An artificial viscosity approach to quasistatic crack growth

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Sunto. – We introduce a new model of irreversible quasistatic crack growth in which the evolution of cracks is the limit of a suitably modified ε-gradient flow of the energy functional, as the "viscosity" parameter ε tends to zero.

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1. – Introduction

In this paper we consider the quasistatic crack growth in brittle materials in the particular case of a preassigned crack path Γ, and propose a new notion of irreversible quasistatic evolution which is based on a local stability criterion for the energy functional, rather than on a global one. To better focus on this aspect we present our approach in the simplest model case of a homogeneous isotropic material subject to antiplane shears.

We assume that the reference configuration Ω is a bounded Lipschitz domain in $\mathbb{R}^2$, and that the crack path Γ is a regular arc with one endpoint on the boundary of Ω. Moreover, we assume that there exists an initial connected crack starting from the boundary point and that the crack remains connected during the evolution. Hence, such a crack will be completely determined by its length σ. The evolution is supposed to be irreversible, so that the length of the crack will be increasing in time, and quasistatic, i.e. at each time the configuration describing the body is in equilibrium. By *configuration* we mean a pair (u, σ) where u represents the displacement orthogonal to the plane of Ω, and σ is the length of the crack.

The choice of the total energy of a configuration (u, σ) is inspired by Griffith's idea [10] that the evolution of cracks in brittle materials is the result of the competition between the elastic energy of the body and the energy needed to extend the crack. In our case the bulk part of the energy is given by the square of the L^2-norm of the gradient of u, while the surface energy will be simply given by the length σ of the crack (i.e. the toughness of the material will be assumed to be equal to one).

The evolution is driven by time-dependent imposed boundary displacements $\psi(t)$ on a part $\partial_D\Omega$ of the boundary, and applied boundary forces $g(t)$ on the remaining

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part $\partial_N\Omega$. Given a crack length σ and a boundary displacement $\psi(t)$, let $AD(\psi(t),\sigma)$ denote the set of *admissible displacements*, i.e. displacements with finite bulk energy, compatible with σ and with $\psi(t)$:

$$AD(\psi(t),\sigma) := \{u \in H^1(\Omega \setminus \Gamma(\sigma)) : u = \psi(t) \text{ on } \partial_D\Omega\}\,,$$

where $\Gamma(\sigma)$ denotes the crack of length σ and the equality on $\partial_D\Omega$ is intended in the sense of traces. The total energy at time t of a configuration (u,σ) with $u \in AD(\psi(t),\sigma)$, denoted by $\mathcal{E}(t)(u,\sigma)$, is the sum of the bulk energy and the surface energy minus the work of the applied forces $g(t)$:

$$\mathcal{E}(t)(u,\sigma) = \int_{\Omega\setminus\Gamma(\sigma)} |Du|^2\,dx + \sigma - \int_{\partial_N\Omega} g(t)u\,d\mathcal{H}^1\,,$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure.

Note that, for fixed t and σ, there exists a unique minimizer $u_{t,\sigma}$ of the energy $\mathcal{E}(t)(u,\sigma)$ in $AD(\psi(t),\sigma)$. Then let us consider the minimal energy $E(t,\sigma)$ corresponding to the boundary data $\psi(t)$ and to the crack length σ, i.e., $E(t,\sigma) := \mathcal{E}(t)(u_{t,\sigma},\sigma)$. The derivative $\partial_\sigma E(t,\sigma)$ can be computed (see Proposition 4) and it is related to the stress intensity factor of the displacement $u_{t,\sigma}$ at the tip of the crack. It plays a crucial rôle in the Griffith's criterion for the propagation of cracks. On the other hand, let us recall that the functional $\mathcal{E}(t)(u,\sigma)$, which depends on σ both through the surface energy term and through the constraint on the set of admissible displacements, is not differentiable, nor convex.

Let us define now the notion of evolution we are interested in. The *irreversible quasistatic evolution problem* consists in finding a left-continuous function of time $t \mapsto (u(t),\sigma(t))$ such that the displacement $u(t)$ at time t belongs to $AD(\psi(t),\sigma(t))$, and the following three conditions are satisfied:

(a) *local unilateral stability:* at every time $t \geq 0$

$$\begin{aligned} &\mathcal{E}(t)(u(t),\sigma(t)) \leq \mathcal{E}(t)(v,\sigma(t)) \qquad \forall\, v \in AD(\psi(t),\sigma(t)) \\ &\partial_\sigma E(t,\sigma(t)) \geq 0\,, \end{aligned}$$

(b) *irreversibility:* the map $t \mapsto \sigma(t)$ is increasing;

(c) *energy inequality:* for every $0 \leq s < t$ we have

$$\mathcal{E}(t)(u(t),\sigma(t)) \leq \mathcal{E}(s)(u(s),\sigma(s)) + \mathrm{Work}(u;s,t)\,,$$

where $\mathrm{Work}(u;s,t)$ denotes the work of external forces. A solution to this problem will be called an *irreversible quasistatic evolution*.

We show that conditions (a)-(c) are enough to ensure that at almost every time t a weak version of Griffith's criterion is satisfied (see Proposition 7). The external forces considered in condition (c) include both the applied surface loads and the surface forces generated by the imposed boundary displacement. Besides the dissipation given by the length, the inequality in condition (c) allows for instantaneous dissipation when the evolution passes from a potential well of the energy to another.

Let us recall that variational models based on an absolute minimality condition were studied, starting from the pioneering paper of Francfort and Marigo [8], e.g., in [4], [2], [7], and, in a very general setting, in [3]. The evolution problem studied therein will be called here *globally stable irreversible quasistatic evolution problem*. It consists in finding an irreversible quasistatic evolution which satisfies the *global stability* condition: at every time $t \geq 0$

$$\mathcal{E}(t)(u(t), \sigma(t)) \leq \mathcal{E}(t)(v, \sigma) \qquad \forall\, \sigma \geq \sigma(t) \quad \forall\, v \in AD(\psi(t), \sigma)\,.$$

In this case condition (c) can be replaced by the *energy balance*: the increment in stored energy plus the energy spent in crack increase equals the work of external forces. Thus the globally stable evolution fits into the general scheme of the continuous-time energetic formulation of rate-independent processes developed by Mielke and his collaborators (see, e.g., [16] and the references therein).

The global minimality condition imposes the comparison, in terms of energy, of a configuration with all admissible configurations with a longer crack, even if they are separated from the initial configuration by a large potential barrier, and hence it might generate jumps in the length of the crack that are not justified by the mechanical interpretation of the problem. That is why we are interested in studying a class of evolutions possibly different from the globally stable ones. We give in Section 4 an example of regular evolution $t \mapsto (u(t), \sigma(t))$ which satisfies Griffith's criterion at every time t and is different from the one provided by the global stability condition. A first attempt to construct an evolution based on a local minimization was made in [5]. There, as in [4], the evolution is obtained as limit of a time discretization procedure when the time step tends to zero, but, in order to localize the minimum points, in the discrete-time problems a penalization term was added to the energy functional. However, at that level of generality it was not possible to prove that the evolution constructed therein coincides with the regular one.

Therefore, the aim of our approach is twofold: to recover the possible regular evolution and to provide a global existence theorem.

We propose in this paper the notion of *approximable irreversible quasistatic evolution* defined as an irreversible quasistatic evolution $t \mapsto (u(t), \sigma(t))$ that is the limit of solutions $t \mapsto (u_\varepsilon(t), \sigma_\varepsilon(t))$ of suitable regular evolution problems. Moreover, we want to choose among possible approximation procedures with a regularizing effect, one such that $t \mapsto (u(t), \sigma(t))$ satisfies the following key property:

($\mathcal{P}$) if on a certain time interval $[t_1, t_2]$ there exists a regular function $\sigma_0(t)$ with $\sigma_0(t_1) = \sigma(t_1)$ and such that

$$\partial_\sigma E(t, \sigma_0(t)) = 0 \quad \text{and} \quad \partial^2_\sigma E(t, \sigma_0(t)) > 0 \qquad \forall\, t \in [t_1, t_2]\,,$$

then $\sigma(t) = \sigma_0(t)$ for every $t \in [t_1, t_2]$.

In other words, property ($\mathcal{P}$) ensures that whenever the Implicit Function Theorem can be applied to $\partial_\sigma E(t, \sigma) = 0$ on a time interval $[t_1, t_2]$, thus providing the existence of a regular evolution $t \mapsto (u_0(t), \sigma_0(t))$, and the evolution $t \mapsto (u(t), \sigma(t))$ coincides with the regular one at time t_1, they coincide on the whole interval where the regular one exists. We prove in Theorem 3 that if $t \mapsto \sigma_\varepsilon(t)$ is strictly increasing on the time interval $[t_1, t_2]$ then the approximable irreversible quasistatic evolution we construct in Section 3 satisfies property $(\mathcal{P})$.

Let us recall that we assume the existence of an initial crack. This hypothesis is used in the construction of the regular evolution and therefore this model is not suited for the study of the crack initiation problem. We also remark that we choose the approximating evolutions on the basis of their mathematical simplicity and the choice we make (see Section 3) does not seem to have any mechanical interpretation. Nevertheless, we think that the notion of approximable irreversible quasistatic evolution proposed here could be the starting point for the study of different approximations with a mechanical justification. For a different approach to the irreversible quasistatic crack growth see also [9].

The plan of the paper is the following. The setting of the problem is the subject of Section 2. In particular, after having introduced all ingredients necessary to define the energy functional $\mathcal{E}$, in Subsection 2.1 we change variables in order to pass to a new functional $\mathcal{F}$ whose domain is fixed (i.e. independent of σ and of t). In Subsection 2.2 we study some properties of the critical points of the energy functional. Using $\mathcal{F}$ we define in Section 3 the approximating regular evolution problem and prove in Theorem 2 the existence of a solution to the approximable quasistatic evolution problem (see Definition 4). As already mentioned, in Section 4 we provide an example of regular evolution different from the globally stable one, while in Section 5 we prove that under the additional hypothesis that $\dot{\sigma}_\varepsilon > 0$, our construction leads to an evolution satisfying property $(\mathcal{P})$ above, that is which coincides with the evolution obtained by the Implicit Function Theorem. In Section 6 we detail our results in the case of monotonically increasing in time imposed boundary displacements, and in Section 7 we provide an explicit example of geometry of the domain and boundary data such that every evolution necessarily has a discontinuity point. Even though it was believed that there exist cases when the energy, as function of the crack length, has at least a concavity interval, this is, at least to our knowledge, the first example in which the existence of a concavity interval is proved.

2. – Setting of the problem

We consider here the case of antiplane shears. Let Ω be a bounded connected open set of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. The set $\overline{\Omega}$ represents the reference configuration of an isotropic homogeneous elastic body. Let $\partial_D\Omega$ be a closed subset of $\partial\Omega$ with $\mathcal{H}^1(\partial_D\Omega) > 0$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, and let $\partial_N\Omega := \partial\Omega \backslash \partial_D\Omega$. On the Dirichlet part of the boundary, $\partial_D\Omega$, we will impose the boundary displacements, while on the Neumann part of the boundary, $\partial_N\Omega$, we will prescribe the boundary forces.

Let Γ be a simple C^3-arc and let $\gamma\colon [0, \overline{\sigma}] \to \Gamma$ be its arc-length parametrization. We assume that $\gamma(0) \in \partial_N\Omega$ and $\gamma(\sigma) \in \Omega$ for $0 < \sigma \leq \overline{\sigma}$. For technical reasons it is convenient to extend Γ until it reaches another point in $\partial_N\Omega$, so that it cuts the reference configuration Ω into two subsets. The extension will still be called Γ, and its arc-length parametrization will now be $\gamma\colon [0, \sigma_{max}] \to \Gamma$. We assume that its intersection with the boundary $\partial\Omega$ is not tangential. Let ν be a continuous unit normal vector field on Γ. Then we denote by Ω^+ the part of $\Omega\setminus\Gamma$ which is positively oriented with respect to ν, and by Ω^- the remaining part, so that $\Omega\setminus\Gamma=\Omega^+\cup\Omega^-$. Both Ω^+ and Ω^- are bounded connected sets with Lipschitz boundary. We assume that $\mathcal{H}^1(\partial_D\Omega\cap\partial\Omega^+)>0$ and $\mathcal{H}^1(\partial_D\Omega\cap\partial\Omega^-)>0$. We make the following simplifying assumption: all admissible cracks are of the form

$$\Gamma(\sigma) := \{\gamma(s) : 0 \le s \le \sigma\} \qquad \text{with } \sigma \le \overline{\sigma}\,.$$

Let $\Gamma(\sigma_0)$ with $0<\sigma_0<\overline{\sigma}$ be the initial crack.

According to Griffith's theory we assume that the energy spent to produce the crack $\Gamma(\sigma)$ is proportional to the length of the crack, and, for simplicity, we take it to be equal to σ.

Given a crack $\Gamma(\sigma)$, an admissible displacement is any function $u\in H^1(\Omega\setminus\Gamma(\sigma))$, and the bulk energy associated to the displacement u is

$$\mathcal{W}(Du) := \int_{\Omega\setminus\Gamma(\sigma)} |Du(x)|^2 dx\,,$$

where Du is the distributional gradient of u and $|\cdot|$ denotes the norm in $\mathbb{R}^2$.

In the following it will be convenient to work on a fixed time interval $[0,T]$ with $T>0$. We impose a time-dependent Dirichlet boundary condition on $\partial_D\Omega$:

$$u=\psi(t) \quad \text{ on } \partial_D\Omega,$$

where the equality on the boundary is considered in the sense of traces. We assume that $\psi(t)$ is the trace on $\partial_D\Omega$ of a bounded Sobolev function, still denoted by $t\mapsto\psi(t)$, with $\psi(t)\in H^1(\Omega)\cap L^\infty(\Omega)$.

We assume also that $\psi\in W^{1,\infty}(0,T;H^1(\Omega))$ and that

$$\sup_{t\in[0,T]}\|\psi(t)\|_{L^\infty(\Omega)}<+\infty.$$

Thus, the time derivative $t\mapsto\dot\psi(t)$ belongs to the space $L^\infty(0,T;H^1(\Omega))$ and its spatial gradient $t\mapsto D\dot\psi(t)$ belongs to the space $L^\infty(0,T;L^2(\Omega;\mathbb{R}^2))$.

We are interested in the case of time-dependent dead loads, in which the density, $g\colon[0,T]\times\partial_N\Omega\to\mathbb{R}$, of the applied surface force per unit area in the reference configuration does not depend on the displacement u. We assume that the function $t\mapsto g(t,\cdot)$ belongs to $W^{1,\infty}(0,T;L^2(\partial_N\Omega,\mathcal{H}^1))$, with time derivative denoted by $t\mapsto\dot g(t,\cdot)$. The associated potential, for a displacement u, is given by

$$\mathcal{G}(t)(u) := \int_{\partial_N\Omega} g(t,x)u(x)d\mathcal{H}^1\,.$$

Moreover, assume that for every $t\in[0,T]$ the support of $g(t,\cdot)$ does not intersect the set Γ.

For every $t\in[0,T]$, the set $AD(\psi(t),\sigma)$ of admissible displacements in Ω with finite energy, corresponding to the crack $\Gamma(\sigma)$ and to the boundary data $\psi(t)$ is given by

$$AD(\psi(t),\sigma) := \{u\in H^1(\Omega\setminus\Gamma(\sigma)) : u=\psi(t) \text{ on } \partial_D\Omega\}\,,$$

where the last equality refers to the traces of u and $\psi(t)$ on $\partial_D\Omega$. The total energy of a configuration (u,σ) with $u \in AD(\psi(t),\sigma)$ is given by

$$\mathcal{E}(t)(u,\sigma) := \mathcal{W}(Du) + \sigma - \mathcal{G}(t)(u)\,.$$

Note that it does not depend on the particular extension $\psi(t)$ chosen, but only on its value on the Dirichlet part of the boundary.

2.1. – *Moving to a fixed domain.* – Let $H^1_{\partial_D\Omega}(\Omega\setminus\Gamma(\sigma))$ denote the space of functions $u \in H^1(\Omega\setminus\Gamma(\sigma))$ whose trace on $\partial_D\Omega$ is zero. We may consider the energy as a functional defined on $H^1_{\partial_D\Omega}(\Omega\setminus\Gamma(\sigma))$ by simply writing $\tilde{u} = u + \psi(t)$ with $\tilde{u} \in AD(\psi(t),\sigma)$ and $u \in H^1_{\partial_D\Omega}(\Omega\setminus\Gamma(\sigma))$. Still the domain of the functional would depend on σ. To transform it into a functional defined on a fixed domain we consider the following change of variables.

For $\sigma \in [\sigma_0,\overline{\sigma}]$, let $\Phi(\cdot,\sigma) = \Phi_\sigma(\cdot) : \Omega \to \Omega$ be a diffeomorphism which coincides with the identity near the boundary of Ω, maps Ω^+ into Ω^+ and Ω^- into Ω^- and transforms $\Gamma(\sigma)$ into the initial crack $\Gamma(\sigma_0)$. Let $\Psi(\cdot,\sigma) = \Psi_\sigma(\cdot) := \Phi^{-1}(\cdot,\sigma)\colon \Omega \to \Omega$. Then

$$\int_{\Omega\setminus\Gamma(\sigma)} |Du + D\psi(t)|^2 dx = \int_{\Omega\setminus\Gamma(\sigma_0)} |Du(\Psi_\sigma(y)) + D\psi(t)(\Psi_\sigma(y))|^2 \det D\Psi_\sigma(y)dy\,.$$

For $u \in H^1_{\partial_D\Omega}(\Omega\setminus\Gamma(\sigma))$ define $v(y,\sigma) := u(\Psi_\sigma(y))$ and let $\tilde{\psi}(t)(y,\sigma) := \psi(t)(\Psi_\sigma(y))$. With these notations

$$\int_{\Omega\setminus\Gamma(\sigma)} |Du+D\psi(t)|^2 dx = \int_{\Omega\setminus\Gamma(\sigma_0)} |((D\Psi_\sigma)^T)^{-1}(y)(Dv(y,\sigma)+D\tilde{\psi}(t)(y,\sigma))|^2 \det D\Psi_\sigma(y)dy,$$

and the last integral can be written also in the form

$$\int_{\Omega\setminus\Gamma(\sigma_0)} \sum_{i,j\in\{1,2\}} a_{ij}(y,\sigma)D_j(v(y,\sigma)+\tilde{\psi}(t)(y,\sigma))D_i(v(y,\sigma)+\tilde{\psi}(t)(y,\sigma))dy\,,$$

with the coefficients a_{ij} given by the change of variables.

Define $A(\sigma) := (a_{ij}(\sigma))_{ij}$ and note that $a_{ij}(\sigma) \in C(\overline{\Omega})$, and $a_{ij}(\sigma) = a_{ji}(\sigma)$, for every $\sigma \in [\sigma_0,\overline{\sigma}]$, and every i,j.

We may assume that $0 < c < \|\det D\Phi_\sigma\|_\infty < C$ independently of $\sigma \in [\sigma_0,\overline{\sigma}]$, where $\|\cdot\|_\infty$ denotes the L^∞-norm on Ω. Since Γ is of class C^3, we may also choose $\Phi(\cdot,\sigma)$ (and hence $\Psi(\cdot,\sigma)$) to depend regularly on σ in such a way that, as functions of σ, the coefficients a_{ij} are of class C^2 on $[\sigma_0,\overline{\sigma}]$, uniformly in $\overline{\Omega}$. In particular, we shall use the fact that there exist positive constants $\lambda,\Lambda,\Lambda',\,L\,,L' > 0$ independent of σ, such that

$$(A(\sigma)\xi|\xi) \geq \lambda|\xi|^2 \qquad \forall\,\xi\in\mathbb{R}^2\,,\quad \forall\,x\in\overline{\Omega}\,, \tag{1}$$

where $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^2$,

$$\|(A(\sigma)\xi|\eta)\|_\infty \leq \Lambda|\xi|\,|\eta| \qquad \forall\,\xi,\eta\in\mathbb{R}^2\,, \tag{2}$$

$$\|(\partial_\sigma A(\sigma)\xi|\eta)\|_\infty \leq \Lambda'|\xi|\,|\eta| \qquad \forall\,\xi,\eta\in\mathbb{R}^2\,, \tag{3}$$

$$\|a_{ij}(\sigma') - a_{ij}(\sigma'')\|_\infty \leq L|\sigma'-\sigma''| \qquad \text{and} \tag{4}$$

$$\|\partial_\sigma a_{ij}(\sigma') - \partial_\sigma a_{ij}(\sigma'')\|_\infty \leq L'|\sigma'-\sigma''| \tag{5}$$

for every $\sigma', \sigma'' \in [\sigma_0, \overline{\sigma}]$ and $i, j = 1, 2$.

Note that, since Ψ_σ coincides with the identity near the boundary of Ω, this change of variables does not have any effect on $\mathcal{G}$:

$$\mathcal{G}(t)(u + \psi(t)) = \mathcal{G}(t)(v + \tilde{\psi}(t)) .$$

Moreover, we can neglect the dependence of $\tilde{\psi}$ on σ since, for every $\sigma \in [\sigma_0, \overline{\sigma}]$, Ψ_σ coincides with the identity near the boundary of Ω, and we may assume that the support of ψ is included in the set where, for every $\sigma \in [\sigma_0, \overline{\sigma}]$, Ψ_σ is the identity. So, from now on we suppose that $\tilde{\psi} = \psi$ and therefore the change of variables influences only the bilinear term in v.

For brevity of notation, let

$$V := H^1_{\partial_D \Omega}(\Omega \setminus \Gamma(\sigma_0)) .$$

On V we consider the norm $\|\cdot\|_V$ defined by $\|v\|_V := \|Dv\|_2$, and the scalar product $(v, w)_V := (Dv, Dw)$, where $\|\cdot\|_2$ and $(\cdot, \cdot)$ denote the norm and, respectively, the scalar product in $L^2(\Omega)$ or $L^2(\Omega \setminus \Gamma(\sigma_0); \mathbb{R}^2)$, depending on the context. Let V' denote its dual space and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between V' and V.

For every $t \in [0, T]$, $v \in V$, and $\sigma \in [\sigma_0, \overline{\sigma}]$ define

$$\begin{aligned}
&\mathcal{F}(t, v, \sigma) := \\
&= \int_{\Omega \setminus \Gamma(\sigma_0)} \sum_{i,j \in \{1,2\}} a_{ij}(\sigma) D_j(v + \psi(t)) D_i(v + \psi(t)) dx + \sigma - \int_{\partial_N \Omega} g(t)(v + \psi(t)) d\mathcal{H}^1 .
\end{aligned}$$

Then the functional $\mathcal{F}$ can be also written as

$$\begin{aligned}
\mathcal{F}(t, v, \sigma) := &\int_{\Omega \setminus \Gamma(\sigma_0)} (A(\sigma) Dv | Dv)\, dx + 2 \int_{\Omega \setminus \Gamma(\sigma_0)} (D\psi(t) | Dv)\, dx - \\
&- \int_{\partial_N \Omega} g(t) v \, d\mathcal{H}^1 + \sigma + \int_\Omega |D\psi(t)|^2 dx - \int_{\partial_N \Omega} g(t) \psi(t)\, d\mathcal{H}^1 ,
\end{aligned}$$

or

$$\begin{aligned}
&\mathcal{F}(t, v, \sigma) := \\
&= (A(\sigma) Dv, Dv) + 2(D\psi(t), Dv) - (g(t), v)_{\partial_N \Omega} + \sigma + \|D\psi(t)\|_2^2 - (g(t), \psi(t))_{\partial_N \Omega}
\end{aligned}$$

where $(\cdot, \cdot)_{\partial_N \Omega}$ denotes the scalar product in $L^2(\partial_N \Omega, \mathcal{H}^1)$. Hence the elastic energy becomes $\mathcal{F}^{el}(t, v, \sigma) := \mathcal{F}(t, v, \sigma) - \sigma$, and there exist four positive constants $\lambda_\mathcal{F}$, $\Lambda_\mathcal{F}$, $\mu_\mathcal{F}$, and $M_\mathcal{F}$, independent of t and σ, such that for every $t \in [0, T]$ and every $\sigma \in [\sigma_0, \overline{\sigma}]$

$$\begin{aligned}
\mathcal{F}^{el}(t, v, \sigma) &\geq \lambda_\mathcal{F} \|v\|_V^2 - \mu_\mathcal{F} \\
\mathcal{F}^{el}(t, v, \sigma) &\leq \Lambda_\mathcal{F} \|v\|_V^2 + M_\mathcal{F},
\end{aligned}$$

for every $v \in V$. Indeed, this follows from the uniform ellipticity of the bilinear part and standard estimates (on Ω^+ and Ω^-).

REMARK **1** The advantage of this change of variables is that now the set of admissible functions v does not depend on t, nor on σ. The same change of variables is considered, in a suitable small neighbourhood of the crack tip, in order to compute the energy release rate, or, equivalently, the stress intensity factor (see, e.g. [12], [1], and [14]).

2.2. – *Critical points of the energy.* – For every $t \in [0,T]$ the function $\mathcal{F}(t,\cdot,\cdot)\colon V\times[\sigma_0,\overline{\sigma}] \to \mathbb{R}$ is twice Fréchet partially differentiable with respect to (v,σ). The partial differential $\partial_v \mathcal{F}(t,v,\sigma)$ belongs to V', while the partial gradient $\mathrm{grad}_v \mathcal{F}(t,v,\sigma)$ is, by definition, the element of V given by

$$(\mathrm{grad}_v \mathcal{F}(t,v,\sigma), w)_V = 2(A(\sigma)Dv, Dw) + 2(\psi(t), w)_V - (g(t), w)_{\partial_N\Omega},$$

for every $w \in V$. The partial differential $\partial_\sigma \mathcal{F}(t,v,\sigma)$ is given by

$$\partial_\sigma \mathcal{F}(t,v,\sigma) = (\partial_\sigma A(\sigma)Dv, Dv) + 1\,.$$

For fixed $v \in V$ and $\sigma \in [\sigma_0,\overline{\sigma}]$, we have that $\mathcal{F}(\cdot,v,\sigma) \in W^{1,\infty}(0,T)$, with

$$\partial_t \mathcal{F}(t,v,\sigma) = 2(D\dot{\psi}(t), Dv + D\psi(t)) - (\dot{g}(t), v + \psi(t))_{\partial_N\Omega} - (g(t), \dot{\psi}(t))_{\partial_N\Omega}\,.$$

Note that by the regularity assumptions on ψ and g it follows also that the map

$$(t,v,\sigma) \mapsto (\mathrm{grad}_v \mathcal{F}(t,v,\sigma), \partial_\sigma \mathcal{F}(t,v,\sigma))$$

is continuous from $]0,T[\times V\times]\sigma_0,\overline{\sigma}[$ into $V\times\mathbb{R}$.

The second order partial differentials with respect to (v,σ) are given by

$$\begin{aligned}\langle\!\langle \partial^2_{(v,\sigma)} \mathcal{F}(t,v,\sigma)(w_1,\tau_1),(w_2,\tau_2)\rangle\!\rangle &= 2(A(\sigma)Dw_1, Dw_2) + 2(\partial_\sigma A(\sigma)Dv, Dw_1)\tau_2 + \\ &\quad + 2(\partial_\sigma A(\sigma)Dv, Dw_2)\tau_1 + (\partial^2_{\sigma\sigma} A(\sigma)Dv, Dv)\tau_1\tau_2\,,\end{aligned}$$

for every $(w_i,\tau_i) \in V\times\mathbb{R}$, $i=1,2$, where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denotes the duality product between $V'\times\mathbb{R}$ and $V\times\mathbb{R}$.

Since, for fixed t and σ, the function $v \mapsto \mathcal{F}(t,v,\sigma)$ is strictly convex, it has a unique critical point $v_{t,\sigma}$, and $v_{t,\sigma}$ is a minimum point. Also the function $u \mapsto \mathcal{E}(t)(u,\sigma)$ is strictly convex and its critical point is the unique minimum point $u_{t,\sigma} \in AD(\psi(t),\sigma)$ of $u \mapsto \mathcal{E}(t)(u,\sigma)$. The function $u_{t,\sigma}$ satisfies

$$2\int_{\Omega\setminus\Gamma(\sigma)} (Du_{t,\sigma}|Dw)dx = \int_{\partial_N\Omega} g(t,x)w\, d\mathcal{H}^1 \qquad \forall\, w \in H^1_{\partial_D\Omega}(\Omega\setminus\Gamma(\sigma))\,.$$

Proposition 1 *For fixed $t \in [0,T]$ critical points of $\mathcal{F}(t,\cdot,\cdot)$ correspond to critical points of $\mathcal{E}(t)$ in the following sense: minimum points $v_{t,\sigma} \in V$ of $v \mapsto \mathcal{F}(t,v,\sigma)$ correspond by the change of variables to minimum points $u_{t,\sigma} \in AD(\psi(t),\sigma)$ of $u \mapsto \mathcal{E}(t)(u,\sigma)$. Moreover, $\partial_\sigma \mathcal{F}(t,v_{t,\sigma},\sigma) = \partial_\sigma E(t,\sigma)$, where $E(t,\sigma) := \mathcal{E}(t)(u_{t,\sigma},\sigma)$.*

Before giving the proof we discuss some properties of the minimizers $u_{t,\sigma}$. The following result provides a useful characterization of the "singular" part of the displacement $u_{t,\sigma}$ near the tip $\gamma(\sigma)$ of the crack. For the proof we refer to [11], [12].

Proposition 2 *Let $\sigma \in [\sigma_0,\overline{\sigma}]$ and $u \in H^1(\Omega\setminus\Gamma(\sigma))$ be such that*

$$\Delta u \in L^2(\Omega\setminus\Gamma(\sigma)) \quad \textit{and} \quad \partial_\nu u = 0 \quad \textit{on } \Gamma(\sigma). \tag{6}$$

Then there exists $\kappa \in \mathbb{R}$ satisfying

$$u - \kappa\sqrt{\frac{2}{\pi}}r^{\frac{1}{2}}\sin\frac{\theta}{2} \in H^2(U\setminus\Gamma(\sigma)), \tag{7}$$

for every $U \subset\subset \Omega$ open. In (7), $r(x) := |x - \gamma(\sigma)|$ and $\theta(x)$ is the continuous function on $U \backslash \Gamma(\sigma)$ which coincides with the counterclockwise oriented angle between $\dot\gamma(\sigma)$ and $x - \gamma(\sigma)$, and vanishes on the points of the form $x = \gamma(\sigma) + h\dot\gamma(\sigma)$ for $h > 0$ sufficiently small.

The coefficient $\kappa\sqrt{2/\pi}$ represents the *stress intensity factor* associated to the displacement u at the tip $\gamma(\sigma)$. We shall use its following characterization.

PROPOSITION **3** *Let $\sigma \in [\sigma_0, \overline{\sigma}]$, $u \in H^1(\Omega\backslash\Gamma(\sigma))$ satisfying (6), and let κ be defined by (7). Then for every $\phi = (\phi_1, \phi_2) \in C_c^\infty(\Omega; \mathbb{R}^2)$ we have*

$$\kappa^2\phi(\gamma(\sigma))\dot\gamma(\sigma) = \int_\Omega \Big[((D_1u)^2 - (D_2u)^2)(D_1\phi_1 - D_2\phi_2)+ \\ +2D_1u\, D_2u(D_1\phi_2 + D_2\phi_1)\Big]\,dx + 2\int_\Omega \Delta u(D_1u\phi_1 + D_2u\phi_2)\,dx.$$

PROOF. For a complete proof we refer to [1, Proposition 2.2], see also [17, Proposition 3.2.3]. The idea is to consider $\eta > 0$ such that $\overline{B}(\gamma(\sigma), \eta) \subset \Omega$, to integrate by parts:

$$\int_{\Omega\backslash B(\gamma(\sigma),\eta)} \Big[((D_1u)^2 - (D_2u)^2)(D_1\phi_1 - D_2\phi_2) + 2D_1u\, D_2u(D_1\phi_2 + D_2\phi_1)\Big]\,dx$$

and to pass to the limit as $\eta \to 0$ using (6). ■

PROPOSITION **4** *The function $\sigma \mapsto E(t,\sigma)$ is differentiable on $[\sigma_0, \overline{\sigma}]$ and*

$$\partial_\sigma E(t,\sigma) = 1 - \kappa_{t,\sigma}^2\,,$$

where $\kappa_{t,\sigma}\sqrt{\frac{2}{\pi}}$ is the stress intensity factor associated to $u_{t,\sigma}$ at $\gamma(\sigma)$.

PROOF. The same arguments of [1, Theorem 3.3] (see also [17, Proposition 3.2.4]) can be used. To compute the partial derivative $\partial_\sigma E(t,\sigma)$ we consider a diffeomorphism similar to Φ_σ and then apply Proposition 3. ■

PROOF OF PROPOSITION 1. It follows from the change of variables, Proposition 3, and Proposition 4. ■

Let us fix $t_0 \in \,]0, T[$ and consider the map $\sigma \mapsto v_{t_0,\sigma}$. Since in this case we are not interested in the dependence on t, let us simplify the notation and set $v_\sigma := v_{t_0,\sigma}$. The following results will be used in Section 5.

PROPOSITION **5** *With the above notation, the map $\sigma \mapsto v_\sigma$ has the same regularity as $\sigma \mapsto A(\sigma)$, hence, under the regularity assumptions we made on $A(\sigma)$, it is of class $C^2(]\sigma_0, \overline{\sigma}[)$.*

PROOF. Standard arguments for elliptic PDE's allow us to obtain that for every $\sigma* \in]\sigma_0, \overline{\sigma}[$ there exists $v'_{\sigma*} \in V$ as strong limit in V of the difference quotient $\frac{v_\sigma - v_{\sigma*}}{\sigma - \sigma*}$, and the map $\sigma \mapsto v'_\sigma$ is continuous in the strong topology of V. The same arguments can be repeated to obtain that there exists $v''_{\sigma*} \in V$ as strong limit in V of the difference quotient $\frac{v'_\sigma - v'_{\sigma*}}{\sigma - \sigma*}$ and that the map $\sigma \mapsto v''_\sigma$ is continuos with respect to the strong topology in V. Note that v'_σ and v''_σ solve the following equations

$$(A(\sigma)Dv'_\sigma, Dw) + (\partial_\sigma A(\sigma)Dv_\sigma, Dw) = 0 \qquad \forall\, w \in V\,, \qquad (8)$$

$$(A(\sigma)Dv''_\sigma, Dw) + 2(\partial_\sigma A(\sigma)Dv'_\sigma, Dw) + (\partial^2_\sigma A(\sigma)Dv_\sigma, Dw) = 0 \quad \forall\, w \in V\,, \qquad (9)$$

respectively. ∎

PROPOSITION **6** *With the same notation as in Proposition 5, $v_\sigma := v_{t_0,\sigma}$, the second order differential, $\partial^2_{(\sigma,v)}\mathcal{F}(t_0, v_\sigma, \sigma)$, of $\mathcal{F}$ with respect to (v, σ) is strictly positive definite if and only if the second order derivative of the function $\sigma \mapsto \mathcal{F}(t_0, v_\sigma, \sigma)$ is strictly positive, when both exist. Moreover, by Proposition 1, this is equivalent to the fact that the second order derivative of $\sigma \mapsto E(t_0, \sigma)$ is strictly positive.*

PROOF. Indeed, as $\partial_v \mathcal{F}(t_0, v_\sigma, \sigma) = 0$, and $\sigma \mapsto v_\sigma$ is, by Proposition 5, a C^2-function, we have

$$\frac{d}{d\sigma}\mathcal{F}(t_0, v_\sigma, \sigma) = \partial_\sigma \mathcal{F}(t_0, v_\sigma, \sigma) + \langle \partial_v \mathcal{F}(t_0, v_\sigma, \sigma), v'_\sigma \rangle = \partial_\sigma \mathcal{F}(t_0, v_\sigma, \sigma)\,,$$

and, using (8),

$$\left\langle \frac{d}{d\sigma}\partial_v \mathcal{F}(t_0, v_\sigma, \sigma), w \right\rangle = \langle \partial_\sigma \partial_v \mathcal{F}(t_0, v_\sigma, \sigma), w\rangle + \langle \partial^2_{vv}\mathcal{F}(t_0, v_\sigma, \sigma)v'_\sigma, w\rangle = 0 \quad \forall\, w \in V\,.$$

Assume that

$$0 < \frac{d}{d\sigma}\partial_\sigma \mathcal{F}(t_0, v_\sigma, \sigma) = \partial^2_{\sigma\sigma}\mathcal{F}(t_0, v_\sigma, \sigma) + \langle \partial_v \partial_\sigma \mathcal{F}(t_0, v_\sigma, \sigma), v'_\sigma \rangle\,.$$

Since in our case $\langle \partial_\sigma \partial_v \mathcal{F}, w\rangle = \langle \partial_v \partial_\sigma \mathcal{F}, w\rangle$, the previous relations imply that

$$\partial^2_{\sigma\sigma}\mathcal{F}(t_0, v_\sigma, \sigma) > \langle \partial^2_{vv}\mathcal{F}(t_0, v_\sigma, \sigma)v'_\sigma, v'_\sigma\rangle\,.$$

Therefore

$$\begin{aligned}
&\langle\!\langle \partial^2_{(v,\sigma)}\mathcal{F}(t_0, v_\sigma, \sigma)(w, \tau), (w, \tau)\rangle\!\rangle = \\
&\quad = \partial^2_{\sigma\sigma}\mathcal{F}(t_0, v_\sigma, \sigma)\tau^2 + 2\langle \partial_\sigma \partial_v \mathcal{F}(t_0, v_\sigma, \sigma), w\rangle\tau + \langle \partial^2_{vv}\mathcal{F}(t_0, v_\sigma, \sigma)w, w\rangle > \\
&\quad > \langle \partial^2_{vv}\mathcal{F}(t_0, v_\sigma, \sigma)v'_\sigma, v'_\sigma\rangle\tau^2 - 2\langle \partial^2_{vv}\mathcal{F}(t_0, v_\sigma, \sigma)v'_\sigma, w\rangle\tau + \langle \partial^2_{vv}\mathcal{F}(t_0, v_\sigma, \sigma)w, w\rangle = \\
&\quad = \langle \partial^2_{vv}\mathcal{F}(t_0, v_\sigma, \sigma)(\tau v'_\sigma - w), (\tau v'_\sigma - w)\rangle \geq 0\,,
\end{aligned}$$

which shows that $\partial^2_{(\sigma,v)}\mathcal{F}(t_0, v_\sigma, \sigma)$ is strictly positive definite.

It is also easy to see that if $\partial^2_{(\sigma,v)}\mathcal{F}(t_0, v_\sigma, \sigma)$ is strictly positive definite then the second order derivative of the function $\sigma \mapsto \mathcal{F}(t_0, v_\sigma, \sigma)$ is strictly positive. ∎

3. – Irreversible quasistatic evolution

Given an initial crack length $\sigma_0 > 0$, and an initial value, u_0, of the displacement, such that the initial configuration is in equilibrium, we want to study a quasistatic evolution of configurations (u, σ) which starts from (u_0, σ_0). We are interested in the evolution until the crack length reaches a certain value $\sigma_1 < \overline{\sigma}$. We cannot avoid the solution to have jumps (even at $t = 0$) to configurations with crack lengths larger than σ_1; if this is the case, then the boundary data are not compatible with a progressive crack growth on the interval $[\sigma_0, \sigma_1]$.

Definition 1 *The* irreversible quasistatic evolution problem *consists in finding a left-continuous map $t \mapsto (u(t), \sigma(t))$, where $\sigma(t)$ represents the length of the crack up to time t, and the displacement $u(t)$ belongs to $AD(\psi(t), \sigma(t))$, which satisfies the following three conditions:*

(a) local unilateral stability: *for every t*

$$\begin{aligned} &\mathcal{E}(t)(u(t), \sigma(t)) \leq \mathcal{E}(t)(u, \sigma(t)) \qquad \forall u \in AD(\psi(t), \sigma(t)) \\ &\partial_\sigma E(t, \sigma(t)) \geq 0, \end{aligned}$$

where $E(t, \sigma)$ is defined in Proposition 1;

(b) irreversibility: *the map $t \mapsto \sigma(t)$ is increasing;*

(c) energy inequality: *for every $0 \leq s < t$ we have*

$$\begin{aligned} &\mathcal{E}(t)(u(t), \sigma(t)) \leq \mathcal{E}(s)(u(s), \sigma(s)) + \\ &+ \int_s^t \Big(2 \int_{\Omega \setminus \Gamma(\sigma(\tau))} (Du(\tau) | D\dot{\psi}(\tau)) dx - \int_{\partial_N \Omega} g(\tau) \dot{\psi}(\tau) \, d\mathcal{H}^1 - \int_{\partial_N \Omega} \dot{g}(\tau) u(\tau) d\mathcal{H}^1 \Big) d\tau \,. \end{aligned}$$

In terms of the functional $\mathcal{F}$, the irreversible quasistatic evolution problem consists in finding a left-continuous function $t \mapsto (v(t), \sigma(t))$ which satisfies the following three conditions:

($a_{\mathcal{F}}$) local unilateral stability: for every t

$$\begin{cases} \operatorname{grad}_v \mathcal{F}(t, v(t), \sigma(t)) = 0, \\ \partial_\sigma \mathcal{F}(t, v(t), \sigma(t)) \geq 0; \end{cases}$$

($b_{\mathcal{F}}$) irreversibility: the map $t \mapsto \sigma(t)$ is increasing;

($c_{\mathcal{F}}$) energy inequality: for every $0 \leq s < t$ we have

$$\mathcal{F}(t, v(t), \sigma(t)) \leq \mathcal{F}(s, v(s), \sigma(s)) + \int_s^t \partial_t \mathcal{F}(\tau, v(\tau), \sigma(\tau)) d\tau \,.$$

A solution, $t \mapsto (v(t), \sigma(t))$, to this problem is called an *irreversible quasistatic evolution for $\mathcal{F}$*.

Let us remark that, by the very construction of the functional $\mathcal{F}$, an evolution for $\mathcal{F}$ is well-defined only for cracks whose length is less than or equal to $\overline{\sigma}$.

In terms of an irreversible quasistatic evolution $t \mapsto (v(t), \sigma(t))$ associated to the functional $\mathcal{F}$, the Griffith's criterion can be expressed as:

$$\begin{cases} \dot{\sigma}(t) \geq 0 \\ \partial_\sigma \mathcal{F}(t, v(t), \sigma(t)) \geq 0 \\ \partial_\sigma \mathcal{F}(t, v(t), \sigma(t))\dot{\sigma}(t) = 0 \end{cases} \tag{10}$$

for a.e. t. Since the first two conditions are included in the definition of an irreversible quasistatic evolution, it remains to prove the last one.

PROPOSITION **7** *Let $t \mapsto (v(t), \sigma(t))$ be an irreversible quasistatic evolution for $\mathcal{F}$. Then for a.e. t we have*

$$\partial_\sigma \mathcal{F}(t, v(t), \sigma(t))\dot{\sigma}(t) = 0\,.$$

PROOF. Since $t \mapsto \sigma(t)$ is increasing, the derivative $\dot{\sigma}(t)$ exists at a.e. t. Fix t_0 such that $\dot{\sigma}(t_0)$ exists. Let us recall that, given $\sigma(t)$, the function $v(t)$ is determined as the unique solution of $\mathrm{grad}_v \mathcal{F}(t, v, \sigma(t)) = 0$. Then the hypotheses we made on $A(\sigma)$ and on the data ψ and g imply that $\dot{v}(t_0)$ exists, as strong limit in V of the difference quotient $\frac{v(t)-v(t_0)}{t-t_0}$.

From the energy inequality ($c_\mathcal{F}$) we deduce that $t \mapsto \mathcal{F}(t, v(t), \sigma(t))$ is a function with bounded variation and that for a.e. t

$$\frac{d}{dt}\mathcal{F}(t, v(t), \sigma(t)) \leq \partial_t \mathcal{F}(t, v(t), \sigma(t))\,.$$

As $\mathrm{grad}_v \mathcal{F}(t, v(t), \sigma(t)) = 0$, it follows that

$$\partial_\sigma \mathcal{F}(t, v(t), \sigma(t))\dot{\sigma}(t) \leq 0\,. \tag{11}$$

Since $\dot{\sigma}(t) \geq 0$ and $\partial_\sigma \mathcal{F}(t, v(t), \sigma(t)) \geq 0$, (11) implies the equality to be proved. ■

Going back to the energy functional $\mathcal{E}$, the Griffith's criterion now reads

$$\begin{cases} \dot{\sigma}(t) \geq 0 \\ 1 - \kappa^2(t) \geq 0 \\ (1 - \kappa^2(t))\dot{\sigma}(t) = 0 \end{cases} \tag{12}$$

for a.e. t, where $\kappa(t)\sqrt{\frac{2}{\pi}}$ is the stress intensity factor associated to the displacement $u(t)$ at the tip $\sigma(t)$ (see Proposition 2). Since by the change of variables we made, $\partial_\sigma \mathcal{F}(t, v(t), \sigma(t)) = 1 - \kappa^2(t)$, the previous proposition shows that during an irreversible quasistatic evolution the Griffith's criterion is satisfied. Note that this can be proved directly for $\mathcal{E}$, following, for instance, the lines of [5, Theorem 6.1].

As mentioned in the introduction, in the context of variational models for quasistatic crack propagation, the evolution of minimum energy configurations was studied (see, e.g. [4], [2], [7], [3]) and existence results were proved in a very general setting (see [3]). This kind of evolution is a solution to the following problem.

Definition 2 *The* globally stable irreversible quasistatic evolution problem *consists in finding a solution to the irreversible quasistatic evolution problem which satisfies the* global stability *condition: for every t*

$$\mathcal{E}(t)(u(t),\sigma(t)) \leq \mathcal{E}(t)(v,\sigma) \qquad \forall\sigma \geq \sigma(t) \quad \forall v \in AD(\psi(t),\sigma)\,.$$

During a globally stable irreversible quasistatic evolution the total energy is an absolutely continuous function of time and the energy inequality (c) becomes an equality.

However, a solution to this problem is not completely satisfactory since, in order to get the global stability, we have to compare, at each time, the energy of a configuration with the energy of all admissible configurations with larger crack lengths. This is why we use another criterion of selection: among all irreversible quasistatic evolutions we choose the *approximable* ones, i.e. those that can be obtained as limits of solutions to a regularized evolution problem.

The regularized problem considered in this paper will be given using the energy functional $\mathcal{F}$. More precisely, we define a modified ε-gradient flow for $\mathcal{F}$ in the following way. Since we are interested in an irreversible crack growth for σ varying in the interval $[\sigma_0,\sigma_1]$, we look for an increasing function $\sigma(t)$. Hence, we consider the positive part of the derivative of $\mathcal{F}$ with respect to σ. Then, we modify the evolution law for the crack length in such a way that it never reaches $\overline{\sigma}$. To this end we introduce a penalization factor $\lambda(\sigma)$ that can be any Lipschitz continuous function of σ which is equal to one for $\sigma \leq \sigma_1$, is strictly positive for $\sigma_1 < \sigma < \overline{\sigma}$, and is equal to zero for $\sigma = \overline{\sigma}$. For instance, let

$$\lambda(\sigma) := \frac{(\overline{\sigma} - (\sigma \vee \sigma_1))^+}{\overline{\sigma} - \sigma_1}\,. \tag{13}$$

In such a way the evolution is the one given by the ε-gradient flow, with the constraint that σ is increasing, on the interval $[\sigma_0,\sigma_1]$ that we are interested in, and it is modified by this artificial penalization term for $\sigma > \sigma_1$, so that we do not consider it meaningful for $\sigma > \sigma_1$.

Definition 3 *A function $t \mapsto (v_\varepsilon(t),\sigma_\varepsilon(t))$ is called a* solution to the initial value problem for the modified ε-gradient flow *for the functional $\mathcal{F}$ on* $[0,T]$

$$\begin{cases} \varepsilon\dot{v}_\varepsilon = -\mathrm{grad}_v\mathcal{F}(t,v_\varepsilon,\sigma_\varepsilon) \\ \varepsilon\dot{\sigma}_\varepsilon = (-\partial_\sigma\mathcal{F}(t,v_\varepsilon,\sigma_\varepsilon))^+\lambda(\sigma_\varepsilon)\,, \\ v_\varepsilon(0) = u_0 \\ \sigma_\varepsilon(0) = \sigma_0\,, \end{cases} \tag{14}$$

where $\lambda(\sigma)$ is given by (13), if $v_\varepsilon \in C^1([0,T];V)$, σ_ε is a C^1 increasing function from $[0,T]$ into $[\sigma_0,\overline{\sigma}]$ and the first equation in (14) is satisfied in the following sense

$$(\varepsilon\dot{v}_\varepsilon, w)_V = -(\mathrm{grad}_v\mathcal{F}(t,v_\varepsilon,\sigma_\varepsilon), w)_V \qquad \forall\, w \in V \quad \forall\, t \in [0,T]\,.$$

Note that (14) is a Cauchy problem for an ordinary differential equation in $V \times \mathbb{R}$.

THEOREM **1** *There exists a solution $(v_\varepsilon, \sigma_\varepsilon)$ to the initial value problem (14) with $\lambda(\sigma)$ given by (13), and the following energy estimate holds: for every s, $t \in [0,T]$ with $s < t$*

$$\varepsilon \int_s^t \|\dot v_\varepsilon(\tau)\|_V^2 d\tau + \varepsilon \int_s^t \frac{|\dot\sigma_\varepsilon(\tau)|^2}{\lambda(\sigma_\varepsilon(\tau))}\, d\tau + \mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \le$$
$$\le \mathcal{F}(s, v_\varepsilon(s), \sigma_\varepsilon(s)) + \int_s^t \partial_t \mathcal{F}(\tau, v_\varepsilon(\tau), \sigma_\varepsilon(\tau))\, d\tau. \tag{15}$$

PROOF. Taking into account the expressions of $\mathrm{grad}_v \mathcal{F}$ and $\partial_\sigma \mathcal{F}$, the equations in (14) can be written as

$$\begin{cases} \varepsilon(\dot v_\varepsilon, w)_V = -2(A(\sigma_\varepsilon)Dv_\varepsilon, Dw) - 2(\psi(t), w)_V + (g(t), w)_{\partial_N\Omega} \quad \forall\, w \in V \\ \varepsilon\dot\sigma_\varepsilon = (-(\partial_\sigma A(\sigma_\varepsilon)Dv_\varepsilon, Dv_\varepsilon) - 1)^+ \lambda(\sigma_\varepsilon)\,. \end{cases} \tag{16}$$

Since the vector field defining the equation (16) depends on t only through the boundary data ψ and g, it is Lipschitz continuous in t. Moreover, for fixed t, standard estimates show that it is Lipschitz continuous and bounded on the bounded subsets of $V \times \mathbb{R}$. Hence classical results on ODE's (see, e.g. [6]) give the local existence and the uniqueness of the solution. Since there exist $\alpha \in C([0,T])$ and $\beta > 0$ such that

$$(-\mathrm{grad}_v \mathcal{F}(t, v, \sigma), v)_V + \sigma(-\partial_\sigma \mathcal{F}(t, v, \sigma))^+ \lambda(\sigma) \le \alpha(t)(\|v\|_V^2 + \sigma^2) + \beta$$

for every $(v, \sigma) \in V \times \mathbb{R}$, the solution is defined on the whole interval $[0,T]$.

The function $t \mapsto \mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t))$ is then Lipschitz continuous on $[0,T]$ with derivative given for a.e. $t \in [0,T]$ by

$$\frac{d}{dt}\mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) = \partial_t \mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) + (\mathrm{grad}_v \mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)), \dot v_\varepsilon(t))_V + $$
$$+ \partial_\sigma \mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t))\dot\sigma_\varepsilon(t)\,.$$

Taking into account the equations satisfied by v_ε and σ_ε, for every s, $t \in [0,T]$ with $s < t$ we have

$$\mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) - \mathcal{F}(s, v_\varepsilon(s), \sigma_\varepsilon(s)) =$$
$$= \int_s^t \left(\partial_t \mathcal{F}(\tau, v_\varepsilon(\tau), \sigma_\varepsilon(\tau)) - \varepsilon\|\dot v_\varepsilon(\tau)\|_V^2 - \varepsilon \frac{(\dot\sigma_\varepsilon(\tau))^2}{\lambda(\sigma_\varepsilon(\tau))} \right) d\tau\,,$$

which implies (15). ■

REMARK **2** Let $t \mapsto (v_\varepsilon(t), \sigma_\varepsilon(t))$ be a solution to (14). Assume $\|v_\varepsilon(t)\|_V \le M$ for some positive constant M independent of t and ε. By (3),

$$\varepsilon\dot\sigma_\varepsilon(t) \le (\Lambda' M^2 + 1)\lambda(\sigma_\varepsilon(t)) \le C(\overline{\sigma} - \sigma_\varepsilon(t))^+\,,$$

for some constant $C > 0$. By classical results on differential inequalities (see, e.g. [13, Theorem I.6.1]) it follows that for every $t \in [0,T]$

$$\sigma_\varepsilon(t) \le \overline{\sigma} - e^{-Ct/\varepsilon}(\overline{\sigma} - \sigma_0)\,,$$

hence σ_ε never reaches $\overline{\sigma}$.

Note that, since the evolution is constrained to cracks with lengths less than or equal to $\overline{\sigma}$, Griffith's criterion is meaningful in this setting only until the length $\overline{\sigma}$ is reached. As the penalization factor $\lambda(\sigma)$ is strictly positive for $\sigma < \overline{\sigma}$, we may replace (10) by

$$\begin{cases} \dot{\sigma}(t) \geq 0 \\ \partial_\sigma \mathcal{F}(t, v(t), \sigma(t))\lambda(\sigma(t)) \geq 0 \\ \partial_\sigma \mathcal{F}(t, v(t), \sigma(t))\dot{\sigma}(t) = 0 \,. \end{cases}$$

for a.e. $t \in [0,T]$. Therefore, also the second line in the local stability condition ($\mathrm{a}_\mathcal{F}$) may be replaced by $\partial_\sigma \mathcal{F}(t, v(t), \sigma(t))\lambda(\sigma(t)) \geq 0$.

We introduce now the following notion of evolution.

Definition 4 *The* approximable irreversible quasistatic evolution problem *on the interval* $[0,T]$ *with initial data* (u_0, σ_0) *consists in finding a left-continuous map* $t \mapsto (v(t), \sigma(t))$ *from* $[0,T]$ *into* $V \times \mathbb{R}$ *which satisfies the following conditions:*

($\mathrm{a}'_\mathcal{F}$) *for every* $t \in [0,T]$

$$\begin{aligned} &\mathrm{grad}_v \mathcal{F}(t, v(t), \sigma(t)) = 0 \\ &\partial_\sigma \mathcal{F}(t, v(t), \sigma(t))\lambda(\sigma(t)) \geq 0 \,; \end{aligned}$$

($\mathrm{b}_\mathcal{F}$) *the map* $t \mapsto \sigma(t)$ *is increasing;*

($\mathrm{c}_\mathcal{F}$) *for every* $0 \leq s < t \leq T$

$$\mathcal{F}(t, v(t), \sigma(t)) \leq \mathcal{F}(s, v(s), \sigma(s)) + \int_s^t \partial_t \mathcal{F}(\tau, v(\tau), \sigma(\tau))d\tau \,;$$

($\mathrm{d}_\mathcal{F}$) *the pair* $(v(t), \sigma(t))$ *is the limit of solutions* $(v_\varepsilon(t), \sigma_\varepsilon(t))$ *of the* modified ε-gradient flow *for* $\mathcal{F}$ *with initial conditions* $v_\varepsilon(0) = u_0$ *and* $\sigma_\varepsilon(0) = \sigma_0$*, along a suitable sequence* $\varepsilon_h \to 0$*, in the sense that, as* $\varepsilon_h \to 0$

$$\begin{aligned} &\sigma_{\varepsilon_h}(t) \to \sigma(t) \quad \textit{and} \\ &v_{\varepsilon_h}(t) \to v(t) \quad \textit{strongly in } V \quad \textit{for a.e. } t \in [0,T] \,. \end{aligned}$$

A solution $t \mapsto (v(t), \sigma(t))$ *to this problem is called an* approximable quasistatic evolution *for* $\mathcal{F}$*.*

We are now in a position to state the main result of this paper.

Theorem 2 *There exists a solution to the approximable irreversible quasistatic evolution problem on* $[0,T]$ *with initial condition* (u_0, σ_0)*.*

Remark 3 The fact that an approximable quasistatic evolution starts from (u_0, σ_0) means only that for every $\varepsilon > 0$, $v_\varepsilon(0) = u_0$ and $\sigma_\varepsilon(0) = \sigma_0$. We may always set $(v(0), \sigma(0)) := (u_0, \sigma_0)$, but in general v and σ are not continuous in $t = 0$. The only case in which (u_0, σ_0) is the initial value for the evolution in a "classical" sense, is when (u_0, σ_0) is the absolute minimum point of $\mathcal{F}(0, \cdot, \cdot)$. Indeed, in this case, by semicontinuity and by the energy inequality ($\mathrm{c}_\mathcal{F}$), it is easy to see that $t \mapsto \mathcal{F}(t, v(t), \sigma(t))$ is continuous in $t = 0$.

Proof of Theorem 2. For $\varepsilon > 0$ let $(v_\varepsilon, \sigma_\varepsilon)$ be the solution of the modified ε-gradient flow with initial data (u_0, σ_0). Let $t \in [0, T]$. The estimates we have on $\mathcal{F}$ together with (15) between $s = 0$ and t imply

$$\lambda_{\mathcal{F}} \|v_\varepsilon(t)\|_V^2 \leq \mu_{\mathcal{F}} + \mathcal{F}(0, u_0, \sigma_0) + \int_0^t (a(\tau)\|v_\varepsilon(\tau)\|_V^2 + b(\tau))d\tau$$

for some functions $a, b \in L^\infty(0, T)$ which depend only on the data ψ and g. Then, by Gronwall's Lemma, there exists a positive constant $C > 0$ independent of t and ε, whose value may change from line to line, such that

$$\|v_\varepsilon(t)\|_V \leq C \qquad \forall t \in [0, T]\,. \tag{17}$$

By (15) we now get

$$\varepsilon \|\dot{v}_\varepsilon\|^2_{L^2(0,T;V)} \leq C \tag{18}$$

$$\varepsilon \|\dot{\sigma}_\varepsilon\|^2_{L^2(0,T)} \leq C\,. \tag{19}$$

Let $\varepsilon_h \to 0$. By Helly's Theorem, there exists a subsequence, still denoted by ε_h, and an increasing function $\sigma\colon [0, T] \to [\sigma_0, \overline{\sigma}]$ such that

$$\sigma_{\varepsilon_h}(t) \to \sigma(t) \qquad \text{for every } t \in [0, T]\,.$$

The estimate (17) implies that there exists a function $v \in L^2(0, T; V)$ such that, for some subsequence that we still denote by ε_h,

$$v_{\varepsilon_h} \rightharpoonup v \qquad \text{weakly in } L^2(0, T; V)\,, \tag{20}$$

while, by (18),

$$\varepsilon_h \dot{v}_{\varepsilon_h} \to 0 \qquad \text{strongly in } L^2(0, T; V)\,. \tag{21}$$

Hence

$$\varepsilon_h(\dot{v}_{\varepsilon_h}(t), w)_V = (-\mathrm{grad}_v \mathcal{F}(t, v_{\varepsilon_h}(t), \sigma_{\varepsilon_h}(t)), w)_V \to (-\mathrm{grad}_v \mathcal{F}(t, v(t), \sigma(t)), w)_V = 0\,,$$

for every $w \in V$ and for a.e. $t \in [0, T]$. In particular, choosing $w = v_{\varepsilon_h}(t)$ and taking into account the explicit form of $\mathrm{grad}_v \mathcal{F}$, the first equality in the last formula gives

$$2\int_0^T (A(\sigma_{\varepsilon_h}(t))Dv_{\varepsilon_h}(t), Dv_{\varepsilon_h}(t))dt = \\ = \int_0^T \Big(- \varepsilon_h(\dot{v}_{\varepsilon_h}(t), v_{\varepsilon_h}(t))_V - 2(\psi(t), v_{\varepsilon_h}(t))_V + (g(t), v_{\varepsilon_h}(t))_{\partial_N \Omega}\Big) dt$$

By (20) and (21) we can pass to the limit in the right-hand side of the above equality and deduce that

$$2\int_0^T (A(\sigma_{\varepsilon_h}(t))Dv_{\varepsilon_h}(t), Dv_{\varepsilon_h}(t))dt \to \int_0^T \Big(- 2(\psi(t), v(t))_V + (g(t), v(t))_{\partial_N \Omega}\Big) dt$$

From $\mathrm{grad}_v \mathcal{F}(t, v(t), \sigma(t)) = 0$ we thus get

$$\int_0^T (A(\sigma_{\varepsilon_h}(t))Dv_{\varepsilon_h}(t), Dv_{\varepsilon_h}(t))dt \to \int_0^T (A(\sigma(t))Dv(t), Dv(t))dt\,, \tag{22}$$

which gives the strong convergence in V of $v_{\varepsilon_h}(t)$ to $v(t)$ for a.e. $t \in [0,T]$. More in detail, as

$$(A(\sigma_{\varepsilon_h}(t))(Dv_{\varepsilon_h}(t) - Dv(t)), (Dv_{\varepsilon_h}(t) - Dv(t))) =$$
$$= (A(\sigma_{\varepsilon_h}(t))Dv_{\varepsilon_h}(t), Dv_{\varepsilon_h}(t)) - 2(A(\sigma_{\varepsilon_h}(t))Dv_{\varepsilon_h}(t), Dv(t)) + (A(\sigma_{\varepsilon_h}(t))Dv(t), Dv(t)),$$

from (22), (20), and the convergence of σ_{ε_h}, we deduce that for a.e. $t \in [0,T]$

$$\lim_{h\to+\infty} (A(\sigma_{\varepsilon_h}(t))(Dv_{\varepsilon_h}(t) - Dv(t)), (Dv_{\varepsilon_h}(t) - Dv(t))) = 0\,,$$

which, by the coerciveness hypothesis (1), implies the desired convergence.

By (19), passing possibly to a further (not relabelled) subsequence, we have that $\varepsilon_h \dot\sigma_{\varepsilon_h}(t) \to 0$ for a.e. $t \in [0,T]$. Taking into account the equation satisfied by σ_{ε_h}, we obtain that $(-\partial_\sigma \mathcal{F}(t, v(t), \sigma(t)))^+ \lambda(\sigma(t)) = 0$ for a.e. $t \in [0,T]$.

When passing to the limit in (15), we neglect the terms containing the norms of the time derivatives of v_{ε_h} and σ_{ε_h}, and thus get that for a.e. s, $t \in [0,T]$ with $s < t$

$$\mathcal{F}(t, v(t), \sigma(t)) \le \mathcal{F}(s, v(s), \sigma(s)) + \int_s^t \partial_t \mathcal{F}(\tau, v(\tau), \sigma(\tau))d\tau\,. \tag{23}$$

(By semicontinuity the estimate holds true for every $t \in [0,T]$.)

Therefore the function $t \mapsto (v(t), \sigma(t))$ satisfies all conditions in the definition of an approximable quasistatic evolution, except possibly for the left-continuity. Hence we modify it in the following way. Since σ is increasing, for every $t \in [0,T]$ there exists the limit $\sigma^\ominus(t) := \lim_{s\to t-} \sigma(s)$. Let $v^\ominus(t)$ be the unique solution to $\mathrm{grad}_v \mathcal{F}(t, v, \sigma^\ominus(t)) = 0$. Then $v(s) \to v^\ominus(t)$ strongly in V as $s \to t-$, $\sigma(t) = \sigma^\ominus(t)$ and $v(t) = v^\ominus(t)$ for a.e. $t \in [0,T]$. By construction, the map $t \mapsto (v^\ominus(t), \sigma^\ominus(t))$ is left-continuous from $[0,T]$ into $V \times [\sigma_0, \overline{\sigma}]$. Moreover, $\partial_\sigma \mathcal{F}(t, v^\ominus(t), \sigma^\ominus(t))\lambda(\sigma^\ominus(t)) \ge 0$ for every $t \in [0,T]$. Let s, $t \in [0,T]$ with $s < t$, and let $s_n \to s-$, $t_n \to t-$ be such that (23) holds for s_n and t_n. Passing to the limit in (23) as $n \to +\infty$ we obtain

$$\mathcal{F}(t, v^\ominus(t), \sigma^\ominus(t)) \le \mathcal{F}(s, v^\ominus(s), \sigma^\ominus(s)) + \int_s^t \partial_t \mathcal{F}(\tau, v^\ominus(\tau), \sigma^\ominus(\tau))d\tau\,,$$

so that we conclude that $t \mapsto (v^\ominus(t), \sigma^\ominus(t))$ is an approximable quasistatic evolution for $\mathcal{F}$ on $[0,T]$ which starts from (u_0, σ_0). ■

Remark 4 Let $t \mapsto (v(t), \sigma(t))$ be an approximable irreversible quasistatic evolution on $[0,T]$. If $\bar t \in [0,T]$ is a discontinuity point of $t \mapsto \mathcal{F}(t, v(t), \sigma(t))$ then

$$\lim_{t\to\bar t+} \mathcal{F}(t, v(t), \sigma(t)) \le \mathcal{F}(\bar t, v(\bar t), \sigma(\bar t))\,.$$

Indeed, note that at every time t the function $t \mapsto \sigma(t)$ has a right limit. Let $\sigma^\oplus(\bar t) := \lim_{t\to\bar t+} \sigma(t)$, and let $v^\oplus(\bar t)$ be the solution to $\mathrm{grad}_v \mathcal{F}(\bar t, v, \sigma^\oplus(\bar t)) = 0$. By the regularity assumptions made on the data, we have that $v(t)$ converges to $v^\oplus(\bar t)$ strongly in V, and hence, using $(\mathrm{c}_\mathcal{F})$, we obtain

$$\lim_{t\to\bar t+} \mathcal{F}(t, v(t), \sigma(t)) = \mathcal{F}(\bar t, v^\oplus(\bar t), \sigma^\oplus(\bar t)) \le \mathcal{F}(\bar t, v(\bar t), \sigma(\bar t))\,.$$

4. – Example: a regular evolution which is not globally stable

The example we provide in this section illustrates the highly expected fact that the notion of evolution given in Definition 1 and the globally stable one may lead to different results. We choose the boundary data in such a way that we can construct a regular evolution during which the crack length grows continuously, while the global minimality condition imposes a jump in the crack length.

Let us start with the following simple setting in which Ω is the unit ball in $\mathbb{R}^2$ and Γ is the segment $[-1,1]\times\{0\}$. Let $\Gamma_1 := [-1,0]\times\{0\}$,

$$w_a(\rho,\theta) := \sqrt{\frac{2}{\pi}}\,\rho^{\frac{1}{2}}\sin\frac{\theta}{2} + a\,\rho^{\frac{3}{2}}\sin\frac{3\theta}{2}\,,$$

where (ρ,θ) are polar coordinates centred in the origin, with $\theta\in[-\pi,\pi]$, and let v_a be the harmonic function in $\Omega^+ := \{(x_1,x_2)\in\Omega : x_2>0\}$ which satisfies the homogeneous Neumann condition on Γ and coincides with w_a on the upper semicircle.

Let us study the behaviour of

$$I_a := \frac{1}{a^2}\Big\{\int_{\Omega\setminus\Gamma_1}|Dw_a|^2 - 2\int_{\Omega^+}|Dv_a|^2\Big\}$$

as a tends to $+\infty$. We note that

$$I_a = \int_{\Omega\setminus\Gamma_1}\Big|D\Big(\frac{w_a}{a}\Big)\Big|^2 - 2\int_{\Omega^+}\Big|D\Big(\frac{v_a}{a}\Big)\Big|^2\,.$$

Moreover, as $a\to+\infty$, the functions $\dfrac{w_a}{a}$ converge strongly in $H^1(\Omega\setminus\Gamma_1)$ to the function w which in polar coordinates is given by $w(\rho,\theta) := \rho^{\frac{3}{2}}\sin\frac{3\theta}{2}$. The function w is the solution of the minimum problem on the domain with the crack Γ_1 corresponding to the Dirichlet condition $w(1,\theta)=\sin\frac{3\theta}{2}$. Analogously, the functions $\dfrac{v_a}{a}$ converge strongly in $H^1(\Omega^+)$ to the function v which solves the Neumann-Dirichlet problem on Ω^+ with $v(1,\theta)=\sin\frac{3\theta}{2}$. Therefore

$$I_a \longrightarrow \int_{\Omega\setminus\Gamma_1}|Dw|^2 - 2\int_{\Omega^+}|Dv|^2 > 0\,.$$

This shows that

$$\int_{\Omega\setminus\Gamma_1}|Dw_a|^2 - 2\int_{\Omega^+}|Dv_a|^2 \to +\infty \qquad \text{as } a\to+\infty\,,$$

and, hence, for a large enough, the total energy corresponding to w_a: $\int_{\Omega\setminus\Gamma_1}|Dw_a|^2+1$ is larger than the total energy corresponding to the complete cut Γ and to the same boundary displacement: $2\int_{\Omega^+}|Dv_a|^2+2$.

Of course in our setting a complete crack is not allowed: all admissible cracks are of the type $[-1,s]\times\{0\}$, with $s\le\overline{s}<1$, but this is not a problem since the results in [4] provide the continuity of the energy with respect to the crack length, so that solutions corresponding to "almost" complete cracks have energy arbitrarily close to $2\int_{\Omega^+}|Dv_a|^2+2$. Therefore the above considerations can still be applied to our problem, provided $\overline{s}$ is "close" enough to 1.

More precisely, in [4] the following situation is considered. Let $\psi_s,\psi\in H^1(\Omega)$, let v_s be the solution to the minimum problem

$$\min\left\{\int_{\Omega\setminus[-1,s]\times\{0\}}|Dv|^2 : v\in H^1(\Omega\setminus[-1,s]\times\{0\}), v=\psi_s \text{ on } \partial\Omega\right\},$$

and let v_+ and v_- be the solutions to the minimum problems

$$\min\left\{\int_{\Omega^+}|Dv|^2 : v\in H^1(\Omega^+), v=\psi \text{ on } \partial\Omega\cap\partial\Omega^+\right\},$$

and

$$\min\left\{\int_{\Omega^-}|Dv|^2 : v\in H^1(\Omega^-), v=\psi \text{ on } \partial\Omega\cap\partial\Omega^-\right\},$$

respectively, where $\Omega^- := \{(x_1,x_2)\in\Omega : x_2<0\}$. Assume $\psi_s\to\psi$ strongly in $H^1(\Omega)$. Then, by Theorem 4.1 in [4], given $\varepsilon>0$ there exists $s_\varepsilon<1$ such that for every $s\in[s_\varepsilon,1[$,

$$\left|\int_{\Omega^+}|Dv_+|^2+\int_{\Omega^-}|Dv_-|^2-\int_{\Omega\setminus[-1,s]\times\{0\}}|Dv_s|^2\right|<\varepsilon\,.$$

Here $\psi_s=\psi=w_a$, and due to the symmetry of the problem, $\int_{\Omega^+}|Dv_+|^2=\int_{\Omega^-}|Dv_-|^2$.

Starting from these simple remarks let us now construct a regular evolution in which the crack length evolves continuously, while, with the same imposed boundary data, the globally stable evolution exhibits a jump in the crack length.

For every $t\in[0,1[$ let $\Gamma(t):=[-1,t]\times\{0\}$ and let $w_a(t)$ be the function given in polar coordinates centred in $(t,0)$ by

$$w_a(t,\rho,\theta)=\sqrt{\frac{2}{\pi}}\rho^{\frac{1}{2}}\sin\frac{\theta}{2}+a\rho^{\frac{3}{2}}\sin\frac{3\theta}{2}\,.$$

Since $w_a(t)$ is the unique solution to the minimum problem

$$\min\left\{\int_{\Omega\setminus\Gamma(t)}|Dv|^2 : v\in H^1(\Omega\setminus\Gamma(t)), v=w_a(t) \text{ on } \partial\Omega\right\},$$

the above arguments show that

$$t\mapsto(w_a(t),1+t)$$

is the unique regular quasistatic evolution.

On the other hand, for a large enough, the globally stable evolution with the same imposed boundary displacement is given by the solution corresponding to the maximum crack length $\overline{s}$ allowed.

5. – Quasistatic evolution and the Implicit Function Theorem

In this section we show that, under suitable regularity assumptions, the solution to the modified ε-gradient flow converges to the continuous solution of the quasistatic evolution problem given by the Implicit Function Theorem.

THEOREM **3** *Assume that in* $(t^0, \sigma^0) \in [0, T[\times [\sigma_0, \sigma_1[$ *the following conditions are satisfied*

$$\begin{aligned} \partial_\sigma E(t^0, \sigma^0) &= 0 \\ \partial^2_\sigma E(t^0, \sigma^0) &> 0\,. \end{aligned}$$

Then there exists a time interval $[t^0, t^1]$ *and a unique Lipschitz continuous function* $\sigma^0 : [t^0, t^1] \to [\sigma^0, \sigma_1]$ *such that*

$$\partial_\sigma E(t, \sigma^0(t)) = 0 \qquad \forall\, t \in [t^0, t^1]\,.$$

Moreover, if $(v_\varepsilon, \sigma_\varepsilon)$ *is the solution to the modified* ε*-gradient flow and the following two conditions are satisfied:*

$$\begin{aligned} &\dot\sigma_\varepsilon(t) > 0 \qquad \forall\, t \in [t^0, t^1] \\ &\sigma_\varepsilon(t^0) \to \sigma^0\,, \end{aligned}$$

then $\sigma_\varepsilon(t) \to \sigma^0(t)$ *and* $E(t, \sigma_\varepsilon(t)) \to E(t, \sigma^0(t))$ *for every* $t \in [t^0, t^1]$.

Proof. The first part of the theorem follows from the Implicit Function Theorem applied to $\partial_\sigma E$ at (t^0, σ^0).

As for the second part, let us remark that even if there are not at the moment general theorems guaranteeing the strict monotonicity of σ_ε during the approximation process, in many cases this will follow, for a suitable choice of the boundary data, from a symmetry argument.

We now prove the theorem in an equivalent form for the functional $\mathcal{F}$. Indeed, since $\partial_\sigma E(t, \sigma) = \partial_\sigma \mathcal{F}(t, v_{t,\sigma}, \sigma)$ (see Proposition 1), if the second order derivative $\partial^2_\sigma E(t^0, \sigma^0) > 0$, then also $\frac{d}{d\sigma}\partial_\sigma \mathcal{F}(t^0, v_{t^0,\sigma^0}, \sigma^0) > 0$, and this last condition is equivalent to the fact that the second order partial differential $\partial^2_{(v,\sigma)}\mathcal{F}(t^0, v_{t^0,\sigma^0}, \sigma^0)$ is strictly positive definite (see Proposition 6).

THEOREM **4** *Assume that in* $(t^0, v^0, \sigma^0) \in [0, T[\times V \times [\sigma_0, \sigma_1[$ *the following conditions are satisfied*

$$\begin{cases} \mathrm{grad}_v \mathcal{F}(t^0, v^0, \sigma^0) = 0\,, \\ \partial_\sigma \mathcal{F}(t^0, v^0, \sigma^0) = 0\,, \end{cases}$$

and the second order differential, $\partial^2_{(v,\sigma)}\mathcal{F}(t^0, v^0, \sigma^0)$*, of* $\mathcal{F}$ *with respect to* (v, σ) *is strictly positive definite, i.e. there exists* $\alpha > 0$ *such that*

$$\langle\!\langle \partial^2_{(v,\sigma)}\mathcal{F}(t^0, v^0, \sigma^0)(w, \tau), (w, \tau) \rangle\!\rangle \geq \alpha(\|w\|^2_V + |\tau|^2) \qquad \forall\, w \in V \quad \forall\, \tau \in \mathbb{R}\,.$$

Then there exist a time interval $[t^0, t^1]$ *and a unique Lipschitz continuous function* $(v^0, \sigma^0) : [t^0, t^1] \to V \times [\sigma^0, \sigma_1]$ *such that*

$$\begin{cases} \mathrm{grad}_v \mathcal{F}(t, v^0(t), \sigma^0(t)) = 0 \\ \partial_\sigma \mathcal{F}(t, v^0(t), \sigma^0(t)) = 0 \end{cases}$$

for every $t \in [t^0, t^1]$.

Let $(v_\varepsilon, \sigma_\varepsilon)$ *be the solution of the modified* ε*-gradient flow for* $\mathcal{F}$ *given by Theorem 1 and assume that*

$$\begin{aligned} v_\varepsilon(t^0) &\to v^0 \qquad \textit{strongly in } V \quad \textit{and} \\ \sigma_\varepsilon(t^0) &\to \sigma^0 \qquad \textit{as } \varepsilon \to 0\,. \end{aligned}$$

Assume that $\dot\sigma_\varepsilon(t) > 0$ *and* $\sigma_\varepsilon(t) < \sigma_1$ *for every* $t \in [t^0, t^1]$*. Then* $v_\varepsilon(t) \to v^0(t)$ *strongly in* V *and* $\sigma_\varepsilon(t) \to \sigma^0(t)$ *for every* $t \in [t^0, t^1]$.

PROOF. By our assumptions on the data, $\partial^2_{(v,\sigma)}\mathcal{F}(t, v, \sigma)$ (see Subsection 2.2) is continuous with respect to $(t, v, \sigma) \in [0, T] \times V \times [\sigma_0, \overline{\sigma}]$. Moreover, the function $t \mapsto \partial_t \mathrm{grad}_v \mathcal{F}(t, v, \sigma)$ belongs to $L^\infty(0, T; V)$, while $\partial_t \partial_\sigma \mathcal{F}(t, v, \sigma) = 0$. By the Implicit Function Theorem (see, e.g., [15]) applied in (t^0, v^0, σ^0) to

$$\begin{cases} \mathrm{grad}_v \mathcal{F}(t, v, \sigma) = 0 \\ \partial_\sigma \mathcal{F}(t, v, \sigma) = 0\,, \end{cases}$$

it follows that there exist a time interval $[t^0, t^1]$ and a unique Lipschitz continuous function $(v^0, \sigma^0)\colon [t^0, t^1] \to V \times [\sigma^0, \sigma_1)$ such that

$$\begin{cases} \mathrm{grad}_v \mathcal{F}(t, v^0(t), \sigma^0(t)) = 0 \\ \partial_\sigma \mathcal{F}(t, v^0(t), \sigma^0(t)) = 0 \end{cases} \tag{24}$$

for every $t \in [t^0, t^1]$. By a compactness argument, changing possibly the value of α, we may assume that there exist $\alpha > 0$ and $r > 0$ such that for every $t \in [t^0, t^1]$, for every $v \in B_r(v^0(t)) \subset V$, and for every $\sigma \in (\sigma^0(t) - r, \sigma^0(t) + r)$

$$\langle\!\langle \partial^2_{(v,\sigma)} \mathcal{F}(t, v, \sigma)(w, \tau), (w, \tau) \rangle\!\rangle \geq \alpha(\|w\|_V^2 + |\tau|^2) \qquad \forall w \in V \quad \forall \tau \in \mathbb{R}\,. \tag{25}$$

Restricting the time interval, if necessary, we have $\sigma^0(t) + r < \sigma_1$ for every $t \in [t^0, t^1]$.

Let $0 < r' < r$ be a number that we shall choose later. For every $\varepsilon > 0$ small enough we have $\|v_\varepsilon(t^0) - v^0\|_V < r'$ and $|\sigma_\varepsilon(t^0) - \sigma^0| < r'$. By continuity, there exists a time interval, depending on ε, on which these inequalities hold. Let τ_ε be the largest time such that for $t < \tau_\varepsilon$, $\|v_\varepsilon(t) - v^0(t)\|_V < r'$ and $|\sigma_\varepsilon(t) - \sigma^0(t)| < r'$. In particular, for $t < \tau_\varepsilon$ we have $\sigma_\varepsilon(t) < \sigma_1$, and hence, $\lambda(\sigma_\varepsilon(t)) = 1$.

We want to prove that $\tau_\varepsilon = t^1$. Assume by contradiction that $\tau_\varepsilon < t^1$. Taking $v_\varepsilon(t) - v^0(t)$ as test function in the equation satisfied by v_ε, multiplying by $\sigma_\varepsilon(t) - \sigma^0(t)$ the equation satisfied by σ_ε, and taking also into account (24), we obtain

$$\begin{aligned} &\frac{\varepsilon}{2}\frac{d}{dt}\|v_\varepsilon(t) - v^0(t)\|_V^2 + \frac{\varepsilon}{2}\frac{d}{dt}|\sigma_\varepsilon(t) - \sigma^0(t)|^2 = \\ &= -(\mathrm{grad}_v \mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) - \mathrm{grad}_v \mathcal{F}(t, v^0(t), \sigma^0(t)), v_\varepsilon(t) - v^0(t))_V + \\ &\qquad + \big(- \partial_\sigma \mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) + \partial_\sigma \mathcal{F}(t, v^0(t), \sigma^0(t))\big)(\sigma_\varepsilon(t) - \sigma^0(t)) - \\ &\qquad - \varepsilon(\dot v^0(t), v_\varepsilon(t) - v^0(t))_V - \varepsilon\dot\sigma^0(t)(\sigma_\varepsilon(t) - \sigma^0(t))\,. \end{aligned}$$

Setting

$$\zeta_\varepsilon(t) := \|v_\varepsilon(t) - v^0(t)\|_V^2 + |\sigma_\varepsilon(t) - \sigma^0(t)|^2 ,$$

from (25) it follows that

$$\begin{aligned}\frac{\varepsilon}{2}\dot\zeta_\varepsilon(t) &\le -\alpha\zeta_\varepsilon(t) - \varepsilon(\dot v^0(t), v_\varepsilon(t) - v^0(t))_V - \varepsilon\dot\sigma^0(t)(\sigma_\varepsilon(t) - \sigma^0(t)) \le \\ &\le -\alpha\zeta_\varepsilon(t) + \frac{\varepsilon}{2}\|\dot v^0(t)\|_V^2 + \frac{\varepsilon}{2}\|v_\varepsilon(t) - v^0(t)\|_V^2 + \frac{\varepsilon}{2}|\dot\sigma^0(t)|^2 + \frac{\varepsilon}{2}|\sigma_\varepsilon(t) - \sigma^0(t)|^2 \le \\ &\le (-\alpha + \frac{\varepsilon}{2})\zeta_\varepsilon(t) + \frac{\varepsilon}{2}\beta \qquad \forall t \in [t^0, \tau_\varepsilon) ,\end{aligned}$$

where β is an upper bound for $\|\dot v^0(t)\|_V^2 + |\dot\sigma^0(t)|^2$ on $[t^0, t^1]$.

Hence

$$\zeta_\varepsilon(t) \le \Big(\zeta_\varepsilon(t^0) - \frac{\beta\varepsilon}{2\alpha - \varepsilon}\Big)e^{(-\frac{2\alpha}{\varepsilon}+1)(t-t^0)} + \frac{\beta\varepsilon}{2\alpha - \varepsilon} \qquad \forall\, t \in [t^0, \tau_\varepsilon) . \tag{26}$$

Therefore, choosing now r' small enough, from (26) we get that also $\|v_\varepsilon(\tau_\varepsilon) - v^0(\tau_\varepsilon)\|_V < r$ and $|\sigma_\varepsilon(\tau_\varepsilon) - \sigma^0(\tau_\varepsilon)| < r$. By continuity, these inequalities hold also for some $t > \tau_\varepsilon$, in contradiction with the maximality of τ_ε. Thus we deduce that $\tau_\varepsilon = t^1$, so that (26) holds for every $t \in [t^0, t^1]$. Passing to the limit in (26) as $\varepsilon \to 0$ we get the conclusion. ■

By the change of variables that defines the functional $\mathcal{F}$, and by the uniqueness of the regular evolution given by the Implicit Function Theorem, it follows that the regular evolution in Theorem 4 corresponds to the one in Theorem 3.

Proof of Theorem 3 continued. Let $(v_\varepsilon, \sigma_\varepsilon)$ be the solution to the modified ε-gradient flow for $\mathcal{F}$. By Theorem 4, $\sigma_\varepsilon(t) \to \sigma^0(t)$ and $v_\varepsilon(t) \to v^0(t)$ strongly in V for every $t \in [t^0, t^1]$. Since the function $v \mapsto \mathrm{grad}_v\mathcal{F}(t, v, \sigma)$ is continuous from V to V with respect to the strong topology, it follows that

$$\mathrm{grad}_v\mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \to \mathrm{grad}_v\mathcal{F}(t, v^0(t), \sigma^0(t)) = 0 .$$

Let $\overline{v}_\varepsilon(t)$ be the element of V associated to $u_{t,\sigma_\varepsilon(t)}$ by the change of variables. As $\mathrm{grad}_v\mathcal{F}(t, \overline{v}_\varepsilon(t), \sigma_\varepsilon(t)) = 0$ we deduce that $v_\varepsilon(t) - \overline{v}_\varepsilon(t) \to 0$ strongly in V. This implies that

$$\mathcal{F}(t, \overline{v}_\varepsilon(t), \sigma_\varepsilon(t)) - \mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \to 0 ,$$

On the other hand,

$$\begin{aligned}&\mathcal{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \to \mathcal{F}(t, v^0(t), \sigma^0(t)) = E(t, \sigma^0(t)) \\ &\mathcal{F}(t, \overline{v}_\varepsilon(t), \sigma_\varepsilon(t)) = E(t, \sigma_\varepsilon(t)) ,\end{aligned}$$

so that we conclude that $E(t, \sigma_\varepsilon(t)) \to E(t, \sigma^0(t))$ for every $t \in [t^0, t^1]$. ■

6. – Monotonically increasing loadings

In this section we detail our study in the case of monotonically increasing loadings. Accordingly, assume $\psi(t) := t\psi_0$, with $\psi_0 \in H^1(\Omega)$, and $g(t) = 0$, and define

$$E(\sigma) := \min\{\|Du\|_2^2 : u \in AD(\psi_0, \sigma)\}.$$

Since $H^1(\Omega \setminus \Gamma(\sigma')) \subset H^1(\Omega \setminus \Gamma(\sigma''))$ for $\sigma' < \sigma''$, we have that $E(\sigma') \geq E(\sigma'')$, so that the function $\sigma \mapsto E(\sigma)$ is decreasing.

Let $(u(\cdot), \sigma(\cdot))$ be an irreversible quasistatic evolution. Recalling that $u(t)$ is the minimum point of $\|Du\|_2$ on $AD(t\psi_0, \sigma(t))$, we have that $\|Du(t)\|_2^2 = t^2 E(\sigma(t))$. We may now express conditions (a), (b) and (c) of Definition 1 of an irreversible quasistatic evolution, in terms of $\sigma(t)$, and, in the case of this particular choice of the data, we obtain:

(a') $1 + t^2 E'(\sigma(t)) \geq 0$ for every $t \geq 0$;

(b') the map $t \mapsto \sigma(t)$ is increasing;

(c') $t^2 E(\sigma(t)) + \sigma(t) \leq s^2 E(\sigma(s)) + \sigma(s) + 2\int_s^t \tau E(\sigma(\tau))\, d\tau$, for every $0 \leq s < t$,

where $E'(\sigma(t))$ denotes the derivative of E with respect to σ computed at $\sigma(t)$.

Remark **5** Let $t \mapsto \sigma(t)$ be a left-continuous map on $[0, T]$ which satisfies condition (c') and define

$$\dot{\sigma}^{\ominus}(t) := \limsup_{s \to t^-} \frac{\sigma(t) - \sigma(s)}{t - s}.$$

Then

$$(1 + t^2 E'(\sigma(t)))\dot{\sigma}^{\ominus}(t) \leq 0 \tag{27}$$

for every $t \in [0, T]$. Indeed, let $t_k \nearrow t$ be such that

$$\lim_{k \to \infty} \frac{\sigma(t) - \sigma(t_k)}{t - t_k} = \dot{\sigma}^{\ominus}(t).$$

Then condition (c') between t_k and t can be written as

$$(t^2 - t_k^2)E(\sigma(t)) + t_k^2(E(\sigma(t)) - E(\sigma(t_k))) + \sigma(t) - \sigma(t_k) \leq 2\int_{t_k}^t \tau E(\sigma(\tau))\, d\tau,$$

and (27) follows dividing by $t - t_k$ and letting $k \to +\infty$.

Remark **6** Let $t \mapsto \sigma(t)$ be a left-continuous map on $[0, T]$ which satisfies conditions (a'), (b'), and (c'), and let $t \geq 0$ be such that $\dot{\sigma}^{\ominus}(t) > 0$. Then, by Remark 5 and conditions (a') and (b'), it follows that

$$E'(\sigma(t)) = \frac{d}{d\sigma} E(\sigma)|_{\sigma=\sigma(t)} = -\frac{1}{t^2},$$

which implies that $\sigma(t)$ does not belong to the concavity intervals of $E(\sigma)$, since $t \mapsto \sigma(t)$ is increasing, and $t \mapsto E'(\sigma(t))$ would be decreasing, while the right-hand side is increasing. More precisely, if there exists an interval $]a, b[\subset [\sigma_0, \overline{\sigma}]$ such that $\sigma \mapsto E'(\sigma)$ is strictly decreasing on $]a, b[$ and there exists $t_0 \geq 0$ such that $\dot{\sigma}(t_0) > 0$ (or $\dot{\sigma}^{\ominus}(t_0) > 0$) and $\sigma(t_0) \in]a, b[$ then we reach a contradiction. Indeed, let $t > t_0$ be such that $\sigma(t) \in]a, b[$. By (b'), $\sigma(t) > \sigma(t_0)$, and by (a') and our assumption on $E'(\sigma)$, we get

$$-\frac{1}{t^2} \leq E'(\sigma(t)) < E'(\sigma(t_0)) = -\frac{1}{t_0^2} < -\frac{1}{t^2},$$

a contradiction.

In order to better specify the monotonicity needed in the above remarks we introduce the following notion. We say that t_0 is a *local left-constancy point* for σ if there exists $\varepsilon > 0$ such that σ is constant on the interval $[t_0 - \varepsilon, t_0]$.

PROPOSITION **8** *Let* $\sigma \colon [0, T] \to [\sigma_0, \overline{\sigma}[$ *be a left-continuous map which satisfies conditions* (a')*,* (b')*, and* (c')*, and let* $t_0 \geq 0$*. If*

(1) t_0 *is not a local left-constancy point for* σ *and*

(2) there exists $]a, b[\subset [\sigma_0, \overline{\sigma}[$ *such that* $E'(\sigma)$ *is strictly decreasing on* $]a, b[$

then $\sigma(t_0) \notin]a, b[$.

PROOF. If t_0 is not a local left-constancy point for σ, then, given $\varepsilon > 0$, there are $t^1_\varepsilon, t^2_\varepsilon \in [t_0 - \varepsilon, t_0]$ such that $\sigma(t^1_\varepsilon) \neq \sigma(t^2_\varepsilon)$. Therefore, there exists $t_\varepsilon \in [t_0 - \varepsilon, t_0[$ such that $\dot{\sigma}^{\ominus}(t_\varepsilon) > 0$. Then (27) together with (a') imply that $1 + t^2_\varepsilon E'(\sigma(t_\varepsilon)) = 0$. By Remark 6, $\sigma(t_\varepsilon) \notin]a, b[$ and we conclude by passing to the limit as $\varepsilon \to 0$ (since σ is left-continuous). ■

PROPOSITION **9** *Let* $\sigma : [0, T] \to [\sigma_0, \overline{\sigma}[$ *be a left-continuous map which satisfies conditions* (a')*,* (b')*, and* (c')*. Assume that* $E(\sigma)$ *is convex on* $]a, b[\subset [\sigma_0, \overline{\sigma}]$*. Then* $\sigma(t)$ *is continuous at every* t *with* $\sigma(t) \in]a, b[$*.*

PROOF. Assume by contradiction that $\sigma(t) < \sigma(t^+)$. Then condition (c') and condition (a') imply

$$\frac{E(\sigma(t^+) - E(\sigma(t)))}{\sigma(t^+) - \sigma(t)} \leq -\frac{1}{t^2} \leq E'(\sigma(t)),$$

a contradiction. ■

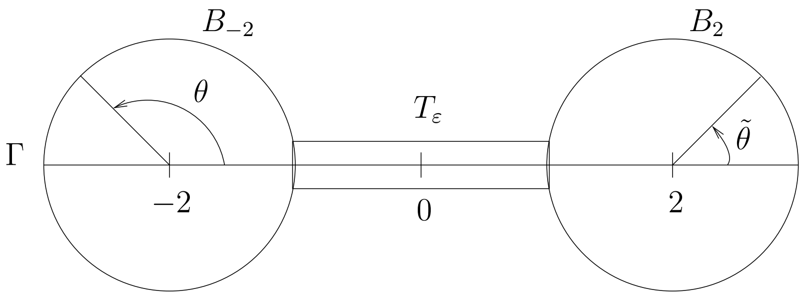

Figure 1: *The set* Ω_ε.

7. – Example: existence of a concavity subinterval for the energy functional

In the same context of Section 6, we consider the energy functional

$$\sigma \mapsto E(\sigma) := \min\{\|Du\|_2^2 : u \in AD(\psi, \sigma)\},$$

and construct an explicit example of Ω and ψ for which $E(\sigma)$ is concave on some subinterval. Let B_{-2} denote the ball of radius 1 centred in $(-2, 0)$, let B_2 denote the ball of radius 1 centred in $(2, 0)$, and let $\Gamma := [-3, 3] \times \{0\}$.

For $\varepsilon > 0$ let

$$T_\varepsilon :=]-2 + \cos\varepsilon, 2 - \cos\varepsilon[\, \times\,]-\sin\varepsilon, \sin\varepsilon[\,, \qquad \Omega_\varepsilon := B_{-2} \cup T_\varepsilon \cup B_2.$$

Further, for every $\sigma \in [-3, 3]$ let

$$\Gamma(\sigma) := [-3, \sigma] \times \{0\}.$$

Let (ρ, θ) and $(\tilde\rho, \tilde\theta)$ be polar coordinates around $(-2, 0)$ and $(2, 0)$, respectively, where the functions θ and $\tilde\theta$ are chosen, as in Proposition 2, such that $\theta(x_1, x_2) \to -\pi$ if $x_2 \to 0-$ and $x_1 < -2$, $\theta(x_1, x_2) \to \pi$ if $x_2 \to 0+$ and $x_1 < -2$, and, analogously, $\tilde\theta(x_1, x_2) \to -\pi$ if $x_2 \to 0-$ and $x_1 < 2$, $\tilde\theta(x_1, x_2) \to \pi$ if $x_2 \to 0+$ and $x_1 < 2$.

On $\partial\Omega_\varepsilon$ we define the boundary data ψ_ε as follows:

$$\psi_\varepsilon(x) := \begin{cases} \sin\frac{\theta(x)}{2} & \text{on } \big(\partial B_{-2} \cap \partial\Omega_\varepsilon\big) \setminus \Gamma(\sigma), \\ \sin\frac{\tilde\theta(x)}{2} & \text{on } \big(\partial B_2 \cap \partial\Omega_\varepsilon\big) \setminus \Gamma(\sigma), \\ \sin\frac{\varepsilon}{2} & \text{on }]-2+\cos\varepsilon, 0[\, \times \{\sin\varepsilon\}, \\ -\sin\frac{\varepsilon}{2} & \text{on }]-2+\cos\varepsilon, 0[\, \times \{-\sin\varepsilon\}, \\ \sin\frac{\varepsilon}{2} + \frac{x_1}{2-\cos\varepsilon}\big(\cos\frac{\varepsilon}{2} - \sin\frac{\varepsilon}{2}\big) & \text{on } [0, 2-\cos\varepsilon[\, \times \{\sin\varepsilon\}, \\ -\sin\frac{\varepsilon}{2} + \frac{x_1}{2-\cos\varepsilon}\big(\sin\frac{\varepsilon}{2} - \cos\frac{\varepsilon}{2}\big) & \text{on } [0, 2-\cos\varepsilon[\, \times \{-\sin\varepsilon\}. \end{cases}$$

For every $\sigma \in]-3, 3[\,$, let $u^\varepsilon(\sigma) \in H^1(\Omega_\varepsilon \setminus \Gamma(\sigma))$ be the solution of the problem:

$$E_\varepsilon(\sigma) := \min\Big\{ \int_{\Omega_\varepsilon \setminus \Gamma(\sigma)} |Du|^2\, dx : u \in AD(\psi_\varepsilon, \sigma) \Big\}.$$

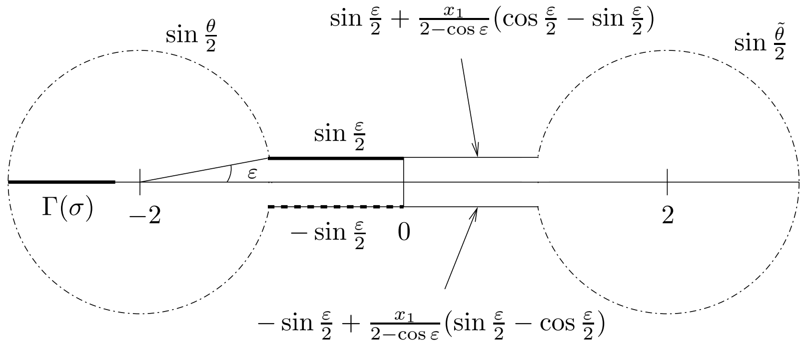

Figure 2: *The boundary datum ψ_ε.*

Our aim is to prove that for ε sufficiently small there exists a subinterval $[a,b]$ of $[-2,2]$ such that $E_\varepsilon(\sigma)$ is concave on $[a,b]$.

As $\sigma\mapsto E_\varepsilon(\sigma)$ is a C^2-function, in order to prove that $E_\varepsilon(\sigma)$ cannot be convex on the whole interval $[-2,2]$, it is enough to show that the following three conditions are satisfied:

(a) $\limsup_{\varepsilon\to0^+}E_\varepsilon(2)$ is finite;

(b) $\liminf_{\varepsilon\to0^+}E_\varepsilon(-2)=\infty$;

(c) $\limsup_{\varepsilon\to0^+}E'_\varepsilon(-2)$ is finite;

where we denote by $'$ the first derivative with respect to σ.

In order to prove condition (a) we construct an admissible function $\tilde u_\varepsilon$ for $E_\varepsilon(2)$ whose energy, $\|D\tilde u_\varepsilon\|_2^2$, is bounded uniformly with respect to ε. We define the open sets B_{-2}^+ and B_{-2}^- by

$$B_{-2}^+=\{(x_1,x_2)\in B_{-2}:x_2>0\}$$
$$B_{-2}^-=\{(x_1,x_2)\in B_{-2}:x_2<0\}.$$

Let v^+ be the solution to the following problem:

$$\begin{cases}\Delta u=0 & \text{on } B_{-2}^+;\\ u(x)=\sin\frac{\theta(x)}{2} & \text{on } \partial B_{-2}^+\cap\partial B_{-2};\\ \partial_\nu u=0 & \text{on }]-3,-1[\times\{0\}.\end{cases}$$

Then the function $v^-(x_1,x_2):=-v^+(x_1,-x_2)$ solves the analogue problem on B_{-2}^-. Let $\tilde u_\varepsilon$ be the function which coincides with the harmonic functions that satisfy the boundary conditions on B_{-2}^+, on B_{-2}^-, and on B_2, respectively, that is, $\tilde u_\varepsilon:=v^+$ on B_{-2}^+, $\tilde u_\varepsilon:=v^-$ on B_{-2}^-, and $\tilde u_\varepsilon:=\tilde\rho^{\frac12}\sin\frac{\tilde\theta}{2}$ on B_2. On $T_\varepsilon\setminus(B_2\cup B_{-2})$ we define $\tilde u_\varepsilon$ in the following way: on the horizontal line $x_2=\sin\theta$, with $\theta\in[-\varepsilon,\varepsilon]$, we set $\tilde u_\varepsilon(x_1,x_2):=\sin\frac{\theta}{2}$ for $x_1\in]-2+\cos\theta,0]$ and then interpolate linearly with the boundary data on $\partial B_2\cap T_\varepsilon$: $\tilde u_\varepsilon(x_1,x_2):=\sin\frac{\theta}{2}+\frac{x_1}{2-\cos\theta}(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})$ for $x_1\in[0,2-\cos\theta[$, if $0<\theta\le\varepsilon$, and $\tilde u_\varepsilon(x_1,x_2):=-\sin\frac{\theta}{2}+\frac{x_1}{2-\cos\theta}(-\sin\frac{\theta}{2}-\cos\frac{\theta}{2})$ for

$x_1 \in [0, 2-\cos\theta[$, if $-\varepsilon \le \theta < 0$. It is easy to check that $\tilde u_\varepsilon \in AD(\psi_\varepsilon, 2)$ and that $D\tilde u_\varepsilon$ is bounded in $L^2(\Omega_\varepsilon \setminus \Gamma; \mathbb{R}^2)$ uniformly with respect to ε. This implies that

$$\limsup_{\varepsilon\to 0^+} E_\varepsilon(2) \le \limsup_{\varepsilon\to 0^+} \int_{\Omega_\varepsilon\setminus\Gamma(2)} |D\tilde u_\varepsilon|^2\,dx < +\infty,$$

and condition (a) is satisfied.

We continue by proving condition (b), i.e., $E_\varepsilon(-2)$ tends to infinity as ε goes to zero. Let us first consider the model problem

$$\min\Big\{\int_{R_\varepsilon} |Du|^2\,dx : u \ge \frac{1}{2} \text{ on } \partial_1 R_\varepsilon, u \le -\frac{1}{2} \text{ on } \partial_2 R_\varepsilon\Big\} \tag{28}$$

where

$$R_\varepsilon := \,]0,1[\,\times\,]-\varepsilon,\varepsilon[, \quad \partial_1 R_\varepsilon := [0,1]\times\{\varepsilon\}, \quad \partial_2 R_\varepsilon := [0,1]\times\{-\varepsilon\}. \tag{29}$$

It is easy to see that problem (28) admits a solution and that it is equivalent to

$$\min\Big\{\int_{R_\varepsilon} |Du|^2\,dx : u = \frac{1}{2} \text{ on } \partial_1 R_\varepsilon, u = -\frac{1}{2} \text{ on } \partial_2 R_\varepsilon\Big\},$$

which admits the affine solution $u^a(x_1,x_2) := \frac{1}{2\varepsilon}\,x_2$ for every $x = (x_1,x_2) \in R_\varepsilon$.

Going back to the domain Ω_ε, let us consider the same problem with different constants: the rectangle R_ε is defined now by

$$R_\varepsilon := \,]A_\varepsilon, 2-\cos\varepsilon[\,\times\,]-\sin\varepsilon, \sin\varepsilon[\, \subset T_\varepsilon\,,$$

where A_ε is a positive constant such that $\psi_\varepsilon(x) \ge \frac{1}{2}$ on $\partial_1 R_\varepsilon := [A_\varepsilon, 2-\cos\varepsilon]\times\{\sin\varepsilon\}$, (and $\psi_\varepsilon(x) \le -\frac{1}{2}$ on $\partial_2 R_\varepsilon := [A_\varepsilon, 2-\cos\varepsilon]\times\{-\sin\varepsilon\}$), when ε is sufficiently small. Then

$$E_\varepsilon(-2) = \int_{\Omega_\varepsilon\setminus\Gamma(-2)} |Du^\varepsilon(-2)|^2 dx \ge \int_{R_\varepsilon} |Du^\varepsilon(-2)|^2 dx \ge \int_{R_\varepsilon} |Du^a|^2 dx\,.$$

Since $\int_{R_\varepsilon} |Du^a|^2 dx \to +\infty$ as $\varepsilon \to 0$, condition (b) is proved.

It remains to show that condition (c) is satisfied, i.e., that the first derivative of $\sigma \mapsto E_\varepsilon(\sigma)$ at $\sigma = -2$ is bounded as ε goes to zero. Since

$$E'_\varepsilon(\sigma) = -\kappa_\varepsilon^2(\sigma)\,, \tag{30}$$

see, e.g. [12, Theorem 6.4.1], where $\kappa_\varepsilon(\sigma)\sqrt{\frac{2}{\pi}}$ is the stress intensity factor associated to $u^\varepsilon(\sigma)$ at the tip $(\sigma,0)$, see Proposition 2, it is enough to show that $\kappa_\varepsilon(\sigma)$ remains bounded when, for instance, $-\frac{5}{2} \le \sigma \le -\frac{3}{2}$.

For $\sigma \in [-5/2, -3/2]$, let $v(\sigma)$ be the solution of the following problem:

$$\min\Big\{\int_{B_{-2}\setminus\Gamma(\sigma)} |Du|^2\,dx : u \in H^1(B_{-2}\setminus\Gamma(\sigma)),\ u = \sin\frac{\theta}{2} \text{ on } \partial B_{-2}\setminus\Gamma(\sigma)\Big\}. \tag{31}$$

Let us extend $v(\sigma)$ to $\mathbb{R}\times[-1,1]$ constantly on the horizontal lines and denote now by $v(\sigma)$ this extension.

We claim that

$$u^\varepsilon(\sigma) \to v(\sigma) \quad \text{strongly in } H^1(B_{-2} \setminus \Gamma(\sigma)). \tag{32}$$

Assuming the claim true, we now use the following characterization of κ_ε (see Proposition 3):

$$\kappa_\varepsilon^2(\sigma) = \int_{B_{-2}\setminus\Gamma(\sigma)} \Big[((D_1u^\varepsilon)^2 - (D_2u^\varepsilon)^2)D_1\varphi + 2D_1u^\varepsilon D_2u^\varepsilon D_2\varphi\Big]\, dx \tag{33}$$

with $\varphi \in C_c^1(B_{-2})$ such that $\varphi(\sigma, 0) = 1$. By (32) and the definition of $v(\sigma)$, we can pass to the limit in the right-hand side as $\varepsilon \to 0^+$ and define in such a way the quantity:

$$\kappa^2(\sigma) := \int_{B_{-2}\setminus\Gamma(\sigma)} \Big[((D_1v(\sigma))^2 - (D_2v(\sigma))^2)D_1\varphi + 2D_1v(\sigma)D_2v(\sigma)D_2\varphi\Big]\, dx. \tag{34}$$

Therefore, by (30),

$$\limsup_{\varepsilon\to 0^+} E'_\varepsilon(\sigma) = -\kappa^2(\sigma) \quad \text{for every } -\frac{5}{2} \le \sigma \le -\frac{3}{2}\,. \tag{35}$$

As, by (34), $\kappa(\sigma)$ is bounded, formula (35) concludes the proof of condition (c).

Proof of the claim. Let $\tilde\Omega_\varepsilon := T_\varepsilon \cup B_2$ and let w_ε be the solution of the following problem:

$$\min\Big\{ \int_{\tilde\Omega_\varepsilon} |Du|^2\, dx : u \in H^1(\tilde\Omega_\varepsilon),\ u = \psi_\varepsilon \text{ on } \partial\Omega_\varepsilon \cap \partial\tilde\Omega_\varepsilon \Big\}.$$

We consider a cut-off function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi(x_1) = 1$ for $x_1 \le -\frac{2}{3}$, and $\varphi(x_1) = 0$ for $x_1 \ge -\frac{1}{3}$. Then the function $\zeta := \varphi v(\sigma) + (1-\varphi)w_\varepsilon$ belongs to $AD(\psi_\varepsilon, \sigma)$ and

$$E_\varepsilon(\sigma) = \int_{\Omega_\varepsilon\setminus\Gamma(\sigma)} |Du^\varepsilon(\sigma)|^2\, dx \le \int_{\Omega_\varepsilon\setminus\Gamma(\sigma)} |D\zeta|^2\, dx. \tag{36}$$

By convexity, we have

$$\begin{aligned}\int_{\Omega_\varepsilon\setminus\Gamma(\sigma)} |D\zeta|^2\, dx \le \int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx + \int_{\tilde\Omega_\varepsilon} |Dw_\varepsilon|^2\, dx + \int_{T_\varepsilon} |Dv(\sigma)|^2\, dx + \\ + \int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} \big(2D\varphi(\varphi Dv(\sigma) + (1-\varphi)Dw_\varepsilon)(v(\sigma) - w_\varepsilon) + |D\varphi|^2(v(\sigma) - w_\varepsilon)^2\big)\, dx.\end{aligned} \tag{37}$$

Now

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \int_{T_\varepsilon} |Dv(\sigma)|^2\, dx = 0, \\ &\lim_{\varepsilon\to 0^+} \int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} |D\varphi|^2(v(\sigma) - w_\varepsilon)^2\, dx = 0,\end{aligned} \tag{38}$$

and, for any $\eta > 0$,

$$\int_{T_\varepsilon \cap (\mathrm{supp} D\varphi)} 2D\varphi(\varphi Dv(\sigma) + (1-\varphi)Dw_\varepsilon)(v(\sigma) - w_\varepsilon)\, dx \leq$$
$$\leq 2 \int_{T_\varepsilon \cap (\mathrm{supp} D\varphi)} D\varphi\varphi Dv(\sigma)(v(\sigma) - w_\varepsilon)\, dx + \tag{39}$$
$$+\frac{1}{\eta}\int_{T_\varepsilon \cap (\mathrm{supp} D\varphi)} |D\varphi|^2 |v(\sigma) - w_\varepsilon|^2\, dx + \eta \int_{T_\varepsilon \cap (\mathrm{supp} D\varphi)} |Dw_\varepsilon|^2 (1-\varphi)^2\, dx\,.$$

Since the first two terms in the right-hand side tend to zero, it remains to prove that

$$\lim_{\varepsilon \to 0^+} \int_{T_\varepsilon \cap (\mathrm{supp} D\varphi)} |Dw_\varepsilon|^2\, dx = 0\,. \tag{40}$$

As in the proof of condition (b), we consider first a model problem. Similarly to (29), we now set

$$R_\varepsilon :=]-1,0[\times]-\varepsilon,\varepsilon[\,, \quad \partial_1 R_\varepsilon := [-1,0]\times\{\varepsilon\}, \quad \partial_2 R_\varepsilon := [-1,0]\times\{-\varepsilon\},$$

and define h_ε as the solution to the following problem:

$$\begin{cases} \Delta h_\varepsilon = 0 & \text{on } R_\varepsilon, \\ h_\varepsilon = \frac{\varepsilon}{2} & \text{on } \partial_1 R_\varepsilon, \\ h_\varepsilon = -\frac{\varepsilon}{2} & \text{on } \partial_2 R_\varepsilon, \\ \|h_\varepsilon\|_\infty \leq 1. \end{cases} \tag{41}$$

We claim that

$$\lim_{\varepsilon \to 0^+} \int_{\tilde{R}_\varepsilon} |Dh_\varepsilon|^2\, dx = 0, \tag{42}$$

where

$$\tilde{R}_\varepsilon := \left]-\frac{4}{5}, -\frac{1}{5}\right[\times]-\varepsilon,\varepsilon[\subset R_\varepsilon.$$

Indeed, note that the function $z_\varepsilon(x_1, x_2) := \frac{1}{2}x_2$ solves (41) (for $\varepsilon \leq 1$). By a Cacciopoli type estimate we obtain

$$\int_{\tilde{R}_\varepsilon} |D(h_\varepsilon - z_\varepsilon)|^2 dx \leq C \int_{R_\varepsilon} |h_\varepsilon - z_\varepsilon|^2 dx \leq C_1 |R_\varepsilon|\,,$$

for some positive constants C and C_1 which do not depend on ε, hence (42) holds.

Applying this argument with

$$R_\varepsilon = \left]-1, 0\right[\times]-\sin\varepsilon, \sin\varepsilon[\quad \text{and} \quad \tilde{R}_\varepsilon = \left]-\frac{4}{5}, -\frac{1}{5}\right[\times]-\sin\varepsilon, \sin\varepsilon[$$

it follows that (40) holds true.

From (36), (37), (38), (39), and (40) we deduce that

$$\int_{\Omega_\varepsilon \setminus \Gamma(\sigma)} |Du^\varepsilon(\sigma)|^2\, dx \leq \int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx + \int_{\tilde{\Omega}_\varepsilon} |Dw_\varepsilon|^2\, dx + o(1).$$

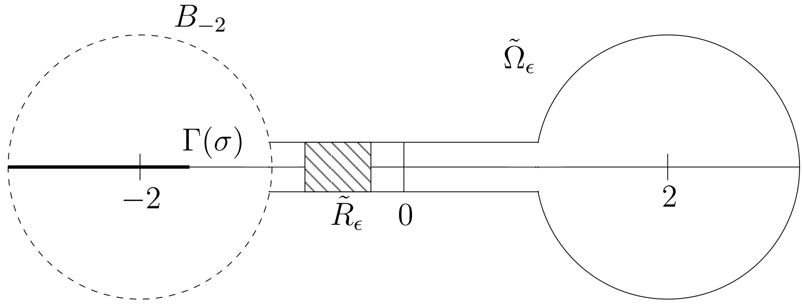

Figure 3: *The rectangle $\tilde{R}_\epsilon$ where we apply a Cacciopoli type estimate in order to obtain (40).*

Since

$$\int_{\tilde{\Omega}_\varepsilon} |Du^\varepsilon(\sigma)|^2 dx \geq \int_{\tilde{\Omega}_\varepsilon} |Dw_\varepsilon|^2 dx$$

we obtain

$$\int_{B_{-2}\setminus\Gamma(\sigma)} |Du^\varepsilon(\sigma)|^2\, dx \leq \int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx + o(1) \leq C \tag{43}$$

uniformly with respect to ε. Thus, there exists $u^*(\sigma) \in H^1(B_{-2} \setminus \Gamma(\sigma))$ such that

$$u^\varepsilon(\sigma) \rightharpoonup u^*(\sigma) \quad \text{weakly on } H^1(B_{-2} \setminus \Gamma(\sigma)), \tag{44}$$

and

$$u^*(\sigma) = \sin\frac{\theta}{2} \quad \text{on } \partial B_{-2} \setminus \Gamma(\sigma). \tag{45}$$

As $(Du^\varepsilon(\sigma), D\varphi) = 0$ for every $\varphi \in H^1(B_{-2} \setminus \Gamma(\sigma))$ with $\varphi = 0$ on $\partial B_{-2} \setminus \Gamma(\sigma)$, by (44) we obtain that $(Du^*(\sigma), D\varphi) = 0$. By (31), this fact, together with (45), implies that

$$u^*(\sigma) = v(\sigma)\,. \tag{46}$$

In addition, by the lower semicontinuity and by (43), we have

$$\int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx \leq \liminf_{\varepsilon\to 0^+} \int_{B_{-2}\setminus\Gamma(\sigma)} |Du^\varepsilon(\sigma)|^2\, dx \leq \int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx. \tag{47}$$

By (44), (46), and (47), we deduce that (32) holds.

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