Homogenization of degenerate elliptic-parabolic equations

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Abstract

In this paper we give a result of G-convergence for a class of strongly degenerate parabolic equations in the case of periodic coefficients. The operators have the form $\mu(x)\partial_t - \operatorname{div}(a(x,t)\cdot D)$ where the quadratic form associated to $a(x,t)$ is degenerating as a Muckenhoupt weight and the coefficient μ is greater or equal to zero, possibly $\mu \equiv 0$, that is the operator may be elliptic, parabolic or elliptic-parabolic.

1 Introduction

The problem of homogenization (see [10], [18], [32], [3]) for partial differential equations was studied since the last century (see [1] and the references therein), and in the last thirty years especially in connection with G-convergence of differential operators and Γ-convergence of functionals (see [27] and [10] as first papers dealing with this problem, and for instance [6], [38], [21]).
Suppose Ω a bounded open set with Lipschitz boundary, $T > 0$, some functions $\mu : \mathbf{R}^n \to \mathbf{R}$ and $a_{ij} : \mathbf{R}^n \times [0,T] \to \mathbf{R}$, $i,j = 1, ...n$, such that μ and $a_{ij}(\cdot,t)$ are $\mathcal{Q}$-periodic functions for a. e. $t \in [0,T]$ where $\mathcal{Q} = [0,1]^n$. Suppose the function $\mu \geqslant 0$ and $\mathcal{Q}_+$ and $\mathcal{Q}_0$ with Lipschitz boundaries, with $\mathcal{Q}_+ \cup \mathcal{Q}_0 = \mathcal{Q}$, such that $\mu(x) > 0$ for a. e. $x \in \mathcal{Q}_+$ and $\mu(x) = 0$ for a. e. $x \in \mathcal{Q}_0$. Now, if $\alpha = (\alpha_1, ...\alpha_n)$ with $\alpha_j \in \mathbf{Z}$, denote $\tau_\alpha \mathcal{Q}_+ := \{x \in \mathbf{R}^n \mid x - \alpha \in \mathcal{Q}_+\}$ and $\tau_\alpha \mathcal{Q}_0 := \{x \in \mathbf{R}^n \mid x - \alpha \in \mathcal{Q}_0\}$, $\Pi_+ := \cup_{\alpha \in \mathbf{Z}^n} \tau_\alpha \mathcal{Q}_+$ and $\Pi_0 := \cup_{\alpha \in \mathbf{Z}^n} \tau_\alpha \mathcal{Q}_0$, for $h \in \mathbf{N}$ $\Pi_{+,h} := \{x \in \mathbf{R}^n \mid hx \in \Pi_+\}$ and $\Pi_{0,h} := \{x \in \mathbf{R}^n \mid hx \in \Pi_0\}$, and finally $\Omega_{+,h} = \Omega \cap \Pi_{+,h}$ and $\Omega_{0,h} = \Omega \cap \Pi_{0,h}$. If $\mu_h(x) = \mu(hx)$ we have that $\mu_h > 0$ a. e. in $\Pi_{+,h}$ and $\mu_h = 0$ a. e. in $\Pi_{0,h}$. Then we denote by $\Omega_{+,h}$ the set $\Omega \cap \mathcal{Q}_{+,h}$ and by $\Omega_{0,h}$ the set $\Omega \cap \mathcal{Q}_{0,h}$.
Note that the equations

$$\mu(hx)\frac{\partial u}{\partial t} - \operatorname{div}(a(hx,t)\cdot Du) = f$$

are parabolic in $\Omega_{+,h}$, elliptic in $\Omega_{0,h}$.
We want to study the limit behaviour ($h \to +\infty$) of the solutions of the following problems

$$\begin{cases} \mu(hx)\dfrac{\partial u}{\partial t} - \operatorname{div}(a(hx,t)\cdot Du) = f & \text{on } \Omega_{+,h} \times (0,T) \\ u = 0 & \text{on } \big((\partial\Omega_{+,h} \cap \partial\Omega) \times (0,T)\big) \\ u = \varphi & \text{on } \big(\Omega_{+,h} \times \{0\}\big) \\ -\operatorname{div}(a_h(\cdot,t)\cdot Du) = f(\cdot,t) & \text{on } \Omega_{0,h} \text{ for a. e. } t \in [0,T] \\ u = 0 & \text{on } \partial\Omega_{0,h} \cap \partial\Omega\,. \end{cases}$$

which can be written

$$\begin{cases} \mu(hx)\dfrac{\partial u}{\partial t} - \operatorname{div}(a(hx,t)\cdot Du) = f & \text{on } \Omega\times(0,T) \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = \varphi(x) & \text{on } \Omega_{+,h}\,. \end{cases} \tag{1.1}$$

We will make the following assumption on the matrix a: we will suppose the coefficients

$$a_{ij}(x,t) \qquad \text{such that} \qquad a_{ij}(\cdot,t) \;\; \mathcal{Q}\text{-periodic for a. e. } t\in[0,T]$$

and there exists a function $\lambda > 0$ and some constants L, M, N such that

$$\begin{aligned} &\lambda(y)|\xi|^2 \leqslant \big(a(y,t)\cdot\xi,\xi\big) \leqslant L\lambda(y)|\xi|^2 \\ &\big(a(y,t)\cdot\xi,\eta\big) \leqslant M\big(a(y,t)\cdot\xi,\xi\big)^{1/2}\big(a(y,t)\cdot\xi,\xi\big)^{1/2} \\ &\big|a(y,t)-a(y,s)\big| \leqslant N\lambda(y)|t-s| \end{aligned}$$

for every $\xi,\eta\in\mathbf{R}^n$, for a. e. $y\in\mathbf{R}^n$ and for a. e. $t,s\in[0,T]$. On the function λ and on the function μ we will make the following assumptions:

$$\mu\geqslant 0, \lambda>0 \text{ a. e. in } \mathbf{R}^n\,, \qquad \mu,\lambda\ \mathcal{Q}\text{-periodic functions}\,, \qquad \mu,\lambda,\lambda^{-1}\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$$

and moreover we will assume that the *weights* μ and λ verify the following Muckenhoupt type conditions

$$\sup_{Q\subset\mathbf{R}^n}\left(\frac{1}{|Q|}\int_Q \lambda dx\right)\left(\frac{1}{|Q|}\int_Q \lambda^{-1} dx\right) < +\infty \tag{A_2}$$

$$\sup_{Q\subset[0,1]^n} |Q|^{\alpha r/n}\left(\frac{1}{|Q|}\int_Q \mu^r dx\right)^{1/2}\left(\frac{1}{|Q|}\int_Q \lambda^{-r} dx\right)^{1/2} < +\infty \tag{$A^{\alpha r}_{2,2}$}$$

(for these conditions see, for instance, [13, 28] and Section 3) for suitable constants $0\leqslant\alpha<1$ and $r>1$ and where Q denotes a cube of $\mathbf{R}^n$ with faces parallel to the coordinate planes. The condition $(A^{\alpha r}_{2,2})$ can be weakened ($r=1$) requiring that μ satisfies an additional assumption (see Section 5, (H.2)).
The main result is the following: denote by $u_h(f,\varphi)$ the unique solutions to (1.1), by $\widehat{a}$ the limit matrix defined as in the classical homogenization case (see, for example, [3] and Section 7) and

$$\widehat{\mu} := \fint_{[0,1]^n} \mu\; dy\,.$$

Theorem *For every $f\in L^2(0,T;L^n(\Omega))$ and $\varphi\in W^{1,\infty}_0(\Omega)$ we have that*

$$\begin{aligned} &u_h(f,\varphi)\to u(f,\varphi) && \text{in } L^2\big(0,T;L^1(\Omega)\big) \\ &a_h\cdot Du_h(f,\varphi)\to\widehat{a}\cdot Du(f,\varphi) && \text{in } L^2\big(0,T;(L^1(\Omega))^n\big)\text{−weak,} \end{aligned}$$

where $u(f,\varphi)$ is the unique solution of the following Cauchy-Dirichlet problem (the matrix $\widehat{a}$ is independent of the weight μ, possibly $\mu\equiv 0$)

$$\begin{cases} \widehat{\mu}\dfrac{\partial}{\partial t}u - \displaystyle\sum_{i,j=1}^n \widehat{a}_{ij}(t)\frac{\partial^2}{\partial x_j\partial x_i}u = f & \text{on } \Omega\times(0,T) \\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0)=\varphi(x) & \text{on } \Omega\,. \end{cases}$$

In the second section we give a result of completeness of a weighted Sobolev space with two different weights (inducing two different measures), $\mu \geqslant 0$ and $\lambda > 0$, in such a way the norm will be $\|u\|^2 = \int_\Omega u^2\mu + \int_\Omega |Du|^2\lambda$.
In the section 4 we give some results about the abstract equation

$$Ru' + \mathcal{A}u = f$$

with $R \geqslant 0$ bounded operator. We give existence results for the equation $Ru' + \mathcal{A}u = f \in L^2(0,T;L^n(\Omega))$ and a regularity result for the solution if $f \in H^1(0,T;L^n(\Omega))$. Via this regularity result we prove the main theorem with a regular datum and then we approximate $f \in L^2(0,T;L^n(\Omega))$ with suitable functions in $H^1(0,T;L^n(\Omega))$.
We describe the problem in Section 5, we obtain some useful compactness results in Section 6, the main theorem in Section 7. In the last section we give some examples and show that such a result is not true if homogenization also in time is considerd, i. e. a sequence of matrices

$$a(h^\alpha x, h^\beta t)\,.$$

Notations: we will denote by $C^1_{\rm c}(\Omega)$ the set of functions with compact support in Ω, by $C^1_{\rm per}(Q)$ the set of periodic functions with period the cube Q and by $C^1_\#(Q)$ the set $C^1_{\rm per}(Q)/\mathbf{R}$ or the subset of $C^1_{\rm per}(Q)$ of the functions with mean value equal to zero.
We will denote by Q a generic (open or closed) cube of $\mathbf{R}^n$ with faces parallel to the coordinate planes, by cQ the cube concentric with Q and having side length c times that of Q and and by $B_r(x)$ the open ball of $\mathbf{R}^n$ centered in x with radius r. We recall that a *dyadic* cube is a cube of $\mathbf{R}^n$ whose vertices are in $2^k\mathbf{Z}^n$ with $k \in \mathbf{Z}$. The symbols $(\cdot,\cdot)$, $|E|$ and p' will indicate respectively the scalar product of $\mathbf{R}^n$, the Lebesgue measure of the set E, and the conjugate $p' = p/(p-1)$ of p. The symbol $\displaystyle\fint_E f dx$ will denote the mean value of f on E (i. e. $|E|^{-1}\displaystyle\int_E \lambda(x)dx$); sometimes for a periodic function we will use the symbol $\langle f\rangle$ to denote the mean value of f on its period. If $f : \Omega \to \mathbf{R}$ we will denote by $\overline{f} : \mathbf{R}^n \to \mathbf{R}$ the function extended zero out of Ω.

Acknowledgements
I wish to thank Prof. A. A. Pankov who, during is visit in Rome, was so kind to answer me many questions about one of his papers ([22]).
I also want to thank G. P. Leonardi and V. Recupero for many interesting discussions, not only on the subject.

2 A result on completeness of a weighted Sobolev space

In this section we show a result (Theorem 2.3 and Corollary 2.5) on completeness of some weighted Sobolev spaces.
We consider the class of admissible functions

$$\mathcal{A}(\Omega) = \{\nu : \Omega \to \mathbf{R} \mid \nu \text{ measurable and } \nu \geqslant 0\}.$$

Among the class $\mathcal{A}(\Omega)$ we consider the subclass B_p defined as follows: given an open set Ω and $1 < p < \infty$ we say that

$$\nu \in B_p(\Omega) \qquad \text{if} \quad \nu^{-1/(p-1)} \in L^1_{\rm loc}(\Omega)\,. \tag{2.1}$$

If $\nu \in B_p(\Omega)$ we have that for every $\omega \subset\subset \Omega$

$$\int_\omega |u| dx \leqslant \Big(\int_\omega |u|^p \nu dx \Big)^{1/p} \Big(\int_\omega \nu^{-1/(p-1)} dx \Big)^{(p-1)/p}.$$

Then if $\nu \in B_p(\Omega)$ we have the inclusions

$$L^p(\Omega, \nu) \subset L^1_{\mathrm{loc}}(\Omega) \tag{2.2}$$

where

$$L^p(\Omega, \nu) = \Big\{ u \in L^1_{\mathrm{loc}}(\Omega) \mid u \nu^{1/p} \in L^p(\Omega) \Big\}.$$

The assumption that $\nu \in B_p(\Omega)$ is sharp to obtain (2.2) (see [14]).
Given an open set Ω, $1 < p < +\infty$ and $n+1$ functions in $\mathcal{A}(\Omega)$ $\mu, \lambda_1, ...\lambda_n$ such that $\lambda_i \in B_p(\Omega)$ we define

$$W^{1,p}(\Omega, \mu, \lambda_1, ...\lambda_n) = \Big\{ u \in L^p(\Omega, \mu) \cap W^{1,1}_{\mathrm{loc}}(\Omega) \;\Big|\; D_i u \in L^p(\Omega, \lambda_i) \; i = 1, ...n \Big\}, \tag{2.3}$$

Suppose the set $\{x \in \Omega \mid \mu(x) > 0\}$ to have positive measure; we can endow the space $W^{1,p}(\Omega, \mu, \lambda_1, ...\lambda_n)$ with the norm

$$\|u\|^p_{W^{1,p}(\Omega,\mu,\lambda_1,...\lambda_n)} \overset{def}{=} \int_\Omega |u|^p \mu dx + \sum_{i=1}^n \int_\Omega |D_i u|^p \lambda_i dx \tag{2.4}$$

Now we want to prove that the normed space just defined is a Banach space.

Lemma 2.1 *Let Ω be an open and connected subset of $\mathbf{R}^n$. Consider a point $x_0 \in \Omega$ and a sequence $(u_h)_h \subset C^1(\Omega)$ such that*

$$\begin{aligned} &u_h(x_0) \text{ is a Cauchy sequence in } \mathbf{R}\,, \\ &\nabla u_h \text{ is a Cauchy sequence in } (L^1_{\mathrm{loc}}(\Omega))^n\,. \end{aligned} \tag{2.5}$$

Then $(u_h)_h$ is a Cauchy sequence in $W^{1,1}_{\mathrm{loc}}(\Omega)$.

Proof - By contradiction we suppose that $(u_h)_h$ is not a Cauchy sequence in $L^1_{\mathrm{loc}}(\Omega)$. Then there exists a subsequence $(u_{h_j})_j$, a ball $B \subset\subset \Omega$ and $\epsilon > 0$ such that

$$\int_B |u_{h_j} - u_{h_{j+1}}| \, dx > \epsilon \qquad \text{for every } j. \tag{2.6}$$

Consider ω an open and connected subset of Ω such that

$$B \cup \{x_0\} \subset\subset \omega \subset\subset \Omega$$

and moreover such that ω is diffeomorphic to a ball. Then it is possible to find $\Gamma \in C^1(\overline{B} \times [0,1]; \omega)$, such that

$$\begin{aligned} &\gamma_x(t) := \Gamma(x,t) \qquad \text{is a regular arc} \\ &\gamma_x(0) = \Gamma(x,0) = x_0, \quad \gamma_x(1) = \Gamma(x,1) = x \qquad \text{for every } x \in B, \\ &\omega_t := \{\gamma_x(t) \in \Omega \mid x \in B\} \quad \text{is a simply connected precompact open set}\,, \\ &\frac{\partial \Gamma}{\partial t}(x,0) = 0 \end{aligned} \tag{2.7}$$

(one can find a C^1 diffeomorphism ψ between ω and a ball B_0 and then take $\Gamma(x,t) = \psi^{-1}(\psi(x_0) + t^2(\psi(x) - \psi(x_0))))$. For every $j \in \mathbf{N}$ we can write

$$u_{h_j}(x) = u_{h_j}(x_0) + \int_{\gamma_x} \big(\nabla u_{h_j}(\gamma_x(t)), \gamma_x'(t)\big)\, dt\,. \tag{2.8}$$

By plugging this expression in (2.10) we obtain

$$\int_B |u_{h_j}(x_0) - u_{h_{j+1}}(x_0)|dx + \int_B |\int_{\gamma_x} \big((\nabla u_{h_j}(\gamma_x(t)) - \nabla u_{h_{j+1}}(\gamma_x(t))), \gamma_x'(t)\big)dt|dx > \epsilon$$

and thanks to (2.5) we deduce that definitely

$$\int_B \Big(\int_{\gamma_x} |\nabla u_{h_j}(\gamma_x(t)) - \nabla u_{h_{j+1}}(\gamma_x(t))||\gamma_x'(t)|dt\Big)dx > \frac{\epsilon}{2} \tag{2.9}$$

that is

$$\begin{aligned}
\frac{\epsilon}{2} &< \int_B \int_0^1 |(\nabla u_{h_j} - \nabla u_{h_{j+1}})(\gamma_x(t))||\gamma_x'(t)|dtdx \\
&= \int_0^1 \int_B |(\nabla u_{h_j} - \nabla u_{h_{j+1}})(\gamma_x(t))||\gamma_x'(t)|dxdt \\
&\leqslant \max_{t\in[0,1]} |\Big[\int_B |(\nabla u_{h_j} - \nabla u_{h_{j+1}})(\gamma_x(t))||\gamma_x'(t)|dx\Big] \\
&= \Big[\int_B |(\nabla u_{h_j} - \nabla u_{h_{j+1}})(\gamma_x(\overline{t}))||\gamma_x'(\overline{t})|dx\Big] \\
&\leqslant \max_{x\in\overline{B}} |\gamma_x'(\overline{t})| \int_B |\nabla u_{h_j}(\gamma_x(\overline{t})) - \nabla u_{h_{j+1}}(\gamma_x(\overline{t}))|dy\,.
\end{aligned}$$

Since $|\gamma_x'(0)| = 0$ for every $x \in B$ the maximum point $\overline{t}$ will not be 0 and then $\omega_{\overline{t}}$ will not be $\{x_0\}$ but an open set. Consequently we can estimate the last term and obtain

$$\frac{\epsilon}{2} < \max_{x\in\overline{B}} \Big|\frac{\partial\Gamma}{\partial t}(x,\overline{t})\Big| \max_{x\in\overline{B}} |J_{\frac{\partial\Gamma}{\partial x}(x,\overline{t})}| \int_{\omega_{\overline{t}}} |\nabla u_{h_j}(y) - \nabla u_{h_{j+1}}(y)|dy$$

where $|J_{\frac{\partial\Gamma}{\partial x}}|$ is the Jacobian. Since $(\nabla u_h)_h$ is a Cauchy sequence in $(L^1(\omega))^n$ the last inequality is impossible. $\square$

Remark 2.2 - The lemma also holds for a sequence

$$(u_h)_h \subset C^1_{\rm c}(\Omega)\,, \qquad \nabla u_h \text{ is a Cauchy sequence in } (L^1_{\rm loc}(\Omega))^n\,, \qquad \Omega \text{ bounded}$$

or a sequence (Q is a cube of $\mathbf{R}^n$)

$$(u_h)_h \subset C^1_{\rm per}(Q)\,, \qquad ⨍_Q u_h \to \alpha \in \mathbf{R}\,, \qquad \nabla u_h \text{ is a Cauchy sequence in } (L^1(Q))^n\,.$$

In fact if $(u_h)_h \subset C^1_{\rm c}(\Omega)$ we have the existence of a constant $c > 0$ such that

$$\int_\Omega |u_h| \leqslant c \int_\Omega |\nabla u_h|.$$

If, by contradiction, we suppose that u_h is not converging in $L^1(\Omega)$ we have that, for $\epsilon > 0$, there exist a ball B and a subsequence $(u_{h_j})_j$ such that

$$\int_B |u_{h_{j+1}} - u_{h_j}| > \epsilon\,.$$

In particular we would have

$$\epsilon < \int_B |u_{h_{j+1}} - u_{h_j}| \leqslant c \int_B |\nabla u_{h_{j+1}} - \nabla u_{h_j}| \to 0\,.$$

The same is obtained if u_h is periodic and its average is converging, using the inequality

$$\int_Q |u_h - \fint_Q u_h| \leqslant c \int_Q |\nabla u_h|\,.$$

Theorem 2.3 *Consider Ω a connected open set of $\mathbf{R}^n$, $\mu, \lambda_1, \ldots\lambda_n \in \mathcal{A}(\Omega)$ such that $A = \{x \in \Omega \mid \mu(x) > 0\}$ has positive measure and $\lambda_i \in B_p(\Omega)$ for $i = 1, \ldots n$. Then the space of functions $W^{1,p}(\Omega, \mu, \lambda_1, \ldots\lambda_n)$ is complete with respect to the topology induced by the norm* (2.4).

Proof - Consider a sequence of functions $(u_h)_h$ and suppose it is a Cauchy sequence with respect to the norm (2.4). Then we have that there exist $n+1$ functions $v_0, v_1, \ldots v_n$ such that

$$\begin{array}{ll} u_h \to v_0 & \text{in } L^p(\Omega, \mu) \\ D_i u_h \to v_i & \text{in } L^p(\Omega, \lambda_i). \end{array}$$

What is to prove is that these functions belong to $L^1_{\rm loc}(\Omega)$ and that $v_i = D_i v_0$, or better that there exists a representative of v_0 in $L^1_{\rm loc}(\Omega)$ which has v_i as derivatives.
Since $L^p(\Omega, \lambda_i) \subset L^1_{\rm loc}(\Omega)$ we immediatly have that $v_i \in L^1_{\rm loc}(\Omega)$ for $i = 1, \ldots n$. Let us now see that $v_0 \in L^1_{\rm loc}(\Omega)$. By contradiction suppose that $(u_h)_h$ is not a Cauchy sequence in $L^1_{\rm loc}(\Omega)$. Then there exists a subsequence $(u_{h_j})_j$, a ball $B_r \subset\subset \Omega$ whose radius is r and $\epsilon > 0$ such that

$$\int_{B_r} |u_{h_j} - u_{h_{j+1}}|\, dx > 2\epsilon \qquad \text{for every } j. \tag{2.10}$$

Since $|A| > 0$ and $|A| = \lim_{\delta \to 0^+} |A_\delta|$ where $A_\delta = \{x \in \Omega \mid \mu(x) > \delta\}$, there exists $\delta > 0$ such that $|A_\delta| > 0$ and $L^p(A_\delta, \mu) \subset L^p(A_\delta)$ with continuous embedding. We can also suppose $A_\delta \subset\subset \Omega$ (otherwise consider a subset A'_δ of A_δ such that $A'_\delta \subset\subset \Omega$ and $|A'_\delta \cap A| > 0$) in such a way that $L^p(A_\delta, \mu) \subset L^1(A_\delta)$.
Consider ω an open and connected subset of Ω such that

$$B_r \cup A_\delta \subset\subset \omega \subset\subset \Omega$$

and moreover such that ω is diffeomorphic to a ball.
Since $(u_h)_h \subset W^{1,1}_{\rm loc}(\Omega)$ for every $\sigma > 0$ and for every j there exists $w_j \in C^1(\Omega)$ such that

$$\|w_j - u_{h_j}\|_{W^{1,1}(\omega)} < \sigma\,. \tag{2.11}$$

Choosing $\sigma \leqslant \epsilon/2$ from (2.10) one in particular has

$$\|w_j - w_{j+1}\|_{L^1(B_r)} > \epsilon \qquad \text{for every } j.$$

On the other hand since $L^p(A_\delta, \mu) \subset L^1(A_\delta)$ from (2.11) $(w_j)_j$ is a Cauchy sequence in $L^1(A_\delta)$ and in particular that there exists a set N of measure zero such that, up to subsequences,

$$w_j(x) \to v_0(x) \quad \text{for every } x \in A_\delta \setminus N.$$

Choose $x_0 \in A_\delta \setminus N$ such that

$$w_j(x_0) \to v_0(x_0). \tag{2.12}$$

Applying the previous lemma to the sequence $(w_j)_j$ we conclude that $(w_j)_j$, and then $(u_h)_h$, is a Cauchy sequence in $L^1(\omega)$, but this fact is in contradiction with (2.10).
Therefore $(u_h)_h$ admits a limit v^* in $L^1_{\text{loc}}(\Omega)$ and since $D_i u_h \to v_i$ in $L^1_{\text{loc}}(\Omega)$ we conclude that $(u_h)_h$ is a Cauchy sequence in $W^{1,1}_{\text{loc}}(\Omega)$ and then

$$v_i = D_i v^*.$$

The last thing to be verified is that $v^* = v_0$ in $L^p(\Omega, \mu)$. Since $u_h \to v_0$ in $L^p(\Omega, \mu)$, $u_h \to v_0$ in $L^p(A_\delta)$ for every $\delta > 0$ and consequently in $L^1_{\text{loc}}(A)$. On the other hand $u_h \to v^*$ in $L^1_{\text{loc}}(\Omega)$ and in particular in $L^1_{\text{loc}}(A)$. Then $v_0 = v^*$ a. e. in A and v^* is a representative of v_0 in $L^p(\Omega, \mu)$. □

REMARK 2.4 - Observe that, in the hypotheses of Theorem 2.3, a Cauchy sequence $(u_h)_h$ in $W^{1,p}(\Omega, \mu, \lambda_1, \ldots \lambda_n)$ is a Cauchy sequence in $W^{1,1}_{\text{loc}}(\Omega)$.

This result is an extension of the result contained in [14]. In this paper the authors show the completeness of $W^{1,p}(\Omega, \mu, \lambda_1, \lambda_n)$ with $\mu > 0$ a. e. in Ω and not $\mu \geqslant 0$. The authors in [14] show the sharpness af the assumptions $\lambda_i \in B_p(\Omega)$. As regards the other hypotheses, if we drop the assumption that Ω is connected the previous result is not true in general: it is sufficient to consider, if $n = 1$, $\Omega = (0,1) \cup (1,2)$, $\mu \equiv 1$ on $(0,1)$ and $\mu \equiv 0$ on $(1,2)$, $\lambda \equiv 1$ on Ω. Then the sequence $u_n = 1$ on $(0,1)$ and $u_n = n$ on $(1,2)$ is a Cauchy sequence in $W^{1,p}(\Omega, \mu, \lambda)$ for every p, but it evidently does not admit a limit in $W^{1,1}_{\text{loc}}(\Omega)$. In the same way if we drop $|\{\mu > 0\}| > 0$ and consider $\mu = 0$ a. e. in Ω it is sufficient, for example, to consider a sequence of constants going to infinity.

To define a Sobolev space with more than one weak derivative, consider a family M of multi-indexes whose lenght is less than or equal to $k \in \mathbf{N}$, then a family of weights $\Lambda_M = \{\lambda_\alpha \mid \alpha \in M\}$ and define $W^{k,p}(\Omega, \Lambda_M)$ as the completion of the set of functions $u \in W^{k,1}_{\text{loc}}(\Omega)$ which satisfy

$$||u||^p_{\Lambda_M} \stackrel{def}{=} \sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha u|^p \lambda_\alpha dx < \infty \tag{2.13}$$

with respect to the norm (2.13).

Corollary 2.5 *Consider a connected open set Ω, the family M of multi-indexes α with $|\alpha| = 0, \ldots k$ and a set of weights $\Lambda_M = \{\lambda_\alpha | \alpha \in M\}$. Suppose there exist some subsets of Ω, A_α with $\alpha \in M$ and $|\alpha| < k$, with positive measure such that*

$$\lambda_\alpha > 0 \text{ in } A_\alpha \quad \text{if } |\alpha| < k, \qquad \lambda_\alpha \in B_p(\Omega) \quad \text{if } |\alpha| = k.$$

Then the space $W^{k,p}(\Omega, \Lambda_M)$ is complete.

Proof - By induction we suppose that for every integer value from 1 to $k-1$ the thesis is true. By M_j we denote the set $\{\alpha \in M \ : \ |\alpha| > 0 \text{ and } \alpha_j \neq 0\}$ and by Λ_j the set of weights Λ_{M_j}.
If we consider a sequence $(u_h)_h \subset W^{k,1}_{\text{loc}}(\Omega)$ which is a Cauchy sequence with respect to the norm (2.13) we have that, for every $j = 1, \ldots n$, the sequence $(D_j u_n)_n$ converges in $W^{k-1,p}(\Omega, \Lambda_j)$. In particular $(D_j u_n)_n$ admits a subsequence converging in $L^1_{\text{loc}}(\Omega)$. Now using Lemma 2.1 we conclude. □

Now we return to spaces with only *one* weak derivative. To have

$$C^1_c(\Omega) \subset W^{1,p}(\Omega, \mu, \lambda_1, ...\lambda_n), \tag{2.14}$$

so to define a space of functions which "are" zero on $\partial\Omega$, we need an additional request on our weights. Concerning this we recall the following result, contained in [14].

Lemma 2.6 *The inclusion* (2.14) *holds if and only if* $\mu, \lambda_1, ...\lambda_n \in L^1_{\rm loc}(\Omega)$.

Now if we suppose $\mu, \lambda_1, ...\lambda_n$ to be $L^1_{\rm loc}(\Omega)$ (with $\lambda_1, ...\lambda_n \in B_p(\Omega)$), we can define the space $W^{1,p}_0(\Omega, \mu, \lambda_1, ...\lambda_n)$ as the completion in $W^{1,p}(\Omega, \mu, \lambda_1, ...\lambda_n)$ of $C^1_c(\Omega)$.
In the same way we define $W^{1,p}_{\rm per}(Q, \mu, \lambda_1, ...\lambda_n)$ as the completion of $C^1_{\rm per}(Q)$ where Q is a cube of $\mathbf{R}^n$ and $C^1_{\rm per}(Q)$ the subset of the Q-periodic functions of $C^1(Q)$ and $W^{1,p}_{\#}(\Omega, \mu, \lambda_1, ...\lambda_n)$ as $W^{1,p}_{\rm per}(\Omega, \mu, \lambda_1, ...\lambda_n)/\mathbf{R}$.

Corollary 2.7 *Consider a connected and bounded open set* Ω *of* $\mathbf{R}^n$ *with Lipschitz boundary. Consider* $\mu, \lambda_1, ...\lambda_n \in \mathcal{A}(\Omega)$ *such that* $\mu, \lambda_i \in L^1_{\rm loc}(\Omega)$ *and* $\lambda_i \in B_p(\Omega)$ *for* $i = 1, ...n$ *(possibly* $\mu \equiv 0$*). Then the completion of* $C^1_c(\Omega)$ *and of* $C^1_{\#}(\Omega)$ *if* Ω *is a cube with respect to the norm*

$$\|u\|^p = \int_\Omega |u|^p \mu dx + \sum_{i=1}^n \int_\Omega |D_i u|^p \lambda_i dx$$

is a Banach space.

Proof - The conclusion follows from Theorem 2.3 if $|\{\mu > 0\}| > 0$ and in the same way using Remark 2.2 if $|\{\mu > 0\}| = 0$. □

3 Some recalls

Because of the results of the previous section with the word *weight* we will mean a function ν such that

$$\nu \text{ weight if:} \quad \nu \geqslant 0 \;\text{ a. e. in } \mathbf{R}^n \quad \text{and} \quad \nu \in L^1_{\rm loc}(\mathbf{R}^n).$$

From now on μ and λ will be functions such that

$$\mu, \lambda, \lambda^{-1} \qquad \text{weights} \tag{3.1}$$

and we will define (as in (2.3) with $p = 2$ and $\lambda_i = \lambda$ for every $i = 1, ...n$)

$$H^1(\Omega, \mu, \lambda) = \left\{ u \in W^{1,1}_{\rm loc}(\Omega) \;\middle|\; u \in L^2(\Omega, \mu) \text{ and } |Du| \in L^2(\Omega, \lambda) \right\}.$$

With hypotheses (3.1), by the results of the previous section, the space $H^1(\Omega, \mu, \lambda)$ endowed with the topology induced by the norm

$$\|u\|_{H^1(\Omega,\mu,\lambda)} := \left(\int_\Omega (u^2 \mu + |Du|^2 \lambda) dx \right)^{1/2} \tag{3.2}$$

is a separable Hilbert space and that $C^1_c(\Omega) \subset H^1(\Omega, \mu, \lambda)$. We will denote by $H^1_0(\Omega, \mu, \lambda)$ the closure of $C^1_c(\Omega)$ in the topology of $H^1(\Omega, \mu, \lambda)$ and by $H^{-1}(\Omega, \mu, \lambda)$ its dual space. If Q is a cube of $\mathbf{R}^n$ we denote

$$H^1_{\rm per}(Q, \mu, \lambda) = \text{ completion of } C^1_{\rm per}(Q) \text{ with respect to the norm (3.2),}$$

and

$$H^1_\#(Q,\mu,\lambda) := H^1_{\rm per}(Q,\mu,\lambda)/\mathbf{R} \tag{3.3}$$

endowed with the norm (3.2) with Ω replaced by Q and by $H^{-1}_\#(Q,\mu,\lambda)$ its dual space.

REMARK 3.1 - Let Ω be a bounded open set and Q_Ω be a cube containing Ω. Then $H^1_0(\Omega,\mu,\lambda)$ is a subspace of $H^1_\#(Q_\Omega,\mu,\lambda)$.

The following definition is usually given for pairs (μ,λ) with $\mu,\lambda>0$ a. e.: we extend this to pairs of *weights*.

Definition 3.2 *Let $p,q>1$, $K>0$ be constants, μ,λ two weights, $\lambda>0$ a. e., $\alpha\in[0,n)$ and Q_0 a cube in $\mathbf{R}^n$ (even the whole $\mathbf{R}^n$ if $\alpha=0$). We say that the pair (μ,λ) belongs to the class $A^\alpha_{p,q}(Q_0,K)$ if*

$$|Q|^{\frac{\alpha}{n}}\left(\fint_Q \mu dx\right)^{1/q}\left(\fint_Q \lambda^{-1/(p-1)}dx\right)^{(p-1)/p}\leqslant K \quad \textit{for every cube } Q\subset Q_0. \tag{3.4}$$

We denote by $A^\alpha_{p,q}(Q_0)$ the class $\bigcup_{K>0}A^\alpha_{p,q}(Q_0,K)$. Moreover we denote by $A_{p,q}(Q_0,K)$ the class $A^\alpha_{p,q}(Q_0,K)$ when $\alpha=0$ and by $A_{p,q}(K)$ the same class when $Q_0=\mathbf{R}^n$. Finally we denote by $A_p(Q_0,K)$ the Muckenhoupt class of weights for which (3.4) holds with $\alpha=0$, $p=q$ and $\mu\equiv\lambda$, by $A_p(K)$ the class $A_p(Q_0,K)$ with $Q_0=\mathbf{R}^n$, by $A_p=\bigcup_{K\geqslant1}A_p(K)$ and $A_\infty=\bigcup_{p>1}A_p$.

REMARK 3.3 - By definition we have that $A^\alpha_{p,q}(Q_0,K)\subset A^\beta_{p,q}(Q_0,|Q_0|^{(\beta-\alpha)/n}K)$ if $0\leqslant\alpha<\beta<n$; $A_p(K)\subset A_q(K)$ if $1<p<q<\infty$.

REMARK 3.4 - Consider $0<s<r<+\infty$, $1<q\leqslant p<+\infty$, $\alpha\in[0,n/r)$. Then if $(\mu^r,\lambda^r)\in A^{\alpha r}_{2,2}(Q_0,K)$ then $(\mu^s,\lambda^s)\in A^{\alpha s}_{2,2}(Q_0,K')$ $(K'=K^{s/r}|Q_0|^{\frac{r-s}{r}(\frac{1}{q}-\frac{1}{p})})$.

REMARK 3.5 - Let Q_0 be a cube, $K>0$, $p,q>1$, $\alpha\in[0,n)$ and consider μ,λ Q_0-periodic weights such that $(\mu,\lambda)\in A^\alpha_{p,q}(Q_0,K)$. Denote with μ_h and λ_h the weights defined

$$\mu_h(x)=\mu(hx),\quad \lambda_h(x)=\lambda(hx)\qquad \text{for } h\in\mathbf{N}.$$

Then for every cube $\widetilde{Q}$ there exists a constant $\widetilde{K}=\widetilde{K}(n,p,q,\widetilde{Q},K)$ (depending only on $n,p,q,\widetilde{Q}$ and K) such that $(\mu_h,\lambda_h)\in A^\alpha_{p,q}(\widetilde{Q},\widetilde{K})$ for every $h\in\mathbf{N}$.

A_p weights verify the following higher sommability property (see [4, 7]): for every $K\geqslant1$, for every $p>1$, there exist two positive constants $c=c(n,p,K)$ and $\delta=\delta(n,p,K)$ (depending only on n, p and K) such that

$$\left(\fint_Q \lambda^{1+\delta}dx\right)^{\frac{1}{(1+\delta)}}\leqslant c\left(\fint_Q \lambda dx\right),\quad \left(\fint_Q \lambda^{-\frac{1+\delta}{p-1}}dx\right)^{\frac{1}{(1+\delta)}}\leqslant c\left(\fint_Q \lambda^{-\frac{1}{p-1}}dx\right), \tag{3.5}$$

for every cube Q and $\lambda\in A_p(K)$. A_p weights also verify the *doubling property*, i. e. if $\lambda\in A_p(K)$ one has that for every $t>0$ there exists a constant $c=c(t,n,p,K)$ (depending only on t,n,p,K) such that

$$\lambda(tQ)\leqslant c\lambda(Q) \tag{3.6}$$

for every cube Q of $\mathbf{R}^n$ (see for instance [13]). The condition (3.6) implies the *reverse doubling* property defined as:

$$\text{there exist } \delta,\epsilon\in(0,1) \text{ such that}\quad \lambda(\delta Q)\leqslant\epsilon\lambda(Q)\qquad \text{for every cube } Q \text{ of } \mathbf{R}^n. \tag{3.7}$$

Remark 3.6 - If ν is a weight satisfying (3.7) with constants $\epsilon, \delta \in (0,1)$, then the weights

$$\nu_h(x) := \nu(hx), \qquad h \in \mathbf{N}$$

satisfy (3.7) with the same constants ϵ, δ.

If Q_0 is a cube and $\lambda \in A_p(K)$ (see [12]) there exists a constant $c = c(n,p,K,Q_0)$ (depending only on n, p, K_1, Q_0), such that

$$\int_{Q_0} |u|^p \lambda dx \leqslant c \int_{Q_0} |Du|^p \lambda dx \tag{3.8}$$

for every $u \in C_0^\infty(Q_0)$. This result is generalized in [28] in the following way.

Theorem 3.7 *Suppose Q_0 is a proper cube of $\mathbf{R}^n$, $1 < p \leqslant q < +\infty$, and u is a Lipschitz continuous function defined on Q_0, with either support in Q_0 or $\int_{Q_0} f(x)dx = 0$. Consider $\mu, \lambda > 0$ a. e., weights on $\mathbf{R}^n$. Then*

$$\Big[\int_{Q_0} |u(x)|^q \mu(x) dx\Big]^{1/q} \leqslant \mathcal{C}(\mu,\lambda,Q_0)\Big[\int_{Q_0} |Du(x)|^p \lambda(x) dx\Big]^{1/p}$$

where

$$\mathcal{C}(\mu,\lambda,Q_0) = c(p,r) \sup_{Q \subset 8Q_0} |Q|^{1/n}\Big(⨍_Q \mu^r\Big)^{1/pr}\Big(⨍_Q \lambda^{r(1-p')}\Big)^{(p-1)/pr}$$

for any $r > 1$ if $q = p$,

$$\mathcal{C}(\mu,\lambda,Q_0) = c(p,q) \sup_{Q \subset 8Q_0} |Q|^{1/n-1}\Big(\int_Q \mu\Big)^{1/q}\Big(\int_Q \lambda^{1-p'}\Big)^{(p-1)/p}$$

if $p < q$ and μ satisfies (3.7), or if $p = q$ and both μ and $\lambda^{1-p'}$ satisfy (3.7).

Remark 3.8 - It is possible to extend this theorem to pairs of weights (μ, λ) even when $\mu \geqslant 0$, at least when $p = q$. Indeed consider a pair (μ, λ) with $\mu \geqslant 0$ and $\lambda > 0$ a. e. and suppose $\mathcal{C}(\mu,\lambda,Q_0)$ is finite. It is sufficient to consider the weight

$$\overline{\mu} = \begin{cases} \mu & \text{where } \mu > 0 \\ \lambda & \text{where } \mu = 0 \end{cases} \tag{3.9}$$

and observe that Theorem 3.7 holds for the pair $(\overline{\mu}, \lambda)$ thanks to (3.5) and Remark 3.3.

If the following weighted Poincaré type inequality

$$\int_Q u^2 \mu dx \leqslant c \int_Q |Du|^2 \lambda dx \quad \text{for every } u \in H^1_\#(Q,\mu,\lambda) \tag{3.10}$$

holds for a suitable positive constant c, then the norm

$$\|u\|_{H^1_0(Q,\mu,\lambda)} = \|u\|_{H^1_\#(Q,\mu,\lambda)} := \Big(\int_Q |Du|^2 \lambda dx\Big)^{1/2} \quad u \in H^1_\#(Q,\mu,\lambda) \tag{3.11}$$

is equivalent on $H^1_\#(Q,\mu,\lambda)$ (and on $H^1_0(Q,\mu,\lambda)$) to the one of $H^1(Q,\mu,\lambda)$. Then, by (3.8) and Theorem 1.4 in [29], if Ω is a bounded open set with Lipschitz boundary and $\lambda \in A_2$, we have that for every μ such that the pair (μ,λ) satisfies hypotheses of Theorem 3.7 we have that

$$H^1_0(\Omega,\mu,\lambda)=H^1(\Omega,\lambda,\lambda)\cap W^{1,1}_0(\Omega)\,,\qquad H^1_\#(Q_0,\mu,\lambda)=H^1(Q_0,\lambda,\lambda)\cap W^{1,1}_\#(Q_0)\,. \quad (3.12)$$

If $\mathcal{C}(\mu,\lambda,Q_0)$ is finite, then from Theorem 3.7 and Remark 3.8 inequality (3.10) holds. In this case we will endow $H^1_0(Q_0,\mu,\lambda)$ and $H^1_\#(Q_0,\mu,\lambda)$ with the norm (3.11).

By (3.5), (3.6) and Theorem 3.7 if Ω is a bounded open set and Q_Ω a cube containing Ω with $(\mu,\lambda)\in A^\alpha_{2,2}(Q_\Omega,K_2)$ with $\alpha<1$ (see also Remark 3.3) and $\lambda\in A_2(K_1)$ there exist $\sigma=\sigma(n,K_1)>0$ (depending only on n,K_1) and $c_i=c_i(n,K_1,K_2,\Omega)>0$ (depending only on n,K_1,K_2,Ω), $i=1,2$, such that

$$c_1\ \|\lambda^{-1}\|^{-1/2}_{L^1(Q_\Omega)}\||Du|\|_{L^{1+\sigma}(Q_\Omega)}\leqslant\||Du|\|_{L^2(Q_\Omega,\lambda)}\leqslant c_2\ \|\lambda\|^{1/2}_{L^1(Q_\Omega)}\||Du|\|_{L^{(1+\sigma)'}(Q_\Omega)} \quad (3.13)$$

for every $u\in W^{1,(1+\sigma)'}_\#(Q_\Omega)$, so that

$$\begin{gathered}W^{1,(1+\sigma)'}_0(\Omega)\subset H^1_0(\Omega,\mu,\lambda)\subset W^{1,1+\sigma}_0(\Omega),\quad H^{-1}(\Omega,\mu,\lambda)\subset W^{-1,1+\sigma}(\Omega)\\ W^{1,(1+\sigma)'}_\#(Q_\Omega)\subset H^1_\#(Q_\Omega,\mu,\lambda)\subset W^{1,1+\sigma}_\#(Q_\Omega),\ H^{-1}_\#(Q_\Omega,\mu,\lambda)\subset W^{-1,1+\sigma}_\#(Q_\Omega).\end{gathered}$$

Moreover, since $H^1_0(\Omega,\mu,\lambda)$ continuously embeds in $W^{1,1}(\Omega)$ which continuously embeds in $L^{n/(n-1)}(\Omega)$ if $n\geqslant 2$ (and in $L^2(\Omega)$ if $n=1$), there exists a positive constant $c_3=c_3(n,\Omega)$ (depending only on n and Ω) such that

$$\|f\|_{H^{-1}(\Omega,\mu,\lambda)}\leqslant c_3\ \left(\int_\Omega\lambda^{-1}dx\right)^{1/2}\|f\|_{L^n(\Omega)}. \quad (3.14)$$

We present here an approximation result contained in [23] which can be proved extending Theorem 2.9 and Corollary 2.10 in [23] to pairs (μ,λ) in which $\mu\geqslant 0$ (the proofs are the same).
We fix ρ a radial non-negative function in $C^\infty(\mathbf{R}^n)$, supported in $B(0,1)$, such that $\int_{\mathbf{R}^n}\rho=1$, and and consider $(\rho_\epsilon)_{\epsilon>0}$ a family of mollifiers, i. e. $\rho_\epsilon(x)=\epsilon^{-n}\rho(x/\epsilon)$. We define

$$v_\epsilon(x,t)\stackrel{def}{=}\int_{\mathbf{R}^n}\rho_\epsilon(x-y)\overline{v}(y,t)dy\qquad(x,t)\in\mathbf{R}^n\times[0,T]. \quad (3.15)$$

if $v\in L^2(0,T;H^1_0(\Omega,\mu,\lambda))$.

Proposition 3.9 *Let Ω be a bounded open set of $\mathbf{R}^n$, $T,K_2>0$, $K_1\geqslant 1$, $\alpha\in[0,1)$, Q_Ω a cube union of dyadic cubes and containing Ω. Then there exists a positive constant $c=c(n,\rho,\alpha,K_1,K_2)$ (depending only on n,ρ,α,K_1,K_2) such that:*

$$\int_0^T\int_{Q_\Omega}|v_\epsilon-v|^2\mu dxdt\leqslant c\ \epsilon^{2(1-\alpha)}\int_0^T\int_{Q_\Omega}|Dv|^2\lambda dxdt$$

(a) either for every $\epsilon>0$, for every pair of weights μ,λ such that $\lambda\in A_2(K_1)$ and $(\mu,\lambda)\in A^\alpha_{2,2}(3Q_\Omega,K_2)$, for every $v\in L^2(0,T;H^1_0(\Omega,\mu,\lambda))$,

(b) or for ϵ small enough (i. e. for every $\epsilon\in(0,\epsilon_0]$ for a e_0), for every pair of weights μ,λ such that $\lambda\in A_2(K_1)$ and $(\mu,\lambda)\in A^\alpha_{2,2}(9Q_\Omega,K_2)$, for every $v\in L^2(0,T;H^1_\#(Q_\Omega,\mu,\lambda)$.

4 Strongly degenerate parabolic equations

In this section we give a result of existence (and uniqueness when boundary conditions are required) for the solution to an abstract strongly degenerate evolution equation

$$Ru' + \mathcal{A}u = f\,,$$

where *strongly* refers to the fact that $R \geqslant 0$. The result about existence of such a problem are contained in [22] and now we are going to set out some results contained there. We are interested to the situation $R \geqslant 0$ (and $\mathcal{A}$ linear), even if actually in [22] a more general situation is considered, i. e. R is not necessarily non-negative. Moreover here we are also interested in problem (4.6)-(II), while in [22] only problems with initial datum is considered.

Consider a reflexive Banach space V and a Hilbert space H and identify H' with H in such a way the embeddings

$$V \subset H \subset V'$$

are continuous and dense. Given $T > 0$ we define the spaces

$$\mathcal{H} \;= L^2\big(0,T;H\big), \quad \mathcal{V} \;= L^2\big(0,T;V\big), \quad \mathcal{V}' \;= L^2\big(0,T;V'\big), \tag{4.1}$$

endowed with the standard norms

$$\|u\|_{\mathcal{H}} = \Big(\int_0^T \|u(t)\|^2_H dt\Big)^{1/2}, \quad \|u\|_{\mathcal{V}} = \Big(\int_0^T \|u(t)\|^2_V dt\Big)^{1/2}, \quad \|u\|_{\mathcal{V}'} = \Big(\int_0^T \|u(t)\|^2_{V'} dt\Big)^{1/2}.$$

Let R be a non-negative and linear bounded self-adjoint operator, $R : H \to H$. We denote by H_0 the kernel of R and by H_+ its orthogonal space. Since R is positive, it admits a square root $R^{1/2}$ which is a positive, linear and bounded self-adjoint operator (see, for instance, [26]). Denote by $H_+^{1/2}$ the completion of H_+ with respect to the topology induced by the norm

$$\|u\|_+ := \|R^{1/2}u\|_H$$

which satisfies

$$\|u\|_+ \leqslant \|R\|^{1/2}\|u\|_H\,.$$

Observe that, since $R^{1/2}$ is self-adjoint, we have that

$$\big(Ru, v\big)_H = \big(R^{1/2}u, R^{1/2}v\big)_H = \big(u, v\big)_{H_+}.$$

Observe moreover that Ker $R^{1/2}$ = Ker R. Denote by P_+ the orthogonal projection from H onto H_+ and by R_+ the operator $R|_{H_+} : H_+ \to H$. Observe that

$$Ru = RP_+u \quad \text{ and } \quad R^{1/2}u = R^{1/2}P_+u\,.$$

We extend the operators $R, R^{1/2}, P_+ : H \to H$ to some operators denoted by the same letters $R, R^{1/2}, P_+ : \mathcal{H} \to \mathcal{H}$ in the following way

$$Ru(t) := R(u(t))\,, \qquad R^{1/2}u(t) := R^{1/2}(u(t))\,, \qquad P_+u(t) := P_+(u(t))\,.$$

Consider the space

$$\mathcal{W} = \big\{v \in \mathcal{V} \mid Rv' \in \mathcal{V}'\big\}\,, \tag{4.2}$$

where Rv' stands for $(Rv)'$ and the derivative is to be intended in the sense of ditributions, endowed with the norm

$$\|u\|_{\mathcal{W}} = \|u\|_{\mathcal{V}} + \|Ru'\|_{\mathcal{V}'}\,.$$

The first result in [22] is the following.

Proposition 4.1 *The operator P_+ maps $\mathcal{W}$ into $C([0,T];H_+^{1/2})$ and the mapping is continuous, i. e. there is a constant $c > 0$ such that*

$$\|P_+u\|_{C([0,T];H_+^{1/2})} = \max_{t\in[0,T]} \|R^{1/2}u(t)\|_H \leqslant c\ \|u\|_{\mathcal{W}}\,.$$

Moreover

$$\big\langle Ru',v\big\rangle_{\mathcal{V}'\times\mathcal{V}} + \big\langle Rv',u\big\rangle_{\mathcal{V}'\times\mathcal{V}} = \big(Ru(T),v(T)\big)_H - \big(Ru(0),v(0)\big)_H\,.$$

Thanks to this last result, given $\psi \in H_+^{1/2}$, we can define the space

$$\mathcal{W}^\psi = \big\{u\in\mathcal{W}\mid P_+u(0)=\psi\big\} \qquad\text{and}\qquad \mathcal{W}^{\mathrm{per}} = \big\{u\in\mathcal{W}\mid P_+u(0)=P_+u(T)\big\}$$

($\mathcal{W}^0$ will be the space with $\psi = 0$). Consider now a family of linear operators defined on V

$$A(t): V \longrightarrow V', \qquad 0\leqslant t\leqslant T$$

such that

$$\begin{cases} t\mapsto \big\langle A(t)u,v\big\rangle_{V'\times V} & \text{is measurable on } [0,T] \\ \|u\|_V^2 \leqslant \big\langle A(t)u,u\big\rangle_{V'\times V} \leqslant L\|u\|_V^2 & \\ |\big\langle A(t)u,v\big\rangle_{V'\times V}| \leqslant M\big\langle A(t)u,u\big\rangle_{V'\times V}^{1/2}\big\langle A(t)v,v\big\rangle_{V'\times V}^{1/2} & \end{cases} \tag{4.3}$$

for every $u,v\in V$. Then we can construct the abstract operators

$$\mathcal{A}:\mathcal{V}\longrightarrow\mathcal{V}' \qquad \text{where}\quad \mathcal{A}u\ (t)\stackrel{def}{=}A(t)u(t) \qquad 0\leqslant t\leqslant T, \tag{4.4}$$

$$\mathcal{P}:\ \mathcal{W}\longrightarrow\mathcal{V}', \qquad (\mathcal{P}u)\ (t)\ =\ Ru'(t)+\mathcal{A}u(t) \qquad 0\leqslant t\leqslant T\,. \tag{4.5}$$

Consider the strongly degenerate parabolic (or elliptic-parabolic) problems

$$\text{(I)}\quad \begin{cases} Ru'+\mathcal{A}u=f \\ P_+u(0)=\psi \end{cases} \qquad\qquad \text{(II)}\quad \begin{cases} Ru'+\mathcal{A}u=f \\ P_+u(0)=P_+u(T) \end{cases} \tag{4.6}$$

where $f\in\mathcal{V}'$ and $\psi\in H_+^{1/2}$. The solution to these problems is to intended it the following way: u is a solution if, first of all,

$$u\in\mathcal{W} \quad\text{ and }\quad \big\langle (Ru)'(t)+\mathcal{A}u(t),v\big\rangle_{V'\times V} = \big\langle f(t),v\big\rangle_{V'\times V} \tag{4.7}$$

for almost every $t\in[0,T]$ and for every $v\in V$; moreover $P_+u(0)=\psi$ in $H_+^{1/2}$ in the first case, $P_+u(0)=P_+u(T)$ in $H_+^{1/2}$ in the second case.
Define the unbounded operators $L_i : D(L_i)\subset\mathcal{V}\to\mathcal{V}'$, $i=1,2$, where

$$L_1v = Rv'\,, \qquad D(L_1)=\{v\in\mathcal{W}\mid P_+v(0)=0\}\,,$$

$$L_2v = Rv'\,, \qquad D(L_2)=\{v\in\mathcal{W}\mid P_+v(0)=P_+v(T)\}\,.$$

Proposition 4.2 *The operators L_i are densely defined closed operators. Moreover L_i are maximal monotone.*

Proof - One may adact the corresponding proof in [37] (see Proposition 32.10 in vol. II/B).
□

Theorem 4.3 *For every $f \in \mathcal{V}'$ and every $\psi \in H_+^{1/2}$ the problem* (4.6)-(I) *admits a unique solution. For every $f \in \mathcal{V}'$ the problem* (4.6)-(II) *admits a unique solution.*
Moreover

(*i*) $\mathcal{P} : \mathcal{W} \longrightarrow \mathcal{V}'$ *is linear and continuous and there exist a constant $c = c(L, M)$ (depending only on L, M) such that*

$$\|\mathcal{P}u\|_{\mathcal{V}'} \leqslant c\, \|u\|_{\mathcal{W}} \qquad \text{for every } u \in \mathcal{W},$$

(*ii*) $\langle \mathcal{P}u, u\rangle \geqslant \|u\|^2_{\mathcal{V}}$ *for every $u \in \mathcal{W}^0$ and for every $u \in \mathcal{W}^{\mathrm{per}}$,*

(*iii*) *for every $\psi \in H_+^{1/2}$ if $u \in \mathcal{W}^\psi$ is the solution to* (4.6)-(I) *there exists a constant $c = c(M, L)$ (depending only on M, L), such that*

$$\|u\|_{\mathcal{W}} \leqslant c\,\Big[\|\mathcal{P}u\|_{\mathcal{V}'} + \|R^{1/2}u(0)\|_H\Big]\,,$$

(*iv*) *if $u \in \mathcal{W}^{\mathrm{per}}$ is the solution to* (4.6)-(II) *there exists a constant $c = c(M, L)$ (depending only on M, L), such that*

$$\|u\|_{\mathcal{W}} \leqslant c\, \|\mathcal{P}u\|_{\mathcal{V}'}\,.$$

Proof - For the proof for the problem (4.6)-(I) with $\psi = 0$ and for the problem (4.6)-(II) one can follow, e. g., the proof of Theorem 32.D in [37] and use Proposition 4.2 since R is maximal monotone and $\mathcal{A}$ is bounded, coercive and strictly monotone. For the general problem (4.6)-(I) we refer to [22] (the idea is to consider the equations $(R+\epsilon(\mathrm{Id}_H - P_+))u' + \mathcal{A}u = f$ with an initial datum $u_{0,\epsilon}$ such that $P_+u_{0,\epsilon} \to \psi$ in $H_+^{1/2}$ and then let ϵ go to zero). □

Remark 4.4 - The unbounded operators $L^{(i)} : D(L^{(i)}) \subset \mathcal{V} \to \mathcal{V}'$, $i = 1, 2$, where

$$\begin{aligned} L^{(1)}v &= -Rv'\,, \qquad & D(L^{(1)}) &= \{v \in \mathcal{W} \mid P_+v(T) = 0\}\,, \\ L^{(2)}v &= Rv'\,, \qquad & D(L^{(2)}) &= \{v \in \mathcal{W} \mid P_+v(0) = P_+v(T)\}\,, \end{aligned}$$

satisfy Proposition 4.2 and then the following problems have an unique solution

$$\begin{cases} -Ru' + \mathcal{A}^*u = f \\ P_+u(T) = 0 \end{cases} \qquad\qquad \begin{cases} -Ru' + \mathcal{A}^*u = f \\ P_+u(0) = P_+u(T)\,. \end{cases}$$

Here we present a regularity result of the solution of problem (4.6)-(I). For similar results for the classical equations see, e. g., [16], [15], [35], [2], [11]. The proof we present follows the one contained in [36]. Before we need to introduce a definition.

Definition 4.5 *We say that $B : [0, T] \to \mathcal{L}(X, X')$, X Banach space, is regular if, for every $u, v \in X$ the function*

$$t \mapsto \big\langle B(t)u, v\big\rangle_{X'\times X}$$

is absolutely continuos on $[0, T]$ and there exists a function $b \in L^1(0, T)$ such that

$$\Big|\frac{d}{dt}\big\langle B(t)u, v\big\rangle_{X'\times X}\Big| \leqslant b(t)\|u\|_X\|v\|_X \quad \text{for a. e. } t \in [0, T]\,.$$

Observe that $B'(t) : X \to X'$ is linear for almost every $t \in (0, T)$.

Remark 4.6 - If B is differentiable in the sense just defined the following fomula holds: if $u \in W^{1,p}(0,T;X)$ for any p, then

$$\big(B(t)u(t)\big)' = B'(t)u(t) + B(t)u'(t) \qquad \text{for a. e. } t \in (0,T)\,.$$

In the special case $B(t) = B$ for every t, taking $v \in L^p(0,T;X)$ and $u(t) = \displaystyle\int_0^t v(s)ds$, we also derive from the formula above that

$$B\int_0^t v(s)ds = \int_0^t Bv(s)ds\,. \tag{4.8}$$

About the operator $A : [0,T] \to \mathcal{L}(V,V')$ we will make the additional assumption (besides to (4.3)):

$$\begin{cases} \text{A is differentiable in the sense of Definition 4.5 for a. e. } t \in (0,T) \\ t \mapsto \big\langle A'(t)u, v\big\rangle_{V'\times V} \qquad\qquad \text{is measurable on } [0,T] \\ |\big\langle A'(t)u, v\big\rangle_{V'\times V}| \leqslant M'(t)\|u\|_V\|v\|_V \end{cases} \tag{4.9}$$

with $M' \in L^2(0,T)$, for almost every $t \in (0,T)$ and for every $u, v \in V$ (M' would be sufficient to be in $L^1(0,T)$, but we require $M' \in L^2(0,T)$ to prove the following result).

Theorem 4.7 *Consider the problem* (4.6)-(I). *Assume the operator A satisfies also* (4.9) *and $f \in H^1(0,T;V')$. Given $u_0 \in V$, $\psi = P_+u_0$ and said u the solution of* (4.6)-(I) *suppose moreover that $f(0) - A(0)u_0 \in \operatorname{Im} R$. Then the solution u satisfies*

$$u \in H^1(0,T;V)\,.$$

Proof - We consider the problem

$$\begin{cases} Ru' + \mathcal{A}u = f \\ P_+u(0) = \psi \end{cases}$$

with $\psi \in H^{1/2}_+$ (note that $P_+u(0) = P_+u_0$). From a formal point of view we have that

$$Ru'(0) = f(0) - A(0)u(0)$$

and if we derive the equation, again from a formal point of view, we have

$$Ru'' + \mathcal{A}'u + \mathcal{A}u' = f'\,.$$

Then we consider the following problem

$$\begin{cases} Rv' + \mathcal{A}v = f' - \mathcal{A}'u \\ P_+v(0) = P_+R_+^{-1}\big(f(0) - A(0)u_0\big)\,. \end{cases} \tag{4.10}$$

Define the function

$$w(t) = u_0 + \int_0^t v(s)ds\,.$$

Problem (4.10) has one solution $v \in \mathcal{W}$ and in particular $v \in L^2(0,T;V)$ so $w \in H^1(0,T;V)$. We conclude if we show that $w = u$. Integrating equation in (4.10), by (4.8), we obtain

$$Rv(t) - Rv(0) + \int_0^t (\mathcal{A}v)(s)ds = f(t) - f(0) - \int_0^t (\mathcal{A}'u)(s)ds\,.$$

Observe that $Rv(0) = RP_+v(0) = P_+Rv(0) = P_+\big(f(0) - A(0)u_0\big) = f(0) - A(0)u_0$ and that, thanks to Remark 4.6, $(Rw)' = Rv$. Then equation above is, since $w' = v$,

$$Rw'(t) = -A(0)u_0 + f(t) - \int_0^t (\mathcal{A}v)(s)ds - \int_0^t (\mathcal{A}'u)(s)ds\,.$$

On the other hand we have

$$\begin{aligned}\int_0^t (\mathcal{A}v)(s)ds = \int_0^t (\mathcal{A}w')(s)ds &= \mathcal{A}w(t) - \mathcal{A}w(0) - \int_0^t (\mathcal{A}'w)(s)ds = \\ &= \mathcal{A}w(t) - A(0)u_0 - \int_0^t (\mathcal{A}'w)(s)ds\end{aligned}$$

and then, by the two last equalities, we obtain

$$Rw'(t) = f(t) - \mathcal{A}w(t) + \int_0^t \big(\mathcal{A}'w(s) - \mathcal{A}'u(s)\big)ds\,.$$

Subtracting to this equation $Ru' + \mathcal{A}u = f$ we obtain

$$\begin{cases} R(w-u)'(t) + A(t)\big(w(t) - u(t)\big) = \displaystyle\int_0^t \big(\mathcal{A}'w(s) - \mathcal{A}'u(s)\big)ds \\ P_+w(0) = P_+u(0)\,. \end{cases}$$

This means that the function $w - u$ solves the following problem in $(0,T)$

$$\begin{cases} Ry'(t) + A(t)y(t) = h(s) \\ P_+y(0) = 0 \end{cases}$$

where $h(t) = \int_0^t \big(\mathcal{A}'w(s) - \mathcal{A}'u(s)\big)ds$. By hypotheses on $\mathcal{A}'$ we have that

$$\|\mathcal{A}'(w-u)(s)\|_{V'} \leqslant M'(s)\ \|w(s) - u(s)\|_V$$

for almost every $s \in (0,T)$ and then

$$\|h(t)\|_{V'} \leqslant \int_0^t \|\mathcal{A}'(w-u)(s)\|_{V'}ds \leqslant \int_0^t M'(s)\|w(s) - u(s)\|_V ds\,.$$

Let T' be the greatest value less or equal to T such that $w(t) = u(t)$ for a. e. $t \in [0,T']$. Then $h(t) = 0$ for $t \in [0,T']$. Then by Theorem 4.3, (iii), there is constant $c = c(L,M)$ such that

$$\begin{aligned}\int_0^T \|w(t) - u(t)\|_V^2 dt &= \int_{T'}^T \|w(t) - u(t)\|_V^2 dt \\ &\leqslant c\int_{T'}^T \|h(t)\|_{V'}^2 dt \leqslant c\int_{T'}^T \Big(\int_0^t M'(s)\|w(s) - u(s)\|_V ds\Big)^2 dt \\ &= c\int_{T'}^T \Big[\int_0^t |M'(s)|^2 ds \int_0^t \|w(s) - u(s)\|_V^2 ds\Big] dt \\ &\leqslant c\,(T - T')\|M'\|_{L^2(0,T)}^2 \|w - u\|_{L^2(T',T;V)}^2\end{aligned}$$

that is $1 \leqslant c\,(T-T')\|M'\|^2_{L^2(0,T)}$. Since every estimate is independent of T, letting T go to T' we obtain a contradiction and consequently $T' = T$. $\square$

Corollary 4.8 *In the same hypotheses as in Theorem* 4.7 *the solution* u *of* (4.6)-(I) *satisfies*

$$Ru'' \in L^2(0,T;V')$$

and moreover there exists a positive constant c *depending* (*only*) *on* $M, L, R^{1/2}$ *such that*

$$\|u\|_{\mathcal{V}} + \|u'\|_{\mathcal{V}} + \|Ru''\|_{\mathcal{V}'} + \|u\|_{C([0,T];V)} \leqslant c\,(\|f\|_{\mathcal{V}'} + \|f'\|_{\mathcal{V}'} + \|\psi\|_H))\,.$$

Proof - The thesis follows from the fact that the function v by which the function w in the proof of the preceeding theorem is defined is the solution to (4.10) and from the estimates of Theorem 4.3. $\square$

The result contained in [17] can be generalized as follows. In the sixth section we do not extend this result to moving spaces as done in [25] or in [23] (see Theorem 6.5), in general an analogous result is not true (see an example in Section 6), even if such a result will turn to be true for the solutions to the problems we will consider (see next section and Theorem 7.1).

Consider two reflexive Banach spaces B_0, B_1, B a Hilbert space such that $B_0 \subset B \subset B_1$ with continuous embeddings. Let $R : B \to B$ be a non-negative and linear bounded self-adjoint operator and P_+ the othogonal projection from B into $B_+ = (\mathrm{Im} R)^\perp$.

Lemma 4.9 *Suppose* $B_0 \subset B$ *is compact. For every* $\eta > 0$ *there exists* $c_\eta > 0$ *such that for every* $v \in V$

$$\|P_+ v\|_B \leqslant \eta \|v\|_{B_0} + c_\eta \|Rv\|_{B_1}\,.$$

Using this lemma one can prove the following theorem.

Theorem 4.10 *Suppose moreover that* $B \subset B_1$ *is compact. Fix two numbers* $p_0, p_1 \in (1,\infty)$ *and* $T > 0$. *The space* $\{v \in L^{p_0}(0,T;B_0) \mid (Rv)' \in L^{p_1}(0,T;B_1)\}$ *compactly embeds in* $L^{p_0}(0,T;B_+)$.

The corrisponing *classical* results is obtained when

$$R = \mathrm{Id}_B, \qquad \text{the identity of } B$$

and for the classical result corresponding to Proposition 4.1 we remand to Theorem 3.1, chap. 1 in [19], to Theorem 4.3 we remand to Theorems 4.1 and Theorems 6.1, chap. 3, in [19], to Lemma 4.9 and Theorem 4.10 we remand to Lemma 5.1 and Theorem 5.1 in [17], or, to quote some more recent books, see e. g. [37], chap. 23 or [36], sections 25 - 27.

5 Hypotheses and position of the problem

We denote with $\mathcal{Q}$ the cube $[0,1]^n$ and consider a $\mathcal{Q}$-periodic weight μ and a matrix $a = [a_{ij}(y,t)]_{i,j=1}^n$ defined in $\mathbf{R}^n \times [0,T]$ whose coefficients $a_{ij}(\cdot,t)$ are $\mathcal{Q}$-periodic functions and such that for a. e. $(y,t),(y,s) \in \mathcal{Q} \times [0,T]$ and for every $\xi, \eta \in \mathbf{R}^n$

$$
\begin{aligned}
&\lambda(y)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(y,t)\xi_i\xi_j \leqslant L\lambda(y)|\xi|^2 \\
&\Big|\sum_{i,j=1}^{n} a_{ij}(y,t)\xi_j\eta_j\Big| \leqslant M\Big(\sum_{i,j=1}^{n} a_{ij}(y,t)\xi_i\xi_j\Big)^{1/2}\Big(\sum_{i,j=1}^{n} a_{ij}(y,t)\eta_i\eta_j\Big)^{1/2} \\
&\big|a(y,t) - a(y,s)\big| \leqslant N\lambda(y)|t-s|
\end{aligned}
\tag{5.1}
$$

hold for some constants $L, M \geqslant 1$, $N \geqslant 0$ and λ weight in A_2. Observe that, even if λ were not periodic, it would be sufficient to consider, for example, $\lambda^* = a_{11}$, so that $\lambda \leqslant \lambda^* \leqslant L\lambda$, and then consider $L^{-1}\lambda^*$ instead of λ and L^2 instead of L. Indeed the space $\mathcal{V}_{\mu,\lambda}$ and $\mathcal{V}_{\mu,\lambda^*}$ are the same. Therefore we will suppose

$$\mu, a_{ij} \text{ and also } \lambda \text{ to be } \mathcal{Q}\text{-periodic.}$$

Moreover, as far as the weights μ and λ are concerned, we will assume either that there exist some constants $K_1 \geqslant 1$, $K_2 > 0$, $0 \leqslant \alpha < 1$ and $r > 1$ (and of course $\alpha r < n$) such that

$$
\begin{aligned}
&\lambda \in A_2(K_1), \\
&(\mu^r, \lambda^r) \in A^{\alpha r}_{2,2}(\mathcal{Q}, K_2)
\end{aligned}
\tag{H.1}
$$

or that there exist some constants $K_1 \geqslant 1$, $K_2 > 0$ such that

$$
\begin{aligned}
&\lambda \in A_2(K_1), \\
&(\mu, \lambda) \in A^{\alpha}_{2,2}(\mathcal{Q}, K_2), \\
&\mu \text{ is reverse doubling, i. e. satisfies (3.7).}
\end{aligned}
\tag{H.2}
$$

Observe that

a) both hypotheses (H.1) and (H.2) on the weight μ imply $(\mu, \lambda) \in A^1_{2,2}(\mathcal{Q})$ (see Remark 3.3 and Remark 3.4) which we require so that Theorem 3.7 holds,

b) $\alpha < 1$ is needed to prove Theorem 6.6, (ii) and (iii), by Proposition 3.9,

c) moreover for every open and bounded set Ω, denoted by Q_Ω a cube containing Ω, there exists $\widetilde{K}_2$ such that $(\mu^s, \lambda^s) \in A^s_{2,2}(Q_\Omega, \widetilde{K}_2)$, $s \geqslant 1$, (see Remark 3.5 with $h = 1$).

Finally observe that if you consider the sequences of weights

$$\mu_h(x) := \mu(hx), \quad \lambda_h(x) := \lambda(hx)$$

then the pairs (μ_h, λ_h) satisfy a), c) and Remark 3.6.

From now on we will denote by Ω a bounded open set with Lipschitz boundary, Q_Ω **a cube whose vertices are points in $\mathbf{Z}^n$**, so that Q_Ω results to be union of dyadic cubes and μ and λ are also Q_Ω-periodic weights (in this way Proposition 3.9 holds). We define

$$\mathcal{Q}_+ = \{x \in \mathcal{Q} \mid \mu(x) > 0\}, \qquad \mathcal{Q}_0 = \{x \in \mathcal{Q} \mid \mu(x) = 0\}.$$

and suppose $\mathcal{Q}_0$ to be a set with Lipschitz boundary, possibly $\mathcal{Q}_0 = \mathcal{Q}$, and $\mathcal{Q}_+ = \mathcal{Q} \setminus \mathcal{Q}_0$, $\mathcal{Q}_+$ with Lipschitz boundary. Extending by periodicity μ (and a_{ij} and λ too) to $\mathbf{R}^n$ we denote, if

$\alpha = (\alpha_1, ...\alpha_n)$ with $\alpha_j \in \mathbf{Z}$, $\tau_\alpha \mathcal{Q}_+ := \{x \in \mathbf{R}^n \mid x - \alpha \in \mathcal{Q}_+\}$ and $\tau_\alpha \mathcal{Q}_0 := \{x \in \mathbf{R}^n \mid x - \alpha \in \mathcal{Q}_0\}$, $\Pi_+ := \cup_{\alpha \in \mathbf{Z}^n} \tau_\alpha \mathcal{Q}_+$ and $\Pi_0 := \cup_{\alpha \in \mathbf{Z}^n} \tau_\alpha \mathcal{Q}_0$, $\Omega_+ := \Pi_+ \cap \Omega$ and $\Omega_0 := \Pi_0 \cap \Omega$. For $h \in \mathbf{N}$ we define $\Pi_{+,h} := \{x \in \mathbf{R}^n \mid hx \in \Pi_+\}$ and $\Pi_{0,h} := \{x \in \mathbf{R}^n \mid hx \in \Pi_0\}$, and finally $\Omega_{+,h} = \Omega \cap \Pi_{+,h}$ and $\Omega_{0,h} = \Omega \cap \Pi_{0,h}$. If $\mu_h(x) = \mu(hx)$ we have that $\mu_h > 0$ a. e. in $\Pi_{+,h}$ and $\mu_h = 0$ a. e. in $\Pi_{0,h}$. Then we denote by $\Omega_{+,h}$ the set $\Omega \cap \mathcal{Q}_{+,h}$ and by $\Omega_{0,h}$ the set $\Omega \cap \mathcal{Q}_{0,h}$.
We want to formalize the following elliptic-parabolic problem: given $f \in L^2(0,T; L^n(\Omega))$ (we will approximate f by functions in $H^1(0,T; L^n(\Omega)))$ and $\varphi \in W_0^{1,\infty}(\Omega)$ (precisely we will consider b$\varphi \in W_0^{1,(1+\sigma)'}(\Omega)$, σ being the constant appearing in (3.13)) consider

$$\begin{cases} \mu \dfrac{\partial}{\partial t} u - \operatorname{div}(a \cdot Du) = f & \text{on } \Omega_+ \times (0,T) \\ u = 0 & \text{on } \big((\partial\Omega_+ \cap \partial\Omega) \times (0,T)\big) \\ u = \varphi & \text{on } \big(\Omega_+ \times \{0\}\big) \\ -\operatorname{div}(a(\cdot,t) \cdot Du) = f(\cdot,t) & \text{on } \Omega_0 \text{ for a. e. } t \in [0,T] \\ u = 0 & \text{on } \partial\Omega_0 \cap \partial\Omega \end{cases} \tag{5.2}$$

To write this problem in abstract form we define

$$\overline{\mu} = \begin{cases} \mu & \text{where } \mu > 0 \text{ (in } \Omega_+) \\ \lambda & \text{where } \mu = 0 \text{ (in } \Omega_0) \end{cases} \tag{5.3}$$

and the operator

$$\begin{array}{rccc} R: & L^2(\Omega, \overline{\mu}) & \longrightarrow & L^2(\Omega, \overline{\mu}) \\ & u & \mapsto & \chi_{\Omega_+} u \end{array} \tag{5.4}$$

where $\chi_{\Omega_+}(x) = 1$ if $x \in \Omega_+$, $\chi_{\Omega_+}(x) = 0$ if $x \in \Omega_0$. In this way R is bounded, even if μ is not. Note as in this case, if H denotes $L^2(\Omega, \overline{\mu})$, the orthogonal space H_+ of the kernel of R is isomorphic to $L^2(\Omega, \mu)$, $H_+^{1/2} = H_+$ and the orthogonal projection $P_+ : H \to H_+$ coincides with R (see the previous section).
Consider the space $H_0^1(\Omega, \overline{\mu}, \lambda)$ and the triplet

$$H_0^1(\Omega, \overline{\mu}, \lambda) \subset L^2(\Omega, \overline{\mu}) \subset H^{-1}(\Omega, \overline{\mu}, \lambda) \,.$$

As a matter of fact (see Theorem 3.7 and Remark 3.8 and also 3.8) we have that

$$H_0^1(\Omega, \overline{\mu}, \lambda) = H_0^1(\Omega, \mu, \lambda) \,,$$

and then the triplet (even if $\mu \equiv 0$)

$$H_0^1(\Omega, \mu, \lambda) \subset L^2(\Omega, \overline{\mu}) \subset H^{-1}(\Omega, \mu, \lambda) \,. \tag{5.5}$$

Consider the family of elliptic operators defined as

$$A(t) = -\operatorname{div}\ (a(\cdot,t) \cdot D) : H_0^1(\Omega, \overline{\mu}, \lambda) \to H^{-1}(\Omega, \overline{\mu}, \lambda).$$

We denote by $\mathcal{H}_{\overline{\mu},\lambda}, \mathcal{V}_{\mu,\lambda}, \mathcal{V}'_{\mu,\lambda}$ the abstrac spaces defined in (4.1) corresponding to the triplet considered in (5.5), by $\mathcal{W}_{\mu,\lambda}$ the abstract space corresponding to (4.2) and by $\mathcal{W}^0_{\mu,\lambda} = \{u \in \mathcal{W}_{\mu,\lambda} \mid P_+ u(0) = Ru(0) = 0\}$. Then we denote by

$$\mathcal{A}_a : \mathcal{V}_{\mu,\lambda} \to \mathcal{V}'_{\mu,\lambda}, \qquad \mathcal{P}_{\mu,a} = \frac{d}{dt} + \mathcal{A} : \mathcal{W}_{\mu,\lambda} \to \mathcal{V}'_{\mu,\lambda}$$

the corresponding abstract operators as done in (4.4). Note as we have, if $u \in \mathcal{W}_{\mu,\lambda}(0,T;\Omega)$ and $\varphi \in C_c^\infty(\Omega \times (0,T))$, that

$$\langle u', \varphi \rangle_{\mu,\lambda} = \int_0^T \int_{\Omega_+} u(x,t)\frac{\partial \varphi}{\partial t}(x,t)\mu(x)dxdt + \int_0^T \int_{\Omega_0} u(x,t)\frac{\partial \varphi}{\partial t}(x,t)\lambda(x)dxdt\,,$$

but

$$\langle Ru', \varphi \rangle_{\mu,\lambda} = \Big(R^{1/2}u, R^{1/2}\frac{\partial \varphi}{\partial t}\Big)_{\mathcal{H}_{\overline{\mu},\lambda}} = \int_0^T \int_{\Omega_+} u(x,t)\frac{\partial \varphi}{\partial t}(x,t)\mu(x)dxdt\,.$$

The abstract problem corresponding to (5.2) is then

$$\begin{cases} Rv' + \mathcal{A}_a v = f \\ v \in \mathcal{W}_{\mu,\lambda}^{P+\varphi} \end{cases}$$

with $f \in \mathcal{V}'_{\mu,\lambda}$. The analogous abstract formulation can be made for the problem (4.6)-(II).

Definition 5.1 *Let $L, M, \geqslant 1$, $N, T > 0$, Ω, a, μ, λ and f as above. We call solution of the Cauchy-Dirichlet problem* (5.2) *a function $u \in \mathcal{W}^0_{\mu,\lambda}$ such that*

$$(\mathcal{P}_{\mu,a}u)(t) = (Ru)'(t) + (\mathcal{A}_a u)(t) = f(t) \qquad \text{for a. e. } t \in [0,T].$$

Analogously we define the solution of the same equation with periodic conditions instead of Cauchy type ones.

Remark 5.2 - Given a matrix a satisfying (5.1) with μ and λ satisfying (H.1) or (H.2), by Theorem 3.7 and Theorem 4.3 we obtain that for every $f \in \mathcal{V}'_{\mu,\lambda}$ there exists a unique solution of the problem (5.2), or the analogous with periodic conditions.
Moreover observe that $A(t) = -\text{div}(a \cdot D)$ is differentiable in the sense of the Definition 4.5 and

$$A'(t) = -\text{div}(a' \cdot D)$$

where a' denotes the matrix whose entries are $\partial_t a_{ij}(x,t)$ (the coefficients $a_{ij}(x,t)$ for almost every $x \in \Omega$ are differentiable almost everywhere in t).

Now we consider the sequences

$$\mu_h(x) := \mu(hx), \quad \lambda_h(x) := \lambda(hx), \quad a_{ij,h}(x,t) := a_{ij}(hx,t), \qquad h \in \mathbf{N}\,. \tag{5.6}$$

Then a_h satisfies (5.1) with the weight λ_h and with the same constants L, M. Moreover there exists a constant depending on n and K_2, which for simplicity we still denote with K_2, such that if μ and λ satisfy hypotheses (H.1) (see Remark 3.5) we have that

$$\lambda_h \in A_2(K_1), \quad (\mu_h^r, \lambda_h^r) \in A_{2,2}^r(\mathcal{Q}, K_2),$$

if μ and λ satisfy hypotheses (H.2) we have that (see Remarks 3.5 and 3.6)

$$\lambda_h \in A_2(K_1), \quad (\mu_h, \lambda_h) \in A_{2,2}^1(\mathcal{Q}, K_2), \quad \mu_h \text{ are equi} - \text{reverse doubling.}$$

We choose two data f and φ in this way (see (3.13)):

$$f \in L^2(0,T; L^n(\Omega))\,, \qquad \varphi \in W_0^{1,(1+\sigma)'}(\Omega)$$

spaces respectively dense in $\mathcal{V}_h'$ and $H_0^1(\Omega,\mu_h,\lambda_h)$ for every $h\in\mathbf{N}$. Once denoted by $\Omega_{+,h}$ the set $\{x\in\Omega \mid \mu_h>0\}$ and by $\Omega_{0,h}$ the set $\{x\in\Omega \mid \mu_h=0\}$, we will study the asymptotic behaviour of the solutions u_h when $h\to\infty$ of the following sequence of degenerate elliptic-parabolic problems

$$\begin{cases} \mu_h\dfrac{\partial}{\partial t}u-\operatorname{div}(a_h\cdot Du)=f & \text{on } \Omega_{+,h}\times(0,T)\\ u=0 & \text{on } \big((\partial\Omega_{+,h}\cap\partial\Omega)\times(0,T)\big)\\ u=\varphi & \text{on } \big(\Omega_{+,h}\times\{0\}\big)\\ -\operatorname{div}(a_h(\cdot,t)\cdot Du)=f(\cdot,t) & \text{on } \Omega_{0,h} \text{ for a. e. } t\in[0,T]\\ u=0 & \text{on } \partial\Omega_{0,h}\cap\partial\Omega \end{cases} \tag{5.7}$$

where the triplets are defined analogously to (5.5).

Notations: for every $h\in\mathbf{N}$ we will simply use the subindex h to denote the weighted abstract spaces related to the weights μ_h and λ_h defined in (5.6), V_h to denote the space $H_0^1(\Omega,\mu_h,\lambda_h)$, $\mathcal{V}_h$ and $\mathcal{W}_h$ the correponding abstract spaces, $A_h(t)$ the operators -div$\big(a_h(\cdot,t)D\big)$: $V_h\to V_h'$, $\mathcal{A}_h:\mathcal{V}_h\to\mathcal{V}_h'$ the operator defined as $\mathcal{A}_hu(t)=A_h(t)u(t)$. Denote R_h (and $P_{+,h}$ which in this case coincides with it) the operator defined by $R_hv(x):=\chi_{\Omega_{+,h}}(x)v(x)$ and $\mathcal{P}_h:\mathcal{W}_h\to\mathcal{V}_h'$ will be the operator defined as $\mathcal{P}_hu=R_hu'+\mathcal{A}_hu$. Then the problems (5.7) written in abstract form become

$$\text{find } v\in\mathcal{W}^{\varphi_h}_{\mu_h,\lambda_h} \text{ such that} \qquad \begin{cases} R_hv'+\mathcal{A}_hv=f\\ P_{+,h}v=P_{+,h}\varphi=:\varphi_h \end{cases} \tag{5.8}$$

We will denote

$$\mathcal{V}:=L^2(0,T;H_0^1(\Omega)),\quad \mathcal{V}':=L^2(0,T;H^{-1}(\Omega))$$
$$\mathcal{W}:=\{u\in\mathcal{V}\mid u'\in\mathcal{V}'\},\quad \langle f,u\rangle:=\langle f,u\rangle_{\mathcal{V}'\times\mathcal{V}}$$

The symbols

$$\langle\mu\rangle,\ \langle\lambda\rangle,\ \langle\lambda^{-1}\rangle,\ \langle a_{ij}(t)\rangle$$

will denote respectively the mean value of $\mu,\lambda,\lambda^{-1},a_{ij}(\cdot,t)$ on $\mathcal{Q}$.

6 Compactness type results in weighted Sobolev spaces

The results stated from now on hold with the hypotheses and the notations of the preceeding section. Therefore we will consider the constants r,K_1,K_2,α, the open set Ω and the cube Q_Ω fixed.
We first recall the following proposition (see Corollary 4.7 in [23]) whose proof does not depend on the fact that here the weight μ can also be equal to zero in a non zero measure set, while in [23] the weight was positive almost everywhere (the only thing we need is a Poincaré inequality).

Proposition 6.1 *Consider two sequences of functions $v_h,w_h\in L^2(0,T;H^1_\#(Q_\Omega,\mu_h,\lambda_h))$, $h\in\mathbf{N}$. Suppose there exist a positive constant c such that*

$$\int_0^T\int_{Q_\Omega}|Dv_h|^2\lambda_h dxdt\leqslant c,\quad \int_0^T\int_{Q_\Omega}|Dw_h|^2\lambda_h dxdt\leqslant c$$

and two function v, w (which will be in $L^2(0,T;H^1(Q_\Omega))$) such that

$$v_h \to v \quad \textit{and} \quad w_h \to w \qquad\qquad \textit{in } L^2(0,T;L^1(Q_\Omega)).$$

Then, up to subsequences, we have that for every $\varphi \in L^\infty(Q_\Omega \times (0,T))$

$$\int_0^T \int_{Q_\Omega} v_h w_h \varphi \mu_h dxdt \ \to \ \langle\mu\rangle \int_0^T \int_{Q_\Omega} vw\varphi dxdt\,.$$

Remark 6.2 - In the preceeding proposition it is not possible to weaken hypotheses in the following way (even if $\mu_h > 0$):

$$\int_0^T \int_{Q_\Omega} |Dv_h|^2 \lambda_h dxdt \leqslant c, \quad \int_0^T \int_{Q_\Omega} w_h^2 \mu_h dxdt \leqslant c\,.$$

Call w the weak limit of w_h in $L^2(0,T;L^1(Q_\Omega))$-weak and suppose $(\mu_h)_h$ bounded in $L^r(Q_\Omega)$ with $r > 1$ (so we suppose $\mu_h \to \mu$ in $L^1(Q_\Omega)$-weak). One might expect to have, up to a subsequence, the following result

$$\int_0^T \int_{Q_\Omega} v_h w_h \varphi \mu_h dxdt \ \to \ \langle\mu\rangle \int_0^T \int_{Q_\Omega} vw\varphi dxdt\,.$$

for every $\varphi \in L^\infty(Q_\Omega \times (0,T))$, while instead we have

$$\int_0^T \int_{Q_\Omega} v_h w_h \varphi \mu_h dxdt \ \to \ \langle\mu\rangle \int_0^T \int_{Q_\Omega} vz\varphi dxdt\,.$$

where z is such that $w_h\mu_h \to z\langle\mu\rangle$ in $L^2(0,T;L^1(Q_\Omega))$-weak. Indeed if $\mu \in L^r_{\mathrm{loc}}(Q_\Omega)$ and $u \in L^2(\Omega,\mu)$ for any $r > 1$ we have that

$$\int_\Omega |u\mu|^p dx \leqslant \Big[\int u^2 \mu dx\Big]^{1/2} \Big[\int \mu^{p/(2-p)} dx\Big]^{1/2}$$

and then there exists $p \in (1,2)$ ($p = 2r/(1+r)$) such that $w_h\mu_h$ equibounded in $L^p(Q_\Omega)$. The following example shows that z may be different from w: consider $w_h(x) = \sin(hx)$ in $(0,2\pi)$ and $\mu_h(x) = \sin(hx) + 2$. Then $w_h \to 0$ and $\mu_h \to 2$, but $w_h\mu_h \to z\mu = \pi$.

Remark 6.3 - The space $L^2(0,T;H^1_\#(Q_\Omega,\mu_h,\lambda_h))$ contains both $L^2(0,T;H^1_0(\Omega,\mu_h,\lambda_h))$ and $L^2(0,T;H^1_\#(\mathcal{Q},\mu_h,\lambda_h))$. Moreover observe that choosing in Proposition 6.1 first $w_h = v_h$ and $\varphi \equiv 1$, then $w_h \equiv 1$, we obtain

$$(i) \ \int_0^T \int_{Q_\Omega} u_h^2 \mu_h dxdt \to \langle\mu\rangle \int_0^T \int_{Q_\Omega} u^2 dxdt;$$

$$(ii) \ \int_0^T \int_{Q_\Omega} u_h \varphi \mu_h dxdt \to \langle\mu\rangle \int_0^T \int_{Q_\Omega} u\varphi dxdt \quad \text{for every } \varphi \in L^\infty(Q_\Omega \times (0,T)).$$

In the fifth section we gave an extension (Theorem 4.10) of the following classical result:

Theorem 6.4 *Consider three reflexive Banach spaces B_0, B, B_1 such that $B_0 \subset B \subset B_1$. Suppose that the embedding $B_0 \hookrightarrow B$ is compact. Then even the embedding of $\big\{u \in L^2(0,T;B_0) \mid u' \in L^2(0,T;B_1)\big\}$ endowed with the norm $\|u\|^2 = \|u\|^2_{L^2(0,T;B_0)} + \|u'\|^2_{L^2(0,T;B_1)}$ in $L^2(0,T;B)$ is compact.*

In [23] the following extension to moving spaces is given.

Theorem 6.5 *Consider Ω an open bounded set of $\mathbf{R}^n$, T, K_1, K_2, r positive constants with $K_1 \geqslant 1$, $r > 1$. Consider the sequences $(\mu_h)_h$ and $(\lambda_h)_h$ defined in* (5.6) *and a sequence $v_h \in \{w \in \mathcal{V}_h \mid w' \in \mathcal{V}_h'\}$ for which there exists a constant $c > 0$ such that*

$$\|v_h\|_{\mathcal{V}_h} + \|v_h'\|_{\mathcal{V}_h'} \leqslant c \qquad \textit{for every } h \in \mathbf{N}.$$

Then there exist $p > 1$, p depending (only) on n, K_1, and a function $v \in L^2(0,T;L^2(\Omega))$ such that, up to a subsequence,

$$v_h \to v \qquad \textit{in} \quad L^2(0,T;L^p(\Omega))\,.$$

In the case of strongly degenerate parabolic equations a similar compactness result does not hold in general, as the following example shows. Consider the functions

$$u_1(x) = \begin{cases} 0 & \text{if } x \in [-2,-1] \\ x+1 & \text{if } x \in [-1,0] \\ -x+1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [1,2] \end{cases} \qquad u_2(x) = \begin{cases} 0 & \text{if } x \in [-2,-1] \\ -x-1 & \text{if } x \in [-1,0] \\ x-1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [1,2] \end{cases}$$

define $\overline{u} : [-1,1] \times [0,1] \to \mathbf{R}$,

$$\overline{u}(x,t) = \begin{cases} u_1(x) & \text{if } t \in [0,1/2] \\ u_2(x) & \text{if } t \in (1/2,1] \end{cases}$$

and extend it by periodicity (in time). Then consider $R : L^2(-2,2) \to L^2(-2,2)$ defined by $Rv := \chi_{(-2,-1)\cup(1,2)}(x)v(x)$. We have that $\overline{u} \in \{v \in L^2(0,1;H_0^1(-2,2)) \mid (Rv)' \in L^2(0,1;H^{-1}(-2,2))\}$ and the sequence $u_h(x,t) := \overline{u}(x,ht)$ satisfies $\|u_h\|_{L^2(0,1;H_0^1(-2,2))} + \|(Ru_h)'\|_{L^2(0,1;H^{-1}(-2,2))} \leqslant c$, but u_h does not strongly converge in $L^2(0,1;L^1(-2,2))$.

This example shows that is not possible to obtain a general compactness result for a sequence $v_h \in \mathcal{W}_{\mu_h,\lambda_h}$ with $\|v_h\|_{\mathcal{W}_{\mu_h,\lambda_h}} \leqslant c$, i. e. generalizing Theorem 4.10 to moving spaces to obtain a result analogous to Theorem 6.5, but this can be obtained if v_h are solutions of (5.8), at least when the datum f is regular enough.

Theorem 6.6 *Consider Ω an open bounded set of $\mathbf{R}^n$, T, K_1, K_2, r positive constants with $K_1 \geqslant 1$, $r > 1$, $\alpha \in [0,1)$. Consider the sequences $(\mu_h)_h$ and $(\lambda_h)_h$ defined in* (5.6) *(satisfying or* (H.1) *either* (H.2)*) and the sequence $u_h \in \{w \in \mathcal{V}_h \mid w' \in \mathcal{V}_h'\}$ of solutions to* (5.8)*. Then there exists a constant $c > 0$ such that*

$$\|u_h\|_{\mathcal{W}_h} = \|u_h\|_{\mathcal{V}_h} + \|R_h u_h'\|_{\mathcal{V}_h'} \leqslant c \qquad \textit{for every } h \in \mathbf{N}.$$

Suppose moreover that the datum f in (5.8) *belongs to $H^1(0,T;L^n(\Omega))$. Then there exists a function $u \in H^1(0,T;H_0^1(\Omega))$ such that, up to subsequences,*

$$u_h \to u \qquad \textit{in} \quad C([0,T];L^1(\Omega))\,.$$

and

$$U_h(t) = \int_\Omega u_h^2(x,t)\mu_h(x)dx \;\to\; U(t) = \langle\mu\rangle \int_\Omega u^2(x,t)dx \qquad \textit{in } C\big([0,T]\big).$$

Proof - The solutions are bounded, that is there exists a constant such that $\|u_h\|_{\mathcal{W}_h} \leqslant c$, from Theorem 4.3, (iii).
Consider the problems (5.7)-(5.8) with the data $f \in H^1(0,T;L^n(\Omega))$ and $\varphi \in C^1_c(\Omega)$. We define a sequence of functions $\tilde{u}_{0,h} \in H^1_0(\Omega_h, \mu_h, \lambda_h)$ as the solutions to the elliptic problems

$$\begin{cases} A_h(0)u = f(0) & \text{on } \Omega_{0,h} \\ u = \varphi & \text{on } \partial\Omega_{0,h} \,, \end{cases}$$

and than we extend them to the whole Ω in this way

$$u_{0,h} := \begin{cases} \tilde{u}_{0,h} & \text{in } \Omega_{0,h} \\ \varphi & \text{in } \Omega_{+,h} \,. \end{cases} \tag{6.1}$$

We can look at the problems

$$\begin{cases} R_h v' + \mathcal{A}_h v = f \\ P_{+,h} v = P_{+,h}\varphi =: \varphi_h \,, \end{cases}$$

in the following way, thinking the initial datum as the restriction of $u_{0,h}$:

$$\begin{cases} R_h v' + \mathcal{A}_h v = f \\ P_{+,h} v = P_{+,h} u_{0,h} = P_{+,h}\varphi =: \varphi_h \,. \end{cases}$$

Then we have that $f(0) - A_h(0)u_{0,h} \in \operatorname{Im} R_h$ and the hypotheses of Theorem 4.7 and Corollary 4.8 are satisfied. Then in particular there exists a constant $c > 0$ (depending only on M, L, f, φ) such that

$$\|u_h\|_{\mathcal{V}_h} + \|u_h'\|_{\mathcal{V}_h} \leqslant c \,.$$

In particular, by Corollary 4.8 and (3.13), we have that $(u_h)_h$ is bounded in $H^1(0,T;W_0^{1,1+\sigma}(\Omega))$ and this space compactly embeds in $L^2(0,T;L^1(\Omega))$ (by Theorem 6.4). But we have more: indeed, since (see, e. g., [11]) for a function $w \in W^{1,p}(0,T;B)$ with B Banach space

$$w(t) - w(s) = \int_s^t w'(\tau)d\tau$$

we have

$$\|u_h(t) - u_h(s)\|_{L^1(\Omega)} \leqslant \int_s^t \|u_h'(\tau)\|_{L^1(\Omega)} d\tau \leqslant |t-s|^{1/2} \|u_h'\|_{L^2(0,T;W^{1,1}(\Omega))} \,,$$

so $(u_h)_h$ is equicontinuous valued in $L^1(\Omega)$, and moreover the sets $\{u_h(t) \mid h \in \mathbf{N}\}$ are relatively compact in $L^1(\Omega)$ for almost every $t \in [0,T]$. Then by Lemma 1 in [30] we obtain $(u_h)_h$ relatively compact in $C([0,T];L^1(\Omega))$. Finally, $u, u' \in L^2([0,T];H^1_0(\Omega))$ from Proposition 3.8, (ii), in [25] (thanks to 3.12).
Now consider the functions U_h. Integrating the abstract equation

$$R_h u_h' + \mathcal{A}_h u_h = f$$

in $(t_1, t_2) \subset (0,T)$ we have that

$$U_h(t_2) - U_h(t_1) = 2\int_{t_1}^{t_2} \Big[-\int_\Omega (a_h Du_h, Du_h)dxdt + \int_\Omega f u_h dx \Big] dt$$

and then

$$|U_h(t_2) - U_h(t_1)| \leqslant \int_{t_1}^{t_2} \Big[2L \max_{t\in[0,T]} \|u_h(t)\|_{V_h} + \max_{t\in[0,T]} \|f(t)\|_{L^n(\Omega)} \max_{t\in[0,T]} \|u_h(t)\|_{W^{1,1}(\Omega)} \Big] dt \,.$$

From continuity of the integral we have that $(U_h)_h$ is relatively compact in $C([0,T])$ and since by Remark 6.3 $U_h \to U$ almost everywhere in $[0,T]$ we conclude. $\square$

We recall this lemma, useful in the next section for homogenization and also to prove the theorem which follows, a result of compensated compactness (for some classical result about compensated compactness see, e. g., [20], [34], [33] ,[32], [31]).

Lemma 6.7 *Consider two sequences $w_h \in L^2(0,T; H^1_\#(Q_\Omega, \mu_h, \lambda_h))$ and $A_h \in L^2(0,T; (L^2(Q_\Omega, \lambda_h^{-1}))^n)$, $h \in \mathbf{N}$, for which there exist $c > 0$, $w \in L^2(0,T; L^1(Q_\Omega))$ and $A \in L^2(0,T; (L^1(Q_\Omega))^n)$ such that*

$$\begin{aligned} &\|A_h\|_{L^2(0,T;(L^2(Q_\Omega,\lambda_h^{-1}))^n)} \leqslant c \quad \textit{and} \quad A_h \to A \ \textit{in} \ L^2(0,T; (L^1(Q_\Omega))^n) - \textit{weak}, \\ &\int_0^T \int_{Q_\Omega} |Dw_h|^2 \lambda_h dxdt \leqslant c \qquad \textit{and} \quad w_h \to w \ \textit{in} \ L^2(0,T; L^1(Q_\Omega)). \end{aligned}$$

Then for every $\varphi \in C_0^\infty(Q_\Omega \times (0,T))$ we have

$$\int_0^T \int_{Q_\Omega} (A_h, D\varphi) w_h dxdt \to \int_0^T \int_{Q_\Omega} (A, D\varphi) w \ dxdt.$$

Theorem 6.8 *Consider a sequence of functions $v_h \in \mathcal{W}_h$ $(h = 1, 2, ...)$, a function $v \in \mathcal{W}$, a constant c_1 such that*

$$\|v_h\|_{\mathcal{W}_h} \leqslant c_1 \quad \textit{for every } h, \quad v_h \to v \quad \textit{in } L^2(0,T; L^1(\Omega)), \tag{6.2}$$

and a sequence of vectorial functions $a_h \in L^2(0,T; (L^2(\Omega, \lambda_h^{-1}))^n)$ $(h = 1, 2, ...)$, $a \in L^2(0,T; (L^2(\Omega))^n)$, a constant c_2 such that

$$\|a_h\|_{L^2(0,T;(L^2(\Omega,\lambda_h^{-1}))^n)} \leqslant c_2 \qquad\qquad a_h \to a \ \textit{in } L^2(0,T; (L^1(\Omega))^n) - \textit{weak}. \tag{6.3}$$

Moreover suppose that

$$u_h' - \operatorname{div}\,(a_h) = f \in L^2(0,T; L^n(\Omega)) \quad \textit{in } C^1_c(\Omega \times (0,T)). \tag{6.4}$$

Then

$$\big(a_h, Du_h\big) \to \big(a, Du\big) \quad \textit{in } \mathcal{D}'(\Omega \times (0,T)).$$

7 Homogenization

Consider a vector $\xi \in \mathbf{R}^n$, the function ψ_ξ solution to

$$\begin{cases} -\mathrm{div}_y\big(a(y,t) D_y \psi_\xi\big) = -\mathrm{div}_y\big(a(y,t)\cdot\xi\big) \quad \text{on } \mathcal{Q}, \quad \text{ for a. e. } t \in (0,T) \\ \psi_\xi \in L^2(0,T; H^1_\#(\mathcal{Q}, \mu, \lambda))\,, \end{cases}$$

the matrix and the weight defined respectively

$$\widehat{a}(t) \cdot \xi = \Big\langle a(y,t)\cdot\xi - a(y,t)\cdot D\psi_\xi \Big\rangle_y \,, \qquad\qquad \widehat{\mu} = \langle \mu \rangle \tag{7.1}$$

and the Cauchy-Dirichlet problems

$$\begin{cases} \widehat{\mu}\dfrac{\partial v}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \widehat{a}_{ij}(t)\frac{\partial^2 v}{\partial x_i \partial x_j} = f & \text{on } \Omega \times (0,T) \\ v = 0 & \text{on } \partial\Omega \times (0,T) \\ v = \varphi & \text{on } \Omega \times \{0\} \end{cases} \tag{7.2}$$

with $f \in L^2(0,T;L^n(\Omega))$ and $\varphi \in W_0^{1,(1+\sigma)'}(\Omega)$, σ being the positive constant appearing in (3.13).

Theorem 7.1 *For every pair of weights μ and λ verifying* (H.1) *or* (H.2) (*see Section 3*), *and for every $f \in L^2(0,T;L^n(\Omega))$ and $\varphi \in W_0^{1,(1+\sigma)'}(\Omega)$, the solutions u_h of the problems* (5.7) (*see Definition* 5.1) *satisfy*

$$\begin{aligned} & u_h \to u && \textit{in } L^2\big(0,T;L^1(\Omega)\big) \\ & a_h \cdot Du_h \to \widehat{a} \cdot Du && \textit{in } L^2\big(0,T;(L^1(\Omega))^n\big)\textit{-weak}\,, \end{aligned}$$

where $\widehat{a}$ is the matrix defined in (7.1) *and u is the solution of* (7.2).

Remark 7.2 - Note that if hypothesis (H.1) or (H.2) with $\alpha = 0$ are satisfied it is possible to choose the datum $\varphi \in L^p(\Omega)$ for some $p \in (1,\infty)$. This is possible estimating as in (7.3) using Theorem 4.3, (iii), since $\mu \in L^r(\Omega)$ with $r > 1$ (if (H.1) holds it is straight forward, otherwise it follows from example d) in Section 8).

Before proving this theorem we need a result (see also [21]). Define p the polynomial $p(y) = \xi_1 y_1 + ... + \xi_n y_n$ and consider ψ_h ($h \in \mathbf{N}$) and ψ the solutions to the two problems

$$\begin{cases} \big[-\dfrac{1}{h^\alpha}\mu(y)\dfrac{\partial}{\partial t} - \text{div }(a^*(y,t)\cdot D)\big]v = -\text{div }(a^*(y,t)\cdot Dp) \\ v \in \mathcal{W}^T_{\mu,\lambda,\#}(0,T;\mathcal{Q})\,, \end{cases}$$

$$\begin{cases} -\text{div }(a^*(y,t)\cdot Dv) = -\text{div }(a^*(y,t)\cdot Dp) \\ v \in L^2(0,T;H^1_\#(\mathcal{Q},\mu,\lambda))\,, \end{cases}$$

where $\alpha > 0$ and where $\mathcal{W}^T_{\mu,\lambda,\#}(0,T;\mathcal{Q}) = \{u \in \mathcal{W}_{\mu,\lambda,\#}(0,T;\mathcal{Q}) \mid P_+u(T) = 0\}$ (see also Remark 4.4).

Lemma 7.3 *The functions ψ_h converge to ψ in $L^2(0,T;H^1_\#(\mathcal{Q},\mu,\lambda))$.*

Proof - It is sufficient to choose $v_h \in C^1(0,T;H^1_\#(\mathcal{Q},\mu,\lambda))$, $v_h(T) = 0$, bounded in $\mathcal{W}$ and such that $\|v_h - \psi\| = \|v_h - \psi\|_{L^2(0,T;H^1_\#(\mathcal{Q},\mu,\lambda))} \to 0$. Then

$$\begin{aligned} \|\psi_h - v_h\|^2 &\leqslant \big\langle \mathcal{A}^*(\psi_h - v_h), \psi_h - v_h \big\rangle \\ &\leqslant \big\langle -h^{-\alpha}(\psi_h - v_h)', \psi_h - v_h \big\rangle + \big\langle \mathcal{A}^*(\psi_h - v_h), \psi_h - v_h \big\rangle \\ &= \big\langle h^{-\alpha}v_h' - \mathcal{A}^*v_h + \mathcal{A}^*p, \psi_h - v_h \big\rangle \\ &= \big\langle h^{-\alpha}v_h' - \mathcal{A}^*v_h + \mathcal{A}^*\psi, \psi_h - v_h \big\rangle \\ &\leqslant \Big[h^{-\alpha}\|v_h\|_{\mathcal{V}'} + \|\mathcal{A}^*(v_h - \psi)\|_{\mathcal{V}'}\Big]\|\psi_h - v_h\| \end{aligned}$$

□

Proof of Theorem 7.1 - First consider problem (5.7) with a datum

$$f \in H^1(0,T;L^n(\Omega))$$

and denote by u_h its solution ($h \in \mathbf{N}$). We define by the functions ψ_h of the lemma above

$$w_h(x,t) := p(x) - \frac{1}{h}\psi_h(hx,t)\,.$$

One can verify that

$$-w_h' + \mathcal{A}_h^* w_h = 0 \quad \text{ in } \mathcal{D}'(\mathbf{R}^{n+1}) \qquad \text{and} \qquad w_h \to p \text{ in } L^2(0,T;L^{1+\sigma}(\mathcal{Q}))\,.$$

Now fix $\phi \in C_c^\infty(\Omega \times (0,T))$ and subtract $\langle -w_h' + \mathcal{A}_h^* w_h, \phi u_h\rangle$ to $\langle u_h' + \mathcal{A}_h u_h, \phi w_h\rangle$ to obtain

$$\langle u_h' + \mathcal{A}_h u_h, \phi w_h\rangle - \langle -w_h' + \mathcal{A}_h^* w_h, \phi u_h\rangle = \langle f, \phi w_h\rangle\,.$$

Obviously

$$\int_0^T \int_\Omega f\phi w_h dxdt \to \int_0^T \int_\Omega f\phi p dxdt\,.$$

Since w_h are bounded in $\mathcal{V} = L^2(H^1_\#)$, by Proposition 6.1 and Theorem 6.6 we have that the terms involving the temporal derivative, up to a subsequence, satisfy

$$-\int_0^T \int_\Omega u_h w_h \frac{\partial \phi}{\partial t}\mu_h dxdt \to -\langle \mu\rangle \int_0^T \int_\Omega up\frac{\partial \phi}{\partial t} dxdt\,.$$

Since $a_h \cdot Du_h$ and $a_h^t \cdot Dw_h$ are bounded in $L^2(0,T;L^2(\Omega,\lambda_h^{-1}))$, thanks to (3.13) we can suppose the existence of two vectors $\eta, \theta \in \mathbf{R}^n$ such that, up to a subsequence,

$$a_h \cdot Du_h \to \eta, \quad a_h^t \cdot Dw_h \to \theta \quad \text{ in } L^2(0,T;L^{1+\sigma}(\Omega)^n)\,.$$

Now using Lemma 6.7, taking the limit we obtain

$$\int_0^T \int_\Omega \Big[-\langle \mu\rangle up\frac{\partial \phi}{\partial t} dxdt + (\xi, D\phi)pdxdt - (\eta, D\phi)u\Big] dxdt = \int_0^T \int_\Omega fp\phi \; dxdt$$

and

$$u' - \operatorname{div} \xi = f\,.$$

Now multiplying this last equality by ϕp and proceeding as in the classical homogenization (see, e. g., [3]) and using Lemma 7.3 and the results of the previous section one obtain that the limit function $u \in L^2(0,T;H_0^1(\Omega))$ with $u' \in L^2(0,T;H^{-1}(\Omega))$ and satisfies the limit equation

$$\widehat{\mu}\frac{\partial u}{\partial t} - \sum_{i,j=1}^n \widehat{a}_{ij}(t)\frac{\partial^2 u}{\partial x_i \partial x_j} = f$$

if $f \in H^1(0,T;L^n(\Omega))$. Now we prove that the initial datum is φ if $f \in H^1(0,T;L^n(\Omega))$ and $\varphi \in W_0^{1,(1+\sigma)'}(\Omega)$. Now we define a sequence of functions $\tilde{u}_{0,h} \in H_0^1(\Omega_h, \mu_h, \lambda_h)$ as the solutions to the elliptic problems

$$\begin{cases} A_h(0)u = f(0) & \text{on } \Omega_{0,h} \\ u = \varphi & \text{on } \partial\Omega_{0,h}\,, \end{cases}$$

and consider $u_{0,h}$ as in (6.1). We have that $\|\tilde u_{0,h}\|_{H^1_0(\Omega_h,\mu_h,\lambda_h)} \leqslant c$ with c depending (only) on L, M, f, φ. Then the sequence $\|u_{0,h}\|_{H^1_0(\Omega,\mu_h,\lambda_h)}$ turns to be bounded, then $(u_{0,h})_h$ turns to be weakly compact in $W^{1,1}(\Omega)$ and then, up to a subsequence, strongly convergent to a function u_0 in $L^1(\Omega)$. Therefore we gave that, up to a subsequence, $u_{0,h}$ strongly conveges to a function u_0 in $L^1(\Omega)$. Moreover we have, if $\chi_h(x) = 1$ for $x \in \Omega_{+,h}$ and $\chi_h(x) = 0$ for $x \in \Omega_{0,h}$,

$$u_{0,h} = \chi_h u_{0,h} + (1-\chi_h)u_{0,h} = \chi_h \varphi + (1-\chi_h)u_{0,h} \to \frac{|\mathcal{Q}_+|}{|\mathcal{Q}|}\varphi + \frac{|\mathcal{Q}_0|}{|\mathcal{Q}|}u_0$$

where the convergence is in $L^1(\Omega)$-weak. Then $u_0 = |\mathcal{Q}_+|\varphi + |\mathcal{Q}_0|u_0$ implies that

$$u_0 = \varphi\,,$$

i. e. the initial datum for the limit problem is the function φ.
Then we have proved the theorem for $f \in H^1(0,T;L^n(\Omega))$. Now we show the convergence of the solutions and of the coerresponding momenta even for $f \in L^2(0,T;L^n(\Omega))$. Suppose $f \in L^2(0,T;L^n(\Omega))$ and, for $\epsilon > 0$, fix $g \in H^1(0,T;L^n(\Omega))$ such that $\|f-g\|_{L^2(0,T;L^n(\Omega))} < \epsilon$. We will denote by $u_h(f)$ and $u_h(g)$ the solutions to (5.7) and by $u(f)$ and $u(g)$ the solutions to (7.2) with respectively data f, φ and g, φ. By Theorem 4.3, point (ii), we have that

$$\|u_h(f) - u_h(g)\|_{\mathcal{V}_h} \leqslant \|f-g\|_{\mathcal{V}'_h}\,.$$

By (3.14) and (3.13) we have that

$$\|u_h(f) - u_h(g)\|_{L^2(0,T;L^1(\Omega))} \leqslant c\ \|f-g\|_{L^2(0,T;L^n(\Omega))} \leqslant c\ \epsilon$$

and analogously $\|u(f)-u(g)\|_{L^2(0,T;L^1(\Omega))} \leqslant c\ \epsilon$. Then

$$\begin{aligned} \|u_h(f) - u(f)\|_{L^2(0,T;L^1(\Omega))} \ &\leqslant\ \|u_h(f) - u_h(g)\|_{L^2(0,T;L^1(\Omega))} + \\ &\quad + \|u_h(g) - u(g)\|_{L^2(0,T;L^1(\Omega))} + \\ &\quad + \|u(g) - u(f)\|_{L^2(0,T;L^1(\Omega))}\,. \end{aligned} \tag{7.3}$$

Taking the limit and since for $g \in H^1(0,T;L^n(\Omega))$ we have $u_h(g) \to u(g)$ in $L^2(0,T;L^1(\Omega))$, we conclude $\limsup_h \|u_h(f)-u(f)\|_{L^2(0,T;L^1(\Omega))} \leqslant 2\ c\ \epsilon$. As regards the *momenta* for a fixed $\Phi \in (C^\infty_c(\Omega \times (0,T)))^n$ we write

$$\begin{aligned} (a_h \cdot Du_h(f), \Phi) - (a \cdot Du(f), \Phi) = \Big[(a_h \cdot Du_h(f), \Phi) - (a_h \cdot Du_h(g), \Phi)\Big] + \\ + \Big[(a_h \cdot Du_h(g), \Phi) - (a \cdot Du(g), \Phi)\Big] + \Big[(a \cdot Du(g), \Phi) - (a \cdot Du(f), \Phi)\Big]\,. \end{aligned}$$

As before the central term is going to zero since $g \in \mathcal{X}$. We estimate, for example, the first term, the last being similar. By (5.1) we have

$$\Big|\int_0^T \int_\Omega (a_h \cdot Du_h(g), \Phi) - (a_h \cdot Du_h(f), \Phi)\Big| \leqslant\ ML\||D\Phi|\|_\infty \Big[T\int_\Omega \lambda_h dx\Big]^{1/2} \|u_h(g) - u_h(f)\|_{\mathcal{V}_h}$$

and this quantity can be estimated as before to find that it is $O(\epsilon)$.
Now take a sequence $(g_h)_h \subset H^1(0,T;L^n(\Omega))$ such that

$$g_h \to f \text{ in } L^2(0,T;L^n(\Omega)) \qquad \text{and} \qquad g_h(0) \text{ compact in } L^n(\Omega)\,.$$

Arguing as before we obtain that the functions

$$u_{0,h} := \begin{cases} \tilde u_{0,h} & \text{in } \Omega_{0,h} \\ \varphi & \text{in } \Omega_{+,h}\,. \end{cases} \qquad \text{where} \qquad \begin{cases} A_h(0)\tilde u_{0,h} = g_h(0) & \text{on } \Omega_{0,h} \\ u = \varphi & \text{on } \partial\Omega_{0,h}\,, \end{cases}$$

converge to φ. □

8 Some examples and comparison with the classical result

In this last section we want to observe how the limit matrix $\widehat{a}(t)$ in (7.1) is independent of the weight μ, that a similar result is not true if we consider homogenization also in time (i. e. $a_h(x,t) = a(h^\alpha x, h^\beta t)$) and we want to show some possible choices of μ.
By $\widehat{a}(t)$ we will denote the matrix defined in (7.1).

a) If we consider $\mu \equiv 1$ we have the classical (degenerate) parabolic problems (see for instance [25] and [23])

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a_h \cdot Du) = f & \text{on } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = \varphi & \text{on } \Omega \times \{0\}\,. \end{cases}$$

Then the solutions converge in $L^2(0,T;L^1(\Omega))$ to the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \widehat{a}_{ij}(t) \frac{\partial^2 u}{\partial x_i \partial x_j} = f & \text{on } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = \varphi & \text{on } \Omega \times \{0\}\,. \end{cases}$$

b) If we consider $\mu \equiv 0$ we have the classical (degenerate) elliptic problems (see for instance [8] and [9])

$$\begin{cases} -\operatorname{div}(a(\cdot,t) \cdot Du) = f(\cdot,t) & \text{on } \Omega \text{ for a. e. } t \in [0,T] \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

In this case the solutions converge in $L^2(0,T;L^1(\Omega))$ to the solution of

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \widehat{a}_{ij}(t) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(\cdot,t) & \text{on } \Omega \text{ for a. e. } t \in [0,T] \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Before showing other possible choices we want to stress that the limit matrix is the same for the parabolic and for the elliptic case.
Consider the classical case (not degenerate), i. e. a sequence of parabolic operators like

$$\frac{\partial}{\partial t} - \operatorname{div}(a \cdot D) \tag{8.1}$$

with

$$\lambda_0 |\xi|^2 \leqslant \big(a(x,t) \cdot \xi, \xi\big) \leqslant \Lambda_0 |\xi|^2\,, \qquad \text{for a. e. } (x,t) \in \Omega \times (0,T)\,,$$

where $\lambda_0, \Lambda_0 \in \mathbf{R}$, $0 < \lambda_0 < \Lambda_0$, the situation consider by F. Colombini and S. Spagnolo in [5]. This corrisponds to consider $\lambda(x) = \lambda_0$ for every $x \in \Omega$ and $\mu \equiv 1$. In [5] the authors obtain a result of compactness with respect to G-convergence for a sequence of parabolic operators like (8.1) defined by matrices satisfying the following condition

$$\lim_{\tau\to 0}\sup_h \int_a^b\int_\omega |a_h(x,t+\tau)-a_h(x,t)|dxdt = 0 \tag{H}$$

for every $(a,b)\subset\subset(0,T)$ and $\omega\subset\subset\Omega$.
F. Colombini and S. Spagnolo prove that (see [5] for the definitions and the result) given a sequence $(a_h)_h$ satisfying hypothesis (H) the sequence of parabolic operators $P_h=\frac{\partial}{\partial t}-\mathrm{div}(a_h(x,t)\cdot D)$ G-converges to $P=\frac{\partial}{\partial t}-\mathrm{div}(a(x,t)\cdot D)$ if and only if the family of sequences of elliptic operators $A_h(t)=-\mathrm{div}(a_h(\cdot,t)\cdot D)$ G-converges to $A(t)=-\mathrm{div}(a(\cdot,t)\cdot D)$ for almost every $t\in(0,T)$.
Moreover they give an example of a sequence $(a_h(x,t))_h$ which does not satisfy (H) such that the parabolic G-limit is $\frac{\partial}{\partial t}-\mathrm{div}(a_1(x,t)\cdot D)$, the elliptic G-limit is $-\mathrm{div}(a_2(\cdot,t)\cdot D)$ for almost every $t\in(0,T)$, but $a_1\neq a_2$.
Observe that even if $a(x,t)\in C^1(\overline{\Omega}\times[0,T])$ and we define $a_h(x,t)=a(h^\alpha x,h^\beta t)$ with $\beta>0$, the sequence $(a_h)_h$ does not satisfy condition (H).
This clarifies that the result of the previous section, it is not possible to obtain a result of G-convergence for a general sequence

$$\mu_h\frac{\partial}{\partial t}-\mathrm{div}(a_h\cdot D)$$

with the limit matrix independent of the coefficients μ_h. In the homogenization case it is possible only for homogenization in space, i. e. $a_h(x,t)=a(hx,t)$. In this sense the result of the previous section is sharp.

c) Interesting are also examples in which the problem is mixed. Take, for example, the simpler situation, that is

$$\mu=\begin{cases}0 & \text{in } \mathcal{Q}_0\\ 1 & \text{in } \mathcal{Q}_+\end{cases}$$

where $\mathcal{Q}_+$ and $\mathcal{Q}_0$ are open sets with Lipschitz boundaries such that $\mathcal{Q}_+\cup\mathcal{Q}_0=[0,1]^n$. In this case the solution of the mixed problems

$$\begin{cases}\mu_h\dfrac{\partial u}{\partial t}-\mathrm{div}(a_h\cdot Du)=f & \text{on } \Omega_{+,h}\times(0,T)\\ u=0 & \text{on } \big((\partial\Omega_{+,h}\cap\partial\Omega)\times(0,T)\big)\\ u=\varphi & \text{on } \big(\Omega_{+,h}\times\{0\}\big)\\ -\mathrm{div}(a(\cdot,t)\cdot Du)=f(\cdot,t) & \text{on } \Omega_{0,h} \text{ for a. e. } t\in[0,T]\\ u=0 & \text{on } \partial\Omega_{0,h}\cap\partial\Omega\end{cases}$$

converge in $L^2(0,T;L^1(\Omega))$ to the solution of

$$\begin{cases}|\mathcal{Q}_+|\dfrac{\partial u}{\partial t}-\displaystyle\sum_{i,j=1}^n\widehat{a}_{ij}(t)\frac{\partial^2 u}{\partial x_i\partial x_j}=f & \text{on } \Omega\times(0,T)\\ u=0 & \text{on } \partial\Omega\times(0,T)\\ u=\varphi & \text{on } \Omega\times\{0\}\,.\end{cases}$$

d) A general interesting sistuation is the following: if $\alpha = 0$ (H.1) or (H.2) are equivalent to the existence of a constant $c > 0$ (see [23], section 6) such that

$$0 \leqslant \mu \leqslant c\,\lambda$$

(see also [24] for $\mu = \lambda$).

References

[1] I. Babuška, *Homogeneization and its application. Mathematical and computational problems*, in Numerical Solution of Part. Diff. Eq., Proc. Third Symp. (SYNSPADE), B. Hubbard, ed., New York - London, 1976, Acad. Press, pp. 89–116.

[2] V. Barbu - T. Precupanu, *Convexity and Optimization in Banach Spaces*, Editura Academiei Republicii Socialiste România, Bucaresti, România - Sijthoff Noordhoff International Publishers, The Netherlands, 1978.

[3] A. Bensoussan - J. L. Lions - G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978.

[4] R. R. Coifman - C. Fefferman, *Weighted norm inequalities for maximal function and singular integrals*, Studia Math., 51 (1974), pp. 241–250.

[5] F. Colombini - S. Spagnolo, *Sur la convergence de solutions d'équations paraboliques*, J. Math. Pur. Appl., 56 (1977), pp. 263–306.

[6] G. Dal Maso, *An introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[7] R. De Arcangelis, *Compactness and convergence of minimum points for a class of nonlinear nonequicoercive functionals*, Nonlinear Anal., 15 (1990), pp. 363–380.

[8] R. De Arcangelis - F. Serra Cassano, *On the homogeneization of degenerate elliptic equations in divergence form*, J. Math. Pur. Appl., 71 (1992), pp. 119–138.

[9] R. De Arcangelis - F. Serra Cassano, *On the convergence of solutions of degenerate elliptic equations in divergence form*, Ann. Mat. Pura Appl., 167 (IV) (1994), pp. 1–23.

[10] E. De Giorgi - S. Spagnolo, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*, Boll. Un. Mat. Ital., 8 (4) (1973), pp. 391–411.

[11] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, U. S. A., 1998.

[12] E. B. Fabes - C. E. Kenig - R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations, 7 (1) (1982), pp. 77–116.

[13] J. Garcia Cuerva - J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.

[14] A. Kufner - B. Opic, *How to define reasonably weighted Sobolev spaces*, Commentationes Mathematicae Universitatis Carolinae, 25 (3) (1984), pp. 537–554.

[15] O. A. Ladyzenskaia - V. A. Solonnikov - N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*.

[16] J. L. Lions, *Equations differentielles operationelles et problemes aux limites*, Springer - Verlag, Berlin - Göttingen - Heidelberg, 1961.

[17] J. L. Lions, *Quelques methodes de résolution des problème aux limites non linéaires*, Gauthier-Villars, Paris, 1969.

[18] J. L. Lions, *Asymptotic problems - An introduction to homogeneization*, NATO course, Liege, 1976.

[19] J. L. Lions - E. Magenes, *Problemes aux limites non homogenes et applications, vol. 1*, Dunod, Paris, 1968.

[20] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) 5 (1978), pp. 489–507.

[21] A. Pankov, *G-convergence and Homogenization of Nonlinear Partial Differential Operators*, Kluwer, 1997.

[22] A. A. Pankov - T. E. Pankova, *Nonlinear evolution equations with non-invertible operator coefficient at the derivative*, Doklady Akad. Nauk Ukr., 9 (1993), pp. 18–20. (written in Russian).

[23] F. Paronetto, *Homogenization of a class of degenerate parabolic equations*, Asymptotic Anal., 21 (3,4) (1999), pp. 275–302.

[24] F. Paronetto, *Remarks on the convergence of degenerate elliptic and parabolic equations*, To appear on Journal of Convex Anal., (2001).

[25] F. Paronetto - F. Serra Cassano, *On the convergence of a class of degenerate parabolic equations*, J. Math. Pur. Appl., 77 (1998), pp. 735–759.

[26] W. Rudin, *Functional Analysis*, McGraw-Hill, 1991.

[27] E. Sanchez Palencia, *Comportements local et macroscopique d'un type de milieux physiques hétérogènes.*, Intern. Jour. Engin. Sci., 12 (1974), pp. 331–351.

[28] E. Sawyer - L. Wheeden, *Weighted inequalities for fractional integrals on euclidean and homogeneous spaces*, Amer. Jour. Math., 114 (1992), pp. 813–878.

[29] F. Serra Cassano, *Un'estensione della G-convergenza alla classe degli operatori ellittici degeneri*, Ricerche Mat., 38 (II) (1989), pp. 167–197.

[30] J. Simon, *Compact Sets in the Space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.

[31] S. Spagnolo, *Convergence of parabolic equations*, Boll. Un. Mat. Ital., 14-B (4) (1977), pp. 547–568.

[32] L. Tartar, *Cours Peccot*, Collège de France, 1977. Partially written in: F. Murat, *H-convergence - Séminaire d'Analyse Fonctionnelle et Numérique*, Université d'Alger, 1977-78. English translation: F. Murat - L. Tartar: *H-Convergence*, in *Topics in the Mathematical Modelling of Composite Materials* (A. Cherkaev, R. Kohn, editors), Birkhäuser, Boston, 1997, 21–43.

[33] L. Tartar, *Homogéneization et compacité par compensation*, Séminaire Schwartz Exposé, n9, 1978.

[34] L. Tartar, *Compensated compactness and applications to partial differential equations, nonlinear analysis and mechanics*, in Heriott-Watt Symposium, Vol IV, Res. Notes Math., vol. 39, Pitman, London, 1979, pp. 136–211.

[35] R. Temam, *Infinite-Dimensional Dynimical Systems in Mechanics and Phisics*, Springer-Verlag, New York, 1997.

[36] J. Wloka, *Partial differential equations*, Cambridge University Press, Cambridge, Great Britain, 1987.

[37] E. Zeidler, *Nonlinear Functional Analysis and its Applications, vol. II A and II B*, Springer Verlag, New York, 1990.

[38] V. V. Zhikov - S. M. Kozlov - O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer Verlag, Berlin Heidelberg, 1994.

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