3D-2D ASYMPTOTIC ANALYSIS FOR INHOMOGENEOUS THIN FILMS

ANDREA BRAIDES, IRENE FONSECA, AND GILLES FRANCFORT

Abstract. A dimension reduction analysis is undertaken using Γ-convergence techniques within a relaxation theory for 3D nonlinear elastic thin domains of the form

$$\Omega_\varepsilon := \{(x_1, x_2, x_3) : (x_1, x_2) \in \omega, |x_3| < \varepsilon f_\varepsilon(x_1, x_2)\},$$

where ω is a bounded domain of $\mathbb{R}^2$ and f_ε is an ε-dependent profile. An abstract representation of the effective 2D energy is obtained, and specific characterizations are found for nonhomogeneous plate models, periodic profiles, and within the context of optimal design for thin films.

Keywords : dimension reduction, Γ-convergence, plate models, periodicity, relaxation.
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Contents

1. Introduction

Dimensional reduction through asymptotic analysis is by now a well established theory in a linear setting. Specifically, the work of Ciarlet et al. [?], has paved the way for a variety of studies ranging from linearly elastic plates [?] to various beam models [?], [?], [?], or shells [?], and also spanning various constitutive behaviors [?].

There have, however, been comparatively few studies in a nonlinear setting (other than the semi-linear setting of [?], [?], [?] in the case of rods). To our knowledge, a quasi-exhaustive list can be readily drawn: in [?], fully nonlinear homogeneous elastic plate models are obtained, thereby providing a rigorous mathematical framework for prior work [?]. Note the absence, in that work, of the well thought of requirement that the energy density become infinite as the jacobian of the transformation tends to 0. The only attempt in that direction is to be found in [?]. In [?], fully nonlinear beam models are obtained while in [?] a very thorough investigation of the monotone, albeit not necessarily variational, case is undertaken, again in a 3D-1D setting. Nonlinear shell models are also discussed in [?], [?] in the footstep of [?].

Finally, a general study of Γ-convergence and dimensional reduction is proposed in [?].

As emphasized in [?], thin film technology has drastically improved as of late and a precise control over thickness as well as material composition of a film is possible. This motivated in part the study in [?] of the optimal(ly worst!) design of a two-phase nonlinearly elastic thin film. In a different direction, "optimal" stiffeners for a linearly elastic plate with fixed average thickness are analyzed in [?] under directional restrictions on the stiffeners, and the existence of a KIRCHHOFF-like plate model is established (at least formally) as limit of 3D domains with (locally) periodic profiles, i.e., profiles of the form $\{|x_3| < \varepsilon f(x, x/\varepsilon^\tau)\}$, where ε is the thickness of the domain, $0 < \tau < \infty$ determines the period of the oscillations, and $f(x, .)$ is periodic.

In the present paper, we propose, in the context of fully nonlinear elasticity, a general approach that allows for material heterogeneity as well as rapidly varying profiles. We show in Theorem **??** that membrane-type models of the form firstly derived in [?] are generic; of course, Theorem **??** is a mere abstract existence result and a more precise determination of the membrane energy density in the spirit of [?] is unfeasible with such a degree of generality. We then proceed in the remainder of the paper to specialize the obtained energy density to more specific settings. Section **??** is devoted to revisiting the model obtained in [?] for transversally inhomogeneous thin domains. Section **??** examines a typical homogenization type problem, namely that in which both microstructure and profile periodically oscillate on a scale that is comparable to that of the thickness of the domain. Finally, Section **??** investigates the optimal problem discussed in [?] without the restriction that the mixtures be of cylindrical type, that is allowing for any kind of two-phase mixture, provided of course that the resulting volume fraction of each material be independent of the transverse variable x_3, a must if one is to hope for a plate-like behavior.

It is worthwhile at this point to be somewhat more specific, so as to achieve a better understanding of the scope and limitations of the model. A profiled 3D thin domain $\Omega(\varepsilon)$ is considered; it is of the form

$$\Omega(\varepsilon) := \{(x_1, x_2, x_3) : (x_1, x_2) \in \omega \text{ and } |x_3| < \varepsilon f_\varepsilon(x_1, x_2)\},$$

where ω is a bounded domain of $\mathbb{R}^2$ and $f_\varepsilon(x_1, x_2)$ determines the ε-dependent profile $x_3 = \pm f_\varepsilon(x_1, x_2)$. This domain is filled with an elastic material with elastic energy density $\mathcal{W}(\varepsilon)(x_1, x_2, x_3; \cdot)$. Let us assume, for the sake of illustration, that $\Omega(\varepsilon)$ is clamped on its lateral boundary and subject to body loads $F(\varepsilon)(x_1, x_2, x_3)$, so that, for fixed ε, in order to reach equilibrium the transformation field $u(\varepsilon)(x_1, x_2, x_3)$ seeks to minimize

$$w \mapsto \int_{\Omega(\varepsilon)} \mathcal{W}(\varepsilon)(x_1, x_2, x_3; Dw)\, dx - \int_{\Omega(\varepsilon)} F(\varepsilon) \cdot w\, dx,$$

among all kinematically admissible fields w.

Note that the displacement field $u(\varepsilon)(x) - x$ can be used in lieu of the transformation field $u(\varepsilon)$ at the expense of an obvious change in the expression for the energy density, namely

$$W(\varepsilon)(x_1, x_2, x_3; Dw) := \mathcal{W}(\varepsilon)(x_1, x_2, x_3; I + Dw).$$

We will denote the displacement field by $u(\varepsilon)$ as well and will not refer any further to transformation fields.

It is tempting to reformulate this problem on a "fixed" domain through a $1/\varepsilon$-dilation in the transverse direction x_3. Set

$$\begin{array}{rcl} \Omega & := & \omega \times (-1,1), \\ \Omega_\varepsilon & := & \{(x_1,x_2,x_3) : (x_1,x_2,\varepsilon x_3) \in \Omega(\varepsilon)\}, \\ u_\varepsilon(x_1,x_2,x_3) & := & u(\varepsilon)(x_1,x_2,\varepsilon x_3), \\ W_\varepsilon(x_1,x_2,x_3;\cdot) & := & W(\varepsilon)(x_1,x_2,\varepsilon x_3;\cdot), \\ F_\varepsilon(x_1,x_2,x_3) & := & F(\varepsilon)(x_1,x_2,\varepsilon x_3). \end{array}$$

Equivalently, u_ε wants to minimize

$$v \mapsto \int_{\Omega_\varepsilon} W_\varepsilon\left(x_1,x_2,x_3;D_1v\Big|D_2v\Big|\frac{1}{\varepsilon}D_3v\right)\,dx - \int_{\Omega_\varepsilon} F_\varepsilon\cdot v\,dx$$

among all kinematically admissible fields v on Ω_ε, where $(\xi_1|\xi_2|\xi_3)$, with $\xi_i \in \mathbb{R}^3$, $i=1,2,3$, stands for the 3×3 matrix with columns ξ_1,ξ_2,ξ_3. Under appropriate coercivity assumptions on W_ε (or $W(\varepsilon)$), it is easily checked (cf. Remark **??**) that, for a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$, there exists $u \in W^{1,p}(\omega;\mathbb{R}^3)$ such that $(u_{\varepsilon_k} - u)\chi_{\Omega_{\varepsilon_k}} \to 0$ strongly in $L^p(\mathbb{R}^3)$, where $\chi_{\Omega_{\varepsilon_k}}$ denotes the characteristic function of Ω_{ε_k}, provided that $\{F_\varepsilon\chi_{\Omega_\varepsilon}\}$ (resp. $\{F(\varepsilon)\chi_{\Omega(\varepsilon)}\}$) is bounded in $L^p(\mathbb{R}^3)$, with $p>1$. It then makes sense to investigate the $\Gamma(L^p)$–limit of the functionals

$$v \mapsto E_\varepsilon(v;\omega) := \int_{\Omega_\varepsilon} W_\varepsilon\left(x_1,x_2,x_3;D_1v\Big|D_2v\Big|\frac{1}{\varepsilon}D_3v\right)\,dx$$

since minimizers of E_ε – if they exist – will L^p-converge to minimizers of that $\Gamma(L^p)$–limit, and thus a characterization of the latter will entail an asymptotic effective energy for equilibria states of Ω_ε. This is what the present paper undertakes.

This approach depends on the adopted scaling in a non trivial way. Indeed, a different kind of estimate on the loads – or, as the language of asymptotics would have it, a different scaling on the loads – will render the subsequent analysis obsolete. In particular, note that the usual scaling of linearized elasticity, that is loads such that $\frac{1}{\varepsilon}F(\varepsilon)_3$ is of the same order as $F(\varepsilon)_1, F(\varepsilon)_2$, is not amenable to the proposed setting; a rescaling of $u(\varepsilon)_3$ as

$$(u_\varepsilon)_3(x_1,x_2,x_3) := \varepsilon u(\varepsilon)_3(x_1,x_2,\varepsilon x_3)$$

is that proposed in linearized elasticity (cf. e.g. [**?**]). It can be shown, however, to prohibit local models in the limit [**?**].

We now close this introduction with a few remarks of a mathematical nature. Firstly, it should be noted that there is nothing in the analysis that precludes a higher (or lower) number of horizontal and vertical directions, the setting being then of mappings from $\mathbb{R}^N$ into $\mathbb{R}^d$ with $N,d \in \mathbb{N}$ arbitrary, although the physical meaning becomes dubious. The reader's attention should be drawn to the pervading problem of the explicit appearance of the parameter ε in the functional. This is a source of numerous difficulties and it prompts extreme caution when extracting subsequences (see e.g. the extraction of the subsequence $\{\varepsilon^{\mathcal{R}}\}$ in the proof of Theorem **??**). We also have to appeal to both Γ–limits and Γ–liminfs. Let us recall that if $\{E_n\}$ is a sequence of functions from a Banach space X into $\overline{\mathbb{R}}$ and E is a function from X into $\overline{\mathbb{R}}$, then

Γ1. E is the $\Gamma(X)$–$\liminf$ of E_n if, for any x in X,

$$E(x) = \inf_{\{x_n\}}\{\liminf_{n\to 0^+} E_n(x_n) : x_n \to x \text{ in } X\},$$

Γ2. E is the $\Gamma(X)$–$\limsup$ of E_ε if, for every x in X,

$$E(x) = \inf_{\{x_n\}}\{\limsup_{n\to 0^+} E_n(x_n) : x_n \to x \text{ in } X\}.$$

Also

Γ3. if $\Gamma(X)\text{–}\liminf E_n = \Gamma(X)\text{–}\limsup E_n$ then the common value is called the $\Gamma(X)$-limit of E_n.
Therefore, $E(u) = \Gamma\text{–}\lim E_n(u)$ if and only if
i) whenever $u_n \to u$ in X then

$$E(u) \le \liminf_{n\to+\infty} E_n(u_n),$$

ii) there exists a sequence $\{u_n\}$ such that $u_n \to u$ in X and

$$E(u) = \liminf_{n\to+\infty} E_n(u_n).$$

Moreover, given a family of maps $E_\varepsilon : X \to \overline{\mathbb{R}}$, $\varepsilon > 0$, and if $u \in X$ then we say that

Γ4. $\Gamma(X)\text{–}\lim E_\varepsilon(u) = E(u)$ if $E(u) = \Gamma(X)\text{–}\lim E_{\varepsilon_n}(u)$ for every sequence $\varepsilon_n \to 0^+$.
Hence, it can be shown that $\Gamma(X)\text{–}\lim E_\varepsilon(u) = E(u)$ if and only if
i) for every sequences $\{u_n\}$ and $\{\varepsilon_n\}$ such that $u_n \to u$ in X and $\varepsilon_n \to 0^+$ then

$$E(u) \le \liminf_{n\to+\infty} E_{\varepsilon_n}(u_n),$$

ii) for every sequence $\{\varepsilon_n\}$ converging to 0^+ there exists a sequence $\{u_n\}$ such that $u_n \to u$ in X and

$$E(u) = \lim_{n\to+\infty} E_{\varepsilon_n}(u_n).$$

Finally, we adopt the following notation: Greek letters will always run from 1 to 2 when taken as indices. Thus coordinates will be denoted by x_α, x_3. The notation $(F_\alpha|F_3)$ will refer to the 3×3 matrix with column elements F_1, F_2, F_3 (3 vectors in $\mathbb{R}^3$). We will identify $W^{1,p}(\Omega) \cap \left\{u : \frac{\partial u}{\partial x_3} = 0\right\}$ with $W^{1,p}(\omega)$ ($\Omega := \omega \times (-1,1)$). Also, attention will be paid to the order in which limits are taken. As a last point, $\to$ will always denote strong convergence whereas $\rightharpoonup$ (resp. $\overset{*}{\rightharpoonup}$) will denote weak (resp. weak-*) convergence.

2. A compactness result in a general setting

In all that follows, $\{\varepsilon\}$ is any decreasing sequence of real numbers with limit 0. We assume that $\{W_\varepsilon(x;F)\}_\varepsilon$ is a sequence of Carathéodory functions on $\Omega \times \mathbb{R}^{3\times 3}$ such that, for a.e. x in Ω, and any F in $\mathbb{R}^{3\times 3}$,

$$\beta'|F|^p - \frac{1}{\beta'} \le W_\varepsilon(x;F) \le \beta(1+|F|^p), \quad 0 < \beta' \le \beta < \infty,\ 1 \le p < \infty. \tag{2.1}$$

For each ε let $f_\varepsilon(x_\alpha)$ be a continuous function on ω such that, for some $\gamma > 0$ independent of ε,

$$0 < \gamma \le f_\varepsilon(x_\alpha) \le 1, \quad \text{for all } x_\alpha \in \omega, \tag{2.2}$$

and set, for any open subset A of ω,

$$A_\varepsilon := \{(x_\alpha, x_3) : x_\alpha \in A, |x_3| < f_\varepsilon(x_\alpha)\},$$

and

$$\partial_t A_\varepsilon := \{(x_\alpha, x_3) :\ |x_3| < f_\varepsilon(x_\alpha),\ x_\alpha \in \partial A\}.$$

Note that $\omega_\varepsilon = \Omega_\varepsilon$. Define, for any u in $L^p(\Omega;\mathbb{R}^3)$,

$$E_\varepsilon(v;A) := \begin{cases} \int_{A_\varepsilon} W_\varepsilon\left(x_\alpha, x_3; D_\alpha v \middle| \frac{1}{\varepsilon} D_3 v\right)\, dx_\alpha dx_3, & \text{if } v \in W^{1,p}(A_\varepsilon;\mathbb{R}^3),\\ +\infty, & \text{otherwise,}\end{cases}$$

and, for any u in $L^p(\Omega;\mathbb{R}^3)$,

$$(2.3)\quad J_{\{\varepsilon\}}(u;A) := \inf_{\{v_\varepsilon\}} \Big\{ \liminf_{\varepsilon\to 0^+} E_\varepsilon(v_\varepsilon;A) \ : \ v_\varepsilon \in W^{1,p}(A_\varepsilon;\mathbb{R}^3) \text{ and } (v_\varepsilon - u)\chi_{A_\varepsilon} \to 0 \text{ in } L^p(\Omega;\mathbb{R}^3)\Big\}.$$

Remark 2.1. $J_{\{\varepsilon\}}(u;\cdot)$ is an increasing function on open subsets of ω.

Remark 2.2. If $u \in W^{1,p}(\omega;\mathbb{R}^3)$, i.e., u does not depend upon x_3, then (**??**) implies that $J_{\{\varepsilon\}}(u;\omega) < \infty$, as immediately seen upon inserting $v_\varepsilon = u$ in the definition (**??**) of $J_{\{\varepsilon\}}$.

Remark 2.3. Assume that $p > 1$. We claim that energy bounded sequences are compact in L^p in the sense of (**??**) below, and with limit in $W^{1,p}(\omega;\mathbb{R}^3)$. Indeed, let $\{v_\varepsilon\}$ be a sequence in $W^{1,p}(\Omega_\varepsilon;\mathbb{R}^3)$ with, say, $v_\varepsilon = 0$ on $\partial_t\omega_\varepsilon$ and

$$\sup_\varepsilon \int_{\Omega_\varepsilon} W_\varepsilon\left(x_\alpha, x_3; D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon} D_3 v_\varepsilon\right)\, dx_\alpha dx_3 < \infty$$

Note that f_ε must be such that the trace of v_ε is meaningful on $\partial_t\omega_\varepsilon$.

We must show that there exist u in $W_0^{1,p}(\omega;\mathbb{R}^3)$ and a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ such that

$$(2.4)\qquad (v_{\varepsilon_k} - u)\chi_{\Omega_{\varepsilon_k}} \to 0 \text{ in } L^p(\Omega;\mathbb{R}^3).$$

In view of (**??**), (**??**),

$$(2.5)\qquad \int_{\omega\times(-\gamma,\gamma)} \left(|D_\alpha v_\varepsilon|^p + \frac{1}{\varepsilon^p}|D_3 v_\varepsilon|^p\right) dx_\alpha dx_3 \le \int_{\omega_\varepsilon}\left(|D_\alpha v_\varepsilon|^p + \frac{1}{\varepsilon^p}|D_3 v_\varepsilon|^p\right) dx_\alpha dx_3 < \infty,$$

so that Poincaré's inequality and Rellich's theorem imply the existence of an element u in $W^{1,p}(\omega\times(-\gamma,\gamma);\mathbb{R}^3)$ and of a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ such that

$$(2.6)\qquad \begin{cases} v_{\varepsilon_k} \rightharpoonup u & \text{in } W^{1,p}(\omega\times(-\gamma,\gamma);\mathbb{R}^3),\\ v_{\varepsilon_k}(x_\alpha,\pm\gamma) \to u(x_\alpha,\pm\gamma) & \text{in } L^p(\omega;\mathbb{R}^3).\end{cases}$$

Further, (**??**) implies that $D_3 u = 0$, i.e., that u lies in $W_0^{1,p}(\omega;\mathbb{R}^3)$. Finally,

$$\begin{aligned}
\int_{\omega_\varepsilon} |v_{\varepsilon_k} - u|^p\, dx_\alpha dx_3 &= \int_{\omega\times(-\gamma,\gamma)} |v_{\varepsilon_k} - u|^p\, dx_\alpha dx_3 \\
&\quad + \int_\omega\int_\gamma^{f_\varepsilon(x_\alpha)} |v_{\varepsilon_k} - u|^p\, dx_\alpha dx_3 + \int_\omega\int_{-f_\varepsilon(x_\alpha)}^{-\gamma} |v_{\varepsilon_k} - u|^p\, dx_\alpha dx_3 \\
&= \int_{\omega\times(-\gamma,\gamma)} |v_{\varepsilon_k} - u|^p\, dx_\alpha dx_3 \\
&\quad + \int_\omega\int_\gamma^{f_\varepsilon(x_\alpha)} \left|\int_\gamma^{x_3} D_3 v_{\varepsilon_k}(x_\alpha,s)\,ds + v_{\varepsilon_k}(x_\alpha,\gamma) - u(x_\alpha)\right|^p dx_\alpha dx_3 \\
&\quad + \int_\omega\int_{-f_\varepsilon(x_\alpha)}^{-\gamma} \left|\int_{-\gamma}^{x_3} D_3 v_{\varepsilon_k}(x_\alpha,s)\,ds + v_{\varepsilon_k}(x_\alpha,-\gamma) - u(x_\alpha)\right|^p dx_\alpha dx_3 \\
&\le \int_{\omega\times(-\gamma,\gamma)} |v_{\varepsilon_k} - u|^p\, dx_\alpha dx_3 \\
&\quad + C\left\{\int_\omega |v_{\varepsilon_k}(x_\alpha,\pm\gamma) - u(x_\alpha)|^p\, dx_\alpha + \int_{\omega_\varepsilon} |D_3 v_{\varepsilon_k}|^p\, dx_\alpha dx_3\right\},
\end{aligned}$$

so that (**??**) and (**??**) imply (**??**).

Remark 2.4. Note that given a function $u \in W^{1,p}(\omega;\mathbb{R}^3)$ then $J_{\{\varepsilon\}}(u;\Omega) < \infty$, and, conversely, if $(u_\varepsilon - u)\chi_{\Omega_\varepsilon} \to 0$ in L^1 and $\{u_\varepsilon\}$ is an energy-bounded sequence, then we may assume that $u \in W^{1,p}(\omega;\mathbb{R}^3)$, where we have used the fact that the sequence $\{f_\varepsilon\}$ is uniformly bounded away from zero (see (**??**)).

Introduce a countable collection $\mathcal{C}$ of subsets of ω such that, for any $\delta > 0$ and any open subset A of ω, there exists a finite union C_A of disjoint elements of $\mathcal{C}$ satisfying

$$\begin{cases} \overline{C}_A \subset A, \\ \mathcal{L}^2(A) \le \mathcal{L}^2(C_A) + \delta. \end{cases}$$

Denote by $\mathcal{R}$ the countable collection of all finite unions of elements of $\mathcal{C}$, i.e.,

$$\mathcal{R} := \left\{ \cup_{i=1}^k C_i : \ k \in \mathbb{N},\ C_i \in \mathcal{C} \right\}.$$

A diagonalization argument, together with a simple argument of Γ-convergence based on the separable and metrizable character of $L^p(\Omega;\mathbb{R}^3)$ – see Proposition 7.9 in [**?**] – permits to assert the existence, for any sequence $\{\varepsilon\} \searrow 0^+$, of a subsequence $\{\varepsilon^{\mathcal{R}}\}$ such that, upon setting

$$(2.7)\quad J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) := \inf_{\{v_{\varepsilon^{\mathcal{R}}}\}} \Big\{ \liminf_{\varepsilon^{\mathcal{R}} \to 0^+} E_{\varepsilon^{\mathcal{R}}}(v_{\varepsilon^{\mathcal{R}}};A) : \ v_{\varepsilon^{\mathcal{R}}} \in W^{1,p}(A_{\varepsilon^{\mathcal{R}}};\mathbb{R}^3) \text{ and } (v_{\varepsilon^{\mathcal{R}}} - u)\chi_{A_{\varepsilon^{\mathcal{R}}}} \to 0 \text{ in } L^p(\Omega;\mathbb{R}^3) \Big\},$$

then, for each u in $L^p(\Omega;\mathbb{R}^3)$ and each C in $\mathcal{R}$, there exists a sequence $\{v^C_{\varepsilon^{\mathcal{R}}}\}$ in $W^{1,p}(C_{\varepsilon^{\mathcal{R}}};\mathbb{R}^3)$ such that

$$(2.8)\quad \begin{cases} (v_{\varepsilon^{\mathcal{R}}} - u)\chi_{\varepsilon^{\mathcal{R}}} \to 0 \text{ in } L^p(\Omega;\mathbb{R}^3) \\ J_{\{\varepsilon^{\mathcal{R}}\}}(u;C) = \lim\limits_{\varepsilon^{\mathcal{R}} \to 0+} E_{\varepsilon^{\mathcal{R}}}(v_{\varepsilon^{\mathcal{R}}};C). \end{cases}$$

In other words, $J_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;C)$ is the $\Gamma(L^p)$–limit of $E_{\varepsilon^{\mathcal{R}}}(\cdot;C)$, for every $C \in \mathcal{R}$. We then prove the following

Theorem 2.5. *For any $u \in W^{1,p}(\omega;\mathbb{R}^3)$, any open subset A of ω, and any decreasing sequence $\{\varepsilon\} \searrow 0^+$, $J_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;A)$ defined in (**??**) is the $\Gamma(L^p)$–limit of $E_{\varepsilon^{\mathcal{R}}}(\cdot;A)$.*

Furthermore, there exists a Carathéodory function $W_{\{\varepsilon^{\mathcal{R}}\}} : \mathbb{R}^2 \times \mathbb{R}^{3\times 2} \to \mathbb{R}$ such that

$$(2.9)\quad J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = 2\int_A W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha; D_\alpha u)\, dx_\alpha.$$

Proof. We extend to the present framework the so-called direct methods of the theory of Γ-convergence (see [**?**] Part II).

The proof is divided into four steps. A first step is devoted to a lemma which will be used in the sequel. The second step establishes the claim that $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A)$ is the $\Gamma(L^p)$–limit of $E_{\varepsilon^{\mathcal{R}}}(u;A)$. The third step ensures that $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ is a finite nonnegative Radon measure. The fourth, and final step, is a mere application of a result in [**?**] (see Theorem 4.3.2) ensuring the integral representation (**??**).

Step 1. In this step we observe that approximating sequences may as well take the value u on the lateral boundary of A_ε.

Lemma 2.6. *Let $u \in W^{1,p}(A;\mathbb{R}^3)$ where A is an open subset of ω. If $\{\overline{\varepsilon}\} \subset \{\varepsilon^{\mathcal{R}}\}$ and $v_{\overline{\varepsilon}} \in W^{1,p}(A_{\overline{\varepsilon}};\mathbb{R}^3)$ are such that*

$$\begin{cases} (v_{\overline{\varepsilon}} - u)\chi_{A_{\overline{\varepsilon}}} \to 0 \quad \text{in} \quad L^p(\Omega;\mathbb{R}^3), \\ J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = \lim\limits_{\overline{\varepsilon} \to 0^+} E_{\overline{\varepsilon}}(v_{\overline{\varepsilon}};A), \end{cases}$$

then there exists a sequence $\{w_{\overline{\varepsilon}}\} \subset W^{1,p}(A_{\overline{\varepsilon}};\mathbb{R}^3)$ *which satisfies (??) and is such that*

$$w_{\overline{\varepsilon}} = u \text{ in } \{(x_\alpha, x_3) : \; x_\alpha \in A \setminus K^{\overline{\varepsilon}} \text{ and } |x_3| < f_{\overline{\varepsilon}}(x_\alpha)\}$$

for some compact set $K^{\overline{\varepsilon}} \subset A$.

Proof. The proof relies on De Giorgi's slicing argument, and on the possibility of considering cut-off functions which are independent of the variable x_3. Set

$$C := \sup_{\overline{\varepsilon}} \int_{A_{\overline{\varepsilon}}} \left(1 + |D_\alpha v_{\overline{\varepsilon}}|^p + \frac{1}{\overline{\varepsilon}^p} |D_3 v_{\overline{\varepsilon}}|^p \right) \; dx_\alpha dx_3,$$

and note that $C < \infty$ by virtue of (??). Define

$$K(\overline{\varepsilon}) := \left\| \left[\frac{1}{||v_{\overline{\varepsilon}} - u||^{1/2}_{L^p(A_{\overline{\varepsilon}})}} \right] \right\|, \tag{2.10}$$

where $[\![a]\!]$ stands for the integer part of the number a, and $M(\overline{\varepsilon}) := [\![\sqrt{K(\overline{\varepsilon})}]\!]$; set also

$$A(\overline{\varepsilon}) := \left\{ x_\alpha \in A : \; \mathrm{dist}(x_\alpha, \partial A) < \frac{M(\overline{\varepsilon})}{K(\overline{\varepsilon})} \right\}.$$

Note that, in view of (??), $K(\overline{\varepsilon}) \nearrow \infty$ while $\mathcal{L}^2(A(\overline{\varepsilon})) \searrow 0^+$ as $\overline{\varepsilon} \searrow 0^+$. Subdivide $A(\overline{\varepsilon})$ into $M(\overline{\varepsilon})$ disjoint subsets,

$$A_i^{\overline{\varepsilon}} := \left\{ x_\alpha \in A : \; \mathrm{dist}(x_\alpha, \partial A) \in \left[\frac{i}{K(\overline{\varepsilon})}, \frac{i+1}{K(\overline{\varepsilon})} \right) \right\}, \quad i = 0, \ldots, M(\overline{\varepsilon}) - 1.$$

Then, there exists $i(\overline{\varepsilon}) \in \{0, ..., M(\overline{\varepsilon}) - 1\}$ such that

$$\int_{(A_i^{\overline{\varepsilon}})_{\overline{\varepsilon}}} \left(1 + |D_\alpha v_{\overline{\varepsilon}}|^p + \frac{1}{\overline{\varepsilon}^p} |D_3 v_{\overline{\varepsilon}}|^p \right) \; dx_\alpha dx_3 \leq \frac{C}{M(\overline{\varepsilon})}, \tag{2.11}$$

where $(A^{\overline{\varepsilon}}_{i(\overline{\varepsilon})})_{\overline{\varepsilon}} := \{(x_\alpha, x_3) : x_\alpha \in A_i^{\overline{\varepsilon}}, \; |x_3| < f_{\overline{\varepsilon}}(x_\alpha)\}$. Consider $\phi(\overline{\varepsilon}) \in C_0^\infty(A)$ such that

$$\begin{cases} 0 \leq \phi(\overline{\varepsilon}) \leq 1, \quad ||D_\alpha \phi(\overline{\varepsilon})||_{L^\infty} \leq 2K(\overline{\varepsilon}), \\ \phi(\overline{\varepsilon}) = \begin{cases} 1, & \text{if } \mathrm{dist}(x_\alpha, \partial A) > \frac{i(\overline{\varepsilon})+1}{K(\overline{\varepsilon})}, \\ \phi(\overline{\varepsilon}) = 0, & \text{if } \mathrm{dist}(x_\alpha, \partial A) \leq \frac{i(\overline{\varepsilon})}{K(\overline{\varepsilon})}, \end{cases} \end{cases} \tag{2.12}$$

and set

$$w_{\overline{\varepsilon}} := \phi(\overline{\varepsilon}) v_{\overline{\varepsilon}} + (1 - \phi(\overline{\varepsilon}))u. \tag{2.13}$$

Note that $w_{\overline{\varepsilon}} = u$ in $\{(x_\alpha, x_3) : x_\alpha \in A \setminus K^{\overline{\varepsilon}}, |x_3| < f_{\overline{\varepsilon}}(x_\alpha)\}$, with $K^{\overline{\varepsilon}} := \left\{ x_\alpha \in A : \mathrm{dist}(x_\alpha, \partial A) \geq \frac{i(\overline{\varepsilon})}{K(\overline{\varepsilon})} \right\}$, and that $w_{\overline{\varepsilon}} \in W^{1,p}(A_{\overline{\varepsilon}};\mathbb{R}^3)$. Furthermore, in view of (??),

$$(w_{\overline{\varepsilon}} - u)\chi_{A_{\overline{\varepsilon}}} \to 0 \text{ in } L^p(\Omega;\mathbb{R}^3). \tag{2.14}$$

Then, by virtue of the bound from above in (**??**), together with (**??**), (**??**),

(2.15)
$$
\begin{aligned}
&J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) \geq \\
&\limsup_{\overline{\varepsilon}\to 0^+} \int_{A_{\overline{\varepsilon}}\cap\left[\left\{x_\alpha:\ \mathrm{dist}(x_\alpha,\partial A)>\frac{i(\overline{\varepsilon})+1}{K(\overline{\varepsilon})}\right\}\times(-1,1)\right]} W_{\overline{\varepsilon}}\left(x_\alpha,x_3;D_\alpha v_{\overline{\varepsilon}}\Big|\frac{1}{\overline{\varepsilon}}D_3 v_{\overline{\varepsilon}}\right)\,dx_\alpha dx_3 \\
&\geq \limsup_{\overline{\varepsilon}\to 0^+}\left\{\int_{A_{\overline{\varepsilon}}} W_{\overline{\varepsilon}}\left(x_\alpha,x_3;D_\alpha w_{\overline{\varepsilon}}\Big|\frac{1}{\overline{\varepsilon}}D_3 w_{\overline{\varepsilon}}\right)\,dx_\alpha dx_3\right. \\
&-C\int_{A_{\overline{\varepsilon}}\cap\left[\left\{x_\alpha:\ \mathrm{dist}(x_\alpha,\partial A)<\frac{i(\overline{\varepsilon})}{K(\overline{\varepsilon})}\right\}\times(-1,1)\right]}(1+|D_\alpha u|^p)\,dx_\alpha dx_3 \\
&-\beta\int_{(A^{\overline{\varepsilon}}_{i(\overline{\varepsilon})})_{\overline{\varepsilon}}}\left(1+|D_\alpha v_{\overline{\varepsilon}}|^p+\frac{1}{\overline{\varepsilon}^p}|D_3 v_{\overline{\varepsilon}}|^p\right)\,dx_\alpha dx_3 \\
&\left.-C|K(\overline{\varepsilon})|^p\int_{(A^{\overline{\varepsilon}}_{i(\overline{\varepsilon})})_{\overline{\varepsilon}}}|v_{\overline{\varepsilon}}-u|^p\,dx_\alpha dx_3\right\} \\
&\geq \limsup_{\overline{\varepsilon}\to 0^+} E_{\overline{\varepsilon}}(w_{\overline{\varepsilon}};A) - C\liminf_{\overline{\varepsilon}\to 0^+}\mathcal{L}^2(A(\overline{\varepsilon})) - C\beta\liminf_{\overline{\varepsilon}\to 0^+}\frac{1}{M(\overline{\varepsilon})} \\
&-\beta\liminf_{\overline{\varepsilon}\to 0^+}||v_{\overline{\varepsilon}}-u||^{p/2}_{L^p(A_{\overline{\varepsilon}})} = \limsup_{\overline{\varepsilon}\to 0^+} E_{\overline{\varepsilon}}(w_{\overline{\varepsilon}};A),
\end{aligned}
$$

where (**??**) and (**??**) have been used in deriving the last inequality in (**??**). But the very definition (**??**) of $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A)$, together with (**??**), imply that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) \leq \liminf_{\overline{\varepsilon}\to 0^+} E_{\overline{\varepsilon}}(w_{\overline{\varepsilon}};A),$$

which, in view of (**??**), yields the desired result. □

Step 2. Let $u \in W^{1,p}(A;\mathbb{R}^3)$, let A be an open subset of ω. In order to prove that $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A)$ is the Γ–limit of $E_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}};A)$ it suffices, in view of its definition (**??**), to prove that property Γ3ii) (see the Introduction) holds. To this end, fix $\delta > 0$ and choose a subset C^δ of A in $\mathcal{R}$ such that

$$\begin{cases} \overline{C^\delta} \subset A, \\ \displaystyle\int_{A\setminus C^\delta}(1+|Du_\alpha|)\,dx_\alpha < \frac{\delta}{2\beta}. \end{cases}$$

Consider a sequence $v^{C^\delta}_{\varepsilon^{\mathcal{R}}}$ satisfying

$$\lim_{\varepsilon^{\mathcal{R}}\to 0^+} E_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}};C^\delta) = J_{\{\varepsilon^{\mathcal{R}}\}}(u;C^\delta).$$

In view of Lemma **??**, we may extend $v^{C^\delta}_{\varepsilon^{\mathcal{R}}}$ as u outside $C^\delta_{\varepsilon^{\mathcal{R}}}$ so as to belong to $W^{1,p}(A_{\varepsilon^{\mathcal{R}}};\mathbb{R}^3)$, and since $J_{\{\varepsilon^{\mathcal{R}}\}}(u;C^\delta) \leq J_{\{\varepsilon^{\mathcal{R}}\}}(u;A)$ for all $\delta > 0$, we have

$$
\begin{aligned}
&\limsup_{\delta\to 0^+}\limsup_{\varepsilon^{\mathcal{R}}\to 0^+} E_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}};A) \\
&\quad\leq \limsup_{\delta\to 0^+}\lim_{\varepsilon^{\mathcal{R}}\to 0^+}\left\{E_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}};C^\delta)+2\beta\int_{A\setminus C^\delta}(1+|D_\alpha u|^p)\,dx_\alpha dx_3\right\} \\
&\quad= \limsup_{\delta\to 0^+} J_{\{\varepsilon^{\mathcal{R}}\}}(u;C^\delta) \\
&\quad\leq J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) \\
&\quad\leq \liminf_{\delta\to 0^+}\liminf_{\varepsilon^{\mathcal{R}}\to 0^+} E_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}};A).
\end{aligned}
$$

Lemma **??** in the Appendix permits to conclude the existence of a decreasing sequence $\{\delta(\varepsilon^{\mathcal{R}})\} \searrow 0^+$ such that

$$\begin{cases} (v_{\varepsilon^{\mathcal{R}}}^{C^{\delta(\varepsilon^{\mathcal{R}})}} - u)\chi_{A_{\varepsilon^{\mathcal{R}}}} \to 0 \text{ in } L^p(\Omega;\mathbb{R}^3) \\ J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = \lim\limits_{\varepsilon^{\mathcal{R}}\to 0^+} E_{\varepsilon^{\mathcal{R}}}\left(v_{\varepsilon^{\mathcal{R}}}^{C^{\delta(\varepsilon^{\mathcal{R}})}};A\right), \end{cases}$$

which, together with (**??**), asserts that $J_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;A)$ is the $\Gamma(L^p)$–limit of $E_{\varepsilon^{\mathcal{R}}}(\cdot;A)$.

Step 3. Let u be an element of $W^{1,p}(\omega;\mathbb{R}^3)$. Implicit in the proof of Step 2 above is the inner regularity of $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A)$, namely, for any $\delta > 0$ there exists $C^\delta \in \mathcal{R}$ such that

$$\begin{cases} \overline{C}^\delta \subset A, \\ J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) \le J_{\{\varepsilon^{\mathcal{R}}\}}(u;C^\delta) + \delta. \end{cases} \tag{2.16}$$

Remark 2.7. Note that (**??**), together with the trivial inequality

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) \ge J_{\{\varepsilon^{\mathcal{R}}\}}(u;A\setminus\overline{C}^\delta) + J_{\{\varepsilon^{\mathcal{R}}\}}(u;C^\delta),$$

immediately implies that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A\setminus\overline{C}^\delta) \le \delta. \tag{2.17}$$

Remark also that (**??**) is obtained simply upon choosing u as a test function in the definition (**??**) of $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A\setminus\overline{C}^\delta)$.

We now show that $J_{\{\varepsilon^{\mathcal{R}}\}}$ is subadditive, that is that for every open subsets C, B, A of ω with $C \subset\subset B \subset A$,

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) \le J_{\{\varepsilon^{\mathcal{R}}\}}(u;B) + J_{\{\varepsilon^{\mathcal{R}}\}}(u;A\setminus\overline{C}). \tag{2.18}$$

To this effect we consider, for any small enough $\delta > 0$, B^δ, D^δ two elements of $\mathcal{R}$ with $B^\delta \subset B, D^\delta \subset A\setminus\overline{C}$, such that

$$\int_{A\setminus(B^\delta\cup D^\delta)} (1+|D_\alpha u|)^p \, dx_\alpha < \delta. \tag{2.19}$$

Note that a small enough δ ensures that $B^\delta \cap D^\delta \neq \emptyset$. Then, there exist two sequences $\left\{v_{\varepsilon^{\mathcal{R}}}^{B^\delta}\right\}, \left\{v_{\varepsilon^{\mathcal{R}}}^{D^\delta}\right\}$ such that (**??**) is satisfied for B^δ and D^δ, respectively, and (see Lemma **??**)

$$v_{\varepsilon^{\mathcal{R}}}^{B^\delta} = u \text{ on } \partial_t B^\delta_{\varepsilon^{\mathcal{R}}}, \quad v_{\varepsilon^{\mathcal{R}}}^{D^\delta} = u \text{ on } \partial_t D^\delta_{\varepsilon^{\mathcal{R}}}.$$

Consider the sequence of Radon measures

$$\lambda_{\varepsilon^{\mathcal{R}}} := \left\{1 + \left|D_\alpha v_{\varepsilon^{\mathcal{R}}}^{B^\delta}\right|^p + \left|D_\alpha v_{\varepsilon^{\mathcal{R}}}^{D^\delta}\right|^p \right. \\ \left. + \left(\frac{1}{\varepsilon^{\mathcal{R}}}\right)^p \left(\left|D_3 v_{\varepsilon^{\mathcal{R}}}^{B^\delta}\right|^p + \left|D_3 v_{\varepsilon^{\mathcal{R}}}^{D^\delta}\right|^p\right)\right\}\chi_{(B^\delta\cap D^\delta)_{\varepsilon^{\mathcal{R}}}}\,\mathcal{L}^3$$

where, as usual, $(B^\delta\cap D^\delta)_{\varepsilon^{\mathcal{R}}} = \{x_\alpha, x_3; x_\alpha \in B^\delta\cap D^\delta, |x_3| < f_{\varepsilon^{\mathcal{R}}}(x_\alpha)\}$. By virtue of the coercivity hypothesis in (**??**), $\{\lambda_{\varepsilon^{\mathcal{R}}}\}$ is a bounded sequence of finite nonnegative Radon measures on $\mathbb{R}^3$, hence there exists a finite nonnegative Radon measure λ such that a subsequence of $\{\lambda_{\varepsilon^{\mathcal{R}}}\}$ – denoted by $\{\lambda_{\overline{\varepsilon}}\}$ – satisfies

$$\lambda_{\overline{\varepsilon}} \overset{*}{\rightharpoonup} \lambda \text{ weakly-* in the sense of measures.} \tag{2.20}$$

Set $\hat{\lambda}(X) := \lambda(X\times[-1,1])$ for any Borel subset X of ω. Define, for $0 < \eta < 1$,

$$S^\delta_\eta := \{x \in B^\delta\cap D^\delta : \operatorname{dist}(x_\alpha, \partial B^\delta) = \eta\}.$$

The family $\{S^\delta_\eta\}_\eta$ is made up of pairwise disjoint elements, thus there exists $\eta_0 \in (0,1)$ such that

$$\hat\lambda(S^\delta_{\eta_0}) = 0. \tag{2.21}$$

For L^δ_ζ, a layer of thickness ζ around $S^\delta_{\eta_0}$, i.e.,

$$L^\delta_\zeta := \{x_\alpha \in B^\delta \cap D^\delta : \operatorname{dist}(x_\alpha, S^\delta_{\eta_0}) \le \zeta\},$$

consider a smooth cut-off function $\phi^\delta_\zeta \in C_0^\infty(\mathbb{R}^2)$ such that

$$\begin{cases} ||\phi^\delta_\zeta||_{L^\infty} \le 1, \quad ||D_\alpha \phi^\delta_\zeta||_{L^\infty} \le C/\zeta, \\ \phi^\delta_\zeta = \begin{cases} 0 & \text{if } x_\alpha \in B^\delta \text{ and } \operatorname{dist}(x_\alpha, \partial B^\delta) \ge \eta_0 + \zeta, \\ 1 & \text{if } x_\alpha \notin B^\delta \text{ or } \operatorname{dist}(x_\alpha, \partial B^\delta) \le \eta_0 - \zeta. \end{cases} \end{cases} \tag{2.22}$$

Setting

$$v_{\zeta,\overline{\varepsilon}} := \phi^\delta_\zeta v^{D^\delta}_{\overline{\varepsilon}} + (1-\phi^\delta_\zeta) v^{B^\delta}_{\overline{\varepsilon}} + \chi_{(A\setminus(B^\delta\cup D^\delta))_{\overline{\varepsilon}}} u,$$

then $v^\delta_{\zeta,\overline{\varepsilon}} \in W^{1,p}(A_{\overline{\varepsilon}};\mathbb{R}^3)$, and

$$(v_{\zeta,\overline{\varepsilon}} - u)\chi_{A_{\overline{\varepsilon}}} \to 0 \text{ in } L^p(\Omega;\mathbb{R}^3).$$

Thus, by the very definition (**??**) and in view of (**??**),

$$\begin{aligned} J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) &\le \liminf_{\overline{\varepsilon}\to 0^+} E_{\overline{\varepsilon}}(v_{\zeta,\overline{\varepsilon}};A) \\ &\le J_{\{\varepsilon^{\mathcal{R}}\}}(u;B^\delta) + J_{\{\varepsilon^{\mathcal{R}}\}}(u;D^\delta) \\ &+2\beta \int_{A\setminus(B^\delta\cup D^\delta)} (1+|D_\alpha u|)\,dx_\alpha \\ &+\beta\left\{\limsup_{\overline{\varepsilon}\to 0^+} \lambda_{\overline{\varepsilon}}(L^\delta_\zeta \times (-1,1)) + \frac{C}{\zeta^p}\limsup_{\overline{\varepsilon}\to 0^+} \int_{(L^\delta_\zeta)_{\overline{\varepsilon}}} \left| v^{B^\delta}_{\overline{\varepsilon}} - v^{D^\delta}_{\overline{\varepsilon}}\right|^p \, dx_\alpha dx_3\right\}, \end{aligned} \tag{2.23}$$

where $(L^\delta_\zeta)_{\overline{\varepsilon}} := \{(x_\alpha, x_3) : x_\alpha \in L^\delta_\zeta \text{ and } |x_3| < f_{\overline{\varepsilon}}(x_\alpha)\}$. Since $\left|v^{B^\delta}_{\overline{\varepsilon}} - v^{D^\delta}_{\overline{\varepsilon}}\right| \chi_{(L^\delta_\zeta)_{\overline{\varepsilon}}} \le \left|v^{B^\delta}_{\overline{\varepsilon}} - u\right| \chi_{B^\delta{}_{\overline{\varepsilon}}} + \left|v^{D^\delta}_{\overline{\varepsilon}} - u\right| \chi_{D^\delta{}_{\overline{\varepsilon}}}$, the last term in the last expression in (**??**) is 0, while, by virtue of (**??**),

$$\limsup_{\overline{\varepsilon}\to 0^+} \lambda_{\overline{\varepsilon}}(L_\zeta \times (-1,1)) \le \hat\lambda(\overline{L}_\zeta).$$

But, as ζ tends to 0, $\hat\lambda(\overline{L^\delta_\zeta})$ goes to $\hat\lambda(S^\delta_{\eta_0}) = 0$ (cf. (**??**)), therefore, upon letting ζ tend to 0 in (**??**), and by (**??**), we obtain

$$\begin{aligned} J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) &\le \liminf_{\delta\to 0^+} \left[J_{\{\varepsilon^{\mathcal{R}}\}}(u;B^\delta) + J_{\{\varepsilon^{\mathcal{R}}\}}(u;D^\delta) + \beta\delta\right] \\ &\le J_{\{\varepsilon^{\mathcal{R}}\}}(u;B) + J_{\{\varepsilon^{\mathcal{R}}\}}(u;A\setminus\overline{C}), \end{aligned}$$

and this proves (**??**).

Finally, the definition (**??**) of $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\omega)$ implies the existence of a subsequence $\{\overline{\varepsilon}\}$ of $\{\varepsilon^{\mathcal{R}}\}$ and of an associated subsequence $\{v_{\overline{\varepsilon}}\}$ in $W^{1,p}(\omega_{\overline{\varepsilon}};\mathbb{R}^3)$ such that

$$\begin{cases} (v_{\overline{\varepsilon}} - u)\chi_{\omega_{\overline{\varepsilon}}} \to 0 \text{ in } L^p(\Omega;\mathbb{R}^3), \\ J_{\{\varepsilon^{\mathcal{R}}\}}(u;\omega) = \lim_{\overline{\varepsilon}\to 0^+} E_{\overline{\varepsilon}}(v_{\overline{\varepsilon}};\omega). \end{cases} \tag{2.24}$$

For a well chosen subsequence of $\{\overline{\varepsilon}\}$, still denoted by $\{\overline{\varepsilon}\}$, there exists a finite Radon measure μ such that

$$W_{\overline{\varepsilon}}(x_\alpha, x_3; D_\alpha v_{\overline{\varepsilon}} | 1/\overline{\varepsilon} D_3 v_{\overline{\varepsilon}})\chi_{\omega_{\overline{\varepsilon}}}\mathcal{L}^3 \overset{*}{\rightharpoonup} \mu \text{ weakly-* in the sense of measures .} \tag{2.25}$$

Set, for any Borel subset $X \subset \mathbb{R}^3$, $\hat{\mu}(X) := \mu(X \times [-1,1])$. Then, by virtue of (**??**), (**??**),

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;\omega) \geq \hat{\mu}(\mathbb{R}^2), \tag{2.26}$$

while, clearly for all open subsets $A \subset \omega$,

$$\begin{aligned} J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) &\leq \liminf_{\overline{\varepsilon}\to 0^+} E_{\overline{\varepsilon}}(v_{\overline{\varepsilon}};A) \\ &= \liminf_{\overline{\varepsilon}\to 0^+} \int_{A_{\overline{\varepsilon}}} W_{\overline{\varepsilon}}\left(x_\alpha, x_3; D_\alpha v_{\overline{\varepsilon}} \Big| \frac{1}{\overline{\varepsilon}} D_3 v_{\overline{\varepsilon}}\right)\, dx_\alpha dx_3 \\ &\leq \mu(\overline{A}\times[-1,1]) = \hat{\mu}(\overline{A}). \end{aligned} \tag{2.27}$$

In view of (**??**), (**??**), (**??**), (**??**), Lemma **??** in the Appendix allows us to conclude that $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ is the trace on the open subsets of ω of a finite nonnegative Radon measure. The bound from above in (**??**) immediately implies that it is absolutely continuous with respect to $\mathcal{L}^2\lfloor\omega$.

Step 4. In view of the preceding considerations we are now in a position to apply Theorem 4.3.2 in [**?**], guaranteeing the existence of an energy density $W_{\{\varepsilon^{\mathcal{R}}\}}$ satisfying (**??**). Indeed, $J_{\{\varepsilon^{\mathcal{R}}\}}$ maps any pair (u,A), $u \in W^{1,p}(\omega;\mathbb{R}^3)$, A an open subset of ω, into $\mathbb{R}$, and, furthermore,

(i) $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = J_{\{\varepsilon^{\mathcal{R}}\}}(v;A)$ whenever $u = v$, a.e. on $\mathbb{R}^2$,
(ii) $J_{\{\varepsilon^{\mathcal{R}}\}}(u;\cdot)$ is a finite nonnegative Radon measure,
(iii) $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) \leq 2\beta\int_A(1+|D_\alpha u|^p)dx_\alpha$,
(iv) $J_{\{\varepsilon^{\mathcal{R}}\}}(u+c;A) = J_{\{\varepsilon^{\mathcal{R}}\}}(u;A), \quad c \in \mathbb{R}$.

The proof of Theorem **??** is complete. □

Remark 2.8. It follows immediately from the growth condition (**??**) and the lower-semicontinuity of the L^p-norm that the density function $W_{\{\varepsilon^{\mathcal{R}}\}}$ in Theorem **??** still satisfies (**??**).

Remark 2.9. The conclusions of Theorem **??** are valid for more general domains Ω_ε, since their particular form is not used in the course of the proof. Namely, we may choose in place of Ω_ε any open set $\Omega'_\varepsilon \subset \omega \times (-1,1)$, and consider the set

$$A_\varepsilon := (A \times (-1,1)) \cap \Omega'_\varepsilon$$

in the definition of $E_\varepsilon(u;A)$. Of course, the price to pay for such a degree of generality may be reflected in the possible degeneracy of the limit energy. In fact, Remarks **??**, **??**, and **??** do not hold true in general; hence, (**??**) may fail to describe fully the Γ–limit of $E_{\{\varepsilon^{\mathcal{R}}\}}$, which may be finite also outside $W^{1,p}(\omega;\mathbb{R}^3)$.

On one end of the spectrum of this degeneracy we have the case where $\Omega'_\varepsilon := \emptyset$, for which the Γ–limit reduces to 0 on the whole $L^p(\Omega;\mathbb{R}^3)$. The same conclusion holds if we take $\Omega'_\varepsilon := \omega \times (-r_\varepsilon, r_\varepsilon)$ with $\lim_\varepsilon r_\varepsilon = 0$.

Another type of degeneracy may be found when Ω'_ε is not connected. As an example, take $\Omega'_\varepsilon := \omega \times ((-1,-1/2)\cup(1/2,1))$. It is clear that the Γ–limit is given by a functional defined on pairs of functions in $W^{1,p}(\omega;\mathbb{R}^3)$, the necessary changes in the statement and proof of the corresponding Theorem **??** being straightforward.

Finally, it may also be possible that, even though Ω'_ε is connected for all ε, the domain of the Γ–limit is all of $W^{1,p}(\Omega;\mathbb{R}^3)$. An example of this phenomenon, obtained by taking Ω'_ε to be a domain with a periodical array of cracks, has been studied in detail by Bhattacharya and Braides [**?**].

3. First application – Nonhomogeneous plate models

In [**?**], a nonlinear plate model is derived from a 3D domain of the form $\omega \times (-\varepsilon,\varepsilon)$ occupied by a nonlinearly elastic material upon letting the thickness 2ε tend to 0.

Specifically, under the assumption that the elastic energy density W is homogeneous and satisfies

$$\beta'|F|^p - \frac{1}{\beta'} \le W(F) \le \beta(1+|F|^p), \quad 0<\beta'\le\beta<\infty,\ 1\le p<\infty, \tag{3.1}$$

it is shown that, for any $u \in W^{1,p}(\omega;\mathbb{R}^3)$, any A open subset of ω, and any sequence $\{\varepsilon\} \searrow 0^+$,

$$J_{\{\varepsilon\}}(u;A) := \inf_{\{v_\varepsilon\}} \left\{ \liminf_{\varepsilon\to 0^+} \int_{A\times(-1,1)} W\left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon} D_3 v_\varepsilon\right) dx_\alpha dx_3 : \right.$$
$$\left. v_\varepsilon \in W^{1,p}(A\times(-1,1);\mathbb{R}^3),\ v_\varepsilon \to u \text{ in } L^p(A\times(-1,1);\mathbb{R}^3) \right\},$$

is given by

$$J_{\{\varepsilon\}}(u;A) = 2\int_A Q\overline{W}(D_\alpha u)dx_\alpha,$$

where

$$\overline{W}(\overline{F}) := \inf_{z\in\mathbb{R}^3} W(\overline{F}|z), \quad \overline{F}\in\mathbb{R}^{3\times 2},$$

and

$$Q\overline{W}(\overline{F}) := \inf_{\phi\in W_0^{1,p}(Q';\mathbb{R}^3)} \int_{Q'} \overline{W}(\overline{F}+D_\alpha\phi)\,dx_\alpha,$$

where Q' is the unit cube $(0,1)^2$ in $\mathbb{R}^2$, and $Q\overline{W}$ is the 2D *quasiconvexification* of $\overline{W}$. Here we propose to extend this result to the nonhomogeneous case where W is also function of x_3.

We thus assume that $W(x_3;F)$ is a Carathéodory function on $(-1,1)\times\mathbb{R}^{3\times 3}$ such that

$$\beta'|F|^p - \frac{1}{\beta'} \le W(x_3;F) \le \beta(1+|F|^p), \quad 0<\beta'\le\beta<\infty, \text{ for a.e. } x_3\in(-1,1),$$

or, in other words, that W_ε defined in Section **??** is independent of ε, and that $f_\varepsilon(x_\alpha)\equiv 1,\ x_\alpha\in\omega$.

Direct application of Theorem **??** permits to assert the existence, for any sequence $\{\varepsilon\}\searrow 0^+$, of a subsequence $\{\varepsilon^{\mathcal{R}}\}\searrow 0^+$ such that $J_{\{\varepsilon^{\mathcal{R}}\}}(u;A)$ defined in (**??**) is given by

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u;A) = 2\int_A W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;D_\alpha u)dx_\alpha. \tag{3.2}$$

It remains to identify $W_{\{\varepsilon^{\mathcal{R}}\}}$. To this effect, we define, for any $\overline{F}\in\mathbb{R}^{3\times 2}$,

$$\underline{W}(\overline{F}) := \inf_{\lambda>0}\inf_{\phi} \left\{ \frac{1}{2}\int_{Q'\times(-1,1)} W(x_3;\overline{F}+D_\alpha\phi|\lambda D_3\phi)\,dx_\alpha dx_3 : \right. \tag{3.3}$$
$$\left. \phi\in W^{1,p}(Q'\times(-1,1);\mathbb{R}^3),\ \phi=0 \text{ on } \partial Q'\times(-1,1) \right\}.$$

Then, the following theorem holds true:

Theorem 3.1. *For almost any $x_\alpha\in\omega$ and for all $\overline{F}\in\mathbb{R}^{3\times 2}$, $W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\overline{F}) = \underline{W}(\overline{F})$. Consequently, for all $u\in W^{1,p}(\omega;\mathbb{R}^3)$, any A open subset of ω,*

$$\Gamma(L^p)\text{–}\lim E_\varepsilon(u;A) = J_{\{\varepsilon\}}(u;A) = 2\int_A \underline{W}(D_\alpha u)dx_\alpha.$$

Proof. Consider any sequence $\{\varepsilon\} \searrow 0^+$ and let $\{\varepsilon^{\mathcal{R}}\}$ be as (??), (??). Fix $\overline{F} \in \mathbb{R}^{3\times 2}$ and let x_0 be a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;\overline{F})$. Then,

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) = \lim_{q\to\infty} q^2 \int_{Q'(x_0;1/q)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\overline{F})\,dx_\alpha, \tag{3.4}$$

where $Q'(x_0;1/q)$ is the cube of $\mathbb{R}^2$ of center x_0 and side length $1/q$, and q is large enough so that $Q'(x_0;1/q) \subset \omega$. In view of (??), (??) also reads as

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) = \lim_{q\to\infty} \frac{q^2}{2} J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x; Q'(x_0;1/q)). \tag{3.5}$$

For q large enough, let $\{v^q_{\varepsilon^{\mathcal{R}}}\} \subset W^{1,p}(Q'(x_0;1/q)\times(-1,1);\mathbb{R}^3)$ be such that

$$\begin{cases} v^q_{\varepsilon^{\mathcal{R}}} \to 0 \text{ in } L^p(Q'(x_0;1/q)\times(-1,1);\mathbb{R}^3),\\ J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x; Q'(x_0;\frac{1}{q})) = \\ \qquad \displaystyle\lim_{\varepsilon^{\mathcal{R}}\to 0^+} \int_{Q'\left(x_0;\frac{1}{q}\right)\times(-1,1)} W\left(x_3; \overline{F} + D_\alpha v^q_{\varepsilon^{\mathcal{R}}} \Big| \frac{1}{\varepsilon^{\mathcal{R}}} D_3 v^q\right) dx_\alpha dx_3. \end{cases} \tag{3.6}$$

Such a sequence exists according to Theorem ??. Set

$$v_{q,\varepsilon^{\mathcal{R}}}(x_\alpha,x_3) := q v^q_{\varepsilon^{\mathcal{R}}}\left(x_0 + \frac{x_\alpha}{q}, x_3\right), \quad x_\alpha \in Q'.$$

Thus, by virtue of (??), (??) reads as

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) = \frac{1}{2}\lim_{q\to\infty}\lim_{\varepsilon^{\mathcal{R}}\to 0^+} \int_{Q'\times(-1,1)} W\left(x_3; \overline{F} + D_\alpha v_{q,\varepsilon^{\mathcal{R}}} \Big| \frac{1}{q\varepsilon^{\mathcal{R}}} D_3 v_{q,\varepsilon^{\mathcal{R}}}\right) dx_\alpha dx_3. \tag{3.7}$$

The inequality

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) \geq \underline{W}(\overline{F}) \tag{3.8}$$

would then be immediate if $v_{q,\varepsilon^{\mathcal{R}}} = 0$ on $\partial Q' \times (-1,1)$, because in such a case Fubini's theorem would imply that, for a.e. $x_3 \in (-1,1)$, $v_{q,\varepsilon^{\mathcal{R}}} \in W^{1,p}_0(Q';\mathbb{R}^3)$ and (??) would become

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) \geq \frac{1}{2}\int_{-1}^{1} \underline{W}(\overline{F})\,dx_3 = \underline{W}(\overline{F}).$$

Unfortunately, such may not be the case and we have to modify $v_{q,\varepsilon^{\mathcal{R}}}$ accordingly. To this effect we firstly note that, at the expense of extracting a subsequence of $\{q,\varepsilon^{\mathcal{R}}\}$, still labeled $\{q,\varepsilon^{\mathcal{R}}\}$, we are always at liberty, in view of the coercive character of W (cf. (??)), to assume that the sequence $\{\lambda_{q,\varepsilon^{\mathcal{R}}}\}$ of nonnegative Radon measures

$$\lambda_{q,\varepsilon^{\mathcal{R}}} := \left(1 + |D_\alpha v_{q,\varepsilon^{\mathcal{R}}}|^p + |\frac{1}{q\varepsilon^{\mathcal{R}}} D_3 v_{q,\varepsilon^{\mathcal{R}}}|^p\right)\chi_{Q'\times(-1,1)}\mathcal{L}^3$$

converges weak-* in the sense of measures to a nonnegative finite Radon measure λ as $\{q,\varepsilon^{\mathcal{R}}\} \to (\infty,0)$. We then define, for all Borel sets B of $\mathbb{R}^2$, $\hat{\lambda}(B) := \lambda(B\times[-1,1])$.

We now introduce, for $k \geq 2$,

$$w_{k,q,\varepsilon^{\mathcal{R}}} := \phi_k v_{q,\varepsilon^{\mathcal{R}}},$$

where $\phi_k \in C^\infty_0(Q')$ is such that

$$\begin{cases} 0 \leq \phi_k \leq 1, \quad ||D_\alpha \phi_k||_{L^\infty} \leq Ck^2,\\ \phi_k = \begin{cases} 1 & \text{if } x_\alpha \in Q'(0,1-1/k),\\ 0 & \text{if } x_\alpha \notin Q'(0,1-1/(k+1)). \end{cases} \end{cases}$$

Note that $w_{k,q,\varepsilon^{\mathcal{R}}}(\cdot, x_3) \in W_0^{1,p}(Q';\mathbb{R}^3)$ for a.e. $x_3 \in (-1,1)$. Thus, recalling (**??**),

$$
\begin{aligned}
(3.9)\quad & W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) \geq \\
& \frac{1}{2}\liminf_{q\to\infty}\liminf_{\varepsilon^{\mathcal{R}}\to 0^+}\int_{Q'(0,1-\frac{1}{k})\times(-1,1)} W\left(x_3;\overline{F}+D_\alpha w_{k,q,\varepsilon^{\mathcal{R}}}\Big|\frac{1}{q\varepsilon^{\mathcal{R}}}D_3 w_{k,q,\varepsilon^{\mathcal{R}}}\right)\,dx_\alpha dx_3 \\
& \geq \frac{1}{2}\liminf_{q\to\infty}\liminf_{\varepsilon^{\mathcal{R}}\to 0^+}\left\{\int_{Q'\times(-1,1)} W\left(x_3;\overline{F}+D_\alpha w_{k,q,\varepsilon^{\mathcal{R}}}\Big|\frac{1}{q\varepsilon^{\mathcal{R}}}D_3 w_{k,q,\varepsilon^{\mathcal{R}}}\right)\,dx_\alpha dx_3\right. \\
& -\beta\int_{(Q'\setminus Q'(0,1-\frac{1}{k}))\times(-1,1)}(1+|\overline{F}|^p)\,dx_\alpha dx_3 \\
& -Ck^{2p}\int_{(Q'(0,1-\frac{1}{k+1})\setminus Q'(0,1-\frac{1}{k}))\times(-1,1)}|v_{q,\varepsilon^{\mathcal{R}}}|^p\,dx_\alpha dx_3 \\
& \left.-C\int_{(Q'(0,1-\frac{1}{k+1})\setminus Q'(0,1-\frac{1}{k}))\times(-1,1)}\left(1+|D_\alpha v_{q,\varepsilon^{\mathcal{R}}}|^p+\left|\frac{1}{q\varepsilon^{\mathcal{R}}}D_3 v_{q,\varepsilon^{\mathcal{R}}}\right|^p\right)dx_\alpha dx_3\right\} \\
& \geq \underline{W}(\overline{F})-\frac{C}{k^2}-\limsup_{q\to\infty}\limsup_{\varepsilon^{\mathcal{R}}\to 0^+}Ck^{2p}\int_{Q'\times(-1,1)}|v_{q,\varepsilon^{\mathcal{R}}}|^p\,dx_\alpha dx_3 \\
& -C\limsup_{q\to\infty}\limsup_{\varepsilon^{\mathcal{R}}\to 0^+}\lambda_{q,\varepsilon^{\mathcal{R}}}\left(\left(Q'\left(0,1-\frac{1}{k+1}\right)\setminus Q'\left(0,1-\frac{1}{k}\right)\right)\times(-1,1)\right).
\end{aligned}
$$

Now, in view of (**??**),

$$
\begin{aligned}
&\limsup_{q\to\infty}\limsup_{\varepsilon^{\mathcal{R}}\to 0^+}\int_{Q'\times(-1,1)}|v_{q,\varepsilon^{\mathcal{R}}}|^p\,dx_\alpha dx_3 = \\
&\qquad \limsup_{q\to\infty}q^{p+2}\lim_{\varepsilon^{\mathcal{R}}\to 0^+}\int_{Q'(x_0,\frac{1}{q})\times(-1,1)}|v^q_{\varepsilon^{\mathcal{R}}}|^p\,dx_\alpha dx_3 = 0,
\end{aligned}
$$

while

$$
\begin{aligned}
&\limsup_{q\to\infty}\limsup_{\varepsilon^{\mathcal{R}}\to 0^+}\lambda_{q,\varepsilon^{\mathcal{R}}}\left(\left(Q'(0,1-\frac{1}{k+1}\right)\setminus Q'\left(0,1-\frac{1}{k}\right)\times(-1,1)\right) \\
&\quad \leq \lambda\left(\overline{\left(Q'\left(0,1-\frac{1}{k+1}\right)\setminus Q'\left(0,1-\frac{1}{k}\right)\right)\times(-1,1)}\right) \\
&\quad \leq \hat{\lambda}\left(Q'\setminus Q'\left(0,1-\frac{1}{k-1}\right)\right).
\end{aligned}
$$

Thus, (**??**) becomes

$$
W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) \geq \underline{W}(\overline{F})-\frac{C}{k^2}-C\hat{\lambda}\left(Q'\setminus Q\left(0,1-\frac{1}{k-1}\right)\right),
$$

and (**??**) is obtained by letting k tend to ∞ since $Q'(0,1-1/(k-1))$ is an increasing sequence of open sets with set limits Q'.

Conversely, for any given $\eta > 0$, let $\lambda > 0$, $\phi \in W^{1,\infty}(Q'\times(-1,1);\mathbb{R}^3)$ with $\phi = 0$ on $\partial Q'\times(-1,1)$, be such that

$$
(3.10)\qquad \underline{W}(\overline{F})+\eta \geq \frac{1}{2}\int_{Q'\times(-1,1)} W(x_3;\overline{F}+D_\alpha\phi|\lambda D_3\phi)\,dx_\alpha dx_3.
$$

This is legitimate because of the density of $W^{1,\infty}(Q'\times(-1,1);\mathbb{R}^3)$ into $W^{1,p}(Q'\times(-1,1);\mathbb{R}^3)$ – both with zero trace on the boundary $\partial Q'\times(-1,1)$ – and of the bound from above in (**??**). Set

$$
v_{\varepsilon^{\mathcal{R}}}(x_\alpha,x_3) := \overline{F}x_\alpha+\lambda\varepsilon^{\mathcal{R}}\phi\left(\frac{x_\alpha}{\lambda\varepsilon^{\mathcal{R}}},x_3\right),
$$

where ϕ has been laterally extended by Q'-periodicity. Then,

$$v_{\varepsilon^{\mathcal{R}}} \to \overline{F} x_\alpha \text{ in } L^p(\Omega;\mathbb{R}^3).$$

Furthermore, for any open set $A \subset \omega$,

$$\begin{aligned}(3.11)\quad J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;A) &\le \liminf_{\varepsilon^{\mathcal{R}}\to 0^+} \int_{A\times(-1,1)} W\left(x_3; D_\alpha v_{\varepsilon^{\mathcal{R}}} \middle| \frac{1}{\varepsilon^{\mathcal{R}}} D_3 v_{\varepsilon^{\mathcal{R}}}\right) dx_\alpha dx_3 \\ &= \liminf_{\varepsilon^{\mathcal{R}}\to 0^+} \int_{A\times(-1,1)} W\left(x_3; \overline{F} + D_\alpha\phi\left(\frac{x_\alpha}{\lambda\varepsilon^{\mathcal{R}}}, x_3\right) \middle| \lambda D_3\phi\left(\frac{x_\alpha}{\lambda\varepsilon^{\mathcal{R}}}, x_3\right)\right) dx_\alpha dx_3.\end{aligned}$$

Since $\int_1^1 W(x_3;\overline{F} + D_\alpha\phi(\cdot,x_3)|\lambda D_3\phi(\cdot,x_3))\,dx_3$ is a periodic function in $L^\infty(\mathbb{R}^2)$, it converges weak-* to its average and (**??**) becomes, in view of (**??**),

$$\begin{aligned}J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;A) &\le \mathcal{L}^2(A)\int_{-1}^1\int_{Q'} W(x_3;\overline{F}+D_\alpha\phi|\lambda D_3\phi)\,dx_\alpha dx_3 \\ &\le 2\mathcal{L}^2(A)\,\underline{W}(\overline{F}) + 2\eta\mathcal{L}^2(A).\end{aligned}$$

Letting η tend to 0^+ yields

$$(3.12)\qquad J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;A) \le 2\mathcal{L}^2(A)\,\underline{W}(\overline{F}).$$

But, according to Theorem **??**, (**??**) also reads as

$$\int_A W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F})dx_\alpha \le \mathcal{L}^2(A)\,\underline{W}(\overline{F}),$$

so that, upon choosing $x_0 \in \Omega$ to be a Lebesgue point for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot;\overline{F})$ and A to be a small ball centered at x_0 and of vanishing radius, we obtain

$$W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) \le \underline{W}(\overline{F}).$$

Since finally $W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F})$ does not depend upon the choice of sequence $\{\varepsilon^{\mathcal{R}}\}$, we conclude that there is no need to extract a subsequence from $\{\varepsilon\}$.

In light of Proposition 7.11 in [?], the proof of Theorem **??** is now complete. □

Remark 3.2. Since $W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha;\cdot)$ is the integrand of a lower semicontinuous functional on $W^{1,p}(\omega;\mathbb{R}^3)$ – namely the $\Gamma(L^p)$–limit of $E_{\varepsilon^{\mathcal{R}}}(\cdot,\omega)$ –, it is quasiconvex (see Statement II.5 in [?]). Thus $\underline{W}(\overline{F})$ defined in (**??**) is actually quasiconvex.

Remark 3.3. Note that, if W does not depend upon x_3, then

$$\underline{W}(\overline{F}) = Q\overline{W}(\overline{F}),\ \overline{F}\in\mathbb{R}^{3\times 2}.$$

In other words, the result of [?] is recovered in the homogeneous case. Indeed, clearly,

$$\begin{aligned}\underline{W}(\overline{F}) &\ge \inf_\phi\left\{\frac{1}{2}\int_{-1}^1\int_{Q'}\overline{W}(\overline{F}+D_\alpha\phi)\,dx_\alpha dx_3 : \phi\in W^{1,p}(Q'\times(-1,1);\mathbb{R}^3),\right. \\ &\qquad\qquad \left.\phi = 0 \text{ on } \partial Q'\times(-1,1)\right\} \\ &\ge \frac{1}{2}\int_{-1}^1 Q\overline{W}(\overline{F})\,dx_3 = Q\overline{W}(\overline{F}),\end{aligned}$$

so that

$$\underline{W}(\overline{F}) \ge Q\overline{W}(\overline{F}).$$

Conversely, since $\underline{W}(\overline{F})$ is quasiconvex according to Remark **??**, it suffices to prove that

$$\underline{W}(\overline{F}) \le \overline{W}(\overline{F}).$$

The continuity and growth of W at infinity imply the existence of $z \in \mathbb{R}^3$ such that

$$\overline{W}(\overline{F}) = W(\overline{F}|z).$$

The density of $W_0^{1,\infty}(Q';\mathbb{R}^3)$ in $L^p(Q';\mathbb{R}^3)$ implies in turn the existence of $\xi^\eta \in W_0^{1,\infty}(Q';\mathbb{R}^3)$ such that

$$\xi^\eta \overset{\eta\searrow 0}{\longrightarrow} z, \text{ strongly in } L^p(Q';\mathbb{R}^3).$$

Extend ξ^η Q'-periodically to $\mathbb{R}^2$ and set

$$\phi_n^\eta(x_\alpha, x_3) := \frac{1}{n^2} x_3 \xi^\eta(n x_\alpha).$$

Then,

$$\begin{aligned}
\underline{W}(\overline{F}) &\leq \frac{1}{2} \liminf_{n\to+\infty} \int_{Q'\times(-1,1)} W(\overline{F} + D_\alpha \phi_n^\eta | n^2 D_3 \phi_n^\eta)\, dx_\alpha dx_3 \\
&= \frac{1}{2} \liminf_{n\to+\infty} \int_{Q'\times(-1,1)} W\left(\overline{F} + \frac{1}{n} x_3 D_\alpha \xi^\eta(n x_\alpha) \Big| \xi^\eta(n x_\alpha)\right) dx_\alpha dx_3 \\
&\leq \frac{1}{2} \liminf_{n\to+\infty} \int_{Q'\times(-1,1)} W(\overline{F}|\xi^\eta(n x_\alpha))\, dx_\alpha dx_3,
\end{aligned}$$

where the uniformly continuous character of W on compact sets has been used in the last inequality (see e.g. the proof of Lemma 4.1 in [?] for more details). But $W(\overline{F}|\xi^\eta(\cdot))$ is a periodic function in $L^\infty(\mathbb{R}^2)$, thus weak-* converges to its average and we obtain

$$\underline{W}(\overline{F}) \leq \int_{Q'} W(\overline{F}|\xi^\eta(x_\alpha))\, dx_\alpha.$$

The result is obtained through direct application of Lebesgue's dominated convergence theorem upon letting η tend to 0.

Remark 3.4. We believe that the result of Theorem **??**, appropriately extended, still holds true in the case of an energy density that also depends upon x_α, although we are not at present in a position to offer a full proof in such a setting.

4. Second application –The periodic case

In this section it is assumed that $W(x_\alpha, x_3; F)$ is a Carathéodory function from $Q' \times (-1,1) \times \mathbb{R}^9$ into $\mathbb{R}$ satisfying

$$\beta' |F|^p - \frac{1}{\beta'} \leq W(x_\alpha, x_3; F) \leq \beta(1 + |F|^p),$$

with $1 \leq p < \infty$ and $\beta', \beta > 0$. The function W is extended by Q'-periodicity to $\mathbb{R}^2 \times (-1,1) \times \mathbb{R}^9$ and we set

$$W_\varepsilon(x_\alpha, x_3; F) := W\left(\frac{x_\alpha}{\varepsilon}, x_3; F\right).$$

Also, we assume that f is a continuous function from Q' into $[0,1]$ with $0 < \gamma \leq \min f$ and we set

$$f_\varepsilon(x_\alpha) := f\left(\frac{x_\alpha}{\varepsilon}\right).$$

We define, for any $\overline{F} \in \mathbb{R}^{3\times 2}$,

$$W_{\rm hom}(\overline{F}) := \liminf_{t\nearrow\infty} g(t),$$

where, for any $t > 0$,

$$g(t) := \frac{1}{t^2} \inf_{\phi} \left\{ \int_{(tQ')^f} W(x_\alpha, x_3; \overline{F} + D_\alpha\phi | D_3\phi)\, dx_\alpha dx_3 : \phi \in W^{1,p}((tQ')^f; \mathbb{R}^3), \right.$$
$$\left. \phi(x_\alpha, x_3) = 0 \text{ if } x_\alpha \in \partial(tQ'),\ |x_3| < f(x_\alpha) \right\},$$

and where, for $A \subset \mathbb{R}^2$, $A^f := \{(x_\alpha, x_3) : x_\alpha \in A, |x_3| < f(x_\alpha)\}$.

Remark 4.1. It is easily shown that

$$W_{\text{hom}}(\overline{F}) = \inf_{t>0} g(t),$$

and also that, in the definition of $g(t)$, periodic boundary conditions on the test functions can be imposed in lieu of Dirichlet boundary conditions.

The following theorem holds true:

Theorem 4.2. *If $u \in W^{1,p}(\omega; \mathbb{R}^3)$ and if A is an open subset of ω, then*

$$\Gamma(L^p)\text{–}\lim E_\varepsilon(u; A) = \int_A W_{\text{hom}}(D_\alpha u)\, dx_\alpha.$$

Proof. Consider a sequence $\{\varepsilon\}$, with $\varepsilon \searrow 0^+$. Application of Theorem **??** permits to assert the existence of a subsequence $\{\varepsilon^{\mathcal{R}}\}$ of $\{\varepsilon\}$ and of a Carathéodory function $W_{\{\varepsilon^{\mathcal{R}}\}}$ such that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(u; A) = \int_A W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha; D_\alpha u)\, dx_\alpha.$$

We firstly show that $W_{\{\varepsilon^{\mathcal{R}}\}}$ is independent of x_α.

Fix $\overline{F} \in \mathbb{R}^{3\times 2}$, and let $x_0, y_0 \in \omega$ be Lebesgue points for $W_{\{\varepsilon^{\mathcal{R}}\}}(\cdot; \overline{F})$, so that

$$\begin{aligned} W_{\{\varepsilon^{\mathcal{R}}\}}(x_0; \overline{F}) &= \lim_{\delta\to 0^+} \frac{1}{\delta^2} \int_{Q'(x_0,\delta)} W_{\{\varepsilon^{\mathcal{R}}\}}(x_\alpha; \overline{F})\, dx_\alpha \\ &= \lim_{\delta\to 0^+} \frac{1}{\delta^2} J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha; Q'(x_0, \delta)), \end{aligned} \tag{4.1}$$

– idem for y_0. Assume that δ is small enough. According to Lemma **??**, there exists a sequence $\{\psi^\delta_{\varepsilon^{\mathcal{R}}}\}$ with $\psi^\delta_{\varepsilon^{\mathcal{R}}} = 0$ on $\{(x_\alpha, x_3) : |x_3| < f(x_\alpha/\varepsilon^{\mathcal{R}}), x_\alpha \in \partial Q'(x_0, \delta)\}$ and $\psi^\delta_{\varepsilon^{\mathcal{R}}}\chi_{Q'(x_0,\delta)_{\varepsilon^{\mathcal{R}}}} \to 0$ in $L^p(\Omega; \mathbb{R}^3)$ (with, as usual, $Q'(x_0, \delta)_{\varepsilon^{\mathcal{R}}} := \{(x_\alpha, x_3) : |x_3| < f(x_\alpha/\varepsilon^{\mathcal{R}}), x_\alpha \in Q'(x_0, \delta)\}$), such that

$$J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha; Q'(x_0, \delta)) = \lim_{\varepsilon^{\mathcal{R}}\to 0^+} E_{\varepsilon^{\mathcal{R}}}(\overline{F}x_\alpha + \psi^\delta_{\varepsilon^{\mathcal{R}}}; Q'(x_0, \delta)). \tag{4.2}$$

Define the vector $\tau_{\varepsilon^{\mathcal{R}}} \in \varepsilon^{\mathcal{R}}\mathbb{Z}^N$ as

$$(\tau_{\varepsilon^{\mathcal{R}}})_i := \varepsilon^{\mathcal{R}} \left\| \left[\frac{(y_0 - x_0)_i}{\varepsilon^{\mathcal{R}}} \right] \right\|, \quad \text{for } i = 1, \dots, N.$$

Cleary $\tau_{\varepsilon^{\mathcal{R}}} \to y_0 - x_0$ as $\varepsilon^{\mathcal{R}} \to 0^+$. Let

$$\phi^\delta_{\varepsilon^{\mathcal{R}}}(x_\alpha, x_3) := \psi^\delta_{\varepsilon^{\mathcal{R}}}(x_\alpha - \tau_{\varepsilon^{\mathcal{R}}}, x_3),$$

where we have extended $\psi^\delta_{\varepsilon^{\mathcal{R}}}$ by 0 to $[\mathbb{R}^2 \setminus Q'(x_0, \delta)]_{\varepsilon^{\mathcal{R}}}$.

Fix $r > 1$ and consider $\varepsilon^{\mathcal{R}}$ small enough so that

$$Q'(y_0 - \tau_{\varepsilon^{\mathcal{R}}}, \delta) \subset Q'(x_0, r\delta). \tag{4.3}$$

Since $\phi^\delta_{\bar\varepsilon}\chi_{Q'(y_0,\delta)_{\varepsilon^{\mathcal{R}}}} \to 0$ in $L^p(\Omega;\mathbb{R}^3)$, we have

$$\begin{aligned}
&J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;Q'(y_0,\delta)) \le \\
&\liminf_{\varepsilon^{\mathcal{R}}\to 0^+}\int_{Q'(y_0,\delta)_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}},x_3;\overline{F}+D_\alpha\phi^\delta_{\varepsilon^{\mathcal{R}}}\middle|\frac{1}{\varepsilon^{\mathcal{R}}}D_3\phi^\delta_{\varepsilon^{\mathcal{R}}}\right)dx_\alpha dx_3 \\
&\le \liminf_{\varepsilon^{\mathcal{R}}\to 0^+}\int_{Q'(y_0-\tau_{\varepsilon^{\mathcal{R}}},\delta)_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_\alpha+\tau_{\varepsilon^{\mathcal{R}}}}{\varepsilon^{\mathcal{R}}},x_3;\overline{F}+D_\alpha\phi^\delta_{\bar\varepsilon}(x_\alpha+\tau_{\varepsilon^{\mathcal{R}}},x_3)\right| \\
&\qquad\qquad \left.\frac{1}{\varepsilon^{\mathcal{R}}}D_3\phi^\delta_{\varepsilon^{\mathcal{R}}}(x_\alpha-\tau_{\varepsilon^{\mathcal{R}}},x_3)\right)dx_\alpha dx_3 \\
&\le \liminf_{\varepsilon^{\mathcal{R}}\to 0^+}\int_{Q'(x_0,r\delta)_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}},x_3;\overline{F}+D_\alpha\psi^\delta_{\bar\varepsilon}(x_\alpha,x_3)\right| \\
&\qquad\qquad \left.\frac{1}{\varepsilon^{\mathcal{R}}}D_3\psi^\delta_{\varepsilon^{\mathcal{R}}}(x_\alpha,x_3)\right)dx_\alpha dx_3 \\
&\le J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;Q'(x_0,\delta)) + 2\beta(1+|\overline{F}|^p)\mathcal{L}^2(Q'(x_0,r\delta)\setminus Q'(x_0,\delta))
\end{aligned}$$

where we have used (**??**), (**??**) and the periodicity of $W(\cdot,x_3)$. Letting $r\to 1$, we finally obtain

$$J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;Q'(y_0,\delta)) \le J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;Q'(x_0,\delta)),$$

hence, in view of (**??**),

$$W_{\{\varepsilon^{\mathcal{R}}\}}(y_0;\overline{F}) = \lim_{\delta\to 0^+}\frac{1}{\delta^2}J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;Q'(y_0,\delta)) \le W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}).$$

Given the arbitrariness of x_0 and y_0 we conclude that

$$W_{\{\varepsilon^{\mathcal{R}}\}}(y_0;\overline{F}) = W_{\{\varepsilon^{\mathcal{R}}\}}(x_0;\overline{F}) =: W_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}).$$

We now identify $W_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F})$. Assuming, without loss of generality, that $0\in\omega$ and $Q'\subset\omega$, by virtue of Lemma **??** there exists a sequence $\{\psi_{\varepsilon^{\mathcal{R}}}\}$ with $\psi_{\varepsilon^{\mathcal{R}}} = 0$ on $\{(x_\alpha,x_3): |x_3| < f(x_\alpha/\varepsilon^{\mathcal{R}}), x_\alpha\in\partial Q'\}$ and $\psi_{\varepsilon^{\mathcal{R}}}\chi_{Q'_{\varepsilon^{\mathcal{R}}}} \to 0$ in $L^p(\Omega;\mathbb{R}^3)$, such that

$$W_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}) = J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}x_\alpha;Q') = \lim_{\varepsilon^{\mathcal{R}}\to 0^+}E_{\varepsilon^{\mathcal{R}}}(\overline{F}x_\alpha+\psi_{\varepsilon^{\mathcal{R}}};Q').$$

Define

$$\phi_{\varepsilon^{\mathcal{R}}}(x_\alpha,x_3) := \frac{1}{\varepsilon^{\mathcal{R}}}\psi_{\varepsilon^{\mathcal{R}}}(\varepsilon^{\mathcal{R}}x_\alpha,x_3).$$

Then, $\phi_{\varepsilon^{\mathcal{R}}}\in W^{1,p}\left(\left[(0,1/\varepsilon^{\mathcal{R}})^2\right]^f;\mathbb{R}^3\right)$, and it is equal to 0 as soon as $x_\alpha\in\partial(0,1/\varepsilon^{\mathcal{R}})^2$; thus it is an admissible test function in the definition of $g(1/\varepsilon^{\mathcal{R}})$ and

$$\begin{aligned}
&\limsup_{\varepsilon^{\mathcal{R}}\to 0^+} g\left(\frac{1}{\varepsilon^{\mathcal{R}}}\right) \le \\
&\limsup_{\varepsilon^{\mathcal{R}}\to 0^+}{\varepsilon^{\mathcal{R}}}^2\int_{\left[(0,\frac{1}{\varepsilon^{\mathcal{R}}})^2\right]^f} W\left(x_\alpha,x_3;\overline{F}+D_\alpha\phi_{\varepsilon^{\mathcal{R}}}|D_3\phi_{\varepsilon^{\mathcal{R}}}\right)dx_\alpha dx_3 \\
&= \limsup_{\varepsilon^{\mathcal{R}}\to 0^+}\int_{Q'_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}},x_3;\overline{F}+D_\alpha\psi_{\varepsilon^{\mathcal{R}}}\middle|\frac{1}{\varepsilon^{\mathcal{R}}}D_3\psi_{\varepsilon^{\mathcal{R}}}\right)dx_\alpha dx_3 \\
&= W_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}).
\end{aligned} \tag{4.4}$$

Conversely, consider $\lambda_n\nearrow\infty$ such that $g(\lambda_n)\to\liminf_{t\nearrow\infty}g(t)$. For each n, take $\phi_n\in W^{1,p}(\{(0,\lambda_n)^2\times(-1,1): |x_3| < f(x_\alpha)\};\mathbb{R}^3)$ with $\phi_n = 0$ if $x_\alpha\in\partial(0,\lambda_n)^2$, and such that

$$g(\lambda_n) + \frac{1}{\lambda_n^3} \geq \frac{1}{\lambda_n^2} \int_{[(0,\lambda_n)^2]^f} W(x_\alpha, x_3; \overline{F} + D_\alpha \phi_n | D_3 \phi_n)\, dx_\alpha dx_3. \tag{4.5}$$

Set $\psi^n_{\varepsilon^{\mathcal{R}}} := \varepsilon^{\mathcal{R}} \phi_n \left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}}, x_3 \right)$, where ϕ_n has been extended by zero to $([\![\lambda_n]\!]+1)^2$, and then to the whole of $\mathbb{R}^2$ by $([\![\lambda_n]\!]+1)^2$-periodicity. Then,

$$\begin{aligned}
J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F} x_\alpha; Q') &\leq \liminf_{\varepsilon^{\mathcal{R}} \to 0^+} E_{\varepsilon^{\mathcal{R}}}(\overline{F} x_\alpha + \psi^n_{\varepsilon^{\mathcal{R}}}; Q') \\
&= \liminf_{\varepsilon^{\mathcal{R}} \to 0^+} \int_{Q'_{\varepsilon^{\mathcal{R}}}} W\left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}}, x_3; \overline{F} + D_\alpha \psi^n_{\varepsilon^{\mathcal{R}}} \Big| \frac{1}{\varepsilon^{\mathcal{R}}} D_3 \psi^n_{\varepsilon^{\mathcal{R}}} \right) dx_\alpha dx_3 \\
&= \liminf_{\varepsilon^{\mathcal{R}} \to 0^+} \int_{Q'} \left[\int_{-f\left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}}\right)}^{f\left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}}\right)} W\left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}}, x_3; \overline{F} + D_\alpha \phi_n \left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}}, x_3 \right) \Big| \right.\right. \\
&\qquad\qquad \left.\left. D_3 \phi_n \left(\frac{x_\alpha}{\varepsilon^{\mathcal{R}}}, x_3 \right) \right) dx_3 \right] dx_\alpha \\
&= \frac{1}{([\![\lambda_n]\!]+1)^2} \int_{(0,\lambda_n)^2} \left[\int_{-f(x_\alpha)}^{f(x_\alpha)} W(x_\alpha, x_3; \overline{F} + D_\alpha \phi_n(x_\alpha, x_3) | \right. \\
&\qquad\qquad \left. D_3 \phi_n(x_\alpha, x_3))\, dx_3 \right] dx_\alpha \\
&\quad + \frac{1}{([\![\lambda_n]\!]+1)^2} \int_{[(0,[\![\lambda_n]\!]+1)^2 \setminus (0,\lambda_n)^2]^f} W(x_\alpha, x_3; \overline{F} | 0)\, dx_\alpha dx_3 \\
&\leq \frac{\lambda_n^2}{([\![\lambda_n]\!]+1)^2} \left(g(\lambda_n) + \frac{1}{\lambda_n^3} \right) + O\left(\frac{1}{n} \right),
\end{aligned}$$

where we have used (**??**) as well as the $([\![\lambda_n]\!]+1)^2$-periodic character of

$$\int_{f(\cdot)}^{f(\cdot)} W(\cdot, x_3; \overline{F} + D_\alpha \phi^n(\cdot, x_3) | D_3 \phi_n(\cdot, x_3))\, dx_3.$$

Thus, letting n tend to ∞,

$$J_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F} x_\alpha; Q') \leq \liminf_{t \nearrow \infty} g(t),$$

or still

$$W_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}) \leq \liminf_{t \nearrow \infty} g(t). \tag{4.6}$$

Recalling (**??**), (**??**), we obtain

$$\liminf_{t \nearrow \infty} g(t) \leq \limsup_{\varepsilon^{\mathcal{R}} \to 0^+} g\left(\frac{1}{\varepsilon^{\mathcal{R}}} \right) \leq W_{\{\varepsilon^{\mathcal{R}}\}}(\overline{F}) \leq \liminf_{t \nearrow \infty} g(t),$$

which proves the desired result. Since the $\Gamma(L^p)$–limit of $E_{\varepsilon^{\mathcal{R}}}(u; A)$ is independent of the specific sequence $\{\varepsilon^{\mathcal{R}}\}$, in light of Proposition 7.11 in [**?**] we conclude that $E_\varepsilon(u; A)$ $\Gamma(L^p)$-converges to $\int_A W_{\text{hom}}(D_\alpha u)\, dx_\alpha$. □

Remark 4.3. Theorem **??** still holds if we only assume that $0 \leq f \leq 1$. In general, the description of the Γ–limit is not complete, as there may exist a $u \notin W^{1,p}(\omega; \mathbb{R}^3)$ such that $J_{\{\varepsilon^{\mathcal{R}}\}}(u; \Omega) < +\infty$. Nevertheless, some degenerate cases can be dealt with in the spirit of the homogenization of domain with soft inclusions. This can be done, for example, if we suppose that for some $\gamma > 0$ the set $B_\gamma := \{x_\alpha \in \mathbb{R}^2 : \gamma < f(x_\alpha)\}$ contains a periodic connected Lipschitz set (see related work in [**?**] and [**?**] Chapter 19).

5. Third application – Optimal design of a thin film

The kind of dimensional reduction performed in this paper has proved relevant in the analysis and design of thin films. We refer the interested reader to [?] and references therein for a detailed motivation of the problem considered below and for relevant results in the so-called cylindrical case (see Remark ?? below).

It is thus assumed in this section that $W_i(F),\ i = 1, 2$, is a continuous real-valued function on $\mathbb{R}^{3\times 3}$ such that

$$\beta'|F|^p - \frac{1}{\beta'} \le W_i(F) \le \beta(1 + |F|^p), 0 < \beta' \le \beta < \infty,\ 1 \le p < \infty, \tag{5.1}$$

Our goal is to compute , for any $v \in W^{1,p}(\omega;\mathbb{R}^3)$, any $\theta \in L^\infty(\omega \times (-1,1);[0,1])$, any open subdomain $A \subset \mathbb{R}^2$,

$$J(v;\theta;A) := \inf_{\{\varepsilon\}\searrow 0^+} J_{\{\varepsilon\}}(v;\theta;A),$$

where

$$\begin{aligned}J_{\{\varepsilon\}}(v;\theta;A) := \inf_{\{\chi_\varepsilon\},\{v_\varepsilon\}} \Bigg\{ & \liminf_{\varepsilon\to 0^+} \int_{A\times(-1,1)} (\chi_\varepsilon W_1 + (1-\chi_\varepsilon)W_2) \left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon} D_3 v_\varepsilon \right) dx_\alpha dx_3 : \\ & v_\varepsilon \in W^{1,p}(A\times(-1,1);\mathbb{R}^3), \chi_\varepsilon \in L^\infty(A\times(-1,1);\{0,1\}), \\ & v_\varepsilon \to v \text{ in } L^p(A\times(-1,1);\mathbb{R}^3), \chi_\varepsilon \overset{*}{\rightharpoonup} \theta \text{in } L^\infty(A\times(-1,1);[0,1]) \Bigg\}.\end{aligned}$$

Let us define, for any $\overline{F} \in \mathbb{R}^{3\times 2},\ \theta \in [0,1]$,

$$\begin{aligned}\widehat{W}(\theta,\overline{F}) :=& \\ & \inf_{\lambda>0}\inf_{\phi,\chi} \Bigg\{ \frac{1}{2}\int_{Q'\times(-1,1)} (\chi W_1 + (1-\chi)W_2)(\overline{F} + D_\alpha\phi|\lambda D_3\phi)\ dx_\alpha dx_3 : \\ & \phi \in W^{1,p}(Q'\times(-1,1);\mathbb{R}^3), \phi = 0 \text{ on } \partial Q'\times(-1,1), \\ & \chi \in L^\infty(Q'\times(-1,1);\{0,1\}),\ \frac{1}{2}\int_{Q'\times(-1,1)} \chi\ dx_\alpha dx_3 = \theta \Bigg\} \\ & = \inf_{k>0}\inf_{\phi,\chi} \Bigg\{ \frac{1}{2k}\int_{Q'\times(-k,k)} (\chi W_1 + (1-\chi)W_2)(\overline{F} + D_\alpha\phi|D_3\phi)\ dx_\alpha dx_3 : \\ & \phi \in W^{1,p}(Q'\times(-k,k);\mathbb{R}^3), \phi = 0 \text{ on } \partial Q'\times(-k,k), \\ & \chi \in L^\infty(Q'\times(-k,k);\{0,1\}),\ \frac{1}{2k}\int_{Q'\times(-k,k)} \chi\ dx_\alpha dx_3 = \theta \Bigg\}.\end{aligned} \tag{5.2}$$

Remark 5.1. It is easily proved that $\widehat{W}$ is an upper-semicontinuous function of $(\theta,\overline{F}) \in [0,1]\times\mathbb{R}^{3\times 2}$. The proof is a strict analogue to that of Proposition 2.9 in [?].

The following theorem holds true:

Theorem 5.2.

$$J(v;\theta;A) \ge 2\int_A \widehat{W}\left(\frac{1}{2}\int_{-1}^1 \theta(x_\alpha,s)\ ds, Dv(x_\alpha)\right)\ dx_\alpha.$$

Further, equality holds if $\theta \in L^\infty(\omega;[0,1])$ *and if* W *satisfies the following symmetry property:*

$$W(\overline{F}|F_3) = W(\overline{F}|-F_3),\ \overline{F} \in \mathbb{R}^{3\times 2}, F_3 \in \mathbb{R}^3. \tag{5.3}$$

Remark 5.3. In contrast with the setting investigated in [?] the material distribution – the characteristic functions χ_ε –is not restricted to cylindrical geometries; in other words χ_ε may also depend on the transverse variable x_3. This is the only difference between Theorem **??** and Theorem 2.3 in [?]. Note that the equality in Theorem **??** holds if θ is independent of x_3, that is if we are truly in a plate-like setting.

Remark 5.4. The section heading is somewhat misleading since, in truth, Theorem **??** is not a mere application of the results of Section **??**; specifically, Theorem **??** cannot be invoked in the current setting because of the presence of the additional field χ_ε. The proof of Theorem 2.3 in [?] can however be revisited in the light of the method used to prove Theorem **??** so as to prove Theorem **??**. This is the object of the proof of Theorem **??** below.

Remark 5.5. We conjecture that the symmetry condition (**??**), although satisfied in many applications, is not necessary, but confess to our inability at doing away with it at present. We remark that this hypothesis is not required in the case of cylindrical inclusions (see [?]).

Proof. The first part of Theorem **??** is contained in the more precise Lemma **??** stated below which is in turn the strict analogue of (part of) Theorem **??** in the current setting.

Lemma 5.6. *For any sequence* $\{\varepsilon\} \searrow 0^+$*, there exists a subsequence* $\{\varepsilon^{\mathcal{R}}\}$ *such that*

$$J_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;A) \geq 2\int_A \widehat{W}\left(\frac{1}{2}\int_{-1}^{1}\theta(x_\alpha,s)\,ds, Dv(x_\alpha)\right)dx_\alpha,$$

thus, in particular,

$$J(v;\theta;A) \geq 2\int_A \widehat{W}\left(\frac{1}{2}\int_{-1}^{1}\theta(x_\alpha,s)\,ds, Dv(x_\alpha)\right)dx_\alpha.$$

Proof. The subsequence $\{\varepsilon^{\mathcal{R}}\}$ is defined exactly as in Section **??** (see argument leading to (**??**)). The proof is then divided into two steps which will be sketched below. Only those parts of the argument that differ from analogous parts in the proofs of Theorem **??** or of Theorem 2.3 in [?] will be detailed. The first step is devoted to a proof that $J_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;\cdot)$ is a finite nonnegative Radon measure which is absolutely continuous with respect to $\mathcal{L}^2\lfloor\omega$. A second step establishes that the Radon-Nykodim derivative $\frac{dJ_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;\cdot)}{d\mathcal{L}^2}(x_0)$ is, for suitable x_0's in ω, greater than or equal to $2\widehat{W}\left(1/2\int_{-1}^{1}\theta(x_0,s)\,ds, D_\alpha v(x_0)\right)$.

Step 1. Step 1 is a *near verbatim* reproduction of Steps 1–3 in the proof of Theorem **??**. Firstly, it is observed, exactly as in Lemma **??**, that approximating sequences for v may as well take the value v on the lateral boundary of $A\times(-1,1)$. The proof is identical to that of Lemma **??**. Then the inner regularity of $J_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;\cdot)$ is established exactly as for (**??**). We now address the subadditive character of $J_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;\cdot)$. Once again the proof is nearly identical to that of (**??**). Note however that the recovery sequences for B^δ, D^δ are pairs $\left(\chi^{B^\delta}_{\varepsilon^{\mathcal{R}}}, v^{B^\delta}_{\varepsilon^{\mathcal{R}}}\right) \in L^\infty(B^\delta\times(-1,1);\{0,1\})\times W^{1,p}(B^\delta\times(-1,1);\mathbb{R}^3)$ – idem for D^δ –, with

$$\begin{cases}\chi^{B^\delta}_{\varepsilon^{\mathcal{R}}} \overset{*}{\rightharpoonup} \theta & \text{in } L^\infty(B^\delta\times(-1,1);[0,1]),\\ v^{B^\delta}_{\varepsilon^{\mathcal{R}}} \to v & \text{in } L^p(B^\delta\times(-1,1);\mathbb{R}^3);\end{cases}$$

idem for D^δ. A new pair-sequence is defined as

$$\begin{cases} \chi_{\overline{\varepsilon}} & := \chi^\delta_{\eta_0}\chi^{D^\delta}_{\overline{\varepsilon}} + (1-\chi^\delta_{\eta_0})\chi^{B^\delta}_{\overline{\varepsilon}} + \chi_{(A\setminus(B^\delta\cup D^\delta))_{\overline{\varepsilon}}} h_{\overline{\varepsilon}} \\ v_{\zeta,\overline{\varepsilon}} & := \phi^\delta_\zeta v^{D^\delta}_{\overline{\varepsilon}} + (1-\phi^\delta_\zeta) v^{B^\delta}_{\overline{\varepsilon}} + \chi_{(A\setminus(B^\delta\cup D^\delta))_{\overline{\varepsilon}}} u, \end{cases}$$

where ϕ_ζ was defined in (**??**), $\{h_{\overline{\varepsilon}}\}$ is any sequence in $L^\infty(\Omega;\{0,1\})$ converging to θ weakly-* in $L^\infty(\Omega;[0,1])$, and $\chi^\delta_{\eta_0}$ is a characteristic function such that

$$\chi^\delta_{\eta_0} = \begin{cases} 1 & \text{if } x_\alpha \notin B^\delta \text{ or } \mathrm{dist}(x_\alpha, \partial B^\delta) \le \eta_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly, as $\overline{\varepsilon} \searrow 0^+$,

$$\begin{cases} \chi_{\overline{\varepsilon}} \overset{*}{\rightharpoonup} \theta & \text{in } L^\infty(A\times(-1,1);[0,1]), \\ v_{\zeta,\overline{\varepsilon}} \to v & \text{in } L^p(A\times(-1,1);\mathbb{R}^3), \end{cases}$$

and the remainder of the proof of (**??**) proceeds as before. Finally, (**??**), (**??**) are unchanged and Lemma **??** permits to conclude that $J_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;\cdot)$ is a finite nonnegative Radon measure. The bound from above in (**??**) immediately implies that it is absolutely continuous with respect to $\mathcal{L}^2\lfloor\omega$.

Step 2. From the definition of Radon-Nykodim derivative, for almost every $x_0 \in \omega$,

$$\begin{aligned} \frac{dJ_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;\cdot)}{d\mathcal{L}^2}(x_0) &= \lim_{\delta\to 0^+} \frac{1}{\delta^2} J_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;Q'(x_0,\delta)) \\ &= \lim_{\delta\to 0^+} \frac{1}{\delta^2} \lim_{\overline{\varepsilon}\to 0^+} \int_{Q'(x_0,\delta)\times(-1,1)} (\chi_{\overline{\varepsilon}} W_1 + (1-\chi_{\overline{\varepsilon}})W_2)\left(D_\alpha v_{\overline{\varepsilon}} \Big| \frac{1}{\overline{\varepsilon}} D_3 v_{\overline{\varepsilon}}\right)\, dx_\alpha dx_3, \end{aligned} \tag{5.4}$$

where $\{\overline{\varepsilon}\}$ is a subsequence of $\{\varepsilon^{\mathcal{R}}\}$ and

$$\begin{cases} \chi_{\overline{\varepsilon}} \overset{*}{\rightharpoonup} \theta & \text{in } L^\infty(Q'(0,\delta)\times(-1,1);[0,1]), \\ v_{\overline{\varepsilon}} \to v & \text{in } L^p(Q'(0,\delta)\times(-1,1);\mathbb{R}^3). \end{cases}$$

Take $x_0 \in \omega$ to be a Lebesgue point for $\int_{-1}^1 \theta(\cdot,s)\, ds$ and a point of approximate differentiability for v. Setting

$$\begin{cases} \chi_{\delta,\overline{\varepsilon}} := \chi_{\overline{\varepsilon}}(x_0+\delta x_\alpha, x_3), \\ v_{\delta,\overline{\varepsilon}} := \dfrac{v_{\overline{\varepsilon}}(x_0+\delta x_\alpha, x_3) - v(x_0)}{\delta} \end{cases}$$

(**??**) now reads as

$$\frac{dJ_{\{\varepsilon^{\mathcal{R}}\}}(v;\theta;\cdot)}{d\mathcal{L}^2}(x_0) = \lim_{\delta\to 0^+}\lim_{\overline{\varepsilon}\to 0^+} \int_{Q'\times(-1,1)} (\chi_{\delta,\overline{\varepsilon}} W_1 + (1-\chi_{\delta,\overline{\varepsilon}})W_2)\left(D_\alpha v_{\delta,\overline{\varepsilon}} \Big| \frac{\delta}{\overline{\varepsilon}} D_3 v_{\delta,\overline{\varepsilon}}\right)\, dx_\alpha dx_3. \tag{5.5}$$

Note that

(5.6)

$$\begin{aligned} \lim_{\delta\to 0^+}\lim_{\overline{\varepsilon}\to 0^+} \frac{1}{2}\int_{Q'\times(-1,1)} \chi_{\delta,\overline{\varepsilon}}\, dx_\alpha dx_3 &= \frac{1}{2}\lim_{\delta}\lim_{\overline{\varepsilon}} \frac{1}{\delta^2}\int_{Q'(x_0,\delta)\times(-1,1)} \chi_{\overline{\varepsilon}}\, dx_\alpha dx_3 \\ &= \frac{1}{2}\lim_{\delta\to 0^+} \frac{1}{\delta^2}\int_{Q'(x_0,\delta)\times(-1,1)} \theta\, dx_\alpha dx_3 \\ &= \frac{1}{2}\lim_{\delta\to 0^+} \frac{1}{\delta^2}\int_{Q'(x_0,\delta)} \left(\int_{-1}^1 \theta(x_\alpha,s)\, ds\right) dx_\alpha \\ &= \frac{1}{2}\int_{-1}^1 \theta(x_0,s)\, ds, \end{aligned}$$

because x_0 is a Lebesgue point for $\int_{-1}^{1} \theta(x_\alpha, s)\, ds$.

Set $A_{\delta,\overline{\varepsilon}} := \{(x_\alpha, x_3) \in Q' \times (-1,1) : \chi_{\delta,\overline{\varepsilon}}(x_\alpha, x_3) = 1\}$. Then (**??**) implies that

$$\lim_{\delta\to 0^+} \lim_{\overline{\varepsilon}\to 0^+} \frac{1}{2}\mathcal{L}^3(A_{\delta,\overline{\varepsilon}}) = \frac{1}{2}\int_{-1}^{1} \theta(x_0, s)\, ds =: \Theta.$$

The remaining of the proof would then be obvious from (**??**) and the definition (**??**) of $\widehat{W}(\Theta, Dv(x_0))$ if

$$\frac{1}{2}\mathcal{L}^3(A_{\delta,\overline{\varepsilon}}) = \Theta$$

and if $v_{\delta,\overline{\varepsilon}} = D_\alpha(x_0)x_\alpha$ on $\partial Q' \times (-1,1)$. Unfortunately, there is no guarantee that the above holds true and the sequence $\{\chi_{\delta,\overline{\varepsilon}}, v_{\delta,\overline{\varepsilon}}\}$ must be modified accordingly. This procedure is identical to that described in the proof of Lemma 3.1 in [**?**] – up to changing the names of the indices q, n to $\delta, \overline{\varepsilon}$, and also up to replacing $\theta(x_0)$ by Θ – and the interested reader is invited to consult pages 185 to 189 of that paper. Note that in the case where $\Theta \equiv 1$ one should remark, in translating the proof into our context, that, by virtue of Remark **??** and also of (**??**), (**??**),

$$Q\overline{W}_1(\overline{F}) = \underline{W}_1(\overline{F}) = \widehat{W}(1, \overline{F}).$$

The proof of Lemma **??** is complete. □

We now address the second part of Theorem **??** and thus assume, from now onward, that θ is independent of x_3.

The proof is divided into two steps. In a first step, it is assumed that $\theta \equiv \theta_\infty$ and $v \equiv v_\infty$ are, respectively, constant and affine functions and we get the following result:

Lemma 5.7. *Let T be a triangle on the plane. Then there exists a sequence $\{v_n, \chi_n\} \in W^{1,p}(T \times (-1,1); \mathbb{R}^3) \times L^\infty(T \times (-1,1); \{0,1\})$ with $v_n = v_\infty$ on $\partial T \times (-1,1)$ such that, as $n \nearrow \infty$,*

$$\begin{cases} \chi_n \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(T \times (-1,1); [0,1]), \\ v_n \to v_\infty & \text{in } L^p(T \times (-1,1); \mathbb{R}^3), \end{cases}$$

and

$$\begin{aligned} J(v_\infty; \theta_\infty; T) &= \lim_{n\to+\infty} \int_{T\times(-1,1)} (\chi_n W_1 + (1-\chi_n)W_2)(D_\alpha v_n | n D_3 v_n)\, dx_\alpha dx_3 \\ &= 2\mathcal{L}^2(T)\, \widehat{W}(\theta_\infty, D_\alpha v_\infty). \end{aligned}$$

A second step addresses the case of general domains $A \subset \omega$, general θ's and general v's, and yields Theorem **??**.

Proof of Lemma **??**. The proof is a blow-up argument in the spirit of [**?**]. From the definition (**??**) of $\widehat{W}$, together with the density of $W^{1,\infty}(Q' \times (-1,1); \mathbb{R}^3)$ into $W^{1,p}(Q' \times (-1,1); \mathbb{R}^3)$, we deduce that, for any $\eta > 0$, there exist $\lambda^\eta > 0$, $\phi^\eta \in W^{1,\infty}(Q' \times (-1,1); \mathbb{R}^3)$, with $\phi^\eta = 0$ on $\partial Q' \times (-1,1)$, and $\chi^\eta \in L^\infty(Q' \times (-1,1); \{0,1\})$ with $1/2 \int_{Q'\times(-1,1)} \chi^\eta\, dx_\alpha dx_3 = \theta_\infty$ such that

(5.7)

$$2\widehat{W}(\theta_\infty, D_\alpha v_\infty) \geq \int_{Q'\times(-1,1)} (\chi^\eta W_1 + (1-\chi^\eta)W_2)(D_\alpha v_\infty + D_\alpha \phi^\eta | \lambda^\eta D_3 \phi^\eta)\, dx_\alpha dx_3 - \eta.$$

Extend ϕ^η, χ^η to be $Q' \times (-2,2)$-periodic by setting

$$\overline{\phi}^\eta, \overline{\chi}^\eta(x_\alpha, x_3) := \begin{cases} \phi^\eta, \chi^\eta(x_\alpha, x_3), & -1 \leq x_3 \leq 1, \\ \phi^\eta, \chi^\eta(x_\alpha, -2-x_3), & -2 \leq x_3 \leq -1, \\ \phi^\eta, \chi^\eta(x_\alpha, 2-x_3), & 1 \leq x_3 \leq 2, \end{cases}$$

then extend $\overline{\phi}^\eta, \overline{\chi}^\eta$ by $Q' \times (-2,2)$-periodicity to $\mathbb{R}^3$. Note that, in view of (**??**), (**??**) also reads as

(5.8)

$$4\widehat{W}(\theta_\infty, D_\alpha v_\infty) \geq \int_{Q'\times(-2,2)} (\overline{\chi}^\eta W_1 + (1-\overline{\chi}^\eta)W_2)(D_\alpha v_\infty + D_\alpha \overline{\phi}^\eta | \lambda^\eta D_3 \overline{\phi}^\eta)\, dx_\alpha dx_3 - 2\eta.$$

Remark 5.8. This is the only instance where assumption (**??**) is used in this study.

Set

$$\begin{cases} \chi_n^\eta(x_\alpha, x_3) & := \overline{\chi}^\eta(n^2 x_\alpha, n x_3), \\ v_n^\eta(x_\alpha, x_3) & := v_\infty(x_\alpha) + \frac{1}{n^2}\overline{\phi}^\eta(n^2 x_\alpha, n\lambda^\eta x_3). \end{cases}$$

If $Q'(a,r)$ is a square on $\mathbb{R}^2$, then it is easily checked that, as $n \nearrow \infty$,

$$\begin{cases} \chi_n^\eta \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(Q'(a,r)\times(-1,1);[0,1]), \\ v_n^\eta \to v_\infty & \text{in } L^p(Q'(a,r)\times(-1,1);\mathbb{R}^3). \end{cases} \tag{5.9}$$

Further, in view of (**??**), together with the periodic character of v_n^η, χ_n^η,

$$\begin{aligned} & J(v_\infty;\theta_\infty;Q'(a,r)) \leq \\ & \liminf_{\eta\to 0^+} \lim_{n\to+\infty} \int_{Q'(a,r)\times(-1,1)} (\chi_n^\eta W_1 + (1-\chi_n^\eta)W_2)(D_\alpha v_n^\eta | n D_3 v_n^\eta)\, dx_\alpha dx_3 \\ & \leq \limsup_{\eta\to 0^+} \lim_{n\to+\infty} \int_{Q'(a,r)\times(-1,1)} (\chi_n^\eta W_1 + (1-\chi_n^\eta)W_2)(D_\alpha v_n^\eta | n D_3 v_n^\eta)\, dx_\alpha dx_3 \\ & \leq 2\mathcal{L}^2(Q'(a,r))\, \widehat{W}(\theta_\infty, D_\alpha v_\infty). \end{aligned}$$

But, in view of Lemma **??**,

$$J(v_\infty;\theta_\infty;Q'(a,r)) \geq 2\mathcal{L}^2(Q'(a,r))\, \widehat{W}(\theta_\infty, D_\alpha v_\infty),$$

so that

$$\begin{aligned} & J(v_\infty;\theta_\infty;Q'(a,r)) = \\ & \lim_{\eta\to 0^+} \lim_{n\to+\infty} \int_{Q'(a,r)\times(-1,1)} (\chi_n^\eta W_1 + (1-\chi_n^\eta)W_2)(D_\alpha v_n^\eta | n D_3 v_n^\eta)\, dx_\alpha dx_3 \\ & = 2\mathcal{L}^2(Q'(a,r))\, \widehat{W}(\theta_\infty, D_\alpha v_\infty), \end{aligned}$$

hence, by virtue of Lemma **??** in the Appendix, there exists a sequence $\{\eta(n)\}_n$ such that $v_n := v_n^{\eta(n)}, \chi_n := \chi_n^{\eta(n)}$ satisfy (**??**) as well as

(5.10)

$$\begin{aligned} J(v_\infty;\theta_\infty;Q'(a,r)) &= \lim_{n\to+\infty} \int_{Q'(a,r)\times(-1,1)} (\chi_n W_1 + (1-\chi_n)W_2)(D_\alpha v_n | n D_3 v_n)\, dx_\alpha dx_3 \\ &= 2\mathcal{L}^2(Q'(a,r))\, \widehat{W}(\theta_\infty, D_\alpha v_\infty). \end{aligned}$$

Consider now a triangle T covered with squares of the type $Q'(a,r), a \in \mathbb{R}^2, r > 0$, up to a set of measure $1/m$, i.e.,

$$T_m := \cup_{i=1}^{N(m)} Q'(a_i^m, r_i^m) \subset\subset T,$$

with $\mathcal{L}^2(T \setminus T_m) \leq 1/m$. An easy construction, identical to that of Step 2 in Lemma 4.2 in [**?**] (see Remark 4.4 in [**?**]) yields (**??**) for T in lieu of $Q'(a,r)$. The proof of Lemma **??** is complete. □

In view of Lemma **??** – and in particular of the boundary condition $v_n = v_\infty$ for the corresponding sequence –, Lemma **??** also holds true if T is replaced by a triangulation of the plane on which v_∞ is continuous and piecewise affine and θ_∞ piecewise constant. Then, let v, θ be arbitrary elements of $W^{1,p}(\omega;\mathbb{R}^3)$ and

$L^\infty(\omega;[0,1])$, respectively, and consider $\{w_k,\theta_k\}$ a piecewise affine/piecewise constant pair defined on triangulations of the plane such that

$$\begin{cases} \theta_k \to \theta & \text{a.e. in } \omega, \\ w_k \to v & \text{in } W^{1,p}(\omega;\mathbb{R}^3). \end{cases}$$

For each pair (w_k,θ_k) there exists a pair-sequence $\{v_k^n,\theta_k^n\}$ defined on the same triangulation satisfying the properties of Lemma **??** for that triangulation and for $v_\infty := w_k, \theta_\infty := \theta_k$. A diagonalization process (with n replaced by $n(k)$) immediately yields a sequence $\{v_k,\chi_k\} \in W^{1,p}(A\times(-1,1);\mathbb{R}^3)\times L^\infty(A\times(-1,1);\{0,1\})$ such that

$$\begin{cases} \chi_k \overset{*}{\rightharpoonup} \theta & \text{in } L^\infty(A\times(-1,1);[0,1]), \\ v_k \to v & \text{in } L^p(A\times(-1,1);\mathbb{R}^3), \end{cases} \tag{5.11}$$

and

(5.12)

$$\begin{aligned} J(v;\theta;A) &\le \liminf_{k\to+\infty} \int_{A\times(-1,1)} (\chi_k W_1 + (1-\chi_k)W_2)(D_\alpha v_k | n(k) D_3 v_k)\, dx_\alpha dx_3 \\ &= \liminf_{k\to+\infty} J(v_k;\theta_k;A) \\ &= \liminf_{k} 2\int_A \widehat{W}(\theta_k, D_\alpha v_k)\, dx_\alpha. \end{aligned}$$

The bound from above for W_i, the first convergence in (**??**), Fatou's lemma, and the upper semicontinuity property of $\widehat{W}$ (see Remark **??**) imply that

$$\liminf_{k\to+\infty}\int_A \{\beta(1+|D_\alpha v_k|^p) - \widehat{W}(\theta_k, D_\alpha v_k)\}dx_\alpha \ge \int_A \{\beta(1+|D_\alpha v|^p) - \widehat{W}(\theta, D_\alpha v)\}dx_\alpha,$$

thus

$$\int_A \widehat{W}(\theta, D_\alpha v)\, dx_\alpha \ge \limsup_{k\to+\infty}\int_A \widehat{W}(\theta_k, D_\alpha v_k)\, dx_\alpha,$$

which, together with (**??**), yields

$$J(v;\theta;A) \le 2\int_A \widehat{W}(\theta, D_\alpha v)\, dx_\alpha.$$

Lemma **??** provides the other inequality and the proof of Theorem **??** is complete. □

6. Final Remarks

This paper provides some insight into the characterization of effective energies for thin structures with varying profiles within a nonlinear setting, and some of our results have already been used and referred to in the literature on equilibria of thin structures, such as the papers by Ansini and Braides [?], Braides and Fonseca [?] and Shu [?]. It is, by no means, a completed subject, as we have pointed out throughout the text. From the technical point of view, we believe that Theorem **??** may be extended to the case where the energy density also depends upon x_α (see Remark **??**), and condition (**??**) should not be requested for proving Theorem **??** (see Remarks **??** and **??**).

Finally, although Theorem **??** holds for arbitrary sets Ω_ε (see Remark **??**), in order to have a complete description of the limit problem it is now known that some geometrical and structural conditions need to be imposed on Ω_ε, as illustrated by the example of Braides and Bhattacharya [?] where the limit problem is 3D and there is no dimensional reduction in the resulting effective energy.

7. Appendix

Lemma **??** and Lemma **??** are trivial diagonalization lemmata.

Lemma 7.1. *Let $a_{k,j}$ be a doubly indexed sequence of real numbers ($k, j \nearrow \infty$). If*

$$\lim_k \lim_j a_{k,j} = L,$$

then there exists a subsequence $\{k(j)\}_j \nearrow \infty$ such that

$$\lim_j a_{k(j),j} = L.$$

Lemma 7.2. *Let $a_{k,j}$ be a doubly indexed sequence of real numbers ($k, j \nearrow \infty$). If*

$$\sup_k \lim_j a_{k,j} = L,$$

then there exists a subsequence $\{k(j)\}_j \nearrow \infty$ such that

$$\lim_j a_{k(j),j} = L.$$

Let $a_{k,j}$ be a doubly indexed sequence of real numbers ($k, j \nearrow \infty$). If

$$\lim_k \limsup_j a_{k,j} = L = \lim_k \liminf_j a_{k,j},$$

then there exists a subsequence $\{k(j)\}_j \nearrow \infty$ such that

$$\lim_j a_{k(j),j} = L.$$

The third lemma in this appendix provides sufficient conditions for a mapping $\pi : \mathcal{O} \to [0, +\infty)$ to be the trace of a Radon measure, where $\mathcal{O}$ is the set of open subsets of ω and ω is an open subset of $\mathbb{R}^N$. It is close in spirit to De Giorgi-Letta's criterion [**?**].

Lemma 7.3. *Let π be a mapping from $\mathcal{O}$ into $\mathbb{R}^+$ and μ be a finite nonnegative* Radon *measure on $\mathbb{R}^N$. If, for any $A, B, C \in \mathcal{O}$,*

(i) $\pi(A) \leq \pi(A \setminus \overline{C}) + \pi(B)$, $\overline{C} \subset B \subset A$,
(ii) *for any $\varepsilon > 0$, there exists $C_\varepsilon \in \mathcal{O}$ with $\overline{C}_\varepsilon \subset A$ and $\pi(A \setminus \overline{C}_\varepsilon) \leq \varepsilon$,*
(iii) $\pi(\omega) \geq \mu(\mathbb{R}^N)$,
(iv) $\pi(A) \leq \mu(\overline{A})$,

then, π is the restriction of μ to $\mathcal{O}$.

Proof. Recalling (ii), consider $d_\varepsilon := \text{dist}(\overline{C}_\varepsilon, \partial A) > 0$. Then, $\overline{C}_\varepsilon \subset B_\varepsilon := \{x \in A;\ \text{dist}(x, \partial A) > d_\varepsilon/2\}$, while $\overline{B}_\varepsilon \subset A$. Thus, by (i, iv), and since $\overline{C}_\varepsilon \subset B_\varepsilon \subset A$,

$$\pi(A) \leq \varepsilon + \pi(B_\varepsilon) \leq \varepsilon + \mu(\overline{B}_\varepsilon) \leq \varepsilon + \mu(A).$$

Hence, letting ε tend to 0,

$$\pi(A) \leq \mu(A).$$

Conversely, since μ is a Radon measure, it is inner regular, so that for any $\varepsilon > 0$, there exists $C_\varepsilon \in \mathcal{O}$ with $\overline{C}_\varepsilon \subset A$ and $\mu(A) \leq \varepsilon + \mu(C_\varepsilon)$. Hence, with the help of (iii) and of the previously derived inequality,

$$\mu(A) \leq \varepsilon + \mu(\overline{C}_\varepsilon) = \varepsilon + \mu(\omega) - \mu(\omega \setminus \overline{C}_\varepsilon) \leq \varepsilon + \mu(\mathbb{R}^N) - \pi(\omega \setminus \overline{C}_\varepsilon) \leq \varepsilon + \pi(\omega) - \pi(\omega \setminus \overline{C}_\varepsilon),$$

and, since $\overline{C}_\varepsilon \subset A \subset \omega$, (i) implies that $\mu(A) \leq \varepsilon + \pi(A)$, so that the result is obtained upon letting ε tend to 0. □

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(Andrea Braides) S.I.S.S.A, 34014 Trieste, Italy

(Irene Fonseca) Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
E-mail address, I. Fonseca: `fonseca@andrew.cmu.edu`

(Gilles Francfort) L.P.M.T.M., Université Paris-Nord, 93430 Villetaneuse, France