *has a unique solution defined for every $t \in [0, +\infty)$. Moreover the solution $(e_\varepsilon, z_\varepsilon)$ of* (3.2.17) *with initial condition $e_\varepsilon(0) = e_0$ and $z_\varepsilon(0) = z_0 > 0$ satisfies $z_\varepsilon(t) > 0$ for every $t \in [0, +\infty)$.*

Proof. As the right-hand sides are locally Lipschitz with respect to e and z by Lemma 1.1, to get global existence it is enough to prove that for every $T > 0$ there is a constant $M_{T,\varepsilon} > 0$ such that $|e_\varepsilon(t)| \le M_{T,\varepsilon}$ and $|z_\varepsilon(t)| \le M_{T,\varepsilon}$ for every $t \in [0, T]$. Since $0 \in K(\zeta)$ for every $\zeta \in R$ by (1.3.3), by (2.3.4) we have $|\mathbb{C}e_\varepsilon(t) - \pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t))| \le |\mathbb{C}e_\varepsilon(t)| \le 2\beta_Q |e_\varepsilon(t)|$ and $|\pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t))| \le |\mathbb{C}e_\varepsilon(t)| \le 2\beta_Q |e_\varepsilon(t)|$ for every $t \in [0, +\infty)$. Therefore, given $T > 0$, from the first equation in (3.2.17) we have

$$|e_\varepsilon(t)| \le A_T + \frac{2\beta_Q}{\varepsilon} \int_0^t |e_\varepsilon(s)|\, ds \qquad \text{for every } t \in [0, T].$$

with $A_T := |e_0| + \int_0^T |\dot{\xi}(s)|\, ds$. It follows from the Gronwall inequality that

$$|e_\varepsilon(t)| \le A_T \exp(T\, \beta_Q / \varepsilon) \qquad \text{for every } t \in [0, T].$$

Then the second equation in (3.2.17) allows easily to obtain a constant $M_{T,\varepsilon} > 0$ such that $|z_\varepsilon(t)| \le M_{T,\varepsilon}$ for every $t \in [0, T]$.

To prove the second part of the statement, we argue by contradiction. Let T be the first time such that $z_\varepsilon(T) = 0$ and suppose by contradiction that $T < +\infty$. Fix $\hat{t} < T$ such that $T - \hat{t} < \frac{\varepsilon}{2M_{T,\varepsilon} M_K}$, where M_K is given by (1.3.4) and let $t_0 < T$ be a maximum point for $z_\varepsilon(t)$ in $[\hat{t}, T]$. We shall have, by (3.2.17) and (1.3.4)

$$\begin{aligned}
0 = \varepsilon z_\varepsilon(t_0) + \varepsilon \int_{t_0}^T \dot{z}_\varepsilon(s)\, ds = \\
= \varepsilon z_\varepsilon(t_0) + \int_{t_0}^T [\operatorname{tr}(\mathbb{C}e_\varepsilon(s))^2 - \operatorname{tr}(\mathbb{C}e_\varepsilon(s)) \operatorname{tr}(\pi_{K(z_\varepsilon(s))}(\mathbb{C}e_\varepsilon(s)))]\, ds \ge \\
\ge \varepsilon z_\varepsilon(t_0) - \int_{t_0}^T |\operatorname{tr}(\mathbb{C}e_\varepsilon(s))|\, |\operatorname{tr}(\pi_{K(z_\varepsilon(s))}(\mathbb{C}e_\varepsilon(s)))|\, ds \ge \\
\ge \varepsilon z_\varepsilon(t_0) - M_{T,\varepsilon} M_K \int_{t_0}^T z_\varepsilon(s)\, ds \ge \\
\ge z_\varepsilon(t_0)[\varepsilon - (T - t_0) M_{T,\varepsilon} M_K] \ge \tfrac{\varepsilon}{2} z_\varepsilon(t_0),
\end{aligned}$$

a contradiction. $\square$

Introducing the dual variable σ, the system becomes

$$\begin{cases} \varepsilon \dot{\sigma}_\varepsilon(t) = \varepsilon \mathbb{C}\dot{\xi}(t) + \mathbb{C}[\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)) - \sigma_\varepsilon(t)], \\ \varepsilon \dot{z}_\varepsilon(t) = \operatorname{tr}(\sigma_\varepsilon(t)) \operatorname{tr}(\sigma_\varepsilon(t) - \pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t))). \end{cases} \tag{3.2.18}$$

Since we want to consider a system which is initially in the elastic regime, for every $\varepsilon > 0$ we will consider an initial condition satisfying $(\sigma_0, z_0) \in \operatorname{int} K$; in particular, we shall have $z_0 > 0$. For every ε the solution of (3.2.18) is trivially given by

$$(\sigma(t), z(t)) = (\sigma_0 + \mathbb{C}(\xi(t) - \xi(0)), z_0) \tag{3.2.19}$$

for t small; actually, this formula gives the solution in the time interval $[0, t_1]$, where

$$t_1 = \inf\{t > 0 : \ (\sigma_0 + \mathbb{C}(\xi(t) - \xi(0)), z_0) \in \partial K\}. \tag{3.2.20}$$

In terms of the function ϱ defined by (3.2.3), for every t such that $\varrho(\sigma_\varepsilon(t), z_\varepsilon(t)) > 0$, equations (3.2.18) become

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}_\varepsilon(t) = \frac{1}{\varepsilon}\varrho(\sigma_\varepsilon(t), z_\varepsilon(t))\, \mathbb{C}\, \nu_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t), z_\varepsilon(t)), \\ \dot{z}_\varepsilon(t) = \frac{1}{\varepsilon}\varrho(\sigma_\varepsilon(t), z_\varepsilon(t))\ \mathrm{tr}(\sigma_\varepsilon(t))\ \mathrm{tr}(\nu_{K(z_\varepsilon(t))}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t))). \end{cases} \tag{3.2.21}$$

Given the solution of (3.2.18) with the prescribed initial data we define

$$\varrho_\varepsilon(t) := \varrho(\sigma_\varepsilon(t), z_\varepsilon(t)); \tag{3.2.22}$$

notice that $\varrho_\varepsilon(t)$ is Lipschitz continuous, thus differentiable, for almost every t; in particular it is differentiable for every t such that $\varrho_\varepsilon(t) > 0$, and we have, by a direct computation, taking into account (3.2.21) and (3.2.7), that

$$\frac{d}{dt}\varrho_\varepsilon(t) = \Phi(t, \sigma_\varepsilon(t), z_\varepsilon(t)) + \frac{\varrho_\varepsilon(t)}{\varepsilon}\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \quad \text{whenever } \varrho_\varepsilon(t) > 0, \tag{3.2.23}$$

where

$$\Phi(t, \sigma, z) : \ = \nu_{K(z)}(\pi_{K(z)}(\sigma)) \cdot \mathbb{C}\dot{\xi}(t), \tag{3.2.24}$$

$$\begin{aligned} \Psi(\sigma, z) : \ &= -\nu_{K(z)}(\pi_{K(z)}(\sigma)) \cdot \mathbb{C}\nu_{K(z)}(\pi_{K(z)}(\sigma)) - \\ &- \frac{\mathrm{tr}(\sigma)\ \mathrm{tr}(\nu_{K(z)}(\pi_{K(z)}(\sigma)))}{z}[\pi_{K(z)}(\sigma) \cdot \nu_{K(z)}(\pi_{K(z)}(\sigma))]. \end{aligned} \tag{3.2.25}$$

The function Φ is defined on $[0, +\infty) \times \{[\mathbb{M}^{N\times N}_{sym} \times (0, +\infty)] \setminus \mathrm{int}\, K\}$ and is continuous, while Ψ is defined on $[\mathbb{M}^{N\times N}_{sym} \times (0, +\infty)] \setminus \mathrm{int}\, K$ and is of class C^1. In what follows, it is often convenient to consider extensions of Φ and Ψ to $[0, +\infty) \times \mathbb{M}^{N\times N}_{sym} \times (0, +\infty)$ and $\mathbb{M}^{N\times N}_{sym} \times (0, +\infty)$ of class C^0 and C^1, respectively. Notice that the partial derivatives of Ψ at each point of ∂K do not depend on the extension.

We will sometimes refer to Φ as to the *elastic-inelastic indicator*, while Ψ will be called *slow-fast indicator*, for reasons that will become clear in the following. Even if, for mathematical reasons, the two indicators are defined on the whole space, we will also see that what only matters are the values they attain on the yield surface.

Remark 3.4. By positive definiteness of $\mathbb{C}$ and by (3.2.8) it is immediate to deduce that, for every (σ, z) such that $\mathrm{tr}(\sigma)\ \mathrm{tr}(\nu_{K(z)}(\pi_{K(z)}(\sigma))) \geq 0$, the indicator Ψ is strictly negative; as we are going to see in what follows, this reflects the fact that, as long as we are in the hardening regime, the evolution does not present discontinuities.

In general, it is easy to verify, taking into account (2.3.3) and (1.3.4), that the following bounds on Ψ hold: from above, we have, for every $(\sigma, z) \in [\mathbb{M}^{N\times N}_{sym} \times (0, +\infty)] \setminus \mathrm{int}\, K$,

$$\Psi(\sigma, z) \leq -\min\{\kappa, 2\mu\} + M_K\sqrt{N}|\mathrm{tr}(\sigma)|, \tag{3.2.26}$$

while from below

$$\Psi(\sigma, z) \geq -\max\{\kappa, 2\mu\} - M_K\sqrt{N}|\mathrm{tr}(\sigma)| \tag{3.2.27}$$

where $k, 2\mu$ are defined by (2.3.3) and M_K is as in (1.3.4); clearly we may assume that any extension of Ψ we will consider preserves these bounds in the whole space. Notice that, by (3.2.26) and (1.3.4), if z is sufficiently close to 0, and $(\sigma, z) \in K$, then the indicator Ψ is strictly negative uniformly in σ; according to what we shall see in the following sections, this means that when the internal variable is sufficiently small the evolution is continuous.

In what follows we shall define, for every $\sigma \in \mathbb{M}^{N\times N}_{sym}$,

$$\lambda(\sigma) := \max\{\kappa, 2\mu\} + M_K\sqrt{N}|\mathrm{tr}(\sigma)|. \tag{3.2.28}$$

3.3 Continuous evolution

3.3.1 The equation of the slow dynamics

In this section we study in detail the equation

$$\begin{cases} \dot\sigma_{sl}(t) &= \frac{\Phi(t,\sigma_{sl}(t),z_{sl}(t))}{\Psi(\sigma_{sl}(t),z_{sl}(t))}\,\mathbb{C}\,\nu_{K(z_{sl}(t))}(\sigma_{sl}(t)) + \mathbb{C}\dot\xi(t),\\ \dot z_{sl}(t) &= -\frac{\Phi(t,\sigma_{sl}(t),z_{sl}(t))}{\Psi(\sigma_{sl}(t),z_{sl}(t))}\,\mathrm{tr}(\sigma_{sl}(t))\;\mathrm{tr}(\nu_{K(z_{sl}(t))}(\sigma_{sl}(t))), \end{cases} \tag{3.3.1}$$

defined on the open submanifold $\partial K \cap \{\Psi(\sigma,z) \neq 0\} \setminus \{(0,0)\}$. This will be called the equation of the slow dynamics: observe that this is a well-defined equation, since, for every $t \in [0,+\infty)$, the vector field

$$\chi_t(\sigma,z) = (\mathbb{C}\dot\xi(t) + \tfrac{\Phi(t,\sigma,z)}{\Psi(\sigma,z)}\,\mathbb{C}\,\nu_{K(z)}(\sigma), \tfrac{-\Phi(t,\sigma,z)}{\Psi(\sigma,z)}\mathrm{tr}(\sigma)\,\mathrm{tr}(\nu_{K(z)}(\sigma)))$$

is a tangent vector field to $\partial K \cap \{\Psi(\sigma,z) \neq 0\} \setminus \{(0,0)\}$; indeed, by (3.2.6), it suffices to show that $\chi_t(\sigma,z)\cdot(z\,\nu_{K(z)}(\sigma), -\sigma\cdot\nu_{K(z)}(\sigma)) = 0$, which follows by a direct computation, recalling (3.2.24), and (3.2.25).

Remark 3.5. Let $(\sigma(t), z(t))$ be a solution of (3.3.1) and define $e(t)$, $p(t)$ through the constitutive relations in (3.1.1); we have that $\dot p(t) = -\frac{\Phi(t,\sigma(t),z(t))}{\Psi(\sigma(t),z(t))}\nu_{K(z(t))}(\sigma(t))$, thus the flow rule in (3.1.1) is satisfied as long as $-\frac{\Phi(t,\sigma(t),z(t))}{\Psi(\sigma(t),z(t))} \geq 0$; that is, in our case, as long as Φ does not become negative along the trajectory. We will see indeed that equation (3.3.1) appears in the limit of (3.2.18) when the slow-fast indicator Ψ is negative.

Viceversa, let $(\sigma(t), z(t))$ be a C^1 function with values on ∂K satisfying (3.1.1) in a certain interval of time; if we suppose $\Psi(\sigma(t), z(t)) \neq 0$, the flow rule and the condition

$$0 = \nu_K((\sigma(t), z(t)))\cdot(\dot\sigma(t), \dot z(t))\,,$$

with the help of (3.2.6), easily imply that $(\sigma(t), z(t))$ satisfies (3.3.1) and that it must be $-\frac{\Phi(t,\sigma(t),z(t))}{\Psi(\sigma(t),z(t))} \geq 0$.

We endow equation (3.3.1) with initial data $(\sigma_1, z_1) \in \partial K$ at a time $t_1 > 0$, with $z_1 > 0$ and $\Psi(\sigma_1, z_1) \neq 0$. We may thus apply all standard results about local existence and uniqueness and the existence of a maximal interval where solutions to (3.3.1) are defined. So, let (t_1, t_2) be the maximal interval of existence for the Cauchy problem associated to (3.3.1) with datum (σ_1, z_1). As said in (3.2.9), we denote the spherical and the deviatoric part of $\sigma_{sl}(t)$ with $x_{sl}(t)$ and $y_{sl}(t)$, and the spherical and the deviatoric part of $\xi(t)$ with $\eta(t)$ and $\gamma(t)$. Using the identity $\mathrm{tr}(\mathbb{C}\sigma) = \kappa N \mathrm{tr}(\sigma)$, from (3.3.1) we obtain

$$\kappa \dot{z}_{sl}(t) = x_{sl}(t)(\kappa N \dot{\eta}(t) - \dot{x}_{sl}(t)). \tag{3.3.2}$$

The next Proposition shows an useful consequence of this equation.

Proposition 3.6. *Assume* (1.3.1)-(1.3.4), (2.3.3), *and* (3.2.2). *Let* Φ, Ψ *be as in* (3.2.24), *and* (3.2.25), *respectively. Let* $(\sigma_{sl}(t), z_{sl}(t))$ *be the unique solution to the Cauchy problem associated to* (3.3.1) *with Cauchy data* $(\sigma_1, z_1) \in \partial K$ *at a time* $t_1 > 0$, *with* $z_1 > 0$ *and* $\Psi(\sigma_1, z_1) \neq 0$, *and let* $[t_1, t_2)$ *be its maximal interval of existence. If* $t_2 < +\infty$, *there exists a positive constant* M *such that*

$$|(\sigma_{sl}(t), z_{sl}(t))| < M \textit{ for every } t \in [t_1, t_2) \tag{3.3.3}$$

Proof. By (1.3.4), it suffices to show that $z_{sl}(t)$ is bounded. Let $L > 0$ such that $|\dot{\eta}(t)| < L$ for every $t \in [t_1, t_2]$: by (3.3.2), and (1.3.4) we have, for every $t \in [t_1, t_2)$

$$\begin{aligned} \kappa(z_{sl}(t) - z_{sl}(t_1)) &= \kappa \int_{t_1}^{t} \dot{z}_{sl}(s)\, ds = \\ = -\int_{t_1}^{t} x_{sl}(s)\dot{x}_{sl}(s)\, ds &+ \kappa N \int_{t_1}^{t} \dot{\eta}(s) x_{sl}(s)\, ds \leq \\ \leq \tfrac{1}{2}[x_{sl}^2(t_1) - x_{sl}^2(t)] &+ \kappa N \int_{t_1}^{t} |\dot{\eta}(s)||x_{sl}(s)|\, ds \leq \\ \leq \tfrac{1}{2} x_{sl}^2(t_1) &+ \kappa L N M_K \int_{t_1}^{t} z_{sl}(s)\, ds \end{aligned}$$

and the conclusion follows by Gronwall's inequality. □

By the use of (3.3.2) we are also able to show that $z_{sl}(t)$ cannot vanish at $t = t_2$.

Proposition 3.7. *Assume* (1.3.1)-(1.3.4), (2.3.3), *and* (3.2.2). *Let* Φ, Ψ *be as in* (3.2.24), *and* (3.2.25), *respectively. Let* $(\sigma_{sl}(t), z_{sl}(t))$ *be the unique solution to the Cauchy problem associated to* (3.3.1) *with Cauchy data* $(\sigma_1, z_1) \in \partial K$ *at a time* $t_1 > 0$, *with* $z_1 > 0$ *and* $\Psi(\sigma_1, z_1) \neq 0$, *and let* $[t_1, t_2)$ *be its maximal interval of existence. If* $t_2 < +\infty$, *then*

$$\liminf_{t \to t_2} z_{sl}(t) > 0. \tag{3.3.4}$$

Proof. Suppose by contradiction that $\liminf\limits_{t \to t_2} z_{sl}(t) = 0$; we first show that this liminf is a limit. Let $L > 0$ such that $|\dot{\eta}(t)| < L$ for every $t \in (t_1, t_2)$, and M_K as in (1.3.4), and let $c := \limsup\limits_{t \to t_2} z_{sl}(t)$; if we suppose $c > 0$, we may fix $\hat{t} < t_2$ such that

1) $LNM_K(t_2 - \hat{t}) < \frac{1}{8}$;

2) $z_{sl}(t) < 2c$ for every $t > \hat{t}$;

3) $z_{sl}(\hat{t}) > \frac{c}{2}$.

We shall then have, by (3.3.2), (1.3.4), and the previous assumptions, that, for every $t > \hat{t}$

$$\begin{aligned}
\kappa z_{sl}(t) = \kappa z_{sl}(\hat{t}) + \int_{\hat{t}}^{t} \dot{z}_{sl}(s)\, ds = \\
= \kappa z_{sl}(\hat{t}) - \int_{\hat{t}}^{t} x_{sl}(s)\dot{x}_{sl}(s)\, ds + \kappa N \int_{\hat{t}}^{t} \dot{\eta}(s) x_{sl}(s)\, ds \geq \\
\geq \kappa \tfrac{c}{2} + \tfrac{1}{2}[x_{sl}^2(\hat{t}) - x_{sl}^2(t)] - \kappa N \int_{\hat{t}}^{t} |\dot{\eta}(s)||x_{sl}(s)|\, ds \geq \\
\geq \kappa \tfrac{c}{2} - \tfrac{1}{2} x_{sl}^2(t) - \kappa N L M_K \int_{\hat{t}}^{t} z_{sl}(s)\, ds \geq \\
\geq \kappa \tfrac{c}{2} - \tfrac{1}{2} x_{sl}^2(t) - \kappa \tfrac{c}{4}.
\end{aligned}$$

So, let t_n a sequence converging to t_2 realizing the liminf; by (1.3.4) we shall get that $\lim_{n\to+\infty} x_{sl}(t_n) = 0$. As $t_n > \hat{t}$ for n sufficiently large, we shall have

$$\kappa z_{sl}(t_n) \geq \kappa \tfrac{c}{4} - \tfrac{1}{2} x_{sl}^2(t_n),$$

which in the limit yields $\frac{c}{4} \leq 0$, a contradiction. We thus have that $\lim_{t\to t_2} z_{sl}(t) = 0$, which immediately implies, by (1.3.4), that $\lim_{t\to t_2} x_{sl}(t) = 0$. We now fix $\bar{t} < t_2$ such that $LNM_K(t_2 - \bar{t}) < \frac{1}{2}$; as $z_{sl}(t) > 0$ in (t_1, t_2) and $\lim_{t\to t_2} z_{sl}(t) = 0$, there exists a maximum point t_3 for $z_{sl}(t)$ in $[\bar{t}, t_2)$. Repeating the previous estimates, we shall have, for every $t > t_3$, that

$$\kappa z_{sl}(t) \geq \kappa z_{sl}(t_3) - \tfrac{1}{2} x_{sl}^2(t) - \kappa N L M_K z_{sl}(t_3)(t_2 - \bar{t}) \geq \kappa \tfrac{z_{sl}(t_3)}{2} - \tfrac{1}{2} x_{sl}^2(t),$$

which in the limit as $t \to t_2$ gives $z_{sl}(t_3) \leq 0$, a contradiction. □

By the previous results, we now may show that the solutions (3.3.1) are globally defined unless the slow-fast indicator vanishes along the trajectory. In the proof we use the following elementary Lemma about differential equations, which can be found in [23, Chapter 1, Lemma 3.1]; we state it for the reader's convenience.

Lemma 3.8. *Let E be a subset of $\mathbb{R} \times \mathbb{R}^n$, let $f\colon E \to \mathbb{R}^n$ a continuous function, and let $u(t)$ a solution of the ODE $\dot{v}(t) = f(t, v(t))$ on an interval $[a, \delta)$ or $(\delta, a]$ where $|\delta| < +\infty$. If there exists a sequence t_k converging to δ such that $u(t_k) \to \bar{u} \in \mathbb{R}^n$ and $f(t, v)$ is bounded on the intersection of E with an open neighborhood of the point $(\delta, \bar{u})$, then*

$$\lim_{t\to\delta} u(t) = \bar{u}.$$

Proposition 3.9. *Assume* (1.3.1)-(1.3.4), (2.3.3), *and* (3.2.2)*; let Φ, Ψ be as in* (3.2.24), *and* (3.2.25), *respectively. Let $(\sigma_{sl}(t), z_{sl}(t))$ be the unique solution to the Cauchy problem associated to* (3.3.1) *with Cauchy data $(\sigma_1, z_1) \in \partial K$ at a time $t_1 > 0$, with $z_1 > 0$ and such that $\Psi(\sigma_1, z_1) \neq 0$, and let $[t_1, t_2)$ be its maximal interval of existence. If $t_2 < +\infty$, then*

$$\lim_{t\to t_2^-} \Psi(\sigma_{sl}(t), z_{sl}(t)) = 0 \tag{3.3.5}$$

Proof. Suppose by contradiction that there exists a sequence $t_k \to t_2$ such that

$$\lim_{k\to+\infty} \Psi(\sigma_{sl}(t_k), z_{sl}(t_k)) \neq 0. \tag{3.3.6}$$

By Proposition 3.6, we may assume that $(\sigma_{sl}(t_k), z_{sl}(t_k))$ tends to a finite limit (σ_2, z_2) as $k \to +\infty$; by Proposition 3.7 we have that $z_2 > 0$. By continuity of Ψ, (3.3.6) implies that $\Psi(\sigma_2, z_2) \neq 0$; it follows now from Lemma 3.8 that

$$\lim_{t\to t_2} (\sigma_{sl}(t), z_{sl}(t)) = (\sigma_2, z_2);$$

we may then solve the Cauchy problem associated to (3.3.1) with data (σ_2, z_2) at time t_2, contradicting the maximality of $[t_1, t_2)$. □

In the next Proposition, we use Lemma 3.8 to prove that, if Ψ vanishes at time $t_2 < +\infty$, then $(\sigma_{sl}(t), z_{sl}(t))$ have a limit at $t = t_2$; the proof is obtained by observing that in this case $z_{sl}(t)$ must be monotone in a neighborhood of t_2. We also need the additional hypothesis that the elastic-inelastic indicator is not vanishing at t_2, that is to say

$$\liminf_{t\to t_2^-} |\Phi(t, \sigma_{sl}(t), z_{sl}(t))| > 0. \tag{3.3.7}$$

Proposition 3.10. *Assume* (1.3.1)-(1.3.4), (2.3.3), *and* (3.2.2)*; let* Φ, Ψ *be as in* (3.2.24), *and* (3.2.25), *respectively. Let* $(\sigma_{sl}(t), z_{sl}(t))$ *be the unique solution to the Cauchy problem associated to* (3.3.1) *with Cauchy data* (σ_1, z_1) *at a time* $t_1 > 0$, *with* $z_1 > 0$ *and such that* $\Psi(\sigma_1, z_1) \neq 0$, *and let* $[t_1, t_2)$ *be its maximal interval of existence. If* $t_2 < +\infty$, *and* (3.3.7) *holds, then there exists*

$$\lim_{t\to t_2^-} (\sigma_{sl}(t), z_{sl}(t)) := (\sigma_2, z_2) \in \partial K. \tag{3.3.8}$$

Proof. By Proposition 3.9 we have $\lim_{t\to t_2^-} \Psi(\sigma_{sl}(t), z_{sl}(t)) = 0$; as seen in Remark 3.4, this implies that

$$\liminf_{t\to t_2^-} x_{sl}(t) < 0 \quad \text{and} \quad \liminf_{s\to t_2^-} \operatorname{tr}(\nu_{K(z_{sl}(t))}(\sigma_{sl}(t))) > 0;$$

if not, in both cases we may find a sequence t_n converging to t_2 along which

$$\limsup_{n\to+\infty} \Psi(\sigma_{sl}(t_n), z_{sl}(t_n)) \leq -\min\{\kappa, 2\mu\} < 0,$$

a contradiction. By (3.3.1), (3.3.5), and (3.3.7) we easily get that there exists a left neighborhood of t_2, denoted with $(\hat{t}, t_2)$, where $\dot{z}_{sl}(t) \neq 0$; thus $z_{sl}(t)$ is invertible in this interval, with inverse $t(z)$, and converges to a limit z_2, which is finite by Proposition 3.6. We now suppose, for instance, that $z_{sl}(t)$ is strictly decreasing, the proof in the other case being completely analogous. We put $\hat{z} := z_{sl}(\hat{t})$ and we express σ in function of z; by (3.3.1), we then get that

$$-\sigma'_{sl}(z) = \tfrac{1}{\operatorname{tr}(\sigma_{sl}(z)) \operatorname{tr}(\nu_{K(z)}(\sigma_{sl}(z)))} [\mathbb{C}\, \nu_{K(z)}(\sigma_{sl}(z)) - \mathbb{C}\, \chi(z) \tfrac{\Psi(\sigma_{sl}(z), z)}{\Phi(t(z), \sigma_{sl}(z), z)}] \tag{3.3.9}$$

for every $z \in (z_2, \hat{z})$; here we have put: $\chi(z) := \dot{\xi}(t(z))$. So, as

$$\liminf_{z\to z_2} |\operatorname{tr}(\sigma_{sl}(z)) \operatorname{tr}(\nu_{K(z)}(\sigma_{sl}(z)))| > 0$$

by the previous discussion, and taking into account (1.3.4) and (3.3.7), $|\sigma'_{sl}(z)|$ remains uniformly bounded in this interval. The conclusion follows. □

Remark 3.11. If the inequalities $\Phi(t_1,\sigma_1,z_1)>0$ and $\Psi(\sigma_1,z_1)<0$ are satisfied, we will see in the next subsection that the solutions of (3.2.18) uniformly converge to the solution of (3.3.1) in a right neighborhood of t_1. In general, $[t_1,t_2)$ may not be the maximal interval of convergence, as positivity of Φ may fail before of t_2. We will show that this convergence holds on $[t_1,t_2)$ whenever

$$\Phi(t,\sigma_{sl}(t),z_{sl}(t))>0 \qquad \text{for every } t<t_2. \tag{3.3.10}$$

Assume this inequality, as well as (3.3.7), suppose that $t_2<+\infty$, and let (σ_2,z_2) be as in (3.3.8); then

$$\Psi(\sigma_2,z_2)=0. \tag{3.3.11}$$

Let us prove that

$$\nabla\Psi(\sigma_2,z_2)\cdot\big(\tfrac{-\mathbb{C}\,\nu_{K(z_2)}(\sigma_2)}{\operatorname{tr}(\sigma_2)\operatorname{tr}(\nu_{K(z_2)}(\sigma_2))},\,1\big)\le 0. \tag{3.3.12}$$

Indeed, as seen in Proposition 3.10 $z_{sl}(t)$ is strictly decreasing in a left neighborhood of t_2, with inverse $t(z)$. If we define $\sigma_{sl}(z):=\sigma_{sl}(t(z))$, we shall then have that $\Psi(\sigma_{sl}(z),z)<0$ in a right neighborhood of z_2, which yields

$$\lim_{z\to z_2}\frac{d}{dz}\Psi(\sigma_{sl}(z),z)\le 0;$$

a direct computation involving (3.3.9) and (3.3.11) gives us condition (3.3.12).

We claim that the vector $\big(\frac{-\mathbb{C}\,\nu_{K(z_2)}(\sigma_2)}{\operatorname{tr}(\sigma_2)\operatorname{tr}(\nu_{K(z_2)}(\sigma_2))},\,1\big)$ is tangent to ∂K at (σ_2,z_2). To prove that, by (3.2.6), it suffices to show that

$$\big(\tfrac{-\mathbb{C}\,\nu_{K(z_2)}(\sigma_2)}{\operatorname{tr}(\sigma_2)\operatorname{tr}(\nu_{K(z_2)}(\sigma_2))},\,1\big)\cdot\big(z_2\,\nu_{K(z_2)}(\sigma_2),-\sigma_2\cdot\nu_{K(z_2)}(\sigma_2)\big)=0\,.$$

Recalling (3.2.25), the left-hand side is equal to $\frac{z_2\Psi(\sigma_2,z_2)}{\operatorname{tr}(\sigma_2)\operatorname{tr}(\nu_{K(z_2)}(\sigma_2))}$, and the conclusion follows by (3.3.11). Thus the left-hand side of (3.3.12) is a tangential derivative and depends only on the values Ψ attains on ∂K.

Due to the presence of the forcing term $\mathbb{C}\dot\xi(t)$, the sign of $\dot z_{sl}(t)$ may change, causing the alternance of hardening and softening regime; we end this subsection by presenting a simple condition that prevents this phenomenon. To be definite, we consider the case where the spherical part of $\xi(t)$ is constant, as in [12]. Observe that here we are assuming (3.2.10)-(3.2.11), in order to apply Proposition 3.2.

Proposition 3.12. *Assume that* (1.3.1)-(1.3.4), (3.2.2), (2.3.3), *and* (3.2.10)-(3.2.11) *are satisfied; let* Φ, Ψ *be as in* (3.2.24), *and* (3.2.25), *respectively. Let* $(\sigma_{sl}(t),z_{sl}(t))$ *the unique solution to* (3.3.1) *with Cauchy data* (σ_1,z_1) *at a time* $t_1>0$, *with* $z_1>0$ *and* $\Psi(\sigma_1,z_1)<0$, *and let* $[t_1,t_2)$ *be its maximal interval of existence. Let* $\hat t\in[t_1,t_2)$ *such that*

$$\Phi(t,\sigma_{sl}(t),z_{sl}(t))>0 \qquad \textit{for every } t\in[t_1,\hat t] \tag{3.3.13}$$

and suppose that $\dot\eta(t)=0$ *for every* $t\in[t_1,\hat t]$. *If there exists* $\bar t\in(t_1,\hat t)$ *such that* $\dot z_{sl}(\bar t)=0$, *then* $\dot z_{sl}(t)=0$ *for every* $t\in[t_1,\hat t]$.

Proof. As $\hat{t} < +\infty$, by the same arguments as in Proposition 3.6 and Proposition 3.7, we may assume that $Z := \inf_{t\in[t_1,\hat{t}]} z_{sl}(t) > 0$ and that $|x_{sl}(t)|$ is bounded by a finite constant M. By (3.3.1) we have that

$$\dot{x}_{sl}(t) = \sqrt{N}\,\tfrac{\Phi(t,\sigma_{sl}(t),z_{sl}(t))}{\Psi(\sigma_{sl}(t),z_{sl}(t))}\,\kappa\,\mathrm{tr}(\nu_{K(z_{sl}(t))}(\sigma_{sl}(t))), \tag{3.3.14}$$

while (3.3.2) reduces to

$$\kappa\dot{z}_{sl}(t) = -x_{sl}(t)\dot{x}_{sl}(t). \tag{3.3.15}$$

By (3.3.14), (3.3.13), (3.2.5), and (3.2.15), we have that

$$\dot{x}_{sl}(t) = 0 \Longleftrightarrow x_{sl}(t) + a\,z_{sl}(t) = 0, \tag{3.3.16}$$

where $a > 0$ is as in (3.2.10). Let us prove that $x_{sl}(t) \neq 0$ for every $t \in (t_1, \hat{t}]$; indeed, by (3.2.2), which is equivalent to (3.2.13), if the value 0 is assumed, it is a maximum value for $x_{sl}(t)$, thus, if for some $t \in (t_1, \hat{t}]$ we have $x_{sl}(t) = 0$, it must be also $\dot{x}_{sl}(t) = 0$, but this is excluded by (3.3.16), as $z_{sl}(t) > 0$.

Suppose that there exists $\bar{t} \in (t_1, \hat{t})$ such that $\dot{z}_{sl}(\bar{t}) = 0$; as $x_{sl}(\bar{t}) \neq 0$, by (3.3.15) we must have $\dot{x}_{sl}(\bar{t}) = 0$, that is to say $x_{sl}(\bar{t}) + a\,z_{sl}(\bar{t}) = 0$. Let $f(t) := x_{sl}(t) + a\,z_{sl}(t)$; under our hypotheses, by (3.3.14) and (3.3.15) there exists a positive constant W such that

$$|\dot{f}(t)| \leq W|\mathrm{tr}(\nu_{K(z_{sl}(t))}(\sigma_{sl}(t))| \qquad \text{for every } t \in [t_1, \hat{t}];$$

(3.2.5) and (3.2.15) imply that

$$|\mathrm{tr}(\nu_{K(z_{sl}(t))}(\sigma_{sl}(t))| \leq \tfrac{F}{Z}|x_{sl}(t) + a\,z_{sl}(t)|,$$

where $F > 0$ is as in (3.2.14). We conclude that

$$|\dot{f}(t)| \leq W\tfrac{F}{Z}|f(t)| \quad \text{for every } t \in [t_1, \hat{t}];$$

as $f(\bar{t}) = 0$, Gronwall's inequality implies that $f(t) = 0$ for every $t \in [t_1, \hat{t}]$, which in its turn entails that $\dot{x}_{sl}(t) = 0$ for every $t \in [t_1, \hat{t}]$, and conclusion follows by (3.3.15). ☐

3.3.2 Convergence to the slow dynamics

In this subsection we examine how to recover equation (3.3.1) from (3.2.18) in the limit as ε goes to 0, under suitable hypotheses on the sign of the indicators Φ and Ψ: the arguments used here are reminiscent of [47, Section 3], where another model of plasticity with softening in the spatially homogenoeus case was considered.

Throughout this part of the chapter, $\hat{t}$ denotes a time such that there exist a left continuous function $t \mapsto (\sigma(t), z(t))$ defined on $[0, \hat{t})$ with values in $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ and an element $(\hat{\sigma}, \hat{z})$ of $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ satisfying the following properties:

$$(\sigma_\varepsilon(t), z_\varepsilon(t)) \to (\sigma(t), z(t)) \quad \text{for a.e. } t \in [0, \hat{t})\,, \tag{3.3.17}$$

$$\text{there exists } \hat{t}_\varepsilon \to \hat{t} \text{ such that } (\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon)) \to (\hat{\sigma}, \hat{z})\,, \tag{3.3.18}$$

$$(\hat{\sigma}, \hat{z}) \in \partial K \quad \text{and} \quad \hat{z} > 0\,, \tag{3.3.19}$$

$$\Phi(\hat{t}, \hat{\sigma}, \hat{z}) > 0\,. \tag{3.3.20}$$

For instance, we can take $\hat{t} = t_1$ defined by (3.2.20), if $t_1 < +\infty$ and, setting

$$(\sigma_1, z_1) := (\sigma_0 + \mathbb{C}(\xi(t_1) - \xi(0)), z_0). \tag{3.3.21}$$

we have

$$\Phi(t_1, \sigma_1, z_1) > 0; \tag{3.3.22}$$

notice that in general we have $\Phi(t_1, \sigma_1, z_1) \geq 0$, as the solution was in K at all previous times, thus we are only excluding the degenerate case when equality holds. The case $\Phi(t_1, \sigma_1, z_1) = 0$ will be discussed in the next subsection.

Lemma 3.13. *Assume* (1.3.1)-(1.3.4), (2.3.3)*, and* (3.2.2)*. Let* Φ *be as in* (3.2.24)*. Let* $\hat{t} > 0$ *satisfy* (3.3.17)-(3.3.20)*, and let* $\hat{t}_\varepsilon$ *be as in* (3.3.18)*; then, for every* $t^* > \hat{t}$*, the set* $\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, t^*]$ *is nonempty, when* ε *is sufficiently small.*

Proof. Assume on the contrary that along a suitable subsequence, that we shall not relabel, one has $\varrho_\varepsilon(t) = 0$ for every $t \in [\hat{t}_\varepsilon, t^*]$; we then get

$$(\sigma_\varepsilon(t), z_\varepsilon(t)) = (\sigma_\varepsilon(\hat{t}_\varepsilon) + \mathbb{C}(\xi(t) - \xi(\hat{t}_\varepsilon)), z_\varepsilon(t_\varepsilon)) \in K \tag{3.3.23}$$

for every $t \in [\hat{t}_\varepsilon, t^*]$. In the limit we obtain that $(\hat{\sigma} + \mathbb{C}(\xi(t) - \xi(\hat{t})), \hat{z}) \in K$ for every $t \in [\hat{t}, t^*]$; by (3.3.19) we easily deduce that it must be $\Phi(\hat{t}, \hat{\sigma}, \hat{z}) \leq 0$, contradicting (3.3.20). ☐

Remark 3.14. Notice that if $\hat{t} = t_1$, the statement of the Lemma holds with $\hat{t}_\varepsilon = t_1$.

We fix an open neighborhood $U_\delta := (\hat{t} - \delta, \hat{t} + \delta) \times B_\delta(\hat{\sigma}, \hat{z})$, where $B_\delta(\hat{\sigma}, \hat{z})$ denotes the open ball of radius $\delta > 0$ centered at $(\hat{\sigma}, \hat{z})$, in a way that there exists a positive constant $\gamma_2 > 0$ such that

$$\Phi(t, \sigma, z) \geq \gamma_2 > 0 \qquad \text{for every } (t, \sigma, z) \in U_\delta. \tag{3.3.24}$$

We may clearly assume that $\delta < \frac{\max\{\kappa, 2\mu\}}{2M_K\sqrt{N}}$, where k and μ are defined by (2.3.3) and M_K is as in (1.3.4), in a way that, for every $(\sigma, z) \in B_\delta(\hat{\sigma}, \hat{z})$, the following holds:

$$\tfrac{\lambda(\sigma)}{\lambda(\hat{\sigma})} < \tfrac{3}{2}, \tag{3.3.25}$$

where $\lambda(\sigma)$ is defined as in (3.2.28). We define

$$a_\varepsilon := \inf\{t \in (\hat{t}_\varepsilon, \hat{t}_\varepsilon + \delta) : \ (\sigma_\varepsilon(t), z_\varepsilon(t)) \in \partial B_\delta(\hat{\sigma}, \hat{z})\}, \tag{3.3.26}$$

where $\hat{t}_\varepsilon$ is given by (3.3.18). The following lemma shows that, thanks to (3.3.24), the function $\frac{1}{\varepsilon}\varrho_\varepsilon(t)$ becomes greater than a fixed positive constant after a time t_ε converging to $\hat{t}$ as $\varepsilon \to 0$, while the motion is still in $B_\delta(\hat{\sigma}, \hat{z})$; we shall see that this implies a transition to the inelastic regime.

Lemma 3.15. *Assume* (1.3.1)-(1.3.4), (2.3.3)*, and* (3.2.2)*. Let* Φ *be as in* (3.2.24)*. Let* $\hat{t} > 0$ *satisfy* (3.3.17)-(3.3.20)*, let* $\hat{t}_\varepsilon$ *be as in* (3.3.18)*, and let* δ*,* a_ε*, and* γ_2*, be as in* (3.3.24) *and* (3.3.26)*. Let* $\varepsilon > 0$ *and* $\varrho_\varepsilon(t)$ *be as in* (3.2.22)*. Define*

$$t_\varepsilon := \inf\{t \in (\hat{t}_\varepsilon, \hat{t}_\varepsilon + \delta) : \ \tfrac{1}{\varepsilon}\varrho_\varepsilon(t) \geq \tfrac{\gamma_2}{3\lambda(\hat{\sigma})}\}. \tag{3.3.27}$$

Then:

a) $t_\varepsilon - \hat{t} \to 0$ *as* $\varepsilon \to 0^+$*;*

b) $t_\varepsilon < a_\varepsilon$ *for* ε *sufficiently small;*

c) $\frac{1}{\varepsilon}\varrho_\varepsilon(t) \geq \frac{\gamma_2}{3\lambda(\hat{\sigma})}$ *for every* $t \in [t_\varepsilon, a_\varepsilon]$.

Proof. Concerning part a) and part b) of the statement, we may clearly suppose that $t_\varepsilon > \hat{t}_\varepsilon$. Let $s_\varepsilon := t_\varepsilon \wedge a_\varepsilon$. We first claim that, for small ε, in $(\hat{t}_\varepsilon, s_\varepsilon)$ one has $\varrho_\varepsilon(t) > 0$.

Indeed, we first observe that if the set $\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, s_\varepsilon]$ is empty along a suitable subsequence (unrelabelled), then clearly $s_\varepsilon = a_\varepsilon$, and (3.3.23) holds for every $t \in [\hat{t}_\varepsilon, t^*]$; we then easily get that $\liminf a_\varepsilon > \hat{t}$, and this contradicts Lemma 3.13. Then, for ε sufficiently small, the set $\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, s_\varepsilon]$ has positive measure. Now, observe that $\dot{\varrho}_\varepsilon(t) = 0$ a.e. in $\{\varrho_\varepsilon(t) = 0\} \cap [\hat{t}_\varepsilon, s_\varepsilon]$, while in the set $\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, s_\varepsilon]$ one has

$$\dot{\varrho}_\varepsilon(t) \geq \tfrac{\gamma_2}{2} \tag{3.3.28}$$

by (3.2.23), (3.3.24), (3.2.27), and (3.3.25). Then, by the fundamental theorem of calculus and by Lemma 3.13, we get

$$\varrho_\varepsilon(\tau) = \int_{\{\varrho_\varepsilon(t)>0\}\cap[\hat{t}_\varepsilon,\tau]} \dot{\varrho}_\varepsilon(t)\,dt \geq \tfrac{\gamma_2}{2}\mathcal{L}^1(\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, \tau]) > 0$$

for every $\tau \in [\hat{t}_\varepsilon, s_\varepsilon]$, which proves our claim. Therefore $\{\varrho_\varepsilon(t) > 0\} \cap (\hat{t}_\varepsilon, s_\varepsilon] = (\hat{t}_\varepsilon, s_\varepsilon]$ so that the previous estimate and the definition of s_ε yield

$$\varepsilon\tfrac{\gamma_2}{3\lambda(\hat{\sigma})} \geq \varrho_\varepsilon(s_\varepsilon) \geq \tfrac{\gamma_2}{2}(s_\varepsilon - \hat{t}_\varepsilon),$$

which implies, by (3.3.18), that

$$s_\varepsilon - \hat{t} \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{3.3.29}$$

Now suppose, by contradiction, that $s_\varepsilon = a_\varepsilon$ as $\varepsilon \to 0$ along a suitable sequence. Then $a_\varepsilon - \hat{t}_\varepsilon \to 0$ as $\varepsilon \to 0^+$ and

$$\sup_{t\in[\hat{t}_\varepsilon,a_\varepsilon]} \tfrac{1}{\varepsilon}\varrho_\varepsilon(t) \leq \tfrac{\gamma_2}{3\lambda(\hat{\sigma})};$$

by the definition of a_ε, (3.2.21), and (3.3.18), this implies

$$\begin{aligned}
\delta + o(1) &= |(\sigma_\varepsilon(a_\varepsilon), z_\varepsilon(a_\varepsilon)) - (\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon))| \\
&\leq |(\sigma_\varepsilon(a_\varepsilon) - \sigma_\varepsilon(\hat{t}_\varepsilon), 0)| + |(0, z_\varepsilon(a_\varepsilon) - z_\varepsilon(\hat{t}_\varepsilon))| \\
&\leq \int_{\hat{t}_\varepsilon}^{a_\varepsilon} |\dot{\sigma}_\varepsilon(t)| + |\dot{z}_\varepsilon(t)|\,dt \qquad (3.3.30)\\
&\leq (|\mathbb{C}| + |\mathrm{tr}(\hat{\sigma})| + \delta + o(1))\int_{\hat{t}_\varepsilon}^{a_\varepsilon} \tfrac{\varrho_\varepsilon(t)}{\varepsilon}\,dt + |\mathbb{C}|\int_{\hat{t}_\varepsilon}^{a_\varepsilon} |\dot{\xi}(t)|\,dt \\
&\leq [(|\mathbb{C}| + |\mathrm{tr}(\hat{\sigma})| + \delta + o(1))\tfrac{\gamma_2}{3\lambda(\hat{\sigma})}](a_\varepsilon - t_1) + |\mathbb{C}|\int_{\hat{t}_\varepsilon}^{a_\varepsilon}|\dot{\xi}(t)|\,dt,
\end{aligned}$$

a contradiction, since the right-hand side tends to 0 as $\varepsilon \to 0$. This proves part a) and part b) of the statement.

Observe now that (3.3.28) yields $\dot{\varrho}_\varepsilon(t_\varepsilon) \geq \frac{\gamma_2}{2}$. Thus, if c) is false, let t^1_ε be the first time in $(t_\varepsilon, a_\varepsilon)$ such that $\varrho_\varepsilon(t^1_\varepsilon) = \frac{\gamma_2}{3\lambda(\sigma_1)}$; then $\dot{\varrho}_\varepsilon(t^1_\varepsilon) \leq 0$. Repeating the proof of (3.3.28) we find $\dot{\varrho}_\varepsilon(t^1_\varepsilon) \geq \frac{\gamma_2}{2} > 0$, a contradiction. □

Remark 3.16. Notice that if $\hat{t} = t_1$, the statement of the Lemma holds with $\hat{t}_\varepsilon = t_1$.

We now focus on the case where the slow-fast indicator is negative at $(\hat{\sigma}, \hat{z})$. As in [47], this allows us to show that, in a neighborhood of $\hat{t}$, the function $\frac{1}{\varepsilon}\varrho_\varepsilon(t)$ remains uniformly bounded. This is the key ingredient to prove that the limit evolution is continuous.

For a suitable choice of δ in the definition of the neighborhood U_δ satisfying (3.3.24), we may assume that there exists a positive constant γ_1 such that

$$\Psi(\sigma, z) \leq -\gamma_1 \qquad \text{for every } (\sigma, z) \in B_\delta(\hat{\sigma}, \hat{z}). \tag{3.3.31}$$

We now state an auxiliary lemma, analogous to [47, Lemma 3.6], which will be used also in Section 3.4. Notice that in the statement of the lemma we make no assumption on the sign of the indicator Φ.

Lemma 3.17. *Assume* (1.3.1)-(1.3.4), (2.3.3), *and* (3.2.2)*; let* Ψ *be as in* (3.2.25). *Let* $\tilde{t} > 0$, $(\tilde{\sigma}, \tilde{z}) \in \partial K$, *and let* $\tilde{t}_\varepsilon$ *be a sequence such that*

$$\tilde{t}_\varepsilon \to \tilde{t} \text{ as } \varepsilon \to 0^+ ,$$
$$(\sigma_\varepsilon(\tilde{t}_\varepsilon), z_\varepsilon(\tilde{t}_\varepsilon)) \to (\tilde{\sigma}, \tilde{z}) \text{ as } \varepsilon \to 0^+ .$$

Suppose that there exist two constants $\eta > 0$, $\gamma > 0$ *such that, for every* (σ, z) *satisfying* $|(\sigma, z) - (\tilde{\sigma}, \tilde{z})| < \eta$, *one has*

$$\Psi(\sigma, z) < -\gamma.$$

Let

$$b_\varepsilon^\eta := \inf\{t \in (\tilde{t}_\varepsilon, \tilde{t} + \eta) : \ (\sigma_\varepsilon(t), z_\varepsilon(t)) \in \partial B_\eta(\tilde{\sigma}, \tilde{z})\}.$$

Then there exist $L > 0$ *and a sequence* $\tilde{s}_\varepsilon$, *which may be taken equal to* $\tilde{t}_\varepsilon$ *whenever* $\limsup\limits_{\varepsilon\to 0} \frac{\varrho_\varepsilon(\tilde{t}_\varepsilon)}{\varepsilon} < +\infty$, *such that*

a) $\tilde{s}_\varepsilon \to \tilde{t}$ *as* $\varepsilon \to 0^+$,

b) $(\sigma_\varepsilon(\tilde{s}_\varepsilon), z_\varepsilon(\tilde{s}_\varepsilon)) \to (\tilde{\sigma}, \tilde{z})$ *as* $\varepsilon \to 0^+$,

c) $\frac{\varrho_\varepsilon(t)}{\varepsilon} \leq \frac{L}{\gamma}$ *for every* $t \in [\tilde{s}_\varepsilon, b_\varepsilon^\eta]$,

d) $\liminf\limits_{\varepsilon\to 0} b_\varepsilon^\eta \geq \tilde{t} + C(\tilde{\sigma}, \eta, \gamma)$,

where $C(\sigma, \eta, \gamma) := \min\{\eta, \frac{\eta\gamma}{L[(1+\gamma)|\mathbb{C}|+|\mathrm{tr}(\sigma)|+\eta]}\}$.

Proof. Choose L such that $|\mathbb{C}\dot{\xi}(t)| < L$ for every $t \in [\tilde{t} - \eta, \tilde{t} + \eta]$. Observe that, from (3.2.23) and the hypotheses, we get

$$\dot{\varrho}_\varepsilon(t) < -\gamma \frac{\varrho_\varepsilon(t)}{\varepsilon} + L \qquad \text{for a.e. } t \in [\tilde{t}_\varepsilon, b_\varepsilon^\eta]; \tag{3.3.32}$$

indeed the inequality holds true also in the set $\{\varrho_\varepsilon(t) = 0\}$, as $\dot{\varrho}_\varepsilon(t) = 0$ almost everywhere in this set. Notice also that it is everywhere satisfied when $\varrho_\varepsilon(t) > 0$.

Let $M = \limsup\limits_{\varepsilon\to 0} \frac{\varrho_\varepsilon(\tilde{t}_\varepsilon)}{\varepsilon}$; we may assume, up to a subsequence, that this limsup is actually a limit. If $M < +\infty$, we may always assume, suitably enlarging the constant L, that

$M < \frac{L}{\gamma}$. If $M = +\infty$, fix $\vartheta > 0$ and define $s_\varepsilon^\eta := \inf\{t \in [\tilde{t}_\varepsilon, b_\varepsilon^\eta] | \frac{\varrho_\varepsilon(t)}{\varepsilon} \leq \frac{L+\vartheta}{\gamma}\}$; then (3.3.32) yields

$$\dot{\varrho}_\varepsilon(t) < -\vartheta \quad \text{for every } t \in [\tilde{t}_\varepsilon, s_\varepsilon^\eta]\,; \tag{3.3.33}$$

integrating, we get

$$(s_\varepsilon^\eta - \hat{t}_\varepsilon)\vartheta < \varrho_\varepsilon(\tilde{t}_\varepsilon) - \varrho_\varepsilon(s_\varepsilon^\eta)\,. \tag{3.3.34}$$

As $\varrho_\varepsilon(\tilde{t}_\varepsilon) \to 0$, we conclude that $s_\varepsilon^\eta \to \hat{t}$ as $\varepsilon \to 0^+$. From this fact and (3.3.34), we also get that $\lim_{\varepsilon\to 0} \varrho_\varepsilon(s_\varepsilon^\eta) = 0$, hence, integrating (3.3.32), we obtain

$$\lim_{\varepsilon\to 0} \int_{\tilde{t}_\varepsilon}^{s_\varepsilon^\eta} \frac{\varrho_\varepsilon(s)}{\varepsilon}\, ds = 0\,. \tag{3.3.35}$$

We can then argue as in (3.3.30), and for every $t \in [\tilde{t}_\varepsilon, s_\varepsilon^\eta]$ we have

$$|\sigma_\varepsilon(t) - \sigma_\varepsilon(\tilde{t}_\varepsilon)| + |\zeta_\varepsilon(t) - \zeta_\varepsilon(\tilde{t}_\varepsilon)| \leq$$
$$\leq (|\mathbb{C}| + |\mathrm{tr}(\tilde{\sigma})| + \eta + o(1)) \int_{\tilde{t}_\varepsilon}^{s_\varepsilon^\eta} \frac{\varrho_\varepsilon}{\varepsilon}(s)\, ds + |\mathbb{C}| \int_{\tilde{t}_\varepsilon}^{s_\varepsilon^\eta} |\dot{\xi}(s)|\, ds\,.$$

From this and (3.3.35) we get

$$\lim_{\varepsilon\to 0} \sup_{\tilde{t}_\varepsilon \leq t \leq s_\varepsilon^\eta} |\sigma_\varepsilon(t) - \sigma_\varepsilon(\tilde{t}_\varepsilon)| + |\zeta_\varepsilon(t) - \zeta_\varepsilon(\tilde{t}_\varepsilon)| = 0\,.$$

In particular we have $s_\varepsilon^\eta < b_\varepsilon^\eta$, when ε is sufficiently small.

So we put $\tilde{s}_\varepsilon := s_\varepsilon^\eta$ when $M = +\infty$, while we put $\tilde{s}_\varepsilon := \tilde{t}_\varepsilon$ otherwise; up to suitably enlarging the constant L, we have, for every ε, $\frac{\varrho_\varepsilon(\tilde{s}_\varepsilon)}{\varepsilon} \leq \frac{L}{\gamma}$ and $\dot{\varrho}_\varepsilon(\tilde{s}_\varepsilon) < 0$. Now, if c) is false, let s_ε^1 be the first time in $(\tilde{s}_\varepsilon, b_\varepsilon^\eta)$ such that $\varrho_\varepsilon(s_\varepsilon^1) = \frac{L}{\gamma}$; then $\dot{\varrho}_\varepsilon(t_\varepsilon^1) \geq 0$. On the other hand (3.3.32) yields $\dot{\varrho}_\varepsilon(t_\varepsilon^1) < -L + L = 0$, a contradiction.

It remains to prove only part d) of the statement. We can suppose $b_\varepsilon^\eta < \tilde{t} + \eta$, otherwise the result is trivial. Again we can argue as in (3.3.30), and we have the estimate

$$\eta = |(\sigma_\varepsilon(b_\varepsilon^\eta) - \sigma_\varepsilon(\tilde{s}_\varepsilon), \zeta_\varepsilon(t) - \zeta_\varepsilon(\tilde{s}_\varepsilon)| \leq$$
$$\leq (|\mathbb{C}| + |\mathrm{tr}(\tilde{\sigma})| + \eta) \int_{\tilde{s}_\varepsilon}^{b_\varepsilon^\eta} \frac{\varrho_\varepsilon}{\varepsilon}(s)\, ds + |\mathbb{C}| \int_{\tilde{s}_\varepsilon}^{b_\varepsilon^\eta} |\dot{\xi}(s)|\, ds\,,$$

which implies, by part c) of the statement,

$$\eta < [(1+\gamma)|\mathbb{C}| + |\mathrm{tr}(\tilde{\sigma})| + \eta]\tfrac{L}{\gamma}(b_\varepsilon^\eta - \tilde{s}_\varepsilon)\,;$$

since $\hat{s}_\varepsilon \to \hat{t}$ as $\varepsilon \to 0^+$, this concludes the proof. $\square$

We are now ready to prove the main result of this section.

Theorem 3.18. *Assume* (1.3.1)-(1.3.4), (2.3.3), *and* (3.2.2), *and define* Φ, *and* Ψ *as in* (3.2.24), *and* (3.2.25), *respectively. Let* $\hat{t} > 0$ *satisfy* (3.3.17)-(3.3.20), *let* $\hat{t}_\varepsilon$ *be as in* (3.3.18), *and suppose that* (3.3.31) *holds. Let* $(\sigma_{sl}(s), z_{sl}(s))$ *be the unique solution to the equation of the slow dynamics* (3.3.1) *with Cauchy datum* $(\hat{\sigma}, \hat{z})$ *at* $\hat{t}$, *and let* $t_2 > \hat{t}$ *be as in* (3.3.5). *Let* $\bar{t} < t_2$ *and suppose that there exists a constant* $\gamma_3 > 0$ *such that*

$$\Phi(s, \sigma_{sl}(s), z_{sl}(s)) \geq \gamma_3 \qquad \textit{for every } s \in [\hat{t}, \bar{t}]. \tag{3.3.36}$$

Then $(\sigma_\varepsilon, z_\varepsilon)$ *converges uniformly to* (σ_{sl}, z_{sl}) *as* $\varepsilon \to 0^+$ *on compact subsets of* $(\hat{t}, \bar{t}]$.

Proof. Let δ, γ_2, γ_1, $\hat{t}_\varepsilon$, and a_ε be given by (3.3.24), (3.3.31), (3.3.18), and (3.3.26), respectively. We put $t^* = \liminf_{\varepsilon\to 0^+} a_\varepsilon$, and we apply Lemma 3.17 with $\tilde{t} = \hat{t}$, $\tilde{t}_\varepsilon = \hat{t}_\varepsilon$, and $b_\varepsilon^\eta = a_\varepsilon$; we have that $t^* > \hat{t}$, and, by part c) of the Lemma, we may assume that there exists a nonnegative function $\omega(t)$ such that, for every $\eta > 0$, $\frac{\varrho_\varepsilon(t)}{\varepsilon}$ w^*-converges in $L^\infty((\hat{t}+\eta, t^*))$ to $\omega(t)$.

We write equation (3.2.18) in the form

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}(t) = \omega_1^\varepsilon(t, \sigma(t), z(t)) \\ \dot{z}(t) = \omega_2^\varepsilon(t, \sigma(t), z(t)), \end{cases}$$

where

$$\omega_1^\varepsilon(t, \sigma(t), z(t)) := \tfrac{\varrho_\varepsilon(t)}{\varepsilon} h_1(\sigma(t), z(t)) \tag{3.3.37}$$

$$\omega_2^\varepsilon(t, \sigma(t), z(t)) := \tfrac{\varrho_\varepsilon(t)}{\varepsilon} h_2(\sigma(t), z(t))\,; \tag{3.3.38}$$

here $h_1(\sigma, z)$ and $h_2(\sigma, z)$ denote two C_c^1 functions coinciding with $\mathbb{C}\nu_{K(z)}(\pi_{K(z)}(\sigma))$, and $\mathrm{tr}(\sigma)\mathrm{tr}(\nu_{K(z)}(\pi_{K(z)}(\sigma)))$, respectively, in $B_\delta(\hat{\sigma}, \hat{z}) \setminus \mathrm{int}\, K$. Corollary 1.16 now provides the uniform convergence of the solutions of (3.2.18) to the solution of the problem

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}(t) = \omega(t) h_1(\sigma(t), z(t)) \\ \dot{z}(t) = \omega(t) h_2(\sigma(t), z(t)), \end{cases} \tag{3.3.39}$$

with the required Cauchy data, on the compact subintervals of $(\hat{t}, t^*]$.

Now, Lemma 3.17, part c), implies that $(\sigma(t), z(t)) \in K$ for every $t \in (\hat{t}, t^*]$, while Lemma 3.15 entails that, for every $t \in (\hat{t}, t^*]$, the points $(\sigma_\varepsilon(t), z_\varepsilon(t))$ do not belong to K when ε is sufficiently small; this proves that $(\sigma(t), z(t)) \in \partial K$ for every $t \in (\hat{t}, t^*]$. Thus, for every $t \in (\hat{t}, t^*]$, the functions $h_1(\sigma(t), z(t))$ and $h_2(\sigma(t), z(t))$ coincide with $\mathbb{C}\nu_{K(z)}(\sigma)$ and $\mathrm{tr}(\sigma)\mathrm{tr}(\nu_{K(z)}(\sigma))$, respectively. Since $(\sigma(t), z(t)) \in \partial K$, we must have, for every $t \in (\hat{t}, t^*]$

$$0 = \nu_K((\sigma(t), z(t))) \cdot (\dot{\sigma}(t), \dot{z}(t));$$

this in turn, recalling (3.2.6), is equivalent to

$$0 = (z\, \nu_{K(z)}(\sigma), -\sigma \cdot \nu_{K(z)}(\sigma)) \cdot (\dot{\sigma}(t), \dot{z}(t)).$$

Then (3.3.39), (3.2.24), and (3.2.25) imply that

$$0 = \omega(t)\Psi(\sigma(t), z(t)) + \Phi(t, \sigma(t), z(t)). \tag{3.3.40}$$

Therefore (3.3.39) coincides with (3.3.1). We conclude that the solutions of (3.2.18) converge uniformly on compact subintervals of $(\hat{t}, t^*]$ to the solution of the equation (3.3.1) with Cauchy data $(\hat{\sigma}, \hat{z})$ at $\hat{t}$, and by uniqueness, the limit is exactly $(\sigma_{sl}(t), z_{sl}(t))$.

Now, let $t^\dagger$ the maximal time such that $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly to (σ_{sl}, z_{sl}) as $\varepsilon \to 0^+$ on compact subintervals of $(\hat{t}, t^\dagger)$; to conclude the proof, we have to show that $t^\dagger > \bar{t}$. Let us argue by contradiction, supposing $t^\dagger \le \bar{t}$. Define $(\sigma^\dagger, z^\dagger) := (\sigma_{sl}(t^\dagger), z_{sl}(t^\dagger))$ and observe that, by the hypotheses, there exist two constants $\eta > 0$ and $\gamma > 0$ such that, for every $(t, \sigma, z) \in [t^\dagger - \eta, t^\dagger + \eta] \times B_\eta(\sigma^\dagger, z^\dagger)$, one has $\Psi(\sigma, z) < -\gamma$ and $\Phi(t, \sigma, z) > \gamma$. We define $c(\frac{\eta}{2}, \gamma)$ as the infimum in $B_{\frac{\eta}{2}}(\sigma^\dagger, z^\dagger)$ of $C(\sigma, \frac{\eta}{2}, \gamma)$, where the latter is the constant defined in Lemma 3.17. Now we may fix $t^\dagger - \frac{\eta}{2} < t_1^\dagger < t_2^\dagger < t^\dagger < t_3^\dagger < t_1^\dagger + c(\frac{\eta}{2}, \gamma)$ in a way that $(\sigma_{sl}(t_1^\dagger), z_{sl}(t_1^\dagger)) \in B_{\frac{\eta}{2}}(\sigma^\dagger, z^\dagger)$ and we shall have that for every $(t, \sigma, z) \in [t_1^\dagger - \frac{\eta}{2}, t_1^\dagger + \frac{\eta}{2}] \times B_{\frac{\eta}{2}}(\sigma_{sl}(t_1^\dagger), z_{sl}(t_1^\dagger))$,

$$\Psi(\sigma, z) < -\gamma \quad \text{and} \quad \Phi(t, \sigma, z) > \gamma\,. \tag{3.3.41}$$

By Lemma 3.17, applied with $\tilde{t} = \tilde{t}_\varepsilon = t_1^\dagger$, we have that there exists $L > 0$ such that for ε sufficiently small $\frac{\varrho_\varepsilon(t)}{\varepsilon} \le \frac{L}{\gamma}$ for every $t \in [t_2^\dagger, t_3^\dagger]$. By Lemma 3.15, applied with $\hat{t} = \hat{t}_\varepsilon = t_1^\dagger$, and $a_\varepsilon = b_\varepsilon^{\frac{\eta}{2}}$ we get that

$$\frac{\varrho_\varepsilon(t)}{\varepsilon} \ge \frac{\gamma}{3\lambda(\sigma_{sl}(t_1^\dagger))} \qquad \text{for every } t \in [t_2^\dagger, t_3^\dagger], \tag{3.3.42}$$

when ε is sufficiently small; here $\lambda(\sigma)$ is defined by (3.2.28). We repeat the arguments of the previous step of the proof, and we also notice that we are in position to apply Theorem 1.15 in place of Corollary 1.16, to get that the solutions of (3.2.18) converges uniformly in the interval $[t_2^\dagger, t_3^\dagger]$ to the solution of the problem (3.3.1) with Cauchy data $(\sigma(t_2^\dagger), z(t_2^\dagger)) = (\sigma_{sl}(t_2^\dagger), z_{sl}(t_2^\dagger))$, that is, by uniqueness, to $(\sigma_{sl}(t), z_{sl}(t))$. This contradicts the maximality of $t^\dagger$. $\square$

Remark 3.19. A slight adaptation of the proof, taking into account Remark 3.16, easily shows that in the particular case $\hat{t} = t_1$ the conclusion of the Theorem holds on the whole closed interval $[t_1, \bar{t}]$.

The previous theorem shows that, if one has

$$\Phi(t, \sigma_{sl}(t), z_{sl}(t)) > 0 \qquad \text{for every } \hat{t} \le t < t_2, \tag{3.3.43}$$

then $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly to (σ_{sl}, z_{sl}) as $\varepsilon \to 0^+$ on compact subintervals of $(\hat{t}, t_2)$. On the contrary, if

$$\Phi(\bar{t}, \sigma_{sl}(\bar{t}), z_{sl}(\bar{t})) = 0 \tag{3.3.44}$$

for some $\hat{t} < \bar{t} < t_2$, then the elastic behavior may re-appear starting from the point $(\bar{\sigma}, \bar{z}) := (\sigma_{sl}(\bar{t}), z_{sl}(\bar{t})) \in \partial K$, as we are going to discuss in the next subsection.

In the last section of the chapter we will consider the case when (3.3.43) holds, and $t_2 < +\infty$; we will show that a transition from the slow to the fast dynamics occurs at time t_2 when (3.3.7) and (3.3.12) hold with strict inequality.

3.3.3 Elastic regime

Another possibility for a continuous evolution is having an elastic regime, where the internal variable is constant and the stress evolves trivially following the linearized elasticity equation, without production of plastic flow. It is obvious that this situation occurs if we a priori know that at a certain time the stress is in the interior of the elastic domain. Here instead we focus on the case where *the stress is on the yield surface*, after a previous branch of elastic regime, or after following the slow dynamics equations, or after a jump along the fast dynamics trajectory (this situation will be discussed in the next section). As the discussion of the previous subsection has clarified, negativeness of the indicator Ψ leads to a continuous evolution, while positiveness of the indicator Φ is responsible for the production of plastic flow. Throughout this subsection, while keeping the hypothesis on Ψ we will deal with the case $\Phi \leq 0$. If the strict inequality holds, it is really easy to prove that the system is switching to the elastic regime, while if $\Phi = 0$ we need to add some suitable assumptions to prove this result. However, also this case is really interesting because it is what happens for instance when (3.3.44) holds.

To be definite, $\bar{t}$ denotes a time such that there exist a left continuous function $t \mapsto (\sigma(t), z(t))$ defined on $[0, \bar{t})$ with values in $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ and an element $(\bar{\sigma}, \bar{z})$ of $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ satisfying the following properties:

$$(\sigma_\varepsilon(t), z_\varepsilon(t)) \to (\sigma(t), z(t)) \quad \text{for a.e. } t \in [0, \hat{t})\,, \tag{3.3.45}$$

$$\text{there exists } \bar{t}_\varepsilon \to \bar{t} \text{ such that } (\sigma_\varepsilon(\bar{t}_\varepsilon), z_\varepsilon(\bar{t}_\varepsilon)) \to (\bar{\sigma}, \bar{z})\,, \tag{3.3.46}$$

$$(\bar{\sigma}, \bar{z}) \in \partial K \quad \text{and} \quad \hat{z} > 0\,, \tag{3.3.47}$$

$$\Phi(\bar{t}, \bar{\sigma}, \bar{z}) \leq 0 \text{ and } \Psi(\bar{\sigma}, \bar{z}) < 0\,. \tag{3.3.48}$$

Denoting with $(\sigma_{sl}(t; \bar{t}), z_{sl}(t; \bar{t}))$ the unique solution of (3.3.1) issuing from $(\bar{\sigma}, \bar{z})$ at time $\bar{t}$, we will also assume that there exists a sequence $t_n \to \bar{t}$ such that

$$\Phi(t_n, \sigma_{sl}(t_n; \bar{t}), z_{sl}(t_n; \bar{t})) < 0 \tag{3.3.49}$$

and that there exists $\eta > 0$ such that, for every $(t, s, \sigma, z) \in (\bar{t}, \bar{t}+\eta) \times (0, \eta) \times (B_\eta(\bar{\sigma}, \bar{z})) \cap \partial K$ satisfying $\Phi(t, \sigma, z) \leq 0$,

$$(\sigma + \mathbb{C}(\xi(t+s) - \xi(t)), z) \in \operatorname{int} K\,. \tag{3.3.50}$$

It is obvious that if the strict inequality holds in (3.3.48), (3.3.49) and (3.3.50) are trivially satisfied. About the meaning of these two additional conditions, we observe that, according to the discussion in Remark 3.5, (3.3.49) has the role of preventing the system from following the slow dynamics equation, while (3.3.50) is a suitable enforcement of the trivial necessary condition for having an elastic regime while keeping the stress constraint.

Remark 3.20. When ξ is at least C^2 regular and $\Phi(\bar{t}, \bar{\sigma}, \bar{z}) = 0$ the inequality

$$\mathbb{C}\ddot{\xi}(\bar{t}) \cdot \nu_{K(\bar{z})}(\bar{\sigma}) + \mathbb{C}\dot{\xi}(\bar{t}) \cdot [\nabla_\sigma \nu_{K(\bar{z})}(\bar{\sigma})]\, \mathbb{C}\dot{\xi}(\bar{t}) < 0 \tag{3.3.51}$$

implies both (3.3.49)) and (3.3.50). It follows from the definition of Φ, from (3.3.44), and from (3.2.6), that the vector $\mathbb{C}\dot{\xi}(\bar{t})$ is tangent to $\partial K(\bar{z})$ at $\bar{\sigma}$, hence $\mathbb{C}\dot{\xi}(\bar{t}) \cdot [\nabla_\sigma \nu_{K(\bar{z})}(\bar{\sigma})]\, \mathbb{C}\dot{\xi}(\bar{t})$ is exactly the second fundamental form of $\partial K(\bar{z})$ at $\bar{\sigma}$, applied to the tangent vector $\mathbb{C}\dot{\xi}(\bar{t})$. We omit the proof of (3.3.51), which is based on elementary facts in differential geometry; the interested reader may refer to [14, Remark 3.19] and [47, Remark 3.12].

The next theorem shows that the hypotheses we made actually guarantee that the system is going to follow the elastic regime in a right neighborhood of $\bar{t}$.

Theorem 3.21. *Assume* (1.3.1)-(1.3.4)*,* (2.3.3)*, and* (3.2.2)*, and define* Φ *and* Ψ *as in* (3.2.24)*, and* (3.2.25)*, respectively. Let* $\bar{t}$ *and* $(\bar{\sigma}, \bar{\zeta})$ *satisfy* (3.3.45)-(3.3.48)*, and assume that* (3.3.49) *and* (3.3.50) *hold. Let*

$$(\sigma_{el}(t), z_{el}(t)) := (\bar{\sigma} + \mathbb{C}(\xi(t) - \xi(\bar{t})), \bar{z})$$

and

$$\tau := \sup\{t > \bar{t} \,|\, (\sigma_{el}(s), z_{el}(s)) \in \operatorname{int} K \textit{ for every } s \in (\bar{t}, t)\}.$$

Then $(\sigma_\varepsilon, z_\varepsilon)$ *converge uniformly to* (σ_{el}, z_{el}) *as* $\varepsilon \to 0^+$ *on compact subsets of* $[\bar{t}, \tau)$*.*

Proof. Observe that, if 3.3.50 holds, τ is strictly larger than t and $\tau - \bar{t} \geq \eta$, where η is given by 3.3.50. As in Theorem 3.18 we denote with $h_1(\sigma, z)$ and $h_2(\sigma, z)$ denote two C^1_c functions coinciding with $\mathbb{C}\nu_{K(z)}(\pi_{K(z)}(\sigma))$, and $\mathrm{tr}(\sigma)\mathrm{tr}(\nu_{K(z)}(\pi_{K(z)}(\sigma)))$, respectively, in $B_eta(\hat{\sigma}, \hat{z}) \setminus \operatorname{int} K$ Since $\Psi(\bar{\sigma}, \bar{z}) < 0$, we can apply Lemma 3.17 as in the proof of Theorem 3.18 to get that there there exists $\vartheta > 0$ such that the solutions of (3.2.18) converge, up to a subsequence, to a function $(\sigma(t), z(t)$ with values on K solving the problem

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}(t) = \omega(t) h_1(\sigma(t), z(t)) \\ \dot{z}(t) = \omega(t) h_2(\sigma(t), z(t)), \end{cases} \tag{3.3.52}$$

with the Cauchy data $(\bar{\sigma}, \bar{\zeta})$, for some suitable nonnegative bounded function $\omega(t)$, on the compact subintervals of $(\bar{t}, \bar{t} + \vartheta]$; thus we may fix $\delta < \eta$ such that $(\sigma(t), z(t)) \in B_\eta(\bar{\sigma}, \bar{z})$ for every $t \in [\bar{t}, \bar{t} + \delta]$. By (3.3.48) we can clearly assume that

$$\Psi(\sigma(t), z(t)) < 0 \tag{3.3.53}$$

for every $t \in [\bar{t}, \bar{t} + \delta]$.

Now, we first prove that the open set $A_{int} := \{t \in (\bar{t}, \bar{t} + \delta) : (\sigma(t), \zeta(t)) \in \operatorname{int} K\}$ must be nonempty. If not, $(\sigma(t), z(t)) \in \partial K$ for every $t \in [\bar{t}, \bar{t} + \delta]$. Then we can prove exactly as in Theorem 3.18 that (3.3.40) is satisfied for every $t \in [\bar{t}, \bar{t} + \delta]$; by uniqueness, this implies $(\sigma(t), z(t)) = (\sigma_{sl}(t; \bar{t}), z_{sl}(t; \bar{t}))$, but then (3.3.49) and (3.3.53) contradicts the nonnegativeness of $\omega(t)$. It is easily seen, as $(\sigma_\varepsilon, z_\varepsilon)$ converge uniformly to (σ, z) on the compact subintervals of $(\bar{t}, \bar{t} + \delta]$ that

$$\omega(t) \equiv 0 \quad \text{for every } t \in A_{int}\,. \tag{3.3.54}$$

We now show that A_{int} is connected. Indeed, let $\hat{t} \in A_{int}$ and let $(\hat{t}_1, \hat{t}_2)$ the connected component containing $\hat{t}$. In $(\hat{t}_1, \hat{t}_2)$, we have, by (3.3.54), that $(\sigma(t), z(t)) = (\sigma(\hat{t}_1) + \mathbb{C}(\xi(t) - \xi(\hat{t}_1), z(\hat{t}_1))$. Notice that, as $(\sigma(\hat{t}_1), z(\hat{t}_1)) \in \partial K$ by maximality, we have that $\Phi(\hat{t}_1, \sigma(\hat{t}_1), \zeta(\hat{t}_1)) \leq 0$, if not the trajectory goes outside of K. Then, 3.3.50 implies that $\hat{t}_2 = \bar{t} + \delta$, thus proving that A_{int} is connected, that is $A_{int} = (\hat{t}_1, \bar{t} + \delta)$. Now, if $\hat{t}_1 > \bar{t}$, for every $t \in [\bar{t}, \bar{t}_1]$ we must have $(\sigma(t), z(t)) \in \partial K$. In this case, again (3.3.40), (3.3.53) and (3.3.49) give us a contradiction. Thus the statement of the theorem is proved in $(\bar{t}, \bar{t} + \delta]$; as, for every $t < \tau$, $(\sigma_{el}(t), z_{el}(t)) \in \operatorname{int} K$, it is easily seen that the maximal interval such that the theorem holds is $(\bar{t}, \tau)$. $\square$

The statements of Theorems 3.18 and 3.21 can be efficiently joined together in the next one. Here we have in mind the case where, at a certain time $\hat{t}$ the stress reaches a point of the yield surface where the indicator Ψ is negative, so that the evolution is continuous, and we assume that Φ is strictly positive, since the case $\Phi \leq 0$ is covered by the previous Theorem. Then, in a right neighborhood of $\hat{t}$, either the evolution follows the slow dynamics, or a continuous combination between the slow dynamics and the elastic regime. To prepare the statement, we introduce some notation.

Definition 3.22. For every $(\hat{\sigma}, \hat{z}) \in \partial K$ satisfying $\Psi(\hat{\sigma}, \hat{z}) \neq 0$, and every $\hat{t} > 0$ we define $(\sigma_{sl}, z_{sl})(t; \hat{\sigma}, \hat{z}, \hat{t})$ as the unique solution to (3.3.1) starting from the point $(\hat{\sigma}, \hat{z})$ at time $\hat{t}$, and t_{sl}^{max} as the supremum of the maximal open interval of existence for $(\sigma_{sl}, z_{sl})(t; \hat{\sigma}, \hat{z}, \hat{t})$.

Theorem 3.23. *Assume* (1.3.1)-(1.3.4), (2.3.3), *and* (3.2.2), *and define* Φ *and* Ψ *as in* (3.2.24), *and* (3.2.25), *respectively. Let* $\hat{t} > 0$ *and* $(\hat{\sigma}, \hat{z}) \in \partial K$ *be as in Theorem 3.18 and define* $(\sigma_{sl}, z_{sl})(t; \hat{\sigma}, \hat{z}, \hat{t})$, *and* $t_{sl}^{max} > \hat{t}$ *as in Definition 3.22. Let*

$$\bar{t} := \inf\{t \in (\hat{t}, t_{sl}^{max}) : \Phi(t, (\sigma_{sl}, z_{sl})(t; \hat{\sigma}, \hat{z}, \hat{t})) \leq 0\}\,.$$

Assume that (3.3.49) *and* (3.3.50) *hold at* $(\bar{\sigma}, \bar{z}) := (\sigma_{sl}(\bar{t}), z_{sl}(\bar{t}))$. *Let*

$$(\sigma_{el}(t), z_{el}(t)) := (\bar{\sigma} + \mathbb{C}(\xi(t) - \xi(\bar{t})), \bar{z})$$

and

$$\tau := \sup\{t > \bar{t}\,|\,(\sigma_{el}(s), z_{el}(s)) \in \operatorname{int} K \textit{ for every } s \in (\bar{t}, t)\}.$$

Then $(\sigma_\varepsilon, z_\varepsilon)$ *converges uniformly on compact subsets of* $(\hat{t}, \tau)$ *to the function* (σ, z) *defined by*

$$(\sigma(t), z(t)) := \begin{cases} (\sigma_{sl}, z_{sl})(t; \hat{\sigma}, \hat{z}, \hat{t}) & \textit{for } \hat{t} < t \leq \bar{t}, \\ (\sigma_{el}(t), z_{el}(t)) & \textit{for } \bar{t} \leq t < \tau. \end{cases} \tag{3.3.55}$$

Proof. Since $\hat{t}$, $\hat{\sigma}$, and $\hat{z}$ are fixed and there is no risk of ambiguity, throughout the proof we will write $(\sigma_{sl}(t), z_{sl}(t))$ in place of $(\sigma_{sl}, z_{sl})(t; \hat{\sigma}, \hat{z}, \hat{t})$. Let $\hat{\tau}$ be the maximal time such that $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly to (σ, z) on compact subintervals of $(\hat{t}, \hat{\tau})$; we have to show that $\hat{\tau} = \tau$. By Theorem 3.18, it follows that $\hat{\tau} \geq \bar{t}$. As in Theorem 3.21, it is easy to see that $\hat{\tau} = \tau$ when $\hat{\tau} > \bar{t}$, therefore we have only to exclude $\hat{\tau} = \bar{t}$.

In this case, there exist two constants $\eta > 0$ and $\gamma > 0$ such that, for every $(\sigma, z) \in B_\eta(\bar{\sigma}, \bar{z})$, one has $\Psi(\sigma, z) < -\gamma$. We define $c(\frac{\eta}{2}, \gamma)$ as the infimum in $B_{\frac{\eta}{2}}(\bar{\sigma}, \bar{z})$ of $C(\sigma, \frac{\eta}{2}, \gamma)$, where the latter is the constant defined in Lemma 3.17. Now we may fix $\bar{t} - \frac{\eta}{2} < \bar{t}_1 < \bar{t}_2 < \bar{t} < \bar{t}_3 < \bar{t}_1 + c(\frac{\eta}{2}, \gamma)$ in a way that $(\sigma_{sl}(\bar{t}_1), z_{sl}(\bar{t}_1)) \in B_{\frac{\eta}{2}}(\bar{\sigma}, \bar{z})$ and we shall have that for every $(\sigma, z) \in B_{\frac{\eta}{2}}(\sigma_{sl}(\bar{t}_1), z_{sl}(\bar{t}_1))$,

$$\Psi(\sigma, z) < -\gamma.$$

By Lemma 3.17, applied with $\tilde{t} = \tilde{t}_\varepsilon = \bar{t}_1$, we have that there exists $L > 0$ such that for ε sufficiently small $\frac{\varrho_\varepsilon(t)}{\varepsilon} \leq \frac{L}{\gamma}$ for every $t \in [\bar{t}_2, \bar{t}_3]$, thus we may assume $\frac{\varrho_\varepsilon(t)}{\varepsilon}$ w^*-converges in $L^\infty((\bar{t}_2, \bar{t}_3))$ to some nonnegative function $\omega(t)$. By (3.3.37), (3.3.38), and Theorem 1.15 the sequence $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly in $[\bar{t}_2, \bar{t}_3]$ to a continuous function $(\tilde{\sigma}, \tilde{z})$. Theorem 3.18 gives $(\tilde{\sigma}, \tilde{z}) = (\sigma_{sl}, z_{sl})$ in $[\bar{t}_2, \bar{t})$, while Theorem 3.21 gives $(\tilde{\sigma}, \tilde{z}) = (\sigma_{el}, z_{el})$ in $[\bar{t}, \bar{t}_3]$, thus $(\tilde{\sigma}, \tilde{z}) = (\sigma, z)$ in $[\bar{t}_2, \bar{t}_3]$. This contradicts the maximality of $\hat{\tau}$, when $\hat{\tau} = \bar{t}$. $\square$

3.4 Softening with discontinuities

3.4.1 The equation of the fast dynamics

The goal of this section is a qualitative study of the equation

$$\begin{cases} \dot{\sigma}_f(s) = \mathbb{C}(\pi_{K(z_f(s))}(\sigma_f(s)) - \sigma_f(s)) \\ \dot{z}_f(s) = \mathrm{tr}(\sigma_f(s))\,\mathrm{tr}\big(\sigma_f(s) - \pi_{K(z_f(s))}(\sigma_f(s))\big); \end{cases} \tag{3.4.1}$$

this is called the fast dynamics equation and appears, as we shall see, as limit of a rescaled version of (3.2.18) near a discontinuity point of a viscosity solution.

Under suitable conditions, we shall see the viscosity solution will jump between the two endpoints of a heteroclinic orbit of (3.4.1), whose existence, together with other properties, is the object of this subsection.

In order to prove the main theorem of this subsection, we need a preliminary lemma, showing that the internal variable is constant along the unique solution of (3.4.1), with an initial condition $(\bar{\sigma}, \bar{z})$ satisfying

$$(\bar{\sigma}, \bar{z}) \notin K \quad \text{and} \quad \mathrm{tr}\big(\nu_{K(\bar{z})}(\pi_{K(\bar{z})}(\bar{\sigma}))\big) = 0. \tag{3.4.2}$$

We preliminarly observe that taking an initial condition outside K easily implies that we can never reach K in finite time, as the set K is made of critical points of the autonomous equation (3.4.1). Through the decomposition (3.2.9) we identify $\mathbb{M}^{N\times N}_{sym}$ with $\mathbb{R} \times \mathbb{M}^{N\times N}_{D}$; in particular $\sigma_f(s)$ is identified with the pair $(x_f(s), y_f(s))$ of its spherical and deviatoric parts. Introducing the function ϱ defined by (3.2.3), which is positive by the previous remark, we may rewrite equation (3.4.1) in the form

$$\begin{cases} \dot{x}_f(s) = -\kappa\sqrt{N}\,\varrho(x_f(s), y_f(s), z_f(s))\,\mathrm{tr}\Big(\nu_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big)\Big), \\ \dot{y}_f(s) = -2\mu\,\varrho(x_f(s), y_f(s), z_f(s))\,n^D_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big), \\ \dot{z}_f(s) = \sqrt{N}x_f(s)\,\varrho(x_f(s), y_f(s), z_f(s))\mathrm{tr}\Big(\nu_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big)\Big). \end{cases} \tag{3.4.3}$$

Here κ and μ are defined in (2.3.3) and $n^D_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big)$ is the deviatoric part of $\nu_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big)$.

Lemma 3.24. *Let* $(\bar{\sigma}, \bar{z}) \in [\mathbb{M}^{N\times N}_{sym} \times (0, +\infty)] \setminus K$ *satisfying* (3.4.2)*, and let* $\bar{x}$ *and* $\bar{y}$ *the spherical and the deviatoric part of* $\bar{\sigma}$*, respectively. Then, for every* $t \in \mathbb{R}$*, the unique solution to equation* (3.4.3) *with Cauchy data* $(x_f(0), y_f(0), z_f(0)) = (\bar{x}, \bar{y}, \bar{z})$ *is given by*

$$(x_f(s), y_f(s), z_f(s)) = (\bar{x}, y(s), \bar{z})$$

where $y(s)$ *solves the equation*

$$\dot{y}(s) = -2\mu\,\varrho(\bar{x}, y(s), \bar{z})N^D_{K(\bar{z})}(\pi_{K(\bar{z})}(\bar{x}, \bar{y})) \tag{3.4.4}$$

with Cauchy condition $y(0) = \bar{y}$*.*

Proof. Let $y(s)$ be the unique solution to (3.4.4) with Cauchy condition $y(0) = \bar{y}$. Then, for every $s' > 0$

$$
\begin{aligned}
(\bar{x}, y(s')) &= \Big(\bar{x}, \bar{y} - 2\mu \int_0^{s'} \varrho(\bar{x}, y(s), \bar{z}) n^D_{K(\bar{z})}(\pi_{K(\bar{z})}(\bar{x}, \bar{y}))\, ds\Big) = \\
&= \Big(\bar{x}, \bar{y} - 2\mu\, n^D_{K(\bar{z})}(\pi_{K(\bar{z})}(\bar{x}, \bar{y})) \int_0^{s'} \varrho(\bar{x}, y(s), \bar{z})\, ds\Big) = \\
&= \Big((\bar{x}, \bar{y}) - 2\mu\, \nu_{K(\bar{z})}(\pi_{K(\bar{z})}(\bar{x}, \bar{y})) \int_0^{s'} \varrho(\bar{x}, y(s), \bar{z})\, ds\Big),
\end{aligned}
$$

Therefore $\pi_{K(\bar{z})}(\bar{x}, y(s')) = \pi_{K(\bar{z})}(\bar{x}, \bar{y})$, provided $(\bar{x}, y(s'), \bar{z}) \notin K$; this allows us to check that $(\bar{x}, y(s), \bar{z})$ solves (3.4.3), at least for small $|s|$. The conclusion for every s follows, as solutions to (3.4.3) can never reach K in finite time. $\square$

Now we are able to prove the existence of a heteroclinic orbit of (3.4.1) starting from a point $(\hat{\sigma}, \hat{z}) \in \partial K$ under suitable hypotheses on the slow-fast indicator Ψ.

Theorem 3.25. *Assume that* (1.3.1)-(1.3.4)*,* (2.3.3)*,* (3.2.2)*, and* (3.2.10)-(3.2.11) *are satisfied. Let* Φ*,* Ψ *be as in* (3.2.24) *and* (3.2.25)*, respectively. Let* $(\hat{\sigma}, \hat{z}) \in \partial K$ *and suppose that*

$$\Psi(\hat{\sigma}, \hat{z}) > 0 \tag{3.4.5}$$

or

$$\Psi(\hat{\sigma}, \hat{z}) = 0 \quad \textit{and} \quad \nabla\Psi(\hat{\sigma}, \hat{z}) \cdot \Big(\frac{-\mathbb{C}\, \nu_{K(\hat{z})}(\hat{\sigma})}{\operatorname{tr}(\hat{\sigma}) \operatorname{tr}(\nu_{K(\hat{z})}(\hat{\sigma}))}, 1\Big) < 0. \tag{3.4.6}$$

Then equation (3.4.1) *has a unique solution* $(\hat{\sigma}_f(s), \hat{z}_f(s))$ *(up to time-translations) satisfying*

$$\lim_{s \to -\infty} (\hat{\sigma}_f(s), \hat{z}_f(s)) = (\hat{\sigma}, \hat{z}). \tag{3.4.7}$$

Moreover, the limit

$$(\sigma_\infty, z_\infty) := \lim_{s \to +\infty} (\hat{\sigma}_f(s), \hat{z}_f(s)) \tag{3.4.8}$$

exists and satisfies the following conditions

$$(\sigma_\infty, z_\infty) \in \partial K,\ z_\infty > 0, \tag{3.4.9}$$

$$\Psi(\sigma_\infty, z_\infty) \le 0, \tag{3.4.10}$$

$$\operatorname{tr}(\sigma_\infty) < 0,\ \operatorname{tr}(\nu_{K(z_\infty)}(\sigma_\infty)) > 0. \tag{3.4.11}$$

Proof. We first observe that, by (3.2.2), (3.2.8), and by (3.2.25), both (3.4.5) and (3.4.6) imply that

$$\operatorname{tr}(\hat{\sigma}) < 0,\ \operatorname{tr}(\nu_{K(\hat{z})}(\hat{\sigma})) > 0. \tag{3.4.12}$$

Moreover, due to our regularity assumptions on K we may assume that in a suitably small neighborhood of $(\hat{\sigma}, \hat{z})$ an oriented distance function r from ∂K is well defined; this is a C^1 extension of the function ϱ, defined by (3.2.3), to the interior of K. In view of the same assumptions, we may also locally define a minimal distance projection onto $\partial K(z)$, denoted by $\pi_{\partial K(z)}$, which obviously coincides with $\pi_{K(z)}$ outside of $K(z)$. For all these reasons, the Cauchy problem

$$\begin{cases} \sigma'(z) = \dfrac{-\mathbb{C}\,\nu_{K(z)}(\sigma(z))}{\operatorname{tr}(\sigma(z))\operatorname{tr}(\nu_{K(z)}(\pi_{\partial K(z)}(\sigma(z))))} \\ \sigma(\hat{z}) = \hat{\sigma} \end{cases} \tag{3.4.13}$$

is well defined and admits a unique solution, which shall be denoted by $\hat{\sigma}(z)$. For z sufficiently close to $\hat{z}$ we then have that $\operatorname{tr}(\hat{\sigma}(z)) < 0$ and $\operatorname{tr}(\nu_{K(z)}(\pi_{\partial K(z)}(\sigma(z))) > 0$; moreover for $z < \hat{z}$, sufficiently close to $\hat{z}$ we can prove that $(\hat{\sigma}(z), z) \notin K$. Indeed, as $r(\hat{\sigma}, \hat{z}) = 0$, it suffices to show that in a left open neighborhood of $\hat{z}$ one has

$$\tfrac{d}{dz} r(\hat{\sigma}(z), z) < 0. \tag{3.4.14}$$

By a direct computation, similar to that in (3.2.7), exploiting (3.4.13) and (3.2.25) we get:

$$\frac{d}{dz} r(\hat{\sigma}(z), z) = \frac{\Psi(\hat{\sigma}(z), z)}{\operatorname{tr}(\hat{\sigma}(z))\operatorname{tr}(\nu_{K(z)}(\pi_{\partial K(z)}(\hat{\sigma}(z))))}. \tag{3.4.15}$$

Then (3.4.5) implies that $\frac{d}{dz} r(\hat{\sigma}(z), z) < 0$ for $z = \hat{z}$, thus (3.4.14) follows; if (3.4.6) holds, deriving $\Psi(\hat{\sigma}(z), z)$, we get that

$$\tfrac{d}{dz} r(\hat{\sigma}(\hat{z}), \hat{z}) = 0 \text{ and } \tfrac{d^2}{dz^2} r(\hat{\sigma}(\hat{z}), \hat{z}) > 0,$$

which in its turn implies (3.4.14). We thus may fix $\bar{z} < \hat{z}$ such that, for every $z \in [\bar{z}, \hat{z})$, the following three hold

$$\varrho(\hat{\sigma}(z), z) > 0\,, \tag{3.4.16}$$

$$\operatorname{tr}(\hat{\sigma}(z)) < 0\,, \tag{3.4.17}$$

$$\operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z)))) > 0\,; \tag{3.4.18}$$

we may indeed replace $\pi_{\partial K}$ with π_K as $(\hat{\sigma}(z), z) \notin K$. Now, let $\hat{z}_f(s)$ the unique solution to the autonomous Cauchy problem

$$\begin{cases} \dot{z}_f(s) = \operatorname{tr}(\hat{\sigma}(z_f(s)))\operatorname{tr}(\hat{\sigma}(z_f(s)) - \pi_{K(z_f(s))}(\hat{\sigma}(z_f(s)))) \\ z_f(0) = \hat{z}; \end{cases}$$

by (3.4.16)-(3.4.18), we have that $\operatorname{tr}(\hat{\sigma}(z))\operatorname{tr}(\hat{\sigma}(z) - \pi_{K(z)}(\hat{\sigma}(z)) < 0$, for every $z \in [\bar{z}, \hat{z})$, with equality in $z = \hat{z}$; the theory of autonomous equations implies that $\hat{z}_f(s)$ is defined for every $s \leq 0$ and satisfies

$$\lim_{t\to-\infty} \hat{z}_f(s) = \hat{z}, \qquad \dot{\hat{z}}_f(s) < 0 \text{ for every } t \leq 0;$$

it now suffices to put $\hat{\sigma}_f(s) := \hat{\sigma}(\hat{z}_f(s))$, to get a solution to (3.4.1) satifying (3.4.7).

To prove uniqueness, let $(\sigma(s), z(s))$ a solution to (3.4.1) satisfying (3.4.7); (3.4.12) implies that there exists $\bar{s} \in \mathbb{R}$ such that, for every $s \leq \bar{s}$, one has $\dot{z}(s) < 0$. Then $z(s)$ is invertible in $(-\infty, \bar{t})$ with inverse $s(z)$. If we put $\sigma(z) := \sigma(s(z))$, it is easy to see that $\sigma(z)$ solves (3.4.13), thus coincides with $\hat{\sigma}(z)$; the theory of autonomous equation now implies that $(\sigma(s), z(s))$ and $(\hat{\sigma}_f(s), \hat{z}_f(s))$ may only differ by a time translation, thus the first part of the statement is proven.

Now, let $(-\infty, S)$ the maximal interval of definition for $(\hat\sigma_f(s), \hat z_f(s))$; observe that, as orbits can never reach K in finite time, $(\hat\sigma_f(s), \hat z_f(s))$ also solves (3.4.3). We split $\hat\sigma_f(s)$ in its spherical part $\hat x_f(s)$ and in its deviatoric part $\hat y_f(s)$ as in (3.2.9), and we observe that, by (3.4.3), the following equality holds:

$$\kappa \dot{\hat z}_f(s) = -\hat x_f(s)\dot{\hat x}_f(s). \tag{3.4.19}$$

Moreover, (3.4.12) implies that there exist $\bar s < S$ such that $\dot{\hat x}_f(s) < 0$ for every $s \le \bar s$. Let us prove that $\dot{\hat x}_f(s) < 0$ for every $s < S$. Indeed, if there exists $s_1 < S$ such that $\dot{\hat x}_f(s_1) = 0$, by (3.4.3), as $\varrho(\hat x_f(s_1), \hat y_f(s_1), z_f(s_1) > 0$, it must be

$$\operatorname{tr}(\nu_{K(\hat z_f(s_1))}(\hat\sigma_f(s_1))) = 0;$$

by Lemma 3.24, this implies $\hat x_f(s) = \hat x_f(s_1)$ for all s, a contradiction. In particular there exists

$$x_S := \lim_{s\to S} \hat x_f(s) < \hat x < 0, \tag{3.4.20}$$

where $\hat x$ is the spherical part of $\hat\sigma$. Now (3.4.19) implies that $\dot{\hat z}_f(s) < 0$ for every $s < S$. In particular there exists $z_S := \lim_{s\to S} \hat z_f(s) < \hat z$.

We now show that z_S is greater than zero. Indeed, by (3.4.3), the fact that $\dot{\hat x}_f(s) < 0$ for every $s < S$ is equivalent to the inequality

$$\operatorname{tr}(\nu_{K(\hat z_f(s))}(\pi_{K(\hat z_f(s))}(\hat x_f(s), \hat y_f(s)))) > 0 \text{ for every } s < S, \tag{3.4.21}$$

and also, as $\varrho(\hat x_f(s), \hat y_f(s), \hat z_f(s)) > 0$, to the inequality

$$\operatorname{tr}(\pi_{K(\hat z_f(s))}(\hat\sigma_f(s)) < \operatorname{tr}(\hat\sigma_f(s)) = \sqrt{N}\hat x_f(s) \text{ for every } s < S. \tag{3.4.22}$$

By (3.2.5) and (3.2.16), (3.4.21) is equivalent to

$$\operatorname{tr}(\pi_{K(\hat z_f(s))}(\hat\sigma_f(s))) + a\sqrt{N}\hat z_f(s) > 0,$$

where a is the positive constant defined by (3.2.10); thus, by (3.4.22) we conclude that

$$\hat x_f(s) + a\hat z_f(s) > 0 \text{ for every } s < S \tag{3.4.23}$$

which in the limit gives $z_S > \frac{|x_S|}{a} > 0$, as claimed.

We now show that $(\hat\sigma_f(s), \hat z_f(s))$ is bounded, which in particular implies that $S = +\infty$. Clearly, it suffices to prove that $\hat y_f(s)$ is bounded. We have, by (3.4.1), the negativeness of $\hat x_f(s)$ and (3.4.22), that

$$\begin{aligned}
\tfrac{d}{ds}\tfrac{|\hat y_f(s)|^2}{2} &= \hat y_f(s)\cdot\dot{\hat y}_f(s) = \\
&= 2\mu\,\hat y_f(s)\cdot\big(\pi_{K(\hat z_f(s))}(\hat\sigma_f(s)) - \hat\sigma_f(s)\big) = \\
&= 2\mu\,\hat\sigma_f(s)\cdot\big(\pi_{K(\hat z_f(s))}(\hat\sigma_f(s)) - \hat\sigma_f(s)\big) - \\
&\quad - 2\tfrac{\mu}{\sqrt N}\,\hat x_f(s)\operatorname{tr}\big(\pi_{K(\hat z_f(s))}(\hat\sigma_f(s)) - \hat\sigma_f(s)\big) \le \\
&\le 2\mu\,\hat\sigma_f(s)\cdot\big(\pi_{K(\hat z_f(s))}(\hat\sigma_f(s)) - \hat\sigma_f(s)\big) \le 0\,,
\end{aligned}$$

as a consequence of (3.2.8); this proves that $|\hat y_f(s)|^2$ is decreasing, thus $\hat y_f(s)$ is bounded.

Thus $S = +\infty$ and z_S is the limit of $\hat{z}_f(s)$ at $+\infty$, which shall be denoted with z_∞ from now on; by the previous discussion, we also have that $z_\infty > 0$, as required by (3.4.9). Now we prove that $\hat{\sigma}_f(s)$ has a limit at $+\infty$. To do that, we observe that $\hat{z}_f(s)$ is strictly decreasing, thus globally invertible; we thus express $\hat{\sigma}$ in function of z and we have to show that there exists $\lim\limits_{z\to z_\infty} \hat{\sigma}(z)$. We already know that $\hat{\sigma}(z)$ is bounded and that its derivative satisfies

$$\hat{\sigma}'(z) = \frac{-\mathbb{C}\,\nu_{K(z)}(\hat{\sigma}(z))}{\operatorname{tr}(\hat{\sigma}(z))\operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z))))} \tag{3.4.24}$$

thus the claim will follow once we get that

$$\liminf_{z\to z_\infty} \operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z)))) > 0. \tag{3.4.25}$$

Suppose that (3.4.25) is false; first, observe that in this case the liminf must be a limit, as a consequence of the boundedness of $\hat{\sigma}(z)$ and of Lemma 3.8. Therefore we will have, exploiting (3.2.25),

$$\lim_{z\to z_\infty} \Psi(\hat{\sigma}(z), z) = -2\mu. \tag{3.4.26}$$

Moreover, observe that by (3.2.5) and (3.2.15),

$$\lim_{z\to z_\infty} \operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z)))) = 0 \Leftrightarrow \lim_{z\to z_\infty} \tfrac{1}{z}[\tfrac{\operatorname{tr}(\pi_{K(z)}(\hat{\sigma}(z)))}{\sqrt{N}} + az] = 0; \tag{3.4.27}$$

on the other hand, clearly $\lim_{z\to z_\infty} \operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z)))) = 0$ implies that

$$\lim_{z\to z_\infty} [\operatorname{tr}(\pi_{K(z)}(\hat{\sigma}(z))) - \sqrt{N}\hat{x}(z)] = 0, \tag{3.4.28}$$

thus combining (3.4.27) and (3.4.28), we get that

$$\lim_{z\to z_\infty} \hat{x}(z) = -az_\infty. \tag{3.4.29}$$

Now, by (3.2.5), (3.2.15), (3.2.16), and (3.4.22), we have that

$$\begin{aligned}|\operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z))))| &\le |\tfrac{1}{z}[\tfrac{\operatorname{tr}(\pi_{K(z)}(\hat{\sigma}(z)))}{\sqrt{N}} + az]| \le \\ &\le \tfrac{1}{z_\infty}[\tfrac{\operatorname{tr}(\pi_{K(z)}(\hat{\sigma}(z)))}{\sqrt{N}} + az] \le \tfrac{1}{z_\infty}[\hat{x}(z) + az]\,.\end{aligned} \tag{3.4.30}$$

By (3.4.24), $\hat{x}'(z) = \frac{-\kappa}{\hat{x}(z)}$; this fact, together with (3.4.29) and (3.4.30), yields that

$$\limsup_{z\to z_\infty} \frac{|\operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z))))|}{z - z_\infty} \le \frac{1}{z_\infty}(\frac{\kappa}{az_\infty} + a). \tag{3.4.31}$$

Since (3.4.15) gives

$$\tfrac{d}{dz}\varrho(\hat{\sigma}(z), z) = \tfrac{\Psi(\hat{\sigma}(z),z)}{\operatorname{tr}(\hat{\sigma}(z))\operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z))))}, \tag{3.4.32}$$

recalling that $\operatorname{tr}(\nu_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z)))) > 0$ for all $z > z_\infty$, we conclude by (3.4.26), (3.4.29), and (3.4.31), that

$$\liminf_{z\to z_\infty} (z - z_\infty)\tfrac{d}{dz}\varrho(\hat{\sigma}(z), z) \ge \tfrac{2\mu z_\infty}{\sqrt{N}(\kappa + az_\infty)} > 0.$$

This finally implies that

$$\lim_{z\to z_\infty} \varrho(\hat{\sigma}(z), z) = -\infty,$$

contradicting the nonnegativeness of ϱ.

We thus have that there exists

$$\sigma_\infty := \lim_{z \to z_\infty} \hat{\sigma}(z)\,,$$

thus the proof of (3.4.8) is concluded. It is obvious that $(\sigma_\infty, z_\infty) \in \partial K$ as it must be a critical point of (3.4.1), thus (3.4.9) is proved. Concerning (3.4.11), it immediately follows from (3.4.25) and (3.4.20). Finally, as $\varrho(\hat{\sigma}(z), z) \geq 0$ for $z > z_\infty$, we must have $\frac{d}{dz}\varrho(\hat{\sigma}(z), z) \geq 0$ for $z = z_\infty$; observing that $\operatorname{tr}(\sigma_\infty)\operatorname{tr}(\nu_{K(z_\infty)}(\sigma_\infty)) < 0$ by (3.4.11), from (3.4.32) we immediately get (3.4.10). □

Remark 3.26. It is easy to show that, if an orbit of the system (3.4.1) has $(\hat{\sigma}, \hat{z})$ as an α-limit point, then $(\hat{\sigma}, \hat{z})$ is indeed its unique α-limit point; indeed, by the same arguments used in the proof of the previous theorem we can show that in this case $z(s)$ is strictly decreasing in a neighborhood of $-\infty$, thus it has $\hat{z}$ as a limit; the rest of the proof follows from (3.4.24), and Lemma 3.8.

We end up this analysis of equation (3.4.1) by remarking that there are some cases where we can improve (3.4.10), that is showing that $\Psi(\sigma_\infty, z_\infty) < 0$. We omit the details of the following example, which can be found in [14, Example 4.4].

Example 3.27. We suppose that for every $z \in (0, +\infty)$, $K(z)$ is an ellipsoid of the form

$$K(z) := \{\sigma \in \mathbb{M}^{N\times N}_{sym} | (x+z)^2 + \tfrac{|y|^2}{b^2} = z^2\}, \tag{3.4.33}$$

where x and y are as in (3.2.9). Notice that $K(1)$ satisfies (3.2.10)-(3.2.11) with $a = 1$. Suppose that, if κ and μ are as in (2.3.3) and b as in (3.4.33) the following condition holds:

$$\kappa N \geq \tfrac{2\mu}{b^2}\,. \tag{3.4.34}$$

Let $(\hat{\sigma}(z), z)$ be the heteroclinic trajectory joining the points $(\hat{\sigma}, \hat{z})$ and $(\sigma_\infty, z_\infty)$ whose existence is guaranteed by the previous theorem. Then, if (3.4.34) holds, by [14, Example 4.4] one has

$$\Psi(\sigma_\infty, z_\infty) < 0\,.$$

3.4.2 Convergence to the fast dynamics

We want now to investigate how equation (3.4.1) governs the jump of our viscosity solution when it reaches a point on the yield surface where the elastic-inelastic indicator is strictly positive (which means that we are in the inelastic regime), while the slow-fast indicator satisfies (3.4.5), or (3.4.6); we will see how a rescaled version of the solution converges to a heteroclinic solution of the auxiliary system (3.4.1), whose asymptotic values at $s = \pm\infty$ give the asymptotic values of the viscosity solution before and after the jump time. Both the cases where (3.4.5) and (3.4.6) hold will be treated simultaneously.

Throughout this part of the chapter, $\hat{t}$ denotes a time such that there exist a left continuous function $t \mapsto (\sigma(t), z(t))$ defined on $[0, \hat{t})$ with values in $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ and an element $(\hat\sigma,\hat z)$ of $\mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$ satisfying the following properties:

$$(\sigma_\varepsilon(t),z_\varepsilon(t))\to(\sigma(s),z(t))\quad\text{for a.e. } t\in[0,\hat t)\,, \tag{3.4.35}$$
$$(\sigma(t),z(t))\to(\hat\sigma,\hat z)\quad\text{as } t\to\hat t^-, \tag{3.4.36}$$
$$(\hat\sigma,\hat z)\in\partial K\quad\text{and}\quad\hat z>0, \tag{3.4.37}$$
$$\Psi(\hat\sigma,\hat z)\text{ satisfies (3.4.5) or (3.4.6)}, \tag{3.4.38}$$
$$\Phi(\hat t,\hat\sigma,\hat z)>0. \tag{3.4.39}$$

For instance, we can take $\hat t=t_1$ defined by (3.2.20), if (3.3.22) holds and $\Psi(\sigma_1,z_1)>0$, or $\hat t=t_2$ defined by (3.3.5), provided that (3.4.6) holds for $(\hat\sigma,\hat z)=(\sigma_2,z_2)$ defined in Proposition 3.10. In the latter case we have $\Psi(\sigma_2,z_2)=0$ and in general, by Remark 3.11, we have the weak inequality

$$\nabla\Psi(\sigma_2,z_2)\cdot\big(\tfrac{-\mathbb{C}\,\nu_{K(z_2)}(\sigma_2)}{\mathrm{tr}(\sigma_2)\,\mathrm{tr}(\nu_{K(z_2)}(\sigma_2))},1\big)\le 0;$$

thus, assuming (3.4.6), we are excluding the degenerate case when equality holds.

By (3.4.35) and (3.4.36) we also may fix a sequence $\hat t_\varepsilon\to\hat t$ such that

$$(\sigma_\varepsilon(\hat t_\varepsilon),z_\varepsilon(\hat t_\varepsilon))\to(\hat\sigma,\hat z); \tag{3.4.40}$$

Indeed, by (3.4.39), and Lemma 3.15 we can find another sequence, still denoted by $\hat t_\varepsilon$, which preserves (3.4.40), and satisfies in addition, for every $\varepsilon>0$,

$$\varrho(\sigma_\varepsilon(\hat t_\varepsilon),z_\varepsilon(\hat t_\varepsilon))>c\varepsilon, \tag{3.4.41}$$

where c is a positive constant independent of ε.

We finally recall, as we have already dicussed in Remark 3.4 and in Proposition 3.10, that in the case $\hat t=t_2$ the internal variable z is strictly decreasing in a left neighborhood of t_2, thus discontinuities can appear only in the softening regime.

We start by fixing an open neighborhood $U_{\delta_1}:=(\hat t-\delta_1,\hat t+\delta_1)\times B_{\delta_1}(\hat\sigma,\hat z)$ of $(\hat t,\hat\sigma,\hat z)$, in a way that (3.3.24) holds. If (3.4.5) holds, we may assume for a suitable choice of δ_1 there exists a positive constant γ_1 such that

$$\Psi(\sigma,z)\ge\gamma_1\qquad\text{for every } (\sigma,z)\in B_{\delta_1}(\hat\sigma,\hat z); \tag{3.4.42}$$

if instead (3.4.6) holds, we may assume that there exists a positive constant γ_4 such that

$$\nabla\Psi(\sigma,z)\cdot\big(\tfrac{-\mathbb{C}\,\nu_{K(z)}(\pi_{K(z)}(\sigma))}{\mathrm{tr}(\sigma)\,\mathrm{tr}(\nu_{K(z)}(\pi_{K(z)}(\sigma)))},1\big)\le-\gamma_4 \tag{3.4.43}$$

for every $(\sigma,z)\in B_{\delta_1}(\hat\sigma,\hat z)\setminus\mathrm{int}\,K$.

We now define the exit time from $B_{\delta_1}(\hat\sigma,\hat z)$

$$b^1_\varepsilon:=\inf\{t\in(\hat t_\varepsilon,\hat t_\varepsilon+\delta_1):\ (\sigma_\varepsilon(t),z_\varepsilon(t))\in\partial B_{\delta_1}(\hat\sigma,\hat z)\}; \tag{3.4.44}$$

by the previous assumptions for small ε we will trivially have $\hat t_\varepsilon<b^1_\varepsilon$. We then fix a positive decreasing sequence $\delta_k\searrow 0^+$, starting from δ_1, and consequently we define, for every $k\in\mathbb{N}$,

$$b^k_\varepsilon:=\sup\{t\in(\hat t_\varepsilon,b^1_\varepsilon):\ (\sigma_\varepsilon(t),z_\varepsilon(t))\in\partial B_{\delta_k}(\hat\sigma,\hat z)\}. \tag{3.4.45}$$

Next lemma, which will be crucial in the remainder of the section, shows that the exit times b_ε^k tend to $\hat{t}$ when ε goes to 0 and that the difference $b_\varepsilon^1 - b_\varepsilon^k$ is of order ε for fixed k.

Lemma 3.28. *Assume* (1.3.1)-(1.3.4), (2.3.3), *and* (3.2.2), *and let* Φ, Ψ *be as in* (3.2.24), *and* (3.2.25), *respectively. Let* $\hat{t} > 0$ *satisfy* (3.4.35)-(3.4.39). *Let* b_ε^1 *be given by* (3.4.44) *and* b_ε^k *be given for every* $k \in \mathbb{N}$, $k > 1$ *by* (3.4.45). *Then, for every* $k \in \mathbb{N}$:

a) $b_\varepsilon^k \to \hat{t}$ *as* $\varepsilon \to 0^+$;

b) $\sup_{\varepsilon>0} \frac{b_\varepsilon^1 - b_\varepsilon^k}{\varepsilon} \le c_k < +\infty$,

where c_k *is a constant depending on* k. *Moreover, for every* $k \in \mathbb{N}$, *there exists a constant* m_k *such that*

$$\varrho(\sigma_\varepsilon(b_\varepsilon^k), z_\varepsilon(b_\varepsilon^k)) > m_k. \tag{3.4.46}$$

Proof. As we already observed in the proof of Theorem 3.25, both (3.4.5) and (3.4.6) imply $\mathrm{tr}(\hat{\sigma}) < 0$; by (3.2.26) this means that

$$\mathrm{tr}(\hat{\sigma}) < -\tfrac{\min\{\kappa, 2\mu\}}{M_K\sqrt{N}},$$

where M_K is as in (1.3.4) and $\kappa, 2\mu$ as in (2.3.3). Provided we have chosen δ_1 suitably small, we may clearly assume throughout the proof that

$$\mathrm{tr}(\sigma_\varepsilon(t)) < -\tfrac{\min\{\kappa, 2\mu\}}{2M_K\sqrt{N}} \qquad \text{for every } t \in (\hat{t}_\varepsilon, b_\varepsilon^1); \tag{3.4.47}$$

Concerning part a) of the statement, it clearly suffices to show this is true for b_ε^1. As $\hat{t}_\varepsilon \to \hat{t}$ this will be proved once we get:

$$\limsup_{\varepsilon \to 0^+} (b_\varepsilon^1 - \hat{t}_\varepsilon) = 0. \tag{3.4.48}$$

By Lemma 3.15 we have that $\varrho_\varepsilon(t) > 0$ for every $t \in (\hat{t}_\varepsilon, b_\varepsilon^1)$, hence (3.2.23) holds.

First we prove the lemma assuming that (3.4.5) holds, which implies on its turn (3.4.42). With this condition, with the help of (3.3.24) and (3.2.23), we get that $\dot{\varrho}_\varepsilon(t) \ge \gamma_1 \frac{1}{\varepsilon} \varrho_\varepsilon(t)$; dividing by $\varrho_\varepsilon(t)$, we get

$$\frac{\dot{\varrho}_\varepsilon(t)}{\varrho_\varepsilon(t)} \ge \frac{\gamma_1}{\varepsilon} \qquad \text{for every } t \in (\hat{t}_\varepsilon, b_\varepsilon^1)\,. \tag{3.4.49}$$

Integrating (3.4.49) between $\hat{t}_\varepsilon$ and b_ε^1, using (3.4.41) and (3.4.44) we finally get the inequality

$$b_\varepsilon^1 - \hat{t}_\varepsilon \le \frac{\varepsilon}{\gamma_1} \log(\frac{\delta_1}{c\varepsilon})$$

which implies (3.4.48). Concerning part b), we fix $k \in \mathbb{N}$; applying (3.4.49) and neglecting the negative term $\varrho_\varepsilon(\hat{t}_\varepsilon)$, we get

$$\begin{aligned}
\delta_k + o(1) &= |(\sigma_\varepsilon(b^k_\varepsilon), z_\varepsilon(b^k_\varepsilon)) - (\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon))| \\
&\le |(\sigma_\varepsilon(b^k_\varepsilon) - \sigma_\varepsilon(\hat{t}_\varepsilon), 0)| + |(0, \zeta_\varepsilon(b^k_\varepsilon) -, z_\varepsilon(\hat{t}_\varepsilon))| \\
&= |\int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} \dot{\sigma}_\varepsilon(s)\, ds| + |\int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} \dot{z}_\varepsilon(s)\, ds| \\
&\le \int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} |\mathbb{C}\dot{\xi}(s) - \dot{\sigma}_\varepsilon(s)|\, ds + |\mathbb{C}| \int_{\hat{t}_\varepsilon}^{a^k_\varepsilon} |\dot{\xi}(s)|\, ds + \int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} |\dot{z}_\varepsilon(s)|\, ds \qquad (3.4.50) \\
&\le (|\mathbb{C}| + |\mathrm{tr}(\hat{\sigma})| + \delta_1) \int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} \frac{\varrho_\varepsilon}{\varepsilon}(s)\, ds + |\mathbb{C}| \int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} |\dot{\xi}(s)|\, ds \\
&\le \tfrac{1}{\gamma_1}(|\mathbb{C}| + |\mathrm{tr}(\hat{\sigma})| + \delta_1) \int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} \dot{\varrho}_\varepsilon(s)\, ds + |\mathbb{C}| \int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} |\dot{\xi}(s)|\, ds \\
&\le \tfrac{1}{\gamma_1}(|\mathbb{C}| + |\mathrm{tr}(\hat{\sigma})| + \delta_1) \varrho_\varepsilon(b^k_\varepsilon) + |\mathbb{C}| \int_{\hat{t}_\varepsilon}^{b^k_\varepsilon} |\dot{\xi}(s)|\, ds\,.
\end{aligned}$$

From this and part a) of the statement, we get (3.4.46). Then, integrating (3.4.49) between b^k_ε and b^1_ε we get that for ε small enough

$$\tfrac{b^1_\varepsilon - b^k_\varepsilon}{\varepsilon} \le \tfrac{1}{\gamma_1} \log(\tfrac{\varrho_\varepsilon(b^1_\varepsilon)}{\varrho_\varepsilon(b^k_\varepsilon)}) \le \tfrac{1}{\gamma_1} \log(\tfrac{\delta_1}{m_k}),$$

and the conclusion then follows.

Assume instead (3.4.6), which implies (3.4.43). As (3.4.6) implies $\mathrm{tr}(\nu_{K(z)}(\hat{\sigma})) > 0$ we may assume

$$\mathrm{tr}(\nu_{K(z_\varepsilon(t))}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t))) > 0 \qquad \text{for every } t \in (\hat{t}_\varepsilon, b^1_\varepsilon).$$

By this, facts, (3.2.21) and (3.4.47) we then easily get the existence of a positive constant C such that

$$\dot{z}_\varepsilon(t) \le -C \frac{\varrho_\varepsilon(t)}{\varepsilon} \qquad \text{for every } t \in (\hat{t}_\varepsilon, b^1_\varepsilon). \qquad (3.4.51)$$

In particular, for fixed $\varepsilon > 0$, the function $\dot{z}_\varepsilon(t)$ never vanishes in the prescribed interval. We also immediately get, as $z_\varepsilon(t) < \hat{z} + \delta_1$ for every $t \in (\hat{t}_\varepsilon, b^1_\varepsilon)$ that there exists a positive constant $\tilde{R}$ independent of ε such that:

$$\int_{\hat{t}_\varepsilon}^{b^1_\varepsilon} \frac{\varrho_\varepsilon(t)}{\varepsilon}\, dt \le \tilde{R}\,. \qquad (3.4.52)$$

Differentiating the function Ψ along the trajectories, we get

$$\begin{aligned}
\tfrac{d}{dt}\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) &= \nabla\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \cdot (\dot{\sigma}_\varepsilon(t), \dot{z}_\varepsilon(t)) = \\
&= \nabla\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \cdot (\mathbb{C}\dot{\xi}(t), 0) + \\
&+ \dot{z}_\varepsilon(t) \nabla\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \cdot (-\tfrac{\mathbb{C}(\dot{\xi}(t) - \dot{\sigma}_\varepsilon(t))}{\dot{z}_\varepsilon(t)}, 1) = \\
&= \nabla\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \cdot (\mathbb{C}\dot{\xi}(t), 0) + \\
&+ \dot{z}_\varepsilon(t) \nabla\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \cdot (\tfrac{-\mathbb{C}\, \nu_{K(z_\varepsilon(t))}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)))}{\mathrm{tr}(\sigma_\varepsilon(t))\mathrm{tr}(\nu_{K(z_\varepsilon(t))}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)))}, 1)\,;
\end{aligned}$$

this equality, together with (3.4.51)) and (3.4.43), implies that there exist two positive constants L and R such that

$$\frac{d}{dt}\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \geq R\frac{\varrho_\varepsilon(t)}{\varepsilon} - L|\mathbb{C}||\dot{\xi}(t)| \qquad \text{for every } t \in (\hat{t}_\varepsilon, b^1_\varepsilon)\,. \tag{3.4.53}$$

We denote with M_ξ the supremum of $|\dot{\xi}(t)|$ in $(\hat{t}_\varepsilon - \delta_1, \hat{t}_\varepsilon + \delta_1)$, and we fix $0 < \eta < \frac{R\gamma_2}{4L|\mathbb{C}|M_\xi}$, where γ_2 is the constant given by (3.3.24). For ε small enough, by the definition of $\hat{t}_\varepsilon$, we shall have

$$\Psi(\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon)) \geq -\eta\,.$$

We then define:

$$\hat{t}^1_\varepsilon := \inf\{\in (\hat{t}_\varepsilon, b^1_\varepsilon):\ \frac{\varrho_\varepsilon(t)}{\varepsilon} \geq \frac{\gamma_2}{4\eta}\}$$
$$\hat{t}^2_\varepsilon := \inf\{t \in (\hat{t}_\varepsilon, b^1_\varepsilon):\ \Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \leq -2\eta\}.$$

Now, let $\tilde{t}_\varepsilon := \hat{t}^1_\varepsilon \wedge \hat{t}^2_\varepsilon \wedge b^1_\varepsilon$; exploiting (3.2.23), the same argument used to prove (3.3.29) shows that $\hat{t}_\varepsilon \to \hat{t}$ when ε goes to 0. Moreover $\tilde{t}_\varepsilon < b^1_\varepsilon$; to get this, it suffices show that

$$\sup_{t\in[\hat{t}_\varepsilon, \tilde{t}_\varepsilon]} |(\sigma_\varepsilon(t), z_\varepsilon(t)) - (\hat{\sigma}, \hat{z})| \to 0 \tag{3.4.54}$$

as ε goes to 0, and this can be proved proceeding exactly as in the proof of (3.3.30), since $\frac{\varrho_\varepsilon(t)}{\varepsilon}$ is equibounded in the time interval $[\hat{t}_\varepsilon, \tilde{t}_\varepsilon]$.

Next we show that for small ε one has $\tilde{t}_\varepsilon < \hat{t}^2_\varepsilon$, which in his turn implies $\tilde{t}_\varepsilon = \hat{t}^1_\varepsilon$, so $\hat{t}^1_\varepsilon \to \hat{t}$ when ε goes to 0 and

$$\Psi(\sigma_\varepsilon(\hat{t}^1_\varepsilon), z_\varepsilon(\hat{t}^1_\varepsilon)) > -2\eta\,. \tag{3.4.55}$$

Suppose by contradiction that along some infinitesimal subsequence $\tilde{t}_\varepsilon = \hat{t}^2_\varepsilon$, that is to say $\Psi(\sigma_\varepsilon(\tilde{t}_\varepsilon), z_\varepsilon(\tilde{t}_\varepsilon)) = -2\eta$. Then, integrating (3.4.53)), since the function $\varrho_\varepsilon(t)$ is positive we get

$$-\eta = \Psi(\sigma_\varepsilon(\hat{t}^2_\varepsilon), z_\varepsilon(\hat{t}^2_\varepsilon)) - \Psi(\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon)) \geq -L|\mathbb{C}| \int_{\hat{t}_\varepsilon}^{\hat{t}^2_\varepsilon} |\dot{\xi}(t)|\,dt$$

which in the limit gives, by absolute continuity of the integral, that $-\eta \geq 0$, a contradiction.

So (3.4.55) holds, and $\frac{\varrho_\varepsilon(\hat{t}^1_\varepsilon)}{\varepsilon} = \frac{\gamma_2}{4\eta}$. Actually, we have

$$\frac{\varrho_\varepsilon(t)}{\varepsilon} > \frac{\gamma_2}{4\eta} \qquad \text{for every } t \in (\hat{t}^1_\varepsilon, b^1_\varepsilon). \tag{3.4.56}$$

Also this can be proved by contradiction. We observe that $\dot{\varrho}_\varepsilon(\hat{t}^1_\varepsilon) \geq -\frac{\gamma_2}{2} + \gamma_2 > 0$, by (3.2.23) and (3.3.24). If (3.4.56) is false, let $\hat{t}^3_\varepsilon$ be the first time in $(\hat{t}^1_\varepsilon, b^1_\varepsilon)$ such that $\varrho_\varepsilon(\hat{t}^3_\varepsilon) = \frac{\gamma_2}{4\eta}$; then $\dot{\varrho}_\varepsilon(\hat{t}^3_\varepsilon) \leq 0$. But, by (3.4.53), for every $t \in (\hat{t}^1_\varepsilon, \hat{t}^3_\varepsilon)$ we shall have $\frac{d}{dt}\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \geq 0$, hence, by (3.4.55),

$$\Psi(\sigma_\varepsilon(t), z_\varepsilon(t) > -2\eta \qquad \text{for every } t \in (\hat{t}^1_\varepsilon, \hat{t}^3_\varepsilon)\,;$$

by this, (3.3.24), and (3.2.23) we infer that $\dot{\varrho}_\varepsilon(\hat{t}^3_\varepsilon) \geq \gamma_2 - \frac{\gamma_2}{4} > 0$, which is a contradiction. Then, by (3.4.56) and (3.4.52), for ε sufficiently small we conclude that

$$\gamma_2(b^1_\varepsilon - \hat{t}^1_\varepsilon) \leq 4\eta\tilde{R}; \tag{3.4.57}$$

as $\hat{t}^1_\varepsilon - \hat{t}_\varepsilon \to 0$, we get that $\limsup_{\varepsilon\to 0} \gamma_2(b^1_\varepsilon - \hat{t}_\varepsilon) \le 4\eta\tilde{R}$, and by the arbitrariness of η, (3.4.48) follows, so part a) of the statement is proved.

Concerning part b), we fix $k \in \mathbb{N}$, $k > 1$. By the definition of b^k_ε and $\hat{t}_\varepsilon$ we shall have, for any $t \in [b^k_\varepsilon, b^1_\varepsilon]$, that

$$\delta_k + o(1) = |(\sigma_\varepsilon(t), \zeta_\varepsilon(t) - (\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon))|\,;$$

it follows, proceeding as in (3.4.50), that there exists a positive constant W such that

$$\delta_k + o(1) \le W\Big(\int_{\hat{t}_\varepsilon}^t \frac{\varrho_\varepsilon(s)}{\varepsilon}\, ds + \int_{\hat{t}_\varepsilon}^t |\dot{\xi}(s)|\, ds\Big). \tag{3.4.58}$$

This in turn implies, by (3.4.53) and the fundamental theorem of calculus that, up to redefining the constant W,

$$\delta_k + o(1) \le W[\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) - \Psi(\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon)) + \int_{\hat{t}_\varepsilon}^t |\dot{\xi}(s)|\, ds]\,. \tag{3.4.59}$$

By the definition of $\hat{t}_\varepsilon$, $\Psi(\sigma_\varepsilon(\hat{t}_\varepsilon), \zeta_\varepsilon(\hat{t}_\varepsilon)) = o(1)$; the absolute continuity of the integral and part a) of the statement now yield that, for ε small enough,

$$\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \ge \frac{\delta_k}{2W}$$

for every $t \in [b^k_\varepsilon, b^1_\varepsilon]$. Substituting in (3.2.23), this gives

$$\dot{\varrho}_\varepsilon(t) \ge \frac{\delta_k}{2W}\frac{\varrho_\varepsilon(t)}{\varepsilon} \qquad \text{for every } t \in [b^k_\varepsilon, b^1_\varepsilon]\,, \tag{3.4.60}$$

and we conclude that, for ε small enough

$$\frac{b^1_\varepsilon - b^k_\varepsilon}{\varepsilon} \le \frac{2W}{\delta_k}\log\Big(\frac{\varrho_\varepsilon(b^1_\varepsilon)}{\varrho_\varepsilon(b^k_\varepsilon)}\Big) \le \frac{2W}{\delta_k}\log\Big(\frac{\delta_1}{\varrho_\varepsilon(b^k_\varepsilon)}\Big)\,.$$

It then follows that part b) of the statement is immediate, once we get (3.4.46). To get a lower bound for $\varrho_\varepsilon(b^k_\varepsilon)$ we observe that a fortiori (3.4.60) holds, with δ_{k+1} in place of δ_k, for any $t \in [b^{k+1}_\varepsilon, b^k_\varepsilon]$. Since clearly

$$\delta_k - \delta_{k+1} \le |(\sigma_\varepsilon(b^k_\varepsilon), z_\varepsilon(b^k_\varepsilon) - (\sigma_\varepsilon(b^{k+1}_\varepsilon), z_\varepsilon(b^{k+1}_\varepsilon)|\,,$$

proceeding as in (3.4.58), we obtain that there exists a positive constant $\tilde{W}$ such that

$$\delta_k - \delta_{k+1} \le \tilde{W}\Big(\int_{b^{k+1}_\varepsilon}^{b^k_\varepsilon} \frac{\varrho_\varepsilon(s)}{\varepsilon}\, ds + \int_{b^{k+1}_\varepsilon}^{b^k_\varepsilon} |\dot{\xi}(s)|\, ds\Big). \tag{3.4.61}$$

Applying (3.4.60), with δ_{k+1} in place of δ_k, and the fundamental theorem of calculus, and neglecting the negative term $-\varrho_\varepsilon(b^{k+1}_\varepsilon)$, we get, up to redefining the constant $\tilde{W}$, that for ε small enough

$$\varrho_\varepsilon(b^k_\varepsilon) \ge (\delta_k - \delta_{k+1})\frac{\delta_{k+1}}{2\tilde{W}} := m_k,$$

and conclusion then follows. $\square$

We are now ready to prove the main result of this section. Notice that in the statement of the Theorem we are arbitrarily selecting one of the infinitely many solutions of (3.4.1) satisfying (3.4.7), which can differ by a time translation.

Theorem 3.29. *Assume that* (1.3.1)-(1.3.4), (2.3.3), (3.2.2), *and* (3.2.10)-(3.2.11) *are satisfied. Let* Φ, Ψ *be as in* (3.2.24), *and* (3.2.25), *respectively. Let* $\hat{t} > 0$, $(\hat{\sigma}, \hat{z}) \in \partial K$, *such that* (3.4.5) *or* (3.4.6) *hold. Assume that* $\Phi(\hat{t}, \hat{\sigma}, \hat{z}) > 0$. *Let* $(\sigma_f(s), z_f(s)$ *be a fixed solution of the problem:*

$$\begin{cases} \dot{\sigma}_f(s) = \mathbb{C}(\pi_{K(z_f(s))}(\sigma_f(s)) - \sigma_f(s)) \\ \dot{z}_f(s) = \operatorname{tr}(\sigma_f(s)) \operatorname{tr}(\sigma_f(s) - \pi_{K(z_f(s))}(\sigma_f(s))) \\ \lim\limits_{s\to-\infty} (\sigma_f(s), z_f(s)) = (\hat{\sigma}, \hat{z}) \end{cases} \tag{3.4.62}$$

Then there exists $\hat{b}^1_\varepsilon \to \hat{t}$ *such that, if we define* $(\hat{\sigma}^1_\varepsilon(s), \hat{z}^1_\varepsilon(s)) := (\sigma_\varepsilon(\hat{b}^1_\varepsilon + \varepsilon s), z_\varepsilon(\hat{b}^1_\varepsilon + \varepsilon s))$ *for every* $s \in \mathbb{R}$, $(\hat{\sigma}^1_\varepsilon(s), \hat{z}^1_\varepsilon(s))$ *converges uniformly on compact subsets of* $\mathbb{R}$ *to* $(\sigma_f(s), z_f(s))$.

Proof. This proof is reminiscent of [55, Lemma 4.3]. Fix $\delta_1 > 0$ as in (3.3.24) and let b^1_ε be given by (3.4.44). We may clearly assume that $\delta_1 < |(\hat{\sigma}, \hat{z}) - (\sigma_\infty, z_\infty)|$ where $(\sigma_\infty, z_\infty)$ satisfies (3.4.8). We also define $\chi_\varepsilon(s) := \dot{\xi}(b^1_\varepsilon + \varepsilon s)$.

First of all, we prove that there exists a sequence c_ε such that

$$(\sigma_\varepsilon(b^1_\varepsilon - \varepsilon c_\varepsilon + \varepsilon s), z_\varepsilon(b^1_\varepsilon - \varepsilon c_\varepsilon + \varepsilon s)) \to (\sigma_f(s), z_f(s)) \tag{3.4.63}$$

as ε goes to 0. To simplify notation, in this part of the proof we write $(\sigma^1_\varepsilon(s), z^1_\varepsilon(s))$ in place of $(\sigma_\varepsilon(b^1_\varepsilon + \varepsilon s), z_\varepsilon(b^1_\varepsilon + \varepsilon s))$. Fix a sequence $\varepsilon_j \to 0$. We start by observing that the function $(\sigma^1_{\varepsilon_j}(s), z^1_{\varepsilon_j}(s))$ solves the problem

$$\begin{cases} \dot{\sigma}^1_{\varepsilon_j}(s) = \mathbb{C}(\pi_{K(z^1_{\varepsilon_j}(s))}(\sigma^1_{\varepsilon_j}(s)) - \sigma^1_{\varepsilon_j}(s)) + \varepsilon \mathbb{C}\, \chi_{\varepsilon_j}(s), \\ \dot{z}^1_{\varepsilon_j}(s) = \operatorname{tr}(\sigma^1_{\varepsilon_j}(s)) \operatorname{tr}(\sigma^1_{\varepsilon_j}(s) - \pi_{K(z^1_{\varepsilon_j}(s))}(\sigma^1_{\varepsilon_j}(s))), \\ (\sigma^1_{\varepsilon_j}(0), z^1_{\varepsilon_j}(0)) = (\sigma_{\varepsilon_j}(b^1_{\varepsilon_j}), z_\varepsilon(b^1_{\varepsilon_j})), \end{cases} \tag{3.4.64}$$

in the interval $[-\frac{b^1_{\varepsilon_j}}{\varepsilon_j}, \frac{\hat{t}+\delta_1-b^1_{\varepsilon_j}}{\varepsilon_j}]$. As $(\sigma_{\varepsilon_j}(b^1_{\varepsilon_j}), z_{\varepsilon_j}(b^1_{\varepsilon_j}))$ belongs to the compact set $\partial B_{\delta_1}(\hat{\sigma}, \hat{z})$ we may assume, possibly passing to a subsequence that $(\sigma_{\varepsilon_j}(b^1_{\varepsilon_j}), z_{\varepsilon_j}(b^1_{\varepsilon_j}))$ converges to $(\hat{\sigma}_1, \hat{z}_1) \in \partial B_{\delta_1}(\hat{\sigma}, \hat{z})$ as $j \to +\infty$. Notice that $(\hat{\sigma}_1, \hat{z}_1)$ has a strictly positive distance from K as a consequence of (3.4.46). Therefore, Lemma 3.28 and the Continuous Dependence Theorem imply that $(\sigma^1_{\varepsilon_j}(s), z^1_{\varepsilon_j}(s))$ converges uniformly on compact subsets of $\mathbb{R}$, as $j \to +\infty$, to the solution $(\sigma^1(s), z^1(s))$ of the problem

$$\begin{cases} \dot{\sigma}^1(s) = \mathbb{C}(\pi_{K(z^1(s))}(\sigma^1(s)) - \sigma^1(s)), \\ \dot{z}^1(s) = \operatorname{tr}(\sigma^1(s)) \operatorname{tr}(\sigma^1(s) - \pi_{K(z^1(s))}(\sigma^1(s))), \\ (\sigma^1(0), z^1(0)) = (\hat{\sigma}_1, \hat{z}_1). \end{cases} \tag{3.4.65}$$

We now show that

$$\lim_{s\to-\infty} (\sigma^1(s), z^1(s)) = (\hat{\sigma}, \hat{z}). \tag{3.4.66}$$

Actually, recalling Remark 3.26, it suffices to show that there exist $s_k \to +\infty$ such that

$$\lim_{k\to+\infty} (\sigma^1(-s_k), z^1(-s_k)) = (\hat{\sigma}, \hat{z}). \tag{3.4.67}$$

To do that, we take δ_k and b^k_ε as in Lemma 3.28, and we define $S^{1,k}_{\varepsilon_j} := \frac{b^1_{\varepsilon_j} - b^k_{\varepsilon_j}}{\varepsilon_j}$; by Lemma 3.28 and a diagunal argument, we may suppose, eventually passing to a subsequence, that for every $k \in \mathbb{N}$ there exists

$$s_k := \lim_{j \to +\infty} S^{1,k}_{\varepsilon_j} \in \mathbb{R}_+ \,.$$

We define $(\sigma^k_{\varepsilon_j}(s), z^k_{\varepsilon_j}(s)) := (\sigma_{\varepsilon_j}(b^k_{\varepsilon_j} + \varepsilon_j s), z_{\varepsilon_j}(b^k_{\varepsilon_j} + \varepsilon_j s))$; by repeating the above arguments we may suppose that for every $k \in \mathbb{N}$ there exists $(\hat\sigma_k, \hat z_k) \in \partial B_{\delta_k}(\hat\sigma, \hat z) \setminus K$ such that $(\sigma^k_{\varepsilon_j}(s), z^k_{\varepsilon_j}(s))$ converges, as $j \to +\infty$, uniformly on compact subsets of $\mathbb{R}$, to the solution $(\sigma^k(s), z^k(s))$ of the problem

$$\begin{cases} \dot\sigma^k(s) = \mathbb{C}(\pi_{K(z^k(s))}(\sigma^k(s)) - \sigma^k(s)), \\ \dot z^k(s) = \operatorname{tr}(\sigma^k(s)) \operatorname{tr}(\sigma^k(s) - \pi_{K(z^k(s))}(\sigma^k(s))), \\ (\sigma^k(0), z^k(0)) = (\hat\sigma_k, \hat z_k). \end{cases} \tag{3.4.68}$$

Moreover, equality $(\sigma^k_{\varepsilon_j j}(S^{1,k}_{\varepsilon_j}), z^k_{\varepsilon_j}(S^{1,k}_{\varepsilon_j})) = (\sigma_{\varepsilon_j}(b^1_\varepsilon), z_{\varepsilon_j}(b^1_{\varepsilon_j}))$ implies that

$$(\sigma^k(s_k), z^k(s_k)) = (\hat\sigma_1, \hat z_1)$$

for every k, hence by the uniqueness of solutions for Cauchy problems we get

$$(\sigma^k(s), z^k(s)) = (\sigma^1(s - s_k), z^1(s - s_k)). \tag{3.4.69}$$

It follows that

$$(\sigma^1(-s_k), z^1(-s_k)) = (\hat\sigma_k, \hat z_k). \tag{3.4.70}$$

As $\delta_k \to 0$, we have that $(\hat\sigma_k, \hat z_k) \to (\hat\sigma, \hat z)$ as k goes to $+\infty$, hence

$$\lim_{k \to +\infty} (\sigma^1(-s_k), z^1(-s_k)) = (\hat\sigma, \hat z); \tag{3.4.71}$$

since $(\hat\sigma, \hat z)$ is an equilibrium point for equation (3.4.1), necessarily $s_k \to +\infty$ as $k \to +\infty$; so, (3.4.67) is proved. By Theorem (3.25), there exists a constant $c \in \mathbb{R}$ such that

$$(\sigma^1(s), z^1(s)) = (\sigma_f(s + c), z_f(s + c))$$

By (3.4.65), we have that c belongs to the set

$$C := \{s \in \mathbb{R} : (\sigma_f(s), z_f(s)) \in \partial B_{\delta_1}(\hat\sigma, \hat z)\} \tag{3.4.72}$$

which easily turns out to be compact, thanks to (3.4.7) and the assumption $\delta_1 < |(\hat\sigma, \hat z) - (\sigma_\infty, z_\infty)|$.

So far we have proved that for every $\varepsilon_j \to 0$ there exists a subsequence ε_{j_h} and a constant $c \in C$, possibly depending on ε_{j_h}, such that

$$(\sigma_{\varepsilon_{j_h}}(b^1_{\varepsilon_{j_h}} + \varepsilon_{j_h} s), z_\varepsilon(b^1_{\varepsilon_{j_h}} + \varepsilon_{j_h} s)) \to (\sigma_f(s + c), z_f(s + c)) \,,$$

which is to say

$$(\sigma_{\varepsilon_{j_h}}(b^1_{\varepsilon_{j_h}} + \varepsilon_{j_h}(s - c)), z_\varepsilon(b^1_{\varepsilon_{j_h}} + \varepsilon_{j_h}(s - c))) \to (\sigma_f(s), z_f(s)) \,.$$

From this (3.4.63) easily follows; moreover, we can take $c_\varepsilon \in C$ for every ε, so that, setting $\hat b^1_\varepsilon := b^1_\varepsilon - \varepsilon c_\varepsilon$, by compactness of C we have $\hat b^1_\varepsilon \to \hat t$ when ε goes to 0, therefore again (3.4.63) gives immediately the conclusion. $\square$

3.5 Statement of the main result

We collect the results of the previous sections in the next theorem, which gives a procedure to construct a viscosity solution to our evolution problem under quite general assumptions; in fact, if these assumptions are satisfied at every step of the construction, the viscosity solution is also unique. The theorem will determine a possibly infinite sequence of times $t_0 < t_1 < \cdots < t_i < \ldots$ such that in each interval $(t_{i-1}, t_i]$ the solution, denoted here by (σ_{i-1}, z_{i-1}) is continuous and satisfies either the slow dynamics, or the elastic regime, or a combination of the two. A jump may occur at time t_i if the value $(\sigma_{i-1}(t_i), z_{i-1}(t_i))$ satisfies (3.4.5) or (3.4.6). In this case the new starting point (σ_i^+, z_i^+) for the solution in the interval $(t_i, t_{i+1}]$ is determined by taking the limit as $s \to +\infty$ of the solution of the fast dynamics originating from $(\sigma_{i-1}(t_i), z_{i-1}(t_i))$ at $s = -\infty$. To prepare the technical statement of the theorem it is convenient to introduce some notation.

Definition 3.30. For every $(\hat\sigma, \hat z) \in \partial K$ satisfying $\Psi(\hat\sigma, \hat z) \neq 0$, and every $T > 0$ we define $(\sigma_{sl}, z_{sl})(t; \hat\sigma, \hat z, T)$ as the unique solution to (3.3.1) starting from the point $(\hat\sigma, \hat z)$ at time T. For every $(\hat\sigma, \hat z) \in \partial K$ we define $(\sigma_{el}, z_{el})(t; \hat\sigma, \hat z, T) = (\hat\sigma + \mathbb{C}(\xi(t) - \xi(T)), \hat z)$. For every $(\hat\sigma, \hat z) \in \partial K$ satisfying (3.4.5) or (3.4.6) we define $(\sigma_f, z_f)(s; \hat\sigma, \hat z)$ as the unique solution to (3.4.1) having $(\hat\sigma, \hat z)$ as an α-limit point.

To simplify our notation, in the statement of the theorem we also put

$$\partial K_f := \{(\sigma, z) \in \partial K : (\sigma, z) \text{ satisfy (3.4.5) or (3.4.6) }\}.$$

Theorem 3.31. *Let* $(\sigma_0, z_0) \in \operatorname{int} K$*, let* $t_0 = 0$*,* t_1 *as in* (3.2.20)*, and* $(\sigma_0(t), z_0(t)) = (\sigma_0 + \mathbb{C}(\xi(t) - \xi(0)), z_0)$*. For every* $i \geq 1$ *with* $t_i < +\infty$ *define*

$$(\sigma_i^+, z_i^+) = \begin{cases} (\sigma_{i-1}, z_{i-1})(t_i) & \text{if } \Psi(\sigma_{i-1}(t_i), z_{i-1}(t_i)) < 0, \\ \lim_{s \to +\infty} (\sigma_f, z_f)(s; \sigma_{i-1}(t_i), z_{i-1}(t_i)) & \text{if } (\sigma_{i-1}(t_i), z_{i-1}(t_i)) \in \partial K_f. \end{cases}$$

If $\Psi(\sigma_i^+, z_i^+) < 0$*, let* $\hat t_i$ *be the maximal time of existence for* $(\sigma_{sl}, z_{sl})(t; \sigma_i^+, z_i^+, t_i)$*, and*

$$\bar t_i := \inf\{t \geq t_i : \Phi(t, (\sigma_{sl}, z_{sl})(t; \sigma_i^+, z_i^+, t_i)) \leq 0\}.$$

If $\hat t_i = \bar t_i$*, put* $t_{i+1} := \hat t_i$*, and*

$$(\sigma_i(t), z_i(t)) = (\sigma_{sl}, z_{sl})(t; \sigma_i^+, z_i^+, t_i)$$

for every $t_i \leq t \leq t_{i+1}$*; if instead* $\hat t_i > \bar t_i$*, put* $(\bar\sigma_i, \bar z_i) := (\sigma_{sl}, z_{sl})(\bar t_i, \sigma_i^+, z_i^+, t_i)$*,*

$$t_{i+1} := \sup\{t > \bar t_i \,|\, (\sigma_{el}, z_{el})(t; \bar\sigma_i, \bar z_i, \bar t_i) \in \operatorname{int} K \text{ for every } s \in (\bar t_i, t)\},$$

and

$$(\sigma_i(t), z_i(t)) = \begin{cases} (\sigma_{sl}, z_{sl})(t; \sigma_i^+, z_i^+, t_i) & \text{for } t_i < t \leq \bar t_i, \\ (\sigma_{el}, z_{el})(t; \bar\sigma_i, \bar z_i, \bar t_i) & \text{for } \bar t_i \leq t \leq t_{i+1}. \end{cases}$$

Define $(\sigma(t), z(t)) := \sum_{i \geq 1} 1_{(t_{i-1}, t_i]}(\sigma_{i-1}(t), z_{i-1}(t))$*. Assume that*

$$\Phi(\sigma(t_i), z(t_i)) > 0 \qquad \text{for every } i \geq 1, \tag{3.5.1}$$

$$\text{(3.3.49) and (3.3.50) hold at } \bar t_i \qquad \text{for every } i \text{ with } t_i \leq \bar t_i < \hat t_i \tag{3.5.2}$$

$$\liminf_{t \to t_{i+1}^-} \Phi(t, \sigma(t), z(t)) > 0 \qquad \text{for every } i \text{ with } t_{i+1} = \hat t_i < +\infty. \tag{3.5.3}$$

Define $e(t)$ *and* $p(t)$ *through the consitutive relations in* (3.1.1)*, and put* $T := \sup_i t_i$. *Then* $(e(t), p(t), \sigma(t), z(t))$ *is the unique viscosity solution of* (3.1.1) *in* $[0, T)$.

Proof. The result follows from Theorems 3.18, 3.21, 3.23, and 3.29. □

Remark 3.32. Notice that assumption (3.5.3) ensures that whenever $t_{i+1} = \hat{t}_i < +\infty$ we can extend by continuity (σ_{sl}, z_{sl}) in t_{i+1} thanks to Proposition 3.10, hence at every step the left limit $(\sigma(t_i), z(t_i))$ is well-defined. Notice that the statement of the theorem covers also the case when $\bar{t}_i = t_i$, which is likely to happen for instance after a jump. Concerning the other assumptions in the theorem, observe that by construction and Theorem 3.25, we always have at least the weak inequality $\Psi(\sigma_i^+, z_i^+) \leq 0$. By construction we also have $\Phi(\sigma(t_i), z(t_i)) \geq 0$ for every i, since at time t_i either we reach the yield surface from the interior of the elastic domain, or we were following the slow dynamics with postive Φ at previous times. Similarly, the weak inequality in (3.5.3) is always true whenever $t_{i+1} = \hat{t}_i$. Thus our construction works at least for the nondegenerate cases where equality is excluded while a higher-order analysis is needed in the remaining situations to get insight of the limit behavior of the viscous approximations.

Chapter 4

Rescaled viscosity evolution

4.1 Overview of the chapter

The study of the spatially homogeneous case in the previous chapter has higlighted that, for many initial data, we cannot expect in general that an evolution satifying the equations (a)-(f) of Chapter 2, Section 2.2 is smooth in time. This is due to the nonconvexity of the problem, since the model allows for a softening regime. Moreover, we have seen that a way to catch the behavior of the system at jump times is the introduction of a time rescaling and the use of a fast time s, which moves while the original time is frozen. Following this idea (also used in [18, 32, 33] for rate independent dissipative problems in finite dimension), in this chapter we introduce a notion of generalized solution to give a meaning to the evolution after the first discontinuity time. The idea, which will be developed in the next chapter, is to consider a viscoplastic approximation of Perzyna-type as in Chapter 2, Section 2.4 and to take the limit as the viscosity parameter tends to 0 of a suitable time-rescaled version of the solutions. The properties of this limit give rise to the notion of a *rescaled viscosity evolution*, which is expressed in terms of a rescaled time s, related to the original time by the equality $t = t^\circ(s)$, where t° is a nondecreasing locally Lipschitz function. The intervals where t° is constant correspond to time discontinuities in the original variable t. The advantage in this approach is that all these functions will be continuous with respect to s, while continuity cannot be expected with respect to the the original time t.

The definition of rescaled viscosity evolution that we give in this chapter is different from the original one contained in [10, Definition 4.1]. Following the ideas of [11, Sections 3 and 4], in the definition we will replace the energy-dissipation balance and the partial flow rule of [10, Definition 4.1] with a measure-theoretical formulation of the flow rule (e) of Chapter 2. The motivation for this approach is that the structure of this definition really resembles the classical formulation of the problem, and the equations satisfied by a rescaled viscosity evolution are the rigorous counterpart of the classical ones. However, there is also a disadvantage, since condition (ev3$''$)$^\circ$ of Definition 4.5 is formulated in terms of a suitable representative $\hat{\boldsymbol{\sigma}}^\circ(s)$ of the stress $\boldsymbol{\sigma}^\circ(s)$, which has to satisfy the integration by parts formula (4.2.3). The existence itself of this representative is not a priori guaranteed and is part of the proof. Nevertheless, in the last section of the chapter we will show that this representative has an intrinsic character. Indeed, if we assume strict convexity of $K(1)$ it can be obtained in Ω as the limit of averages of the stress $\boldsymbol{\sigma}^\circ(s)$.

The goal of this chapter is to show that the two definitions are indeed equivalent. The proof will also give us the possibility to introduce some tools that will be used in the next chapter to prove the existence of a rescaled viscosity evolution. In this perspective, it should also be noticed that, since [10, Definition 4.1] replaces a differential inclusion with an energy equality, it also requires less a priori information about the time regularity of the involved functions. This is why the existence theorem that will be proved in Chapter 5 will rely on proving the energy-dissipation balance (4.3.1) instead than directly (ev3$''$)$^\circ$.

For all the notation and the assumptions on the model we refer to Chapters 1 and 2.

4.2 Quasistatic evolution

The definition of rescaled viscosity evolution that we are going to give in this section involves a suitable representative of the stress $\boldsymbol{\sigma}$, that we now define. In the definition the measure $[\sigma : p]$ is the one defined by (2.3.10)-(2.3.11). Notice that we are not a priori claiming that such a representative exists for any given admissible stress. We recall that, as in Chapter 2, the space $\Pi_{\Gamma_0}(\Omega)$ of admissible plastic strains is defined as the set of all $p \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$ for which there exist $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^N)$, and $e \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ satisfying (2.3.1).

Definition 4.1. Let $p \in \Pi_{\Gamma_0}(\Omega)$ and $\zeta \in C^0(\overline{\Omega})^+$. Let $\sigma \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ be such that $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^N)$ and $[\sigma\nu] \in L^\infty(\Gamma_1;\mathbb{R}^N)$. Let

$$\mu := \begin{cases} \mathcal{L}^n + |p| & \text{if } \sigma \in \mathcal{K}(\zeta)\,, \\ \mathcal{L}^n & \text{if } \sigma \notin \mathcal{K}(\zeta)\,. \end{cases}$$

We say that a function $\hat\sigma \in L^2_\mu(\Omega \cup \Gamma_0;\mathbb{M}^{N\times N}_{sym})$ is a *precise representative* of σ with respect to p and ζ if $\hat\sigma = \sigma$ $\mathcal{L}^n$-a.e. on Ω, and

$$\sigma \in \mathcal{K}(\zeta) \Rightarrow \hat\sigma \in \mathcal{K}_\mu(\zeta)\,, \tag{4.2.1}$$

$$\sigma \in \mathcal{K}(\zeta) \Rightarrow [\sigma : p] = \Big(\hat\sigma : \frac{p}{\mu}\Big)\,\mu \qquad \text{on } \Omega \cup \Gamma_0\,, \tag{4.2.2}$$

where $\mathcal{K}_\mu(\zeta)$ is defined by (1.3.16), and $\frac{p}{\mu}$ is the Radon-Nikodym derivative of p with respect to μ.

Remark 4.2. Observe that (1.3.4) assures that σ belongs to $L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym})$, and thus to space $\Sigma(\Omega)$ defined by (2.3.8), whenever $\sigma \in \mathcal{K}(\zeta)$, so that $[\sigma : p]$ makes sense. Clearly one can take $\hat\sigma = \sigma$ as a precise representative whenever $\sigma \notin \mathcal{K}(\zeta)$. The choice to contemplate this obvious case in Definition 4.1 will be useful to write condition (ev3$''$)$^\circ$ in a more compact way.

Using Proposition 2.3, we can easily prove that condition (4.2.2) is equivalent to the following integration by parts formula: for every $\varphi \in C^1(\overline{\Omega})$ we have

$$\begin{aligned} \langle \varphi\hat\sigma, p\rangle = &-\langle \hat\sigma, \varphi(e - Ew)\rangle - \langle \hat\sigma, (u-w)\odot\nabla\varphi\rangle + \\ &+ \langle f, \varphi(u-w)\rangle + \langle g, \varphi(u-w)\rangle_{\Gamma_1}\,, \end{aligned} \tag{4.2.3}$$

where $f := -\operatorname{div}\sigma$, $g := [\sigma\nu]$, $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^N)$, and $e \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ satisfy (2.3.1), and the duality product in the left-hand side is the standard duality between a bounded measurable function and a bounded measure.

Throughout the chapter we will assume that

$$\begin{gathered}\boldsymbol{u}^\circ\colon [0,+\infty) \to BD(\Omega) \quad \text{is weakly}^* \text{ continuous}\,,\\ \boldsymbol{e}^\circ\colon [0,+\infty) \to L^2(\Omega;\mathbb{M}^{N\times N}_{sym}) \quad \text{is weakly continuous}\,,\\ \boldsymbol{p}^\circ\colon [0,+\infty) \to M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym}) \quad \text{is 1-Lipschitz}\,,\\ \boldsymbol{z}^\circ\colon [0,+\infty) \to C^0(\overline{\Omega})^+ \quad \text{is locally Lipschitz}\,,\\ t^\circ\colon [0,+\infty) \to [0,+\infty) \quad \text{is nondecreasing, surjective, and locally Lipschitz.}\end{gathered} \tag{4.2.4}$$

As usual, we set

$$\boldsymbol{\sigma}^\circ(s) := \mathbb{C}\boldsymbol{e}^\circ(s) \quad \text{and} \quad \boldsymbol{\zeta}^\circ(s) := V(\boldsymbol{z}^\circ(s)) \quad \text{for every } s \in [0,+\infty)\,. \tag{4.2.5}$$

Since now $\boldsymbol{p}^\circ(s) \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ (see [50] for some examples showing that we cannot expect $\boldsymbol{p}^\circ(s) \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$), the derivative of $\boldsymbol{p}^\circ$ with respect to s can be defined only in the weak* sense given by Theorem 1.8, namely

$$\dot{\boldsymbol{p}}^\circ(s) := w^*\text{-}\lim_{h\to 0}\frac{\boldsymbol{p}^\circ(s+h)-\boldsymbol{p}^\circ(s)}{h} \qquad (w^*\text{-topology of } M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym}))\,. \tag{4.2.6}$$

We define

$$B^\circ := \{s\in[0,+\infty) : \boldsymbol{\sigma}^\circ(s) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s))\} \text{ and } A^\circ := [0,+\infty)\setminus B^\circ\,. \tag{4.2.7}$$

Remark 4.3. The continuity properties of $\boldsymbol{\sigma}^\circ$ and $\boldsymbol{\zeta}^\circ$ imply that A° is open. Indeed, by convexity, for every $\zeta \in C^0(\bar\Omega)^+$ the function $\sigma \mapsto d_2(\sigma,\mathcal{K}(\zeta))$ is weakly lower semicontinuous in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$. From (1.3.7), we deduce that

$$|d_2(\sigma,\mathcal{K}(\zeta_1)) - d_2(\sigma,\mathcal{K}(\zeta_2))| \le 2M_K\|\zeta_1-\zeta_2\|_2$$

for every $\sigma \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and every $\zeta_1,\zeta_2 \in C^0(\bar\Omega)^+$. It follows that $(\sigma,\zeta)\mapsto d_2(\sigma,\mathcal{K}(\zeta))$ is lower semicontinuous with respect to the weak topology in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and to the strong topology of $C^0(\overline{\Omega})$. Since $\boldsymbol{\sigma}^\circ = \mathbb{C}\boldsymbol{e}^\circ$ and since $\boldsymbol{e}^\circ$ is continuous for the weak topology of $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and $\boldsymbol{\zeta}^\circ$ is continuous for the strong topology of $C^0(\overline{\Omega})$ by (5.4.44), it follows that $s\mapsto d_2(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s)))$ is lower semicontinuous on $[0,+\infty)$. Therefore the set A°_S is open.

The data of the problem $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ appear only through the composite functions $\boldsymbol{f}(t^\circ(s))$, $\boldsymbol{g}(t^\circ(s))$, and $\boldsymbol{w}(t^\circ(s))$. We will frequently use the shorthands $\boldsymbol{f}^\circ(s)$, $\boldsymbol{g}^\circ(s)$, and $\boldsymbol{w}^\circ(s)$ in place of $\boldsymbol{f}(t^\circ(s))$, $\boldsymbol{g}(t^\circ(s))$, and $\boldsymbol{w}(t^\circ(s))$. The displacement $\boldsymbol{u}^\circ(s)$, the elastic and plastic strain $\boldsymbol{e}^\circ(s)$ and $\boldsymbol{p}^\circ(s)$, and the boundary displacement $\boldsymbol{w}^\circ(s)$ are related by the kinematic condition (2.3.1), which reads in this case as

$$\begin{gathered}E\boldsymbol{u}^\circ(s) = \boldsymbol{e}^\circ(s) + \boldsymbol{p}^\circ(s) \quad \text{in } \Omega\,,\\ \boldsymbol{p}^\circ(s) = (\boldsymbol{w}(t^\circ(s)) - \boldsymbol{u}^\circ(s))\odot\nu\mathcal{H}^{n-1} \quad \text{in } \Gamma_0\,,\end{gathered}$$

while the stress $\boldsymbol{\sigma}^\circ(s)$ has to satisfy the equilibrium condition

$$-\operatorname{div}\boldsymbol{\sigma}^\circ(s) = \boldsymbol{f}^\circ(s) \quad \text{in } \Omega\,, \qquad [\boldsymbol{\sigma}^\circ(s)\nu] = \boldsymbol{g}^\circ(s) \quad \text{on } \Gamma_1.$$

It follows from these two conditions and (2.3.42) that for every s such that $\boldsymbol{\sigma}^\circ(s) \in L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym})$ we can define the measure $[\boldsymbol{\sigma}^\circ(s) : \boldsymbol{p}^\circ(s)]$ and the duality $\langle \boldsymbol{\sigma}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ as in (2.3.10)-(2.3.11), and (2.3.12), respectively. If in addition we suppose

$$\boldsymbol{e}^\circ \colon [0,+\infty) \to L^2(\Omega;\mathbb{M}^{N\times N}_{sym}) \quad \text{is strongly continuous and } \mathcal{L}^1\text{-a.e. differentiable}\,, \tag{4.2.8}$$

the stress $\boldsymbol{\sigma}^\circ(s)$ can be put in duality with the rate of plastic strain $\dot{\boldsymbol{p}}^\circ(s)$ at $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$ such that $\boldsymbol{\sigma}^\circ(s) \in L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym})$, as the next remark highlights.

Remark 4.4. Let $s \in [0,+\infty)$ be such that the derivatives $\dot{\boldsymbol{\sigma}}^\circ(s)$, $\dot{\boldsymbol{p}}^\circ(s)$, and $\dot{\boldsymbol{w}}^\circ(s)$ exist. We claim that for every such s, the measure $\dot{\boldsymbol{p}}^\circ(s) \in \Pi_{\Gamma_0}(\Omega)$, so that for every $\chi \in \Sigma(\Omega)$ we can define the measure $[\chi : \dot{\boldsymbol{p}}^\circ(s)]$ and the duality $\langle \chi, \dot{\boldsymbol{p}}^\circ(s)\rangle$ according to (2.3.10)-(2.3.11), and (2.3.12), respectively. Together with (1.3.4), this implies in particular that the duality $\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$ and the measure $[\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)]$ are correctly defined for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$ such that $\boldsymbol{\sigma}^\circ(s) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s))$.

To prove the claim, we notice that, if $s \in [0,+\infty)$ has the required properties, the difference quotients

$$\frac{1}{h} E(\boldsymbol{u}^\circ(s+h) - \boldsymbol{u}^\circ(s))$$

converge weakly* in $M_b(\Omega;\mathbb{M}^{N\times N}_{sym})$ to $\dot{\boldsymbol{e}}^\circ(s) + \dot{\boldsymbol{p}}^\circ(s)$; moreover, using (2.3.1) and the estimate proved in [52, Proposition 2.4 and Remark 2.5], taking also into account the continuity of the trace operator from $H^1(\Omega;\mathbb{R}^N)$ into $L^1(\partial\Omega;\mathbb{R}^N)$, we can prove that there exists a constant C such that

$$\begin{aligned} &\frac{1}{h}\|\boldsymbol{u}^\circ(s+h) - \boldsymbol{u}^\circ(s)\|_1 \le \\ &\le C\Big(\frac{1}{h}\|\boldsymbol{p}^\circ(s+h) - \boldsymbol{p}^\circ(s)\|_1 + \frac{1}{h}\|\boldsymbol{w}^\circ(s+h) - \boldsymbol{w}^\circ(s)\|_{H^1} + \frac{1}{h}\|E(\boldsymbol{u}^\circ(s+h) - \boldsymbol{u}^\circ(s))\|_1\Big)\,. \end{aligned}$$

Therefore the difference quotients of $\boldsymbol{u}^\circ$ are bounded in $BD(\Omega)$, thus converge weakly* in $BD(\Omega)$, up to a subsequence, to a function $\dot{\boldsymbol{u}}^\circ(s)$. We can easily prove, arguing for instance as in [13, Lemma 2.1], that $(\dot{\boldsymbol{u}}^\circ(s), \dot{\boldsymbol{e}}^\circ(s), \dot{\boldsymbol{p}}^\circ(s), \dot{\boldsymbol{w}}^\circ(s))$ satisfy (2.3.1), hence $\dot{\boldsymbol{p}}^\circ(s) \in \Pi_{\Gamma_0}(\Omega)$, as required. It also follows that the limit $\dot{\boldsymbol{u}}^\circ(s)$ is uniquely determined by $(\dot{\boldsymbol{e}}^\circ(s), \dot{\boldsymbol{p}}^\circ(s), \dot{\boldsymbol{w}}^\circ(s))$ and hence does not depend on the chosen subsequence.

Finally, to give the definition of rescaled viscosity evolution we need a suitable extension of the notion of normal cone. Indeed, The classical Prandtl-Reuss flow rule is usually formulated in terms of a differential inclusion involving the normal cone $N_C(\xi)$ to a convex set C at some point $\xi \in C$. In convex analysis, the normal cone is extended by setting $N_C(\xi) = \emptyset$ whenever $\xi \notin C$. For our purposes we find it convenient to consider a different extension. Given a Hilbert space X and a convex closed subset C of X, for every $\xi \in X$ we define the *extended normal cone* $N^{ext}_C(\xi)$ to C at ξ in the following way:

$$N^{ext}_C(\xi) := \begin{cases} N_C(\xi) & \text{if } \xi \in C, \\ \{\lambda(\xi - \pi_C(\xi)) : \lambda \ge 0\} & \text{if } \xi \notin C\,, \end{cases} \tag{4.2.9}$$

where π_C denotes the minimal distance projection. Unlike the normal cone of convex analysis, N^{ext}_C is not a monotone operator. However, the multi-valued map $\xi \mapsto N^{ext}_C(\xi)$

has closed graph, that is, if $\xi_j \to \xi$ and $v_j \in N_C^{ext}(\xi_j)$ for every j, then any limit point v of v_j belongs to $N_C^{ext}(\xi)$. The simple verification is left to the reader.

We are now finally ready to define our notion of generalized solution. Notice that, here and henceforth, among the hypotheses on the data of the problem, we are in particular assuming that the uniform safe-load condition (2.3.45)-(2.3.48) is satisfied.

Definition 4.5. Assume that $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{w}$ satisfy (2.3.42)-(2.3.48), and let u_0, e_0, p_0, z_0 be as in (2.3.53)-(2.3.57). Consider $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ satisfying (4.2.4), define $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$ and $\dot{\boldsymbol{p}}^\circ$ as in (4.2.5)-(4.2.6), let A° and B° be as in (4.2.7) and set

$$\boldsymbol{\mu}(s) := \begin{cases} \mathcal{L}^n & \text{if } s \in A^\circ, \\ \mathcal{L}^n + |\dot{\boldsymbol{p}}^\circ(s)| & \text{if } s \in B^\circ. \end{cases} \tag{4.2.10}$$

We say that $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ is a *rescaled viscosity evolution* with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ and initial condition $(u_0, e_0, p_0, z_0, 0)$ if $\boldsymbol{e}^\circ$ satisfies (4.2.8) and the following conditions hold:

(ev0)$^\circ$ *Initial condition*: $(\boldsymbol{u}^\circ(0), \boldsymbol{e}^\circ(0), \boldsymbol{p}^\circ(0), \boldsymbol{z}^\circ(0), t^\circ(0)) = (u_0, e_0, p_0, z_0, 0)$.

(ev1)$^\circ$ *Weak kinematic admissibility*: for every $s \in [0, +\infty)$

$$\begin{gathered} E\boldsymbol{u}^\circ(s) = \boldsymbol{e}^\circ(s) + \boldsymbol{p}^\circ(s) \quad \text{in } \Omega, \\ \boldsymbol{p}^\circ(s) = (\boldsymbol{w}(t^\circ(s)) - \boldsymbol{u}^\circ(s)) \odot \nu \mathcal{H}^{n-1} \quad \text{in } \Gamma_0. \end{gathered} \tag{4.2.11}$$

(ev2)$^\circ$ *Equilibrium condition*: for every $s \in [0, +\infty)$

$$-\operatorname{div} \boldsymbol{\sigma}^\circ(s) = \boldsymbol{f}(t^\circ(s)) \quad \text{in } \Omega, \qquad [\boldsymbol{\sigma}^\circ(s)\nu] = \boldsymbol{g}(t^\circ(s)) \quad \text{on } \Gamma_1. \tag{4.2.12}$$

(ev3$'$)$^\circ$ *Partial stress constraint*:

$$\boldsymbol{\sigma}^\circ(s) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s)) \quad \text{for every } s \in [0, +\infty) \setminus U^\circ, \tag{4.2.13}$$

where

$$U^\circ := \{s \in (0, +\infty) : t^\circ \text{ is constant in a neighbourhood of } s\}. \tag{4.2.14}$$

(ev3$''$)$^\circ$ *Flow rule*: for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$ we have $\dot{\boldsymbol{p}}^\circ(s) \ll \boldsymbol{\mu}(s)$ and there exists a precise representative $\hat{\boldsymbol{\sigma}}^\circ(s)$ of $\boldsymbol{\sigma}^\circ(s)$ with respect to $\dot{\boldsymbol{p}}^\circ(s)$ and $\boldsymbol{\zeta}^\circ(s)$ such that

$$\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)} \in N^{ext}_{\mathcal{K}_{\boldsymbol{\mu}(s)}(\boldsymbol{\zeta}^\circ(s))}(\hat{\boldsymbol{\sigma}}^\circ(s)), \tag{4.2.15}$$

where $\mathcal{K}_{\boldsymbol{\mu}(s)}(\boldsymbol{\zeta}^\circ(s))$ and $N^{ext}_{\mathcal{K}_{\boldsymbol{\mu}(s)}(\boldsymbol{\zeta}^\circ(s))}$ are defined by (1.3.16), and (4.2.9) respectively.

(ev4)$^\circ$ *Evolution law for the internal variable*: for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$ the strong $C^0(\overline{\Omega})$-limit

$$\dot{\boldsymbol{z}}^\circ(s) := s\text{-}\lim_{h \to 0} \frac{\boldsymbol{z}^\circ(s+h) - \boldsymbol{z}^\circ(s)}{h} \tag{4.2.16}$$

exists, and

$$\dot{\boldsymbol{z}}^\circ(s) = \rho_1 \star \big((\rho_2 \star \operatorname{tr} \boldsymbol{\sigma}^\circ(s)) \operatorname{tr} \dot{\boldsymbol{p}}^\circ(s)\big) \quad \text{in } \overline{\Omega} \text{ for } \mathcal{L}^1\text{-a.e. } s \in (0, +\infty); \tag{4.2.17}$$

moreover, if we define

$$\boldsymbol{w}^\circ(s) := \boldsymbol{w}(t^\circ(s)) \quad \text{and} \quad \boldsymbol{\chi}^\circ(s) := \boldsymbol{\chi}(t^\circ(s))\,, \tag{4.2.18}$$

where $\boldsymbol{\chi}$ is given by (2.3.45)-(2.3.48), we have that

$$\|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty + \|E\dot{\boldsymbol{w}}^\circ(s)\|_2 \leq 1\,. \tag{4.2.19}$$

Notice that, if (4.2.8) and (4.2.11) hold, by Remark 4.4 we have that $\dot{\boldsymbol{p}}^\circ(s) \in \Pi_{\Gamma_0}(\Omega)$ for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$, therefore it makes sense to speak of a precise representative according to Definition 4.1.

Remark 4.6. If $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ is a rescaled viscosity evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$, then $(\boldsymbol{u}^\circ\circ\varphi, \boldsymbol{e}^\circ\circ\varphi, \boldsymbol{p}^\circ\circ\varphi, \boldsymbol{z}^\circ\circ\varphi, \varphi^{-1}\circ t^\circ\circ\varphi)$ is a rescaled viscosity evolution with data $\boldsymbol{f}\circ\varphi$, $\boldsymbol{g}\circ\varphi$, and $\boldsymbol{w}\circ\varphi$ for every C^1 bijective increasing function $\varphi\colon [0,+\infty) \to [0,+\infty)$.

We shall frequently use the inclusion

$$A^\circ \subset U^\circ \tag{4.2.20}$$

which trivially follows from (4.2.13) and (4.2.7).

4.3 Equivalent formulation in energetic form

The goal of this section is to state and prove the main theorem of the chapter, showing that Definition 4.5 is indeed equivalent to another one, where the measure theoretical formulation of the flow rule (4.2.15) is replaced by the energy-dissipation balance (4.3.1) and the partial flow rule (4.3.2), which accounts for the behavior of the system at jump times. This formulation, which shares some features with the so-called energetic formulation for rate-independent processes (for this notion we refer to [31]), does not require the additional information (4.2.8) on the regularity in time of the stress $\boldsymbol{\sigma}^\circ(s)$ and is the one that we will use in the next chapter to prove the existence of an evolution satisfying Definition 4.5. We now state the announced result.

Theorem 4.7. *Assume that $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{w}$ satisfy* (2.3.42)-(2.3.48)*, that u_0, e_0, p_0, and z_0 are as in* (2.3.53)-(2.3.57)*, and that* (4.2.19) *holds. Let $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, and t° satisfy* (4.2.4)*. Let $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, and $\dot{\boldsymbol{p}}^\circ$ be defined as in* (4.2.5)-(4.2.6)*, and let A° and B° as in* (4.2.7)*. Then the following conditions are equivalent:*

(a) $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ *is a rescaled viscosity evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$, and initial condition $(u_0, e_0, p_0, z_0, 0)$, according to Definition 4.5;*

(b) conditions (ev0)°, (ev1)°, (ev2)°, (ev3′)°, (ev4)° *of Definition 4.5 are satisfied, as well as the following two properties:*

Energy-dissipation balance: for every $S \in [0, +\infty)$

$$\begin{aligned}\mathcal{Q}(\boldsymbol{e}^\circ(S)) - \mathcal{Q}(e_0) + \int_0^S \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds + \int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds = \\ = \int_0^S \Big(\langle \boldsymbol{\sigma}^\circ(s), E\dot{\boldsymbol{w}}(t^\circ(s))\rangle - \langle \boldsymbol{L}(t^\circ(s)), \dot{\boldsymbol{w}}(t^\circ(s))\rangle \Big)\, \dot{t}^\circ(s)\, ds - \qquad (4.3.1) \\ - \int_0^S \langle \dot{\boldsymbol{L}}(t^\circ(s)), \boldsymbol{u}^\circ(s)\rangle\, \dot{t}^\circ(s)\, ds + \langle \boldsymbol{L}(t^\circ(S)), \boldsymbol{u}^\circ(S)\rangle - \langle \boldsymbol{L}(0), u_0\rangle\,,\end{aligned}$$

where d_2 *is defined in* (1.3.17)*, and* $\boldsymbol{L}(t)$ *in* (2.3.43)*.*

Partial flow-rule: for $\mathcal{L}^1$*-a.e.* $s \in [0, +\infty)$ *with* $\boldsymbol{\sigma}^\circ(s) \notin \mathcal{K}(\boldsymbol{\zeta}^\circ(s))$ *we have* $\dot{\boldsymbol{p}}^\circ(s) \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ *and*

$$\langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle = \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\,. \qquad (4.3.2)$$

Remark 4.8. For every $\zeta \in C^0(\overline{\Omega})^+$ the function $s \mapsto \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)$ is measurable on $[0, +\infty)$ by Theorem 1.8. Approximating $s \mapsto \boldsymbol{\zeta}^\circ(s)$ by piecewise constant functions, we find that $s \mapsto \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))$ is measurable on $[0, +\infty)$, so the first integral in (4.3.1) makes sense.

Let φ_i be a dense sequence in the unit ball of $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$, composed of continuous functions with compact support. Since, taking into account (1.2.1),

$$\|\dot{\boldsymbol{p}}^\circ(s)\|_2 = \sup_i \langle \varphi_i, \dot{\boldsymbol{p}}^\circ(s)\rangle,$$

the function $s \mapsto \|\dot{\boldsymbol{p}}^\circ(s)\|_2$ is measurable, so the second integral in (4.3.1) makes sense.

It easily follows from (4.3.2) that there exists a measurable function $\lambda\colon A^\circ \to [0, +\infty)$ such that

$$\dot{\boldsymbol{p}}^\circ(s) = \lambda(s)\big(\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s))\big) \qquad (4.3.3)$$

for $\mathcal{L}^1$-a.e. $s \in A^\circ$. This justifies the choice of the name *flow-rule* for condition (4.3.2).

Remark 4.9. Notice that the energy-dissipation balance, together with (2.3.42) and (4.2.4), implies that the function $s \mapsto \mathcal{Q}(\boldsymbol{e}^\circ(s))$ is continuous. As the quadratic form $\mathcal{Q}$ is coercive, the weak continuity of $s \mapsto \boldsymbol{e}^\circ(s)$ from $[0, +\infty)$ to $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ implies the strong continuity. Together with the Lipschitz continuity of $s \mapsto \boldsymbol{p}^\circ(s)$, this gives that also $\boldsymbol{u}^\circ$ is continuous from $[0, +\infty)$ to $BD(\Omega)$ with respect to the norm topology.

We also observe that for every s_1, $s_2 \in [0, +\infty)$ we have

$$\boldsymbol{\chi}^\circ(s_2) - \boldsymbol{\chi}^\circ(s_1) = \int_{s_1}^{s_2} \dot{\boldsymbol{\chi}}^\circ(s)\, ds\,, \qquad (4.3.4)$$

where the last term is a Bochner integral in the Banach space $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$. Since $\|\cdot\|_\infty$ is convex and lower semicontinuous in $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$, the Jensen inequality and (4.2.19) imply that

$$\|\boldsymbol{\chi}^\circ(s_2) - \boldsymbol{\chi}^\circ(s_1)\|_\infty \le |s_2 - s_1|\,. \qquad (4.3.5)$$

However we remark that (4.3.4) and (4.3.5) do not imply the time differentiability of $\boldsymbol{\chi}^\circ$ with respect the L^∞ norm.

Before starting the proof of the theorem, we first notice that the energy-dissipation balance (4.3.1) can be expressed in terms of the function $\boldsymbol{\chi}$ that appears in the safe load condition (2.3.45)-(2.3.48). This will be useful in the proof of Theorem 4.7, and also to get the existence of a rescaled viscosity evolution in the next chapter.

Proposition 4.10. *Let $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ be as in* (2.3.42). *Assume that $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, and t° satisfy* (4.2.4), (4.2.11), *and* (4.2.12), *and that the safe load condition* (2.3.45)-(2.3.48) *holds. For every $s \in [0, +\infty)$ let us define*

$$\boldsymbol{w}^\circ(s) := \boldsymbol{w}(t^\circ(s)) \quad \textit{and} \quad \boldsymbol{\chi}^\circ(s) := \boldsymbol{\chi}(t^\circ(s))\,. \tag{4.3.6}$$

Then (4.3.1) *is equivalent to*

$$\begin{aligned} &\mathcal{Q}(\boldsymbol{e}^\circ(S)) - \mathcal{Q}(e_0) + \int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\Big)\, ds - \\ &- \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \boldsymbol{\chi}(0), p_0\rangle + \int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds = \\ &= \int_0^S \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds + \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{e}^\circ(S)\rangle - \langle \boldsymbol{\chi}(0), e_0\rangle\,, \end{aligned} \tag{4.3.7}$$

where $\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ and $\langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ are defined according to (2.3.12) *for every $s \in [0, +\infty)$.*

Proof. For every $s \in [0, +\infty)$ we put $\boldsymbol{L}^\circ(s) := \boldsymbol{L}(t^\circ(s))$. Since $\boldsymbol{L}^\circ \in H^1_{loc}([0, +\infty); BD(\Omega)')$ and $\boldsymbol{w}^\circ \in H^1_{loc}([0, +\infty); H^1(\Omega; \mathbb{R}^N))$, the scalar function $s \mapsto \langle \boldsymbol{L}^\circ(s), \boldsymbol{w}^\circ(s)\rangle$ belongs to $H^1_{loc}([0, +\infty))$ and its derivative is given by $s \mapsto \langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{w}^\circ(s)\rangle + \langle \boldsymbol{L}^\circ(s), \dot{\boldsymbol{w}}^\circ(s)\rangle$. Therefore we have

$$\begin{aligned} &- \int_0^S \langle \boldsymbol{L}^\circ(s), \dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{u}^\circ(s)\rangle\, ds + \langle \boldsymbol{L}^\circ(S), \boldsymbol{u}^\circ(S)\rangle - \langle \boldsymbol{L}^\circ(0), u_0\rangle = \\ &= \int_0^S \langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle\, ds + \langle \boldsymbol{L}^\circ(S), \boldsymbol{u}^\circ(S) - \boldsymbol{w}^\circ(S)\rangle - \langle \boldsymbol{L}^\circ(0), u_0 - \boldsymbol{w}(0)\rangle\,. \end{aligned} \tag{4.3.8}$$

By (2.3.44), for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$ we have

$$\langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle = \langle \dot{\boldsymbol{f}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle + \langle \dot{\boldsymbol{g}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle_{\Gamma_1}\,. \tag{4.3.9}$$

By (2.3.47) $\dot{\boldsymbol{\chi}}^\circ(s) \in L^\infty(\Omega; \mathbb{M}^{N\times N}_{sym})$, while (2.3.45) gives $-\mathrm{div}\, \dot{\boldsymbol{\chi}}^\circ(s) = \dot{\boldsymbol{f}}^\circ(s)$ in Ω and $[\dot{\boldsymbol{\chi}}^\circ(s)\nu] = \dot{\boldsymbol{g}}^\circ(s)$ on Γ_1 for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$. Therefore we can apply the integration-by-parts formula (2.3.29), which together with (4.3.9) gives

$$\langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)\rangle = -\langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle + \langle \dot{\boldsymbol{\chi}}^\circ(s), E\boldsymbol{w}^\circ(s)\rangle \tag{4.3.10}$$

for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$. This proves that $s \mapsto \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ is measurable; by (2.3.48) and by (2.3.13), we deduce that $s \mapsto \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ belongs to $L^1_{loc}([0, +\infty))$.

Similarly we prove

$$\langle \boldsymbol{L}^\circ(s), \boldsymbol{u}^\circ(s) - \boldsymbol{w}^\circ(s)\rangle = \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle + \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s)\rangle - \langle \boldsymbol{\chi}^\circ(s), E\boldsymbol{w}^\circ(s)\rangle \tag{4.3.11}$$

for every $s \in [0, +\infty)$. By (4.3.8), (4.3.10), and (4.3.11) we have

$$-\int_0^S \langle \boldsymbol{L}^\circ(s), \dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{L}}^\circ(s), \boldsymbol{u}^\circ(s)\rangle\, ds + \langle \boldsymbol{L}^\circ(S), \boldsymbol{u}^\circ(S)\rangle - \langle \boldsymbol{L}^\circ(0), u_0\rangle =$$
$$= -\int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds - \int_0^S \langle \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds +$$
$$+ \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle - \langle \boldsymbol{\chi}(0), p_0\rangle + \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{e}^\circ(S)\rangle - \langle \boldsymbol{\chi}(0), e_0\rangle.$$

Therefore (4.3.1) is equivalent to (4.3.7). □

The proof of Theorem 4.7 needs some preliminary work, where we use some tools that will also be employed in the remainder of the thesis. A first point is to deduce from (4.3.1) some suitable estimates allowing us to improve the time regularity of the stress. In this perspective, taking into account the previous proposition, it will be useful to study some properties of the function $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$, where the duality, thanks to (2.3.45)-(2.3.46) is correctly defined according to (2.3.12). They are collected in the next lemma, where we also prove (4.3.13) and (4.3.14) that need the a-priori information on the time differentiability of $\boldsymbol{\sigma}^\circ$.

Lemma 4.11. *Assume that $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, and t° satisfy* (4.2.4)*,* (4.2.11)*, and* (4.2.12)*, and that the safe load condition* (2.3.45)-(2.3.48) *holds. Define $\boldsymbol{\chi}^\circ$ as in* (4.3.6)*. Then, for every $s' \in [0, +\infty)$ the functions $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s')\rangle$ and $s \mapsto \langle \boldsymbol{\chi}^\circ(s'), \boldsymbol{p}^\circ(s)\rangle$ are globally Lipschitz continuous with Lipschitz constants $\|p_0\|_1 + s'$ and $M_k\zeta_m$, respectively. Therefore $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ is locally Lipschitz continuous. Moreover, for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$,*

$$\frac{d}{ds}\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s')\rangle = \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s')\rangle\,. \tag{4.3.12}$$

If in addition $\boldsymbol{e}^\circ$ is differentiable with respect to the strong topology of $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$, then

$$\frac{d}{ds}\langle \boldsymbol{\chi}^\circ(s'), \boldsymbol{p}^\circ(s)\rangle = \langle \boldsymbol{\chi}^\circ(s'), \dot{\boldsymbol{p}}^\circ(s)\rangle\,, \tag{4.3.13}$$
$$\frac{d}{ds}\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle = \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle + \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \tag{4.3.14}$$

for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$.

Proof. Let us fix $s' \in [0, +\infty)$. The integration-by-parts formula (2.3.29), together with (2.3.45)-(2.3.47), gives

$$\begin{aligned} \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s')\rangle &= \langle \boldsymbol{\chi}^\circ(s), E\boldsymbol{w}^\circ(s') - \boldsymbol{e}^\circ(s')\rangle + \\ &+ \langle \boldsymbol{f}^\circ(s), \boldsymbol{u}^\circ(s') - \boldsymbol{w}^\circ(s')\rangle + \langle \boldsymbol{g}^\circ(s), \boldsymbol{u}^\circ(s') - \boldsymbol{w}^\circ(s')\rangle_{\Gamma_1} \end{aligned} \tag{4.3.15}$$

for every $s, s' \in [0, S]$. In view of the differentiability properties of $\boldsymbol{\chi}^\circ$, $\boldsymbol{f}^\circ$ and $\boldsymbol{g}^\circ$ given by (4.2.19) and (2.3.42), this implies that $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s')\rangle$ is absolutely continuous, as well as (4.3.12). By (4.2.19) and (2.3.13) we also get that $|\langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s')\rangle| \le (\|p_0\|_1 + s')$, therefore the global Lipschitz continuity of $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s')\rangle$ follows.

The Lipschitz continuity of $s \mapsto \langle \boldsymbol{\chi}^\circ(s'), \boldsymbol{p}^\circ(s)\rangle$ and the estimate of the Lipschitz constant are an immediate consequence of the 1-Lipschitz continuity of $\boldsymbol{p}^\circ$, together with (2.3.13) and (2.3.46). To prove (4.3.13), we preliminarly observe that at each $s \in [0,+\infty)$ such that $\boldsymbol{e}^\circ(s)$ is differentiable, by Remark 4.4 the measure $[\boldsymbol{\chi}^\circ(s') : \dot{\boldsymbol{p}}^\circ(s)]$, as well as the duality $\langle \boldsymbol{\chi}^\circ(s'), \dot{\boldsymbol{p}}^\circ(s)\rangle$, are correctly defined according to (2.3.10)-(2.3.11), and (2.3.12), respectively. By Proposition 2.3, for a.e. $s \in [0,+\infty)$ and every $\psi \in C^1(\overline{\Omega})$, with $\psi = 0$ in a neighborhood of $\partial\Omega \setminus \Gamma_0$, we have

$$\begin{aligned} \langle [\boldsymbol{\chi}^\circ(s') : \dot{\boldsymbol{p}}^\circ(s)], \psi\rangle = -\langle \psi\boldsymbol{\chi}^\circ(s'), \dot{\boldsymbol{e}}^\circ(s) - E\dot{\boldsymbol{w}}^\circ(s)\rangle - \\ - \langle \boldsymbol{\chi}^\circ(s'), (\dot{\boldsymbol{u}}^\circ(s) - \dot{\boldsymbol{w}}^\circ(s)) \odot \nabla\psi\rangle + \langle \boldsymbol{f}^\circ(s'), \psi(\dot{\boldsymbol{u}}^\circ(s) - \dot{\boldsymbol{w}}^\circ(s))\rangle\,. \end{aligned} \tag{4.3.16}$$

Now, if $\boldsymbol{e}^\circ$ is differentiable at s, the right-hand side is clearly the limit of the corresponding difference quotients. Therefore (4.3.16) and the analogous formula for the difference quotients, which can be deduced again from Proposition 2.3, imply that at $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$, the derivative of the function $s \mapsto \langle [\boldsymbol{\chi}^\circ(s') : \boldsymbol{p}^\circ(s)], \psi\rangle$ equals to $\langle [\boldsymbol{\chi}^\circ(s') : \dot{\boldsymbol{p}}^\circ(s)], \psi\rangle$, hence for every $0 \le s_1 \le s_2 < +\infty$

$$\langle [\boldsymbol{\chi}^\circ(s') : \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)], \psi\rangle = \int_{s_1}^{s_2} \langle [\boldsymbol{\chi}^\circ(s') : \dot{\boldsymbol{p}}^\circ(s)], \psi\rangle\, ds. \tag{4.3.17}$$

We then consider a sequence $\psi_k \in C^\infty(\bar{\Omega})$, with $0 \le \psi_k \le 1$ in $\bar{\Omega}$ and $\psi_k = 0$ in a neighborhood of $\partial\Omega \setminus \Gamma_0$, such that $\psi_k(x) \to 1$ for every $x \in \Omega \cup \Gamma_0$. We now apply (4.3.17) to ψ_k. Since (2.3.13) implies that the integrands in the right-hand side are uniformly bounded, we can apply the Dominated Convergence Theorem and we finally get

$$\langle \boldsymbol{\chi}^\circ(s'), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle = \int_{s_1}^{s_2} \langle \boldsymbol{\chi}^\circ(s'), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \tag{4.3.18}$$

for every $0 \le s_1 \le s_2 < +\infty$.

To prove (4.3.14) we first observe that, by a direct computation and using (4.3.18) and (4.3.13), for every s and h we have

$$\begin{aligned} &\frac{1}{h}\big(\langle \boldsymbol{\chi}^\circ(s+h), \boldsymbol{p}^\circ(s+h)\rangle - \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\big) = \\ &= \frac{1}{h}\int_s^{s+h} \big(\langle \boldsymbol{\chi}^\circ(s+h), \dot{\boldsymbol{p}}^\circ(\tau)\rangle + \langle \dot{\boldsymbol{\chi}}^\circ(\tau), \boldsymbol{p}^\circ(s)\rangle\big)\, d\tau\,. \end{aligned}$$

Now, using (4.2.4), (4.3.5), and (2.3.13), we easily get that

$$\frac{1}{h}\int_s^{s+h} \langle \boldsymbol{\chi}^\circ(s+h) - \boldsymbol{\chi}^\circ(\tau), \dot{\boldsymbol{p}}^\circ(\tau)\rangle\, d\tau \le h\,.$$

Similarly, using also (4.2.19), we can prove that

$$\frac{1}{h}\int_s^{s+h} \langle \dot{\boldsymbol{\chi}}^\circ(\tau), \boldsymbol{p}^\circ(s) - \boldsymbol{p}^\circ(\tau)\rangle\, d\tau \le h\,.$$

It follows that

$$\begin{aligned} &\tfrac{1}{h}\big(\langle \boldsymbol{\chi}^\circ(s+h), \boldsymbol{p}^\circ(s+h)\rangle - \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\big) = \\ &\int_s^{s+h} \big(\langle \boldsymbol{\chi}^\circ(\tau), \dot{\boldsymbol{p}}^\circ(\tau)\rangle + \langle \dot{\boldsymbol{\chi}}^\circ(\tau), \boldsymbol{p}^\circ(\tau)\rangle\big)\, d\tau + R_h\,, \end{aligned}$$

where the remainder term R_h goes to 0 when h tends to 0. Therefore, (4.3.14) follows from the Lebesgue differentiation Theorem. □

We first want to prove that conditions (b) in the statement of Theorem 4.7 are sufficient for the existence of a rescaled viscosity evolution. To this end, we want to use (4.3.1) and (4.3.2) to deduce (4.2.8). We need two different strategies depending on whether (4.2.13) is satisfied or not. First, we deal with the case when the stress constraint is satisfied. The key estimate contained in the next theorem will help us to prove differentiability of $\boldsymbol{\sigma}^\circ$ in the set B°. In the proof of the theorem, we will make use of the following Gronwall-type inequality, whose proof can be found in [15, Lemma 7.3].

Lemma 4.12. *Let $\phi\colon [0,T] \to [0,+\infty[$ be a bounded measurable function, let $\psi\colon [0,T] \to [0,+\infty[$ be an integrable function, and $\chi(t)$ a positive nondecreasing function. Suppose that*

$$\phi(t)^2 \leq \int_0^t \phi(s)\,\psi(s)\,ds + \Big(\int_0^t \psi(s)\,ds\Big)^2 + \chi(t) \tag{4.3.19}$$

for every $t \in [0,T]$. Then

$$\phi(t) \leq \frac{3}{2}\Big(\int_0^t \psi(s)\,ds + \sqrt{\chi(t)}\Big) \tag{4.3.20}$$

for every $t \in [0,T]$.

Theorem 4.13. *Let $S > 0$, and assume that $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ satisfy the hypotheses of Theorem 4.7. Define $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, and $\dot{\boldsymbol{p}}^\circ$ as in (4.2.5) and (4.2.6). Let A° and B° be as in (4.2.7). Assume that condition (b) of Theorem 4.7 holds. Then there exists $L_S > 0$ such that*

$$\|\boldsymbol{\sigma}^\circ(s_2) - \boldsymbol{\sigma}^\circ(s_1)\|_2 \leq L_S(s_2 - s_1) \tag{4.3.21}$$

for every $0 \leq s_1 < s_2 \leq S$ with $s_1 \in B^\circ$.

Proof. We fix $0 \leq s_1 < s_2 \leq S$ with $s_1 \in B^\circ$. Denoting with $\mathcal{V}(\boldsymbol{p}^\circ, \boldsymbol{\zeta}^\circ(s_1); s_1, s_2)$ the total variation of $\boldsymbol{p}^\circ$ on $[s_1, s_2]$ with respect to the functional $\mathcal{H}(\cdot, \zeta)$ introduced in (1.3.20), Theorem 1.8 implies that

$$\mathcal{V}(\boldsymbol{p}^\circ, \boldsymbol{\zeta}^\circ(s_1); s_1, s_2) = \int_{s_1}^{s_2} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s_1))\,ds\,. \tag{4.3.22}$$

As $\boldsymbol{\zeta}^\circ$ is locally Lipschitz continuous, using the estimate (1.3.14), together with (4.3.22), we get that there exits a positive constant M_S such that

$$\mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_1)) \leq \int_{s_1}^{s_2} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\,ds + M_S(s_2 - s_1)^2\,. \tag{4.3.23}$$

Taking into account the energy-dissipation balance (4.3.7), we get the inequality

$$\begin{aligned} &\mathcal{Q}(\boldsymbol{e}^\circ(s_2)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_1)) + \mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_1)) + \int_{s_1}^{s_2} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\,ds \leq \\ &\leq \int_{s_1}^{s_2} \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s))\,ds - \int_{s_1}^{s_2} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\,ds + \langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{p}^\circ(s_2)\rangle - \\ &\quad - \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_1)\rangle + \langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{e}^\circ(s_2)\rangle - \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_1)\rangle + M_S(s_2 - s_1)^2\,. \end{aligned} \tag{4.3.24}$$

Since $\boldsymbol{\sigma}^\circ(s_1) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s_1))$ and $\boldsymbol{\zeta}^\circ(s_1) \in C^0(\overline{\Omega})$, by Proposition 2.5 we get

$$\langle \boldsymbol{\sigma}^\circ(s_1), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle \leq \mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_1))\,, \tag{4.3.25}$$

where the duality is defined according to (2.3.12). As $\boldsymbol{\sigma}^\circ(s_1)-\boldsymbol{\chi}^\circ(s_1))$ belongs to the space $\Sigma_0(\Omega)$ defined by (2.3.32), by the integration by parts formula proved in Proposition 2.3 and (4.3.25) we get

$$\begin{aligned} &\langle \boldsymbol{\sigma}^\circ(s_1)-\boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_1)-\boldsymbol{e}^\circ(s_2)\rangle = \\ &= \langle \boldsymbol{\sigma}^\circ(s_1)-\boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\rangle - \langle \boldsymbol{\sigma}^\circ(s_1)-\boldsymbol{\chi}^\circ(s_1), E\boldsymbol{w}^\circ(s_2)-E\boldsymbol{w}^\circ(s_1) \le \\ &\le \mathcal{H}(\boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_1)) - \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\rangle - \\ &- \int_{s_1}^{s_2} \langle \boldsymbol{\sigma}^\circ(s_1)-\boldsymbol{\chi}^\circ(s_1), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds\,. \end{aligned} \tag{4.3.26}$$

By a direct computation, and using (4.3.12), we have

$$\langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{p}^\circ(s_2)\rangle - \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_1)\rangle - \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s_1)\rangle = \int_{s_1}^{s_2} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s_2)\rangle\, ds\,, \tag{4.3.27}$$

while the similar equality

$$\langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{e}^\circ(s_2)\rangle - \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_1)\rangle - \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s_1)\rangle = \int_{s_1}^{s_2} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s_2)\rangle\, ds \tag{4.3.28}$$

is straighforward.

Summing (4.3.24) and (4.3.26), by the use of (4.3.27) and (4.3.28) we obtain that

$$\begin{aligned} &\mathcal{Q}(\boldsymbol{e}^\circ(s_2)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_1)) + \langle \boldsymbol{\sigma}^\circ(s_1), \boldsymbol{e}^\circ(s_1)-\boldsymbol{e}^\circ(s_2)\rangle \le \\ &\le \int_{s_1}^{s_2} \|\boldsymbol{\sigma}^\circ(s)-\boldsymbol{\sigma}^\circ(s_1)\|_2 \|E\dot{\boldsymbol{w}}^\circ(s))\|_2 + \int_{s_1}^{s_2} \langle \boldsymbol{\chi}^\circ(s_1)-\boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s))\rangle\, ds + \\ &+ \int_{s_1}^{s_2} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s)\rangle\, ds + \int_{s_1}^{s_2} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s)\rangle\, ds + M_S(s_2-s_1)^2\,. \end{aligned} \tag{4.3.29}$$

The inequality

$$\int_{s_1}^{s_2} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s_2)-\boldsymbol{p}^\circ(s)\rangle\, ds \le (s_2-s_1)^2 \tag{4.3.30}$$

easily follows from (2.3.13), the 1-Lipschitz continuity of $\boldsymbol{p}^\circ$ and (4.2.19). The latter also implies that there exists a positive constant, still denoted by M_S, such that

$$\int_{s_1}^{s_2} \langle \boldsymbol{\chi}^\circ(s_1)-\boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s))\rangle\, ds \le M_S(s_2-s_1)^2. \tag{4.3.31}$$

Moreover, we easily have that

$$\begin{aligned} \int_{s_1}^{s_2} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s)\rangle\, ds \le \int_{s_1}^{s_2} \|\dot{\boldsymbol{\chi}}^\circ(s)\|_2 \|\boldsymbol{e}^\circ(s)-\boldsymbol{e}^\circ(s_1)\|_2\, ds + \\ \|\boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s_1)\|_2 \int_{s_1}^{s_2} \|\dot{\boldsymbol{\chi}}^\circ(s)\|_2\, ds\,. \end{aligned} \tag{4.3.32}$$

Now the left-hand side of (4.3.29) equals $\frac{1}{2}\langle \boldsymbol{\sigma}^\circ(s_2)-\boldsymbol{\sigma}^\circ(s_1), \boldsymbol{e}^\circ(s_2)-\boldsymbol{e}^\circ(s_1)\rangle$, as a direct computation shows. Taking into account the coerciveness and the continuity of the quadratic form Q, from(4.3.30), (4.3.31), and (4.3.32), we obtain that there exists a constant B_S such that, for every $s_2 \in (s_1, S]$:

$$\|\boldsymbol{\sigma}^\circ(s_2) - \boldsymbol{\sigma}^\circ(s_1)\|_2^2 \le B_S\Big(\int_{s_1}^{s_2} \|\boldsymbol{\sigma}^\circ(s) - \boldsymbol{\sigma}^\circ(s_1)\|_2(\|E\dot{\boldsymbol{w}}^\circ(s))\|_2 + \|\dot{\boldsymbol{\chi}}^\circ(s)\|_2)\, ds + \\ + \|\boldsymbol{\sigma}^\circ(s_2) - \boldsymbol{\sigma}^\circ(s_1)\|_2 \int_{s_1}^{s_2} \|\dot{\boldsymbol{\chi}}^\circ(s)\|_2\, ds + (s_2 - s_1)^2\Big).$$

Let $\psi(s)$ be given by $\|E\dot{\boldsymbol{w}}^\circ(s))\|_2 + \|\dot{\boldsymbol{\chi}}^\circ(s)\|_2$. By the previous estimate and the Cauchy inequality the exists a constant C_S such that

$$\|\boldsymbol{\sigma}^\circ(s_2) - \boldsymbol{\sigma}^\circ(s_1)\|_2^2 \le \\ C_S\Big(\int_{s_1}^{s_2} \|\boldsymbol{\sigma}^\circ(s) - \boldsymbol{\sigma}^\circ(s_1)\|_2\, \psi(s)\, ds + \big(\int_{s_1}^{s_2} \psi(s)\, ds\big)^2 + (s_2 - s_1)^2\Big).$$

(4.3.21) now follows immediately by applying Lemma 4.12 with $\phi(s)$ and $\chi(s)$ given by $\|\boldsymbol{\sigma}^\circ(s) - \boldsymbol{\sigma}^\circ(s_1)\|_2$ and $(s - s_1)^2$, respectively, taking into account that $\psi(s)$ is bounded by (4.2.19). ☐

The differentiability of $\boldsymbol{\sigma}^\circ$ as an L^2-valued function for $\mathcal{L}^1$-a.e. $s \in B^\circ$ follows now from the following abstract result. It deals with differentiation of a function $\boldsymbol{v}$ from an interval to a reflexive Banach space, which satisfies the Lipschitz condition $\|\boldsymbol{v}(s_2)-\boldsymbol{v}(s_1)\| \le L(s_2-s_1)$ when one of the points s_1, s_2 belongs to a fixed closed set.

Theorem 4.14. *Let $S > 0$, let A be an open subset of $(0, S)$, and let $B := [0, S] \setminus A$. Let X be a reflexive Banach space, let $L > 0$, and $\boldsymbol{v}\colon [0, S] \to X$ be a function such that*

$$\|\boldsymbol{v}(s_2) - \boldsymbol{v}(s_1)\| \le L(s_2 - s_1) \tag{4.3.33}$$

for every $0 \le s_1 < s_2 \le S$ with $s_1 \in B$. Then for $\mathcal{L}^1$-a.e. $s_0 \in B$ there exists

$$\dot{\boldsymbol{v}}(s_0) := s\text{-}\lim_{h\to 0} \frac{\boldsymbol{v}(s_0 + h) - \boldsymbol{v}(s_0)}{h}, \tag{4.3.34}$$

where the limit is taken in the strong topology of X.

Proof. Let $\tilde{\boldsymbol{v}}\colon [0, S] \to X$ be the function defined by $\tilde{\boldsymbol{v}}(s) := \boldsymbol{v}(s)$ if $s \in B$, and

$$\tilde{\boldsymbol{v}}(s) := \frac{s-a}{b-a}\boldsymbol{v}(b) + \frac{b-s}{b-a}\boldsymbol{v}(a)$$

if $s \in A$ and (a, b) is the connected component of A containing s. It follows easily from (4.3.33) that $\tilde{\boldsymbol{v}}$ satisfies the Lipschitz estimate

$$\|\tilde{\boldsymbol{v}}(s_2) - \tilde{\boldsymbol{v}}(s_1)\| \le L(s_2 - s_1) \tag{4.3.35}$$

for every $0 \le s_1 < s_2 \le S$. It follows from the general theory of absolutely continuous functions with values in reflexive Banach spaces (see, e.g., [4, Appendix]) that for $\mathcal{L}^1$-a.e. $s_0 \in [0, S]$ there exists the limit

$$\dot{\tilde{\boldsymbol{v}}}(s_0) := s\text{-}\lim_{h\to 0} \frac{\tilde{\boldsymbol{v}}(s_0 + h) - \tilde{\boldsymbol{v}}(s_0)}{h}. \tag{4.3.36}$$

On the other hand, by the Lebesgue Differentiation Theorem for $\mathcal{L}^1$-a.e. $s_0 \in B$ we have

$$\lim_{h \to 0^+} \frac{\mathcal{L}^1(A \cap (s_0 - h, s_0 + h))}{h} = 0\,. \tag{4.3.37}$$

Let us fix s_0 satisfying (4.3.36) and (4.3.37). We want to prove that (4.3.34) holds. Fisrt of all we observe that, by (4.3.37), s_0 cannot be an endpoint of a connected component of A. For every h with $s_0 + h \in [0, S]$ we define η_h in the following way. If $s_0 + h \in B$, then $\eta_h := h$; if $s_0 + h \in A$ and (a_h, b_h) is the connected component of A containing $s_0 + h$, we set $\eta_h = a_h - s_0$. Note that $s_0 + \eta_h \in B$, $\eta_h \leq h$, and η_h has the same sign as h.

Let us prove that

$$\eta_h \to 0 \quad \text{as} \quad h \to 0\,. \tag{4.3.38}$$

If $h > 0$ this is obvious, since $0 < \eta_h \leq h$ by construction. To prove that $\eta_h \to 0$ as $h \to 0^-$, we assume by contradiction that $\eta_{h_i} \to \eta < 0$ for some sequence $h_i \to 0^-$. By construction the interval $(s_0 + \eta_{h_i}, s_0 + h_i)$ is contained in A. It follows that $(s_0 + \eta, s_0) \subset A$. This contradicts the fact that s_0 is not an endpoint of a connected component of A, and concludes the proof of (4.3.38).

If $s_0 + h \in A$ and $h > 0$, then

$$h - \eta_h = \mathcal{L}^1((a_h, s_0 + h)) \leq \mathcal{L}^1(A \cap (s_0 - h, s_0 + h))\,,$$

where the last inequality follows from the inclusion $(a_h, s_0 + h) \subset A$ and the inequality $s_0 \leq a_h$. By (4.3.37), this implies that

$$\lim_{h \to 0^+} \frac{\eta_h}{h} = 1\,. \tag{4.3.39}$$

On the other hand, if $s_0 + h \in A$ and $h < 0$, then $(s_0 + \eta_h, s_0 + h) \subset A$, thus

$$0 < h - \eta_h = \leq \mathcal{L}^1(A \cap (s_0 - \eta_h, s_0 + \eta_h))\,.$$

By (4.3.37) and (4.3.38), we conclude that

$$\lim_{h \to 0^-} \frac{\eta_h}{h} = 1\,. \tag{4.3.40}$$

As $s_0 + \eta_h \in B$, (4.3.36) implies that

$$\dot{\tilde{\ }}(s_0) := s\text{-}\lim_{h \to 0} \frac{\boldsymbol{v}(s_0 + \eta_h) - \boldsymbol{v}(s_0)}{\eta_h}\,. \tag{4.3.41}$$

To prove (4.3.35), it is enough to show that

$$\frac{\boldsymbol{v}(s_0 + h) - \boldsymbol{v}(s_0)}{h} - \frac{\boldsymbol{v}(s_0 + \eta_h) - \boldsymbol{v}(s_0)}{\eta_h} \tag{4.3.42}$$

tends to 0. We write (4.3.42) as I_h and II_h, where

$$\begin{aligned} I_h &:= \frac{\boldsymbol{v}(s_0 + h) - \boldsymbol{v}(s_0)}{h} - \frac{\boldsymbol{v}(s_0 + \eta_h) - \boldsymbol{v}(s_0)}{h}\,, \\ II_h &:= \frac{\boldsymbol{v}(s_0 + \eta_h) - \boldsymbol{v}(s_0)}{h} - \frac{\boldsymbol{v}(s_0 + \eta_h) - \boldsymbol{v}(s_0)}{\eta_h}\,. \end{aligned}$$

As $s_0 + \eta_h \in B$ and $\eta_h \leq h$, by (4.3.33) we have

$$\|I_h\| \leq L\frac{h-\eta_h}{h} \to 0\,, \tag{4.3.43}$$

where the convergence to 0 follows from (4.3.39) and (4.3.40). As for the second term, we have

$$\|II_h\| = \big(\frac{\eta_h}{h} - 1\big)\frac{\|\boldsymbol{v}(s_0+\eta_h) - \boldsymbol{v}(s_0)\|}{\eta_h} \to 0\,, \tag{4.3.44}$$

where the convergence to 0 follows from (4.3.39), (4.3.40), and (4.3.41). From (4.3.43) and (4.3.44) we deduce that (4.3.42) tends to 0, and this concludes the proof. □

We turn to the case when $s \in A^\circ$, that is when (4.2.13) is not satisfied. Assuming (4.3.2) and strong continuity of $\boldsymbol{\sigma}^\circ$, we are able to prove that in the set A°, $\boldsymbol{p}^\circ$ is L^2-valued up to constant measure and Lipschitz continuous in the L^2 topology. This enables us to deduce the Lipschitz continuity of $\boldsymbol{\sigma}^\circ$ in A°. Since A° is open, this implies that $\boldsymbol{\sigma}^\circ$ is differentiable with respect to time for $\mathcal{L}^1$-a.e. $s \in A^\circ$. This is the object of the following three lemmas. We premit that absolute continuity of $\boldsymbol{p}^\circ$ could also be deduced from the $\leq$ inequality in (4.3.1), as we will be forced to do in the next chapter. Here we prefer a different statement since, in the present form, Lemma 4.15 proves useful for both the implications of Theorem 4.7.

Lemma 4.15. *Assume that* $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ *satisfy the hypotheses of Theorem 4.7, and conditions* (ev0)°, (ev1)°, (ev2)°, (ev3′)°, (ev4)° *of Definition 4.5. Define* $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, *and* $\dot{\boldsymbol{p}}^\circ$ *as in* (4.2.5) *and* (4.2.6). *Let* A° *and* B° *be as in* (4.2.7). *Let* (a,b) *be a connected component of* A°, *and let* $c \in (a,b)$. *If* (4.3.2) *holds and* $s \mapsto \boldsymbol{e}^\circ(s)$ *is strongly continuous as a function from* $[0,S]$ *to* $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, *then* $\boldsymbol{p}^\circ - \boldsymbol{p}^\circ(c) \in Lip_{loc}((a,b);L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))$. *In particular, for* $\mathcal{L}^1$*-a.e.* $s \in (a,b)$, $\dot{\boldsymbol{p}}^\circ(s)$ *is the strong limit in* $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, *as* $h \to 0$, *of the difference quotient* $\frac{1}{h}(\boldsymbol{p}^\circ(s+h) - \boldsymbol{p}^\circ(s))$, *and* $\dot{\boldsymbol{p}}^\circ \in L^\infty_{loc}((a,b);L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))$. *Moreover, for every* $s_1, s_2 \in (a,b)$, *we have*

$$\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1) \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym}) \quad \textit{and} \quad \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1) = \int_{s_1}^{s_2} \dot{\boldsymbol{p}}^\circ(s)\,ds\,, \tag{4.3.45}$$

where the last term is a Bochner integral in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$.

Proof. By (4.3.2) for $\mathcal{L}^1$-a.e. $s \in (a,b)$ we have $\dot{\boldsymbol{p}}^\circ(s) \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and there exists a measurable function $\lambda\colon A^\circ \to [0,+\infty)$ such that

$$\dot{\boldsymbol{p}}^\circ(s) = \lambda(s)\big(\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s))\big) \tag{4.3.46}$$

for $\mathcal{L}^1$-a.e. $s \in (a,b)$.

We now show that $\lambda(s)$ is locally bounded in (a,b). To this aim, we fix $a < s_1 < s_2 < b$. Observe that by our hypotheses the function $s \mapsto \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s))$ is continuous also with respect to the $L^1(\Omega;\mathbb{M}^{N\times N}_{sym})$ norm topology, therefore there exists $\eta > 0$ such that $\|\boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s))\|_1 \geq \eta$ for every $s \in [s_1,s_2]$. Since $\|\dot{\boldsymbol{p}}^\circ(s)\|_1 \leq 1$ for $\mathcal{L}^1$-a.e. s, we get from (4.3.46) that $\lambda(s) \leq \frac{1}{\eta}$ for $\mathcal{L}^1$-a.e. $s \in [s_1,s_2]$. It then follows, again using (4.3.46) that there exits $C(s_1,s_2)$ such that

$$\|\dot{\boldsymbol{p}}^\circ(s)\|_2 \leq C(s_1,s_2) \tag{4.3.47}$$

for $\mathcal{L}^1$-a.e $s \in [s_1, s_2]$.

This fact and the measurability of $s \mapsto \langle \varphi, \dot{\boldsymbol{p}}^\circ(s)\rangle$ for every $\varphi \in C_0^0(\Omega; \mathbb{M}^{N\times N}_{sym})$ imply that $s \mapsto \langle \psi, \dot{\boldsymbol{p}}^\circ(s)\rangle$ is measurable for every $\psi \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$, hence $\dot{\boldsymbol{p}}^\circ\colon [a_1, b_1] \to L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ is weakly measurable. By Pettis Theorem it is also strongly measurable, so that (4.3.47) implies that $\dot{\boldsymbol{p}}^\circ \in L^\infty_{loc}((a,b); L^2(\Omega; \mathbb{M}^{N\times N}_{sym}))$. For every $\varphi \in C_0^0(\Omega; \mathbb{M}^{N\times N}_{sym})$, the function $s \mapsto \langle \varphi, \dot{\boldsymbol{p}}^\circ(s)\rangle$ is measurable and bounded, hence, for every $s_1, s_2 \in (a,b)$, we have

$$\langle \varphi, \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle = \int_{s_1}^{s_2} \langle \varphi, \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds = \Big\langle \varphi, \int_{s_1}^{s_2} \dot{\boldsymbol{p}}^\circ(s)\, ds \Big\rangle\,,$$

where the last equality follows from the fact that the Bochner integral of $\dot{\boldsymbol{p}}^\circ$ in the last term is well defined in $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$. By the arbitrariness of φ, this proves (4.3.45). The inclusion $\boldsymbol{p}^\circ - \boldsymbol{p}^\circ(c) \in Lip_{loc}((a,b); L^2(\Omega; \mathbb{M}^{N\times N}_{sym}))$ follows now from (4.3.45), as well as the statement about the difference quotients, thanks to the Differentiation Theorem for Bochner integrals. $\square$

Lemma 4.16. *Assume that* $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ *satisfy the hypotheses of Theorem 4.7 and conditions* (ev0)°, (ev1)°, (ev2)°, and (ev3′)° *of Definition 4.5. Let* (a,b) *be as in Lemma 4.15 and assume that* (4.3.45) *holds. Then, for every* $a < s_1 < s_2 < b$,

$$\boldsymbol{u}^\circ(s_2) - \boldsymbol{u}^\circ(s_1) \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^N)\,, \tag{4.3.48}$$

where $H^1_{\Gamma_0}(\Omega; \mathbb{R}^N)$ *is defined by* (1.2.3).

Proof. Let us fix $a < s_1 < s_2 < b$. From the weak kinematic admissibility (4.2.11), we have

$$E\boldsymbol{u}^\circ(s_2) - E\boldsymbol{u}^\circ(s_1) = \boldsymbol{e}^\circ(s_2) - \boldsymbol{e}^\circ(s_1) + \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1) \quad \text{in } \Omega\,, \tag{4.3.49}$$

$$\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1) = ((\boldsymbol{w}^\circ(s_2) - \boldsymbol{w}^\circ(s_1)) - (\boldsymbol{u}^\circ(s_2) - \boldsymbol{u}^\circ(s_1))) \odot \nu\mathcal{H}^{n-1} \quad \text{in } \Gamma_0\,. \tag{4.3.50}$$

As the measure $\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)$ belongs to $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$, it does not charge Γ_0, so that the left-hand side of (4.3.50) is 0; since $\boldsymbol{w}^\circ(s)$ is constant in (a,b) by (4.2.20), we get $\boldsymbol{u}^\circ(s_2) - \boldsymbol{u}^\circ(s_1) = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_0. Moreover, the right-hand side of (4.3.49) belongs to $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$. By (1.2.2) we have $\boldsymbol{u}^\circ(s_2) - \boldsymbol{u}^\circ(s_1) \in H^1(\Omega; \mathbb{R}^N)$. $\square$

Lemma 4.17. *Under the assumptions of Lemma 4.16, the function* $\boldsymbol{e}^\circ$ *belongs to the space* $AC_{loc}((a,b); L^2(\Omega; \mathbb{M}^{N\times N}_{sym}))$ *and*

$$\alpha_Q \|\dot{\boldsymbol{e}}^\circ(s)\|_2 \le \beta_Q \|\dot{\boldsymbol{p}}^\circ(s)\|_2 \tag{4.3.51}$$

for $\mathcal{L}^1$*-a.e.* $s \in (a,b)$. *In particular, under the assumptions of Lemma 4.15, the function* $\boldsymbol{e}^\circ$ *belongs to* $\mathrm{Lip}_{loc}((a,b); L^2(\Omega; \mathbb{M}^{N\times N}_{sym}))$

Proof. Let us fix $a < s_1 < s_2 < b$. By (4.2.20) and by (4.2.12) we have that $\boldsymbol{\sigma}^\circ(s_2) - \boldsymbol{\sigma}^\circ(s_1)$ belongs to the set $\Sigma_0(\Omega)$ defined by (2.3.32), so that from (2.3.33), (4.3.48), and (4.3.49), we get

$$\langle \boldsymbol{\sigma}^\circ(s_2) - \boldsymbol{\sigma}^\circ(s_1), \boldsymbol{e}^\circ(s_2) - \boldsymbol{e}^\circ(s_1)\rangle = \langle \boldsymbol{\sigma}^\circ(s_2) - \boldsymbol{\sigma}^\circ(s_1), \boldsymbol{p}^\circ(s_1) - \boldsymbol{p}^\circ(s_2)\rangle\,; \tag{4.3.52}$$

by (2.3.4) this yields $2\alpha_Q \|\boldsymbol{e}^\circ(s_2) - \boldsymbol{e}^\circ(s_1)\|_2 \le 2\beta_Q \|\boldsymbol{p}^\circ(s_1) - \boldsymbol{p}^\circ(s_2)\|_2$, and the conclusion follows from (4.3.45). $\square$

As a second step towards the proof of Theorem 4.7 we use the previous results to prove that condition (b) in the statement of Theorem 4.7 implies a weak formulation of the flow rule

$$\dot{p}^\circ(s,x) \in N_{K(\zeta^\circ(s,x))}(\sigma^\circ(s,x))$$

for almost all s such that $\boldsymbol{\sigma}^\circ(s) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s))$. This weak formulation is a measure-theoretical counterpart of the so-called maximal dissipation principle (see [28]).

Theorem 4.18. *Let* $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ *be a rescaled viscosity evolution with data* $\boldsymbol{f}$, $\boldsymbol{g}$, *and* $\boldsymbol{w}$ *satisfying* (2.3.42)-(2.3.48) *and initial condition* $(u_0, e_0, p_0, z_0, 0)$ *as in* (2.3.53)-(2.3.57), *and define* $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, *and* $\dot{\boldsymbol{p}}^\circ$ *as in* (4.2.5) *and* (4.2.6). *Let* A° *and* B° *as in* (4.2.7). *Then, for* $\mathcal{L}^1$*-a.e.* $s \in B^\circ$

$$\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = \langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s) \rangle \,, \tag{4.3.53}$$

where the duality in the right-hand side is defined according to (2.3.12).

Proof. As $\dot{\boldsymbol{\sigma}}^\circ(s)$ exists for $\mathcal{L}^1$-a.e. $s \in B^\circ$ by Theorems 4.13 and 4.14, applying Remark 4.4 we obtain that $\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s) \rangle$ is well-defined at $\mathcal{L}^1$-a.e. $s \in B^\circ$. Moreover, by Proposition 2.5, we have the inequality

$$\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s) \rangle \le \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) \,. \tag{4.3.54}$$

To prove the converse inequality, we first observe that, for every $s > 0$ such that the derivatives $\dot{\boldsymbol{\chi}}^\circ(s)$ and $\dot{\boldsymbol{e}}^\circ(s)$ exist, the function $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s) \rangle$ is trivially differentiable and

$$\frac{d}{ds} \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s) \rangle = \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s) \rangle + \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{e}}^\circ(s) \rangle \,. \tag{4.3.55}$$

By (4.3.7), for every $s \in [0, +\infty)$ and $h > 0$ we have the energy inequality

$$\begin{aligned} &\mathcal{Q}(\boldsymbol{e}^\circ(s+h)) - \mathcal{Q}(\boldsymbol{e}^\circ(s)) + \int_s^{s+h} \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(\tau), \boldsymbol{\zeta}^\circ(\tau)) + \langle \dot{\boldsymbol{\chi}}^\circ(\tau), \boldsymbol{p}^\circ(\tau) \rangle \Big)\, d\tau - \\ &- \langle \boldsymbol{\chi}^\circ(s+h), \boldsymbol{p}^\circ(s+h) \rangle + \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s) \rangle \le \int_s^{s+h} \langle \boldsymbol{\sigma}^\circ(\tau) - \boldsymbol{\chi}^\circ(\tau), E\dot{\boldsymbol{w}}^\circ(\tau) \rangle \, d\tau - \\ &- \int_s^{s+h} \langle \dot{\boldsymbol{\chi}}^\circ(\tau), \boldsymbol{e}^\circ(\tau) \rangle \, d\tau + \langle \boldsymbol{\chi}^\circ(s+h), \boldsymbol{e}^\circ(s+h) \rangle - \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s) \rangle \,. \end{aligned}$$

Dividing by h and taking the limit as h tends to 0, by (4.3.14), (4.3.55), and the Lebesgue Differentiation Theorem, we get

$$\langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{e}}^\circ(s) \rangle + \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) - \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s) \rangle \le \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s) \rangle \tag{4.3.56}$$

for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$. As $\boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s)$ belongs to the set $\Sigma_0(\Omega)$ defined by (2.3.32), the integration by parts formula given by Proposition 2.3 implies that, when $s \in B^\circ$, (4.3.56) is equivalent to

$$\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s) \rangle \ge \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) \,,$$

as required. ☐

Remark 4.19. As a technical point, notice that Theorem 4.18 could not be proved directly after Theorem 4.14 as it may seem at a first glance, since the use of (4.3.14) requires the time differentiability of $\boldsymbol{e}^\circ$ for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$, and not only in B°.

We are ready to prove one implication in Theorem 4.7.

Proof of Theorem 4.7: part one. Assume (b). Then (4.2.8) follows from Remark 4.9, Theorems 4.13, 4.14 and Lemma 4.17. It remains to show that (ev3$''$)$^\circ$. Using Definition 4.1 and (4.2.9) it is easy to see that (4.3.2) is equivalent to (ev3$''$)$^\circ$ when $s \in A^\circ$ with $\hat{\boldsymbol{\sigma}}^\circ(s) := \boldsymbol{\sigma}^\circ(s)$, therefore we have only to deal with the case $s \in B^\circ$. By Proposition 2.5, the measure $H(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) - [\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)]$, which is correctly defined for $\mathcal{L}^1$-a.e. $s \in B^\circ$ thanks to Remark 4.4, is a nonnegative measure on $\Omega \cup \Gamma_0$. Therefore equality (4.3.53) implies that

$$H(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = [\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)] \quad \text{on } \Omega \cup \Gamma_0. \tag{4.3.57}$$

for $\mathcal{L}^1$-a.e. $s \in B^\circ$.

Let us fix $s \in B^\circ$ such that $\dot{\boldsymbol{p}}^\circ(s)$ exists and (4.3.57) holds. Let $E(s) \subset \Omega$ and $F(s) \subset \Omega \cup \Gamma_0$ be two disjoint Borel sets such that $E(s) \cup F(s) = \Omega \cup \Gamma_0$ and $\boldsymbol{\mu}_s(s)(E(s)) = \mathcal{L}^n(F(s)) = 0$, where $\boldsymbol{\mu}_s(s)$ denotes the singular part of $\boldsymbol{\mu}(s)$ with respect to the Lebesgue measure. By (2.3.26) and (2.3.27) we have that

$$[\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)] = \boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}_a^\circ(s) \text{ on } E(s)\,, \tag{4.3.58}$$

where $\dot{\boldsymbol{p}}_a^\circ(s)$ denotes the absolutely continuous part of $\dot{\boldsymbol{p}}^\circ(s)$ with respect to the Lebesgue measure. We define

$$\hat{\sigma}^\circ(s,x) := \sigma^\circ(s,x) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in E(s)\,, \tag{4.3.59}$$

$$\hat{\sigma}^\circ(s,x) := \partial_0 H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x), \zeta^\circ(s,x)\Big) \qquad \text{for } \boldsymbol{\mu}_s(s)\text{-a.e. } x \in F(s)\,, \tag{4.3.60}$$

where $\partial_0 H(\xi, \zeta^\circ(s,x))$ denotes the element of $\partial_\xi H(\xi, \zeta^\circ(s,x))$ with minimum norm. Observe that by the definition of $\boldsymbol{\mu}(s)$, for $\boldsymbol{\mu}(s)$-a.e. $x \in F(s)$ we have

$$\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x) = \frac{\dot{\boldsymbol{p}}^\circ(s)}{|\dot{\boldsymbol{p}}^\circ(s)|}(x).$$

Therefore, thanks to the continuity of $x \mapsto \zeta^\circ(s,x)$, [48, Lemma 3.16] yields that for $\mathcal{L}^1$-a.e. $s \in B^\circ$ the function $\hat{\boldsymbol{\sigma}}^\circ(s)$ belongs to $L^\infty_{\boldsymbol{\mu}(s)}(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$. It is obvious that $\hat{\sigma}^\circ(s,x) = \sigma^\circ(s,x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$.

We now prove that $\hat{\boldsymbol{\sigma}}^\circ(s)$ is a precise representative $\hat{\boldsymbol{\sigma}}^\circ(s)$ of $\boldsymbol{\sigma}^\circ(s)$ with respect to $\dot{\boldsymbol{p}}^\circ(s)$ and $\boldsymbol{\zeta}^\circ(s)$. Since $\sigma^\circ(s,x) \in K(\zeta^\circ(s,x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$ when $s \in B^\circ$, and $\partial_0 H(\xi,\zeta) \subseteq K(\xi)$ for every $\xi \in \mathbb{M}^{N\times N}_{sym}$ and $\zeta \in (0,+\infty)$ by [40, Corollary 23.5.3], we get (4.2.1). Since $\hat{\sigma}^\circ(s,x) = \sigma^\circ(s,x)$ for $|\dot{\boldsymbol{p}}_a^\circ(s)|$-a.e. $x \in E(s)$, using (4.3.58) we can easily deduce that

$$[\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)] = \Big(\hat{\boldsymbol{\sigma}}^\circ(s) : \frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}\Big)\, \boldsymbol{\mu}(s) \qquad \text{on } E(s)\,. \tag{4.3.61}$$

By (4.3.60) and the Euler identity, we get

$$\hat{\sigma}^\circ(s,x) : \frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x) = H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x), \zeta^\circ(s,x)\Big) \qquad \text{for } \boldsymbol{\mu}(s)\text{-a.e. } x \in F(s)\,,$$

therefore (4.3.57) implies that

$$[\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)] = \Big(\hat{\boldsymbol{\sigma}}^\circ(s) : \frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}\Big)\, \boldsymbol{\mu}(s) \qquad \text{on } F(s)\,.$$

From this and (4.3.61) we get (4.2.2).

It only remains to prove (4.2.15). Using [40, Theorem 23.5] we can prove that, when $s \in B^\circ$, condition (4.2.15) is equivalent to

$$\hat{\sigma}^\circ(s,x) \in \partial_p H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x), \zeta^\circ(s,x)\Big) \qquad \text{for } \boldsymbol{\mu}(s)\text{-a.e. } x \in \Omega \cup \Gamma_0\,,$$

where ∂_p denotes the subdifferential with respect to the first variable. Since $\partial_p \mathcal{H}$ is positively homogeneous of degree 0 and $\boldsymbol{\mu}(s) \ll \mathcal{L}^n$ on $E(s)$, taking into account (4.3.59) this is equivalent to the fact that both the following inclusions are satisfied:

$$\sigma^\circ(s,x) \in \partial_p H(\dot{\boldsymbol{p}}^\circ_a(s)(x), \zeta^\circ(s,x)) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega\,, \tag{4.3.62}$$

$$\hat{\sigma}^\circ(s,x) \in \partial_p H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x), \zeta^\circ(s,x)\Big) \qquad \text{for } \boldsymbol{\mu}(s)\text{-a.e. } x \in F(s)\,, \tag{4.3.63}$$

where, in (4.3.62), we also used the fact that $\mathcal{L}^n(F(s)) = 0$. By construction, (4.3.63) is satisfied. Taking the absolutely continuous part in (4.3.57), we easily get that

$$\sigma^\circ(s,x) : \dot{\boldsymbol{p}}^\circ_a(s)(x) = H(\dot{\boldsymbol{p}}^\circ_a(s)(x), \zeta^\circ(s,x)) \tag{4.3.64}$$

for $\mathcal{L}^n$-a.e. $x \in \Omega$. At every $x \in \Omega$ such that $\sigma^\circ(s,x) \in K(\zeta^\circ(s,x))$, by the definition of H for every $\xi \in \mathbb{M}^{N\times N}_{sym}$ we get that $\sigma^\circ(s,x) : \xi \le H(\xi, \zeta^\circ(s,x))$. Combining this inequality with (4.3.64) we get

$$\sigma^\circ(s,x) : (\xi - \dot{\boldsymbol{p}}^\circ_a(s)(x)) \le H(\xi, \zeta^\circ(s,x)) - H(\dot{\boldsymbol{p}}^\circ_a(s)(x), \zeta^\circ(s,x))$$

for every $\xi \in \mathbb{M}^{N\times N}_{sym}$ and $\mathcal{L}^n$-a.e. $x \in \Omega$. This implies (4.3.63). Therefore we have shown that (b) $\Longrightarrow$ (a). The proof of the converse implication, which needs some other preliminary lemmas, will restart after Lemma 4.22. $\square$

To prove the implication (a) $\Longrightarrow$ (b) in Theorem 4.7 we need some other preliminary lemmas, and to fix some notation. We define

$$\mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)) := \mathcal{Q}(\boldsymbol{e}^\circ(s)) - \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\,, \tag{4.3.65}$$

$$\boldsymbol{\tau}^\circ(s) := \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s) \tag{4.3.66}$$

for every $s \in [0,+\infty)$. It is easily seen that $\boldsymbol{\tau}^\circ$ belongs to the space $\Sigma_0(\Omega)$ defined by (2.3.32). For every fixed $S > 0$ we set $A^\circ_S := A^\circ \cap [0,S]$ and $B^\circ_S := B^\circ \cap [0,S]$. A key point for proving Theorem 4.7 is showing that, if $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ is a rescaled viscosity evolution, the function $s \mapsto \mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s))$ is absolutely continuous. This is the object of the following three lemmas.

Lemma 4.20. *Let $S > 0$. Assume that the hypotheses of Theorem 4.7 and that conditions* (ev0)$^\circ$, (ev1)$^\circ$, (ev2)$^\circ$, *and* (4.2.19) *of Definition 4.5 are satisfied. Then there exists a constant L_S such that*

$$|\mathcal{Q}_{\boldsymbol{\chi}}(s_2, \boldsymbol{e}^\circ(s_2)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_1, \boldsymbol{e}^\circ(s_1))| \le L_S |s_2 - s_1| \tag{4.3.67}$$

for every s_1 and s_2 in B°_S.

Proof. Let s_1 and s_2 be as in the statement of the lemma. Since $\boldsymbol{\tau}^\circ(s) \in \Sigma_0(\Omega)$ for every s, a direct algebraic computation and Proposition 2.3 give

$$\begin{gathered}\mathcal{Q}_{\boldsymbol{\chi}}(s_2, \boldsymbol{e}^\circ(s_2)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_1, \boldsymbol{e}^\circ(s_1)) + \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_2) - \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_1) + \boldsymbol{e}^\circ(s_2)\rangle = \\ = \tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_2) + \boldsymbol{\tau}^\circ(s_1), \boldsymbol{e}^\circ(s_2) - \boldsymbol{e}^\circ(s_1)\rangle = \\ = \tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1) + \boldsymbol{\tau}^\circ(s_2), E\boldsymbol{w}^\circ(s_2) - E\boldsymbol{w}^\circ(s_1)\rangle - \tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1) + \boldsymbol{\tau}^\circ(s_2), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle\,.\end{gathered}$$

Let ζ_S be an upper bound for $\|\boldsymbol{\zeta}^\circ(s)\|_\infty$ on $[0,S]$; by (1.3.4) and (2.3.46), we get that

$$\|\boldsymbol{\tau}^\circ(s_i)\|_\infty \le M_K(\zeta_m + \zeta_S) \tag{4.3.68}$$

for $i = 1, 2$. Therefore (4.2.4) and (2.3.13) give

$$|\tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1) + \boldsymbol{\tau}^\circ(s_2), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle| \le M_K(\zeta_m + \zeta_S)|s_2 - s_1|\,.$$

By (4.3.68) and (4.2.19)

$$|\tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1) + \boldsymbol{\tau}^\circ(s_2), E\boldsymbol{w}^\circ(s_2) - E\boldsymbol{w}^\circ(s_1)\rangle| \le \mathcal{L}^n(\Omega)^{\frac{1}{2}} M_K(\zeta_m + \zeta_S)|s_2 - s_1|\,.$$

Denoting with C_S an upper bound for $\|\boldsymbol{e}^\circ(s)\|_2$ on $[0,S]$, (4.3.5) yields

$$|\tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_2) - \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_1) + \boldsymbol{e}^\circ(s_2)\rangle| \le \mathcal{L}^n(\Omega)^{\frac{1}{2}} C_S|s_2 - s_1|\,.$$

The conclusion follows easily from the previous inequalities. □

Lemma 4.21. *Let $S > 0$, assume that the hypotheses of Theorem 4.7 hold, and that $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ is a rescaled viscosity evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ and initial condition $(u_0, e_0, p_0, z_0, 0)$. Then*

$$\int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds < +\infty\,. \tag{4.3.69}$$

If we define the function $g(s)$ by

$$g(s) := -\int_0^s \mathcal{H}(\boldsymbol{p}^\circ(\tau), \boldsymbol{\zeta}^\circ(\tau))\, d\tau - \int_0^s \|\dot{\boldsymbol{p}}^\circ(\tau)\|_2\, d_2(\boldsymbol{\sigma}^\circ(\tau), \boldsymbol{\zeta}^\circ(\tau))\, d\tau + \langle \boldsymbol{\chi}^\circ(\tau), \boldsymbol{p}^\circ(\tau)\rangle\,,$$

for every $s \ge 0$, we have that the function g is absolutely continuous. Moreover, if (a,b) is a connected component of A°, then the equality

$$\mathcal{Q}_{\boldsymbol{\chi}}(s_2, \boldsymbol{e}^\circ(s_2)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_1, \boldsymbol{e}^\circ(s_1)) = g(s_2) - g(s_1) \tag{4.3.70}$$

holds for every $a \le s_1 \le s_2 \le b$.

Proof. Fix a and b as in the statement. As $\boldsymbol{\mu}(s) = \mathcal{L}^n$ when $s \in A^\circ$, it is easily seen that (4.2.15) is equivalent to (4.3.2). Therefore the hypotheses of Lemma 4.15 are satisfied, hence $\|\dot{\boldsymbol{p}}^\circ(s)\|_2$ is locally bounded in (a,b) and (4.3.45) holds. By Lemma 4.17 this implies that $\boldsymbol{e}^\circ$ is Lipschitz continuous in $[s_1, s_2]$ for every $a < s_1 < s_2 < b$. It follows that the function $s \mapsto \mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s))$ is Lipschitz continuous on $[s_1, s_2]$. Moreover, since $\boldsymbol{\chi}^\circ$ is constant on $[a,b]$, we have

$$\tfrac{d}{ds}\mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)) = \langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle \tag{4.3.71}$$

for $\mathcal{L}^1$-a.e. $s \in [s_1, s_2]$. By (4.3.45) we also get that

$$\tfrac{d}{ds}\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle = \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \tag{4.3.72}$$

for $\mathcal{L}^1$-a.e. $s \in [s_1, s_2]$, where in this case the right-hand side can be equivalently regarded as the generalized duality defined by (2.3.12), or the usual scalar product of L^2.

By definition of the extended normal cone, $N^{ext}_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)) \subset N_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s))$, where the latter is the usual normal cone of Convex Analysis. As $\boldsymbol{\mu}(s) = \mathcal{L}^n$ for every $s \in (a, b)$, inclusion (4.2.15) and [40, Theorem 23.5] give that

$$\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)) \in \partial_p \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta^\circ(s))$$

for $\mathcal{L}^1$-a.e. $s \in (a, b)$. Therefore, the Euler identity implies that

$$\langle \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle = \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) \tag{4.3.73}$$

for $\mathcal{L}^1$-a.e. $s \in (a, b)$. As we already noticed, when $s \in A^\circ$, (4.2.15) is equivalent to (4.3.2), which in its turn is equivalent (4.3.3). The latter gives

$$\|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle$$

for $\mathcal{L}^1$-a.e. $s \in A^\circ_S$, so that using (4.3.73) we get

$$\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = \langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\,. \tag{4.3.74}$$

Recalling that $E\dot{\boldsymbol{w}}^\circ(s) = 0$ in A°_S, by an integration by parts argument using (4.3.48), we get

$$\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle = \langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle + \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle = -\langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle + \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\,. \tag{4.3.75}$$

With this, integrating (4.3.74) between s_1 and s_2 and taking into account (4.3.71) and (4.3.72), we obtain that

$$\begin{aligned} &\mathcal{Q}_{\boldsymbol{\chi}}(s_2, \boldsymbol{e}^\circ(s_2)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_1, \boldsymbol{e}^\circ(s_1)) + \int_{s_1}^{s_2} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds + \\ &+ \int_{s_1}^{s_2} \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds = \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle\,, \end{aligned} \tag{4.3.76}$$

where we used the fact that $\boldsymbol{\chi}^\circ$ is constant in $[a, b]$.

So far, (4.3.76) holds for $a < s_1 \le s_2 < b$; by continuity of the function $s \mapsto \boldsymbol{e}^\circ(s)$, it is also true when $a = s_1$ and $b = s_2$. Therefore, since a and b belong to B°_S and taking into account (2.3.13) and (4.3.67), we get

$$\begin{aligned} &\int_a^b \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds \le \\ &\le \int_a^b \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds + L_S(b-a) + \|\boldsymbol{\chi}^\circ(a)\|_\infty \|\boldsymbol{p}^\circ(b) - \boldsymbol{p}^\circ(a)\|_1\,. \end{aligned}$$

By the Lipschitz continuity of $\boldsymbol{p}^\circ$ and recalling that $\|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = 0$ in B°_S, this immediately implies (4.3.69). Taking into account that $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ is locally Lipschitz continuous by Lemma 4.11, it follows that $g(s)$, as defined in the statement, is absolutely continuous. By (4.3.76), the proof is concluded. ☐

We are now in position to prove that $\mathcal{Q}_{\boldsymbol{\chi}}$ is absolutely continuous.

Lemma 4.22. *Under the assumptions of Lemma 4.21, the function $s \mapsto \mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s))$ belongs to $AC([0,S])$. Moreover*

$$\tfrac{d}{ds}\mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)) = \langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle \tag{4.3.77}$$

for $\mathcal{L}^1$-a.e. $s \in [0,S]$.

Proof. The absolute continuity of $\mathcal{Q}_{\boldsymbol{\chi}}$ trivially follows from Lemmas 4.20 and 4.21. Since $\boldsymbol{e}^\circ$ is almost everywhere differentiable, a direct computation gives (4.3.77). □

After these preliminary lemmas, we can now complete the proof of Theorem 4.7.

Proof of Theorem 4.7: part 2. Assume (a). When $s \in A^\circ$, it is easy to see that condition (4.2.15) is equivalent to (4.3.2), therefore only (4.3.1) has to be proved. Fix $S > 0$. Since (4.2.15) is equivalent to (4.3.63) for $\mathcal{L}^1$-a.e. $s \in B^\circ_S$, the Euler identity and (4.2.2) give that

$$\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = \langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$$

for $\mathcal{L}^1$-a.e. $s \in B^\circ_S$. This equality can be also written in the form

$$\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = \langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\,, \tag{4.3.78}$$

observing that the additional term equals 0 in B°_S. By (4.3.74) this equality holds true indeed for $\mathcal{L}^1$-a.e. $s \in [0,S]$, provided that the duality in the right-hand side is understood as the generalized duality defined by (2.3.12) when $s \in B^\circ_S$ and as the usual L^2 duality product when $s \in A^\circ_S$.

Integrating by parts as in Proposition 2.3 and recalling the definition of $\boldsymbol{\tau}^\circ$, we get that

$$\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle = -\langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle + \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle + \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \tag{4.3.79}$$

for $\mathcal{L}^1$-a.e. $s \in B^\circ_S$. By (4.3.75), recalling that $E\dot{\boldsymbol{w}}^\circ(s) = 0$ in A°_S, this equality holds true for $\mathcal{L}^1$-a.e. $s \in [0,S]$.

Summing (4.3.77) with (4.3.78) and taking into account (4.3.79), we get

$$\begin{aligned}\tfrac{d}{ds}\mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)) + \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s)) =\\ = \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle + \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\end{aligned}$$

for $\mathcal{L}^1$-a.e. $s \in [0,S]$. Using (4.3.14) we obtain

$$\begin{aligned}\tfrac{d}{ds}\mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)) + \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle + \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s)) =\\ = \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle + \tfrac{d}{ds}\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\end{aligned}$$

for $\mathcal{L}^1$-a.e. $s \in [0,S]$. Integrating between 0 and S, by the absolute continuity of the function $s \mapsto \mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s))$ and recalling the definitions of $\mathcal{Q}_{\boldsymbol{\chi}}$ and $\boldsymbol{\tau}^\circ$, we obtain (4.3.7), which is equivalent to (4.3.1) by Proposition 4.10. This concludes the proof. □

4.4 Intrinsic charachter of the precise representative

The main difficulty in Theorem 4.7 is the choice of a representative $\hat{\boldsymbol{\sigma}}^\circ(s)$ that satisfies simultaneously the flow rule and (4.2.2). However, this representative has an intrinsic character, provided we assume that the elastic domain is strictly convex. Indeed, if $K(1)$ is strictly convex, i.e., $\lambda\xi_1 + (1-\lambda)\xi_2$ is an interior point of $K(1)$ for every $0 < \lambda < 1$ and every pair of distinct points ξ_1, $\xi_2 \in K(1)$, then, for fixed s and x, $H(\xi, \zeta^\circ(s,x))$ is differentiable with respect to ξ at all points $\xi \neq 0$ (see, e.g., [40, Corollary 23.5.4 and Theorem 25.1]) and we keep the notation $\partial_\xi H(\xi, \zeta^\circ(s,x))$ for the partial gradient. Under this hypothesis, for $\mathcal{L}^1$-a.e. $s \in B^\circ$ the representative $\hat{\boldsymbol{\sigma}}^\circ(s)$ of $\boldsymbol{\sigma}^\circ(s)$ is uniquely determined $\boldsymbol{\mu}(s)$-a.e. on $\Omega \cup \Gamma_0$ by (4.3.63) as

$$\hat{\sigma}^\circ(s,x) = \sigma^\circ(s,x) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega\,, \tag{4.4.1}$$

$$\hat{\sigma}^\circ(s,x) = \partial_\xi H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x), \zeta^\circ(s,x)\Big) \qquad \text{for } \boldsymbol{\mu}(s)\text{-a.e. } x \in \Omega \cup \Gamma_0\,. \tag{4.4.2}$$

The following theorem shows that, under the same hypothesis, $\hat{\boldsymbol{\sigma}}^\circ(s)$ can be obtained in Ω as the limit of $\boldsymbol{\sigma}^\circ_r(s)$ as $r \to 0$, where for every $r > 0$ and every $s \in [0, +\infty)$ the spherical averages $\boldsymbol{\sigma}^\circ_r(s) \in C(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym})$ are defined by

$$\sigma^\circ_r(s,x) := \frac{1}{\mathcal{L}^n(B(x,r)\cap\Omega)} \int_{B(x,r)\cap\Omega} \sigma^\circ(s,y)\,dy\,. \tag{4.4.3}$$

Theorem 4.23. *Assume that $K(1)$ is strictly convex. Let $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ be a rescaled viscosity evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfying* (2.3.42)-(2.3.48) *and initial condition $(u_0, e_0, p_0, z_0, 0)$ as in* (2.3.53)-(2.3.57), *and define $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, and $\dot{\boldsymbol{p}}^\circ$ as in* (4.2.5) *and* (4.2.6). *Let $\hat{\boldsymbol{\sigma}}^\circ(s)$ be a representative of $\boldsymbol{\sigma}^\circ(s)$ as in (ev3′)°, and let $\boldsymbol{\sigma}^\circ_r$ be defined by* (4.4.3). *Then $\boldsymbol{\sigma}^\circ_r(s) \to \hat{\boldsymbol{\sigma}}^\circ(s)$ strongly in $L^1_{\boldsymbol{\mu}(s)}(\Omega; \mathbb{M}^{N\times N}_{sym})$ for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$.*

Proof. This proof closely follows that of [13, Theorem 6.6], which was in its turn inspired by [2, Theorem 3.7]. Let A° and B° as in (4.2.7). We observe that $\boldsymbol{\sigma}^\circ_r(s) \to \boldsymbol{\sigma}^\circ(s)$ strongly in $L^1(\Omega; \mathbb{M}^{N\times N}_{sym})$ for every s. When $s \in A^\circ$, we have $\boldsymbol{\mu}(s) = \mathcal{L}^n$ and $\hat{\boldsymbol{\sigma}}^\circ(s) = \boldsymbol{\sigma}^\circ(s)$ $\boldsymbol{\mu}(s)$-a.e., so that the result is obvious in this case.

When $s \in B^\circ$, $\|\boldsymbol{\sigma}^\circ_r(s)\|_\infty$ is bounded uniformly with respect to r, so it is enough to prove that $\boldsymbol{\sigma}^\circ_r(s) \to \hat{\boldsymbol{\sigma}}^\circ(s)$ strongly in $L^1_{|\dot{\boldsymbol{p}}^\circ(s)|}(U; \mathbb{M}^{N\times N}_{sym})$ for every open set $U \subset\subset \Omega$.

Let us fix U. Since $\boldsymbol{\sigma}^\circ_r(s) \to \boldsymbol{\sigma}^\circ(s)$ strongly in $L^2(U; \mathbb{M}^{N\times N}_{sym})$, div $\boldsymbol{\sigma}^\circ_r(s) \to$ div $\boldsymbol{\sigma}^\circ(s)$ strongly in $L^n(U; \mathbb{R}^N)$, and $\boldsymbol{\sigma}^\circ_r(s)$ is bounded in $L^\infty(U; \mathbb{M}^{N\times N}_{sym})$, by (2.3.24) we have

$$\langle[\boldsymbol{\sigma}^\circ_r(s) : \dot{\boldsymbol{p}}^\circ(s)], \varphi\rangle \to \langle[\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)], \varphi\rangle \tag{4.4.4}$$

for every $\varphi \in C^0_0(U)$ and for $\mathcal{L}^1$-a.e. $s \in B^\circ$. As $x \mapsto \sigma^\circ_r(s,x)$ is a continuous function, by (2.3.22) we have

$$\langle[\boldsymbol{\sigma}^\circ_r(s) : \dot{\boldsymbol{p}}^\circ(s)], \varphi\rangle = \langle\varphi\boldsymbol{\sigma}^\circ_r(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\,, \tag{4.4.5}$$

where the duality in the right-hand side is the standard duality between a continuous function and a measure. By (4.3.57), we also have $H(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = [\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)]$ on $\Omega \cup \Gamma_0$. Therefore the definition (1.3.19) of $H(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))$, (4.4.4), and (4.4.5) together with the boundedness of $\boldsymbol{\sigma}_r^\circ(s)$, imply that

$$\boldsymbol{\sigma}_r^\circ(s) : \frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)} \rightharpoonup H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}, \boldsymbol{\zeta}^\circ(s)\Big) \qquad \text{weakly}^* \text{ in } L^\infty_{\boldsymbol{\mu}(s)}(U) \tag{4.4.6}$$

for $\mathcal{L}^1$-a.e. $s \in B^\circ$.

Let us fix $s \in B^\circ$ such that (4.4.2) and (4.4.6) hold. Since $\boldsymbol{\sigma}_r^\circ(s)$ is bounded in $L^\infty_{\boldsymbol{\mu}(s)}(U; \mathbb{M}^{N\times N}_D)$, there exists a sequence $r_j \to 0$ such that $\boldsymbol{\sigma}_{r_j}^\circ(s) \rightharpoonup \sigma^*$ for some $\sigma^* \in L^\infty_{\boldsymbol{\mu}(s)}(U; \mathbb{M}^{N\times N}_{sym})$. From (4.4.6) we deduce that

$$\sigma^* : \frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)} \;=\; H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}, \boldsymbol{\zeta}^\circ(s)\Big) \qquad \boldsymbol{\mu}(s)\text{-a.e. on } U\,. \tag{4.4.7}$$

Let us fix $\xi \in \mathbb{M}^{N\times N}_{sym}$ and $\varepsilon > 0$. We denote the unit ball of $\mathbb{M}^{N\times N}_{sym}$ by $B_{\mathbb{M}^{N\times N}_{sym}}$. As the function $x \mapsto \zeta^\circ(s,x)$ is continuous, arguing as kn the proof of Lemma 2.4 for j large enough, only depending on ε and U, we have $\sigma_{r_j}^\circ(s,x) \in K(\zeta^\circ(s,x)) + \varepsilon B_{\mathbb{M}^{N\times N}_{sym}}$ for every $x \in U$. We then have $\sigma_{r_j}^\circ(s,x) : \xi \le H(\xi, \zeta^\circ(s,x)) + \varepsilon|\xi|$ for every x in U. As $\boldsymbol{\sigma}_{r_j}^\circ(s) : \xi \rightharpoonup \boldsymbol{\sigma}^* : \xi$ weakly* in $L^\infty_{\boldsymbol{\mu}(s)}(U)$, by the arbitrariness of ε we have also $\sigma^*(x) : \xi \le H(\xi, \zeta^\circ(s,x))$ for $\boldsymbol{\mu}(s)$-a.e. x in U, where $\sigma^*(x)$ denotes the value of $\boldsymbol{\sigma}^*$ at the point x. Taking (4.4.7) into account, we get

$$\sigma^* : \Big(\xi - \frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}\Big) \;\le\; H(\xi, \boldsymbol{\zeta}^\circ(s)) - H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}, \boldsymbol{\zeta}^\circ(s)\Big) \qquad \boldsymbol{\mu}(s)\text{-a.e. on } U\,. \tag{4.4.8}$$

In view of the differentiability properties of H, this implies

$$\sigma^* = \partial_\xi H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}, \boldsymbol{\zeta}^\circ(s)\Big) \qquad \boldsymbol{\mu}(s)\text{-a.e. on } U\,.$$

By (4.4.2) we deduce that $\sigma^* = \hat{\boldsymbol{\sigma}}^\circ(s)$ $\boldsymbol{\mu}(s)$-a.e. on U. Since the limit does not depend on the sequence r_j, we conclude that

$$\boldsymbol{\sigma}_r^\circ(s) \rightharpoonup \hat{\boldsymbol{\sigma}}^\circ(s) \qquad \text{weakly}^* \text{ in } L^\infty_{\boldsymbol{\mu}(s)}(U; \mathbb{M}^{N\times N}_D)\,.$$

Since $\dot{\boldsymbol{p}}^\circ(s) \ll \boldsymbol{\mu}(s)$, we get

$$\boldsymbol{\sigma}_r^\circ(s) \rightharpoonup \hat{\boldsymbol{\sigma}}^\circ(s) \qquad \text{weakly}^* \text{ in } L^\infty_{|\dot{\boldsymbol{p}}^\circ(s)|}(U; \mathbb{M}^{N\times N}_D)\,. \tag{4.4.9}$$

Now, as $\Big|\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x)\Big| > 0$ for $|\dot{\boldsymbol{p}}^\circ(s)|$-a.e. x on $\Omega \cup \Gamma_0$, and $N_K(\xi) = \{0\}$ if ξ is in the interior of K, for $\mathcal{L}^1$-a.e. $s \in B^\circ$ we deduce from (4.2.15) that $\hat\sigma^\circ(s,x) \in \partial K(\zeta^\circ(s,x))$ for $|\dot{\boldsymbol{p}}^\circ(s)|$-a.e. $x \in U$. On the other hand we easily have that $\sigma_r^\circ(s,x) \in K^r(s,x)$ for every $x \in U$, where $K^r(s,x)$ is the closed convex set defined by

$$K^r(s,x) := \overline{\mathrm{conv}}\Big(\bigcup_{y \in B(x,r)\cap\Omega} K(\zeta^\circ(s,y))\Big)\,.$$

When r tends to 0, by the continuity of the function $x \mapsto \zeta^\circ(s,x)$ we have that $K^r(s,x) \to K(\zeta^\circ(s,x))$ in the Hausdorff distance, uniformly for $x \in U$. Therefore the strict convexity of $K(\zeta^\circ(s,x))$ and [54, Corollary 2] can be used to improve the weak* convergence in (4.4.9) and to obtain strong convergence in $L^1_{|\dot{\boldsymbol{p}}^\circ(s)|}(U; \mathbb{M}^{N\times N}_{sym})$. $\square$

We are finally in position to give another equivalent definition of a rescaled viscosity evolution under hypotheses of strict convexity. To be definite, in the next theorem we show that, if $K(1)$ is strictly convex, it is enough that (4.2.15) is satisfied in Ω with $\hat{\boldsymbol{\sigma}}^\circ(s)$ equal to the limit of the spherical averages, provided that a different form of the flow rule holds at the boundary (see (4.4.11) below).

Theorem 4.24. *Assume that $K(1)$ is strictly convex. Let $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfy* (2.3.42)-(2.3.48)*, let u_0, e_0, p_0, and z_0 be as in* (2.3.53)-(2.3.57) *and suppose that* (4.2.19) *holds. Let $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, and t° satisfy* (4.2.4)*, let $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, and $\dot{\boldsymbol{p}}^\circ$ be defined as in* (4.2.5)-(4.2.6)*, let $\boldsymbol{\sigma}^\circ_r$ be defined as in* (4.4.3)*, let A° and B° be as in* (4.2.7)*, and let $\boldsymbol{\mu}(s)$ be as in* (4.2.10)*. Then the following conditions are equivalent:*

(a) $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ is a rescaled viscosity evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$, and initial condition $(u_0, e_0, p_0, z_0, 0)$, according to Definition 4.5;

(b) the function $\boldsymbol{e}^\circ \colon [0,+\infty) \to L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ is strongly continuous on $[0,+\infty)$ and differentiable $\mathcal{L}^1$-a.e. on $[0,+\infty)$, conditions (ev0)°, (ev1)°, (ev2)°, (ev3′)°, (ev4)° *of Definition 4.5 are satisfied, for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$ we have that $\dot{\boldsymbol{p}}^\circ(s) \ll \boldsymbol{\mu}(s)$, the sequence $\boldsymbol{\sigma}^\circ_r(s)$ converges strongly in $L^1_{\boldsymbol{\mu}(s)}(\Omega;\mathbb{M}^{N\times N}_{sym})$ to a function $\hat{\boldsymbol{\sigma}}^\circ(s)$ as $r \to 0^+$, and*

$$\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)} \in N^{ext}_{\mathcal{K}^\Omega_{\boldsymbol{\mu}(s)}(\boldsymbol{\zeta}^\circ(s))}(\hat{\boldsymbol{\sigma}}^\circ(s)) \quad \text{in } L^2_{\boldsymbol{\mu}(s)}(\Omega;\mathbb{M}^{N\times N}_{sym}), \tag{4.4.10}$$

$$[\boldsymbol{\sigma}^\circ(s)\nu]\cdot(\dot{\boldsymbol{w}}^\circ(s) - \dot{\boldsymbol{u}}^\circ(s)) = H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}, \boldsymbol{\zeta}^\circ(s)\Big)\frac{\boldsymbol{\mu}(s)}{\mathcal{H}^{n-1}} \quad \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_0, \tag{4.4.11}$$

where for every $\zeta \in C^0(\bar{\Omega})$ and every $\mu \in M^+_b(\Omega)$

$$\mathcal{K}^\Omega_\mu(\zeta) := \{\sigma \in L^2_\mu(\Omega;\mathbb{M}^{N\times N}_{sym}) : \sigma(x) \in K(\zeta(x)) \text{ for } \mu\text{-a.e. } x \in \Omega\}$$

and $N^{ext}_{\mathcal{K}^\Omega_\mu(\zeta)}$ is the corresponding extended normal cone in $L^2_\mu(\Omega;\mathbb{M}^{N\times N}_{sym})$ according to (4.2.9)*.*

Notice that, under the assumptions in (b), the existence of

$$\dot{\boldsymbol{u}}^\circ(s) := w^*\text{-}\lim_{h\to 0}\frac{\boldsymbol{u}^\circ(s+h) - \boldsymbol{u}^\circ(s)}{h} \qquad (w^*\text{-topology of } BD(\Omega))$$

follows from Remark 4.4. The same remark assures that $(\dot{\boldsymbol{u}}^\circ(s), \dot{\boldsymbol{e}}^\circ(s), \dot{\boldsymbol{p}}^\circ(s), \dot{\boldsymbol{w}}^\circ(s))$ satisfy the weak kinematic admissibility condition. In particular, $\dot{\boldsymbol{p}}^\circ(s) = (\dot{\boldsymbol{w}}^\circ(s) - \dot{\boldsymbol{u}}^\circ(s)) \odot \nu\mathcal{H}^{n-1}$ on Γ_0, so that $\boldsymbol{\mu}(s) \ll \mathcal{H}^{n-1}$ on Γ_0. Moreover, under the same assuptions, $\hat{\boldsymbol{\sigma}}^\circ(s) \in L^\infty_{\boldsymbol{\mu}(s)}(\Omega;\mathbb{M}^{N\times N}_{sym})$ for $\mathcal{L}^1$-a.e. $s \in B^\circ$. On the other hand, when $s \in A^\circ$ we have $\boldsymbol{\mu}(s) = \mathcal{L}^n$, so that, by the Lebesgue Differentiation Theorem, $\hat{\boldsymbol{\sigma}}^\circ(s) = \boldsymbol{\sigma}^\circ(s) \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$.

Proof of Theorem 4.24. Assume (a). Taking into account Definition 4.5 and Theorem 4.23, in order to obtain (b) it only remains to prove (4.4.11). By Remark 4.4 we have $\dot{\boldsymbol{p}}^\circ(s) = (\dot{\boldsymbol{w}}^\circ(s) - \dot{\boldsymbol{u}}^\circ(s)) \odot \nu\mathcal{H}^{n-1}$ on Γ_0 for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$. On the other hand $\dot{\boldsymbol{p}}^\circ(s) \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ for $\mathcal{L}^1$-a.e. $s \in A^\circ$ by (ev3″)°. We deduce that for these values of s we have $\dot{\boldsymbol{w}}^\circ(s) - \dot{\boldsymbol{u}}^\circ(s) = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_0. Since $\frac{\boldsymbol{\mu}(s)}{\mathcal{H}^{n-1}} = \frac{\mathcal{L}^n}{\mathcal{H}^{n-1}} = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_0 for every $s \in A^\circ$, we obtain (4.4.11) for $\mathcal{L}^1$-a.e. $s \in A^\circ$.

We now consider the case $s \in B^\circ$. By (4.3.57), we have

$$H(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = [\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)] \quad \text{on } \Gamma_0 .$$

On the other hand, by Remark 4.4 and (2.3.11), we have

$$[\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)] = [\boldsymbol{\sigma}^\circ(s)\nu] \cdot (\dot{\boldsymbol{w}}^\circ(s) - \dot{\boldsymbol{u}}^\circ(s))\mathcal{H}^{n-1} \quad \text{on } \Gamma_0 . \tag{4.4.12}$$

The last two equalities imply (4.4.11).

Conversely, assume (b). Since $\mathcal{L}^n \ll \boldsymbol{\mu}(s)$, by the Lebesgue Differentiation Theorem we get $\hat{\boldsymbol{\sigma}}^\circ(s) = \boldsymbol{\sigma}^\circ(s)$ $\mathcal{L}^n$-a.e. on Ω. For $\mathcal{L}^1$-a.e. $s \in B^\circ$ we extend $\hat{\boldsymbol{\sigma}}^\circ(s)$ to $\Omega \cup \Gamma_0$ by setting

$$\hat{\sigma}^\circ(s,x) = \partial_\xi H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x), \zeta^\circ(s,x)\Big) \qquad \text{for } \boldsymbol{\mu}(s)\text{-a.e. } x \text{ on } \Gamma_0 . \tag{4.4.13}$$

As $\partial_\xi H(\xi,\zeta) \subseteq K(\zeta)$ (see [40, Corollary 23.5.3]), we have $\sigma^\circ(s,x) \in K(\zeta^\circ(s,x))$ for $\boldsymbol{\mu}(s)$-a.e. $x \in \Gamma_0$. On the other hand, for every $s \in B^\circ$ we have $\hat{\sigma}^\circ(s,x) \in K(\zeta^\circ(s,x))$ for $\mathcal{L}^n$-a.e. x on Ω. Since $\boldsymbol{\zeta}^\circ(s)$ is continuous, arguing as in the proof of Theorem 4.23 we find that $\hat{\sigma}^\circ(s,x) \in K(\zeta^\circ(s,x))$ for $\boldsymbol{\mu}(s)$-a.e. x on Ω . This gives (4.2.1).

Since (4.4.13) is equivalent to $\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}(x) \in N_{K(\zeta^\circ(s,x))}(\hat{\sigma}^\circ(s,x))$ for $\boldsymbol{\mu}(s)$-a.e. $x \in \Gamma_0$ (see [40, Theorem 23.5]), combining this with (4.4.10) we get (4.2.15).

We claim that $\hat{\boldsymbol{\sigma}}^\circ(s)$ satisfies (4.2.2). Let us fix an open set $U \subset\subset \Omega$. Arguing as in the proof of Theorem 4.23 we find that

$$\langle \varphi \boldsymbol{\sigma}_r^\circ(s), \dot{\boldsymbol{p}}^\circ(s) \rangle \to \langle [\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)], \varphi \rangle$$

for every $\varphi \in C_0(U)$ and for $\mathcal{L}^1$-a.e. $s \in B^\circ$. Here the duality in the left-hand side is the standard duality between a continuous function and a measure. It follows that

$$[\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)] = \Big(\hat{\boldsymbol{\sigma}}^\circ(s) : \frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}\Big)\, \boldsymbol{\mu}(s) \qquad \text{on } \Omega . \tag{4.4.14}$$

To prove the same equality on Γ_0, we first observe that (4.4.11) and (4.4.12) give

$$[\boldsymbol{\sigma}^\circ(s) : \dot{\boldsymbol{p}}^\circ(s)] = H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}, \boldsymbol{\zeta}^\circ(s)\Big)\boldsymbol{\mu}(s) \quad \boldsymbol{\mu}(s)\text{-a.e. on } \Gamma_0 .$$

By (4.4.13) and the Euler identity we have

$$H\Big(\frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)}, \boldsymbol{\zeta}^\circ(s)\Big) = \hat{\boldsymbol{\sigma}}^\circ : \frac{\dot{\boldsymbol{p}}^\circ(s)}{\boldsymbol{\mu}(s)} \quad \boldsymbol{\mu}(s)\text{-a.e. on } \Gamma_0 ,$$

so that (4.2.2) follows from (4.4.14) and the last two equalities. This concludes the proof. $\square$

Chapter 5

Existence of a rescaled viscosity evolution

5.1 Overview of the chapter

This chapter is devoted to proving the existence of a rescaled viscosity evolution according to Definition 4.5. We will see that this kind of generalized solution appears in the limit, as the viscosity parameter ε tends to 0, of a suitable time rescaled version of the solutions of the viscoplastic problem considerd in Chapter 2, Section 2.4. Therefore, as a first step in Section 5.2 we investigate the well-posedness of such a problem. We first prove (Theorem 5.3) that for every function $\zeta(t,x)$ in a suitable function space there exists a solution $u^\zeta_\varepsilon(t,x)$, $e^\zeta_\varepsilon(t,x)$, $p^\zeta_\varepsilon(t,x)$, $\sigma^\zeta_\varepsilon(t,x)$ of (a), (b), (c), and (e_ε) (see Chapter 2, Section 2.4), adapting a result obtained by Suquet [50]. Then we prove the existence of a viscoplastic evolution by a fixed point argument (Theorem 5.5).

An energy estimate (Theorem 5.4) allows us to prove the existence of change of variables $t = t^\circ_\varepsilon(s)$, uniformly Lipschitz with respect to s, such that the rescaled functions $p^\circ_\varepsilon(s,x) := p_\varepsilon(t_\varepsilon(s),x)$ are uniformly Lipschitz with respect to s, in a suitable function space. The Ascoli-Arzelà Theorem provides the existence of a subsequence (not relabelled), such that

$$t^\circ_\varepsilon(s) \to t^\circ(s) \quad \text{and} \quad p^\circ_\varepsilon(s,\cdot) \rightharpoonup p^\circ(s,\cdot)\,,$$

the latter in a weak topology. A further argument, based on the uniqueness of the solution to an auxiliary variational problem, shows that

$$e^\circ_\varepsilon(s,\cdot) \rightharpoonup e^\circ(s,\cdot)\,, \quad u^\circ_\varepsilon(s,\cdot) \rightharpoonup u^\circ(s,\cdot)\,, \quad \sigma^\circ_\varepsilon(s,\cdot) \rightharpoonup \sigma^\circ(s,\cdot)\,.$$

The compactness ensured by the presence of the convolutions in the evolution law for the internal variable allows us to prove that

$$z^\circ_\varepsilon(s,x) \to z^\circ(s,x) \quad \text{and} \quad \zeta^\circ_\varepsilon(s,x) \to \zeta^\circ(s,x)\,,$$

uniformly with respect to x. The goal of this chapter is showing that these limit functions satisfy all conditions in (b) of Theorem 4.7, which are equivalent to those in Definition 4.5.

A brief outline of the development of the long proof, which will run through three Sections, will be given in Section 5.3, after the statement of the existence result (Theorem 5.6). The main theoretical difficulty in the proof is that the total variation, with respect to time, of the plastic strain can be controlled only in a nonreflexive Banach space, while no such a control is available for the elastic part. As we are not a priori allowed to take time derivatives of the stress for $\mathcal{L}^1$-a.e. $s \in [0, +\infty)$, we need a delicate approximation argument of the integrals that appear in the energy-dissipation balance (4.3.1). We will consider two different approximations on the set A° defined in (4.2.7), where the stress $\boldsymbol{\sigma}^\circ$ is locally absolutely continuous in time, and on its complement, where instead we have an uniform bound on the spatial L^∞-norm of the stress. Then we will conclude the proof with the help of the results of Chapter 1, Section 1.5.

For all the notation and the assumptions on the model we refer to Chapters 1 and 2.

5.2 The viscoplastic approximations

In this section, given a viscosity parameter $\varepsilon > 0$, we study the existence of a solution to the Perzyna-type viscoplastic evolution problem corresponding to Cam-Clay plasticity.

Definition 5.1. Let $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ be as in (2.3.42), consider $u_0 \in H^1(\Omega; \mathbb{R}^N)$, $e_0 \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$, $p_0 \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$, $z_0 \in C^0(\overline{\Omega})$, and let $\varepsilon > 0$. An *ε-viscoplastic evolution* with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$, and initial condition (u_0, e_0, p_0, z_0) is a function $(\boldsymbol{u}_\varepsilon, \boldsymbol{e}_\varepsilon, \boldsymbol{p}_\varepsilon, \boldsymbol{z}_\varepsilon)$, with

$$\begin{gathered}\boldsymbol{u}_\varepsilon \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^N)), \qquad \boldsymbol{e}_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{N\times N}_{sym})), \\ \boldsymbol{p}_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{N\times N}_{sym})), \qquad \boldsymbol{z}_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega)), \\ \boldsymbol{z}_\varepsilon(t) \in C^0(\overline{\Omega}) \quad \text{for every } t \in [0,+\infty),\end{gathered} \tag{5.2.1}$$

such that, setting

$$\boldsymbol{\sigma}_\varepsilon(t) := \mathbb{C}\boldsymbol{e}_\varepsilon(t) \quad \text{and} \quad \boldsymbol{\zeta}_\varepsilon(t) := V(\boldsymbol{z}_\varepsilon(t)), \tag{5.2.2}$$

the following conditions are satisfied:

$(\mathrm{ev0})_\varepsilon$ *initial condition:* $(\boldsymbol{u}_\varepsilon(0), \boldsymbol{e}_\varepsilon(0), \boldsymbol{p}_\varepsilon(0), \boldsymbol{z}_\varepsilon(0)) = (u_0, e_0, p_0, z_0)$;

$(\mathrm{ev1})_\varepsilon$ *kinematic admissibility:* for every $t \in [0, +\infty)$

$$\begin{gathered}E\boldsymbol{u}_\varepsilon(t) = \boldsymbol{e}_\varepsilon(t) + \boldsymbol{p}_\varepsilon(t) \quad \mathcal{L}^n\text{-a.e. in } \Omega, \\ \boldsymbol{u}_\varepsilon(t) = \boldsymbol{w}(t) \quad \mathcal{H}^{n-1}\text{-a.e. in } \Gamma_0;\end{gathered} \tag{5.2.3}$$

$(\mathrm{ev2})_\varepsilon$ *equilibrium condition:* for every $t \in [0, +\infty)$

$$-\mathrm{div}\,\boldsymbol{\sigma}_\varepsilon(t) = \boldsymbol{f}(t) \quad \text{in } \Omega, \qquad [\boldsymbol{\sigma}_\varepsilon(t)\nu] = \boldsymbol{g}(t) \quad \text{on } \Gamma_1. \tag{5.2.4}$$

$(\mathrm{ev3})_\varepsilon$ *regularized flow rule:* for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$

$$\dot{\boldsymbol{p}}_\varepsilon(t) = \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega, \tag{5.2.5}$$

where $\mathcal{N}^\varepsilon_{\mathcal{K}}$ is defined by (2.4.5).

$(\mathrm{ev4})_\varepsilon$ *evolution law for the internal variable:* for $\mathcal{L}^1$-a.e. $t \in [0,+\infty)$

$$\dot{\boldsymbol{z}}_\varepsilon(t) = \rho_1 \star \big(\rho_2 \star \mathrm{tr}\boldsymbol{\sigma}_\varepsilon(t))\, \mathrm{tr}\, \dot{\boldsymbol{p}}_\varepsilon(t)\big) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,. \tag{5.2.6}$$

Remark 5.2. Let us fix $t \in [0,+\infty)$ such that the derivatives $\dot{\boldsymbol{p}}_\varepsilon(t)$ exists. Then the following conditions are equivalent:

$$\dot{\boldsymbol{p}}_\varepsilon(t) = \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \tag{5.2.7}$$

$$\boldsymbol{\sigma}_\varepsilon(t) \in \partial_p \mathcal{H}_\varepsilon(\dot{\boldsymbol{p}}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \tag{5.2.8}$$

$$\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon(t) \in \partial_p \mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,. \tag{5.2.9}$$

Indeed, by (2.4.7) we have $\partial_\sigma \mathcal{H}^*_\varepsilon(\boldsymbol{\sigma}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t)) = \mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\sigma}_\varepsilon(t), \boldsymbol{\zeta}_\varepsilon(t))$, so that (5.2.7) and (5.2.8) are equivalent by a standard property of conjugate functions (see, e.g., [19, Corollary I.5.2]). The equivalence between (5.2.8) and (5.2.9) follows immediately from (2.4.2).

To prove the existence of an ε-viscoplastic evolution we will use a fixed point argument. To this end, in the next theorem we prove existence and continous dependence on the data for a similar problem with prescribed $\boldsymbol{\zeta}$. We present here a simpler proof than the original one, which was obtained by adapting the arguments of [50]. The one we give here is again an adaptation of a result by Suquet, contained in his Ph.D. thesis [51]. We report it for the reader's convenience.

Theorem 5.3. *Let $\boldsymbol{\zeta} \in C^0([0,+\infty); L^2(\Omega)^+)$ and let $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, u_0, e_0, p_0, ε be as in Definition 5.1. Assume that (u_0, e_0, p_0) satisfies the kinematic admissibility condition at $t = 0$:*

$$\begin{aligned} Eu_0 &= e_0 + p_0 \quad \mathcal{L}^n\text{-a.e. in } \Omega\,,\\ u_0 &= \boldsymbol{w}(0) \quad \mathcal{H}^{n-1}\text{-a.e. in } \Gamma_0\,, \end{aligned} \tag{5.2.10}$$

and that the safe load condition (2.3.45)-(2.3.48) *holds. Then there exists a unique function $(\boldsymbol{u}^\zeta_\varepsilon, \boldsymbol{e}^\zeta_\varepsilon, \boldsymbol{p}^\zeta_\varepsilon)$, with*

$$\begin{gathered} \boldsymbol{u}^\zeta_\varepsilon \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^N))\,, \qquad \boldsymbol{e}^\zeta_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))\,,\\ \boldsymbol{p}^\zeta_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))\,, \end{gathered} \tag{5.2.11}$$

such that setting

$$\boldsymbol{\sigma}^\zeta_\varepsilon(t) = \mathbb{C}\boldsymbol{e}^\zeta_\varepsilon(t),$$

the following conditions are satisfied:

$(\mathrm{ev0})^\zeta_\varepsilon$ initial condition: $(\boldsymbol{u}^\zeta_\varepsilon(0), \boldsymbol{e}^\zeta_\varepsilon(0), \boldsymbol{p}^\zeta_\varepsilon(0)) = (u_0, e_0, p_0)$;

$(\mathrm{ev1})^\zeta_\varepsilon$ kinematic admissibility: for every $t \in [0,+\infty)$

$$\begin{aligned} E\boldsymbol{u}^\zeta_\varepsilon(t) &= \boldsymbol{e}^\zeta_\varepsilon(t) + \boldsymbol{p}^\zeta_\varepsilon(t) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,,\\ \boldsymbol{u}^\zeta_\varepsilon(t) &= \boldsymbol{w}(t) \quad \mathcal{H}^{n-1}\text{-a.e. in } \Gamma_0\,; \end{aligned} \tag{5.2.12}$$

(ev2)$^{\boldsymbol{\zeta}}_{\varepsilon}$ equilibrium condition: for every $t \in [0, +\infty)$

$$-\mathrm{div}\, \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) = \boldsymbol{f}(t) \quad \text{in } \Omega\,, \qquad [\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\nu] = \boldsymbol{g}(t) \quad \text{on } \Gamma_1. \tag{5.2.13}$$

(ev3)$^{\boldsymbol{\zeta}}_{\varepsilon}$ regularized flow rule: for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$

$$\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}}_{\varepsilon}(t) = \mathcal{N}^{\varepsilon}_{\mathcal{K}}(\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \tag{5.2.14}$$

where $\mathcal{N}^{\varepsilon}_{\mathcal{K}}$ is defined by (2.4.5).

Since the right-hand side of (5.2.14) *belongs to $C^0([0, +\infty), L^2(\Omega, \mathbb{M}^{N\times N}_{sym}))$ by* (1.3.7)*, it follows that*

$$\boldsymbol{p}^{\boldsymbol{\zeta}}_{\varepsilon} \in C^1([0, +\infty), L^2(\Omega, \mathbb{M}^{N\times N}_{sym}))\,. \tag{5.2.15}$$

Moreover, for every $T > 0$ there exists a constant $C_{\varepsilon,T}$ such that

$$\max_{t\in[0,T]} \|\boldsymbol{\sigma}^{\boldsymbol{\zeta}_1}_{\varepsilon}(t) - \boldsymbol{\sigma}^{\boldsymbol{\zeta}_2}_{\varepsilon}(t)\|_2 \le C_{\varepsilon,T} \max_{t\in[0,T]} \|\boldsymbol{\zeta}_1(t) - \boldsymbol{\zeta}_2(t)\|_2 \tag{5.2.16}$$

$$\max_{t\in[0,T]} \|\dot{\boldsymbol{p}}^{\boldsymbol{\zeta}_1}_{\varepsilon}(t) - \dot{\boldsymbol{p}}^{\boldsymbol{\zeta}_2}_{\varepsilon}(t)\|_2 \le C_{\varepsilon,T} \max_{t\in[0,T]} \|\boldsymbol{\zeta}_1(t) - \boldsymbol{\zeta}_2(t)\|_2 \tag{5.2.17}$$

for every $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$ in $C^0([0, +\infty); L^2(\Omega)^+)$.

Proof. Let $\mathbb{A} := \mathbb{C}^{-1}$. If a triple $(\boldsymbol{u}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}, \boldsymbol{p}^{\boldsymbol{\zeta}}_{\varepsilon})$ satisfies conditions (ev0)$^{\boldsymbol{\zeta}}_{\varepsilon}$-(ev3)$^{\boldsymbol{\zeta}}_{\varepsilon}$, then for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$

$$E\dot{\boldsymbol{u}}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \mathbb{A}\dot{\boldsymbol{\sigma}}^{\boldsymbol{\zeta}}_{\varepsilon}(t) = \mathcal{N}^{\varepsilon}_{\mathcal{K}}(\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) \quad \mathcal{L}^n\text{-a.e. in } \Omega\,. \tag{5.2.18}$$

Let us define $\boldsymbol{\tau}^{\boldsymbol{\zeta}}_{\varepsilon} \in H^1_{loc}([0, +\infty); L^2(\Omega; \mathbb{M}^{N\times N}_{sym}))$ by

$$\boldsymbol{\tau}^{\boldsymbol{\zeta}}_{\varepsilon}(t) := \boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \boldsymbol{\chi}(t). \tag{5.2.19}$$

By (2.3.32), (2.3.45), and (5.2.13) we have $\boldsymbol{\tau}^{\boldsymbol{\zeta}}_{\varepsilon}(t) \in \Sigma_0(\Omega)$ for every $t \in [0, +\infty)$ and hence, integrating by parts, for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$ we obtain

$$\langle \mathbb{A}\dot{\boldsymbol{\tau}}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \hat{\sigma}\rangle = -\langle \mathcal{N}^{\varepsilon}_{\mathcal{K}}(\boldsymbol{\chi}(t) + \boldsymbol{\tau}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)), \hat{\sigma}\rangle + \langle E\dot{\boldsymbol{w}}(t) - \mathbb{A}\dot{\boldsymbol{\chi}}(t), \hat{\sigma}\rangle \tag{5.2.20}$$

for every $\hat{\sigma} \in \Sigma_0(\Omega)$. The initial condition for $\boldsymbol{\tau}^{\boldsymbol{\zeta}}_{\varepsilon}$ is given by

$$\boldsymbol{\tau}^{\boldsymbol{\zeta}}_{\varepsilon}(0) = \sigma_0 - \boldsymbol{\chi}(0)\,, \tag{5.2.21}$$

where $\sigma_0 := \mathbb{C}e_0$.

Conversely, assume that $\boldsymbol{\tau}^{\boldsymbol{\zeta}}_{\varepsilon} \in H^1_{loc}([0, +\infty); \Sigma_0(\Omega))$ and that (5.2.20) holds for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$. If we define $\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t)$ through (5.2.19), then (5.2.13) follows from (2.3.45). Moreover, for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$, we obtain by (5.2.20) that $\mathbb{A}\dot{\boldsymbol{\sigma}}^{\boldsymbol{\zeta}}_{\varepsilon}(t) + \mathcal{N}^{\varepsilon}_{\mathcal{K}}(\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) - E\dot{\boldsymbol{w}}(t)$ are orthogonal to $\Sigma_0(\Omega)$ in $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$. Therefore, by (2.3.33), for $\mathcal{L}^1$-a.e. $t \in [0, +\infty)$ there exists $\boldsymbol{v}^{\boldsymbol{\zeta}}_{\varepsilon}(t) \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^N)$ such that $E\boldsymbol{v}^{\boldsymbol{\zeta}}_{\varepsilon}(t) = \mathbb{A}\dot{\boldsymbol{\sigma}}^{\boldsymbol{\zeta}}_{\varepsilon}(t) + \mathcal{N}^{\varepsilon}_{\mathcal{K}}(\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t), \boldsymbol{\zeta}(t)) - E\dot{\boldsymbol{w}}(t)$ $\mathcal{L}^n$-a.e. in Ω. If we define

$$\boldsymbol{u}^{\boldsymbol{\zeta}}_{\varepsilon}(t) := \boldsymbol{w}(t) + \int_0^t \boldsymbol{v}^{\boldsymbol{\zeta}}_{\varepsilon}(\tau)\, d\tau + u_0 - \boldsymbol{w}(0)\,, \quad \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}(t) = \mathbb{A}\boldsymbol{\sigma}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\,, \quad \boldsymbol{p}^{\boldsymbol{\zeta}}_{\varepsilon}(t) := E\boldsymbol{u}^{\boldsymbol{\zeta}}_{\varepsilon}(t) - \boldsymbol{e}^{\boldsymbol{\zeta}}_{\varepsilon}(t)\,,$$

then the triple $(\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}})$ satisfies conditions (ev1)$_\varepsilon^{\boldsymbol{\zeta}}$-(ev3)$_\varepsilon^{\boldsymbol{\zeta}}$. If, in addition, (5.2.21) holds, then the initial condition (ev0)$_\varepsilon^{\boldsymbol{\zeta}}$ is satisfied.

Since $\Sigma_0(\Omega)$ is a closed linear subspace of $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, denoting with P_{Σ_0} the linear orthogonal projection onto $\Sigma_0(\Omega)$, it is clear that (5.2.20) is equivalent to the generalized ODE

$$\mathbb{A}\dot{\boldsymbol{\tau}}_\varepsilon^{\boldsymbol{\zeta}}(t) + P_{\Sigma_0}(\mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\chi}(t)+\boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}}(t),\boldsymbol{\zeta}(t))) = P_{\Sigma_0}(E\dot{\boldsymbol{w}}(t) - \mathbb{A}\dot{\boldsymbol{\chi}}(t)) \tag{5.2.22}$$

in $H^1_{loc}([0,+\infty);\Sigma_0(\Omega))$. By the 1-Lipschitz continuity of P_{Σ_0} and (1.3.4) for every $\tau \in \Sigma_0(\Omega)$ and every $t\in[0,+\infty)$ we have

$$\|P_{\Sigma_0}(\mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\chi}(t)+\tau,\boldsymbol{\zeta}(t)))\|_2 \le \tfrac{1}{\varepsilon}(\|\tau\|_2 + \|\boldsymbol{\chi}(t)\|_2 + M_K\|\boldsymbol{\zeta}(t)\|_2)\,. \tag{5.2.23}$$

Using Lemma 1.1, for every τ_1, $\tau_2\in\Sigma_0(\Omega)$ and every $t\in[0,+\infty)$ we get that

$$\|P_{\Sigma_0}(\mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\chi}(t)+\tau_2,\boldsymbol{\zeta}(t))) - P_{\Sigma_0}(\mathcal{N}^\varepsilon_{\mathcal{K}}(\boldsymbol{\chi}(t)+\tau_1,\boldsymbol{\zeta}(t)))\|_2 \le \tfrac{2}{\varepsilon}\|\tau_2-\tau_1\|_2\,. \tag{5.2.24}$$

Since $\boldsymbol{w}\in H^1_{loc}([0,+\infty);H^1(\Omega;\mathbb{R}^N)$ and $\boldsymbol{\chi}\in H^1_{loc}([0,+\infty);L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ we easily have

$$P_{\Sigma_0}(E\dot{\boldsymbol{w}}(t) - \mathbb{A}\dot{\boldsymbol{\chi}}(t)) \in L^2_{loc}([0,+\infty);\Sigma_0(\Omega))\,. \tag{5.2.25}$$

Since $\mathbb{A}$ is bounded and invertible, using (5.2.23), (5.2.24), and (5.2.25) the Cauchy-Lipschitz Theorem gives the existence of a unique solution $\boldsymbol{\tau}_\varepsilon^{\boldsymbol{\zeta}}\in H^1_{loc}([0,+\infty);\Sigma_0(\Omega))$ of (5.2.20) with initial condition (5.2.21).

To prove estimate (5.2.16) we consider two solutions $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ corresponding to $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ in $C^0([0,T];L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))$, respectively. Subtracting (5.2.20) corresponding to $\boldsymbol{\tau}_1 := \boldsymbol{\sigma}_1 - \boldsymbol{\chi}$ and $\boldsymbol{\tau}_2 := \boldsymbol{\sigma}_2-\boldsymbol{\chi}$, taking $\hat\sigma = \boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)$ as test function, and using Lemma 1.1 we obtain

$$\tfrac{d}{dt}\tfrac{1}{2}\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|^2_{\mathbb{A}} \le \tfrac{1}{\varepsilon}\big[\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_2 + 2M_K\|\boldsymbol{\zeta}_1(t)-\boldsymbol{\zeta}_2(t)\|_2\big]\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_2\,.$$

To get (5.2.16) it is enough to apply Gronwall inequality. The other inequality (5.2.17) follows from (ev3)$_\varepsilon^{\boldsymbol{\zeta}}$ using (1.3.7) and (5.2.16). □

The following theorem shows that the modified flow rule (ev3)$_\varepsilon$ can be replaced by a suitable stress constraint and an energy-dissipation balance.

Theorem 5.4. *Let* $\boldsymbol{\zeta}\in C^0([0,+\infty);L^2(\Omega)^+)$ *and let* $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, u_0, e_0, p_0, ε *be as in Definition 5.1. Assume that the safe load condition* (2.3.45)-(2.3.48) *holds. Let* $(\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}})$ *be a function satisfying* (5.2.11)*, the initial condition (ev0)*$_\varepsilon$*, the kinematic admissibility (ev1)*$_\varepsilon^{\boldsymbol{\zeta}}$*, and the equilibrium condition (ev2)*$_\varepsilon^{\boldsymbol{\zeta}}$ *of Theorem 5.3, with* $\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t) := \mathbb{C}\boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(t)$.

Then $(\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}})$ *satisfies the regularized flow rule (ev3)*$_\varepsilon^{\boldsymbol{\zeta}}$ *of Theorem 5.3 if and only if the following properties are simultaneously satisfied:*

(ev3′)$_\varepsilon^{\boldsymbol{\zeta}}$ *modified stress constraint: for* $\mathcal{L}^1$*-a.e.* $t\in[0,+\infty)$

$$\boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(t) \in \mathcal{K}(\boldsymbol{\zeta}(t))\,; \tag{5.2.26}$$

(ev3$''$)$_\varepsilon^\zeta$ energy-dissipation balance: for every $T>0$ we have

$$\begin{gathered}\mathcal{Q}(\boldsymbol{e}_\varepsilon^\zeta(T)) - \mathcal{Q}(e_0) + \int_0^T \Big(\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon^\zeta(t), \boldsymbol{\zeta}(t)) - \langle \boldsymbol{\chi}(t), \dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\rangle\Big)\, dt + \\ +\varepsilon \int_0^T \|\dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\|_2^2\, dt = \int_0^T \langle \boldsymbol{\sigma}_\varepsilon^\zeta(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt - \int_0^T \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon^\zeta(t)\rangle\, dt + \\ + \langle \boldsymbol{\chi}(T), \boldsymbol{e}_\varepsilon^\zeta(T)\rangle - \langle \boldsymbol{\chi}(0), e_0\rangle\,.\end{gathered} \tag{5.2.27}$$

Proof. Suppose that $(\boldsymbol{u}_\varepsilon^\zeta, \boldsymbol{e}_\varepsilon^\zeta, \boldsymbol{p}_\varepsilon^\zeta)$ satisfies (ev3)$_\varepsilon^\zeta$. By (1.3.22) we have $\partial_p \mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon^\zeta(t), \boldsymbol{\zeta}(t)) \subset \mathcal{K}(\boldsymbol{\zeta}(t))$. Therefore (5.2.9) implies (ev3$'$)$_\varepsilon^\zeta$.

Since $\mathcal{H}(\cdot,\zeta)$ is convex and positively homogeneous of degree one, the Euler relation gives $\langle\sigma, p\rangle = \mathcal{H}(p,\zeta)$ whenever $\sigma \in \partial_p\mathcal{H}(p,\zeta)$. Therefore, (5.2.9) implies

$$\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon^\zeta(t), \boldsymbol{\zeta}(t)) = \langle \boldsymbol{\sigma}_\varepsilon^\zeta(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon^\zeta(t), \dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\rangle\,, \tag{5.2.28}$$

which is equivalent to

$$\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon^\zeta(t), \boldsymbol{\zeta}(t)) + \varepsilon\|\dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\|_2^2 = \langle \boldsymbol{\sigma}_\varepsilon^\zeta(t), \dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\rangle\,. \tag{5.2.29}$$

By (5.2.12) we have

$$\langle \boldsymbol{\sigma}_\varepsilon^\zeta(t), \dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\rangle = \langle \boldsymbol{\sigma}_\varepsilon^\zeta(t), E\dot{\boldsymbol{u}}_\varepsilon^\zeta(t)\rangle - \langle \boldsymbol{\sigma}_\varepsilon^\zeta(t), \dot{\boldsymbol{e}}_\varepsilon^\zeta(t)\rangle\,. \tag{5.2.30}$$

Since $\dot{\boldsymbol{u}}_\varepsilon^\zeta(t) - \dot{\boldsymbol{w}}(t) \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^N)$ by (5.2.12), using (2.3.45) and (5.2.13) we obtain

$$\langle \boldsymbol{\sigma}_\varepsilon^\zeta(t), E\dot{\boldsymbol{u}}_\varepsilon^\zeta(t)\rangle = \langle \boldsymbol{\sigma}_\varepsilon^\zeta(t), E\dot{\boldsymbol{w}}(t)\rangle + \langle \boldsymbol{\chi}(t), E\dot{\boldsymbol{u}}_\varepsilon^\zeta(t)\rangle - \langle \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\,. \tag{5.2.31}$$

Combining (5.2.29), (5.2.30), and (5.2.31), we deduce that

$$\langle \boldsymbol{\sigma}_\varepsilon^\zeta(t), \dot{\boldsymbol{e}}_\varepsilon^\zeta(t)\rangle + \mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon^\zeta(t), \boldsymbol{\zeta}(t)) + \varepsilon\|\dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\|_2^2 = \langle \boldsymbol{\sigma}_\varepsilon^\zeta(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle + \langle \boldsymbol{\chi}(t), E\dot{\boldsymbol{u}}_\varepsilon^\zeta(t)\rangle\,.$$

By (5.2.12) we have

$$\begin{aligned}&\langle \boldsymbol{\sigma}_\varepsilon^\zeta(t), \dot{\boldsymbol{e}}_\varepsilon^\zeta(t)\rangle + \mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon^\zeta(t), \boldsymbol{\zeta}(t)) - \langle \boldsymbol{\chi}(t), \dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\rangle + \varepsilon\|\dot{\boldsymbol{p}}_\varepsilon^\zeta(t)\|_2^2 = \\ &= \langle \boldsymbol{\sigma}_\varepsilon^\zeta(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle + \tfrac{d}{dt}\langle \boldsymbol{\chi}(t), \boldsymbol{e}_\varepsilon^\zeta(t)\rangle - \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon^\zeta(t)\rangle\,.\end{aligned} \tag{5.2.32}$$

The energy-dissipation balance (ev3$''$)$_\varepsilon^\zeta$ can be obtained from (5.2.32) by integration.

Conversely, assume that $(\boldsymbol{u}_\varepsilon^\zeta, \boldsymbol{e}_\varepsilon^\zeta, \boldsymbol{p}_\varepsilon^\zeta)$ satisfies conditions (ev3$'$)$_\varepsilon^\zeta$ and (ev3$''$)$_\varepsilon^\zeta$. By differentiating (ev3$''$)$_\varepsilon^\zeta$ we obtain (5.2.32). Thanks to (5.2.30) and (5.2.31), from (5.2.32) we deduce (5.2.29), which is equivalent to (5.2.28). By (ev3$'$)$_\varepsilon^\zeta$ for $\mathcal{L}^1$-a.e. $t \in (0,+\infty)$ we have

$$\boldsymbol{\sigma}_\varepsilon^\zeta(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon^\zeta(t) \in \mathcal{K}(\boldsymbol{\zeta}(t)) = \partial_p\mathcal{H}(0, \boldsymbol{\zeta}(t))\,. \tag{5.2.33}$$

Since $\mathcal{H}(\cdot, \boldsymbol{\zeta}(t))$ is convex and $\mathcal{H}(0,\boldsymbol{\zeta}(t)) = 0$, condition (5.2.9) follows easily from (5.2.28) and (5.2.33). ☐

Theorem 5.5. *Let $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, u_0, e_0, p_0, ε be as in Definition 5.1. Assume that (u_0, e_0, p_0) satisfies the kinematic admissibility condition* (5.2.10) *at $t=0$ and that the safe load condition* (2.3.45)-(2.3.48) *holds. Then there exists an ε-viscoplastic evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ and initial condition (u_0, e_0, p_0, z_0).*

Proof. Let us fix $T > 0$. We will apply a fixed-point argument in $C^0([0,T];L^2(\Omega;I_m))$, where $I_m := [\zeta_m, +\infty)$. Given $\boldsymbol{\zeta} \in C^0([0,T];L^2(\Omega;I_m))$, by Theorem 5.3 we can find a unique function $(\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}})$, satisfying (5.2.11) and (ev0)$_\varepsilon^{\boldsymbol{\zeta}}$-(ev3)$_\varepsilon^{\boldsymbol{\zeta}}$. Define

$$\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}}(t) := \rho_2 \star \operatorname{tr} \boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}(t)\,. \tag{5.2.34}$$

As $\operatorname{tr} \boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}} \in C^0([0,T];L^2(\Omega))$, we deduce from (2.3.41) that $\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}} \in C^0([0,T];C^1(\overline{\Omega}))$. By (5.2.15) we have $\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}} \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}} \in C^0([0,T];L^2(\Omega))$, hence (2.3.41) gives that $\rho_1 \star (\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}} \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}})$ belongs to $C^0([0,T];C^1(\overline{\Omega}))$. Let $\boldsymbol{z}_\varepsilon^{\boldsymbol{\zeta}} \in C^1([0,T];C^1(\overline{\Omega}))$ be defined by

$$\boldsymbol{z}_\varepsilon^{\boldsymbol{\zeta}}(t) = z_0 + \int_0^t \rho_1 \star (\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}}(\tau) \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(\tau))\, d\tau\,. \tag{5.2.35}$$

It satisfies

$$\boldsymbol{z}_\varepsilon^{\boldsymbol{\zeta}}(0) = z_0 \quad \text{and} \quad \dot{\boldsymbol{z}}_\varepsilon^{\boldsymbol{\zeta}}(t) = \rho_1 \star (\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}}(t) \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(t)) \quad \text{for every } t \in [0,T]\,.$$

Let us define the operator $\mathcal{G}\colon C^0([0,T];L^2(\Omega;I_m)) \to C^0([0,T];L^2(\Omega;I_m))$ by

$$\mathcal{G}(\boldsymbol{\zeta}) := V(\boldsymbol{z}_\varepsilon^{\boldsymbol{\zeta}})\,. \tag{5.2.36}$$

It follows from the definitions that, if $\boldsymbol{\zeta}$ is a fixed point of $\mathcal{G}$, then $(\boldsymbol{u}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{p}_\varepsilon^{\boldsymbol{\zeta}}, \boldsymbol{z}_\varepsilon^{\boldsymbol{\zeta}})$ is an ε-viscoplastic evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ and initial condition (u_0, e_0, p_0, z_0).

To find a fixed point we will apply Schauder's theorem. In the rest of the proof C will denote a positive constant, depending only on T, ε, Ω, e_0, $\boldsymbol{w}$, $\boldsymbol{\chi}$, ρ_1, ρ_2, α_Q, and β_Q, which may change from line to line. By (2.3.49) and (5.2.27) in Theorem 5.4 we have

$$\max_{t\in[0,T]} \|\boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_2^2 \le C + C \max_{t\in[0,T]} \|\boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_2\,,$$

which implies

$$\max_{t\in[0,T]} \|\boldsymbol{e}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_2 \le C\,. \tag{5.2.37}$$

Using this inequality in (5.2.27) and taking into account (2.3.49), we obtain

$$\int_0^T \|\dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_2^2\, dt \le \tfrac{C}{\varepsilon}\,. \tag{5.2.38}$$

From (2.3.41), (5.2.34), and (5.2.37) it follows that

$$\max_{t\in[0,T]} \|\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_\infty \le C\,.$$

Thus, $\|\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}}(t) \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_2 \le C \|\dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_2$, and hence, by (2.3.41),

$$\|\dot{\boldsymbol{z}}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_\infty \le C \|\dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_2 \quad \text{and} \quad \|\nabla \dot{\boldsymbol{z}}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_\infty \le C \|\dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}}(t)\|_2\,. \tag{5.2.39}$$

Inequalities (5.2.38) and (5.2.39) imply that the norm of $\dot{\boldsymbol{z}}_\varepsilon^{\boldsymbol{\zeta}}$ in $L^2([0,T];H^1(\Omega))$ is bounded by a constant independent of $\boldsymbol{\zeta}$. Therefore, the norm of $\boldsymbol{z}_\varepsilon^{\boldsymbol{\zeta}} - z_0$ in $H^1([0,T];H^1(\Omega))$ uniformly bounded. It follows that

$$\boldsymbol{z}_\varepsilon^{\boldsymbol{\zeta}} - z_0 \in C^{0,1/2}([0,T];H^1(\Omega))\,,$$

and its norm is bounded by a constant independent of $\boldsymbol{\zeta}$. This implies that there exists a closed ball $\mathcal{B}$ in $H^1(\Omega)$ such that

$$z_\varepsilon^{\boldsymbol{\zeta}} - z_0 \in C^{0,1/2}([0,T];\mathcal{B}) \quad \text{for every } \boldsymbol{\zeta} \in C^0([0,T];L^2(\Omega;I_m))\,.$$

Since $\mathcal{B}$ is compact in $L^2(\Omega)$, the set $\{z_\varepsilon^{\boldsymbol{\zeta}} : \boldsymbol{\zeta} \in C^0([0,T];L^2(\Omega;I_m))\}$ is relatively compact in $C^0([0,T];L^2(\Omega;I_m))$ by the Arzelà-Ascoli Theorem. Therefore the operator $\mathcal{G}$ defined by (5.2.36) maps $C^0([0,T];L^2(\Omega;I_m))$ into a compact subset of $C^0([0,T];L^2(\Omega;I_m))$.

To apply Schauder's Theorem, it is enough to show that the operator $\mathcal{G}$ is continuous from $C^0([0,T];L^2(\Omega;I_m))$ to $C^0([0,T];L^2(\Omega;I_m))$. The continuity of the map $\boldsymbol{\zeta} \mapsto \boldsymbol{\sigma}_\varepsilon^{\boldsymbol{\zeta}}$ follows from (5.2.16). Then (2.3.41) and (5.2.34) imply the continuity of $\boldsymbol{\zeta} \mapsto \boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}}$ from $C^0([0,T];L^2(\Omega;I_m))$ to $C^0([0,T];C^1(\overline{\Omega}))$. Using (2.3.41) and (5.2.17) we obtain the continuity of $\boldsymbol{\zeta} \mapsto \rho_1 \star (\boldsymbol{a}_\varepsilon^{\boldsymbol{\zeta}} \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^{\boldsymbol{\zeta}})$ from $C^0([0,T];L^2(\Omega;I_m))$ to $C^0([0,T];C^1(\overline{\Omega}))$. Then (5.2.35) gives the continuity of $\boldsymbol{\zeta} \mapsto z_\varepsilon^{\boldsymbol{\zeta}}$ from $C^0([0,T];L^2(\Omega;I_m))$ to $C^1([0,T];C^1(\overline{\Omega}))$. The continuity of $\mathcal{G}$ follows now easily from (5.2.36). □

5.3 Statement of the main result

We now state the main result of the chapter.

Theorem 5.6. *Assume that the safe load condition* (2.3.45)-(2.3.48) *holds. Let* $\boldsymbol{f}$, $\boldsymbol{g}$, *and* $\boldsymbol{w}$ *be as in* (2.3.42), *and assume that* u_0, e_0, p_0, z_0 *satisfy* (2.3.53)-(2.3.57). *Then there exists a rescaled viscosity evolution with data* $\boldsymbol{f}$, $\boldsymbol{g}$, *and* $\boldsymbol{w}$, *and initial condition* $(u_0, e_0, p_0, z_0, 0)$.

The proof will be given in Sections 5.4, 5.5, and 5.6 according to the following scheme. In Section 5.4 we introduce an intrinsic rescaling of all the ε-viscoplastic evolutions through a change of variables $t = t_\varepsilon^\circ(s)$, and prove that, up to a subsequence, these rescaled functions, together with t_ε° converge to a function $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ satisfying (4.2.4) and conditions (ev0)$^\circ$, (ev1)$^\circ$, (ev2)$^\circ$, and (ev3$'$)$^\circ$. As a first step towards the proof of the energy-dissipation balance (4.3.7), which is equivalent to (4.3.1) by Proposition 4.10, in Section 5.5 we prove the energy inequality (5.5.1). The proof relies on the lower semicontinuity of the terms

$$\int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\Big)\, ds - \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \chi_0, p_0\rangle\,,$$
$$\int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds\,, \tag{5.3.1}$$

which is proved in Lemmas 5.10- 5.13. At the end of Section 5.5 we also prove the evolution law for the internal variable (ev4)$^\circ$. The proof of the energy-dissipation balance is completed in Section 5.6, where we prove the energy inequality (5.6.2) through a suitable discrete approximation of the integrals that appear in this inequality. Note that (5.6.2) is not the opposite of (5.5.1), since the term (5.3.1) is replaced by

$$\int_{A_S^\circ} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds\,.$$

This finally allows us to deduce the partial flow-rule (4.3.2) from (5.5.1) and (5.6.2).

5.4 Proof of Theorem 5.6: Part one

We start with a technical lemma.

Lemma 5.7. *Let $u \in BD(\Omega)$. Then there exists a sequence u_k of Lipschitz functions from $\overline{\Omega}$ into $\mathbb{R}^N$, with $u_k = 0$ on $\partial\Omega$, such that*

$$u_k \to u \quad \textit{strongly in } L^1(\Omega;\mathbb{R}^N), \tag{5.4.1}$$

$$Eu_k \rightharpoonup (Eu) \llcorner \Omega - u \odot \nu \mathcal{H}^{n-1} \llcorner \Gamma_0 \quad \textit{weakly}^* \textit{ in } M_b(\Omega \cup \Gamma_0;\mathbb{R}^N). \tag{5.4.2}$$

Proof. It is enough to prove the theorem in a neighbourhood of each point of $\partial\Omega$: the global result can be obtained through a partition of unity. Since Ω has Lipschitz boundary, we may assume that it is the subgraph of a Lipschitz function, i.e.,

$$\Omega := \{x \in \mathbb{R}^N : \hat{x} \in A\,,\ a < x_n < h(\hat{x})\} \subset R := A \times (a,b)\,, \tag{5.4.3}$$

where $\hat{x} := (x_1, \ldots, x_{n-1})$, A is an open rectangle in $\mathbb{R}^{n-1}$, a, $b \in \mathbb{R}$, $a < b$, and $h\colon \overline{A} \to (a,b)$ is a Lipschitz function. We may also assume that $\operatorname{supp} u \subset\subset R$ and that $\Gamma_0 \subset R \cap \partial\Omega$.

Since Ω has Lipschitz boundary, by a standard approximation result (see, e.g., [52, Chapter II, Theorem 3.2]) there exists a sequence $v_k \in C^\infty(\bar{\Omega};\mathbb{R}^N)$ such that

$$\begin{gathered} v_k \to u \quad \text{in } L^1(\Omega;\mathbb{R}^N), \\ Ev_k \rightharpoonup (Eu) \llcorner \Omega \quad \text{weakly}^* \text{ in } M_b(\bar{\Omega};\mathbb{R}^N), \\ \|Ev_k\|_1 \to \|Eu\|_1, \end{gathered} \tag{5.4.4}$$

and therefore (see, e.g., [52, Chapter II, Theorem 3.1])

$$v_k \to u \quad \text{in } L^1(\partial\Omega;\mathbb{R}^N). \tag{5.4.5}$$

Since $\operatorname{supp} u \subset\subset R$, we may assume that $\operatorname{supp} v_k \subset\subset R$ for every k.

Using the special form (5.4.3) of Ω, we can define a sequence of Lipschitz functions $\psi_j\colon \overline{\Omega} \to [0,1]$ by $\psi_j(x) := \min\{j(h(\hat{x}) - x_n), 1\}$. Then $\psi_j = 0$ on the graph of h, $\psi_j \to 1$ on Ω, and $\nabla\psi_j \rightharpoonup -\nu\mathcal{H}^{n-1} \llcorner \Gamma_0$ weakly* in $M_b(\Omega \cup \Gamma_0;\mathbb{R}^N)$. Therefore for every k we have

$$\begin{gathered} \psi_j v_k \to v_k \quad \text{in } L^1(\Omega;\mathbb{R}^N), \\ E(\psi_j v_k) \rightharpoonup Ev_k - v_k \odot \nu\mathcal{H}^{n-1} \llcorner \Gamma_0 \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0;\mathbb{R}^N). \end{gathered} \tag{5.4.6}$$

Since the weak* convergence is metrisable on bounded sets of $M_b(\Omega \cup \Gamma_0;\mathbb{R}^N)$, it follows from (5.4.4), (5.4.5) and (5.4.6), that for every k we can select j_k so that (5.4.1) and (5.4.2) are satisfied by $u_k := \psi_{j_k} v_k$, which vanishes on $\partial\Omega$ by the properties of ψ_j and v_k. $\square$

Proof of Theorem 5.6. If we apply Lemma 5.7 to $u = u_0 - \boldsymbol{w}(0)$ we find a sequence u_0^ε in $H^1(\Omega;\mathbb{R}^N)$ such that

$$u_0^\varepsilon = \boldsymbol{w}(0) \quad \mathcal{H}^{n-1}\text{-a.e. in } \Gamma_0, \tag{5.4.7}$$

$$u_0^\varepsilon \rightharpoonup u_0 \quad \text{weakly}^* \text{ in } BD(\Omega), \tag{5.4.8}$$

$$Eu_0^\varepsilon \rightharpoonup (Eu_0) \llcorner \Omega + (\boldsymbol{w}(0) - u_0) \odot \nu\mathcal{H}^{n-1} \llcorner \Gamma_0 \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0;\mathbb{R}^N). \tag{5.4.9}$$

We define $p_0^\varepsilon := Eu_0^\varepsilon - e_0$. From the weak kinematic admissibility condition (2.3.55), and from (5.4.9), we have

$$p_0^\varepsilon \rightharpoonup p_0 \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{R}^N). \tag{5.4.10}$$

By Theorem 5.5 there exists an ε-viscoplastic evolution $(\boldsymbol{u}_\varepsilon, \boldsymbol{e}_\varepsilon, \boldsymbol{p}_\varepsilon, \boldsymbol{z}_\varepsilon)$ with boundary datum $\boldsymbol{w}$ and initial condition $(u_0^\varepsilon, e_0, p_0^\varepsilon, z_0)$. The energy equality (5.2.27), together with (2.3.38) and (2.3.49), implies that for every $T > 0$ there exists a constant C_T, independent of ε, such that

$$\sup_{t\in[0,T]} \|\boldsymbol{e}_\varepsilon(t)\|_2 \le C_T \quad \text{and} \quad \sup_{t\in[0,T]} \|\boldsymbol{\sigma}_\varepsilon(t)\|_2 \le C_T \tag{5.4.11}$$

(see the proof of (5.2.37)). The same equality and the same estimates, together with (5.4.11), give also that for every $T > 0$ there exists a constant M_T, independent of ε, such that

$$\int_0^T \|\dot{\boldsymbol{p}}_\varepsilon(t)\|_1 \, dt \le M_T < +\infty \, . \tag{5.4.12}$$

Let $s_\varepsilon^\circ \colon [0,+\infty) \to [0,+\infty)$ be the absolutely continuous, increasing, and bijective function defined by

$$s_\varepsilon^\circ(t) := \int_0^t (\|\dot{\boldsymbol{p}}_\varepsilon(\tau)\|_1 + \|\dot{\boldsymbol{\chi}}(\tau)\|_\infty + \|E\dot{\boldsymbol{w}}(\tau)\|_2 + 1) \, d\tau. \tag{5.4.13}$$

It is easy to see that

$$s_\varepsilon^\circ(t_2) - s_\varepsilon^\circ(t_1) \ge t_2 - t_1 \qquad \text{for every } 0 \le t_1 < t_2 < +\infty. \tag{5.4.14}$$

Let $t_\varepsilon^\circ : [0,+\infty) \to [0,+\infty)$ be the inverse of s_ε°. By (5.4.14) t_ε° satisfies

$$0 < t_\varepsilon^\circ(s_2) - t_\varepsilon^\circ(s_1) \le s_2 - s_1 \qquad \text{for every } 0 \le s_1 < s_2 < +\infty.$$

By the Arzelà-Ascoli Theorem we may assume that t_ε° converges uniformly on compact sets to a function $t^\circ \colon [0,+\infty) \to [0,+\infty)$ such that

$$0 \le t^\circ(s_2) - t^\circ(s_1) \le s_2 - s_1 \qquad \text{for every } 0 \le s_1 < s_2 < +\infty.$$

We observe that $t^\circ(0) = 0$. Let us prove that

$$t^\circ(s) \to +\infty \text{ when } s \to +\infty \tag{5.4.15}$$

Indeed, by (2.3.48), (5.4.11), (5.4.12), and (5.4.13), for every $T > 0$ there exists a constant s_T, independent of ε, such that $s_\varepsilon^\circ(T) < s_T$. This gives $T \le t_\varepsilon^\circ(s_T)$ for every ε, which implies $T \le t^\circ(s_T)$, and concludes the proof of (5.4.15).

Define the rescaled functions on $[0,+\infty)$ by

$$\begin{gathered} \boldsymbol{u}_\varepsilon^\circ(s) := \boldsymbol{u}_\varepsilon(t_\varepsilon^\circ(s)) \, , \quad \boldsymbol{e}_\varepsilon^\circ(s) := \boldsymbol{e}_\varepsilon(t_\varepsilon^\circ(s)) \, , \quad \boldsymbol{p}_\varepsilon^\circ(s) := \boldsymbol{p}_\varepsilon(t_\varepsilon^\circ(s)) \, , \quad \boldsymbol{z}_\varepsilon^\circ(s) := \boldsymbol{z}_\varepsilon(t_\varepsilon^\circ(s)) \, , \\ \boldsymbol{f}_\varepsilon^\circ(s) := \boldsymbol{f}(t_\varepsilon^\circ(s)) \, , \quad \boldsymbol{g}_\varepsilon^\circ(s) := \boldsymbol{g}(t_\varepsilon^\circ(s)) \, , \quad \boldsymbol{w}_\varepsilon^\circ(s) := \boldsymbol{w}(t_\varepsilon^\circ(s)) \, , \\ \boldsymbol{\sigma}_\varepsilon^\circ(s) := \boldsymbol{\sigma}_\varepsilon(t_\varepsilon^\circ(s)) \, , \quad \boldsymbol{\zeta}_\varepsilon^\circ(s) := \boldsymbol{\zeta}_\varepsilon(t_\varepsilon^\circ(s)) \, , \qquad \boldsymbol{\chi}_\varepsilon^\circ(s) := \boldsymbol{\chi}(t_\varepsilon^\circ(s)) \, . \end{gathered} \tag{5.4.16}$$

Note that by (5.2.2)

$$\boldsymbol{\sigma}_\varepsilon^\circ(s) := \mathbb{C}\boldsymbol{e}_\varepsilon^\circ(s) \quad \text{and} \quad \boldsymbol{\zeta}_\varepsilon^\circ(s) := V(\boldsymbol{z}_\varepsilon^\circ(s)) \tag{5.4.17}$$

for every $s \in [0, +\infty)$. Since $t^\circ_\varepsilon(s) \to t^\circ(s)$ uniformly on compact sets, the continuity properties of $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, and $\boldsymbol{\chi}$ imply that for every $s \in [0, +\infty)$ we have that

$$\begin{aligned} &\boldsymbol{f}^\circ_\varepsilon(s) \to \boldsymbol{f}^\circ(s) \text{ strongly in } L^n(\Omega;\mathbb{R}^N)\,, \quad \boldsymbol{g}^\circ_\varepsilon(s) \to \boldsymbol{g}^\circ(s) \text{ strongly in } L^\infty(\Gamma_1;\mathbb{R}^N)\,, \\ &\boldsymbol{w}^\circ_\varepsilon(s) \to \boldsymbol{w}^\circ(s) \text{ strongly in } H^1(\Omega;\mathbb{R}^N)\,, \quad \boldsymbol{\chi}^\circ_\varepsilon(s) \to \boldsymbol{\chi}^\circ(s) \text{ strongly in } L^2(\Omega;\mathbb{R}^N)\,, \end{aligned} \tag{5.4.18}$$

where

$$\begin{aligned} &\boldsymbol{f}^\circ \in H^1_{loc}([0,+\infty); L^n(\Omega;\mathbb{R}^N))\,, \qquad \boldsymbol{g}^\circ \in H^1_{loc}([0,+\infty); L^\infty(\Gamma_1;\mathbb{R}^N))\,, \\ &\boldsymbol{w}^\circ \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^N))\,, \qquad \boldsymbol{\chi}^\circ \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{N\times N}_{sym})) \end{aligned}$$

are defined by

$$\boldsymbol{f}^\circ(s) := \boldsymbol{f}(t^\circ(s))\,, \quad \boldsymbol{g}^\circ(s) := \boldsymbol{g}(t^\circ(s))\,, \quad \boldsymbol{w}^\circ(s) := \boldsymbol{w}(t^\circ(s))\,, \quad \boldsymbol{\chi}^\circ(s) := \boldsymbol{\chi}(t^\circ(s)). \tag{5.4.19}$$

From the definitions of s°_ε and t°_ε we obtain easily that

$$\|\boldsymbol{p}^\circ_\varepsilon(s_2) - \boldsymbol{p}^\circ_\varepsilon(s_1)\|_1 + \|\boldsymbol{\chi}^\circ_\varepsilon(s_2) - \boldsymbol{\chi}^\circ_\varepsilon(s_1)\|_\infty + \|E\boldsymbol{w}^\circ_\varepsilon(s_2) - E\boldsymbol{w}^\circ_\varepsilon(s_1)\|_2 \le s_2 - s_1 \tag{5.4.20}$$

for every $0 \le s_1 < s_2$, hence

$$\|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_1 + \|\dot{\boldsymbol{\chi}}^\circ_\varepsilon(s)\|_\infty + \|E\dot{\boldsymbol{w}}^\circ_\varepsilon(s)\|_2 \le 1 \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in [0,+\infty)\,. \tag{5.4.21}$$

Let M be an upper bound of $\|p^\varepsilon_0\|_1$ (see (5.4.10)). From (5.4.20) we get

$$\|\boldsymbol{p}^\circ_\varepsilon(s)\|_1 \le M + s \tag{5.4.22}$$

for every $s \in [0, +\infty)$. Passing to the limit in (5.4.20), we obtain

$$\|\boldsymbol{\chi}^\circ(s_2) - \boldsymbol{\chi}^\circ(s_1)\|_\infty + \|E\boldsymbol{w}^\circ(s_2) - E\boldsymbol{w}^\circ(s_1)\|_2 \le s_2 - s_1 \tag{5.4.23}$$

for every $0 \le s_1 < s_2$, hence

$$\|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty + \|E\dot{\boldsymbol{w}}^\circ(s)\|_2 \le 1 \qquad \text{for } \mathcal{L}^1\text{-a.e. } s \in [0,+\infty)\,. \tag{5.4.24}$$

For every $S > 0$, let

$$\mathcal{B}_S := \{p \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym}) : \|p\|_1 \le M + S\}\,.$$

There exists a distance d_S on $\mathcal{B}_S$ inducing the weak* convergence such that

$$d_S(p,q) \le \|p - q\|_1 \quad \text{for every } p, q \in \mathcal{B}_S. \tag{5.4.25}$$

By (5.4.20) we have that $\boldsymbol{p}^\circ_\varepsilon(s) \in \mathcal{B}_S$ for every $s \in [0, S]$ and every $\varepsilon > 0$. By (5.4.20) and (5.4.25), the sequence $\boldsymbol{p}^\circ_\varepsilon(s)$ is equicontinuous on $[0, S]$ with respect to the distance d_S. We then apply the Arzelà-Ascoli Theorem for every $S > 0$ and we find that there exists a subsequence, still denoted by $\boldsymbol{p}^\circ_\varepsilon$, and a function $\boldsymbol{p}^\circ\colon [0, +\infty) \to M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$ such that

$$\boldsymbol{p}^\circ_\varepsilon(s) \rightharpoonup \boldsymbol{p}^\circ(s) \text{ weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym}) \tag{5.4.26}$$

for every $s \in [0, +\infty)$. By lower semicontinuity we obtain from (5.4.20)

$$\|\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\|_1 \leq s_2 - s_1 \tag{5.4.27}$$

for every $0 \leq s_1 < s_2$, hence

$$\|\dot{\boldsymbol{p}}^\circ(s)\|_1 \leq 1 \qquad \text{for } \mathcal{L}^1\text{-a.e. } s \in [0, +\infty)\,. \tag{5.4.28}$$

where the time derivative $\dot{\boldsymbol{p}}^\circ(s)$ is defined as in (4.2.6). Moreover, from (5.4.20) and (5.4.26) we obtain that

$$\boldsymbol{p}^\circ_\varepsilon(s_\varepsilon) \rightharpoonup \boldsymbol{p}^\circ(s) \text{ weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym}) \tag{5.4.29}$$

for every $s \in [0, +\infty)$ and every $s_\varepsilon \to s$.

We now show that for every $s \in [0, +\infty)$ there exist $\boldsymbol{e}^\circ(s) \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ and $\boldsymbol{u}^\circ(s) \in BD(\Omega)$ such that $(\boldsymbol{u}^\circ(s), \boldsymbol{e}^\circ(s), \boldsymbol{p}^\circ(s), \boldsymbol{w}^\circ(s))$ satisfies the weak kinematic admissibility condition (4.2.11), $\boldsymbol{\sigma}^\circ(s) := \mathbb{C}\,\boldsymbol{e}^\circ(s)$ satisfies the equilibrium condition (4.2.12), and

$$\boldsymbol{e}^\circ_\varepsilon(s_\varepsilon) \rightharpoonup \boldsymbol{e}^\circ(s) \text{ weakly in } L^2(\Omega; \mathbb{M}^{N\times N}_{sym})\,, \tag{5.4.30}$$

$$\boldsymbol{u}^\circ_\varepsilon(s_\varepsilon) \rightharpoonup \boldsymbol{u}^\circ(s) \text{ weakly}^* \text{ in } BD(\Omega)\,, \tag{5.4.31}$$

for every $s_\varepsilon \to s$.

Let us fix $s \in [0, +\infty)$. By (5.4.11) the sequence $\|\boldsymbol{e}^\circ_\varepsilon(s)\|_2$ is bounded uniformly with respect to ε, thus there exists a subsequence $\boldsymbol{e}^\circ_{\varepsilon_j}(s)$ of $\boldsymbol{e}^\circ_\varepsilon(s)$, possibly depending on s, and a function $\boldsymbol{e}^\circ(s) \in L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ such that

$$\boldsymbol{e}^\circ_{\varepsilon_j}(s) \rightharpoonup \boldsymbol{e}^\circ(s) \text{ weakly in } L^2(\Omega; \mathbb{M}^{N\times N}_{sym})\,. \tag{5.4.32}$$

By (5.4.26) and (5.4.32), the kinematic admissibility condition (5.2.3) implies that the sequence $\boldsymbol{u}_{\varepsilon_j}(s)$ is bounded in $BD(\Omega)$. Therefore, up to extracting a further subsequence, it converges weakly* in $BD(\Omega)$ to a function $\boldsymbol{u}^\circ(s) \in BD(\Omega)$ such that $E\boldsymbol{u}^\circ(s) = \boldsymbol{e}^\circ(s) + \boldsymbol{p}^\circ(s)$ in Ω. By considering suitable extensions and arguing as in [13, Lemma 2.1] we obtain also that $\boldsymbol{p}^\circ(s) = (\boldsymbol{w}^\circ(s) - \boldsymbol{u}^\circ(s)) \odot \nu\mathcal{H}^{n-1}$ in Γ_0. Therefore weak kinematic admissibility condition (4.2.11) is satisfied.

Passing to the limit in (5.2.4) we obtain the equilibrium condition (4.2.12). This implies

$$\mathcal{Q}(\boldsymbol{e}^\circ(s)) \leq \mathcal{Q}(\boldsymbol{e}^\circ(s) + E\varphi) - \langle \boldsymbol{f}^\circ(s), \varphi\rangle_\Omega - \langle \boldsymbol{g}^\circ(s), \varphi\rangle_{\Gamma_1} \tag{5.4.33}$$

for every $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$. By strict convexity the inequality is strict, unless $E\varphi = 0$ $\mathcal{L}^n$-a.e. in Ω. It remains to prove (5.4.30) and (5.4.31) for an arbitrary sequence $s_\varepsilon \to s$. As in the previous step, we see that $\|\boldsymbol{e}^\circ_\varepsilon(s_\varepsilon)\|_2$ is bounded uniformly with respect to ε. Let $\boldsymbol{e}^\circ_{\hat\varepsilon_j}(s_{\hat\varepsilon_j})$ be a subsequence of $\boldsymbol{e}^\circ_\varepsilon(s_\varepsilon)$ which converges to a function $\hat{\boldsymbol{e}}(s)$ weakly in $L^2(\Omega; \mathbb{R}^n)$. The previous arguments, together with (5.4.29), show that there exists a function $\hat{\boldsymbol{u}}(s) \in BD(\Omega)$ such that $\boldsymbol{u}^\circ_{\hat\varepsilon_j}(s_{\hat\varepsilon_j}) \rightharpoonup \hat{\boldsymbol{u}}(s)$ weakly* in $BD(\Omega)$, $E\hat{\boldsymbol{u}}(s) = \hat{\boldsymbol{e}}(s) + \boldsymbol{p}^\circ(s)$ in Ω, and $\boldsymbol{p}^\circ(s) = (\boldsymbol{w}^\circ(s) - \hat{\boldsymbol{u}}(s)) \odot \nu\mathcal{H}^{n-1}$ in Γ_0. By difference we obtain that $E(\hat{\boldsymbol{u}}(s) - \boldsymbol{u}^\circ(s)) = \hat{\boldsymbol{e}}(s) - \boldsymbol{e}^\circ(s)$ in Ω and $(\hat{\boldsymbol{u}}(s) - \boldsymbol{u}^\circ(s)) \odot \nu = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_0. By (1.2.2) we have $\hat{\boldsymbol{u}}(s) - \boldsymbol{u}^\circ(s) \in H^1(\Omega; \mathbb{R}^n)$ and $\hat{\boldsymbol{u}}(s) - \boldsymbol{u}^\circ(s) = 0$ on Γ_0.

By (5.4.33) we have $\mathcal{Q}(\boldsymbol{e}^\circ(s)) \leq \mathcal{Q}(\hat{\boldsymbol{e}}(s)) - \langle \boldsymbol{f}^\circ(s), \hat{\boldsymbol{u}}(s) - \boldsymbol{u}^\circ(s)\rangle_\Omega - \langle \boldsymbol{g}^\circ(s), \hat{\boldsymbol{u}}(s) - \boldsymbol{u}^\circ(s)\rangle_{\Gamma_1}$. Exchanging the roles of $\boldsymbol{e}^\circ(s)$ and $\hat{\boldsymbol{e}}(s)$ we obtain $\mathcal{Q}(\boldsymbol{e}^\circ(s)) = \mathcal{Q}(\hat{\boldsymbol{e}}(s)) - \langle \boldsymbol{f}^\circ(s), \hat{\boldsymbol{u}}(s) -$

$\boldsymbol{u}^\circ(s)\rangle_\Omega - \langle \boldsymbol{g}^\circ(s), \hat{\boldsymbol{u}}(s) - \boldsymbol{u}^\circ(s)\rangle_{\Gamma_1}$. The strict convexity argument mentioned after (5.4.33) yields $\boldsymbol{e}^\circ(s) = \hat{\boldsymbol{e}}(s)$ $\mathcal{L}^n$-a.e. in Ω, which in turn gives $\boldsymbol{u}^\circ(s) = \hat{\boldsymbol{u}}(s)$ $\mathcal{L}^n$-a.e. in Ω. This shows that the limit does not depend on the subsequence, and concludes the proof of (5.4.30) and (5.4.31).

Let us prove that

$$\boldsymbol{e}^\circ \text{ is weakly continuous in } L^2(\Omega;\mathbb{M}^{N\times N}_{sym}). \tag{5.4.34}$$

Let s_k be a sequence converging to s. For every fixed k, we can apply (5.4.30) with $s_\varepsilon = s_k$ for every ε, and we find $\varepsilon_k > 0$ such that $d_w(\boldsymbol{e}^\circ_{\varepsilon_k}(s_k), \boldsymbol{e}^\circ(s_k)) < \frac{1}{k}$, where d_w is a distance which metrises the weak topology on bounded subsets of $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$. By (5.4.30), $\boldsymbol{e}^\circ_{\varepsilon_k}(s_k) \rightharpoonup \boldsymbol{e}^\circ(s)$ weakly in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, so that the previous inequality gives $\boldsymbol{e}^\circ(s_k) \rightharpoonup \boldsymbol{e}^\circ(s)$ weakly in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$. This concludes the proof of the weak continuity of $\boldsymbol{e}^\circ$. In a similar way we can prove that $\boldsymbol{u}^\circ\colon [0,+\infty) \to BD(\Omega)$ is weakly* continuous.

Define now for every $s \in [0,+\infty)$

$$\boldsymbol{a}^\circ_\varepsilon(s) = \rho_2 \star \operatorname{tr} \boldsymbol{\sigma}^\circ_\varepsilon(s)\,, \tag{5.4.35}$$

$$\boldsymbol{a}^\circ(s) := \rho_2 \star \operatorname{tr} \boldsymbol{\sigma}^\circ(s)\,, \tag{5.4.36}$$

so that, by (5.2.6) and (5.4.16),

$$\dot{\boldsymbol{z}}^\circ_\varepsilon(s) = \rho_1 \star \big(\boldsymbol{a}^\circ_\varepsilon(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\big) \tag{5.4.37}$$

for $\mathcal{L}^1$-a.e. $s \in [0,+\infty)$. Using (2.3.41) and (5.4.11), we can prove that for every $S > 0$ there exists a constant C°_S, independent of ε, such that

$$\sup_{s\in[0,S]} \|\boldsymbol{a}^\circ_\varepsilon(s)\|_\infty \le C^\circ_S\,, \qquad \sup_{s\in[0,S]} \|\nabla \boldsymbol{a}^\circ_\varepsilon(s)\|_\infty \le C^\circ_S\,. \tag{5.4.38}$$

Therefore for every s, the functions $\boldsymbol{a}^\circ_\varepsilon(s)$ are equicontinuous and equibounded on $\overline{\Omega}$. Since $\boldsymbol{\sigma}^\circ_\varepsilon(s) \rightharpoonup \boldsymbol{\sigma}^\circ(s)$ weakly in $L^2(\Omega,\mathbb{M}^{N\times N}_{sym})$, the sequence $\boldsymbol{a}^\circ_\varepsilon(s)$ converges to $\boldsymbol{a}^\circ(s)$ pointwise in $\overline{\Omega}$. It follows that

$$\boldsymbol{a}^\circ_\varepsilon(s) \to \boldsymbol{a}^\circ(s) \quad \text{strongly in } C^0(\overline{\Omega}) \tag{5.4.39}$$

for every $s \in [0,+\infty)$.

By (5.4.28) and (5.4.38) we have $\|\boldsymbol{a}^\circ_\varepsilon(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_1 \le \sqrt{n} C^\circ_S$ for $\mathcal{L}^1$-a.e. $s \in [0,S]$, and hence by (2.3.41) and (5.4.37)

$$\|\dot{\boldsymbol{z}}^\circ_\varepsilon(s)\|_\infty \le \sqrt{n} C^\circ_S \|\rho_1\|_\infty \quad \text{and} \quad \|\nabla \dot{\boldsymbol{z}}^\circ_\varepsilon(s)\|_\infty \le \sqrt{n} C^\circ_S \|\nabla \rho_1\|_\infty\,. \tag{5.4.40}$$

This implies that

$$\|\boldsymbol{z}^\circ_\varepsilon(s_2) - \boldsymbol{z}^\circ_\varepsilon(s_1)\|_\infty + \|\nabla \boldsymbol{z}^\circ_\varepsilon(s_2) - \nabla \boldsymbol{z}^\circ_\varepsilon(s_1)\|_\infty \le M^\circ_S |s_2 - s_1| \tag{5.4.41}$$

for every $s_1, s_2 \in [0,S]$, where $M^\circ_S := \sqrt{n} C^\circ_S (\|\rho_1\|_\infty + \|\nabla \rho_1\|_\infty)$.

We can then apply the Arzelà-Ascoli Theorem as in the proof of (5.4.26). This gives a subsequence, still denoted $\boldsymbol{z}^\circ_\varepsilon$, such that $\boldsymbol{z}^\circ_\varepsilon(s) \rightharpoonup \boldsymbol{z}^\circ(s)$ weakly* in $W^{1,\infty}(\Omega)$ for every $s \in [0,+\infty)$, which implies

$$\boldsymbol{z}^\circ_\varepsilon(s) \to \boldsymbol{z}^\circ(s) \text{ strongly in } C^0(\overline{\Omega})\,. \tag{5.4.42}$$

Using (5.4.41), we deduce that

$$\boldsymbol{z}_\varepsilon^\circ \to \boldsymbol{z}^\circ \text{ strongly in } C^0([0,S];C^0(\overline{\Omega}))\,. \tag{5.4.43}$$

Passing to the limit in (5.4.41), we get

$$\|\boldsymbol{z}^\circ(s_2)-\boldsymbol{z}^\circ(s_1)\|_\infty + \|\nabla\boldsymbol{z}^\circ(s_2)-\nabla\boldsymbol{z}^\circ(s_1)\|_\infty \le M_S^\circ|s_2-s_1| \tag{5.4.44}$$

for every $s_1,s_2\in[0,S]$.

Let us fix $r>n$. Since $W^{1,r}(\Omega)$ is reflexive, it follows from (5.4.44) that the strong $W^{1,r}$ limit

$$\dot{\boldsymbol{z}}^\circ(s) := s\text{-}\lim_{h\to 0}\frac{\boldsymbol{z}^\circ(s+h)-\boldsymbol{z}^\circ(s)}{h} \tag{5.4.45}$$

exists for $\mathcal{L}^1$-a.e. $s\in[0,+\infty)$, and that $\dot{\boldsymbol{z}}^\circ\in L^\infty_{loc}([0,+\infty);W^{1,r}(\Omega))$. Since the embedding of $W^{1,r}(\Omega)$ into $C^0(\overline{\Omega})$ is continuous, the limit in (5.4.45) takes place in $C^0(\overline{\Omega})$ and $\dot{\boldsymbol{z}}^\circ\in L^\infty_{loc}([0,+\infty);C^0(\overline{\Omega}))$. Moreover, from (5.4.41) and (5.4.42) we obtain that

$$\boldsymbol{z}_\varepsilon^\circ(s_\varepsilon) \to \boldsymbol{z}^\circ(s) \text{ strongly in } C^0(\overline{\Omega}) \tag{5.4.46}$$

for every $s\in[0,+\infty)$ and every $s_\varepsilon\to s$.

For every $s\in[0,+\infty)$ let us define

$$\boldsymbol{\zeta}^\circ(s) := V(\boldsymbol{z}^\circ(s))\,. \tag{5.4.47}$$

The initial condition (ev0)$^\circ$ follows easily from the definitions of $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, and t°, thanks to (5.4.8) and (5.4.10).

To prove (ev3$'$)$^\circ$ we need Lemmas 5.8 and 5.9. The proof will be continued after Lemma 5.9. □

We start with an elementary result about the convergence of inverse functions. To this end we introduce some notation. For every $t\in[0,+\infty)$ we set

$$s_-^\circ(t) := \sup\{s\in[0,+\infty) : t^\circ(s)<t\}\,, \tag{5.4.48}$$

$$s_+^\circ(t) := \inf\{s\in[0,+\infty) : t^\circ(s)>t\}\,, \tag{5.4.49}$$

with the convention $\sup\emptyset=0$, so that $s_-^\circ(0)=0$. We also define the set

$$S^\circ := \{t\in[0,+\infty) : s_-^\circ(t)<s_+^\circ(t)\} \tag{5.4.50}$$

Lemma 5.8. *Let s_-° and s_+° be as in* (5.4.48) *and* (5.4.49)*, respectively. Then*

$$s_-^\circ(t)\le s_+^\circ(t) \quad \text{and} \quad t^\circ(s_-^\circ(t)) = t = t^\circ(s_+^\circ(t)) \tag{5.4.51}$$

for every $t\in[0,+\infty)$, and

$$s_-^\circ(t^\circ(s)) \le s \le s_+^\circ(t^\circ(s)) \tag{5.4.52}$$

for every $s\in[0,+\infty)$. Moreover the set S° defined by (5.4.50) *is at most countable, and the set U° introduced in* (4.2.14) *satisfies*

$$U^\circ = \bigcup_{t\in S^\circ}(s_-^\circ(t),s_+^\circ(t))\,. \tag{5.4.53}$$

Finally

$$s^\circ_-(t) \le \liminf_{\varepsilon\to 0} s^\circ_\varepsilon(t) \le \limsup_{\varepsilon\to 0} s^\circ_\varepsilon(t) \le s^\circ_+(t) \tag{5.4.54}$$

for every $t \in [0,+\infty)$.

Proof. All assertions are well-known properties of monotone functions, except for the last one. We only prove the first inequality in (5.4.54). If $s^\circ_-(t) = 0$ the inequality is obvious. If $s^\circ_-(t) > 0$ we fix $0 < s < s^\circ_-(t)$. By the definition of s°_-, we have $t^\circ(s) < t$; for ε small enough, this implies $t^\circ_\varepsilon(s) < t$, hence $s < s^\circ_\varepsilon(t)$. This gives $s \le \liminf_\varepsilon s^\circ_\varepsilon(t)$, and the conclusion follows from the arbitrariness of $s < s^\circ_-(t)$. ☐

Lemma 5.9. *Let* $t \in [0,+\infty) \setminus S^\circ$*, where* S° *is the set defined in* (5.4.50)*. Then*

$$\boldsymbol{u}_\varepsilon(t) \rightharpoonup \boldsymbol{u}^\circ(s^\circ_-(t)) \text{ weakly* in } BD(\Omega)\,, \tag{5.4.55}$$

$$\boldsymbol{e}_\varepsilon(t) \rightharpoonup \boldsymbol{e}^\circ(s^\circ_-(t)) \text{ weakly in } L^2(\Omega;\mathbb{M}^{N\times N}_{sym})\,, \tag{5.4.56}$$

$$\boldsymbol{p}_\varepsilon(t) \rightharpoonup \boldsymbol{p}^\circ(s^\circ_-(t)) \text{ weakly* in } M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})\,, \tag{5.4.57}$$

$$\boldsymbol{z}_\varepsilon(t) \to \boldsymbol{z}^\circ(s^\circ_-(t)) \text{ strongly in } C^0(\bar\Omega)\,. \tag{5.4.58}$$

Proof. Since $t \notin S^\circ$, Lemma 5.8 gives $s^\circ_\varepsilon(t) \to s^\circ_-(t)$. By (5.4.16) we have $\boldsymbol{u}_\varepsilon(t) = \boldsymbol{u}^\circ_\varepsilon(s^\circ_\varepsilon(t))$, $\boldsymbol{e}_\varepsilon(t) = \boldsymbol{e}^\circ_\varepsilon(s^\circ_\varepsilon(t))$, $\boldsymbol{p}_\varepsilon(t) = \boldsymbol{p}^\circ_\varepsilon(s^\circ_\varepsilon(t))$, $\boldsymbol{z}_\varepsilon(t) = \boldsymbol{z}^\circ_\varepsilon(s^\circ_\varepsilon(t))$. Therefore the conclusion follows from (5.4.29), (5.4.30), (5.4.31), and (5.4.46). ☐

Proof of Theorem 5.6 (continuation). By (2.3.38), (2.3.49), (5.2.27), and (5.4.11), for every $T > 0$ we have

$$\varepsilon^2 \int_0^T \|\dot{\boldsymbol{p}}_\varepsilon(t)\|_2^2 dt \to 0\,.$$

This implies that a subsequence, not relabelled, satisfies

$$\varepsilon\dot{\boldsymbol{p}}_\varepsilon(t) \to 0 \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$$

for $\mathcal{L}^1$-a.e. $t \in [0,+\infty)$. This fact, together with Lemma 5.9, yields that

$$\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon(t) \rightharpoonup \boldsymbol{\sigma}^\circ(s^\circ_-(t)) \text{ weakly in } L^2(\Omega;\mathbb{M}^{N\times N}_{sym})\,,$$
$$\boldsymbol{\zeta}_\varepsilon(t) \to \boldsymbol{\zeta}^\circ(s^\circ_-(t)) \text{ strongly in } C^0(\bar\Omega)\,,$$

for $\mathcal{L}^1$-a.e. $t \in [0,+\infty)$. Since K is convex, the inclusion $\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{p}}_\varepsilon(t) \in \mathcal{K}(\boldsymbol{\zeta}_\varepsilon(t))$, established in (5.2.26), passes to the limit and we obtain

$$\boldsymbol{\sigma}^\circ(s^\circ_-(t)) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s^\circ_-(t))). \tag{5.4.59}$$

By (5.4.34), (5.4.44), and the left continuity of s°, (5.4.59) holds for every $t \in [0,+\infty)$. A similar proof shows that

$$\boldsymbol{\sigma}^\circ(s^\circ_+(t)) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s^\circ_+(t))). \tag{5.4.60}$$

Let U° be the set defined in (4.2.14) and let $s \in [0,+\infty) \setminus U^\circ$. By (5.4.52) and (5.4.53) we have

$$\text{either} \quad s = s^\circ_-(t^\circ(s)) \quad \text{or} \quad s = s^\circ_+(t^\circ(s))\,. \tag{5.4.61}$$

The partial stress constraint (ev3$'$)$^\circ$ of Definition 4.5 follows now from (5.4.59), (5.4.60), and (5.4.61).

It remains to prove the energy-dissipation balance (4.3.1), the partial flow-rule (4.3.2), and the evolution law for the internal variable (ev4)$^\circ$. The proof will be continued after Remark 5.14. $\square$

5.5 Proof of Theorem 5.6: energy inequality and evolution law

The goal of the first part of this section is to prove that the functions $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, $\boldsymbol{w}^\circ$, $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, and $\boldsymbol{\chi}^\circ$ introduced in the previous section satisfy the energy inequality

$$\begin{aligned}\mathcal{Q}(\boldsymbol{e}^\circ(S)) - \mathcal{Q}(e_0) + \int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\Big)\, ds -\\ - \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \chi_0, p_0\rangle + \int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds \le \\ \le \int_0^S \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds + \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{e}^\circ(S)\rangle - \langle \chi_0, e_0\rangle\end{aligned} \tag{5.5.1}$$

for every $S>0$, where $\chi_0 := \boldsymbol{\chi}(0) = \boldsymbol{\chi}^\circ(0)$. To this aim we prove four lower semicontinuity results concerning the integrals in the left-hand side of (5.5.1) and the functions $\boldsymbol{p}^\circ_\varepsilon$, $\boldsymbol{\sigma}^\circ_\varepsilon$, $\boldsymbol{\zeta}^\circ_\varepsilon$, and $\boldsymbol{\chi}^\circ_\varepsilon$ defined in (5.4.16).

Lemma 5.10. *For every $S>0$, $\psi \in C^0(\overline{\Omega})^+$, and $\boldsymbol{\zeta} \in C^0([0,+\infty); C^0(\overline{\Omega})^+)$ we have*

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}(s))\, ds \le \liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ_\varepsilon(s), \boldsymbol{\zeta}(s))\, ds\,. \tag{5.5.2}$$

Proof. Since the function $s \mapsto \dot{\boldsymbol{p}}^\circ(s)$ is weakly* measurable from $[0,+\infty)$ to $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$, it is possible to define $\mu_\varepsilon, \mu \in M_b((0,S)\times(\Omega\cup\Gamma_0); \mathbb{M}^{N\times N}_{sym})$ by setting

$$\langle \varphi, \mu_\varepsilon\rangle := \int_0^S \langle \varphi(s,\cdot), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\, ds \quad \text{and} \quad \langle \varphi, \mu\rangle := \int_0^S \langle \varphi(s,\cdot), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds$$

for every $\varphi \in C^0_0((0,S)\times(\Omega\cup\Gamma_0); \mathbb{M}^{N\times N}_{sym})$. If $\varphi \in C^1_c((0,S)\times(\Omega\cup\Gamma_0); \mathbb{M}^{N\times N}_{sym})$, we have

$$\langle \varphi, \mu_\varepsilon\rangle = -\int_0^S \langle \partial_s\varphi(s,\cdot), \boldsymbol{p}^\circ_\varepsilon(s)\rangle\, ds \to -\int_0^S \langle \partial_s\varphi(s,\cdot), \boldsymbol{p}^\circ(s)\rangle\, ds = \langle \varphi, \mu\rangle,$$

by (5.4.22) and (5.4.26). Since $\|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_1 \le 1$ and $\|\dot{\boldsymbol{p}}^\circ(s)\|_1 \le 1$ by (5.4.21) and (5.4.28), by uniform approximation we obtain $\langle \varphi, \mu_\varepsilon\rangle \to \langle \varphi, \mu\rangle$ for every $\varphi \in C^0_0((0,S)\times(\Omega\cup\Gamma_0); \mathbb{M}^{N\times N}_{sym}))$, i.e.,

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{weakly}^* \text{ in } M_b((0,S)\times(\Omega\cup\Gamma_0); \mathbb{M}^{N\times N}_{sym}). \tag{5.5.3}$$

Since $s \mapsto |\dot{\boldsymbol{p}}^\circ(s)|$ is weakly* measurable from $[0,+\infty)$ to $M_b(\Omega\cup\Gamma_0)$, we define $\lambda_\varepsilon, \lambda \in M_b((0,S)\times(\Omega\cup\Gamma_0))$ by setting

$$\langle \phi, \lambda_\varepsilon\rangle := \int_0^S \langle \phi(s,\cdot), |\dot{\boldsymbol{p}}^\circ_\varepsilon(s)|\rangle\, ds \quad \text{and} \quad \langle \phi, \lambda\rangle := \int_0^S \langle \phi(s,\cdot), |\dot{\boldsymbol{p}}^\circ(s)|\rangle\, ds$$

for every $\phi \in C^0_0((0,S)\times(\Omega\cup\Gamma_0))$. It is easy to see that $\mu_\varepsilon \ll \lambda_\varepsilon$ and $\mu \ll \lambda$. Moreover

$$\frac{d\mu_\varepsilon}{d\lambda_\varepsilon}(s,x) = \frac{d\dot{\boldsymbol{p}}^\circ_\varepsilon(s)}{d|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)|}(x) \quad \text{and} \quad \frac{d\mu}{d\lambda}(s,x) = \frac{d\dot{\boldsymbol{p}}^\circ(s)}{d|\dot{\boldsymbol{p}}^\circ(s)|}(x)\,.$$

Using the definition of $\mathcal{H}$, see (1.3.20), it follows that

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}(s))\,ds = \int_{(0,S)\times(\Omega\cup\Gamma_0)} H(\psi(x)\frac{d\mu_\varepsilon}{d\lambda_\varepsilon}(s,x),\boldsymbol{\zeta}(s,x))\,d\lambda_\varepsilon(s,x)\,, \tag{5.5.4}$$

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}(s))\,ds = \int_{(0,S)\times(\Omega\cup\Gamma_0)} H(\psi(x)\frac{d\mu}{d\lambda}(s,x),\boldsymbol{\zeta}(s,x))\,d\lambda(s,x)\,. \tag{5.5.5}$$

By (5.5.3) we can now apply Reshetnyak's lower semicontinuity Theorem [39, Theorem 2] and we obtain

$$\begin{aligned}\int_{(0,S)\times(\Omega\cup\Gamma_0)} &H(\psi(x)\frac{d\mu}{d\lambda}(s,x),\boldsymbol{\zeta}(s,x))\,d\lambda(s,x) \le \\ &\le \liminf_{\varepsilon\to 0}\int_{(0,S)\times(\Omega\cup\Gamma_0)} H(\psi(x)\frac{d\mu_\varepsilon}{d\lambda_\varepsilon}(s,x),\boldsymbol{\zeta}(s,x))\,d\lambda_\varepsilon(s,x)\,.\end{aligned} \tag{5.5.6}$$

Inequality (5.5.2) follows now from (5.5.4), (5.5.5), and (5.5.6). □

Lemma 5.11. *For every $S>0$, and every $\psi \in C^0(\overline{\Omega})^+$, we have*

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s))\,ds \le \liminf_{\varepsilon\to 0}\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ_\varepsilon(s))\,ds\,. \tag{5.5.7}$$

Proof. As $\boldsymbol{\zeta}^\circ \in C^0([0,+\infty);C^0(\overline{\Omega}))$ by (5.4.44) and (5.4.47), we can apply Lemma 5.10 and we obtain

$$\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s))\,ds \le \liminf_{\varepsilon\to 0}\int_0^S \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ(s))\,ds\,,$$

for every $S>0$. Using (1.3.10), (1.3.12), (5.4.21), and the definition of $\mathcal{H}$ we obtain for every $s\in[0,+\infty)$

$$|\mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ(s)) - \mathcal{H}(\psi\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ_\varepsilon(s))| \le M_K\|\psi\|_\infty\|\boldsymbol{\zeta}^\circ_\varepsilon(s)-\boldsymbol{\zeta}^\circ(s)\|_\infty\,.$$

By (5.4.17), (5.4.43), and (5.4.47) $\|\boldsymbol{\zeta}^\circ_\varepsilon(s)-\boldsymbol{\zeta}^\circ(s)\|_\infty \to 0$ uniformly on compact sets, and inequality (5.5.7) follows. □

Lemma 5.12. *For every $S>0$, we have*

$$\begin{aligned}\int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s)) + \langle\dot{\boldsymbol{\chi}}^\circ(s),\boldsymbol{p}^\circ(s)\rangle\Big)\,ds - \langle\boldsymbol{\chi}^\circ(S),\boldsymbol{p}^\circ(S)\rangle + \langle\chi_0,p_0\rangle \le \\ \le \liminf_{\varepsilon\to 0}\int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ_\varepsilon(s)) - \langle\boldsymbol{\chi}^\circ_\varepsilon(s),\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\Big)\,ds\,.\end{aligned} \tag{5.5.8}$$

Proof. We consider a sequence $\psi_k \in C^\infty(\bar{\Omega})$, with $0\le\psi_k\le 1$ in $\bar{\Omega}$ and $\psi_k=0$ in a neighbourhood of $\bar{\Gamma}_1$, such that $\psi_k(x)\to 1$ for every $x\in\Omega\cup\Gamma_0$. By (2.3.49) the function $H(\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ(s)) - \boldsymbol{\chi}^\circ_\varepsilon(s)\colon\dot{\boldsymbol{p}}^\circ_\varepsilon(s)$ is positive $\mathcal{L}^n$-a.e. in Ω for every $s\in[0,+\infty)$, hence

$$\mathcal{H}(\psi_k\,\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ(s)) - \langle\psi_k\,\boldsymbol{\chi}^\circ_\varepsilon(s),\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle \le \mathcal{H}(\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ(s)) - \langle\boldsymbol{\chi}^\circ_\varepsilon(s),\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\,. \tag{5.5.9}$$

Integrating by parts in time, we have

$$\begin{aligned}\int_0^S \langle \psi_k \,\boldsymbol{\chi}^\circ_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle \, ds = -\int_0^S \langle \psi_k \,\dot{\boldsymbol{\chi}}^\circ_\varepsilon(s), \boldsymbol{p}^\circ_\varepsilon(s)\rangle \, ds +\\ + \langle \psi_k \,\boldsymbol{\chi}^\circ_\varepsilon(S), \boldsymbol{p}^\circ_\varepsilon(S)\rangle - \langle \psi_k \,\chi_0, p_0^\varepsilon\rangle \,.\end{aligned} \tag{5.5.10}$$

Performing the change of variables $t = t^\circ_\varepsilon(s)$, we get

$$\int_0^S \langle \psi_k \,\dot{\boldsymbol{\chi}}^\circ_\varepsilon(s), \boldsymbol{p}^\circ_\varepsilon(s)\rangle \, ds = \int_0^{T_\varepsilon} \langle \psi_k \,\dot{\boldsymbol{\chi}}(t), \boldsymbol{p}_\varepsilon(t)\rangle \, dt \,, \tag{5.5.11}$$

where $T_\varepsilon := t^\circ_\varepsilon(S)$. As $\psi_k = 0$ on Γ_1, integrating by parts in space and using (5.2.3), we obtain for every $t \in [0, T_\varepsilon]$

$$\begin{aligned}\langle \psi_k \,\dot{\boldsymbol{\chi}}(t), \boldsymbol{p}_\varepsilon(t)\rangle = -\langle \psi_k \,\dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t) - E\boldsymbol{w}(t)\rangle -\\ - \langle \dot{\boldsymbol{\chi}}(t), (\boldsymbol{u}_\varepsilon(t) - \boldsymbol{w}(t)) \odot \nabla \psi_k\rangle + \langle \dot{\boldsymbol{f}}(t), \psi_k \,(\boldsymbol{u}_\varepsilon(t) - \boldsymbol{w}(t))\rangle \,.\end{aligned}$$

which, thanks to Lemma 5.9 converges to

$$\begin{aligned}-\langle \psi_k \,\dot{\boldsymbol{\chi}}(t), \boldsymbol{e}^\circ(s^\circ_-(t)) - E\boldsymbol{w}(t)\rangle - \langle \dot{\boldsymbol{\chi}}(t), (\boldsymbol{u}^\circ(s^\circ_-(t)) - \boldsymbol{w}(t)) \odot \nabla \psi_k\rangle +\\ + \langle \dot{\boldsymbol{f}}(t), \psi_k \,(\boldsymbol{u}^\circ(s^\circ_-(t)) - \boldsymbol{w}(t))\rangle \,.\end{aligned}$$

By (2.3.28), this expression equals to $\langle [\dot{\boldsymbol{\chi}}(t) : \boldsymbol{p}^\circ(s^\circ_-(t))], \psi_k\rangle$; as $\|\boldsymbol{p}_\varepsilon(t)\|_1$ is bounded by (5.4.10) and (5.4.12), while $\|\dot{\boldsymbol{\chi}}(t)\|_\infty$ is locally integrable by (2.3.48), the Dominated Convergence Theorem yields

$$\lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \langle \psi_k \,\dot{\boldsymbol{\chi}}(t), \boldsymbol{p}_\varepsilon(t)\rangle \, dt = \int_0^T \langle [\dot{\boldsymbol{\chi}}(t) : \boldsymbol{p}(s^\circ_-(t))], \psi_k\rangle \, dt \,.$$

Let $\boldsymbol{\omega}(t) := \dot{\boldsymbol{\chi}}(t)$ if the derivative exists at t, and $\boldsymbol{\omega}(t) = 0$ otherwise. By (1.4.17) and (5.4.19) we get

$$\dot{\boldsymbol{\chi}}^\circ(s) = \boldsymbol{\omega}(t^\circ(s)) \,\dot{t}^\circ(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in [0, S].$$

This equality, together with the change of variables formula (1.4.18), yields

$$\begin{aligned}\int_0^T \langle [\dot{\boldsymbol{\chi}}(t) : \boldsymbol{p}(s^\circ_-(t))], \psi_k\rangle \, dt = \int_0^T \langle [\boldsymbol{\omega}(t) : \boldsymbol{p}(s^\circ_-(t))], \psi_k\rangle \, dt =\\ = \int_0^S \langle [\dot{\boldsymbol{\chi}}^\circ(s) : \boldsymbol{p}^\circ(s^\circ_-(t^\circ(s)))], \psi_k\rangle \, ds = \int_0^S \langle [\dot{\boldsymbol{\chi}}^\circ(s) : \boldsymbol{p}^\circ(s)], \psi_k\rangle \, ds \,,\end{aligned} \tag{5.5.12}$$

where the last equality follows from the fact that $\dot{\boldsymbol{\chi}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s \in U^\circ$ and that $s^\circ_-(t^\circ(s)) = s$ for $\mathcal{L}^1$-a.e. $s \in [0, S] \backslash U^\circ$ (indeed, by (5.4.52) and (5.4.53), the only exceptions are the points of the form $s = s^\circ_+(t)$ for $t \in S^\circ$). We conclude that

$$\lim_{\varepsilon\to 0} \int_0^S \langle \psi_k \,\dot{\boldsymbol{\chi}}^\circ_\varepsilon(s), \boldsymbol{p}^\circ_\varepsilon(s)\rangle \, ds = \int_0^S \langle [\dot{\boldsymbol{\chi}}^\circ(s) : \boldsymbol{p}^\circ(s)], \psi_k\rangle \, ds \,. \tag{5.5.13}$$

Another integration-by-parts argument, using (5.4.7), (5.4.8), (5.4.10), and (5.4.18), shows that

$$\lim_{\varepsilon\to 0} \Big(\langle \psi_k \,\boldsymbol{\chi}^\circ_\varepsilon(S), \boldsymbol{p}^\circ_\varepsilon(S)\rangle - \langle \psi_k \,\chi_0, p_0^\varepsilon\rangle \Big) = \langle [\boldsymbol{\chi}^\circ(S) : \boldsymbol{p}^\circ(S)], \psi_k\rangle - \langle [\chi_0 : p_0], \psi_k\rangle \,. \tag{5.5.14}$$

By (5.5.7), (5.5.9), (5.5.13), and (5.5.14) we finally get

$$\int_0^S \Big(\mathcal{H}(\psi_k\,\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s)) + \langle[\dot{\boldsymbol{\chi}}^\circ(s):\boldsymbol{p}^\circ(s)],\psi_k\rangle\Big)\,ds - \langle[\boldsymbol{\chi}^\circ(S):\boldsymbol{p}^\circ(S)],\psi_k\rangle +$$
$$+\langle[\chi_0:p_0],\psi_k\rangle \le \liminf_{\varepsilon\to 0}\int_0^S \Big(\mathcal{H}(\psi_k\,\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ_\varepsilon(s)) - \langle\psi_k\,\boldsymbol{\chi}^\circ_\varepsilon(s),\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\Big)\,ds \le$$
$$\le \liminf_{\varepsilon\to 0}\int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ_\varepsilon(s),\boldsymbol{\zeta}^\circ_\varepsilon(s)) - \langle\boldsymbol{\chi}^\circ_\varepsilon(s),\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\Big)\,ds\,.$$

Using (1.3.12), (2.3.48), (5.4.27), (5.4.28), and (5.4.44) we can pass to the limit as $k\to\infty$, applying the Dominated Convergence Theorem, and we obtain (5.5.8) . □

We recall that we are adopting convention (1.2.1) about L^p-norms.

Lemma 5.13. *Let $S>0$, and let A° be as in* (4.2.7). *Then*

$$\int_{A^\circ_S}\|\dot{\boldsymbol{p}}^\circ(s)\|_2\,d_2(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\,ds \le \liminf_{\varepsilon\to 0+}\int_{A^\circ_S}\|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_2\,d_2(\boldsymbol{\sigma}^\circ_\varepsilon(s),\mathcal{K}(\boldsymbol{\zeta}^\circ_\varepsilon(s)))\,ds\,, \quad (5.5.15)$$

where

$$A^\circ_S = A^\circ\cap[0,S]\,. \quad (5.5.16)$$

Proof. Since $\boldsymbol{e}^\circ$ is continuous for the weak topology of $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ by (5.4.34), and $\boldsymbol{\zeta}^\circ$ is continuous for the strong topology of $C^0(\overline{\Omega})$ by (5.4.44), by Remark 4.3 A° is open. Observe that we can equivalently define A° as the set of times s such that $d_2(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s)))>0$.

We fix a compact set $C\subset A^\circ_S$ and a continuous function $\psi\colon C\to[0,+\infty)$ such that

$$d_2(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s)))>\psi(s) \qquad \text{for every } s\in C\,. \quad (5.5.17)$$

We claim that, for ε sufficiently small, we have

$$d_2(\boldsymbol{\sigma}^\circ_\varepsilon(s),\mathcal{K}(\boldsymbol{\zeta}^\circ_\varepsilon(s)))>\psi(s) \qquad \text{for every } s\in C\,. \quad (5.5.18)$$

If not, there exist $\varepsilon_k\to 0$ and $s_k\in C$ such that $d_2(\boldsymbol{\sigma}^\circ_{\varepsilon_k}(s_k),\mathcal{K}(\boldsymbol{\zeta}^\circ_{\varepsilon_k}(s_k)))\le\psi(s_k)$. We may assume that $s_k\to s_0\in C$; now, by (5.4.30), (5.4.46), and (5.4.47), thanks to the lower semicontinuity of $d_2(\sigma,\mathcal{K}(\zeta))$ proved in Remark 4.3, the previous inequality gives $d_2(\boldsymbol{\sigma}^\circ(s_0),\mathcal{K}(\boldsymbol{\zeta}^\circ(s_0)))\le\psi(s_0)$, which contradicts (5.5.17). This proves (5.5.18).

By a standard approximation argument from below, in order to prove (5.5.15), it suffices to prove

$$\int_C\|\dot{\boldsymbol{p}}^\circ(s)\|_2\,\psi(s)\,ds \le \liminf_{\varepsilon\to 0+}\int_C\|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_2\,\psi(s)\,ds, \quad (5.5.19)$$

for every compact $C\subset A^\circ_S$ and every continuous function $\psi\colon C\to[0,+\infty)$. To this end, let φ_i be a dense sequence in the unit ball of $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, composed of continuous functions with compact support. Since

$$\|\dot{\boldsymbol{p}}^\circ(s)\|_2 = \sup_i\langle\varphi_i,\dot{\boldsymbol{p}}^\circ(s)\rangle,$$

by the Localisation Lemma (see, e.g., [5, Lemma 2.3.2]) we have

$$\int_C\|\dot{\boldsymbol{p}}^\circ(s)\|_2\,\psi(s)\,ds = \sup\sum_{i=1}^k\int_{C_i}\langle\varphi_i,\dot{\boldsymbol{p}}^\circ(s)\rangle\psi(s)\,ds, \quad (5.5.20)$$

where the supremum is taken over all integers k and over all finite Borel partitions $C_1, \dots, C_k$ of C. For every i the real-valued functions $s \mapsto \langle \varphi_i, \boldsymbol{p}^\circ_\varepsilon(s)\rangle$ are equi-Lipschitz on $[0,S]$ (by (5.4.20)) and converge to $s \mapsto \langle \varphi_i, \boldsymbol{p}^\circ(s)\rangle$ for every s (by (5.4.26)), hence the functions $s \mapsto \langle \varphi_i, \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle$ converge $\langle \varphi_i, \dot{\boldsymbol{p}}^\circ(s)\rangle$ weakly* in $L^\infty([0,S])$. It follows that

$$\sum_{i=1}^k \int_{C_i} \langle \varphi_i, \dot{\boldsymbol{p}}^\circ(s)\rangle \psi(s)\, ds = \lim_{\varepsilon\to 0} \sum_{i=1}^k \int_{C_i} \langle \varphi_i, \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle \psi(s)\, ds \le \liminf_{\varepsilon\to 0} \int_C \|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_2 \psi(s)\, ds.$$

Inequality (5.5.19) follows now from (5.5.20). □

Remark 5.14. Since $d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s)) = 0$ outside of the set A°_S, by (5.5.15) and the nonnegativeness of the integrands we easily get

$$\int_0^S \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds \le \liminf_{\varepsilon\to 0} \int_0^S \|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ_\varepsilon(s), \boldsymbol{\zeta}^\circ_\varepsilon(s))\, ds\,. \tag{5.5.21}$$

We are now in a position to prove the energy inequality (5.5.1).

Proof of Theorem 5.6 (continuation). Let us fix $S > 0$ and define $T_\varepsilon := t^\circ_\varepsilon(S)$. By (5.2.27)

$$\begin{aligned}
&\mathcal{Q}(\boldsymbol{e}_\varepsilon(T_\varepsilon)) - \mathcal{Q}(e_0) + \int_0^{T_\varepsilon} \Big(\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon(t), \boldsymbol{\zeta}(t)) - \langle \boldsymbol{\chi}(t), \dot{\boldsymbol{p}}_\varepsilon(t)\rangle\Big)\, dt + \varepsilon \int_0^{T_\varepsilon} \|\dot{\boldsymbol{p}}_\varepsilon(t)\|_2^2\, dt = \\
&= \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt - \int_0^{T_\varepsilon} \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t)\rangle\, dt + \langle \boldsymbol{\chi}(T_\varepsilon), \boldsymbol{e}_\varepsilon(T_\varepsilon)\rangle - \langle \chi_0, e_0\rangle\,,
\end{aligned}$$

where $\chi_0 := \boldsymbol{\chi}(0)$. By (5.2.5) we have

$$\begin{aligned}
&\mathcal{Q}(\boldsymbol{e}_\varepsilon(T_\varepsilon)) - \mathcal{Q}(e_0) + \int_0^{T_\varepsilon} \Big(\mathcal{H}(\dot{\boldsymbol{p}}_\varepsilon(t), \boldsymbol{\zeta}(t)) - \langle \boldsymbol{\chi}(t), \dot{\boldsymbol{p}}_\varepsilon(t)\rangle\Big)\, dt + \\
&+ \int_0^{T_\varepsilon} \|\dot{\boldsymbol{p}}_\varepsilon(t)\|_2\, d_2(\boldsymbol{\sigma}_\varepsilon(t), \mathcal{K}(\boldsymbol{\zeta}_\varepsilon(t)))\ dt = \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt - \\
&- \int_0^{T_\varepsilon} \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t)\rangle\, dt + \langle \boldsymbol{\chi}(T_\varepsilon), \boldsymbol{e}_\varepsilon(T_\varepsilon)\rangle - \langle \chi_0, e_0\rangle\,.
\end{aligned}$$

Performing the change of variable $t = t^\circ_\varepsilon(s)$ in the left-hand side, we obtain

$$\begin{aligned}
&\mathcal{Q}(\boldsymbol{e}^\circ_\varepsilon(S)) - \mathcal{Q}(e_0) + \int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ_\varepsilon(s), \boldsymbol{\zeta}^\circ_\varepsilon(s)) - \langle \boldsymbol{\chi}^\circ_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\Big)\, ds + \\
&+ \int_0^S \|\dot{\boldsymbol{p}}^\circ_\varepsilon(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ_\varepsilon(s), \mathcal{K}(\boldsymbol{\zeta}^\circ_\varepsilon(s)))\, ds = \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt - \\
&- \int_0^{T_\varepsilon} \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t)\rangle\, dt + \langle \boldsymbol{\chi}(T_\varepsilon), \boldsymbol{e}^\circ_\varepsilon(S)\rangle - \langle \chi_0, e_0\rangle\,.
\end{aligned} \tag{5.5.22}$$

By the lower semicontinuity of $\mathcal{Q}$, in view of (5.4.30) we have

$$\mathcal{Q}(\boldsymbol{e}^\circ(S)) \le \liminf_{\varepsilon\to 0} \mathcal{Q}(\boldsymbol{e}^\circ_\varepsilon(S)). \tag{5.5.23}$$

By (5.4.11) and (5.4.56) we have

$$\int_0^T \langle \boldsymbol{\sigma}^\circ(s^\circ_-(t)) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt = \lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\, dt\,, \tag{5.5.24}$$

where $T := t^\circ(S)$. Let $\boldsymbol{\omega}(t) := E\dot{\boldsymbol{w}}(t)$ if the derivative exists at t, and $\boldsymbol{\omega}(t) = 0$ otherwise. By (1.4.17) and (5.4.19) we get

$$E\dot{\boldsymbol{w}}^\circ(s) = \boldsymbol{\omega}(t^\circ(s))\,\dot{t}^\circ(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in [0,S].$$

This equality, together with the change of variables formula (1.4.18), yields

$$\int_0^T \langle \boldsymbol{\sigma}^\circ(s^\circ_-(t)) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\,dt = \int_0^T \langle \boldsymbol{\sigma}^\circ(s^\circ_-(t)) - \boldsymbol{\chi}(t), \boldsymbol{\omega}(t)\rangle\,dt =$$
$$= \int_0^S \langle \boldsymbol{\sigma}^\circ(s^\circ_-(t^\circ(s))) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\,ds = \int_0^S \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\,ds,$$

where the last equality follows from the fact that $E\dot{\boldsymbol{w}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s \in U^\circ$ and that $s^\circ_-(t^\circ(s)) = s$ for $\mathcal{L}^1$-a.e. $s \in [0,S] \setminus U^\circ$ (see the proof of Lemma 5.12). Therefore, (5.5.24) gives

$$\int_0^S \langle \boldsymbol{\sigma}^\circ(s) - \boldsymbol{\chi}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\,ds = \lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \langle \boldsymbol{\sigma}_\varepsilon(t) - \boldsymbol{\chi}(t), E\dot{\boldsymbol{w}}(t)\rangle\,dt. \tag{5.5.25}$$

Similarly, we prove

$$\int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\,ds = \lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \langle \dot{\boldsymbol{\chi}}(t), \boldsymbol{e}_\varepsilon(t)\rangle\,dt\,. \tag{5.5.26}$$

Inequality (5.5.1) follows now from (5.4.30), (5.5.8), (5.5.21), (5.5.22), (5.5.23), (5.5.25), and (5.5.26).

To prove the evolution law (4.2.17) we need a technical result on the convergence of $\dot{\boldsymbol{p}}^\circ_\varepsilon$ to $\dot{\boldsymbol{p}}^\circ$. The proof of Theorem 5.6 will be continued after the following lemma. □

Lemma 5.15. *Let $S > 0$ and let $\boldsymbol{\varphi}_\varepsilon, \boldsymbol{\varphi} \in L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}))$. Assume that $\boldsymbol{\varphi}_\varepsilon \to \boldsymbol{\varphi}$ strongly in $L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}))$. Then $s \mapsto \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$ is integrable on $[0,S]$ and*

$$\int_0^S \langle \boldsymbol{\varphi}_\varepsilon(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\,ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\,ds \quad \textit{as } \varepsilon \to 0\,.$$

Proof. We start by proving

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ_\varepsilon(s)\rangle\,ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\,ds \quad \text{as } \varepsilon \to 0\,. \tag{5.5.27}$$

By (5.4.27) we have $\boldsymbol{p}^\circ \in C^0([0,S], M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym}))$. Since

$$\langle \boldsymbol{\varphi}(s), \frac{\boldsymbol{p}^\circ(s+h) - \boldsymbol{p}^\circ(s)}{h}\rangle \to \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \quad \text{as } h \to 0$$

for $\mathcal{L}^1$-a.e. $s \in [0,S]$, the function $s \mapsto \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$ is measurable on $[0,S]$. By (5.4.28) we have $|\langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle| \le \|\boldsymbol{\varphi}(s)\|_\infty$ for $\mathcal{L}^1$-a.e. $s \in [0,S]$. Since $s \mapsto \|\boldsymbol{\varphi}(s)\|_\infty$ is integrable on $[0,S]$, the same property holds for $s \mapsto \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$.

If $\boldsymbol{\varphi} \in C^1_c((0,S); C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}))$ we can write

$$\int_0^S \langle \boldsymbol{\varphi}(s), \frac{\boldsymbol{p}^\circ_\varepsilon(s+h) - \boldsymbol{p}^\circ_\varepsilon(s)}{h}\rangle\,ds = \int_0^S \langle \frac{\boldsymbol{\varphi}(s-h) - \boldsymbol{\varphi}(s)}{h}, \boldsymbol{p}^\circ_\varepsilon(s)\rangle\,ds\,.$$

Passing to the limit as $h \to 0$ we obtain

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \rangle \, ds = - \int_0^S \langle \dot{\boldsymbol{\varphi}}(s), \boldsymbol{p}_\varepsilon^\circ(s) \rangle \, ds \, . \tag{5.5.28}$$

A similar formula holds for $\boldsymbol{p}^\circ$. Thus (5.5.27) follows from (5.4.22) and (5.4.26).

Since $\|\dot{\boldsymbol{p}}_\varepsilon^\circ(s)\|_1 \leq 1$ and $\|\dot{\boldsymbol{p}}^\circ(s)\|_1 \leq 1$ by (5.4.21) and (5.4.28), the same conclusion in the case $\boldsymbol{\varphi} \in L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}))$ follows from the density of $C^1_c((0,S); C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}))$ in $L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}))$.

Now, by (5.4.21) we have

$$\Big| \int_0^S \langle \boldsymbol{\varphi}_\varepsilon(s), \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \rangle \, ds - \int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \rangle \, ds \Big| \leq \int_0^S \|\boldsymbol{\varphi}_\varepsilon(s) - \boldsymbol{\varphi}(s)\|_\infty ds \, .$$

Since the right-hand side tends to 0 as $\varepsilon \to 0$, the conclusion follows from (5.5.27). □

We now prove the evolution law (4.2.17).

Proof of Theorem 5.6 (continuation). Let us fix $S > 0$. Define $\boldsymbol{a}_\varepsilon^\circ(s)$ and $\boldsymbol{a}^\circ(s)$ as in (5.4.35), and (5.4.36), respectively. We first prove that

$$\int_0^S \langle \boldsymbol{\varphi}(s), \boldsymbol{a}_\varepsilon^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \rangle \, ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \boldsymbol{a}^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ(s) \rangle \, ds \tag{5.5.29}$$

for every $\boldsymbol{\varphi} \in L^1([0,S]; C^0(\overline{\Omega}))$. We observe that we can write $\langle \boldsymbol{\varphi}(s), \boldsymbol{a}_\varepsilon^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \rangle = \langle \boldsymbol{\varphi}(s) \boldsymbol{a}_\varepsilon^\circ(s) I, \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \rangle$ and $\langle \boldsymbol{\varphi}(s), \boldsymbol{a}^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ(s) \rangle = \langle \boldsymbol{\varphi}(s) \boldsymbol{a}^\circ(s) I, \dot{\boldsymbol{p}}^\circ(s) \rangle$. Therefore (5.5.29) follows from Lemma 5.15, because $\boldsymbol{\varphi}\, \boldsymbol{a}_\varepsilon^\circ I \to \boldsymbol{\varphi}\, \boldsymbol{a}^\circ I$ strongly in $L^1([0,S]; C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym}))$ thanks to (5.4.38) and (5.4.39). Using the equalities

$$\langle \boldsymbol{\varphi}(s), \rho_1 \star \big(\boldsymbol{a}_\varepsilon^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \big) \rangle = \langle \check{\rho}_1 \star \boldsymbol{\varphi}(s), \boldsymbol{a}_\varepsilon^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \rangle$$

and

$$\langle \boldsymbol{\varphi}(s), \rho_1 \star \big(\boldsymbol{a}^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ(s) \big) \rangle = \langle \check{\rho}_1 \star \boldsymbol{\varphi}(s), \boldsymbol{a}^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ(s) \rangle \, ,$$

where $\check{\rho}_1(x) := \rho_1(-x)$, from (2.3.41) and (5.5.29) we obtain

$$\int_0^S \langle \boldsymbol{\varphi}(s), \rho_1 \star \big(\boldsymbol{a}_\varepsilon^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}_\varepsilon^\circ(s) \big) \rangle \, ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \rho_1 \star \big(\boldsymbol{a}^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ(s) \big) \rangle \, ds \tag{5.5.30}$$

for every $\boldsymbol{\varphi} \in L^1([0,S]; L^1(\Omega))$. By (5.4.37) and (5.5.30) we have

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{z}}_\varepsilon^\circ(s) \rangle \, ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \rho_1 \star \big(\boldsymbol{a}^\circ(s) \operatorname{tr} \dot{\boldsymbol{p}}^\circ(s) \big) \rangle \, ds \quad \text{as } \varepsilon \to 0 \tag{5.5.31}$$

for every $\boldsymbol{\varphi} \in L^1([0,S]; L^1(\Omega))$. On the other hand, if $\boldsymbol{\varphi} \in C^1_c((0,S); L^1(\Omega))$, we have

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{z}}_\varepsilon^\circ(s) \rangle \, ds = - \int_0^S \langle \dot{\boldsymbol{\varphi}}(s), \boldsymbol{z}_\varepsilon^\circ(s) \rangle \, ds \, ,$$

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{z}}^\circ(s) \rangle \, ds = - \int_0^S \langle \dot{\boldsymbol{\varphi}}(s), \boldsymbol{z}^\circ(s) \rangle \, ds \, ,$$

so that (5.4.43) gives

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{z}}_\varepsilon^\circ(s) \rangle \, ds \to \int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{z}}^\circ(s) \rangle \, ds \quad \text{as } \varepsilon \to 0 \, .$$

By (5.5.31) this implies

$$\int_0^S \langle \boldsymbol{\varphi}(s), \dot{\boldsymbol{z}}^\circ(s)\rangle\, ds = \int_0^S \langle \boldsymbol{\varphi}(s), \rho_1 \star \big(\boldsymbol{a}^\circ(s)\,\mathrm{tr}\,\dot{\boldsymbol{p}}^\circ(s)\big)\rangle\, ds$$

for every $\boldsymbol{\varphi} \in C^1_c((0,S);L^1(\Omega))$, and hence

$$\dot{\boldsymbol{z}}^\circ(s) = \rho_1 \star \big(\boldsymbol{a}^\circ(s)\,\mathrm{tr}\,\dot{\boldsymbol{p}}^\circ(s)\big) \quad \text{in } \overline{\Omega} \text{ for } \mathcal{L}^1\text{-a.e. } s \in [0,S]\,. \tag{5.5.32}$$

This concludes the proof of (4.2.17).

The proof of Theorem 5.6 will be continued in Section 5.6 after Lemma 5.23. □

5.6 Proof of Theorem 5.6: Conclusion

In this section $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{w}$, u_0, e_0, p_0, and z_0 are as in Definition 4.5 and satisfy the uniform safe-load condition (2.3.45)-(2.3.48). We assume that $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, t°, $\boldsymbol{\sigma}^\circ$, and $\boldsymbol{\zeta}^\circ$ satisfy (4.2.4) and (4.2.5), together with conditions (ev0)°, (ev1)°, (ev2)°, and (ev3′)° of Definition 4.5. Let us fix $S>0$ and let A°_S be the open set defined by (4.2.7) and (5.5.16). We also assume that

$$\int_{A^\circ_S} \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds < +\infty\,, \tag{5.6.1}$$

so that $\dot{\boldsymbol{p}}^\circ(s)$, defined by (4.2.6), belongs to $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ for $\mathcal{L}^1$-a.e. $s \in A^\circ_S$.

The goal of this section is to prove that the functions $\boldsymbol{u}^\circ$, $\boldsymbol{e}^\circ$, $\boldsymbol{p}^\circ$, $\boldsymbol{z}^\circ$, $\boldsymbol{w}^\circ$, $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\zeta}^\circ$, and $\boldsymbol{\chi}^\circ$ satisfy the energy inequality

$$\begin{aligned}
&\mathcal{Q}_{\boldsymbol{\chi}}(S, \boldsymbol{e}^\circ(S)) - \mathcal{Q}_{\boldsymbol{\chi}}(0, e_0) + \int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\Big)\, ds - \\
&- \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \chi_0, p_0\rangle + \int_{A^\circ_S} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \geq \\
&\geq \int_0^S \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds - \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds\,,
\end{aligned} \tag{5.6.2}$$

where $\chi_0 := \boldsymbol{\chi}(0) = \boldsymbol{\chi}^\circ(0)$, and, according to the notation introduced in (4.3.65)-(4.3.66), $\boldsymbol{\tau}^\circ := \boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ$, and $\mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)) := \mathcal{Q}(\boldsymbol{e}^\circ(s)) - \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{e}^\circ(s)\rangle$.

We start by proving that (5.6.1) implies a variant of Lemma 4.15.

Lemma 5.16. *Let (a,b) be a connected component of A°_S, and let $c \in (a,b)$. Then $\boldsymbol{p}^\circ - \boldsymbol{p}^\circ(c) \in AC_{loc}((a,b);L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))$. In particular, for $\mathcal{L}^1$-a.e. $s \in (a,b)$, $\dot{\boldsymbol{p}}^\circ(s)$ is the strong limit in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, as $h \to 0$, of the difference quotient $\frac{1}{h}(\boldsymbol{p}^\circ(s+h) - \boldsymbol{p}^\circ(s))$, and $\dot{\boldsymbol{p}}^\circ \in L^1_{loc}((a,b);L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))$. Moreover, for every $s_1, s_2 \in (a,b)$, we have*

$$\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1) \in L^2(\Omega;\mathbb{M}^{N\times N}_{sym}) \quad \textit{and} \quad \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1) = \int_{s_1}^{s_2} \dot{\boldsymbol{p}}^\circ(s)\, ds\,,$$

where the last term is a Bochner integral in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$.

Proof. In the proof of Lemma 5.13 we have seen that $d_2\big(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s))\big)$ is lower semicontinuous. Therefore for every $[a_1,b_1]\subset(a,b)$, there exists a constant $\eta_1>0$ such that $d_2\big(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s))\big)\geq\eta_1$ for every $s\in[a_1,b_1]$. By (5.6.1) this gives

$$\int_{a_1}^{b_1}||\dot{\boldsymbol{p}}^\circ(s)||_2\,ds<+\infty\,. \tag{5.6.3}$$

This inequality and the measurability of $s\mapsto\langle\varphi,\dot{\boldsymbol{p}}^\circ(s)\rangle$ for every $\varphi\in C^0_0(\Omega;\mathbb{M}^{N\times N}_{sym})$ imply that $s\mapsto\langle\psi,\dot{\boldsymbol{p}}^\circ(s)\rangle$ is measurable for every $\psi\in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, hence $\dot{\boldsymbol{p}}^\circ\colon[a_1,b_1]\to L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ is weakly measurable. By Pettis Theorem it is also strongly measurable, so that (5.6.3) implies that $\dot{\boldsymbol{p}}^\circ\in L^1_{loc}((a,b);L^2(\Omega;\mathbb{M}^{N\times N}_{sym}))$. The rest of the proof follows by the same arguments as in Lemma 4.15. $\square$

Lemma 5.16 implies in particular that the assumptions of Lemmas 4.16 and 4.17 are satisfied, so that $\boldsymbol{e}^\circ\in AC_{loc}((a,b);L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, and (4.3.48) and (4.3.51) hold. However, local absolute continuity of $\boldsymbol{e}^\circ$ is not enough for the approximation argument that we will employ in order to prove (5.6.2). We need to recover at least strong continuity in a and in b of $\boldsymbol{e}^\circ$. This will be done in Lemma 5.18, using a weak L^1-estimate for gradients of solutions of the elliptic system of linearized elasticity proved in the next theorem. We preliminarily recall that, for every measurable set B and for every measurable function f defined on B with values in a finite dimensional Hilbert space, we define

$$\|f\|_{1,w,B}:=\sup_{t>0}t\,\mathcal{L}^n(\{|f|>t\}\cap B)\,. \tag{5.6.4}$$

It is well-known that $\|f\|_{1,w,B}\leq\|f\|_{1,B}$ (Chebychev Inequality) and that $\|f_1+f_2\|_{1,w,B}\leq 2\|f_1\|_{1,w,B}+2\|f_2\|_{1,w,B}$ for every pair of functions f_1,f_2. We now state and prove the announced regularity result.

Theorem 5.17. *For every open set $\Omega'\subset\subset\Omega$ there exists a constant C depending only on Ω', Ω, and $\mathbb{C}$ such that, if $p\in L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ and $u\in H^1_{loc}(\Omega;\mathbb{R}^N)$ satisfies the equation*

$$-\mathrm{div}\,(\mathbb{C}Eu)=-\mathrm{div}\,(\mathbb{C}p)\quad\textit{in}\ \Omega\,, \tag{5.6.5}$$

then we have the estimate

$$\|\nabla u\|_{1,w,\Omega'}\leq C(\|p\|_{1,\Omega}+\|u\|_{1,\Omega})\,, \tag{5.6.6}$$

where $\|\cdot\|_{1,w,\Omega'}$ is defined in (5.6.4).

Proof. Let Ω'' be an open set such that $\Omega'\subset\subset\Omega''\subset\subset\Omega$, and let $\varphi\in C^\infty_c(\Omega)$ be a cutoff function with $\varphi=1$ on Ω'' and $0\leq\varphi\leq1$. Let p and u be as in the statement, and let $q:=\mathbb{C}p$, and $v:=\varphi u$. It turns out that v has compact support and satisfies the equation

$$-\mathrm{div}\,(\mathbb{C}Ev)=-\mathrm{div}\,(\varphi q)+q\nabla\varphi-(\mathbb{C}Eu)\nabla\varphi-\mathrm{div}\,(\mathbb{C}(u\odot\nabla\varphi)\quad\text{in }\mathbb{R}^N\,. \tag{5.6.7}$$

The fundamental solution of the operator $-\mathrm{div}\,(\mathbb{C}Eu)$ is given by

$$G(x):=a\,g(x)\,I+b\,\nabla g(x)\otimes x\,, \tag{5.6.8}$$

where g is the fundamental solution of the Laplace operator, $a = \frac{1}{2(\lambda+2\mu)} + \frac{1}{2\mu}$, and $b = \frac{1}{2(\lambda+2\mu)} - \frac{1}{2\mu}$ (see [38, Section 2.5.2] and [52, Chapter II, formula (1.46)]). Since v has compact support, equation (5.6.7) gives the representation

$$v_i(x) = \sum_{h,k=1}^{n} \int_{\mathbb{R}^N} D_k G_{ih}(x-y)(\varphi q)_{hk}(y)\, dy + \sum_{h=1}^{n} \int_{\mathbb{R}^N} G_{ih}(x-y)(q\nabla\varphi)_h(y)\, dy -$$
$$- \sum_{h=1}^{n} \int_{\mathbb{R}^N} G_{ih}(x-y)(\mathbb{C}Eu\nabla\varphi)_h(y)\, dy + \sum_{h,k=1}^{n} \int_{\mathbb{R}^N} D_k G_{ih}(x-y)(\mathbb{C}(u\odot\nabla\varphi))_{hk}(y)\, dy\,.$$

For a.e. $x \in \Omega'$ it follows that

$$D_j v_i(x) = \alpha(x) + \beta(x) - \gamma(x) + \delta(x)\,,$$

where

$$\alpha(x) := \sum_{h,k=1}^{n} \int_{\mathbb{R}^N} D_j D_k G_{ih}(x-y)(\varphi q)_{hk}(y)\, dy\,,$$
$$\beta(x) := \sum_{h=1}^{n} \int_{\Omega\setminus\Omega''} D_j G_{ih}(x-y)(q\nabla\varphi)_h(y)\, dy\,,$$
$$\gamma(x) := \sum_{h=1}^{n} \int_{\Omega\setminus\Omega''} D_j G_{ih}(x-y)(\mathbb{C}Eu\nabla\varphi)_h(y)\, dy\,,$$
$$\delta(x) := \sum_{h,k=1}^{n} \int_{\Omega\setminus\Omega''} D_j D_k G_{ih}(x-y)(\mathbb{C}(u\odot\nabla\varphi))_{hk}(y)\, dy\,.$$

The function $D_j D_k G_{ih}$ is homogeneous of degree $-n$. Using the explicit expression of G_{ih} given by (5.6.8) we can check that $D_j D_k G_{ih}$ has mean value 0 on the boundary of each ball around the origin. Therefore we can apply the Calderon-Zygmund estimate contained in [49, Chapter II, Theorem 4], obtaining

$$\|\alpha\|_{1,w,\Omega} \le C_1 \|p\|_{1,\Omega}\,, \tag{5.6.9}$$

where the constant C_1 only depends on the function G, and the elasticity tensor $\mathbb{C}$.

To estimate the term $\gamma(x)$ we introduce the cartesian components c^{lm}_{hk} of the tensor $\mathbb{C}$, defined by

$$(\mathbb{C}Eu)_{hk} = \sum_{l,m=1}^{n} c^{lm}_{hk} D_l u_m\,.$$

It follows that

$$\gamma(x) = \sum_{h,k,l,m=1}^{n} c^{lm}_{hk} \int_{\Omega\setminus\Omega''} D_j G_{ih}(x-y) D_l u_m(y) D_k\varphi(y)\, dy\,.$$

For $x \in \Omega'$, the function $y \mapsto G_{ih}(x-y)$ is of class C^∞ in $\Omega\setminus\Omega''$. Integrating by parts, we obtain

$$\gamma(x) = -\sum_{h,k,l,m=1}^{n} c^{lm}_{hk} \int_{\Omega\setminus\Omega''} D_j D_l G_{ih}(x-y) u_m(y) D_k\varphi(y)\, dy -$$
$$- \sum_{h,k,l,m=1}^{n} c^{lm}_{hk} \int_{\Omega\setminus\Omega''} D_j G_{ih}(x-y) u_m(y) D_l D_k\varphi(y)\, dy\,.$$

As $D_j D_l G_{ih}(x-y)$ and $D_j G_{ih}(x-y)$ are uniformly bounded when $x \in \Omega'$ and $y \in \Omega \setminus \Omega''$, we obtain the estimate

$$\|\gamma\|_{\infty,\Omega'} \leq C_3 \|u\|_{1,\Omega}\,, \tag{5.6.10}$$

where the constant C_3 depends on the function G, on the elasticity tensor $\mathbb{C}$, on the pair Ω', Ω'', and on the function φ.

In a similar, and easier, way we prove the estimates

$$\|\beta\|_{\infty,\Omega'} \leq C_2 \|p\|_{1,\Omega} \quad \text{ and } \|\delta\|_{\infty,\Omega'} \leq C_4 \|u\|_{1,\Omega}\,, \tag{5.6.11}$$

where the constants C_2 and C_4 depend on the function G, on the elasticity tensor $\mathbb{C}$, on the pair Ω', Ω'', and on the function φ. Inequality (5.6.6) follows now from (5.6.9), (5.6.10), and (5.6.11). $\square$

With the previous estimate, we can prove a continuity result for the stress that we will use in the sequel.

Lemma 5.18. *Let (a,b) be a connected component of A°_S. Then there exists an increasing sequence $s_k \to b$ such that $\boldsymbol{\sigma}^\circ(s_k) \to \boldsymbol{\sigma}^\circ(b)$ strongly in $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$.*

Proof. First, we prove that there exists an increasing sequence $s_k \to b$ such that

$$d_2(\boldsymbol{\sigma}^\circ(s_k), \mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))) \to 0\,. \tag{5.6.12}$$

If not, there exist $c \in (a,b)$ and $\eta > 0$ such that $d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s))) \geq \eta$ for every $s \in [c,b)$. Then (4.2.5), (5.6.1), and (4.3.51) imply that

$$\int_c^b ||\dot{\boldsymbol{\sigma}}^\circ(s)||_2 \, ds < +\infty.$$

It follows that $\boldsymbol{\sigma}^\circ(s)$ has a strong limit in $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ as $s \to b^-$. Since $\boldsymbol{\sigma}^\circ(s) \rightharpoonup \boldsymbol{\sigma}^\circ(b)$ weakly in $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ as $s \to b^-$, we deduce that $\boldsymbol{\sigma}^\circ(s) \to \boldsymbol{\sigma}^\circ(b)$ strongly in $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$ as $s \to b^-$. Since $\boldsymbol{\sigma}^\circ(b) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(b))$, we conclude that $d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s))) \to 0$ as $s \to b^-$, which contradicts our assumption on η. Thus, (5.6.12) is proved and we can fix such a sequence s_k.

Now, let $h < k$. By Lemma 4.16 we have $\boldsymbol{u}^\circ(s_h) - \boldsymbol{u}^\circ(s_k) \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^N)$, while $\boldsymbol{\sigma}^\circ(s_h) - \boldsymbol{\sigma}^\circ(s_k) \in \Sigma_0(\Omega)$ by (4.2.12), thanks to the inclusion $A^\circ_S \subset U^\circ$ proved in (4.2.20). Then (4.3.49) implies that

$$-\mathrm{div}\,(\mathbb{C}E(\boldsymbol{u}^\circ(s_h) - \boldsymbol{u}^\circ(s_k))) = -\mathrm{div}\,(\mathbb{C}(\boldsymbol{p}^\circ(s_h) - \boldsymbol{p}^\circ(s_k))).$$

Let us fix an open set $\Omega' \subset\subset \Omega$. By (5.6.6) there exists a constant C such that

$$\|E(\boldsymbol{u}^\circ(s_h) - \boldsymbol{u}^\circ(s_k))\|_{1,w,\Omega'} \leq C\|\boldsymbol{p}^\circ(s_h) - \boldsymbol{p}^\circ(s_k)\|_1 + C\|\boldsymbol{u}^\circ(s_h) - \boldsymbol{u}^\circ(s_k)\|_1\,;$$

then (4.2.5), (4.3.49), the Lipschitz continuity of $\boldsymbol{p}^\circ$, and the strong continuity of $\boldsymbol{u}\colon [0,S] \to L^1(\Omega; \mathbb{R}^N)$ entail that $\boldsymbol{\sigma}^\circ(s_k)$ is a Cauchy sequence with respect to convergence in measure in Ω. As $\boldsymbol{\sigma}^\circ(s_k) \rightharpoonup \boldsymbol{\sigma}^\circ(b)$ weakly in $L^2(\Omega; \mathbb{M}^{N\times N}_{sym})$, it follows that $\boldsymbol{\sigma}^\circ(s_k) \to \boldsymbol{\sigma}^\circ(b)$ in measure. We now consider the decomposition

$$\boldsymbol{\sigma}^\circ(s_k) = \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k)) + (\boldsymbol{\sigma}^\circ(s_k) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k))\,. \tag{5.6.13}$$

The sequence $\boldsymbol{\sigma}^\circ(s_k) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k))$ converges to 0 strongly in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$ by (5.6.12). As $\boldsymbol{\sigma}^\circ(s_k) \to \boldsymbol{\sigma}^\circ(b)$ in measure, this implies that $\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k)) \to \boldsymbol{\sigma}^\circ(b)$ in measure. Since $\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k))$ is uniformly bounded in $L^\infty(\Omega;\mathbb{M}^{N\times N}_{sym})$, by the Dominated Convergence Theorem we have $\pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s_k))}(\boldsymbol{\sigma}^\circ(s_k)) \to \boldsymbol{\sigma}^\circ(b)$ strongly in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, therefore (5.6.13) gives $\boldsymbol{\sigma}^\circ(s_k) \to \boldsymbol{\sigma}^\circ(b)$ strongly in $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$, as required. □

The next five lemmas provide a discrete approximation of the integrals in (5.6.2). We start with a result of approximation with Riemann sums for the duality $\langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\langle$.

Lemma 5.19. *Let* $\{s_k^i\}_{0\le i\le i_k}$ *be a sequence of subdivisions of* $[0,S]$ *satisfying* (1.5.2). *Then*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \langle \boldsymbol{\chi}^\circ(s_k^i) - \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle - \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds \Big| = 0\,, \tag{5.6.14}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \Big| \langle \boldsymbol{\chi}^\circ(s_k^i) - \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i})\rangle - \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds \Big| = 0\,, \tag{5.6.15}$$

where all duality products are defined according to (2.3.12) *for every* $s\in[0,S]$.

Proof. As a starting point we observe that (4.3.12) implies

$$\langle \boldsymbol{\chi}^\circ(s_k^i) - \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle = \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s_k^{i-1})\rangle\, ds \tag{5.6.16}$$

for every k and every i. Since $\boldsymbol{p}^\circ$ is 1-Lipschitz continuous, by (2.3.13) for every k we have

$$\begin{aligned} &\Big| \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s) - \boldsymbol{p}^\circ(s_k^{i-1})\rangle\, ds \Big| \le \\ &\le \int_{s_k^{i-1}}^{s_k^i} \|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty \|\boldsymbol{p}^\circ(s) - \boldsymbol{p}^\circ(s_k^{i-1})\|_1\, ds \le \eta_k \int_{s_k^{i-1}}^{s_k^i} \|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty\, ds\,. \end{aligned} \tag{5.6.17}$$

It follows from (5.6.16) and (5.6.17) that

$$\sum_{i=1}^{i_k} \Big| \langle \boldsymbol{\chi}^\circ(s_k^i) - \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle - \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds \Big| \le \eta_k \int_0^S \|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty\, ds.$$

As the right-hand side is finite by (2.3.48), (1.5.2) gives (5.6.14). The same argument proves (5.6.15). □

We introduce the notation

$$B_S^\circ := \{s\in[0,S] : \boldsymbol{\sigma}^\circ(s) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s))\} = [0,S]\setminus A_S^\circ\,. \tag{5.6.18}$$

Since A_S° is open, B_S° is compact.

Lemma 5.20. *For every* $s_1, s_2 \in B_S^\circ$ *with* $s_1 < s_2$ *we have*

$$\begin{aligned} &\tfrac12 \langle \boldsymbol{\tau}^\circ(s_1) + \boldsymbol{\tau}^\circ(s_2), E\boldsymbol{w}^\circ(s_2) - E\boldsymbol{w}^\circ(s_1)\rangle - \tfrac12 \langle \boldsymbol{\chi}^\circ(s_2) - \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_2) + \boldsymbol{e}^\circ(s_1)\rangle \le \\ &\quad \le \mathcal{Q}_{\boldsymbol{\chi}}(s_2, \boldsymbol{e}^\circ(s_2)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_1, \boldsymbol{e}^\circ(s_1)) + \tfrac12 \mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_1)) + \\ &\quad + \tfrac12 \mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_2)) - \tfrac12 \langle \boldsymbol{\chi}^\circ(s_2) + \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle\,. \end{aligned} \tag{5.6.19}$$

Proof. Let s_1 and s_2 be as in the statement of the lemma. Since $\boldsymbol{\tau}^\circ(s) \in \Sigma_0(\Omega)$ for every s by (2.3.45) and (4.2.12), a direct algebraic computation and (2.3.29) give

$$\begin{aligned}\mathcal{Q}_{\boldsymbol{\chi}}(s_2, \boldsymbol{e}^\circ(s_2)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_1, \boldsymbol{e}^\circ(s_1)) + \tfrac{1}{2}\langle \boldsymbol{\chi}^\circ(s_2) - \boldsymbol{\chi}^\circ(s_1), \boldsymbol{e}^\circ(s_1) + \boldsymbol{e}^\circ(s_2)\rangle = \\ = \tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_2) + \boldsymbol{\tau}^\circ(s_1), \boldsymbol{e}^\circ(s_2) - \boldsymbol{e}^\circ(s_1)\rangle = \\ = \tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1) + \boldsymbol{\tau}^\circ(s_2), E\boldsymbol{w}^\circ(s_2) - E\boldsymbol{w}^\circ(s_1)\rangle - \tfrac{1}{2}\langle \boldsymbol{\tau}^\circ(s_1) + \boldsymbol{\tau}^\circ(s_2), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle\,.\end{aligned}$$

Since $\boldsymbol{\sigma}^\circ(s_i) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s_i))$ and $\boldsymbol{\zeta}^\circ(s_i) \in C^0(\overline{\Omega})$ for $i = 1, 2$, by Proposition 2.5 we obtain $\langle \boldsymbol{\sigma}^\circ(s_i), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle \leq \mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_i))$. Therefore

$$\langle \boldsymbol{\tau}^\circ(s_i), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle \leq \mathcal{H}(\boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1), \boldsymbol{\zeta}^\circ(s_i)) - \langle \boldsymbol{\chi}^\circ(s_i), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle$$

and (5.6.19) easily follows from the previous equalities. □

Lemma 5.21. *Let (a, b) be a connected component of A_S° and let $a \leq s_1 < s_2 \leq b$. Then*

$$\begin{aligned}0 \leq \mathcal{Q}_{\boldsymbol{\chi}}(s_2, \boldsymbol{e}^\circ(s_2)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_1, \boldsymbol{e}^\circ(s_1)) + \int_{s_1}^{s_2} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds - \\ - \langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{p}^\circ(s_2)\rangle + \langle \boldsymbol{\chi}^\circ(s_1), \boldsymbol{p}(s_1)\rangle + \int_{s_1}^{s_2} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds\,.\end{aligned} \tag{5.6.20}$$

Proof. We first observe that $\boldsymbol{\chi}^\circ$ is constant on (a, b) by the inclusion $A_S^\circ \subset U^\circ$ proved in (4.2.20). By Lemma 4.11 the function $s \mapsto \langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle$ is absolutely continuous on $[a, b]$. Since the function $s \mapsto \mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s))$ is lower semicontinuous, we can assume that $a < s_1$. Moreover Lemma 5.18 provides a sequence $s_k \to b$ such that $\mathcal{Q}_{\boldsymbol{\chi}}(s_k, \boldsymbol{e}^\circ(s_k)) \to \mathcal{Q}_{\boldsymbol{\chi}}(b, \boldsymbol{e}^\circ(b))$, so that we may also assume $s_2 < b$. Therefore it is enough to prove the inequality on a compact subinterval $[s_1, s_2]$ of (a, b).

By Lemma 5.16, we can apply Lemma 4.17; with this, we get that the function $s \mapsto \mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)))\rangle$ is absolutely continuous on $[s_1, s_2]$ and, since $\boldsymbol{\chi}^\circ$ is constant on $[s_1, s_2]$, we have

$$\tfrac{d}{ds}\mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)))\rangle = \langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle \tag{5.6.21}$$

for a.e. $s \in [s_1, s_2]$. Similarly, by Lemma 5.16

$$\tfrac{d}{ds}\langle \boldsymbol{\chi}^\circ(s), \boldsymbol{p}^\circ(s)\rangle = \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \tag{5.6.22}$$

for a.e. $s \in [s_1, s_2]$, where the right-hand side is the usual scalar product of L^2. In view of (5.6.21) and (5.6.22), inequality (5.6.20) easily follows from the inequality $0 \leq \frac{d}{ds}\mathcal{Q}_{\boldsymbol{\chi}}(s, \boldsymbol{e}^\circ(s)) + \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) - \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle$, which is equivalent to

$$\begin{aligned}0 \leq \langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle + \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) - \\ - \langle \boldsymbol{\chi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\,.\end{aligned} \tag{5.6.23}$$

As $\boldsymbol{\tau}^\circ(s) \in \Sigma_0(\Omega)$ by (2.3.45) and (4.2.12), from (2.3.33), (4.3.48), and (4.3.49) we get

$$\langle \boldsymbol{\tau}^\circ(s), \boldsymbol{e}^\circ(s+h) - \boldsymbol{e}^\circ(s)\rangle = -\langle \boldsymbol{\tau}^\circ(s), \boldsymbol{p}^\circ(s+h) - \boldsymbol{p}^\circ(s)\rangle\,;$$

by Lemmas 4.17 and 5.16, we conclude that $\langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{e}}^\circ(s)\rangle = -\langle \boldsymbol{\tau}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle$, therefore (5.6.23) is equivalent to

$$\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle \leq \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\,;$$

this inequality can be proved by observing that

$$\begin{aligned}\langle \boldsymbol{\sigma}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle &= \langle \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle \leq \\ &\leq \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\,,\end{aligned}$$

where the inequality follows from the definition of $\mathcal{H}$. This concludes the proof. □

Lemma 5.22. *Let $(s^i_k)_{0\leq i\leq i_k}$ be a sequence of subdivisions of* $[0,S]$ *satisfying* (1.5.2) *and* (1.5.13), *with $\boldsymbol{\psi}$ given by the following functions: $\boldsymbol{\sigma}^\circ$, $\boldsymbol{\sigma}^\circ 1_{B^\circ_S}$, $\boldsymbol{\chi}^\circ 1_{B^\circ_S}$, and $1_{B^\circ_S}$, the first three with $X = L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$. Let I^A_k, I^B_k, and J^A_k be defined by* (1.5.20), (1.5.21), *and* (1.5.24), *with $A = A^\circ_S$ and $B = B^\circ_S$. Then*

$$\lim_{k\to\infty}\sum_{i\in I^B_k}\langle \boldsymbol{\tau}^\circ(s^{i-1}_k), E\boldsymbol{w}^\circ(s^i_k) - E\boldsymbol{w}^\circ(s^{i-1}_k)\rangle = \int_0^S \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds\,, \tag{5.6.24}$$

$$\lim_{k\to\infty}\sum_{i\in I^A_k\cup J^A_k}\int_{s^{i-1}_k}^{s^i_k}(\|\boldsymbol{\sigma}^\circ(s)\|_2 + \|\boldsymbol{\chi}^\circ(s)\|_2 + 1)\, 1_{B^\circ_S}(s)\, ds = 0\,, \tag{5.6.25}$$

$$\lim_{k\to\infty}\sum_{i\in I^B_k}\langle \boldsymbol{\chi}^\circ(s^i_k) - \boldsymbol{\chi}^\circ(s^{i-1}_k), \boldsymbol{e}^\circ(s^{i-1}_k)\rangle = \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\, ds\,, \tag{5.6.26}$$

$$\lim_{k\to\infty}\sum_{i\in I^B_k}\langle \boldsymbol{\chi}^\circ(s^i_k) - \boldsymbol{\chi}^\circ(s^{i-1}_k), \boldsymbol{p}^\circ(s^{i-1}_k)\rangle = \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds\,. \tag{5.6.27}$$

These equalities continue to hold if $\boldsymbol{\tau}^\circ(s^{i-1}_k)$, $\boldsymbol{e}^\circ(s^{i-1}_k)$, and $\boldsymbol{p}^\circ(s^{i-1}_k)$ are replaced by $\boldsymbol{\tau}^\circ(s^i_k)$, $\boldsymbol{e}^\circ(s^i_k)$, and $\boldsymbol{p}^\circ(s^i_k)$, respectively.

Proof. Equality (5.6.25) follows from (1.5.27), with $\boldsymbol{\psi}$ given by $\boldsymbol{\sigma}^\circ 1_{B^\circ_S}$, $\boldsymbol{\chi}^\circ 1_{B^\circ_S}$, and $1_{B^\circ_S}$. Now, recalling that $E\dot{\boldsymbol{w}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s \in A^\circ_S$ by the inclusion $A^\circ_S \subset U^\circ$ proved in (4.2.20), and that $\|E\dot{\boldsymbol{w}}^\circ(s)\|_2 \leq 1$ for $\mathcal{L}^1$-a.e. $s\in[0,S]$ by (5.4.24), we get

$$\begin{aligned}&\Big|\sum_{i\in I^B_k}\langle \boldsymbol{\tau}^\circ(s^{i-1}_k), E\boldsymbol{w}^\circ(s^i_k) - E\boldsymbol{w}^\circ(s^{i-1}_k)\rangle - \int_0^S \langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\, ds\Big| \leq \\ &\leq \sum_{i\in I^B_k}\int_{s^{i-1}_k}^{s^i_k}\big|\langle \boldsymbol{\tau}^\circ(s^{i-1}_k) - \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle\big|\, ds + \sum_{i\in I^A_k\cup J^A_k}\int_{s^{i-1}_k}^{s^i_k}|\langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle|\, ds \leq \\ &\leq \sum_{i\in I^B_k}\int_{s^{i-1}_k}^{s^i_k}\|\boldsymbol{\tau}^\circ(s^{i-1}_k) - \boldsymbol{\tau}^\circ(s)\|_2\, ds + \sum_{i\in I^A_k\cup J^A_k}\int_{s^{i-1}_k}^{s^i_k}\|\boldsymbol{\tau}^\circ(s)\|_2 1_{B^\circ_S}(s)\, ds\,.\end{aligned}$$

The first term in the right-hand side vanishes in the limit since $\boldsymbol{\tau}^\circ = \boldsymbol{\sigma}^\circ - \boldsymbol{\chi}^\circ$, $\boldsymbol{\sigma}^\circ$ satisfies (1.5.13), and $\boldsymbol{\chi}^\circ$ is continuous. As the second one tends to 0 by (5.6.25), equality (5.6.24) is proved.

Since $\dot{\boldsymbol{\chi}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s\in A^\circ_S \subset U^\circ$, and $\|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty \leq 1$ for $\mathcal{L}^1$-a.e. $s\in[0,S]$ by (5.4.24), by adapting the previous argument we can prove (5.6.26). We finally observe that, by (2.3.13) and (5.4.24),

$$\sum_{i\in I^A_k\cup J^A_k}\Big|\int_{s^{i-1}_k}^{s^i_k}\langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds\Big| \leq M\sum_{i\in I^A_k\cup J^A_k}\int_{s^{i-1}_k}^{s^i_k} 1_{B^\circ_S}(s)\, ds\,,$$

where M is an upper bound of $\|\boldsymbol{p}^\circ(s)\|_1$ on $[0,S]$, and the right-hand side vanishes in the limit as $k\to\infty$ by (5.6.25). Together with (5.6.14) and (5.6.15), this proves (5.6.27). The last assertion of the lemma can be proved in a similar way. □

Lemma 5.23. *Let $(s_k^i)_{0\le i\le i_k}$, I_k^A, I_k^B, and J_k^A be as in Lemma 5.22. Assume that (s_k^{i-1}, s_k^i) is contained in A_S° for every $i \in J_k^A$. Then there exists a sequence $R_k \to 0$ such that*

$$\sum_{i\in I_k^A\cup J_k^A} \Big(\mathcal{Q}_{\boldsymbol{\chi}}(s_k^i, \boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-1}, \boldsymbol{e}^\circ(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds - \\ - \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i)\rangle + \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle + \\ + \int_{A_k^{i-1,i}} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \Big) \ge -R_k\,, \tag{5.6.28}$$

where $A_k^{i-1,i} := A_S^\circ \cap (s_k^{i-1}, s_k^i)$.

Proof. Define

$$\hat{I}_k^A := \{ i \in I_k^A \cup J_k^A : (s_k^{i-1}, s_k^i) \subset A_S^\circ \} \quad \text{and} \quad \check{I}_k^A := \{ i \in I_k^A : (s_k^{i-1}, s_k^i) \cap B_S^\circ \neq \emptyset \}\,;$$

our assumption on J_k^A implies that $\hat{I}_k^A \cup \check{I}_k^A = I_k^A \cup J_k^A$. By Lemma 5.21, we have

$$\sum_{i\in \hat{I}_k^A} \Big(\mathcal{Q}_{\boldsymbol{\chi}}(s_k^i, \boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-1}, \boldsymbol{e}^\circ(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds - \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i)\rangle + \\ + \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle + \int_{A_k^{i-1,i}} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \Big) \ge 0\,.$$

For every $i \in \check{I}_k^A$, we define $s_k^{i-\frac{2}{3}}$ (respectively $s_k^{i-\frac{1}{3}}$) as the supremum (respectively the infimum) of the connected component of A_S° containing s_k^{i-1} (respectively s_k^i). Notice that both $s_k^{i-\frac{1}{3}}$ and $s_k^{i-\frac{2}{3}}$ belong to the set B_S°. By Lemma 5.21, we have

$$0 \le \sum_{i\in \check{I}_k^A} \Big(\mathcal{Q}_{\boldsymbol{\chi}}(s_k^i, \boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-\frac{1}{3}}, \boldsymbol{e}^\circ(s_k^{i-\frac{1}{3}})) + \int_{s_k^{i-\frac{1}{3}}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds - \\ - \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i)\rangle + \langle \boldsymbol{\chi}^\circ(s_k^{i-\frac{1}{3}}), \boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}})\rangle + \int_{s_k^{i-\frac{1}{3}}}^{s_k^i} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \Big)$$

and

$$0 \le \sum_{i\in \check{I}_k^A} \Big(\mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-\frac{2}{3}}, \boldsymbol{e}^\circ(s_k^{i-\frac{2}{3}})) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-1}, \boldsymbol{e}^\circ(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^{i-\frac{2}{3}}} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds - \\ - \langle \boldsymbol{\chi}^\circ(s_k^{i-\frac{2}{3}}), \boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}})\rangle + \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle + \int_{s_k^{i-1}}^{s_k^{i-\frac{2}{3}}} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \Big).$$

Therefore to prove (5.6.28) it is enough to show that there exists $R_k \to 0$ such that

$$\sum_{i\in \check{I}_k^A} \Big(\mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-\frac{1}{3}}, \boldsymbol{e}^\circ(s_k^{i-\frac{1}{3}})) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-\frac{2}{3}}, \boldsymbol{e}^\circ(s_k^{i-\frac{2}{3}})) + \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds - \\ - \langle \boldsymbol{\chi}^\circ(s_k^{i-\frac{1}{3}}), \boldsymbol{p}^\circ(s_k^{i-\frac{1}{3}})\rangle + \langle \boldsymbol{\chi}^\circ(s_k^{i-\frac{2}{3}}), \boldsymbol{p}^\circ(s_k^{i-\frac{2}{3}})\rangle + \\ + \int_{A_k^{i-\frac{2}{3},i-\frac{1}{3}}} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds \ge -R_k\,, \tag{5.6.29}$$

where $A_k^{i-\frac{2}{3},i-\frac{1}{3}} := A_S^\circ \cap (s_k^{i-\frac{2}{3}}, s_k^{i-\frac{1}{3}})$.

Let $\check{B}_k$ be the union of the intervals (s_k^{i-1}, s_k^i) for $i \in \check{I}_k^A$. By the definition of $\check{I}_k^A$ each point of $\check{B}_k$ has distance from B_S° less than the constant η_k introduced in (1.5.2). Since B_S° is compact, we have $\mathcal{L}^1(\check{B}_k \cap A_S^\circ) \to 0$. By (5.6.1) this implies that

$$\int_{\check{B}_k \cap A_S^\circ} \|\dot{\boldsymbol{p}}^\circ(s)\|_2 \, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds \to 0\,. \tag{5.6.30}$$

By Lemma 5.20 we have

$$\begin{aligned}
&\mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-\frac13}, \boldsymbol{e}^\circ(s_k^{i-\frac13})) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-\frac23}, \boldsymbol{e}^\circ(s_k^{i-\frac23})) + \tfrac12 \mathcal{H}(\boldsymbol{p}^\circ(s_k^{i-\frac13}) - \boldsymbol{p}^\circ(s_k^{i-\frac23}), \boldsymbol{\zeta}^\circ(s_k^{i-\frac23})) + \\
&+ \tfrac12 \mathcal{H}(\boldsymbol{p}^\circ(s_k^{i-\frac13}) - \boldsymbol{p}^\circ(s_k^{i-\frac23}), \boldsymbol{\zeta}^\circ(s_k^{i-\frac13})) - \tfrac12 \langle \boldsymbol{\chi}^\circ(s^{i-\frac13}) + \boldsymbol{\chi}^\circ(s^{i-\frac23}), \boldsymbol{p}^\circ(s^{i-\frac13}) - \boldsymbol{p}^\circ(s^{i-\frac23})\rangle \geq \\
&\geq \tfrac12 \langle \boldsymbol{\tau}^\circ(s_k^{i-\frac23}) + \boldsymbol{\tau}^\circ(s_k^{i-\frac13}), E\boldsymbol{w}^\circ(s_k^{i-\frac13}) - E\boldsymbol{w}^\circ(s_k^{i-\frac23})\rangle - \\
&- \tfrac12 \langle \boldsymbol{\chi}^\circ(s^{i-\frac13}) - \boldsymbol{\chi}^\circ(s^{i-\frac23}), \boldsymbol{e}^\circ(s^{i-\frac13}) + \boldsymbol{e}^\circ(s^{i-\frac23})\rangle\,.
\end{aligned}$$

Now, recalling that $E\dot{\boldsymbol{w}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s \in A_S^\circ \subset U^\circ$, and that $\|E\dot{\boldsymbol{w}}^\circ(s)\|_2 \leq 1$ for $\mathcal{L}^1$-a.e. $s \in [0,S]$ by (5.4.24), we get

$$\begin{aligned}
&\big|\tfrac12 \langle \boldsymbol{\tau}^\circ(s_k^{i-\frac23}) + \boldsymbol{\tau}^\circ(s_k^{i-\frac13}), E\boldsymbol{w}^\circ(s_k^{i-\frac13}) - E\boldsymbol{w}^\circ(s_k^{i-\frac23})\rangle\big| \leq \\
&\leq C_1 \|E\boldsymbol{w}^\circ(s_k^{i-\frac13}) - E\boldsymbol{w}^\circ(s_k^{i-\frac23})\|_2 \leq C_1 \int_{s_k^{i-\frac23}}^{s_k^{i-\frac13}} 1_{B_S^\circ}(s)\, ds \leq C_1 \int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds\,,
\end{aligned}$$

where C_1 is an upper bound of $\|\boldsymbol{\tau}^\circ(s)\|_2$ on $[0,S]$. Similarly, as $\dot{\boldsymbol{\chi}}^\circ(s) = 0$ for $\mathcal{L}^1$-a.e. $s \in A_S^\circ \subset U^\circ$ and $\|\dot{\boldsymbol{\chi}}^\circ(s)\|_\infty \leq 1$ for $\mathcal{L}^1$-a.e. $s \in [0,S]$ by (5.4.24), using (4.3.4) and the Jensen's inequality as in Remark 4.9, we get

$$\big|\tfrac12 \langle \boldsymbol{\chi}^\circ(s^{i-\frac13}) - \boldsymbol{\chi}^\circ(s^{i-\frac23}), \boldsymbol{e}^\circ(s^{i-\frac13}) + \boldsymbol{e}^\circ(s^{i-\frac23})\rangle\big| \leq C_2 \int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds\,,$$

where C_2 is an upper bound of $\|\boldsymbol{e}^\circ(s)\|_1$ on $[0,S]$. Arguing as before, by (2.3.13) and (5.4.24), we can also prove that

$$\big|\tfrac12 \langle \boldsymbol{\chi}^\circ(s^{i-\frac13}) - \boldsymbol{\chi}^\circ(s^{i-\frac23}), \boldsymbol{p}^\circ(s^{i-\frac13}) + \boldsymbol{p}^\circ(s^{i-\frac23})\rangle\big| \leq C_3 \int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds\,,$$

where C_3 is an upper bound of $\|\boldsymbol{p}^\circ(s)\|_1$ on $[0,S]$. By this inequality and a direct computation we deduce that

$$\begin{aligned}
&-\tfrac12 \langle \boldsymbol{\chi}^\circ(s^{i-\frac13}) + \boldsymbol{\chi}^\circ(s^{i-\frac23}), \boldsymbol{p}^\circ(s^{i-\frac13}) - \boldsymbol{p}^\circ(s^{i-\frac23})\rangle \leq \\
&\leq -\langle \boldsymbol{\chi}^\circ(s^{i-\frac13}), \boldsymbol{p}^\circ(s^{i-\frac13})\rangle + \langle \boldsymbol{\chi}^\circ(s^{i-\frac23}), \boldsymbol{p}^\circ(s^{i-\frac23})\rangle + C_3 \int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds\,.
\end{aligned}$$

Therefore, setting $C := C_1 + C_2 + C_3$, from the previous inequalities we obtain that (5.6.29) holds with

$$\begin{aligned}
R_k := {}& C \sum_{i \in \check{I}_k^A} \int_{s_k^{i-1}}^{s_k^i} 1_{B_S^\circ}(s)\, ds + \int_{\check{B}_k \cap A_S^\circ} \|\dot{\boldsymbol{p}}^\circ(s)\|_2 \, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds + \\
&+ \tfrac12 \sum_{i \in \check{I}_k^A} \Big(\mathcal{H}(\boldsymbol{p}^\circ(s_k^{i-\frac13}) - \boldsymbol{p}^\circ(s_k^{i-\frac23}), \boldsymbol{\zeta}^\circ(s_k^{i-\frac23})) - \int_{s_k^{i-\frac23}}^{s_k^{i-\frac13}} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) ds \Big) + \\
&+ \tfrac12 \sum_{i \in \check{I}_k^A} \Big(\mathcal{H}(\boldsymbol{p}^\circ(s_k^{i-\frac13}) - \boldsymbol{p}^\circ(s_k^{i-\frac23}), \boldsymbol{\zeta}^\circ(s_k^{i-\frac13})) - \int_{s_k^{i-\frac23}}^{s_k^{i-\frac13}} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds \Big)\,.
\end{aligned}$$

From Lemma 1.10 and from (1.5.27) and (5.6.30) we obtain $R_k \to 0$, concluding the proof. $\square$

We are finally ready to conclude the proof of Theorem 5.6.

Proof of Theorem 5.6 (conclusion). Let us fix $S>0$ and let A_S° and B_S° be the sets defined in (5.5.16) and (5.6.18). Let $(s_k^i)_{0\le i\le i_k}$, I_k^A, I_k^B, and J_k^A be as in Lemma 5.22. By Remark 1.14 we may assume that $(s_k^{i-1}, s_k^i) \subset A_S^\circ$ for every $i \in J_k^A$. By Lemma 5.22 there exists a sequence $\varrho_k^1 \to 0$ such that

$$\begin{aligned}\int_0^S \Big(\langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\Big) ds \le \\ \le \tfrac12 \sum_{i\in I_k^B} \langle \boldsymbol{\tau}^\circ(s_k^{i-1}) + \boldsymbol{\tau}^\circ(s_k^i), E\boldsymbol{w}^\circ(s_k^i) - E\boldsymbol{w}^\circ(s_k^{i-1})\rangle - \\ - \tfrac12 \sum_{i\in I_k^B} \langle \boldsymbol{\chi}^\circ(s_k^i) - \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{e}^\circ(s_k^{i-1}) + \boldsymbol{e}^\circ(s_k^i)\rangle + \varrho_k^1 .\end{aligned}$$

By Lemma 5.20 we then deduce that

$$\begin{aligned}\int_0^S \Big(\langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\Big) ds \le \sum_{i\in I_k^B} (\mathcal{Q}_{\boldsymbol{\chi}}(s_k^i, \boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-1}, \boldsymbol{e}^\circ(s_k^{i-1}))) + \\ + \tfrac12 \sum_{i\in I_k^B} \mathcal{H}(\boldsymbol{p}^\circ(s_k^i) - \boldsymbol{p}^\circ(s_k^{i-1}), \boldsymbol{\zeta}^\circ(s_k^{i-1})) + \tfrac12 \sum_{i\in I_k^B} \mathcal{H}(\boldsymbol{p}^\circ(s_k^i) - \boldsymbol{p}^\circ(s_k^{i-1}), \boldsymbol{\zeta}^\circ(s_k^i)) + \\ + \tfrac12 \sum_{i\in I_k^B} \langle \boldsymbol{\chi}^\circ(s_k^{i-1}) - \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i) + \boldsymbol{p}^\circ(s_k^{i-1})\rangle - \\ - \sum_{i\in I_k^B} \Big(\langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i)\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle\Big) + \varrho_k^1 ,\end{aligned}$$

where we replaced the term $-\frac12 \langle \boldsymbol{\chi}^\circ(s_k^{i-1}) + \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i) - \boldsymbol{p}^\circ(s_k^{i-1})\rangle$ with the equivalent

$$\tfrac12 \langle \boldsymbol{\chi}^\circ(s_k^{i-1}) - \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i) + \boldsymbol{p}^\circ(s_k^{i-1})\rangle - \langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i)\rangle + \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle .$$

By (5.6.27), Lemma 1.10 provides a sequence $\varrho_k^2 \to 0$ such that

$$\begin{aligned}\int_0^S \Big(\langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle\Big) ds \le \sum_{i\in I_k^B} (\mathcal{Q}_{\boldsymbol{\chi}}(s_k^i, \boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-1}, \boldsymbol{e}^\circ(s_k^{i-1}))) + \\ + \sum_{i\in I_k^B} \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s)), \boldsymbol{\zeta}^\circ(s))\, ds + \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds - \\ - \sum_{i\in I_k^B} \Big(\langle \boldsymbol{\chi}^\circ(s_k^i), \boldsymbol{p}^\circ(s_k^i)\rangle - \langle \boldsymbol{\chi}^\circ(s_k^{i-1}), \boldsymbol{p}^\circ(s_k^{i-1})\rangle\Big) + \varrho_k^1 + \varrho_k^2 .\end{aligned}$$

Adding (5.6.28), where $A_k^{i-1,i} := A_S^\circ \cap (s_k^{i-1}, s_k^i)$, we get

$$\int_0^S \Big(\langle \boldsymbol{\tau}^\circ(s), E\dot{\boldsymbol{w}}^\circ(s)\rangle - \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{e}^\circ(s)\rangle \Big) ds \le \sum_{i=1}^{i_k} (\mathcal{Q}_{\boldsymbol{\chi}}(s_k^i, \boldsymbol{e}^\circ(s_k^i)) - \mathcal{Q}_{\boldsymbol{\chi}}(s_k^{i-1}, \boldsymbol{e}^\circ(s_k^{i-1}))) +$$

$$+ \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s)), \boldsymbol{\zeta}^\circ(s))\, ds + \int_0^S \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle\, ds -$$

$$- \langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \chi_0, p_0\rangle + \sum_{i \in I_k^A \cup J_k^A} \int_{A_k^{i-1,i}} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds + \varrho_k^3 \le$$

$$\le \mathcal{Q}_{\boldsymbol{\chi}}(S, \boldsymbol{e}^\circ(S)) - \mathcal{Q}_{\boldsymbol{\chi}}(0, e_0) + \int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}(s), \boldsymbol{\zeta}^\circ(s)) + \langle \dot{\boldsymbol{\chi}}^\circ(s), \boldsymbol{p}^\circ(s)\rangle \Big)\, ds -$$

$$\langle \boldsymbol{\chi}^\circ(S), \boldsymbol{p}^\circ(S)\rangle + \langle \chi_0, p_0\rangle + \int_{A_S^\circ} \langle \boldsymbol{\sigma}^\circ(s) - \pi_{\mathcal{K}(\boldsymbol{\zeta}^\circ(s))}(\boldsymbol{\sigma}^\circ(s)), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds + \varrho_k^3\,,$$

with

$$\varrho_k^3 := \varrho_k^1 + \varrho_k^2 + R_k + \int_{B_k \cap A_S^\circ} \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds\,, \tag{5.6.31}$$

where B_k is the union of the intervals (s_k^{i-1}, s_k^i) for $i \in I_k^B$. By the definition of I_k^B each point of B_k has distance from B_S° less than the constant η_k introduced in (1.5.2). Since B_S° is compact, we have $\mathcal{L}^1(B_k \cap A_S^\circ) \to 0$. By (5.6.1) this implies that the integral in (5.6.31) tends to 0 as $k \to \infty$. Therefore $\varrho_k^3 \to 0$, and the last chain of inequalities yields (5.6.2). Together with inequality (5.5.1), proved in Section 5.5, this gives (4.3.2) and (4.3.7). By Proposition 4.10, the latter is equivalent to (4.3.1). By Theorem 4.7, this concludes the proof of Theorem 5.6. □

Chapter 6

Viscosity solutions

6.1 Overview of the chapter

In this chapter we consider the behavior of the evolution in terms of the original time variable t. For this purpose, we compose the rescaled viscosity evolution whose existence is provided by Theorem 5.6 with the left-continuous function

$$s^\circ_-(t) := \sup\{s \in [0, +\infty) : t^\circ(s) < t\}\,,$$

which has the property that $t^\circ(s^\circ_-(t)) = t$ for every $t \geq 0$. The composite function obtained in this way is called a *viscosity evolution* (see Definition 6.2). Indeed it has been proved in Lemma 5.9 that the unrescaled viscoplastic approximations considered in the previous chapters converge to this viscosity evolution for every t, except for the countable set of the discontinuity times. We prove that every viscosity evolution satisfies an energy-dissipation balance and an evolution law for the internal variable, that can be expressed in terms of integrals depending only on the original time t (see Theorems 6.7 and 6.14). However, both these integral identities contain terms concentrated on the jump times, whose value can only be determined by looking at the rescaled formulation (see Remarks 6.8 and 6.15). Theorem 6.7 shows in addition that, in the vanishing viscosity limit, the viscous dissipation is concentrated at the discontinuity times.

From a technical point of view, in the proofs we will rely on the notion of "weak*-derivative" for functions of bounded variation with values in the dual of a separable Banach space introduced in Chapter 1, Section 1.4.

6.2 The energy balance in the original time

We start with a simple Remark that will be useful in some proofs of this chapter.

Remark 6.1. Consider two times s_1 and s_2 with the property that the open interval (s_1, s_2) is contained in the set U° defined by (4.2.14), that is the set where t° is locally constant. Therefore, some terms on both sides of the energy-dissipation balance (4.3.1) and of the equivalent (4.3.7) vanish, and the energy-dissipation balance simply reads as

$$\mathcal{Q}(\boldsymbol{e}^\circ(s_2)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_1)) + \int_{s_1}^{s_2} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds + \\ + \int_{s_1}^{s_2} \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds = \langle \boldsymbol{L}(t^\circ(s_2)), \boldsymbol{u}^\circ(s_2) - \boldsymbol{u}^\circ(s_1)\rangle\,, \tag{6.2.1}$$

or equivalently

$$\mathcal{Q}(\boldsymbol{e}^\circ(s_2)) - \mathcal{Q}(\boldsymbol{e}^\circ(s_1)) + \int_{s_1}^{s_2} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds - \langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{p}^\circ(s_2) - \boldsymbol{p}^\circ(s_1)\rangle + \\ + \int_{s_1}^{s_2} \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\, ds = \langle \boldsymbol{\chi}^\circ(s_2), \boldsymbol{e}^\circ(s_2) - \boldsymbol{e}^\circ(s_1)\rangle\,, \tag{6.2.2}$$

for every $0 \le s_1 < s_2 \le S$ such that $(s_1, s_2) \subset U^\circ$.

We now want to consider the behavior of the evolution in terms of the original time t. To this aim we introduce the notion of *viscosity evolution.* In the next definition, given t° as in (4.2.4), the right inverse functions $s^\circ_-(t)$ and $s^\circ_+(t)$ are defined by (5.4.48), and (5.4.49), respectively.

Definition 6.2. Assume that $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfy (2.3.42)-(2.3.48), and let u_0, e_0, p_0, and z_0 be as in (2.3.53)-(2.3.57). We say that $(\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{z})$ is a *viscosity evolution* with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ and initial condition (u_0, e_0, p_0, z_0) if there exists a rescaled viscosity evolution $(\boldsymbol{u}^\circ, \boldsymbol{e}^\circ, \boldsymbol{p}^\circ, \boldsymbol{z}^\circ, t^\circ)$ with the same data and initial condition $(u_0, e_0, p_0, z_0, 0)$ such that

$$\boldsymbol{u}(t) = \boldsymbol{u}^\circ(s^\circ_-(t)), \quad \boldsymbol{e}(t) = \boldsymbol{e}^\circ(s^\circ_-(t)), \quad \boldsymbol{p}(t) = \boldsymbol{p}^\circ(s^\circ_-(t)), \quad \boldsymbol{z}(t) = \boldsymbol{z}^\circ(s^\circ_-(t)) \tag{6.2.3}$$

for every $t \in [0, +\infty)$. Moreover, we define

$$\boldsymbol{\sigma}(t) := \mathbb{C}\boldsymbol{e}(t) = \boldsymbol{\sigma}^\circ(s^\circ_-(t)), \quad \boldsymbol{\zeta}(t) := V(\boldsymbol{z}(t)) = \boldsymbol{\zeta}^\circ(s^\circ_-(t))\,, \tag{6.2.4}$$

where $\boldsymbol{\sigma}^\circ$ and $\boldsymbol{\zeta}^\circ$ are defined by (4.2.5).

The name viscosity evolution is justified by Lemma 5.9. By Definition 4.5, Remark 4.9 and the left-continuity of s°_-, all functions introduced in Definition 6.2 are left-continuous in the norm topology of their target spaces. Since

$$\lim_{h \to 0^+} s^\circ_-(t+h) = s^\circ_+(t) \tag{6.2.5}$$

for every $t \in [0, +\infty)$, the right limits $\boldsymbol{u}(t+)$, $\boldsymbol{e}(t+)$, $\boldsymbol{p}(t+)$, and $\boldsymbol{z}(t+)$ in the corresponding norm topologies satisfy

$$\boldsymbol{u}(t+) = \boldsymbol{u}^\circ(s^\circ_+(t)), \quad \boldsymbol{e}(t+) = \boldsymbol{e}^\circ(s^\circ_+(t)), \quad \boldsymbol{p}(t+) = \boldsymbol{p}^\circ(s^\circ_+(t)), \quad \boldsymbol{z}(t+) = \boldsymbol{z}^\circ(s^\circ_+(t))\,. \tag{6.2.6}$$

Observe that $\boldsymbol{p}$ has bounded variation as a function from $[0, T]$ to $M_b(\Omega \cup \Gamma_0, \mathbb{M}^{N\times N}_{sym})$, as $\boldsymbol{p}^\circ$ is Lipschitzian and s°_- is nondecreasing; similarly, both $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ have bounded variation as functions from $[0, T]$ to $C^0(\bar{\Omega})$. It follows from (6.2.5) that the (at most countable) set S° defined by (5.4.50) is the jump set of the monotone function s°_-. By construction all the functions defined in (6.2.4) are continuous outside S°.

Remark 6.3. Notice that $\boldsymbol{\sigma}$ has bounded variation as a function from $[0,T]$ to the Banach space $L^2(\Omega;\mathbb{M}^{N\times N}_{sym})$. Indeed, for every $0\le t_1<t_2\le T$, recalling that $s^\circ_-(t_1)\in B^\circ$ by (4.2.13), (4.2.7), and (5.4.53), inequality (4.3.21) yields

$$\|\boldsymbol{\sigma}(t_2)-\boldsymbol{\sigma}(t_1)\|_2\le L_T(s^\circ_-(t_2)-s^\circ_-(t_1))\,, \tag{6.2.7}$$

which easily implies the claim as $s^\circ_-(t)$ is nondecreasing.

Given $\boldsymbol{q}\colon[0,T]\to M_b(\Omega\cup\Gamma_0,\mathbb{M}^{N\times N}_{sym})$, for every $0\le a\le b\le T$ the total variation of $\boldsymbol{q}$ on $[a,b]$, denoted by $\mathrm{Var}(\boldsymbol{q};a,b)$, is defined by (1.4.2). Given $\zeta\in C^0(\overline{\Omega})^+$ the variation of $\boldsymbol{q}$ on $[a,b]$ with respect to the functional $\mathcal{H}(\cdot,\zeta)$ introduced in (1.3.20), denoted by $\mathcal{V}(\boldsymbol{q},\zeta;a,b)$, is defined by (1.5.1).

Given a viscosity solution $(\boldsymbol{u},\boldsymbol{e},\boldsymbol{p},\boldsymbol{z})$ we define μ as the unique Radon measure on $[0,T]$ such that

$$\mu([0,t])=\mathrm{Var}(\boldsymbol{p};0,t) \tag{6.2.8}$$

for every $t\in[0,T]$ where $t\mapsto\mathrm{Var}(\boldsymbol{p};0,t)$ is continuous. The continuity properties of $\boldsymbol{p}$ imply that $\mu(\{t\})=0$ for every $t\notin S^\circ$. It follows that the diffuse part μ_d of μ satisfies

$$\mu_d=\mu-\sum_{\tau\in S^\circ}\mu(\{\tau\})\delta_\tau\,, \tag{6.2.9}$$

where δ_τ is the unit mass at τ.

The goal of this section is to derive the precise form of the energy-dissipation balance in the t variable. We start with a change of variable formula.

Lemma 6.4. *Let $(\boldsymbol{u},\boldsymbol{e},\boldsymbol{p},\boldsymbol{z})$ be a viscosity evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$, let $T>0$, and let $S=s^\circ_-(T)$. Let μ and μ_d be as in* (6.2.8) *and* (6.2.9)*, respectively, and define $\boldsymbol{\nu_p}$ as in Theorem 1.3, with $X=M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ and $Y=C^0_0(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$. Then*

$$\int_{(s_1,s_2)\setminus U^\circ}\langle\varphi,\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds=\int_{t^\circ(s_1)}^{t^\circ(s_2)}\langle\varphi,\boldsymbol{\nu_p}(t)\rangle\,d\mu_d(t) \tag{6.2.10}$$

for every $\varphi\in C^0(\overline{\Omega};\mathbb{M}^{N\times N}_{sym})$ and every $0\le s_1<s_2\le S$, where the duality products $\langle\varphi,\dot{\boldsymbol{p}}^\circ(s)\rangle$ and $\langle\varphi,\boldsymbol{\nu_p}(t)\rangle$ are defined as the integrals of the function φ on the set $\Omega\cup\Gamma_0$ with respect to the measures $\dot{\boldsymbol{p}}^\circ(s)$ and $\boldsymbol{\nu_p}(t)$, respectively.

Proof. We first consider the case $\varphi\in C^0_0(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$. For every $0\le t_1\le t_2\le T$, by (1.4.13) we have that

$$\int_{t_1}^{t_2}\langle\varphi,\boldsymbol{\nu_p}(t)\rangle\,d\mu_d(t)=\langle\varphi,\boldsymbol{p}(t_2)-\boldsymbol{p}(t_1)\rangle-\sum_{\tau\in S^\circ\cap[t_1,t_2)}\langle\varphi,\boldsymbol{p}(\tau+)-\boldsymbol{p}(\tau)\rangle\,.$$

By (6.2.3) and the Lipschitz continuity of $\boldsymbol{p}^\circ$, we have

$$\langle\varphi,\boldsymbol{p}(t_2)-\boldsymbol{p}(t_1)\rangle=\int_{s^\circ_-(t_1)}^{s^\circ_-(t_2)}\langle\varphi,\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds\,.$$

Let $\tau\in S^\circ$. When $t_2\to\tau+$, by (6.2.5), (6.2.6) and the previous equality we obtain that

$$\langle\varphi,\boldsymbol{p}(\tau+)-\boldsymbol{p}(\tau)\rangle=\int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)}\langle\varphi,\dot{\boldsymbol{p}}^\circ(s)\rangle\,ds\,.$$

From the last two equalities and (5.4.53) we get

$$\int_{t_1}^{t_2} \langle \varphi, \boldsymbol{\nu}_{\boldsymbol{p}}(t)\rangle \, d\mu_d(t) = \int_{(s^\circ_-(t_1), s^\circ_-(t_2))\setminus U^\circ} \langle \varphi, \dot{\boldsymbol{p}}^\circ(s)\rangle \, ds \,. \tag{6.2.11}$$

Now fix $0 \le s_1 < s_2 \le S$. By (6.2.11) we have

$$\int_{t^\circ(s_1)}^{t^\circ(s_2)} \langle \varphi, \boldsymbol{\nu}_{\boldsymbol{p}}(t)\rangle \, d\mu_d(t) = \int_{(s^\circ_-(t^\circ(s_1)),\, s^\circ_-(t^\circ(s_2)))\setminus U^\circ} \langle \varphi, \dot{\boldsymbol{p}}^\circ(s)\rangle \, ds \,.$$

By (5.4.52) and (5.4.53), if $s^\circ_-(t^\circ(s_1)) < s_1$, then the open interval $(s^\circ_-(t^\circ(s_1)), s_1)$ is contained in the set U°; a similar property holds for s_2. This concludes the proof when $\varphi \in C^0_0(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$.

Let us consider now the case $\varphi \in C^0(\overline{\Omega}; \mathbb{M}^{N\times N}_{sym})$. We fix a sequence $\psi_k \in C^\infty(\bar{\Omega})$, with $0 \le \psi_k \le 1$ in $\bar{\Omega}$ and $\psi_k = 0$ in a neighbourhood of $\partial\Omega \setminus \Gamma_0$, such that $\psi_k(x) \to 1$ for every $x \in \Omega \cup \Gamma_0$. Since $\psi_k \varphi \in C^0_0(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$, formula (6.2.10) holds with φ replaced by $\psi_k\varphi$. The conclusion can be obtained by passing to the limit as $k \to +\infty$ thanks to the Dominated Convergence Theorem. $\square$

In the next two lemmas we prove a change of variable formula for the integral

$$\int_{(0,S)\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) \, ds \,.$$

We begin with the case of a function ζ indepedent of s.

Lemma 6.5. *Under the assumptions of Lemma 6.4, we have*

$$\int_a^b \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \zeta) \, d\mu_d(t) = \int_{(s^\circ_-(a), s^\circ_-(b))\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta) \, ds \tag{6.2.12}$$

for every $\zeta \in C^0(\bar{\Omega})^+$ and every $0 \le a \le b \le T$.

Proof. Let $\mathcal{K}(\zeta)$ be as in (1.3.15), and let $a = t_0 \le t_1 \le \cdots \le t_{N-1} \le t_N = b$ be a subdivision of $[a, b]$. By (1.4.13) and (6.2.11), for every $1 \le i \le N$ and every $\varphi \in \mathcal{K}(\zeta) \cap C^0_0(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$ we have

$$\begin{aligned}
\langle \varphi, \boldsymbol{p}(t_i) - \boldsymbol{p}(t_{i-1})\rangle &= \int_{t_{i-1}}^{t_i} \langle \varphi, \boldsymbol{\nu}_{\boldsymbol{p}}(t)\rangle \, d\mu_d(t) + \sum_{\tau \in S^\circ \cap [t_{i-1}, t_i)} \langle \varphi, \boldsymbol{p}(\tau+) - \boldsymbol{p}(\tau)\rangle = \\
&= \int_{(s^\circ_-(t_{i-1}), s^\circ_-(t_i))\setminus U^\circ} \langle \varphi, \dot{\boldsymbol{p}}^\circ(s)\rangle \, ds + \sum_{\tau \in S^\circ \cap [t_{i-1}, t_i)} \langle \varphi, \boldsymbol{p}(\tau+) - \boldsymbol{p}(\tau)\rangle \le \\
&\le \int_{(s^\circ_-(t_{i-1}), s^\circ_-(t_i))\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta) \, ds + \sum_{\tau \in S^\circ \cap [t_{i-1}, t_i)} \mathcal{H}(\boldsymbol{p}(\tau)^+ - \boldsymbol{p}(\tau), \zeta) \,,
\end{aligned}$$

where in the last inequality we used the definition of $\mathcal{H}$. Taking the supremum with respect to φ, by Theorem 1.2 we get

$$\mathcal{H}(\boldsymbol{p}(t_i) - \boldsymbol{p}(t_{i-1}), \zeta) \le \int_{(s^\circ_-(t_{i-1}), s^\circ_-(t_i))\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta) \, ds + \sum_{\tau \in S^\circ \cap [t_{i-1}, t_i)} \mathcal{H}(\boldsymbol{p}(\tau+) - \boldsymbol{p}(\tau), \zeta)$$

for every $1 \le i \le N$. Summing over i and taking the supremum over all subdivisions, we obtain

$$\mathcal{V}(\boldsymbol{p}, \zeta; a, b) \le \int_{(s^\circ_-(a), s^\circ_-(b))\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds + \sum_{\tau \in S^\circ \cap [a,b)} \mathcal{H}(\boldsymbol{p}(\tau+) - \boldsymbol{p}(\tau), \zeta)\,,$$

where $\mathcal{V}$ is defined by (1.5.1). Thanks to (1.4.14),this inequality is equivalent to

$$\int_a^b \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \zeta)\, d\mu_d(t) \le \int_{(s^\circ_-(a), s^\circ_-(b))\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds\,. \tag{6.2.13}$$

To get the converse inequality, let $s^\circ_-(a) = s_0 \le s_1 \le \cdots \le s_{N-1} \le s_N = s^\circ_-(b)$ be a subdivision of $[s^\circ_-(a), s^\circ_-(b)]$. Using the definition of $\mathcal{H}$, by (6.2.10) we obtain

$$\begin{aligned}
\langle \varphi, \boldsymbol{p}^\circ(s_i) - \boldsymbol{p}^\circ(s_{i-1})\rangle &\le \int_{(s_{i-1},s_i)\setminus U^\circ} \langle \varphi, \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds + \int_{(s_{i-1},s_i)\cap U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds = \\
&= \int_{t^\circ(s_{i-1})}^{t^\circ(s_i)} \langle \varphi, \boldsymbol{\nu}_{\boldsymbol{p}}(t)\rangle\, d\mu_d(t) + \int_{(s_{i-1},s_i)\cap U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds \le \\
&\le \int_{t^\circ(s_{i-1})}^{t^\circ(s_i)} \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \zeta)\, d\mu_d(t) + \int_{(s_{i-1},s_i)\cap U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds
\end{aligned}$$

for every $1 \le i \le N$ and every $\varphi \in \mathcal{K}$. Taking the supremum over φ and using Theorem 1.2 we have

$$\mathcal{H}(\boldsymbol{p}^\circ(s_i) - \boldsymbol{p}^\circ(s_{i-1}), \zeta) \le \int_{t^\circ(s_{i-1})}^{t^\circ(s_i)} \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \zeta)\, d\mu_d(t) + \int_{(s_{i-1},s_i)\cap U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds$$

for every $1 \le i \le N$. By summing over i and taking the supremum over all subdivisions we get thanks to Theorem 1.8

$$\int_{s^\circ_-(a)}^{s^\circ_-(b)} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds \le \int_a^b \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \zeta)\, d\mu_d(t) + \int_{(s^\circ_-(a), s^\circ_-(b))\cap U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds\,;$$

here we also exploited the fact that $t^\circ(s^\circ_-(t)) = t$ for every t (see (5.4.51)). This is clearly the same as saying

$$\int_a^b \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \zeta)\, d\mu_d(t) \ge \int_{(s^\circ_-(a), s^\circ_-(b))\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta)\, ds\,,$$

which concludes the proof. □

We now extend the previous lemma to the time-dependent function $\boldsymbol{\zeta}^\circ$.

Lemma 6.6. *Under the assumptions of Lemma 6.4, let $\boldsymbol{\zeta}$ be defined by* (6.2.4). *Then we have*

$$\int_0^T \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t))\, d\mu_d(t) = \int_{(0,S)\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds\,. \tag{6.2.14}$$

Proof. Since $t \mapsto \boldsymbol{\zeta}(t)$ is left-continuous and has bounded variation there exists a sequence of left continuous piecewise constant functions $\boldsymbol{\zeta}_k(t) := \zeta^0_k + \sum_{i=1}^{i_k} \zeta^i_k\, 1_{(a_{i-1},a_i]}(t)$, with $\zeta^i_k \in C^0(\overline{\Omega})$ for every i and every k such that

$$\|\boldsymbol{\zeta}_k(t) - \boldsymbol{\zeta}(t)\|_\infty \to 0 \tag{6.2.15}$$

uniformly for $t \in [0,T]$ (see, for instance [34, Proposition 4.6]). Define

$$\boldsymbol{\zeta}^\circ_k(s) := \zeta^0_k + \sum_{i=1}^{i_k} \zeta^i_k \, 1_{(s^\circ_-(a_{i-1}), s^\circ_-(a_i)]}(s)\,.$$

By (6.2.12) we easily get

$$\int_0^T \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \boldsymbol{\zeta}_k(t))\, d\mu_d(t) = \int_{(0,S)\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ_k(s))\, ds\,. \tag{6.2.16}$$

By (5.4.52) and (5.4.53) we have that $s^\circ_-(t^\circ(s)) = s$ for $\mathcal{L}^1$-a.e. $s \in [0,S] \setminus U^\circ$, which implies both $\boldsymbol{\zeta}^\circ_k(s) = \boldsymbol{\zeta}_k(t^\circ(s))$ and $\boldsymbol{\zeta}^\circ(s) = \boldsymbol{\zeta}(t^\circ(s))$. Then, by (6.2.15) we conclude that

$$\|\boldsymbol{\zeta}^\circ_k(s) - \boldsymbol{\zeta}^\circ(s)\|_\infty \to 0 \tag{6.2.17}$$

for $\mathcal{L}^1$-a.e. $s \in [0,S] \setminus U^\circ$. Since $\|\dot{\boldsymbol{p}}^\circ(s)\| \leq 1$, and $\|\boldsymbol{\nu}_{\boldsymbol{p}}(t)\|_1 = 1$ by (1.4.12), passing to the limit in (6.2.16) we get the required equality thanks to (6.2.15) and (6.2.17), using the Lipschitz continuity of $H(\xi,\zeta)$ with respect to ζ (see (1.3.14)). □

The next theorem finally gives the precise form of the energy balance in the variable t.

Theorem 6.7. *Let $(\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{z})$ be a viscosity evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfying (2.3.42) and initial condition (u_0, e_0, p_0, z_0) as in (2.3.53)-(2.3.57), let $T > 0$, and define $\boldsymbol{\sigma}$ and $\boldsymbol{\zeta}$ as in (6.2.4). Let S° be defined in (5.4.50), and let λ be the Radon measure on $[0,+\infty)$ defined by*

$$\lambda := \sum_{\tau \in S^\circ} \big(\mathcal{Q}(\boldsymbol{e}(\tau)) - \mathcal{Q}(\boldsymbol{e}(\tau+)) + \langle \boldsymbol{L}(\tau), \boldsymbol{u}(\tau+) - \boldsymbol{u}(\tau)\rangle\big)\,\delta_\tau. \tag{6.2.18}$$

Let μ and μ_d be as in (6.2.8) and (6.2.9), respectively, and define $\boldsymbol{\nu}_{\boldsymbol{p}}$ as in Theorem 1.3, with $X = M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$ and $Y = C^0_0(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$. Then λ is a positive measure and

$$\begin{aligned}
\mathcal{Q}(\boldsymbol{e}(T)) - \mathcal{Q}(e_0) + \int_0^T \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t), \boldsymbol{\zeta}(t))\, d\mu_d(t) + \lambda([0,T)) &= \\
= \int_0^T \langle E\dot{\boldsymbol{w}}(t), \boldsymbol{\sigma}(t)\rangle\, dt - \int_0^T \langle \boldsymbol{L}(t), \dot{\boldsymbol{w}}(t)\rangle\, dt &- \\
- \int_0^T \langle \dot{\boldsymbol{L}}(t), \boldsymbol{u}(t)\rangle\, dt + \langle \boldsymbol{L}(T), \boldsymbol{u}(T)\rangle - \langle \boldsymbol{L}(0), u_0\rangle&\,.
\end{aligned} \tag{6.2.19}$$

Proof. Let $S = s^\circ_-(T)$ and let U° be as in (4.2.14). We start by looking at the term

$$\int_0^S \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\Big)\, ds$$

in the energy-dissipation balance. We split this integral into two parts:

$$I := \int_{(0,S)\setminus U^\circ} \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s))\, ds\,, \tag{6.2.20}$$

$$II := \int_{(0,S)\cap U^\circ} \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \boldsymbol{\zeta}^\circ(s)) + \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\Big)\, ds\,; \tag{6.2.21}$$

indeed by (4.2.13) the term $\|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s), \mathcal{K}(\boldsymbol{\zeta}^\circ(s)))$ gives a contribution only in U°.

If follows from (5.4.53) that

$$II := \sum_{\tau\in S^\circ\cap[0,T)} \int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s)) + \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\Big)\, ds\,.$$

Since $(s^\circ_-(\tau),s^\circ_+(\tau))\subset U^\circ$ for every $\tau\in S^\circ$, (5.4.51) and (6.2.1) give

$$\int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s),\boldsymbol{\zeta}^\circ(s)) + \|\dot{\boldsymbol{p}}^\circ(s)\|_2\, d_2(\boldsymbol{\sigma}^\circ(s),\mathcal{K}(\boldsymbol{\zeta}^\circ(s)))\Big)\, ds =$$

$$= \mathcal{Q}(\boldsymbol{e}^\circ(s^\circ_-(\tau))) - \mathcal{Q}(\boldsymbol{e}^\circ(s^\circ_+(\tau)) + \langle \boldsymbol{L}(\tau), \boldsymbol{u}^\circ(s^\circ_+(\tau)) - \boldsymbol{u}^\circ(s^\circ_-(\tau))\rangle\,. \tag{6.2.22}$$

By definition $\mathcal{Q}(\boldsymbol{e}^\circ(s^\circ_-(\tau))) = \mathcal{Q}(\boldsymbol{e}(\tau))$ and $\boldsymbol{u}^\circ(s^\circ_-(\tau)) = \boldsymbol{u}(\tau)$. On the other hand, (6.2.6) gives $\mathcal{Q}(\boldsymbol{e}(\tau+)) = \mathcal{Q}(\boldsymbol{e}^\circ(s^\circ_+(\tau))$ and $\boldsymbol{u}^\circ(s^\circ_+(\tau)) = \boldsymbol{u}(\tau+)$. Therefore we conclude that

$$II = \sum_{\tau\in S^\circ\cap[0,T)} \Big(\mathcal{Q}(\boldsymbol{e}(\tau)) - \mathcal{Q}(\boldsymbol{e}(\tau+)) - \langle \boldsymbol{L}(\tau), \boldsymbol{u}(\tau+) - \boldsymbol{u}(\tau)\rangle\Big) = \lambda([0,T))\,. \tag{6.2.23}$$

Moreover, since the left-hand side of (6.2.22) is nonnegative, we have that

$$\mathcal{Q}(\boldsymbol{e}(\tau)) - \mathcal{Q}(\boldsymbol{e}(\tau+)) - \langle \boldsymbol{L}(\tau), \boldsymbol{u}(\tau+) - \boldsymbol{u}(\tau)\rangle \geq 0$$

for every $\tau\in S^\circ$, hence λ is a positive measure.

By (6.2.14) and (6.2.20) we have

$$I = \int_0^T \mathcal{H}(\boldsymbol{\nu}_{\boldsymbol{p}}(t),\boldsymbol{\zeta}(t))\, d\mu_d(t)\,. \tag{6.2.24}$$

Arguing as in (5.5.12) we have also

$$\int_0^S \langle E\dot{\boldsymbol{w}}^\circ(s),\boldsymbol{\sigma}^\circ(s)\rangle\, ds = \int_0^T \langle E\dot{\boldsymbol{w}}(t),\boldsymbol{\sigma}(t)\rangle\, dt\,. \tag{6.2.25}$$

With a similar argument we can also prove that

$$\int_0^S \langle \dot{\boldsymbol{L}}(t^\circ(s)),\boldsymbol{u}^\circ(s)\rangle\, \dot{t}^\circ(s)\, ds = \int_0^T \langle \dot{\boldsymbol{L}}(t),\boldsymbol{u}(t)\rangle\, dt\,, \tag{6.2.26}$$

while the equality

$$\int_0^S \langle \boldsymbol{L}(t^\circ(s)),\dot{\boldsymbol{w}}(t^\circ(s))\rangle\, \dot{t}^\circ(s)\, ds = \int_0^T \langle \boldsymbol{L}(t),\dot{\boldsymbol{w}}(t)\rangle\, dt \tag{6.2.27}$$

is simply (1.4.18) with $\boldsymbol{h}(t)$ and $\varphi(s)$ replaced by $\langle \boldsymbol{L}(t),\dot{\boldsymbol{w}}(t)\rangle$ and $t^\circ(s)$, respectively.

Since, by construction, $\mathcal{Q}(\boldsymbol{e}^\circ(S)) = \mathcal{Q}(\boldsymbol{e}(T))$, $\boldsymbol{L}(t^\circ(S)) = \boldsymbol{L}(T)$ and $\boldsymbol{u}^\circ(S) = \boldsymbol{u}(T)$, the required equality follows from (4.3.1), (6.2.23), (6.2.24), (6.2.25), (6.2.26), and (6.2.27) . $\square$

Remark 6.8. The energy-dissipation balance (6.2.19) shows in particular that the viscous dissipation is concentrated at the jump times. Notice that the exact amount of dissipation occurring at these times can be obtained only from the rescaled formulation, using the equality

$$\lambda := \sum_{t\in S^\circ} \big(\mathcal{Q}(\boldsymbol{e}^\circ(s^\circ_-(t))) - \mathcal{Q}(\boldsymbol{e}^\circ(s^\circ_+(t))) + \langle \boldsymbol{L}(t), \boldsymbol{u}(s^\circ_+(t)) - \boldsymbol{u}(s^\circ_-(t))\rangle\big)\, \delta_t\,,$$

which follows from (6.2.6) and (6.2.18).

6.3 The evolution of the internal variable in the original time

In order to write the evolution law for the internal variable in the original time t, we first show that, given an exponent $r > n$, the variation of $t \mapsto \boldsymbol{z}(t) - z_0$ as a function from $[0,T]$ to $W^{1,r}(\Omega)$ is controlled by the variation of $t \mapsto \boldsymbol{p}(t)$ as a function from $[0,T]$ to the Banach space $M_b(\Omega \cup \Gamma_0)$. To this aim we first notice that by (5.4.44) $s \mapsto \boldsymbol{z}^\circ(s) - z_0$ is locally absolutely continuous as a function from $[0,+\infty)$ to $W^{1,r}(\Omega)$ and, by (2.3.41), (4.2.17), (5.4.38), and (5.4.39), given $S > 0$ there exists a positive constant M_S° such that

$$\|\dot{\boldsymbol{z}}^\circ(s)\|_{1,r} \leq M_S^\circ \|\dot{\boldsymbol{p}}^\circ(s)\|_1 \tag{6.3.1}$$

for $\mathcal{L}^1$-a.e. $s \in [0,S]$, where $\|\cdot\|_{1,r}$ denotes the norm in $W^{1,r}(\Omega)$. Moreover (4.2.17) yields, in particular, that $s \mapsto \rho_1 \star \big((\rho_2 \star \mathrm{tr}\boldsymbol{\sigma}^\circ(s))\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\big)$ is locally Bochner integrable as a function from $[0,+\infty)$ to $W^{1,r}(\Omega)$, and consequently to $C^0(\bar{\Omega})$.

As in (5.4.36), we put

$$\boldsymbol{a}^\circ(s) := \rho_2 \star \mathrm{tr}(\boldsymbol{\sigma}^\circ(s))\,.$$

Similarly we set

$$\boldsymbol{a}(t) := \rho_2 \star \mathrm{tr}(\boldsymbol{\sigma}(t)) = \boldsymbol{a}^\circ(s_-^\circ(t))\,, \tag{6.3.2}$$

where the last equality follows from (6.2.4). We start with the following lemma which is a refinement of Lemma 6.4.

Lemma 6.9. *Under the assumptions of Lemma 6.4, let* $\boldsymbol{\varphi}^\circ\colon [0,S] \to C^0(\bar{\Omega};\mathbb{M}^{N\times N}_{sym})$ *be a bounded measurable function such that* $\boldsymbol{\varphi}(t) := \boldsymbol{\varphi}^\circ(s_-^\circ(t))$ *has bounded variation as a function from* $[0,T]$ *to the Banach space* $C^0(\bar{\Omega};\mathbb{M}^{N\times N}_{sym})$. *Then*

$$\int_{(s_1,s_2)\setminus U^\circ} \langle \boldsymbol{\varphi}^\circ(s), \dot{\boldsymbol{p}}^\circ(s)\rangle\, ds = \int_{t^\circ(s_1)}^{t^\circ(s_2)} \langle \boldsymbol{\varphi}(t), \boldsymbol{\nu}_{\boldsymbol{p}}(t)\rangle\, d\mu_d(t) \tag{6.3.3}$$

for every $0 \leq s_1 < s_2 \leq S$.

Proof. If $\boldsymbol{\varphi}^\circ(s)$ does not depend on s, the result is proved in Lemma 6.4. The general case can be obtained by approximating $\boldsymbol{\varphi}(t)$ with piecewise constant functions, arguing as in Lemma 6.6. □

As a consequence we get the following corollary, for every exponent $r > n$.

Corollary 6.10. *Under the assumptions of Lemma 6.4, let* $\boldsymbol{\sigma}$ *be defined as in* (6.2.4), *and let* $\boldsymbol{a}^\circ(s)$ *and* $\boldsymbol{a}(t)$ *be as in* (5.4.36) *and* (6.3.2). *Then* $t \mapsto \rho_1 \star \big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(t))\big)$ *is* μ_d*-Bochner integrable as a function from* $[0,T]$ *to* $W^{1,r}(\Omega)$, *and consequently to* $C^0(\bar{\Omega})$, *and*

$$\int_{(s_1,s_2)\setminus U^\circ} \rho_1 \star \big(\boldsymbol{a}^\circ(s)\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\big)\, ds = \int_{t^\circ(s_1)}^{t^\circ(s_2)} \rho_1 \star \big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(t))\big)\, d\mu_d(t) \tag{6.3.4}$$

for every $0 \leq s_1 < s_2 \leq S$.

Proof. Observe that $\boldsymbol{a}(t)$ has bounded variation as a consequence of Remark 6.3. Fix $\varphi \in C^0(\overline{\Omega})$ and let I be the identity matrix. We have that

$$\langle \varphi, \boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\rangle = \langle \varphi \boldsymbol{a}(t)I, \boldsymbol{\nu_p}(t)\rangle\,, \tag{6.3.5}$$

and similarly

$$\langle \varphi, \boldsymbol{a}^\circ(s)\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\rangle = \langle \varphi \boldsymbol{a}^\circ(s)I, \dot{\boldsymbol{p}}^\circ(s)\rangle\,, \tag{6.3.6}$$

where the duality product is defined as an integral on $\Omega \cup \Gamma_0$ If $\boldsymbol{a}(t)$ does not depend on t, from (6.3.5) and Theorem 1.3, we get that $t \mapsto \langle \varphi, \boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\rangle$ is μ_d-integrable in $[0,T]$ for every $\varphi \in C^0_0(\Omega \cup \Gamma_0)$. The same result holds when $\boldsymbol{a}(t)$ depends on t. This can be proved by approximating $\boldsymbol{a}(t)$ with piecewise constant functions, arguing as in Lemma 6.6. Moreover, arguing as at the end of Lemma 6.4, we can prove that $t \mapsto \langle \varphi, \boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\rangle$ is μ_d-integrable in $[0,T]$ for every $\varphi \in C^0(\overline{\Omega})$.

Let $\check{\rho}_1(x) := \rho_1(-x)$. Since $\langle \varphi, \rho_1 \star \big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\big)\rangle = \langle \check{\rho}_1 \star \varphi, \boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\rangle$, we deduce that $t \mapsto \langle \varphi, \rho_1 \star \big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\big)\rangle$ is μ_d-integrable in $[0,T]$ for every $\varphi \in C^0(\overline{\Omega})$. It follows that $t \mapsto \rho_1 \star \big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\big)$ is μ_d-weakly measurable from $[0,T]$ to $W^{1,r}(\Omega)$. By Pettis' Theorem and by the boundedness of $\boldsymbol{a}(t)$ and $\boldsymbol{\nu_p}(t)$, this implies that $t \mapsto \rho_1 \star \big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\big)$ is μ_d-Bochner integrable from $[0,T]$ to $W^{1,r}(\Omega)$, and consequently to $C^0(\bar{\Omega})$. On the other hand, as it has been observed at the beginning of the section, $s \mapsto \rho_1 \star \big(\boldsymbol{a}^\circ(s)\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\big)$ is $\mathcal{L}^1$-Bochner integrable as a function from $[0,S]$ to $C^0(\bar{\Omega})$.

We now fix $\varphi \in C^0(\overline{\Omega})$ and define $\boldsymbol{\varphi}^\circ(s) := (\check{\rho}_1 \star \varphi)\boldsymbol{a}^\circ(s)I$ and $\boldsymbol{\varphi}(t) := (\check{\rho}_1 \star \varphi)\boldsymbol{a}(t)I$. Then (6.3.5) and (6.3.6) hold with φ replaced by $\check{\rho}_1 \star \varphi$. Therefore Lemma 6.9 gives

$$\int_{(s_1,s_2)\setminus U^\circ} \langle \varphi, \rho_1 \star \big(\boldsymbol{a}^\circ(s)\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\big)\rangle\, ds = \int_{t^\circ(s_1)}^{t^\circ(s_2)} \langle \varphi, \rho_1 \star \big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu_p}(t))\big)\rangle\, d\mu_d(t)$$

for every $0 \le s_1 < s_2 \le S$. By the arbitrariness of φ and standard properties of the Bochner integral, this equality is equivalent to (6.3.4). ☐

We are now in a position to prove the estimate for the variation of $\boldsymbol{z}$. For every $r > n$ and every $0 \le a < b < +\infty$, we define

$$\mathrm{Var}_{1,r}(\boldsymbol{z}; a, b) := \sup\Big\{ \sum_{i=1}^N \|\boldsymbol{z}(t_i) - \boldsymbol{z}(t_{i-1})\|_{1,r} \colon\, a = t_0 \le t_1 \le \cdots \le t_N = b,\, N \in \mathbb{N}\Big\}, \tag{6.3.7}$$

where $\|\cdot\|_{1,r}$ denotes the norm in $W^{1,r}(\Omega)$.

Theorem 6.11. *Let* $(\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{p}, \boldsymbol{z})$ *be a viscosity evolution with data* $\boldsymbol{f}$, $\boldsymbol{g}$, *and* $\boldsymbol{w}$ *satisfying* (2.3.42)-(2.3.48) *and initial condition* (u_0, e_0, p_0, z_0) *as in* (2.3.53)-(2.3.57), *and let* $T > 0$. *Let* μ *and* μ_d *be as in* (6.2.8) *and* (6.2.9), *respectively, and define* $\boldsymbol{\nu_p}$ *as in Theorem 1.3, with* $X = M_b(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$ *and* $Y = C^0_0(\Omega \cup \Gamma_0; \mathbb{M}^{N\times N}_{sym})$. *Assume that the uniform safe-load condition* (2.3.45)-(2.3.48) *holds. Let* $r > n$ *and let* $\mathrm{Var}(\boldsymbol{p}; a, b)$ *be defined as in* (1.4.2). *Then there exists a positive constant* C_T *such that*

$$\mathrm{Var}_{1,r}(\boldsymbol{z}; a, b) \le C_T\, \mathrm{Var}(\boldsymbol{p}; a, b) \tag{6.3.8}$$

for every $0 \le a < b \le T$.

Proof. Let $S = s^\circ_-(T)$, fix $0 \le a \le t_1 < t_2 \le b \le T$, and let $\boldsymbol{a}^\circ(s)$ and $\boldsymbol{a}(t)$ be as in (5.4.36) and (6.3.2). By the definition of $\boldsymbol{z}$, the Lipschitz continuity of $\boldsymbol{z}^\circ$, and (4.2.17), we have

$$\|\boldsymbol{z}(t_2) - \boldsymbol{z}(t_1)\|_{1,r} = \Big\| \int_{s^\circ_-(t_1)}^{s^\circ_-(t_2)} \dot{\boldsymbol{z}}^\circ(s)\,ds \Big\|_{1,r} \le$$
$$\le \Big\| \int_{(s^\circ_-(t_1), s^\circ_-(t_2))\setminus U^\circ} \rho_1 \star \big(\boldsymbol{a}^\circ(s)\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\big)\,ds \Big\|_{1,r} + \Big\| \int_{(s^\circ_-(t_1), s^\circ_-(t_2))\cap U^\circ} \dot{\boldsymbol{z}}^\circ(s)\,ds \Big\|_{1,r} . \tag{6.3.9}$$

By Corollary 6.10, recalling (5.4.51) we get.

$$\int_{(s^\circ_-(t_1), s^\circ_-(t_2))\setminus U^\circ} \rho_1 \star \big(\boldsymbol{a}^\circ(s)\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\big)\,ds = \int_{t_1}^{t_2} \rho_1 \star \big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(t))\big)\,d\mu_d(t) .$$

Since $\boldsymbol{a}(t)$ is uniformly bounded in $C^0(\overline{\Omega})$ and $\|\boldsymbol{\nu}_{\boldsymbol{p}}(t)\|_1 = 1$ for μ_d-a.e. t (see (1.4.5)), by standard properties of the convolution the integrand in the right-hand side is uniformly bounded in $W^{1,r}(\Omega)$, therefore

$$\Big\| \int_{(s^\circ_-(t_1), s^\circ_-(t_2))\setminus U^\circ} \rho_1 \star \big(\boldsymbol{a}^\circ(s)\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\big)\,ds \Big\|_{1,r} \le C_T \mu_d((t_1, t_2)) \tag{6.3.10}$$

for a suitable constant C_T.

To estimate the second integral in the right-hand side of (6.3.9), let $\zeta_m > 0$ be the constant in (2.3.38), and let $\boldsymbol{\chi}(t)$ be a function satisfying the uniform safe-load condition. Fix $\tau \in S^\circ \cap [t_1, t_2)$ (see (5.4.50)). Observe that by Remark 4.4 for $\mathcal{L}^1$-a.e. $s \in [s^\circ_-(\tau), s^\circ_+(\tau))$ the duality product $\langle \boldsymbol{\chi}(\tau), \dot{\boldsymbol{p}}^\circ(s)\rangle$ is correctly defined according to (2.3.12). Moreover, by (4.2.18) and (5.4.51) we have $\boldsymbol{\chi}(\tau) = \boldsymbol{\chi}^\circ(s^\circ_+(\tau))$. By (2.3.36) we get that

$$r_0 \|\dot{\boldsymbol{p}}^\circ(s)\|_1\,ds \le \mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta_m) - \langle \boldsymbol{\chi}(\tau), \dot{\boldsymbol{p}}^\circ(s)\rangle ,$$

where $r_0 > 0$ is as in (2.3.46). With these facts, (2.3.38), (6.2.2), and (4.3.18) give

$$\begin{aligned}
r_0 \int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \|\dot{\boldsymbol{p}}^\circ(s)\|_1\,ds &\le \int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta_m) - \langle \boldsymbol{\chi}(\tau), \dot{\boldsymbol{p}}^\circ(s)\rangle\Big)\,ds \le \\
&\le \int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \Big(\mathcal{H}(\dot{\boldsymbol{p}}^\circ(s), \zeta^\circ(s)) - \langle \boldsymbol{\chi}(\tau), \dot{\boldsymbol{p}}^\circ(s)\rangle\Big)\,ds = \\
&= \int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \mathcal{H}(\boldsymbol{p}^\circ(s), \zeta^\circ(s))\,ds - \langle \boldsymbol{\chi}(\tau), \boldsymbol{p}^\circ(s^\circ_+(\tau)) - \boldsymbol{p}^\circ(s^\circ_-(\tau))\rangle \le
\end{aligned} \tag{6.3.11}$$

$$\le \mathcal{Q}(\boldsymbol{e}^\circ((s^\circ_-(\tau))) - \mathcal{Q}(\boldsymbol{e}^\circ((s^\circ_+(\tau))) + \langle \boldsymbol{\chi}(\tau), \boldsymbol{e}^\circ(s^\circ_+(\tau)) - \boldsymbol{e}^\circ(s^\circ_-(\tau))\rangle =$$

$$= \frac{1}{2}\langle \boldsymbol{\sigma}^\circ(s^\circ_-(\tau)) + \boldsymbol{\sigma}^\circ(s^\circ_+(\tau)) - 2\,\boldsymbol{\chi}(\tau), \boldsymbol{e}^\circ(s^\circ_-(\tau)) - \boldsymbol{e}^\circ(s^\circ_+(\tau))\rangle .$$

By (2.3.45) and (4.2.12), $\frac{1}{2}(\boldsymbol{\sigma}^\circ(s^\circ_-(\tau)) + \boldsymbol{\sigma}^\circ(s^\circ_+(\tau)) - 2\,\boldsymbol{\chi}(\tau) \in \Sigma_0(\Omega)$, where $\Sigma_0(\Omega)$ is defined by (2.3.32). Since $E\boldsymbol{w}^\circ(s) = E\boldsymbol{w}(\tau)$ for every $s \in [s^\circ_-(\tau), s^\circ_+(\tau)]$, Proposition 2.3 yields that

$$\begin{aligned}
&\tfrac{1}{2}\langle \boldsymbol{\sigma}^\circ(s^\circ_-(\tau)) + \boldsymbol{\sigma}^\circ(s^\circ_+(\tau)) - 2\,\boldsymbol{\chi}(\tau), \boldsymbol{e}^\circ(s^\circ_-(\tau)) - \boldsymbol{e}^\circ(s^\circ_+(\tau))\rangle = \\
&= \tfrac{1}{2}\langle \boldsymbol{\sigma}^\circ(s^\circ_-(\tau)) + \boldsymbol{\sigma}^\circ(s^\circ_+(\tau)) - 2\,\boldsymbol{\chi}(\tau), \boldsymbol{p}^\circ(s^\circ_+(\tau)) - \boldsymbol{p}^\circ(s^\circ_-(\tau))\rangle .
\end{aligned} \tag{6.3.12}$$

By (4.2.13), we have that $\boldsymbol{\sigma}^\circ(s^\circ_+(\tau)) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s^\circ_+(\tau)))$ and $\boldsymbol{\sigma}^\circ(s^\circ_-(\tau)) \in \mathcal{K}(\boldsymbol{\zeta}^\circ(s^\circ_-(\tau)))$, therefore, taking into account (2.3.46)

$$\|\boldsymbol{\sigma}^\circ(s^\circ_-(\tau)) + \boldsymbol{\sigma}^\circ(s^\circ_+(\tau)) - 2\,\boldsymbol{\chi}(\tau)\|_\infty \le 2M_K[L_S(\|z_0\|_\infty + S) + \zeta_m]\,, \tag{6.3.13}$$

where M_K is the constant in (1.3.4), ζ_m is given by (2.3.38), and L_S is the Lipschitz constant of $\boldsymbol{\zeta}^\circ$ on $[0,S]$. By (2.3.13), (6.3.11), (6.3.12), and (6.3.13), we conclude that there exist a constant C_T, independent of t, such that

$$\int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \|\dot{\boldsymbol{p}}^\circ(s)\|_1\,ds \le C_T\|\boldsymbol{p}^\circ(s^\circ_+(\tau)) - \boldsymbol{p}^\circ(s^\circ_-(\tau))\|_1 = C_T\|\boldsymbol{p}(\tau+) - \boldsymbol{p}(\tau)\|_1\,, \tag{6.3.14}$$

where the last equality follows from (6.2.6).

Using (5.4.53), (6.3.1), and (6.3.14) we get that, up to redefining the positive constant C_T

$$\begin{aligned}\Big\|\int_{(s^\circ_-(t_1),s^\circ_-(t_2))\cap U^\circ} \dot{\boldsymbol{z}}^\circ(s)\,ds\Big\|_{1,r} &\le \sum_{\tau\in S^\circ\cap[t_1,t_2)} \int_{s^\circ_-(\tau)}^{s^\circ_+(\tau)} \Big\|\dot{\boldsymbol{z}}^\circ(s)\Big\|_{1,r}\,ds \le \\ &\le C_T \sum_{\tau\in S^\circ\cap[t_1,t_2)} \|\boldsymbol{p}(\tau+) - \boldsymbol{p}(\tau)\|_1\,.\end{aligned} \tag{6.3.15}$$

Finally, putting together (6.3.9), (6.3.10), and (6.3.15), from the definitions of μ and μ_d we get

$$\begin{aligned}\|\boldsymbol{z}(t_2) - \boldsymbol{z}(t_1)\|_{1,r} &\le C_T\Big(\mu_d((t_1,t_2)) + \sum_{\tau\in S^\circ\cap[t_1,t_2)} \|\boldsymbol{p}(\tau+) - \boldsymbol{p}(\tau)\|_1\Big) = \\ &= C_T\Big(\mu_d((t_1,t_2)) + \sum_{\tau\in S^\circ\cap[t_1,t_2)} \mu(\{\tau\})\Big) = C_T\mu([t_1,t_2)) = C_T\mathrm{Var}(\boldsymbol{p};t_1,t_2)\,.\end{aligned}$$

From this the conclusion easily follows. □

The proof of the following lemma could be recovered by repeating the arguments of [34, Sections 6 and 12]; for the reader's convenience we give here an independent proof based on the results in Chapter 1, Section 1.4.

Lemma 6.12. *Under the assumptions of Lemma 6.4, there exists a unique Bochner μ-integrable function $\boldsymbol{\nu}_{\boldsymbol{z},\mu}\colon [0,T] \to C^0(\bar\Omega)$ such that*

$$\boldsymbol{z}(b) - \boldsymbol{z}(a) = \int_a^b \boldsymbol{\nu}_{\boldsymbol{z},\mu}(t)\,d\mu(t) \tag{6.3.16}$$

for every $a, b \in [0,T]$ with $a \le b$, such that $\mu(\{a\}) = \mu(\{b\}) = 0$.

Proof. Fix $r > n$, and $a, b \in [0,T]$, with $a \le b$, such that $\mu(\{a\}) = \mu(\{b\}) = 0$. Let λ be the unique Radon measure on $[0,T]$ such that $\lambda([0,t]) = \mathrm{Var}_{1,r}(\boldsymbol{z};0,t)$ for every $t \in [0,T]$ where $t \mapsto \mathrm{Var}_{1,r}(\boldsymbol{z};0,t)$ is continuous. By (6.3.8), λ is absolutely continuous with respect to μ; in particular, we have $\lambda(\{a\}) = \lambda(\{b\}) = 0$. By Theorem 1.3 and the reflexivity of $W^{1,r}(\Omega)$, there exists a unique weakly λ-measurable function $\boldsymbol{\nu}_z\colon [0,T] \to W^{1,r}(\Omega)$ satisfying

$$\langle y, \boldsymbol{z}(b) - \boldsymbol{z}(a)\rangle = \int_a^b \langle y, \boldsymbol{\nu}_{\boldsymbol{z}}(t)\rangle\,d\lambda(t)$$

for every y in the dual of $W^{1,r}(\Omega)$; since the latter is separable, by Pettis' Theorem $\boldsymbol{\nu}_{\boldsymbol{z}}$ is λ-measurable, therefore it is Bochner integrable with respect to λ by (1.4.5). It then follows that

$$\boldsymbol{z}(b)-\boldsymbol{z}(a)=\int_a^b \boldsymbol{\nu}_{\boldsymbol{z}}(t)\,d\lambda(t)\,.$$

By the Sobolev Imbedding Theorem and (1.4.5) $\boldsymbol{\nu}_{\boldsymbol{z}}\in L^\infty_\lambda([0,T];C^0(\bar\Omega))$. By the Radon-Nikodym Theorem, λ has a density $\frac{d\lambda}{d\mu}(t)$ with respect to μ; therefore the conclusion follows with $\boldsymbol{\nu}_{\boldsymbol{z},\mu}(t)=\boldsymbol{\nu}_{\boldsymbol{z}}(t)\frac{d\lambda}{d\mu}(t)$. ☐

We now study the evolution law for the internal variable in the original time t. The first result concerns only the continuity points of $\boldsymbol{z}$.

Proposition 6.13. *Let* $(\boldsymbol{u},\boldsymbol{e},\boldsymbol{p},\boldsymbol{z})$ *be a viscosity evolution with data* $\boldsymbol{f}$, $\boldsymbol{g}$, *and* $\boldsymbol{w}$ *satisfying* (2.3.42)-(2.3.48) *and initial condition* (u_0,e_0,p_0,z_0) *as in* (2.3.53)-(2.3.57), *and let* $T>0$. *Define* $\boldsymbol{\sigma}$ *as in* (6.2.4). *Let* μ *and* μ_d *be as in* (6.2.8) *and* (6.2.9), *respectively. Define* $\boldsymbol{\nu}_{\boldsymbol{p}}$ *as in Theorem 1.3, with* $X=M_b(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$ *and* $Y=C^0_0(\Omega\cup\Gamma_0;\mathbb{M}^{N\times N}_{sym})$, *and let* $\boldsymbol{\nu}_{\boldsymbol{z},\mu}$ *as in Lemma 6.12. Then*

$$\boldsymbol{\nu}_{\boldsymbol{z},\mu}(t)=\rho_1\star\big((\rho_2\star\mathrm{tr}(\boldsymbol{\sigma}(t)))\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(t))\big) \tag{6.3.17}$$

for μ_d*-a.e.* $t\in[0,T]$.

Proof. It clearly suffices to show that the required equality holds for μ-a.e. $t\in[0,T]\setminus S^\circ$. Fix $t\in[0,T]\setminus S^\circ$; this implies that $\mu(\{t\})=0$. We shall additionally require that the following properties are satisfied in t:

$$\lim_{h\to0^+}\frac{\mu([t-h,t+h])}{\mu_d([t-h,t+h])}=1\,, \tag{6.3.18}$$

$$\lim_{h\to0^+}\frac{1}{\mu([t-h,t+h])}\int_{[t-h,t+h]}\boldsymbol{\nu}_{\boldsymbol{z},\mu}(\tau)\,d\mu(\tau)=\boldsymbol{\nu}_{\boldsymbol{z},\mu}(t)\,, \tag{6.3.19}$$

$$\lim_{h\to0^+}\frac{1}{\mu_d([t-h,t+h])}\int_{t-h}^{t+h}\rho_1\star\big(\boldsymbol{a}(\tau)\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(\tau))\big)\,d\mu_d(\tau)=\rho_1\star\big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(t))\big) \tag{6.3.20}$$

This is not restrictive, as the Besicovitch Differentiation Theorem guarantees that all these properties are satisfied for μ-a.e. $t\in[0,T]\setminus S^\circ$. Notice that, by (6.3.15), the Sobolev Imbedding Theorem, and the definitions of μ and μ_d, we have that

$$\begin{aligned}\frac{1}{\mu_d([t-h,t+h])}\Big\|\int_{(s^\circ_-(t-h),s^\circ_-(t+h))\cap U^\circ}\dot{\boldsymbol{z}}^\circ(s)\,ds\Big\|_\infty\le\\ \le C_T\frac{1}{\mu_d([t-h,t+h])}\sum_{\tau\in S^\circ\cap[t-h,t+h)}\|\boldsymbol{p}(\tau+)-\boldsymbol{p}(\tau)\|_1\le\\ \le C_T\frac{\mu([t-h,t+h])-\mu_d([t-h,t+h])}{\mu_d([t-h,t+h])}\to0\end{aligned} \tag{6.3.21}$$

when $h\to0^+$ thanks to (6.3.18). We fix a sequence $h_j\to0^+$ such that $\mu(\{t+h_j\})=\mu(\{t-h_j\})=0$ for every j. Defining $\boldsymbol{a}^\circ(s)$ and $\boldsymbol{a}(t)$ as in (5.4.36) and (6.3.2), and using the definition of $\boldsymbol{z}$, the Lipschitz continuity of $\boldsymbol{z}^\circ$, and the evolution law in the rescaled time s (4.2.17), by (6.3.4), (6.3.16), (6.3.18), (6.3.19), (6.3.20), and (6.3.21) we finally get

$$
\begin{aligned}
\boldsymbol{\nu}_{\boldsymbol{z},\mu}(t) &= \lim_{j\to+\infty} \frac{1}{\mu_d([t-h_j,t+h_j])} \int_{[t-h_j,t+h_j]} \boldsymbol{\nu}_{\boldsymbol{z},\mu}(t)\,d\mu(t) = \\
&= \lim_{j\to+\infty} \frac{1}{\mu_d([t-h_j,t+h_j])} (\boldsymbol{z}(t+h_j)-\boldsymbol{z}(t-h_j)) = \\
&= \lim_{j\to+\infty} \frac{1}{\mu_d([t-h_j,t+h_j])} \int_{s_-^\circ(t-h_j)}^{s_-^\circ(t+h_j)} \dot{\boldsymbol{z}}^\circ(s)\,ds = \\
&= \lim_{j\to+\infty} \frac{1}{\mu_d([t-h_j,t+h_j])} \int_{(s_-^\circ(t-h_j),s_-^\circ(t+h_j))\setminus U^\circ} \dot{\boldsymbol{z}}^\circ(s)\,ds = \\
&= \lim_{j\to+\infty} \frac{1}{\mu_d([t-h_j,t+h_j])} \int_{(s_-^\circ(t-h_j),s_-^\circ(t+h_j))\setminus U^\circ} \rho_1\star\big(\boldsymbol{a}^\circ(s)\mathrm{tr}(\dot{\boldsymbol{p}}^\circ(s))\big)\,ds = \\
&= \lim_{j\to+\infty} \frac{1}{\mu_d([t-h_j,t+h_j])} \int_{t-h_j}^{t+h_j} \rho_1\star\big(\boldsymbol{a}(\tau)\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(\tau))\big)\,d\mu_d(\tau) = \\
&= \rho_1\star\big(\boldsymbol{a}(t)\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(t))\big)\,,
\end{aligned}
$$

as required. □

We are finally in position to prove the evolution law for the internal variable in the original time t.

Theorem 6.14. *Let $(\boldsymbol{u},\boldsymbol{e},\boldsymbol{p},\boldsymbol{z})$ be a viscosity evolution with data $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfying* (2.3.42)-(2.3.48) *and initial condition (u_0,e_0,p_0,z_0) as in* (2.3.53)-(2.3.57)*, and let $T>0$. Define $\boldsymbol{\sigma}$ as in* (6.2.4)*. Let S° be as in* (5.4.50)*, and let λ_z be the locally bounded $C^0(\overline{\Omega})$-valued Radon measure on $[0,+\infty)$ defined by*

$$
\lambda_z := \sum_{t\in S^\circ} (\boldsymbol{z}(t+)-\boldsymbol{z}(t))\delta_t. \tag{6.3.22}
$$

Let μ and μ_d be as in (6.2.8) *and* (6.2.9)*, respectively, and let $\boldsymbol{\nu}_{\boldsymbol{p}}$ be as in Theorem 1.3. Then*

$$
\boldsymbol{z}(T) = z_0 + \int_0^T \rho_1\star\big((\rho_2\star\mathrm{tr}(\boldsymbol{\sigma}(t)))\mathrm{tr}(\boldsymbol{\nu}_{\boldsymbol{p}}(t))\big)\,d\mu_d(t) + \lambda_z([0,T))\,, \tag{6.3.23}
$$

where the integral is a Bochner integral in the space $C^0(\overline{\Omega})$.

Proof. Using the left continuity of $t\mapsto\boldsymbol{z}(t)$ we deduce from (6.3.16) that

$$
\begin{gathered}
\boldsymbol{z}(T)-z_0 = \int_{[0,T)} \boldsymbol{\nu}_{\boldsymbol{z},\mu}(t)\,d\mu(t)\,,\\
\boldsymbol{\nu}_{\boldsymbol{z},\mu}(t)\,\mu(\{t\}) = \boldsymbol{z}(t+)-\boldsymbol{z}(t) \quad \text{for every } t\in[0,+\infty)\,.
\end{gathered}
$$

Therefore, using the definitions of μ_d and λ_z ((6.2.9) and (6.3.22)), we obtain

$$
\boldsymbol{z}(T)-z_0 = \int_0^T \boldsymbol{\nu}_{\boldsymbol{z},\mu}(t)\,d\mu_d(t) + \lambda_z([0,T))\,.
$$

Equality (6.3.23) follows now from (6.3.17). □

Remark 6.15. By (6.3.23) the value of $\boldsymbol{z}(T)$ is uniquely determined by z_0, $t \mapsto \boldsymbol{\sigma}(t)$, $t \mapsto \boldsymbol{p}(t)$, provided we know the behavior of $\boldsymbol{z}$ at its discontinuity points. This can be deduced from the rescaled formulation, using the equality

$$\lambda_z = \sum_{t \in S^\circ} (\boldsymbol{z}^\circ(s^\circ_+(t)) - \boldsymbol{z}^\circ(s^\circ_-(t)))\delta_t \,,$$

which follows from (6.2.6) and (6.3.22).

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