# Γ-LIMITS OF CONVOLUTION FUNCTIONALS

LUCA LUSSARDI AND ANNIBALE MAGNI

Abstract. We compute the Γ-limit of a sequence of non-local integral functionals depending on a regularization of the gradient term by means of a convolution kernel. In particular, as Γ-limit, we obtain free discontinuity functionals with linear growth and with anisotropic surface energy density.

Keywords: Free discontinuities, Γ-convergence, anisotropy.

2010 Mathematics Subject Classification: 49Q20, 49J45, 49M30.

## 1. Introduction

As it is well known, many variational problems which are recently under consideration, arising for instance from image segmentation, signal reconstruction, fracture mechanics and liquid crystals, involve a *free discontinuity set* (according to a terminology introduced in [19]). This means that the variable function $u$ is required to be smooth outside a surface $K$, depending on $u$, and both $u$ and $K$ enter the structure of the functional, which takes the form given by

$$\mathcal{F}(u,K) = \int_{\Omega\setminus K} \phi(|\nabla u|)\,\mathrm{d}x + \int_{K\cap\Omega} \theta(|u^+ - u^-|, \nu_K)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

being $\Omega$ an open subset of $\mathbb{R}^n$, $K$ is a $(n-1)$-dimensional compact subset of $\mathbb{R}^n$, $|u^+ - u^-|$ the jump of $u$ across $K$, $\nu_K$ the normal direction to $K$, while $\phi$ and $\theta$ given positive functions, whereas $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

The classical weak formulation for such problems can be obtained considering $K$ as the set of the discontinuities of $u$ and thus working in the space of functions with bounded variation. More precisely, the aforementioned weak form of $\mathcal{F}$ takes on $BV(\Omega)$ the general form

$$\mathcal{F}(u) = \int_{\Omega} \phi(|\nabla u|)\,\mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + c_0|D^c u|(\Omega), \tag{1.1}$$

where $Du = \nabla u\,\mathcal{L}^n + (u^+ - u^-)\mathcal{H}^{n-1} + D^c u$ is the decomposition of the measure derivative of $u$ in its absolutely continuous, jump and Cantor part, respectively, $S_u$ denotes the set of discontinuity points of $u$, and $\nu_u$ is a choice of the unit normal at $S_u$.

The main difficulty in the actual minimization of $\mathcal{F}$ comes from the surface integral

$$\int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

which makes it necessary to use suitable approximations guaranteeing the convergence of minimum points and naturally leads to Γ-convergence.

As pointed out in [10], it is not possible to obtain a variational approximation for $\mathcal{F}$ by the typical integral functionals

$$\mathcal{F}_\varepsilon(u) = \int_{\Omega} f_\varepsilon(\nabla u)\,\mathrm{d}x$$

defined on some Sobolev spaces. Indeed, when considering the lower semicontinuous envelopes of these functionals, we would be lead to a convex limit, which conflicts with the non-convexity of $\mathcal{F}$.

Heuristic arguments suggest that, to get rid of the difficulty, we have to prevent that the effect of *large* gradients is concentrated on *small* regions. Several approximation methods fit this requirements. For instance in [7], [12], [24] the case where the functionals $\mathcal{F}_\varepsilon$ are restricted to finite elements spaces on regular triangulations of size $\varepsilon$ is considered. In [1], [2], [23] the implicit constraint on the gradient through the addition of a higher order penalization is investigated. Moreover, it is important to mention the AMBROSIO & TORTORELLI approximation (see [4] and [5]) of the Mumford-Shah functional via elliptic functionals.

The study of non-local models, where the effect of a large gradient is spread onto a set of size $\varepsilon$, was first introduced by BRAIDES & DAL MASO in order to approximate the Mumford-Shah functional (see [10] and also [11], [13], [14], [15], [16]) by means of the family

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u|^2 \,\mathrm{d}y\right)\mathrm{d}x, \quad u \in H^1(\Omega), \tag{1.2}$$

where, for instance, $f(t) = t \wedge 1/2$ and $B_\varepsilon(x)$ denotes the ball of centre $x$ and radius $\varepsilon$. A variant of the method proposed in [10] has been used in [22] to deal with the approximation of a functional $\mathcal{F}$ of the form (1.1), with $\phi$ having linear growth and $\theta$ independent on the normal $\nu_u$ (see also [20] and [21]). More precisely, in [22] the $\Gamma$-limit of the family

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u| \,\mathrm{d}y\right)\mathrm{d}x, \quad u \in W^{1,1}(\Omega),$$

for a suitable concave function $f$, is computed.

In [25] (see also [13]) the case of an anisotropic variant of (1.2), i.e. the presence of a convolution between the gradient and a not radially symmetrical kernel, has been considered. In particular it is proven that the family

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\big(\varepsilon|\nabla u|^p * \rho_\varepsilon\big)\mathrm{d}x, \quad u \in H^1(\Omega), \quad p > 1,$$

$\Gamma$-converges to an anisotropic version of the Mumford-Shah functional.

In this paper we investigate the $\Gamma$-convergence of the family

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\big(\varepsilon|\nabla u| * \rho_\varepsilon\big)\mathrm{d}x, \quad u \in W^{1,1}(\Omega),$$

where the family $(f_\varepsilon)_{\varepsilon>0}$ satisfies some conditions. The main difficulty to overcome is the estimate from below for the lower $\Gamma$-limit in terms of the surface part, while the contribution arising from the volume and Cantor parts has been treated along the same line of the argument already exploited in [25]. The estimate from above has been achieved by density and relaxation arguments. We prove that the $\Gamma$-limit, in the strong $L^1$-topology, is given by

$$\mathcal{F}(u) = \int_\Omega \phi(|\nabla u|)\,\mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + c_0|D^c u|(\Omega),$$

where $\phi(t) \sim \frac{1}{\varepsilon} f_\varepsilon(\varepsilon t)$, as $\varepsilon \to 0^+$, is a convex and non-decreasing function with $\phi(0) = 0$ and with $\phi(t)/t \to c_0 > 0$ as $t \to +\infty$; moreover,

$$\theta(s,\nu) = \inf\left\{\liminf_{j\to+\infty} \frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j) \in W_\nu^{0,s},\ \varepsilon_j \to 0^+\right\},$$

being $f$ the uniform limit, on compact subsets of $[0,+\infty)$, of $f_\varepsilon$, $W_\nu^{a,b}$ the space of all sequences on the cylinder

$$Q_\nu = \{x \in \mathbb{R}^n : |\langle x, \nu\rangle| \le 1, \text{ the orthogonal projection of } x \text{ onto } \nu^\perp \text{ belongs to the unit ball}\}.$$

which converge, shrinking onto the interface, to the function that jumps from $a$ to $b$ around the origin (see paragraph 3.1 for details).

In section 7 we have been able to show that the method used in [22] to write $\theta$ in a more explicit form works only if $n = 1$. In the case $n > 1$ such an argument does not work. Let us briefly discuss the reason. Without loss of generality we can suppose $\nu = \mathbf{e}_1$. Let $P_C^\perp$ be the orthogonal projection of $C$ onto $\{x_1 = 0\}$. Denote by $X$ the space of all functions $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R} \times P_C^\perp)$ which are non-decreasing in the first variable and such that there exist $\xi_0 < \xi_1$ with $v(x) = 0$ if $x_1 < \xi_0$ and

$v(x) = s$ if $x_1 > \xi_1$. Then, exploiting the same argument as in [22], we have $\theta(s, \mathbf{e}_1) \geq \inf_X G$, where

$$G(v) = \int_{-\infty}^{+\infty} f\left( \int_{C(s\mathbf{e}_1)} \partial_1 v(z)\rho(z - t\mathbf{e}_1)\,\mathrm{d}z \right) \mathrm{d}t.$$

The estimate $\theta(s, \mathbf{e}_1) \geq \inf_X G$ turns out to be optimal if $\inf_X G = \inf_Y G$, where $Y$ is the space of all functions $v \in X$ such that $v$ depends only on the first variable. This is due to the fact that proving the inequality $\theta(s, \mathbf{e}_1) \geq \inf_X G$ we lose control on all the derivatives $\partial_i v$ for any $i = 2, \cdots, n$. In the case $C = B_1$ and $\rho = \frac{1}{\omega_n}\chi_{B_1}$, treated in [22], one is able to prove that $\inf_X G = \inf_Y G$ computing directly $\inf_X G$ by a discretization argument (see Prop. 5.7 in [22]). In general, $\inf_X G = \inf_Y G$ does not hold. Indeed proceeding at first as in the proof of Prop. 5.6 in [22], one is able to show that for any $C \subset \mathbb{R}^2$ open, bounded, convex and symmetrical set (i.e. $C = -C$) and for $\rho = \frac{1}{|C|}\chi_C$, it holds

$$\inf_Y G = \int_{-h_1}^{h_1} f\left( \frac{s}{|C|}\mathcal{H}^1(C \cap \{z_1 = t\}) \right) \mathrm{d}t. \tag{1.3}$$

Now if $C$ is the parallelogram $C = \{(x, y) \in \mathbb{R}^2 : -2 \leq y \leq 2,\ x - 1 \leq y \leq x + 1\}$ applying (1.3), we get

$$\inf_Y G' = 2f\left(\frac{2s}{|C|}\right) + 2\int_0^2 f\left(\frac{sr}{|C|}\right) \mathrm{d}r.$$

If we compute $G$ on the function $w$ given by

$$w(x, y) = \begin{cases} 0 & \text{if } y > x - 1 \\ s & \text{if } y \leq x - 1 \end{cases},$$

(to do this we notice that the functional $G$ makes sense also on $BV_{\mathrm{loc}}(\mathbb{R} \times (-2, 2))$ writing $D_1 v$ instead of $\partial_1 v\,\mathrm{d}z$) we obtain

$$G(w) = 2f\left(\frac{4s}{|C|}\right).$$

If $f$ is strictly concave then

$$G(w) < 2f\left(\frac{2s}{|C|}\right) + 2f\left(\frac{2s}{|C|}\right) < 2f\left(\frac{2s}{|C|}\right) + 2\int_0^2 f\left(\frac{sr}{|C|}\right) \mathrm{d}r = \inf_Y G.$$

By a density argument we deduce that $\inf_X G < \inf_Y G$.

As a conclusion, it seems that for a generic anisotropic convolution kernel $\rho_\varepsilon$ the expression for $\theta$ can not be further simplified when $n > 1$.

## 2. Notation and preliminaries

We will denote by $L^p(\Omega)$ and by $W^{k,p}(\Omega)$, for $k \in \mathbb{N}$, $k \geq 1$, and for $1 \leq p \leq +\infty$, respectively the classical Lebesgue and Sobolev spaces on $\Omega$. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ will be denoted by $|A|$, whereas the Hausdorff measure of $A$ of dimension $m < n$ will be denoted by $\mathcal{H}^m(A)$. The ball centered in $x$ with radius $r$ will be denoted by $B_r(x)$, while $B_r$ stands for $B_r(0)$; moreover, we will use the notation $\mathbb{S}^{n-1}$ for the boundary of $B_1$ in $\mathbb{R}^n$. The volume of the unit ball in $\mathbb{R}^n$ will be denoted by $\omega_n$, with the convention $\omega_0 = 1$. Finally $\mathcal{A}(\Omega)$ denotes the set of all open subsets of $\Omega$.

### 2.1. Functions of bounded variation.

**2.1. Functions of bounded variation.** For a thorough treatment of $BV$ functions we refer the reader to [3]. Let $\Omega$ be an open subset of $\mathbb{R}^n$. We recall that the space $BV(\Omega)$ of real *functions of bounded variation* is the space of the functions $u \in L^1(\Omega)$ whose distributional derivative is representable by a measure in $\Omega$, i.e.

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x = -\int_\Omega \varphi\,\mathrm{d}D_i u, \quad \forall\varphi \in C_c^\infty(\Omega),\ \forall i = 1, \ldots, n,$$

for some $\mathbb{R}^n$-valued measure $Du = (D_1 u, \dots, D_n u)$ on $\Omega$. We say that $u$ has *approximate limit* at $x \in \Omega$ if there exists $z \in \mathbb{R}$ such that

$$\lim_{r \to 0^+} \fint_{B_r(x)} |u(y) - z| \, dy = 0.$$

The set $S_u$ where this property fails is called *approximate discontinuity set* of $u$. The vector $z$ is uniquely determined for any point $x \in \Omega \setminus S_u$ and is called the *approximate limit* of $u$ at $x$ and denoted by $\tilde{u}(x)$. We say that $x$ is an *approximate jump point* of the function $u \in BV(\Omega)$ if there exist $a, b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and

$$\lim_{r \to 0^+} \fint_{B_r^+(x,\nu)} |u(y) - a| \, dy = 0, \quad \lim_{r \to 0^+} \fint_{B_r^-(x,\nu)} |u(y) - b| \, dy = 0, \tag{2.1}$$

where $B_r^+(x,\nu) = \{y \in B_r(x) : \langle y - x, \nu \rangle > 0\}$ and $B_r^-(x,\nu) = \{y \in B_r(x) : \langle y - x, \nu \rangle < 0\}$. The set of approximate jump points of $u$ is denoted by $J_u$. The triplet $(a, b, \nu)$, which turns out to be uniquely determined up to a permutation of $a$ and $b$ and a change of sign of $\nu$, is usually denoted by $(u^+(x), u^-(x), \nu_u(x))$. On $\Omega \setminus S_u$ we set $u^+ = u^- = \tilde{u}$. It turns out that for any $u \in BV(\Omega)$ the set $S_u$ is countably $(n-1)$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. Moreover,

$$Du \llcorner J_u = (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u$$

and $\nu_u(x)$ gives the approximate normal direction to $S_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$.

For a function $u \in BV(\Omega)$ let $Du = D^a u + D^s u$ be the Lebesgue decomposition of $Du$ into absolutely continuous and singular part. We denote by $\nabla u$ the density of $D^a u$; the measures $D^j u := D^s u \llcorner J_u$ and $D^c u := D^s u \llcorner (\Omega \setminus S_u)$ are called the *jump part* and the *Cantor part* of the derivative, respectively. It holds $Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u + D^c u$. Let us recall the following important compactness Theorem in $BV$ (see Th. 3.23 and Prop. 3.21 in [3]):

**Theorem 2.1.** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Every sequence $(u_h)$ in $BV(\Omega)$ which is bounded in $BV(\Omega)$ admits a subsequence converging in $L^1(\Omega)$ to a function $u \in BV(\Omega)$.*

We say that a function $u \in BV(\Omega)$ is a *special function of bounded variation*, and we write $u \in SBV(\Omega)$, if $|D^c u|(\Omega) = 0$. We say that a function $u \in L^1(\Omega)$ is a *generalized function of bounded variation*, and we write $u \in GBV(\Omega)$, if $u^T := (-T) \vee u \wedge T$ belongs to $BV(\Omega)$ for every $T \geq 0$. If $u \in GBV(\Omega)$, the function $\nabla u$ given by

$$\nabla u = \nabla u^T \qquad \text{a.e. on } \{|u| \leq T\} \tag{2.2}$$

turns out to be well-defined. Moreover, the set function $T \mapsto S_{u^T}$ is monotone increasing; therefore, if we set $S_u = \bigcup_{T>0} J_{u^T}$, for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ we can consider the functions of $T$ given by $(u^T)^-(x)$, $(u^T)^+(x)$, $\nu_{u^T}(x)$. It turns out that

$$u^-(x) = \lim_{T \to +\infty} (u^T)^-(x), \quad u^+(x) = \lim_{T \to +\infty} (u^T)^+(x), \quad \nu_u(x) = \lim_{T \to +\infty} \nu_{u^T}(x) \tag{2.3}$$

are well-defined for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ Finally, for a function $u \in GBV(\Omega)$, let $|D^c u|$ be the supremum, in the sense of measures, of $|D^c u^T|$ for $T > 0$. It can be proved that for any Borel subset $B$ of $\Omega$

$$|D^c u|(B) = \lim_{T \to +\infty} |D^c u^T|(B) \, . \tag{2.4}$$

2.2. **Slicing.** In order to obtain the estimate from below of the lower $\Gamma$-limit (see next paragraph) we need some basic properties of one-dimensional sections of $BV$-functions. We first introduce some notation. Let $\xi \in \mathbb{S}^{n-1}$, and let $\xi^\perp$ be the vector subspace orthogonal to $\xi$. If $y \in \xi^\perp$ and $E \subseteq \mathbb{R}^n$ we set $E_{\xi,y} = \{t \in \mathbb{R} : \ y + t\xi \in E\}$. Moreover, for any given function $u \colon \Omega \to \mathbb{R}$ we define $u_{\xi,y} \colon \Omega_{\xi,y} \to \mathbb{R}$ by $u_{\xi,y}(t) = u(y + t\xi)$. For the results collected in the following Theorem see [3], section 3.11.

**Theorem 2.2.** *Let $u \in BV(\Omega)$. Then $u_{\xi,y} \in BV(\Omega_{\xi,y})$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \xi^\perp$. For such values of $y$ we have $u'_{\xi,y}(t) = \langle \nabla u(y+t\xi), \xi\rangle$ for a.e. $t \in \Omega_{\xi,y}$ and $J_{u_{\xi,y}} = (J_u)_{\xi,y}$, where $u'_{\xi,y}$ denotes the absolutely continuous part of the measure derivative of $u_{\xi,y}$. Moreover, for every open subset $A$ of $\Omega$ we have*

$$\int_{\xi^\perp} |D^c u_{\xi,y}|(A_{\xi,y})\, \mathrm{d}\mathcal{H}^{n-1}(y) = |\langle D^c u, \xi\rangle|(A).$$

2.3. **Γ-convergence.** For the general theory see [9] and [18]. Let $(X, d)$ be a metric space. Let $(\mathcal{F}_j)$ be a sequence of functions $X \to \overline{\mathbb{R}}$. We say that $(\mathcal{F}_j)$ Γ-*converges*, as $j \to +\infty$, to $\mathcal{F}\colon X \to \overline{\mathbb{R}}$, if for all $u \in X$ we have:

a) For every sequence $(u_j)$ converging to $u$ it holds

$$\mathcal{F}(u) \le \liminf_{j\to+\infty} \mathcal{F}_j(u_j).$$

b) There exists a sequence $(u_j)$ converging to $u$ such that

$$\mathcal{F}(u) \ge \limsup_{j\to+\infty} \mathcal{F}_j(u_j).$$

The *lower* and *upper* Γ-limits of $(\mathcal{F}_j)$ in $u \in X$ are defined as

$$\mathcal{F}'(u) = \inf\bigl\{\liminf_{j\to+\infty} \mathcal{F}_j(u_j) : \ u_j \to u\bigr\}, \quad \mathcal{F}''(u) = \inf\bigl\{\limsup_{j\to+\infty} \mathcal{F}_j(u_j) : \ u_j \to u\bigr\}$$

respectively. We extend this definition of convergence to families depending on a real parameter. Given a family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ of functions $X \to \overline{\mathbb{R}}$, we say that it Γ-converges, as $\varepsilon \to 0$, to $\mathcal{F}\colon X \to \overline{\mathbb{R}}$ if for every positive infinitesimal sequence $(\varepsilon_j)$ the sequence $(\mathcal{F}_{\varepsilon_j})$ Γ-converges to $F$. If we define the lower and upper Γ-limits of $(\mathcal{F}_\varepsilon)$ as

$$\mathcal{F}'(u) = \inf\bigl\{\liminf_{\varepsilon\to0} \mathcal{F}_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\bigr\}, \quad \mathcal{F}''(u) = \inf\bigl\{\limsup_{\varepsilon\to0} \mathcal{F}_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\bigr\}$$

respectively, then $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F}$ in $u$ if and only if $\mathcal{F}'(u) = \mathcal{F}''(u) = \mathcal{F}(u)$. It turns out that both $\mathcal{F}'$ and $\mathcal{F}''$ are lower semicontinuous on $X$. In the estimate of $\mathcal{F}'$ we shall use the following immediate consequence of the definition:

$$\mathcal{F}'(u) = \inf\bigl\{\liminf_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u_j) : \ \varepsilon_j \to 0^+, \ u_j \to u\bigr\}.$$

It turns out that the infimum is attained.

An important consequence of the definition of Γ-convergence is the following result about the convergence of minimizers (see, e.g., [18], Cor. 7.20):

**Theorem 2.3.** *Let $\mathcal{F}_j\colon X \to \overline{\mathbb{R}}$ be a sequence of functions which Γ-converges to some $\mathcal{F}\colon X \to \overline{\mathbb{R}}$; assume that $\inf_{v\in X} \mathcal{F}_j(v) > -\infty$ for every $j$. Let $(\sigma_j)$ be a positive infinitesimal sequence, and for every $j$ let $u_j \in X$ be a $\sigma_j$-minimizer of $\mathcal{F}_j$, i.e.*

$$\mathcal{F}_j(u_j) \le \inf_{v\in X} \mathcal{F}_j(v) + \sigma_j\,.$$

*Assume that $u_j \to u$ for some $u \in X$. Then $u$ is a minimum point of $\mathcal{F}$, and*

$$\mathcal{F}(u) = \lim_{j\to+\infty} \mathcal{F}_j(u_j).$$

**Remark 2.4.** *The following property is a direct consequence of the definition of Γ-convergence: if $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$ then $\mathcal{F}_\varepsilon + \mathcal{G} \xrightarrow{\Gamma} \mathcal{F} + \mathcal{G}$ whenever $\mathcal{G}\colon X \to \overline{\mathbb{R}}$ is continuous.*

2.4. **Supremum of measures.** In order to prove the Γ-liminf inequality we recall the following useful tool, which can be found in [8].

**Lemma 2.5.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ and denote by $\mathcal{A}(\Omega)$ the family of its open subsets. Let $\lambda$ be a positive Borel measure on $\Omega$, and $\mu\colon \mathcal{A}(\Omega) \to [0,+\infty)$ a set function which is superadditive on open sets with disjoint compact closures, i.e. if $A, B \subset\subset \Omega$ and $\overline{A} \cap \overline{B} = \emptyset$, then*

$$\mu(A \cup B) \ge \mu(A) + \mu(B).$$

*Let* $(\psi_i)_{i\in I}$ *be a family of positive Borel functions. Suppose that*

$$\mu(A) \geq \int_A \psi_i \,\mathrm{d}\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } i \in I.$$

*Then*

$$\mu(A) \geq \int_A \sup_i \psi_i \,\mathrm{d}\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega).$$

2.5. **A density result.** The right bound for the upper Γ-limit from above will be first obtained for a suitable dense subset of $SBV(\Omega)$. More precisely, let $\mathcal{W}(\Omega)$ be the space of all functions $w \in SBV(\Omega)$ such that

(a) $\mathcal{H}^{n-1}(\overline{S}_w \setminus S_w) = 0$;
(b) $\overline{S}_w$ is the intersection of $\Omega$ with the union of a finite member of $(n-1)$-dimensional simplexes;
(c) $w \in W^{k,\infty}(\Omega \setminus \overline{S}_w)$ for every $k \in \mathbb{N}$.

Theorem 3.1 in [17] gives us the density property of $\mathcal{W}(\Omega)$ we need; here

$$SBV^2(\Omega) = \{u \in SBV(\Omega) : |\nabla u| \in L^2(\Omega),\, \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

**Theorem 2.6.** *Assume that* $\partial\Omega$ *is Lipschitz. Let* $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$*. Then there exists a sequence* $(w_h)$ *in* $\mathcal{W}(\Omega)$ *such that* $w_h \to u$ *strongly in* $L^1(\Omega)$*,* $\nabla w_h \to \nabla u$ *strongly in* $L^2(\Omega, \mathbb{R}^n)$*, with* $\limsup_{h\to+\infty} \|w_h\|_\infty \leq \|u\|_\infty$ *and such that*

$$\limsup_{h\to+\infty} \int_{S_{w_h}} \psi(w_h^+, w_h^-, \nu_{w_h}) \,\mathrm{d}\mathcal{H}^{n-1} \leq \int_{S_u} \psi(u^+, u^-, \nu_u) \,\mathrm{d}\mathcal{H}^{n-1}$$

*for every upper semicontinuous function* $\psi$ *such that* $\psi(a, b, \nu) = \psi(b, a, -\nu)$ *whenever* $a, b \in \mathbb{R}$ *and* $\nu \in \mathbb{S}^{n-1}$.

2.6. **A relaxation result.** To conclude this section we prove a relaxation result which will be used in the sequel. Recall that given $X$ be a topological space and $\mathcal{F}\colon X \to \mathbb{R} \cup \{\pm\infty\}$, the *relaxed functional* of $\mathcal{F}$, denoted by $\overline{\mathcal{F}}$, is the largest lower semicontinuous functional which is smaller than $F$.

**Theorem 2.7.** *Let* $\phi\colon [0, +\infty) \to [0, +\infty)$ *be a convex, non-decreasing and lower semicontinuous function with* $\phi(0) = 0$ *and with*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = c \in (0, +\infty).$$

*Let* $\theta\colon [0, +\infty) \times \mathbb{S}^{n-1} \to [0, +\infty)$ *be a lower semicontinuous function such that* $\theta(s, \nu) \leq c's$ *for any* $(s, \nu) \in [0, +\infty) \times \mathbb{S}^{n-1}$*, for some* $c' > 0$*. For any* $A \in \mathcal{A}(\Omega)$ *let*

$$\mathcal{F}(u, A) = \begin{cases} \displaystyle\int_A \phi(|\nabla u|) \,\mathrm{d}x + \int_{S_u \cap A} \theta(|u^+ - u^-|, \nu_u) \,\mathrm{d}\mathcal{H}^{n-1} & \textit{if } u \in SBV^2(\Omega) \cap L^\infty(\Omega) \\ +\infty & \textit{otherwise in } L^1(\Omega). \end{cases}$$

*Then the relaxed functional of* $\mathcal{F}$ *with respect to the strong* $L^1$*-topology satisfies*

$$\overline{\mathcal{F}}(u) \leq \int_\Omega \phi(|\nabla u|) \,\mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u) \,\mathrm{d}\mathcal{H}^{n-1} + c|D^c u|(\Omega)$$

*for any* $u \in BV(\Omega)$.

*Proof.* Combining a standard convolution argument with a well known relaxation result (see, for instance, Th. 5.47 in [3]) we can say that the relaxed functional of

$$\mathcal{G}(u, A) = \begin{cases} \displaystyle\int_A \phi(|\nabla u|) \,\mathrm{d}x & \text{if } u \in C^1(\overline{\Omega}) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

is given by

$$\overline{\mathcal{G}}(u,A) = \begin{cases} \displaystyle\int_A \phi(|\nabla u|)\,\mathrm{d}x + c|D^s u|(A) & \text{if } u \in BV(\Omega) \\ \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Since $C^1(\overline{\Omega}) \subseteq SBV^2(\Omega) \cap L^\infty(\Omega)$ then we get $\mathcal{F}(u,A) \leq \mathcal{G}(u,A)$. Hence for any $A \in \mathcal{A}(\Omega)$ and for any $u \in BV(\Omega)$

$$\overline{\mathcal{F}}(u,A) \leq \int_A \phi(|\nabla u|)\,\mathrm{d}x + c|D^s u|(A).$$

We can now conclude using the fact that for every $u \in BV(\Omega)$ the set function $\overline{\mathcal{F}}(u,\cdot)$ is the trace on $\mathcal{A}(\Omega)$ of a regular Borel measure $\mu$. This can be proven exactly along the same line of Prop. 3.3 in [6]. Hence

$$\begin{aligned} \overline{\mathcal{F}}(u) &= \mu(\Omega) = \mu(\Omega \setminus S_u) + \mu(\Omega \cap S_u) \\ &\leq \int_\Omega \phi(|\nabla u|)\,\mathrm{d}x + c|D^c u|(\Omega) + \int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

which is what we wanted to prove. □

## 3. Statement of the main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\phi\colon [0,+\infty) \to [0,+\infty)$ be a convex and non-decreasing function with $\phi(0) = 0$ and

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = c_0 \in (0,+\infty). \tag{3.1}$$

For any $\varepsilon > 0$ let $f_\varepsilon\colon [0,+\infty) \to [0,+\infty)$ be such that:

A1) $f_\varepsilon$ is non-decreasing, continuous, with $f_\varepsilon(0) = 0$.

A2) It holds $$\lim_{(\varepsilon,t)\to(0,0)} \frac{f_\varepsilon(t)}{\varepsilon\phi\left(\frac{t}{\varepsilon}\right)} = 1\,.$$

A3) $f_\varepsilon$ converges uniformly on the compact subsets of $[0,+\infty)$ to a concave function $f$.

**Example 3.1.** *Given $f$ and $\phi$ as above, a possible choice for $f_\varepsilon$ satisfying A1-A3 is given by*

$$f_\varepsilon(t) = \begin{cases} \varepsilon\phi\left(\frac{t}{\varepsilon}\right) & \text{if } 0 \leq t \leq t_\varepsilon \\ \\ f(t - t_\varepsilon) + \varepsilon\phi\left(\frac{t_\varepsilon}{\varepsilon}\right) & \text{if } t > t_\varepsilon \end{cases}$$

*where $t_\varepsilon \to 0$, and $t_\varepsilon/\varepsilon \to +\infty$. The only non-trivial assumption to verify is A2. Since $\varepsilon/t\phi(t/\varepsilon) \to c_0$ as $(\varepsilon,t) \to (0,0)$, with $t \geq t_\varepsilon$, the check amounts to verify that*

$$\lim_{\substack{(\varepsilon,t)\to(0,0)\\ t\geq t_\varepsilon}} \frac{f(t-t_\varepsilon) + \varepsilon\phi\left(\frac{t_\varepsilon}{\varepsilon}\right)}{t} = c_0\,.$$

*This follows immediately from $f(t-t_\varepsilon)/(t-t_\varepsilon) \to c_0$ and $\varepsilon/t_\varepsilon\phi(t_\varepsilon/\varepsilon) \to c_0$ as $(\varepsilon,t) \to (0,0)$, and $t \geq t_\varepsilon$.*

Let $C \subset \mathbb{R}^n$ be open, bounded, and connected with $0 \in C$. Let $\rho\colon C \to (0,+\infty)$ be a continuous and bounded convolution kernel with

$$\int_C \rho\,\mathrm{d}x = 1.$$

For any $\varepsilon > 0$ and for any $x \in \mathbb{R}^n$ we will denote by $C_\varepsilon(x)$ the set $x + \varepsilon C$. For any $x \in \varepsilon C$ let

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right).$$

We consider the family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ of functionals $L^1(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f_\varepsilon(\varepsilon|\nabla u| * \rho_\varepsilon)\,\mathrm{d}x & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases} \tag{3.2}$$

where, for any $x \in \Omega$,

$$|\nabla u| * \rho_\varepsilon(x) = \int_{C_\varepsilon(x)\cap\Omega} |\nabla u(y)|\rho_\varepsilon(y-x)\,\mathrm{d}y \tag{3.3}$$

is a regularization by convolution of $|\nabla u|$ by means of the kernel $\rho_\varepsilon$.

**Remark 3.2.** *Notice that with the choice $C = B_1$ and $\rho = \frac{1}{\omega_n}\chi_{B_1}$ we get*

$$|\nabla u| * \rho_\varepsilon(x) = \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u|\,\mathrm{d}y$$

*and thus the family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ reduces to the case already investigated in* [20], [21] *and* [22].

In order to prove the $\Gamma$-convergence of $\mathcal{F}_\varepsilon$ it is convenient to introduce a localized version of $\mathcal{F}_\varepsilon$: more precisely, for each $A \in \mathcal{A}(\Omega)$ we set

$$\mathcal{F}_\varepsilon(u, A) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_A f_\varepsilon(\varepsilon|\nabla u| * \rho_\varepsilon)\,\mathrm{d}x & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{3.4}$$

Clearly, $\mathcal{F}_\varepsilon\big(\cdot, \Omega\big)$ coincides with the functional $\mathcal{F}_\varepsilon$ defined in (3.2). The lower and upper $\Gamma$-limits of $\big(\mathcal{F}_\varepsilon(\cdot, A)\big)$ will be denoted by $\mathcal{F}'(\cdot, A)$ and $\mathcal{F}''(\cdot, A)$, respectively.

### 3.1. **The anisotropy.**

In this paragraph we define the surface density

$$\theta\colon [0, +\infty) \times \mathbb{S}^{n-1} \to [0, +\infty)$$

which will appear in the expression of the $\Gamma$-limit of $\mathcal{F}_\varepsilon$.

Given $\nu \in \mathbb{S}^{n-1}$ and $a, b \in \mathbb{R}$ let us denote by $u_\nu^{a,b}$ the function $\mathbb{R}^n \to \mathbb{R}$ given by

$$u_\nu^{a,b}(x) = \begin{cases} a & \text{if } \langle x, \nu\rangle < 0 \\ b & \text{if } \langle x, \nu\rangle \geq 0. \end{cases}$$

For any $x \in \mathbb{R}^n$ and any $\nu \in \mathbb{S}^{n-1}$ let $P_\nu^\perp(x)$ be the orthogonal projection of $x$ onto the subspace $\nu^\perp = \{x \in \mathbb{R}^n : \langle x, \nu\rangle = 0\}$. We define the cylinder

$$Q_\nu = \{x \in \mathbb{R}^n : |\langle x, \nu\rangle| \leq 1,\ P_\nu^\perp(x) \in B_1 \cap \nu^\perp\}.$$

Given $\Omega' \subset \mathbb{R}^n$ with $Q_\nu \subset\subset \Omega'$ denote by $W_\nu^{a,b}$ the space of all sequences $(u_j)$ in $W^{1,1}_{\mathrm{loc}}(\Omega')$ such that $u_j \to u_\nu^{a,b}$ in $L^1(\Omega')$, and such that there exist two positive infinitesimal sequences $(a_j), (b_j)$ with $u_j(x) = a$ if $\langle x, \nu\rangle < -a_j$ and $u_j = b$ if $\langle x, \nu\rangle > b_j$. Let

$$\theta(s, \nu) = \frac{1}{\omega_{n-1}} \inf\left\{ \liminf_{j\to+\infty} \frac{1}{\varepsilon_j} \int_{Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j) \in W_\nu^{0,s},\ \varepsilon_j \to 0^+ \right\}. \tag{3.5}$$

Notice that $\theta(s, \nu)$ does not depend on the choice of $\Omega'$. Let us collect some easy properties of $\theta$ which immediately descend from the definition.

**Lemma 3.3.** *The following properties hold:*

$$\theta \text{ is continuous.} \tag{3.6}$$

$$\theta(s, \nu) = \theta(s, -\nu), \quad \forall s \geq 0, \quad \forall \nu \in \mathbb{S}^{n-1}. \tag{3.7}$$

(3.8)
$$\inf\left\{\liminf_{j\to+\infty}\frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j|*\rho_{\varepsilon_j})\,\mathrm{d}x : (u_j)\in W^{0,s}_\nu,\ \varepsilon_j\to 0^+\right\}$$
$$=\inf\left\{\liminf_{j\to+\infty}\frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j|*\rho_{\varepsilon_j})\,\mathrm{d}x : (u_j)\in W^{a,b}_\nu,\ \varepsilon_j\to 0^+\right\}$$
*whenever* $|a-b|=s$.

*Moreover, for any* $x_0\in\mathbb{R}^n$, $\nu\in\mathbb{S}^{n-1}$ *and* $s\geq 0$ *we have*

$$\theta(s,\nu)=\frac{1}{\omega_{n-1}}\inf\left\{\liminf_{j\to+\infty}\frac{1}{\varepsilon_j}\int_{x_0+Q_\nu} f(\varepsilon_j|\nabla u_j|*\rho_{\varepsilon_j})\,\mathrm{d}x : (u_j(\cdot-x_0))\in W^{0,s}_\nu,\ \varepsilon_j\to 0^+\right\}. \tag{3.9}$$

3.2. **Main results.** We are now in position to state the main result of the paper.

**Theorem 3.4.** *Let* $\mathcal{F}_\varepsilon$ *be as in (3.2), with* $f_\varepsilon$ *satisfying conditions A1-A3. Then* $\mathcal{F}_\varepsilon$ Γ*-converges, with respect to the strong* $L^1$*-topology, as* $\varepsilon\to 0$*, to* $\mathcal{F}\colon L^1(\Omega)\to[0,+\infty]$ *given by*

$$\mathcal{F}(u)=\begin{cases}\displaystyle\int_\Omega\phi(|\nabla u|)\,\mathrm{d}x+\int_{S_u}\theta(|u^+-u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+c_0|D^cu|(\Omega) & \text{if } u\in GBV(\Omega)\\ +\infty & \text{otherwise in } L^1(\Omega).\end{cases}$$

**Remark 3.5.** *Notice that for any* $u\in GBV(\Omega)$ *the expression* $\theta(|u^+-u^-|,\nu_u)$ *turns out to be well defined* $\mathcal{H}^{n-1}$*-a.e.* $x\in S_u$*, since* (3.7) *holds.*

The proof of Theorem 3.4 will descend combining Proposition 5.10 (the Γ-liminf inequality) with Proposition 6.3 (the Γ-limsup inequality).

As a typical consequence of a Γ-convergence result, we are able to prove a result of convergence of minima by means of the following compactness result for equibounded (in energy) sequences, which will be proved in §4.

**Theorem 3.6.** *Let* $(\varepsilon_j)$ *be a positive infinitesimal sequence, and let* $(u_j)$ *be a sequence in* $L^1(\Omega)$ *such that* $||u_j||_\infty\leq M$*, and such that* $\mathcal{F}_{\varepsilon_j}(u_j)\leq M$ *for some positive constant* $M$ *independent of* $j$*. Then the sequence* $(u_j)$ *converges, up to a subsequence, in* $L^1(\Omega)$ *to a function* $u\in BV(\Omega)$*.*

**Theorem 3.7.** *Let* $(\varepsilon_j)$ *be a positive infinitesimal sequence and let* $g\in L^\infty(\Omega)$*. For every* $u\in L^1(\Omega)$ *and* $j\in\mathbb{N}$ *let*

$$\mathcal{I}_j(u)=\mathcal{F}_{\varepsilon_j}(u)+\int_\Omega|u-g|\,\mathrm{d}x,\qquad \mathcal{I}(u)=\mathcal{F}(u)+\int_\Omega|u-g|\,\mathrm{d}x\,.$$

*For every* $j$ *let* $u_j\in L^1(\Omega)$ *be such that*

$$\mathcal{I}_j(u_j)\leq\inf_{L^1(\Omega)}\mathcal{I}_j+\varepsilon_j.$$

*Then the sequence* $(u_j)$ *converges, up to a subsequence, to a minimizer of* $\mathcal{I}$ *in* $L^1(\Omega)$*.*

*Proof.* Since $g\in L^\infty(\Omega)$ and since $\mathcal{F}_{\varepsilon_j}$ decreases by truncation, we can assume that $(u_j)$ is equibounded in $L^\infty(\Omega)$; for instance $||u_j||_\infty\leq||g||_\infty$. Applying Theorem 3.6 there exists $u\in BV(\Omega)$ such that (up to a subsequence) $u_j\to u$ in $L^1(\Omega)$. By Theorem 2.3, since $(\mathcal{I}_j)$ Γ-converges to $\mathcal{I}$ (see Th. 3.4 and Remark 2.4), $u$ is a minimum point of $\mathcal{I}$ on $L^1(\Omega)$. □

## 4. Compactness

In this section we prove Theorem 3.6. Let us first recall a useful technical Lemma which can be found in [10], Prop. 4.1. Actually such a Proposition has been proved for $|\nabla u|^2$, but, up to simple modifications, the same proof works for $|\nabla u|$.

For every $A\in\mathcal{A}(\Omega)$ and $\sigma>0$ we set

$$A_\sigma=\{x\in A:\ d(x,\partial A)>\sigma\}\,.$$

**Lemma 4.1.** *Let $g\colon [0,+\infty)\to[0,+\infty)$ be a non-decreasing continuous function such that*

$$\lim_{t\to 0}\frac{g(t)}{t}=c$$

*for some $c>0$. Let $A\in\mathcal{A}(\Omega)$ with $A\subset\subset\Omega$, and let $u\in W^{1,1}(\Omega)\cap L^\infty(\Omega)$. For any $\delta>0$ and for any $\varepsilon>0$ sufficiently small, there exists a function $v\in SBV(A)\cap L^\infty(A)$ such that*

$$(1-\delta)\int_A|\nabla v|\,\mathrm{d}x\le\frac{1}{\varepsilon}\int_A g\bigg(\varepsilon\ \fint_{B_\varepsilon(x)}|\nabla u|\,\mathrm{d}y\bigg)\,\mathrm{d}x,$$

$$\mathcal{H}^{n-1}(S_v\cap A_{6\varepsilon})\le\frac{c'}{\varepsilon}\int_A g\bigg(\varepsilon\ \fint_{B_\varepsilon(x)}|\nabla u|\,\mathrm{d}y\bigg)\,\mathrm{d}x,$$

$$\|v\|_{L^\infty(A)}\le\|u\|_{L^\infty(A)}$$

$$\|v-u\|_{L^1(A_{6\varepsilon})}\le c'\|u\|_{L^\infty(A)}\int_A g\bigg(\varepsilon\ \fint_{B_\varepsilon(x)}|\nabla u|\,\mathrm{d}y\bigg)\,\mathrm{d}x,$$

*where $c'$ is a constant depending only on $n,\delta$ and $g$.*

*Proof of Theorem 3.6.* Let $A\in\mathcal{A}(\Omega)$ with $A\subset\subset\Omega$ and $\partial A$ smooth. Let $r>0$ such that $B_r\subset C$, and let $m=\inf_{B_r}\rho>0$. Then for any $x\in A$ we have $B_{r\varepsilon_j}(x)\subset C_{\varepsilon_j}(x)$ and thus for $j$ sufficiently large,

$$\begin{aligned}|\nabla u_j|*\rho_{\varepsilon_j}(x)&=\int_{C_{\varepsilon_j}(x)}|\nabla u_j(y)|\rho_{\varepsilon_j}(y-x)\,\mathrm{d}y\ge\frac{m}{\varepsilon_j^n}\int_{B_{r\varepsilon_j}(x)}|\nabla u_j(y)|\,\mathrm{d}y\\&=mr^n\omega_n\fint_{B_{r\varepsilon_j}(x)}|\nabla u_j(y)|\,\mathrm{d}y\end{aligned}$$

for any $x\in A$. Fix $\delta>0$. By A2 there exist $t_\delta>0$ and $j_\delta$ such that $f_{\varepsilon_j}(t)\ge(1-\delta)\varepsilon_j\phi(t/\varepsilon_j)$ for any $t\in[0,t_\delta]$ and $j>j_\delta$. Let $\alpha,\beta\in\mathbb{R}$, with $\alpha>0$ and $\beta<0$, be such that $\phi(t)\ge\alpha t+\beta$ everywhere. Then, since $f_{\varepsilon_j}$ is non-decreasing, we have $f_{\varepsilon_j}(t)\ge g^\delta_{\varepsilon_j}(t)$ for any $t\ge 0$, being

$$g^\delta_{\varepsilon_j}(t)=\begin{cases}(1-\delta)\alpha t+\varepsilon_j\beta & \text{if } t\in[0,t_\delta]\\(1-\delta)\alpha t_\delta+\varepsilon_j\beta & \text{if } t>t_\delta.\end{cases}$$

Therefore, letting $h_\delta(t)=g^\delta_{\varepsilon_j}(t)-\varepsilon_j\beta$, we have

$$\begin{aligned}\mathcal{F}_{\varepsilon_j}(u_j,A)&\ge\frac{1}{\varepsilon_j}\int_A h_\delta(|\nabla u_j|*\rho_{\varepsilon_j})\,\mathrm{d}x+\beta|A|\\&\ge\frac{1}{\varepsilon_j}\int_A h_\delta\bigg(mr^n\omega_n\varepsilon_j\fint_{B_{r\varepsilon_j}(x)}|\nabla u_j|\,\mathrm{d}y\bigg)\,\mathrm{d}x+\beta|A|.\end{aligned}\tag{4.1}$$

Let $\eta_j=r\varepsilon_j$ and $g_{\delta,m,r}(t)=\frac{1}{r}g_\delta(mr^{n-1}\omega_n t)$. Notice that, by construction,

$$\lim_{t\to 0}\frac{g_{\delta,m,r}(t)}{t}$$

exists and is finite. Then inequality (4.1) becomes

$$\mathcal{F}_{\varepsilon_j}(u_j,A)-\beta|A|\ge\frac{1}{\eta_j}\int_\Omega g_{\delta,r,m}\bigg(\eta_j\fint_{B_{\eta_j}(x)}|\nabla u_j|\,\mathrm{d}y\bigg)\,\mathrm{d}x.$$

Applying Lemma 4.1 we find a sequence $(v_j)$ in $SBV(A)$ and a constant $C$ independent of $A$ such that $\|v_j\|_{BV(A)}\le C$ and $\|v_j\|_{L^\infty(A)}\le C$. Moreover,

$$\|v_j-u_j\|_{L^1(A)}\to 0.\tag{4.2}$$

Hence, by Theorem 2.1, the sequence $(v_j)$ converges, up to a subsequence not relabeled, to some $u\in BV(A)$, with $\|u\|_{BV(A)}\le C$. By (4.2) also $u_j$ converges to $u$ in $L^1(A)$. The arbitrariness of $A$ and a diagonal argument allow to find a subsequence $(u_{j_k})$ which converges in $L^1_{\mathrm{loc}}(\Omega)$ to a function $u\in BV_{\mathrm{loc}}(\Omega)$, and the uniform bound of $\|u_j\|_{L^\infty(\Omega)}$ implies the convergence is strong in $L^1(\Omega)$. □

## 5. The Γ-liminf inequality

In this section we will prove that for any $u \in L^1(\Omega)$ the inequality

$$\mathcal{F}(u) \le \liminf_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u_j)$$

holds for any $u_j \to u$ in $L^1(\Omega)$. First we will investigate two particular situations.

5.1. **A preliminary estimate from below in terms of the volume and Cantor parts.** In this paragraph we will take into account a simpler family of functionals. Let $\alpha, \beta > 0$ and let $g\colon [0,+\infty) \to [0,+\infty)$ given by $g(t) = \alpha t \wedge \beta$. Let $\mathcal{G}_\varepsilon\colon L^1(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be defined by

$$\mathcal{G}_\varepsilon(u,A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A g(\varepsilon|\nabla u| * \rho_\varepsilon)\,\mathrm{d}x & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

We wish to estimate from below the lower Γ-limit $\mathcal{G}'(\cdot, A)$ in terms of the volume and the Cantor parts of $Du$. To this sake, we apply a slicing procedure, so that at first we will establish a suitable one-dimensional inequality. The idea of the proof is the same as in [25], where the superlinear growth case is treated.

Let $m \in \mathbb{N}$ odd, let $A$ be an open interval in $\mathbb{R}$, and let $(\varepsilon_j)$ be a positive infinitesimal sequence. Let $A_j = \{x \in \varepsilon_j\mathbb{Z} : x \in A\}$. For any $j \in \mathbb{N}$ and for any $x \in A_j$ we define the interval

$$I_j(x) = \left[x - \frac{m\varepsilon_j}{2}, x + \frac{m\varepsilon_j}{2}\right].$$

**Lemma 5.1.** *Let $\alpha', \beta' > 0$ and let $h_j\colon [0,+\infty) \to [0,+\infty)$ given by $h_j(t) = \alpha' t \wedge \frac{\beta'}{\varepsilon_j}$. Let $u \in BV(A)$ and let $u_j \to u$ in $L^1(A)$ with $u_j \in W^{1,1}(A)$ for any $j \in \mathbb{N}$. Then*

$$\liminf_{j\to+\infty} \varepsilon_j \sum_{x\in A_j} h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr) \ge \alpha' \int_A |u'|\,\mathrm{d}y + \alpha'|D^c u|(A). \tag{5.1}$$

*Proof.* For any $j \in \mathbb{N}$ and $i = 0, \dots, m-1$ let $A_j^i = (i\varepsilon_j + m\varepsilon_j\mathbb{Z}) \cap A$. Obviously $A_j$ is the disjoint union of $A_j^i$ for $i \in \{0, \dots, m-1\}$. Then

$$\sum_{x\in A_j} h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr) \ge \frac{1}{m}\sum_{i=0}^{m-1}\sum_{x\in A_j^i} m h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr).$$

Now let

$$\overline{A_j^i} = \left\{x \in A_j^i :\ \fint_{I_j(x)} |u_j'|\,\mathrm{d}x \le \frac{\beta'}{\alpha'\varepsilon_j}\right\}$$

and let $v_j \in SBV(A)$ given by

$$v_j(x) = \begin{cases} u_j(x) & \text{if } x \in \bigcup_{y\in\overline{A_j^i}} I_j(y) \\ 0 & \text{otherwise in } A. \end{cases}$$

Hence

$$\sum_{x\in A_j^i} m\varepsilon_j h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr) \ge \sum_{x\in\overline{A_j^i}} m\varepsilon_j h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr) = \alpha' \sum_{x\in\overline{A_j^i}} \int_{I_j(x)} |u_j'|\,\mathrm{d}y$$
$$= \alpha' \int_A |v_j'|\,\mathrm{d}y.$$

Observe that since we can suppose, without loss of generality, that

$$\varepsilon_j \sum_{x\in A_j} h_j\biggl(\fint_{I_j(x)} |u_j'|\,\mathrm{d}y\biggr) \le M$$

for some $M \geq 0$, we deduce that

$$M \geq \varepsilon_j \sum_{x \in A_j \setminus \bigcup_{i=0}^{m-1} \overline{A_j^i}} h_j\left( \fint_{I_j(x)} |u_j'| \, \mathrm{d}y \right) = \varepsilon_j \frac{\beta'}{\varepsilon_j} \sharp \left( A_j \setminus \bigcup_{i=0}^{m-1} \overline{A_j^i} \right)$$

from which necessarily we have

$$\varepsilon_j \sharp \left( A_j \setminus \bigcup_{i=0}^{m-1} \overline{A_j^i} \right) \to 0, \quad \text{as } j \to +\infty.$$

This implies that $||u_j - v_j||_{L^1(A)} \to 0$ as $j \to +\infty$. Therefore, $v_j \to u$ in $L^1(A)$. Finally, by the superadditivity of the lim inf and by the lower semicontinuity of the total variation, we get

$$\begin{aligned} \liminf_{j \to +\infty} \varepsilon_j \sum_{x \in A_j} h_j\left( \fint_{I_j(x)} |u_j'| \, \mathrm{d}y \right) &\geq \frac{1}{m} \sum_{i=0}^{m-1} \liminf_{j \to +\infty} \sum_{x \in \overline{A_j^i}} m \varepsilon_j h_j \left( \fint_{I_j(x)} |u_j'| \, \mathrm{d}y \right) \\ &\geq \alpha' \liminf_{j \to +\infty} \int_A |v_j'| \, \mathrm{d}y \geq \alpha' |Du|(A) \\ &\geq \alpha' \int_A |u'| \, \mathrm{d}y + \alpha' |D^c u|(A) \end{aligned}$$

which ends the proof. □

Now, by applying the slicing Theorem 2.2, we will reduce the $n$-dimensional inequality to the one-dimensional inequality 5.1. Fix $\xi \in \mathbb{S}^{n-1}$ and $\delta \in (0,1)$; consider an orthonormal basis $\{\mathbf{e}_i\}$ with $\mathbf{e}_n = \xi$. Let

$$Q_\delta^\xi = \left\{ x \in \mathbb{R}^n : |\langle x, \mathbf{e}_i \rangle| \leq \frac{\delta}{2}, \, i = 1, \dots, n \right\}, \quad Q_\delta^\xi(x) = x + Q_\delta^\xi$$

and the lattice $Z_\delta^\xi = \{x \in \mathbb{R}^n : \langle x, \mathbf{e}_i \rangle \in \delta\mathbb{Z}, \, i = 1, \dots, n\}$. In what follows we will denote by $g_j(t) = \frac{1}{\varepsilon_j} g(\varepsilon_j t)$; in particular it holds $g_j(t) = \alpha t \wedge \frac{\beta}{\varepsilon_j}$ and

$$\mathcal{G}_{\varepsilon_j}(u, A) = \int_A g_j(|\nabla u| * \rho_{\varepsilon_j}) \, \mathrm{d}x, \quad u \in W^{1,1}(\Omega).$$

Finally fix $A \in \mathcal{A}(\Omega)$ and let $A_\delta^\xi = \{x \in Z_\delta^\xi : Q_\delta^\xi(x) \subset A\}$. The following Lemma is a standard easy application of the mean value Theorem (see also Lemma 4.2 in [10]).

**Lemma 5.2.** *Let $u \in W^{1,1}(\Omega)$. Then there exists $\tau \in Q_\delta^\xi$ such that*

$$\mathcal{G}_{\varepsilon_j}(u, A) \geq \sum_{x \in A_\delta^\xi} \delta^n g_j(|\nabla u| * \rho_{\varepsilon_j}(x + \tau)).$$

*Proof.* We have

$$\mathcal{G}_{\varepsilon_j}(u, A) \geq \sum_{x \in A_\delta^\xi} \int_{Q_\delta^\xi(x)} g_j(|\nabla u| * \rho_{\varepsilon_j}(y)) \, \mathrm{d}y = \int_{Q_\delta^\xi} \sum_{x \in A_\delta^\xi} g_j(|\nabla u| * \rho_{\varepsilon_j}(y + x)) \, \mathrm{d}y.$$

Applying the mean value Theorem we get

$$\int_{Q_\delta^\xi} \sum_{x \in A_\delta^\xi} g_j(|\nabla u| * \rho_{\varepsilon_j}(y + x)) \, \mathrm{d}y = \sum_{x \in A_\delta^\xi} g_j(|\nabla u| * \rho_{\varepsilon_j}(\tau + x))$$

for some $\tau \in Q_\delta^\xi$, which concludes the proof. □

We are in position to apply the slicing procedure.

**Proposition 5.3.** *Let $u \in BV(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Then*

$$\mathcal{G}'(u, A) \geq \alpha \int_A |\nabla u| \, \mathrm{d}x \quad \textit{and} \quad \mathcal{G}'(u, A) \geq \alpha |D^c u|(A).$$

*Proof.* Fix $\xi \in \mathbb{S}^{n-1}$. For any $\eta > 0$ let $P^\xi_\eta$ be the union of the squares $Q^\xi_\eta(y_i) \subset C$ with $y_i \in Z^\xi_\eta$ for $i = 1, \dots, m$, for some $m \in \mathbb{N}$ depending on $\eta$ and $\xi$. Let $\rho_\eta$ be a non-negative constant function on the squares $Q^\xi_\eta(y_i)$ with $0 < \rho_\eta \leq \rho$ and such that

$$c_\eta = \int_C \rho_\eta \,\mathrm{d}x \to 1, \quad \text{as } \eta \to 0.$$

Let $c_i = \rho_\eta(y_i)$; then we can rewrite $c_\eta$ as $c_\eta = \sum_{i=1}^m c_i \eta^n$. Let $P^\xi_{\eta\varepsilon_j}$ be the union of the squares $Q^\xi_{\eta\varepsilon_j}(y_i) \subseteq C_{\varepsilon_j}$, with $y_i \in Z^\xi_{\eta\varepsilon_j}$, for $i = 1, \dots, m$. Let $A^\xi_j = A^\xi_{\eta\varepsilon_j}$; applying Lemma 5.2, since we can suppose, without loss of generality, that $u_j \in W^{1,1}(\Omega)$, there exists $\tau_j \in Q^\xi_{\eta\varepsilon_j}$ such that

$$\mathcal{G}_{\varepsilon_j}(u_j, A) \geq \sum_{x \in A^\xi_j} (\eta\varepsilon_j)^n g_j(|\nabla u_j| * \rho_{\varepsilon_j}(x + \tau_j)).$$

Let $B \subset\subset A$, and, for any $j$ sufficiently large, let $v_j(y) = u_j(y + \tau_j)$. Then we get $v_j \in W^{1,1}(B)$ and $v_j \to u$ in $L^1(B)$. Thus

$$\mathcal{G}_{\varepsilon_j}(u_j, A) \geq \sum_{x \in B^\xi_j} (\eta\varepsilon_j)^n g(|\nabla v_j| * \rho_{\varepsilon_j}(x))$$

being $B^\xi_j = \{x \in Z^\xi_{\eta\varepsilon_j} : Q^\xi_{\eta\varepsilon_j} \subseteq B\}$. Now, for each $x \in B^\xi_j$, we estimate the term $|\nabla v_j| * \rho_{\varepsilon_j}(x)$; we have, for $j$ large enough,

$$\begin{aligned}|\nabla v_j| * \rho_{\varepsilon_j}(x) &= \int_{C_{\varepsilon_j}} |\nabla v_j(y + x)| \rho_{\varepsilon_j}(y) \,\mathrm{d}y \geq \frac{1}{\varepsilon_j^n} \int_{P^\xi_{\eta\varepsilon_j}} |\nabla v_j(y + x)| \rho_\eta\left(\frac{y}{\varepsilon_j}\right) \mathrm{d}y \\ &\geq \frac{1}{\varepsilon_j^n} \sum_{i=1}^m c_i \int_{Q^\xi_{\eta\varepsilon_j}(y_i)} |\nabla v_j(y + x)| \,\mathrm{d}y = \sum_{i=1}^m \frac{c_i \eta^n}{c_\eta} \fint_{Q^\xi_{\eta\varepsilon_j}(y_i)} c_\eta |\nabla v_j(y + x)| \,\mathrm{d}y.\end{aligned}$$

Since $\sum_{i=1}^m \frac{c_i \eta^n}{c_\eta} = 1$ and since $g_j$ is concave we get, for every $x \in B^\xi_j$,

$$g_j(|\nabla v_j| * \rho_{\varepsilon_j}(x)) \geq \sum_{i=1}^m \frac{c_i \eta^n}{c_\eta} g_j\left(c_\eta \fint_{Q^\xi_{\eta\varepsilon_j}(y_i)} |\nabla v_j(y + x)| \,\mathrm{d}y\right).$$

Thus, reordering the terms, we deduce that

$$\mathcal{G}_{\varepsilon_j}(u_j, A) \geq \sum_{x \in D^\xi_j} (\eta\varepsilon_j)^n g_j\left(c_\eta \fint_{Q^\xi_{\eta\varepsilon_j}(x)} |\nabla v_j| \,\mathrm{d}z\right)$$

for any $D \subset\subset B$ and $j$ sufficiently large, being, as usual, $D^\xi_j = \{x \in Z^\xi_{\eta\varepsilon_j} : Q^\xi_{\eta\varepsilon_j} \subseteq D\}$. For convenience we can suppose $\nabla v_j = 0$ on

$$\mathbb{R}^n \setminus \bigcup_{Q^\xi_{\eta\varepsilon_j} \subseteq D} Q^\xi_{\eta\varepsilon_j}.$$

Let $\langle\xi\rangle$ be the one-dimensional space generated by $\xi$. Let us denote by $Z^{\xi_\parallel}_{\eta\varepsilon_j}$ and by $Z^{\xi_\perp}_{\eta\varepsilon_j}$ the orthogonal projections of $Z^\xi_{\eta\varepsilon_j}$ respectively on $\langle\xi\rangle$ and $\xi^\perp$. Then

$$\begin{aligned}\mathcal{G}_{\varepsilon_j}(u_j, A) &\geq \sum_{x \in Z^\xi_{\eta\varepsilon_j}} (\eta\varepsilon_j)^n g_j\left(c_\eta \fint_{Q^\xi_{\eta\varepsilon_j}(x)} |\nabla v_j| \,\mathrm{d}z\right) \\ &\geq \sum_{x_\perp \in Z^{\xi_\perp}_{\eta\varepsilon_j}} \sum_{x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}} (\eta\varepsilon_j)^n g_j\left(c_\eta \fint_{Q^\xi_{\eta\varepsilon_j}(x_\perp + x_\parallel)} |\nabla v_j| \,\mathrm{d}z\right)\end{aligned}$$

where $x = x_\parallel + x_\perp$ turns out to be the unique decomposition of any $x \in Z^\xi_{\eta\varepsilon_j}$ with $x_\parallel \in Z^{\xi_\parallel}_{\eta\varepsilon_j}$ and $x_\perp \in Z^{\xi_\perp}_{\eta\varepsilon_j}$. Moreover, denoting by $Q^{\xi_\parallel}_{\eta\varepsilon_j}$ and by $Q^{\xi_\perp}_{\eta\varepsilon_j}$ the projections of $Q^\xi_{\eta\varepsilon_j}$ respectively on $\langle\xi\rangle$

and on $\xi^\perp$, applying Jensen's inequality we deduce that

$$\begin{aligned}
\mathcal{G}_{\varepsilon_j}(u_j,A) &\geq \sum_{x_\perp\in Z^{\xi_\perp}_{\eta\varepsilon_j}}\sum_{x_\parallel\in Z^{\xi_\parallel}_{\eta\varepsilon_j}}(\eta\varepsilon_j)^n g_j\left(c_\eta \fint_{Q^{\xi_\perp}_{\eta\varepsilon_j}(x_\perp)}\fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)}|\langle\nabla v_j(z_\perp+z_\parallel),\xi\rangle|\,\mathrm{d}z_\parallel\,\mathrm{d}z_\perp\right)\\
&\geq \sum_{x_\perp\in Z^{\xi_\perp}_{\eta\varepsilon_j}}\sum_{x_\parallel\in Z^{\xi_\parallel}_{\eta\varepsilon_j}}(\eta\varepsilon_j)^n \fint_{Q^{\xi_\perp}_{\eta\varepsilon_j}(x_\perp)} g_j\left(c_\eta \fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)}|\langle\nabla v_j(z_\perp+z_\parallel),\xi\rangle|\,\mathrm{d}z_\parallel\right)\mathrm{d}z_\perp\\
&\geq \sum_{x_\perp\in Z^{\xi_\perp}_{\eta\varepsilon_j}}\int_{Q^{\xi_\perp}_{\eta\varepsilon_j}(x_\perp)}\sum_{x_\parallel\in Z^{\xi_\parallel}_{\eta\varepsilon_j}}\eta\varepsilon_j g_j\left(c_\eta \fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)}|\langle\nabla v_j(z_\perp+z_\parallel),\xi\rangle|\,\mathrm{d}z_\parallel\right)\mathrm{d}z_\perp\\
&\geq \int_{\xi^\perp}\sum_{x_\parallel\in Z^{\xi_\parallel}_{\eta\varepsilon_j}}\eta\varepsilon_j g_j\left(c_\eta \fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)}|\langle\nabla v_j(z_\perp+z_\parallel),\xi\rangle|\,\mathrm{d}z_\parallel\right)\mathrm{d}z_\perp.
\end{aligned}$$

For any $\sigma>0$ small let $D_\sigma=\{x\in D : d(x,\partial D)>\sigma\}$ and $D^{x_\perp}_\sigma=\{x\in D_\sigma : x=x_\perp+x_\parallel\xi,\ x_\parallel\in\mathbb{R}\}$, for $x_\perp\in\xi^\perp$. For $j$ sufficiently large, $v_j(x_\perp+\cdot)\in W^{1,1}(D^{x_\perp}_\sigma)$. Furthermore, $v_j\to u$ in $L^1(D^{x_\perp}_\sigma)$ for a.e. $x_\perp\in\xi^\perp$. Let $h_j(t)=g_j(c_\eta t)$; then, by the very definition of $g$, it is easy to see that $h_j(t)=\alpha c_\eta t\wedge\frac{\beta}{\varepsilon_j}$. We are in position to apply Lemma 5.1 with choice $\alpha'=\alpha c_\eta$ and $\beta'=\beta$. Thus

$$\begin{aligned}
\liminf_{j\to+\infty}&\sum_{x_\parallel\in Z^{\xi_\parallel}_{\eta\varepsilon_j}}\eta\varepsilon_j g_j\left(c_\eta\fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)}|\langle\nabla v_j(z_\perp+z_\parallel),\xi\rangle|\,\mathrm{d}z_\parallel\right)\\
&=\liminf_{j\to+\infty}\sum_{x_\parallel\in Z^{\xi_\parallel}_{\eta\varepsilon_j}}\eta\varepsilon_j h_j\left(\fint_{Q^{\xi_\parallel}_{\eta\varepsilon_j}(x_\parallel)}|\langle\nabla v_j(z_\perp+z_\parallel),\xi\rangle|\,\mathrm{d}z_\parallel\right)\\
&\geq \alpha c_\eta\int_{D^{z_\perp}_\sigma}|\langle\nabla u(z_\perp+z_\parallel),\xi\rangle|\,\mathrm{d}z_\parallel+\alpha c_\eta|\langle D^c u(z_\perp+\cdot),\xi\rangle|(D^{z_\perp}_\sigma).
\end{aligned}$$

Taking into account Theorem 2.2 and Fatou's Lemma we conclude that

$$\liminf_{j\to+\infty}\mathcal{G}_{\varepsilon_j}(u_j,A)\geq c_\eta\alpha\int_{D_\sigma}|\langle\nabla u(z),\xi\rangle|\,\mathrm{d}z+c_\eta\alpha|\langle D^c u,\xi\rangle(D_\sigma).$$

Since $c_\eta\to 1$ as $\eta\to 0$, let $\sigma\to 0$ and $D\nearrow A$. Then

$$\mathcal{G}'(u,A)\geq\alpha\int_A|\langle\nabla u(z),\xi\rangle|\,\mathrm{d}z\quad\text{and}\quad\mathcal{G}'(u,A)\geq\alpha|\langle D^c u,\xi\rangle|(A)\tag{5.2}$$

for any $\xi\in\mathbb{S}^{n-1}$. From the first inequality, using the superadditivity of $\mathcal{G}'$ and Lemma 2.5 we easily deduce that

$$\mathcal{G}'(u,A)\geq\alpha\int_A|\nabla u|\,\mathrm{d}z.$$

Now if $\psi_\xi=\left\langle\frac{dD^c u}{d|D^c u|},\xi\right\rangle$ the second inequality in (5.2) can be rewritten as

$$\mathcal{G}'(u,A)\geq\alpha\int_A|\psi_\xi|\,\mathrm{d}|D^c u|.$$

Another application of Lemma 2.5 yields

$$\mathcal{G}'(u,A)\geq\alpha\int_A\sup_{\xi\in\mathbb{S}^{n-1}}|\psi_\xi|\,\mathrm{d}|D^c u|\geq\alpha\int_A|\sup_{\xi\in\mathbb{S}^{n-1}}\psi_\xi|\,\mathrm{d}|D^c u|=\alpha|D^c u|(A)\,.$$

This concludes the proof. □

5.2. **A preliminary estimate in terms of the surface part.** In this section we will consider the family of functionals $L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ given by

$$\mathcal{E}_\varepsilon(u, A) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_A h(\varepsilon |\nabla u| * \rho_\varepsilon) \, \mathrm{d}x & \text{if } u \in W^{1,1}(\Omega) \\ \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

where $h \colon [0, +\infty) \to [0, +\infty)$ is a non-decreasing concave function with $h(0) = 0$ and with

$$\lim_{t \to 0} = \frac{h(t)}{t} = c' > 0.$$

The aim of this section is to estimate from below the lower Γ-limit of $\mathcal{E}_\varepsilon$ in terms of a surface integral; to do this the main idea, as in [22], is to estimate from below the Radon-Nikodym derivative of the lower Γ-limit $\mathcal{E}'$ with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ by means of a blow-up argument around a jump point; then the result follows applying Besicovitch's Differentiation Theorem in a standard way.

Given $x_0 \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $a, b \in \mathbb{R}$, when considering $\mathcal{E}'$ for the blow up $u_{x_0}^{\nu, a, b} = u_\nu^{a,b}(\cdot - x_0)$ (see paragraph 3.1 for the definition of $u_\nu^{a,b}$) on a unit ball $B_1$ as below (or on a cylinder $Q_\nu$ as in the sequel), we will assume as $\Omega$ any set $\Omega'$ strictly containing $B_1$ (or $Q_\nu$): the lower Γ-limit of $\mathcal{E}_\varepsilon(\cdot, A)$ does not change by replacing $\Omega$ with any $\Omega' \supset\supset A$.

For every $A \in \mathcal{A}(\Omega)$ let $\mathcal{E}'_-(\cdot, A)$ be the inner regular envelope of $\mathcal{E}'$, i.e.

$$\mathcal{E}'_-(\cdot, A) = \sup\{\mathcal{E}'(\cdot, B) : B \in \mathcal{A}(\Omega), B \subset\subset A\}\,.$$

**Proposition 5.4.** *Let $u \in BV(\Omega)$ and let $x_0 \in J_u$. Then*

$$\liminf_{\varrho \to 0} \frac{\mathcal{E}'_-(u, B_\varrho(x))}{\varrho^{n-1}} \geq \mathcal{E}'(u_{x_0}^{\nu_u(x_0), u^+(x_0), u^-(x_0)}, B_1(x_0)).$$

*Proof.* Let $\delta \in (0, 1)$. Then $\mathcal{E}'_-\big(u, B_\varrho(x_0)\big) \geq \mathcal{E}'\big(u, B_{\delta\varrho}(x_0)\big)$ for every $\varrho > 0$. Thus

$$\liminf_{\varrho \to 0} \frac{\mathcal{E}'_-(u, B_\varrho(x_0))}{\varrho^{n-1}} \geq \delta^{n-1} \liminf_{r \to 0} \frac{\mathcal{E}'(u, B_r(x_0))}{r^{n-1}}\,. \tag{5.3}$$

Let us now estimate the lower limit in the right-hand side. Without loss of generality we can assume $x_0 = 0$; moreover, for the sake of simplicity, we will denote by $u_0$ the function $u_0^{\nu_u(0), u^+(0), u^-(0)}$.

Let $(r_k)$ be a decreasing infinitesimal sequence; for every $k \in \mathbb{N}$ there exists $u_j \in W^{1,1}(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$ and

$$\liminf_{j \to +\infty} \mathcal{E}_{\varepsilon_j}(u_j, B_{r_k}) \leq \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{2k}.$$

Let $\bar{j} = j(k)$ be such that $\varepsilon_{\bar{j}}/r_k \leq 1/k$ and

$$\mathcal{E}_{\varepsilon_{\bar{j}}}(u_{\bar{j}}, B_{r_k}) \leq \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{k},$$

$\|u_{\bar{j}} - u\|_{L^1(\Omega)} \leq \frac{1}{k}$ and such that

$$\int_{B_2} |u_{\bar{j}}(r_k x) - u(r_k x)| \, \mathrm{d}x \leq \frac{1}{k}.$$

Let $v_k = u_{j(k)}$. We can suppose that the sequence $j(k)$ is increasing, and we set $\sigma_k = \varepsilon_{j(k)}$. Hence, $v_k \to u$ in $L^1(\Omega)$,

$$\mathcal{E}_{\sigma_k}(v_k, B_{r_k}) \leq \mathcal{E}'(u, B_{r_k}) + \frac{r_k^{n-1}}{k} \tag{5.4}$$

and

$$\int_{B_2} |v_k(r_k x) - u(r_k x)| \, \mathrm{d}x \leq \frac{1}{k}\,. \tag{5.5}$$

Inequality (5.4) gives

$$\liminf_{k\to+\infty}\frac{\mathcal{E}'(u,B_{r_k})}{r_k^{n-1}}\geq\liminf_{k\to+\infty}\frac{\mathcal{E}_{\sigma_k}(v_k,B_{r_k})}{r_k^{n-1}}$$

while from (5.5) we get

$$\int_{B_2}|v_k(r_kx)-u_0(r_kx)|\,\mathrm{d}x\leq\frac{1}{k}+\int_{B_2}|v(r_kx)-u_0(r_kx)|\,\mathrm{d}x\to 0$$

as $k\to+\infty$. Let $w_k(t)=v_k(r_kt)$. Then $w_k\to u_0$ in $L^1(B_2)$; moreover, for every $x\in B_{r_k}$ we have, setting $y=r_kt$ and observing that $|\nabla w_k(t)|=r_k|\nabla v_k(r_kt)|$,

$$\begin{aligned}|\nabla v_k|*\rho_{\sigma_k}(x)&=\int_{C_{\sigma_k}(x)}|\nabla v_k(y)|\rho_{\sigma_k}(y-x)\,\mathrm{d}y=\frac{1}{\sigma_k^n}\int_{C_{\sigma_k}(x)}|\nabla v_k(y)|\rho\Big(\frac{y-x}{\sigma_k}\Big)\,\mathrm{d}y\\&=\frac{r_k^{n-1}}{\sigma_k^n}\int_{C_{\sigma_k/r_k}(x/r_k)}|\nabla w_k(t)|\rho\Big(\frac{t}{\sigma_k/r_k}-\frac{x}{\sigma_k}\Big)\,\mathrm{d}t.\end{aligned}$$

Therefore, setting $x=r_kz$, we obtain

$$\begin{aligned}\frac{\mathcal{E}_{\sigma_k}(v_k,B_{r_k})}{r_k^{n-1}}&=\frac{1}{r_k^{n-1}\sigma_k}\int_{B_{r_k}}h(\sigma_k|\nabla v_k|*\rho_{\sigma_k}(x))\,\mathrm{d}x\\&=\frac{1}{r_k^{n-1}\sigma_k^n}\int_{B_{r_k}}h\bigg(\frac{r_k^{n-1}}{\sigma_k^{n-1}}\int_{C_{\sigma_k/r_k}(x/r_k)}|\nabla w_k(t)|\rho\Big(\frac{t}{\sigma_k/r_k}-\frac{x}{\sigma_k}\Big)\,\mathrm{d}t\bigg)\,\mathrm{d}x\\&=\frac{1}{\sigma_k/r_k}\int_{B_1}h\bigg(\frac{\sigma_k}{r_k}\frac{r_k^n}{\sigma_k^n}\int_{C_{\sigma_k/r_k}(z)}|\nabla w_k(t)|\rho\Big(\frac{t-z}{\sigma_k/r_k}\Big)\,\mathrm{d}t\bigg)\,\mathrm{d}z\\&=\frac{1}{\sigma_k/r_k}\int_{B_1}h\bigg(\frac{\sigma_k}{r_k}|\nabla w_k|*\rho_{\sigma_k/r_k}(z)\bigg)\,\mathrm{d}z.\end{aligned}$$

Since $\sigma_k/r_k\to 0$, and $w_k\to u_0$ in $L^1(B_2)$, by the arbitrariness of $(r_k)$ and the definition of $\mathcal{E}'$, we conclude combining (5.3) with the arbitrariness of $\delta\in(0,1)$. □

Now we estimate from below $\mathcal{E}'(u^{\nu,a,b}_{x_0},B_1(x_0))$. Without loss of generality, we can assume $x_0=0$ and $\nu=\mathbf{e}_1$; we will denote, for the sake of simplicity, by $u^{a,b}$ the function $u_0^{\mathbf{e}_1,a,b}$. In order to estimate from below $\mathcal{E}'(u^{a,b},B_1)$ first we need to consider the problem on a suitable cylinder.

Recall that (see paragraph 3.1) $Q_{\mathbf{e}_1}=\{x\in\mathbb{R}^n:|x_1|<1,\,P^{\perp}_{\mathbf{e}_1}(x)\in B_1\cap\mathbf{e}_1^{\perp}\}$, being $P^{\perp}_{\mathbf{e}_1}(x)$ the orthogonal projection of $x$ onto the subspace $\mathbf{e}_1^{\perp}$; for simplicity of notation we will use $Q$ instead of $Q_{\mathbf{e}_1}$.

**Lemma 5.5.** *For any $A$ open subset of $Q$ there exist a positive infinitesimal sequence $(\varepsilon_j)$ and a sequence $u_j$ in $W^{1,1}(\Omega')$ converging to $u^{a,b}$ in $L^1(\Omega')$ such that*

$$\lim_{j\to+\infty}\mathcal{E}_{\varepsilon_j}(u_j,A)=\mathcal{E}'(u^{a,b},A)\tag{5.6}$$

*and such that*

$$u_j(x)=a,\quad\text{if } x_1\leq -a_j\quad\text{and}\quad u_j(x)=b,\quad\text{if } x_1\geq b_j\tag{5.7}$$

*for some positive infinitesimal sequences $(a_j)$ and $(b_j)$.*

*Proof.* We divide the proof in two steps.

*Step 1.* Fix $A\in\mathcal{A}(Q)$ with $A\subset\subset Q$, $\varepsilon,\sigma>0$ sufficiently small. Let $\varphi$ given by

$$\varphi(x)=\begin{cases}0 & x_1\leq-2\varepsilon-\sigma\\ \text{affine} & -2\varepsilon-\sigma<x_1<-2\varepsilon\\ 1 & x_1\geq-2\varepsilon.\end{cases}$$

Obviously we have $|\nabla\varphi|\leq\frac{1}{\sigma}$. Let

$$A_\varepsilon=\{x\in\mathbb{R}^n:x_1<-2\varepsilon-k_1\varepsilon-\sigma\},\quad B_\varepsilon=\{x\in\mathbb{R}^n:x_1>-2\varepsilon+\varepsilon k_2\}$$

$$S_\varepsilon = \{x \in \mathbb{R}^n : -2\varepsilon - \varepsilon k_1 - \sigma < x_1 < -2\varepsilon + \varepsilon k_2\}$$

where $k_1 = \sup_{x\in C}\langle x, \mathbf{e}_1\rangle$ and $k_2 = -\inf_{x\in C}\langle x, \mathbf{e}_1\rangle$. Let $u_1, u_2 \in W^{1,1}(\Omega')$ and $v = \varphi u_1 + (1-\varphi)u_2$. Then

$$\mathcal{E}_\varepsilon(v, A) = \frac{1}{\varepsilon}\int_{A\cap A_\varepsilon} h(\varepsilon|\nabla u_2| * \rho_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{A\cap B_\varepsilon} h(\varepsilon|\nabla u_1| * \rho_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon|\nabla v| * \rho_\varepsilon)\,\mathrm{d}x.$$

Taking into account the subadditivity of $h$ we get

$$\frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon|\nabla v| * \rho_\varepsilon)\,\mathrm{d}x \le \frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon(\varphi|\nabla u_1|) * \rho_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon((1-\varphi)|\nabla u_2|) * \rho_\varepsilon)\,\mathrm{d}x$$
$$+ \frac{1}{\varepsilon}\int_{A\cap S_\varepsilon} h(\varepsilon(|\nabla\varphi||u_1 - u_2|) * \rho_\varepsilon)\,\mathrm{d}x.$$

Then

$$\mathcal{E}_\varepsilon(v, A) \le \mathcal{E}_\varepsilon(u_1, A\cap(B_\varepsilon \cup S_\varepsilon)) + \mathcal{E}_\varepsilon(u_2, A\cap(A_\varepsilon\cup S_\varepsilon)) + \frac{c'}{\sigma}\int_{A\cap S_\varepsilon} |u_1 - u_2| * \rho_\varepsilon\,\mathrm{d}x$$

where we have used $h(t) \le c't$ for each $t \ge 0$.

*Step 2.* Now let $(\varepsilon_j)$ be a positive infinitesimal sequence and let $(v_j)$ be a sequence in $W^{1,1}(\Omega')$ such that $v_j \to u^{a,b}$ in $L^1(\Omega')$ and

$$\lim_{j\to+\infty} \mathcal{E}_{\varepsilon_j}(v_j, A) = \mathcal{E}'(u^{a,b}, A).$$

Choosing $u_1 = v_j$ and $u_2 = a$ we have, since $\mathcal{E}_{\varepsilon_j}(u_2, A) = 0$,

$$\mathcal{E}_{\varepsilon_j}(\varphi v_j + (1-\varphi)u_2, A) \le \mathcal{E}_{\varepsilon_j}(v_j, A) + \frac{c'}{\sigma}\int_{\{x_1<0\}} |v_j - u_2| * \rho_{\varepsilon_j}\,\mathrm{d}x.$$

By standard properties of the convolution,

$$\int_{\{x_1<0\}} |v_j - u_2| * \rho_{\varepsilon_j}\,\mathrm{d}x \le ||v_j - u_2||_{L^1(\{x_1<0\})} \to 0$$

as $j \to +\infty$. Therefore, by a diagonal argument, if $\sigma_h \to 0$ we can find $j_h \to +\infty$ be such that

$$\lim_{h\to+\infty} \frac{1}{\sigma_h}\int_{\{x_1<0\}} |v_{j_h} - u_2| * \rho_{\varepsilon_{j_h}}\,\mathrm{d}x = 0.$$

Thus

$$\limsup_{h\to+\infty} \mathcal{E}_{\varepsilon_{j_j}}(\varphi v_{j_h} + (1-\varphi)u_2, A) \le \limsup_{h\to+\infty} \mathcal{E}_{\varepsilon_{j_h}}(v_{j_h}, A) = \mathcal{E}'(u^{a,b}, A).$$

Setting

$$u_{j_h} = \begin{cases} a & x_1 \le -2\varepsilon_{j_h} - \sigma_h \\ v_{j_h} & x_1 \ge 0 \end{cases}$$

we easily have $u_{j_h} \to u^{a,b}$ in $L^1(\Omega')$ and $u_{j_h} = a$ for $x_1 \le -a_j$ for a suitable positive infinitesimal sequence $(a_j)$. With the same argument one can prove that $u_{j_h} = b$ for $x_1 \ge b_j$ for another suitable positive infinitesimal sequence $(b_j)$. Thus $(u_{j_h})$ is optimal and (5.7) hold. □

**Proposition 5.6.** *We have* $\mathcal{E}'(u^{a,b}, B_1) \ge \mathcal{E}'(u^{a,b}, Q)$.

*Proof.* Fix $\delta \in (0,1)$. Let $(u_j)$ be given by the previous Lemma, applied with $A = B_1$. Then $u_j(x) = a$ if $x_1 \le -a_j$, and $u_j(x) = b$ if $x_1 \ge b_j$, where $(a_j)$ and $(b_j)$ are suitable positive infinitesimal sequences. Let $S_j = (-a_j, b_j)\times\mathbb{R}^{n-1}$. For $j$ sufficiently large, we have $\delta Q \cap S_j \subset\subset B_1$, from which $\mathcal{E}_{\varepsilon_j}\big(u_j, \delta Q\cap B_1\big) = \mathcal{E}_{\varepsilon_j}\big(u_j, \delta Q\big)$. Then

$$\mathcal{E}_{\varepsilon_j}(u_j, B_1) \ge \mathcal{E}_{\varepsilon_j}(u_j, B_1\cap\delta Q) = \mathcal{E}_{\varepsilon_j}(u_j, \delta Q)\,. \tag{5.8}$$

Let $v_j(x) = u_j(\delta x)$. Then by a simple scaling argument we have $\mathcal{E}_{\varepsilon_j}(u_j, \delta Q) = \delta^{n-1}\mathcal{E}_{\varepsilon_j/\delta}(v_j, Q)$. Passing to the limit in (5.8) we get

$$\mathcal{E}'(u^{a,b}, B_1) \ge \delta^{n-1}\liminf_{j\to+\infty} \mathcal{E}_{\varepsilon_j/\delta}(v_j, Q) \ge \delta^{n-1}\mathcal{E}'(u^{a,b}, Q).$$

We conclude by taking the limit as $\delta \to 1^-$ . □

Now, by an application of the Besicovitch's Differentiation Theorem, we are able to prove the correct estimate from below for the lower Γ-limit of $\mathcal{E}_{\varepsilon_j}$. In order to apply such a Theorem, let us consider the set function $\mathcal{E}'_-(u,\cdot)$. It is well known that an increasing set function $\alpha\colon \mathcal{A}(\Omega) \to [0,+\infty]$ which satisfies $\alpha(\emptyset)=0$, which is subadditive, superadditive and inner regular, can be extended to a Borel measure on $\Omega$ (for instance see [18], Th. 14.23). This result can be applied to $\mathcal{E}'_-(u,\cdot)$, the subadditivity of $\mathcal{E}'_-(u,\cdot)$ being the only condition which is not easy to prove, but it can be recovered as in the proof of Prop. 4.3 and Th. 4.6 of [13]; these results are established in the case $p>1$, but the same arguments work if $p=1$.

Denote by $\mu_u$ the Borel measure on $\Omega$ which extends $\mathcal{E}'_-(u,\cdot)$.

**Lemma 5.7.** *Let $u \in BV(\Omega)$. Then $\mu_u$ is a finite measure.*

*Proof.* Let $(u_h)$ be a sequence in $L^1(\Omega)$ converging weakly* converging to $u$ in $BV(\Omega)$. By definition

$$|Du_h| * \rho_\varepsilon(x) = \int_{C_\varepsilon(x)\cap\Omega} \rho_\varepsilon(x-y)\,\mathrm{d}|Du_h|(y).$$

Since $Du_h \overset{*}{\rightharpoonup} Du$ as measures, by Fatou's Lemma and taking into account that $f$ is non-decreasing and continuous, we get

$$\liminf_{h\to+\infty} \frac{1}{\varepsilon}\int_\Omega h\big(\varepsilon|Du_h| * \rho_\varepsilon\big)\,\mathrm{d}x \geq \frac{1}{\varepsilon}\int_\Omega h\big(\varepsilon \liminf_{h\to+\infty} |Du_h| * \rho_\varepsilon\big)\,\mathrm{d}x \geq \frac{1}{\varepsilon}\int_\Omega h(\varepsilon|Du| * \rho_\varepsilon)\,\mathrm{d}x. \tag{5.9}$$

Now let $u \in BV(\Omega)$ and let $(u_h)$ be a sequence in $L^1(\Omega)$ strictly converging to $u$. In particular, $|Du_h| \to |Du|$ weakly* as measures (see, for instance, Prop. 3.15 in [3]). Note that that $D^c u$ vanishes on the sets with finite $\mathcal{H}^{n-1}$ measure. Moreover, if $S$ is $\sigma$-finite with respect to $\mathcal{H}^{n-1}$, then $\{x\in\Omega : \mathcal{H}^{n-1}(S\cap\partial C_\varepsilon(x))>0\}$ is at most countable. Then (see, for instance, Prop. 1.62 in [3]), we have

$$\lim_{h\to+\infty} |Du_h| * \rho_\varepsilon(x) = |Du| * \rho_\varepsilon(x), \quad \text{a.e. } x\in\Omega.$$

Applying the Dominated Convergence Theorem, we obtain

$$\lim_{h\to+\infty} \frac{1}{\varepsilon}\int_\Omega h\left(\varepsilon|Du_h| * \rho_\varepsilon\right)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_\Omega h\left(\varepsilon|Du| * \rho_\varepsilon\right)\,\mathrm{d}x. \tag{5.10}$$

Combining (5.9) with (5.10) and taking into account that $\mathcal{E}'_-$ is lower semicontinuous, we have

$$\mathcal{E}'_-(u) \leq \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega h\left(\varepsilon|Du| * \rho_\varepsilon\right)\,\mathrm{d}x.$$

Notice that there exists $\gamma > 0$ such that $|C_\varepsilon(x)\cap\Omega| \leq \gamma\varepsilon^n$ for any $x\in\Omega$. Denoting by $M = \sup_C \rho$ and taking into Fubini's Theorem, we get that for sufficiently small $\varepsilon$,

$$\begin{aligned}\int_\Omega h\left(\varepsilon|Du| * \rho_\varepsilon\right)\,\mathrm{d}x &\leq c'\int_\Omega\int_{C_\varepsilon(x)\cap\Omega} \rho_\varepsilon(y-x)\,\mathrm{d}|Du|(y)\,\mathrm{d}x = c'\int_\Omega\int_\Omega \rho_\varepsilon(y-x)\chi_{C_\varepsilon(x)}\,\mathrm{d}x\,\mathrm{d}|Du|(y)\\ &\leq c'M\int_\Omega\int_\Omega \frac{|C_\varepsilon(x)\cap\Omega|}{\varepsilon^n}\,\mathrm{d}|Du|(y) \leq c'M\gamma|Du|(\Omega)\end{aligned}$$

and this yields the conclusion. □

**Proposition 5.8.** *Let $u\in BV(\Omega)$ and $A\in\mathcal{A}(\Omega)$. Then*

$$\mathcal{E}'(u,A) \geq \int_{S_u\cap A} \psi(|u^+-u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

*where*

$$\psi(s,\nu) = \frac{1}{\omega_{n-1}}\inf\left\{\liminf_{j\to+\infty}\frac{1}{\varepsilon_j}\int_{Q_\nu} h(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j)\in W^{0,s}_\nu,\ \varepsilon_j\to 0^+\right\}.$$

*Proof.* For every $k \in \mathbb{N}$ let $S_k = \{x \in S_u : \; |u^+(x) - u^-(x)| > 1/k\}$. Clearly we have $\mathcal{H}^{n-1}(S_k) < +\infty$; let $\lambda_k = \mathcal{H}^{n-1} \llcorner S_k$. Applying the Besicovitch's Differentiation Theorem we deduce that the limit

$$g(x) = \lim_{\varrho \to 0} \frac{\mu_u(B_\varrho(x))}{\lambda_k(B_\varrho(x))}$$

exists and is finite for $\lambda_k$-a.e. $x \in \Omega$, and is $\lambda_k$-measurable. Moreover, the Radon-Nikodym decomposition of $\mu_u$ is given by $\mu_u = g\lambda_k + \mu^s$, with $\mu^s \perp \lambda_k$. By rectifiability for $\mathcal{H}^{n-1}$-a.e. $x \in S_k$ we get

$$\lim_{\varrho \to 0} \frac{\lambda_k(B_\varrho(x))}{\omega_{n-1}\varrho^{n-1}} = 1.$$

Thus, for $\mathcal{H}^{n-1}$-a.e. $x_0 \in S_k$ we have, applying Proposition 5.4, Proposition 5.6 and taking into account (5.7),

$$g(x_0) = \lim_{\varrho \to 0} \frac{\mu_u(B_\varrho(x_0))}{\omega_{n-1}\varrho^{n-1}} = \liminf_{\varrho \to 0} \frac{\mathcal{E}'_-(u, B_\varrho(x_0))}{\omega_{n-1}\varrho^{n-1}}$$
$$\geq \frac{1}{\omega_{n-1}} \inf \left\{ \liminf_{j \to +\infty} \frac{1}{\varepsilon_j} \int_{x_0 + Q_\nu} h(\varepsilon_j |\nabla u_j| * \rho_{\varepsilon_j}) \, \mathrm{d}x : (u_j(\cdot - x_0)) \in W^{u^+(x_0), u^-(x_0)}_{\nu_u(x_0)}, \; \varepsilon_j \to 0^+ \right\}.$$

Taking into account (3.8) and (3.9) (which obviously hold for $h$ instead of $f$) we get

$$\inf \left\{ \liminf_{j \to +\infty} \frac{1}{\varepsilon_j} \int_{x_0 + Q_\nu} h(\varepsilon_j \; |\nabla u_j| * \rho_{\varepsilon_j}) \, \mathrm{d}x : (u_j(\cdot - x_0)) \in W^{u^+(x_0), u^-(x_0)}_{\nu_u(x_0)}, \; \varepsilon_j \to 0^+ \right\}$$
$$= \psi(|u^+(x_0) - u^-(x_0)|, \nu_u(x_0)).$$

Since $\mu^s$ is non-negative, we deduce that

$$\mathcal{E}'_-(u, A) \geq \int_A \psi(|u^+ - u^-|, \nu_u) \, \mathrm{d}\lambda_k = \int_{S_k \cap A} \psi(|u^+ - u^-|, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1}.$$

By considering the supremum for $k \in \mathbb{N}$ we easily obtain

$$\mathcal{E}'_-(u, A) \geq \int_{S_u \cap A} \psi(|u^+ - u^-|, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1}$$

and the conclusion follows by definition of $\mathcal{E}'_-$. □

5.3. **Proof of the Γ-liminf inequality.** We are ready to prove the Γ-liminf inequality for the family $(\mathcal{F}_\varepsilon)_{\varepsilon > 0}$. The main step of the proof consists in combining Proposition 5.3 with Proposition 5.8 and then using a supremum of measures argument.

**Lemma 5.9.** *Let $\mu$ be as in Lemma 2.5. Let $\lambda_1, \lambda_2$ be mutually singular Borel measures, and $\psi_1, \psi_2$ positive Borel functions. Assume that*

$$\mu(A) \geq \int_A \psi_i \, \mathrm{d}\lambda_i$$

*for every $A \in \mathcal{A}(\Omega)$ and $i = 1, 2$. Then it holds*

$$\mu(A) \geq \int_A \psi_1 \, \mathrm{d}\lambda_1 + \int_A \psi_2 \, \mathrm{d}\lambda_2$$

*for every $A \in \mathcal{A}(\Omega)$.*

*Proof.* Let $E \subseteq \Omega$ be such that $\lambda_1(\Omega \setminus E) = 0$ and $\lambda_2(E) = 0$. Then we can suppose that $\psi_1 = 0$ on $\Omega \setminus E$ and $\psi_2 = 0$ on $E$. Then $\max\{\psi_1, \psi_2\} = \psi_1 + \psi_2$. We conclude by applying the Lemma 2.5 with the choice $\lambda = \lambda_1 + \lambda_2$. □

**Proposition 5.10.** *Let $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Then*

$$\mathcal{F}'(u, A) \geq \int_A \phi(|\nabla u|) \, \mathrm{d}x + \int_{S_u \cap A} \theta(|u^+ - u^-|, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1} + c_0 |D^c u|(A).$$

*Proof.* First notice that we can suppose $u \in GBV(\Omega)$. Indeed, if $(\mathcal{F}_{\varepsilon_j}(u_j))$ is bounded and $u_j \to u$ in $L^1(\Omega)$ then $u \in GBV(\Omega)$: it suffices to apply Theorem 3.6 to $u_j^T = -T \vee u_j \wedge T$, hence we get $u^T \in BV(\Omega)$ which means $u \in GBV(\Omega)$.

Now the key point of the proof is the construction of a suitable family of functions below $f_{\varepsilon_j}$.

*Step 1.* Let $\delta \in (0,1)$. We claim that there exists $t_\delta > 0$ and for any $h \in \mathbb{N}$ and for any $\varepsilon > 0$ there exist $c_h^\delta > 0$, $d_h^\delta < 0$ and $g_h^\delta \colon [t_\delta, +\infty) \to \mathbb{R}$ such that if we let

$$f_\varepsilon^{h,\delta}(t) = \begin{cases} c_h^\delta t + \varepsilon d_h^\delta & \text{if } t \in [0, t_\delta] \\ \\ c_h^\delta t_\delta + \varepsilon d_h^\delta + g_h^\delta(t) & \text{if } t > t_\delta \end{cases}$$

we have:

$$\sup_h (c_h^\delta t + d_h^\delta) = (1-\delta)\phi(t), \quad \forall t \geq 0. \tag{5.11}$$

$$f_\varepsilon(t) \geq f_\varepsilon^{h,\delta}(t),\ \forall t \geq 0,\ \forall h \in \mathbb{N}, \text{ for } \varepsilon \text{ sufficiently small.} \tag{5.12}$$

$$f_\varepsilon^{h,\delta} \text{ is continuous, non-decreasing and concave for any } \varepsilon > 0 \text{ and any } h \in \mathbb{N}. \tag{5.13}$$

$$f_\varepsilon^{h,\delta} - \varepsilon d_h^\delta \text{ converges to } (1-\delta)f \text{ uniformly on compact sets of } [0,+\infty) \text{ as } h \to +\infty. \tag{5.14}$$

First of all we point out that

$$\lim_{t \to 0} \frac{f(t)}{t} = c_0. \tag{5.15}$$

Indeed, by A2 for any $\sigma \in (0,1)$ there exist $t_\sigma, \varepsilon_\sigma > 0$ such that $f_\varepsilon(t) \leq (1+\sigma)\varepsilon\phi(t/\varepsilon)$ for each $t \in [0, t_\sigma]$ and for each $\varepsilon \in (0, \varepsilon_\sigma]$. Since $\phi(s) \leq c_0 s$ for any $s \geq 0$, we have $f_\varepsilon(t)/t \leq (1+\sigma)c_0$. By A3 the previous inequality reduces to $f(t)/t \leq (1+\sigma)c_0$. On the other hand there exist $t'_\sigma, \varepsilon'_\sigma > 0$ such that $f_\varepsilon(t) \geq (1-\sigma)\varepsilon\phi(t/\varepsilon)$ for each $t \in [0, t'_\sigma]$ and for each $\varepsilon \in (0, \varepsilon'_\sigma]$. Since $\phi(s) \geq c_0 s - q$, for a suitable $q > 0$, we have $f_\varepsilon(t)/t \geq (1-\sigma)(c_0 t - \varepsilon q)$. We thus get $f(t)/t \geq (1-\sigma)c_0$. By the arbitrariness of $\sigma > 0$ we have (5.15).

Formula (5.15) is useful in order to construct the family $(f_\varepsilon^{h,\delta})$ as follows. By A2 there exists $t_\delta > 0$ such that $f_\varepsilon(t) \geq (1-\delta)\varepsilon\phi(t/\varepsilon)$ for each $t \in [0, t_\delta]$ and for each $\varepsilon$ sufficiently small. Fix $h \in \mathbb{N}$ with $h > 0$ and let $(\ell_h)_{h \in \mathbb{N}}$ be a family of affine functions such that $\sup_h \ell_h(t) = \phi(t)$ for any $t \geq 0$ (recall that $\phi$ is convex); we let $\ell_h(t) = c_h t + d_h$. Let $c_h^\delta = (1-\delta)c_h$ and $d_h^\delta = (1-\delta)d_h$. Then (5.11) holds and we obtain $f_\varepsilon(t) \geq c_h^\delta t + \varepsilon d_h^\delta$ for all $t \in [0, t_\delta]$. Now it is easy to conclude the construction of $f_\varepsilon^{h,\delta}$ in such a way (5.12), (5.13) and (5.14) hold: for instance connecting the graphic of the affine piece with a suitable rotation and truncation of the graph of $f$ (see also (5.15)).

*Step 2.* Let $\delta \in (0,1)$ and let $(f_{\varepsilon_j}^{h,\delta})$ be the family constructed in step 1. Let $\psi_h^\delta = f_{\varepsilon_j}^{h,\delta} - \varepsilon_j d_h^\delta$. Then we get

$$\mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_A \psi_h^\delta \left(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)\right) \mathrm{d}x + d_h^\delta |A| \tag{5.16}$$

for any $u \in W^{1,1}(\Omega)$ and $A \in \mathcal{A}(\Omega)$. Let $A', A''$ be open disjoint subsets of $A$ such that $|A''| < \delta$, $S_u \subset A''$. Therefore,

$$\mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_{A'} \psi_h^\delta \left(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)\right) \mathrm{d}x + \frac{1}{\varepsilon_j} \int_{A''} \psi_h^\delta \left(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)\right) \mathrm{d}x + d_h^\delta |A'| + \delta d_h^\delta. \tag{5.17}$$

In particular we get

$$\mathcal{F}_{\varepsilon_j}(u, A) \geq \frac{1}{\varepsilon_j} \int_{A'} \psi_h^\delta \left(\varepsilon_j |\nabla u| * \rho_{\varepsilon_j}(x)\right) \mathrm{d}x + d_h^\delta |A'|$$

Notice that $\psi_h^\delta$ is linear in $[0, t_\delta]$. Applying Proposition 5.3 with the choice $g = \psi_h^\delta \wedge \psi_h^\delta(t_\delta)$ we obtain

$$\mathcal{F}'(u, A) \geq c_h^\delta \int_{A'} |\nabla u| \,\mathrm{d}x + c_h^\delta |D^c u|(A) + d_h^\delta |A'| = (1-\delta) \int_{A'} \ell_h(|\nabla u|) \,\mathrm{d}x + (1-\delta)c_h |D^c u|(A').$$

Since $\mathcal{F}'(u,\cdot)$ is a superadditive function on open sets of $\Omega$ with disjoint compact closures, by applying Lemma 2.5 and (5.11) we get, by the arbitrariness of $A'$ and $\delta$,

$$\mathcal{F}'(u,A) \geq \int_A \phi(|\nabla u|)\mathrm{d}x + c_0|D^c u|(A). \tag{5.18}$$

Now (5.17) implies also

$$\mathcal{F}_{\varepsilon_j}(u,A) \geq \frac{1}{\varepsilon_j}\int_{A''} \psi_h^\delta\left(\varepsilon_j|\nabla u| * \rho_{\varepsilon_j}(x)\right)\,\mathrm{d}x.$$

Applying now Proposition 5.8 with the choice $h = \psi_h^\delta$ we deduce that

$$\mathcal{F}'(u,A) \geq \int_{S_u\cap A''} \theta_h^\delta(|u^+ - u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

being

$$\theta_h^\delta(s,\nu) = \frac{1}{\omega_{n-1}}\inf\left\{\liminf_{j\to+\infty}\frac{1}{\varepsilon_j}\int_{Q_\nu}\psi_h^\delta(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x : (u_j)\in W_\nu^{0,s},\ \varepsilon_j\to 0^+\right\}.$$

Using (5.14) and the arbitrariness of $\delta$, it follows that $\theta_h^\delta \to \theta$ as $h\to+\infty$ and $\delta\to 0$. Applying once more Lemma 2.5, by the arbitrariness of $A''$, we have

$$\mathcal{F}'(u,A) \geq \int_{S_u\cap A} \theta(|u^+ - u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{5.19}$$

Applying Lemma 5.9 choosing $\lambda_1 = \mathcal{L}^n$, $\lambda_2 = \mathcal{H}^{n-1} \llcorner J_u$, $\lambda_3 = |D^c u|$ and taking into account (5.18) and (5.19), we immediately obtain $\mathcal{F}'(u) \geq \mathcal{F}(u)$ for any $u \in BV(\Omega)$.

Let us now consider the case $u \in GBV(\Omega)$. Let $(u_j)$ be a sequence in $W^{1,1}(\Omega)$ converging to $u$ in $L^1(\Omega)$ and such that

$$\lim_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j) = \mathcal{F}'(u).$$

Define $u_j^T = (-T)\vee u_j \wedge T$, and $u^T = (-T)\vee u\wedge T$. Since $u_j^T \to u^T$ in $L^1(\Omega)$, and $u^T \in BV(\Omega)$, we have

$$\mathcal{F}'(u) = \liminf_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j) \geq \liminf_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j^T) \geq \mathcal{F}(u^T).$$

Applying (2.2), (2.3) and (2.4) and taking into account the continuity of $\theta$ we obtain

$$\lim_{T\to+\infty}\left(\int_\Omega \phi(|\nabla u^T|)\,\mathrm{d}x + \int_{S_{u^T}}\theta\big(|(u^T)^+ - (u^T)^-|,\nu_{u^T}\big)\,\mathrm{d}\mathcal{H}^{n-1} + c_0|D^c u^T|(\Omega)\right) = \mathcal{F}(u)$$

so we are done. □

## 6. The Γ-limsup inequality

In this section we will prove that $\mathcal{F}''(u) \leq \mathcal{F}(u)$ for any $u \in L^1(\Omega)$; since, by definition, $\mathcal{F}(u) = +\infty$ for any $u \in L^1(\Omega)\setminus GBV(\Omega)$, it is sufficient to consider the case $u \in GBV(\Omega)$.

**Lemma 6.1.** *Let $(\varepsilon_j)$ be a positive infinitesimal sequence, $\nu\in\mathbb{S}^{n-1}$ and $s\geq 0$. Let $(u_j)\in W_\nu^{0,s}$ be such that*

$$\omega_{n-1}\theta(s,\nu) = \lim_{j\to+\infty}\frac{1}{\varepsilon_j}\int_{Q_\nu} f(\varepsilon_j|\nabla u_j| * \rho_{\varepsilon_j})\,\mathrm{d}x.$$

*Then for any $r>0$ there exists a positive infinitesimal sequence $\sigma_j$ and $(v_j)\in W_\nu^{0,s}$ such that for any $\sigma>0$ it holds*

$$\omega_{n-1}r^{n-1}\theta(s,\nu) = \lim_{j\to+\infty}\frac{1}{\sigma_j}\int_{rQ_\nu^\sigma} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x,$$

*where $Q_\nu^\sigma = \{x\in Q_\nu : |\langle x,\nu\rangle| < \sigma\}$.*

*Proof.* Let $\sigma_j = r\varepsilon_j$ and $v_j(x) = u_j(rx)$. Then by the change of variables $x = rz$ and $y = rt$ we get

$$\frac{1}{\sigma_j}\int_{rQ_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x = \frac{r^n}{\sigma_j}\int_{Q_\nu} f\biggl(\frac{\sigma_j}{r}\int_{C_{\sigma_j/r}} |\nabla v_j(rz-rt)|\rho_{\sigma_j/r}(t)\,\mathrm{d}t\biggr)\,\mathrm{d}z$$
$$= \frac{r^{n-1}}{\varepsilon_j}\int_{Q_\nu} f\biggl(\varepsilon_j\int_{C_{\varepsilon_j}} |\nabla u_j(z-t)|\rho_{\varepsilon_j}(t)\,\mathrm{d}t\biggr)\,\mathrm{d}z.$$

Passing to the limit as $j \to +\infty$ we get

$$\lim_{j\to+\infty}\frac{1}{\sigma_j}\int_{rQ_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x = r^{n-1}\theta(s,\nu).$$

Since the transition set of the optimal sequence $(u_j)$ shrinks onto the interface (see (5.7) or the definition of $W^{0,s}_\nu$) we deduce that

$$\lim_{j\to+\infty}\frac{1}{\sigma_j}\int_{rQ_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x = \lim_{j\to+\infty}\frac{1}{\sigma_j}\int_{rQ^\sigma_\nu} f(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x$$

for any $\sigma > 0$, hence we conclude. □

**Proposition 6.2.** *For any $u \in \mathcal{W}(\Omega)$ it holds $\mathcal{F}''(u) \le \mathcal{F}(u)$.*

*Proof.* By the very definition of $\mathcal{W}(\Omega)$ (see paragraph 2.5) the set $S_u$ is contained in the union of a finite collection $K_1, \dots, K_m$ of $(n-1)$-dimensional simplexes; it will not be restrictive to assume $m = 1$ and $K = K_1 \subseteq \{x \in \mathbb{R}^n : x_1 = 0\}$. Fix $h \in \mathbb{N}$, $h \ge 1$. Let $\Omega_h = \{x \in \Omega \setminus K : d(x,K) > 1/h\}$. Let $S$ be the relative boundary of $K$; obviously it holds $\mathcal{H}^{n-1}(S) = 0$. Let $K_h = \{x \in K : d(x,S) > 1/h\}$. Let $k \in \mathbb{N}$, $k \ge 1$, $x_1, \dots, x_k \in K_h$ and $r \ge 0$ be such that $B_r(x_i)$ are pairwise disjoint, $B_r(x_i) \cap \{x_1 = 0\} \subseteq K_h$ for any $i = 1, \dots, k$ and

$$\mathcal{H}^{n-1}\biggl(K_h \setminus \biggl(\bigcup_{i=1}^k B_r(x_i) \cap \{x_1 = 0\}\biggr)\biggr) < \frac{1}{h}. \tag{6.1}$$

Let $Q_h = \{x \in rQ_{\mathbf{e}_1} : |x_1| < 1/h\}$ and $Q_h(x) = x + Q_h$ for any $x \in \mathbb{R}^n$. Moreover, let $Q_h^+ = Q_h \cap \{x_1 > 0\}$ and $Q_h^- = Q_h \cap \{x_1 < 0\}$. At this point we divide the proof in two steps.

*Step 1.* Take a function $v \in \mathcal{W}(\Omega)$ with $S_v \subseteq K$ and such that $v$ is constant in any $x_i + Q_h^+$ and in any $x_i + Q_h^-$. Denote by $v_i^+$ the value of $v$ in $x_i + Q_h^+$ and by $v_i^-$ the value of $v$ in $x_i + Q_h^-$. We claim that

$$\mathcal{F}''(v) \le \int_\Omega \phi(|\nabla v|)\,\mathrm{d}x + \sum_{i=1}^k \int_{K\cap B_r(x_i)} \theta(|v_i^+ - v_i^-|, \mathbf{e}_1)\,\mathrm{d}\mathcal{H}^{n-1} + c|Dv|(\Omega'_h), \tag{6.2}$$

for some $c > 0$, where

$$\Omega'_h = \Omega \setminus \biggl(\Omega_h \cup \bigcup_{i=1}^k (x_i + Q_h)\biggr).$$

Let $(\varepsilon_j)$ be a positive infinitesimal sequence and let $\delta \in (0,1)$. Accordingly to Lemma 6.1, let us define $v_j \in \mathcal{W}(\Omega)$ be such that we have

$$\lim_{j\to+\infty} \mathcal{F}_{\sigma_j}(v_j, x_i + \delta Q_h) = (\delta r)^{n-1}\theta(|v_i^+ - v_i^-|, \mathbf{e}_1), \tag{6.3}$$

where $\sigma_j = r\varepsilon_j$. Otherwise in $\Omega$ we set $v_j = v$. Then, using the same argument as in the proof of Lemma 5.7, we deduce that

$$\frac{1}{\sigma_j}\int_\Omega f_{\sigma_j}(\sigma_j|\nabla v_j| * \rho_{\sigma_j})\,\mathrm{d}x \le \mathcal{F}_{\sigma_j}(v, \Omega_h) + \sum_{i=1}^k \mathcal{F}_{\sigma_j}(v_j, x_i + \delta Q_h) + c|Dv|(\Omega'_{h,\delta}), \tag{6.4}$$

being

$$\Omega'_{h,\delta} = \Omega \setminus \biggl(\Omega_h \cup \bigcup_{i=1}^k (x_i + \delta Q_h)\biggr).$$

The first term on the right-hand side of (6.4) is given by

$$\frac{1}{\sigma_j}\int_{\Omega_h} f_{\sigma_j}(\sigma_j|\nabla v|*\rho_{\sigma_j})\,\mathrm{d}x.$$

By standard properties of the convolution we have $|\nabla v|*\rho_{\sigma_j}\to|\nabla v|$ in $L^1(\Omega)$ and a.e. in $\Omega$. From A2 we deduce that

$$\lim_{\varepsilon\to 0}\frac{f_\varepsilon(\varepsilon t_\varepsilon)}{\varepsilon}=\phi(t) \tag{6.5}$$

whenever $t_\varepsilon\to t$, for each $t\geq 0$. By the Dominated Convergence Theorem we get

$$\lim_{j\to+\infty}\frac{1}{\sigma_j}\int_{\Omega_h} f_{\sigma_j}(\sigma_j|\nabla v|*\rho_{\sigma_j})\,\mathrm{d}x=\int_{\Omega_h}\phi(|\nabla v|)\,\mathrm{d}x\leq\int_{\Omega}\phi(|\nabla v|)\,\mathrm{d}x.$$

Passing to the limsup in (6.4), using (6.3) and using the arbitrariness of $\delta\in(0,1)$ we get (6.2).

*Step 2.* For any $i=1,\dots,k$ let

$$u_i^+=⨍_{B_r(x_i)\cap K}u^+\,\mathrm{d}\mathcal{H}^{n-1},\quad u_i^-=⨍_{B_r(x_i)\cap K}u^-\,\mathrm{d}\mathcal{H}^{n-1}$$

and

$$u_i(x)=\begin{cases}u_i^+ & \text{if } (x_i)_1-x_1>0\\ u_i^- & \text{if } (x_i)_1-x_1\leq 0\end{cases},\quad x\in B_r(x_i).$$

For any $h\in\mathbb{N}$, $h\geq 1$, let $u_h=u_i$ on $Q_h(x_i)$ and $u_h=u$ otherwise in $\Omega$. Applying step 1 with the choice $v=u_h$ we get

$$\mathcal{F}''(u_h)\leq\int_\Omega\phi(|\nabla u|)\,\mathrm{d}x+\sum_{i=1}^k\int_{K\cap B_r(x_i)}\theta(|u_i^+-u_i^-|,\mathbf{e}_1)\,\mathrm{d}\mathcal{H}^{n-1}+c|Du|(\Omega_h').$$

Now $|\Omega_h'|\to 0$. Furthermore, taking into account (6.1) we deduce that $\mathcal{H}^{n-1}(S_u\cap\Omega_h')\to 0$ as $h,k\to+\infty$. Hence $|Du|(\Omega_h')\to 0$ as $h,k\to+\infty$. Exploiting the uniform continuity of the traces of $u$ and the continuity of $\theta$, we also get

$$\sum_{i=1}^k\int_{K\cap B_r(x_i)}\theta(|u_i^+-u_i^-|,\mathbf{e}_1)\,\mathrm{d}\mathcal{H}^{n-1}\overset{h,k\to+\infty}{\longrightarrow}\int_{S_u}\theta(|u^+-u^-|,\mathbf{e}_1)\,\mathrm{d}\mathcal{H}^{n-1}$$

and the lower semicontinuity of $\mathcal{F}''$ yields the conclusion. □

**Proposition 6.3.** *Let $u\in GBV(\Omega)$. Then it holds $\mathcal{F}''(u)\leq\mathcal{F}(u)$.*

*Proof.* First let $u\in SBV^2(\Omega)\cap L^\infty(\Omega)$. We can apply Theorem 2.6, choosing

$$\psi(a,b,\nu)=\theta(|a-b|,\nu)$$

(see (3.6) and (3.7)). Then there exists a sequence $w_j\to u$ in $L^1(\Omega)$, with $w_j\in\mathcal{W}(\Omega)$, such that $\nabla w_j\to\nabla u$ strongly in $L^2(\Omega,\mathbb{R}^n)$ and

$$\limsup_{j\to+\infty}\int_{S_{w_j}}\theta(|w_j^+-w_j^-|,\nu_{w_j})\,\mathrm{d}\mathcal{H}^{n-1}\leq\int_{S_u}\theta(|u^+-u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{6.6}$$

By the lower semicontinuity of $\mathcal{F}''$ and by Proposition 6.2 we deduce that, applying the Dominated Convergence Theorem and (6.6),

$$\mathcal{F}''(u)\leq\liminf_{j\to+\infty}\mathcal{F}''(w_j)\leq\int_\Omega\phi(|\nabla u|)\,\mathrm{d}x+\int_{S_u}\theta(|u^+-u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}.$$

Using relaxation Theorem 2.7 we get

$$\mathcal{F}''(u)\leq\int_\Omega\phi(|\nabla u|)\,\mathrm{d}x+\int_{J_u}\theta(|u^+-u^-|,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+c_0|D^cu|(\Omega)$$

for each $u \in BV(\Omega)$. Finally, let $u \in GBV(\Omega)$ and, for any $T > 0$, $u^T = -T \vee u \wedge T$. Then $u^T \in BV(\Omega)$ for each $T > 0$ and $u^T \to u$ in $L^1(\Omega)$ as $T \to +\infty$. Taking into account (2.2), (2.3) and (2.4) we obtain, exploiting again the lower semicontinuity of $\mathcal{F}''$ and the continuity of $\theta$,

$$\begin{aligned}\mathcal{F}''(u) &\le \limsup_{T\to+\infty}\left(\int_\Omega \phi(|\nabla u^T|)\,\mathrm{d}x + \int_{S_{u^T}} \theta(|(u^T)^+ - (u^T)^-|, \nu_{u^T})\,\mathrm{d}\mathcal{H}^{n-1} + c_0|D^c u^T|(\Omega)\right)\\ &= \int_\Omega \phi(|\nabla u|)\,\mathrm{d}x + \int_{S_u} \theta(|u^+ - u^-|, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + c_0|D^c u|(\Omega)\end{aligned}$$

which is what we wanted to prove. □

## 7. Computation of $\theta$ in the one-dimensional case

In this section we are able to give an explicit formula for $\theta$ if $n = 1$ along the same line of the discretization argument used in [22].

Let $n = 1$, then we can set $\Omega = (a, b)$, $C = I$ to be an open interval around 0, $\rho\colon I \to (0, +\infty)$ continuous and bounded with

$$\int_I \rho\,\mathrm{d}t = 1.$$

For any $\varepsilon > 0$ let $\rho_\varepsilon(t) = 1/\varepsilon\rho(t/\varepsilon)$ and $I_\varepsilon(x) = x + \varepsilon I$.

**Theorem 7.1.** *It holds*

$$\theta(s) = \int_{-\infty}^{+\infty} f(s\rho(t))\,\mathrm{d}t.$$

*Proof.* In the one-dimensional setting the expression for $\theta$ given by (3.5) reads

$$\theta(s) = \inf\left\{\liminf_{j\to+\infty}\frac{1}{\varepsilon_j}\int_{-1}^{1} f(\varepsilon_j|u_j'| * \rho_{\varepsilon_j})\,\mathrm{d}t : (u_j) \in W^{0,s},\ \varepsilon_j \to 0^+\right\},$$

being $W^{0,s}$ the space of all sequences $(u_j)$ in $W^{1,1}_{\mathrm{loc}}(\Omega')$, $(-1, 1) \subset \Omega'$, such that $u_j \to s\chi_{(0,+\infty)}$ in $L^1(\Omega')$, and such that there exist two positive infinitesimal sequences $(a_j), (b_j)$ with $u_j(t) = 0$ if $t < -a_j$ and $u_j = s$ if $t > b_j$. Let $(u_j) \in W^{0,s}$ and

$$v_j(t) = \int_{-1}^{t} (u_j'(r))^+\,\mathrm{d}r.$$

Moreover, let $w_j = 0 \vee v_j \wedge s$. Then $(w_j) \in W^{0,s}$ and by the change of variables $y = \varepsilon_j z$ and $t = \varepsilon_j r$ we get

$$\begin{aligned}\frac{1}{\varepsilon_j}\int_{-1}^{1} f(\varepsilon_j|u_j'| * \rho_{\varepsilon_j})\,\mathrm{d}t &\ge \frac{1}{\varepsilon_j}\int_{-1}^{1} f\left(\int_{I_{\varepsilon_j}} w_j'(t+y)\rho\left(\frac{y}{\varepsilon_j}\right)\right)\mathrm{d}t\\ &= \frac{1}{\varepsilon_j}\int_{-1}^{1} f\left(\varepsilon_j\int_I w_j'(t+\varepsilon_j z)\rho(z)\right)\mathrm{d}t = \int_{-1/\varepsilon_j}^{1/\varepsilon_j} f\left(\varepsilon_j\int_I w_j'(\varepsilon_j r+\varepsilon_j z)\rho(z)\right)\mathrm{d}r\\ &= \int_{-1/\varepsilon_j}^{1/\varepsilon_j} f\left(\int_I \tilde{w}_j'(r+z)\rho(z)\right)\mathrm{d}r,\end{aligned}$$

where $\tilde{w}_j(t) = w_j(\varepsilon_j t)$. Since $(w_j) \in W^{0,s}$ then the previous inequality becomes

$$\frac{1}{\varepsilon_j}\int_{-1}^{1} f(\varepsilon_j|u_j'| * \rho_{\varepsilon_j})\,\mathrm{d}t \ge \int_{-\infty}^{+\infty} f\left(\int_I \tilde{w}_j'(t+z)\rho(z)\,\mathrm{d}z\right)\mathrm{d}t.$$

Denoting by $X$ the space of all functions $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ which are non-decreasing and such that there exist $\xi_0 < \xi_1$ with $v(t) = 0$ if $t < t_0$ and $v = s$ if $t > t_1$, we are led to solve the minimization problem $\inf_X G$, being

$$G(v) = \int_{-\infty}^{+\infty} f\left(\int_{I(t)} v'(x)\rho(x-t)\,\mathrm{d}x\right)\mathrm{d}t, \quad v \in X.$$

By a simple regularization argument it is not restrictive to assume $f \in C^2(0,+\infty)$ and $f$ strictly concave. For each $k \in \mathbb{N}$, with $k \geq 1$, we now consider a discrete version $G_k$ of $G$ defined on the space of the functions on $\mathbb{R}$ which are constant on each interval of the form

$$J_i^k = \left[\frac{i}{k}, \frac{i+1}{k}\right), \quad i \in \mathbb{Z}.$$

We define $X_k$ as the set of the functions $v\colon \mathbb{R} \to [0,s]$, such that:

a) $v$ is constant on any $J_i^k$; denote by $v^i$ the value of $v$ on $J_i^k$.
b) $v^i \leq v^{i+1}$ for any $i \in \mathbb{Z}$.
c) $v^i = 0$ if $i < i_0$ and $v^i = s$ if $i > i_1$ for some $i_0 < i_1$.

Let $I^k = \{i \in \mathbb{Z} : J_i^k \subset I\}$. Finally, let $G_k\colon X_k \to \mathbb{R}$ be defined by

$$G_k(v) = \frac{1}{k}\sum_{i\in\mathbb{Z}} f\Big(\sum_{h\in I^k}(v^{i+h+1} - v^{i+h})\rho_h^k\Big), \quad \rho_h^k = \fint_{J_h^k}\rho(z)\,\mathrm{d}z.$$

Obviously $G_k$ admit minimizers on $X_k$. We claim that each minimizer of $G_k$ on $X_k$ takes only the values 0 and $s$.

Let $v$ be a minimizer of $G_k$ on $X_k$. Suppose, by contradiction, that there exists $i_0 \in \mathbb{Z}$ with $v^{i_0} = c \in (0,s)$. We can assume that for a suitable $r \in \mathbb{N}$ it holds

$$v^{i_0-1} < c\,, \quad c = v^{i_0} = v^{i_0+1} = \cdots = v^{i_0+r}\,, \quad v^{i_0+r+1} > c.$$

Given $t \in \mathbb{R}$ sufficiently small, we define $v_t \in X_k$ letting $v_t^{i_0+l} = c + t$, if $0 \leq l \leq r$, and $v_t = v$ otherwise. It is easy to see that for some $\alpha_i^k, \beta_i^k \neq 0$ which do not depend on $t$, we have

$$G_k(v_t) = \frac{1}{k}\sum_{i\in J} f(\alpha_i^k + t\beta_i^k)$$

for some finite set $J \subset \mathbb{Z}$. The function $t \mapsto G_k(v_t)$ is twice continuously differentiable in $t = 0$, due to the smoothness of $f$ and we have

$$\frac{d^2}{dt^2}G_k(v_t)\Big|_{t=0} = \frac{1}{k}\sum_{i\in J} f''(\alpha_i^k)(\beta_i^k)^2 < 0$$

by the strict concavity of $f$. This contradicts the fact that $v$ is a minimizer for $G_k$ on $X_k$.

Since $G_k$ is invariant under translation, we have already shown that

$$\min_{X_k} G_k = G_k(\hat{v})$$

where $v = s\chi_{(0,+\infty)}$. Since

$$G_k(\hat{v}) = \frac{1}{k}\sum_{i\in\mathbb{Z}} f(s\rho_{-i}^k).$$

by the definition of the Riemann integral as the limit of the Riemann sums, we deduce that

$$\liminf_{k\to+\infty}\min_{X_k} G_k \geq \int_{-\infty}^{+\infty} f(s\rho(t))\,\mathrm{d}t.$$

Given $\sigma > 0$ let $v_\sigma \in X$ be such that $\inf_X G \geq G(v_\sigma) - \sigma$. Let $w_\sigma\colon \mathbb{R} \to [0,s]$ given by

$$w_\sigma(t) = w_\sigma^i = \fint_{J_i^k} v_\sigma(r)\,\mathrm{d}r, \quad t \in J_i^k.$$

Notice that $w_\sigma \in X_k$. Let $k$ be sufficiently large such that $G(v_\sigma) \geq G_k(w_\sigma) - \sigma$. Hence

$$G(v_\sigma) \geq \liminf_{k\to+\infty}\min_{X_k} G_k - \sigma \geq \int_{-\infty}^{+\infty} f(s\rho(t))\,\mathrm{d}t - \sigma.$$

By the arbitrariness of $\sigma > 0$ we obtain

$$\theta(s) \geq \inf_X G \geq \int_{-\infty}^{+\infty} f(s\rho(t))\,\mathrm{d}t.$$

If we let

$$u_j(t) = \begin{cases} 0 & \text{if } \leq -\varepsilon_j \\ \dfrac{s}{\varepsilon_j} t + s & \text{if } t \in (-\varepsilon_j, 0) \\ s & \text{if } t \geq 0 \end{cases}$$

for $\varepsilon_j \to 0^+$, we have $(u_j) \in W^{0,s}$ and a straightforward computation shows that

$$\lim_{j \to +\infty} \frac{1}{\varepsilon_j} \int_{-1}^{1} f(\varepsilon_j |u_j'| * \rho_{\varepsilon_j})\, \mathrm{d}t = \int_{-\infty}^{+\infty} f(s\rho(t))\, \mathrm{d}t$$

and this yields the conclusion. □

**Remark 7.2.** *Observe that when* $I = (-1, 1)$ *and* $\rho = \frac{1}{2}\chi_{(-1,1)}$ *we get*

$$\theta(s) = 2f\left(\frac{s}{2}\right).$$

*Hence we recover the case investigated in* [21].

## Acknowledgments

A. Magni have been supported by the research group "Forschergruppe 718 der Deutschen Forschungsgemeinschaft: Analysis and Stochastics in Complex Physical Systems". Both the authors would like to thank the Biomathematic group of the University of Dortmund, which made the collaboration possible.

## References

[1] R. Alicandro, A. Braides, and M. S. Gelli: *Free-discontinuity problems generated by singular perturbation.* Proc. Roy. Soc. Edinburgh Sect. A, **6** (1998), 1115-1129.
[2] R. Alicandro and M. S. Gelli: *Free discontinuity problems generated by singular perturbation: the n-dimensional case.* Proc. Roy. Soc. Edinburgh Sect. A, **130**(3) (2000), 449-469.
[3] L. Ambrosio, N. Fusco and D. Pallara: "Functions of Bounded Variation and Free Discontinuity Problems". Oxford University Press, 2000.
[4] L. Ambrosio and V.M. Tortorelli: *Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence.* Comm. Pure Appl. Math., **XLIII** (1990), 999-1036.
[5] L. Ambrosio and V.M. Tortorelli: *On the approximation of free discontinuity problems.* Boll. Un. Mat. Ital. B (7), **VI**(1) (1992), 105-123.
[6] G. Bouchitté, A. Braides and G. Buttazzo: *Relaxation results for some free discontinuity problems.* J. Reine Angew. Math., **458** (1995) 1-18.
[7] B. Bourdin and A. Chambolle: *Implementation of an adaptive finite-element approximation of the Mumford-Shah functional.* Numer. Math., **85**(4) (2000), 609-646.
[8] A. Braides: *Approximation of free-discontinuity problems.* Volume 1694 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1998.
[9] A. Braides: "Γ-convergence for beginners". Oxford University Press, 2002.
[10] A. Braides and G. Dal Maso: *Non-local approximation of the Mumford-Shah functional.* Calc. Var., **5**(4) (1997) 293-322.
[11] A. Braides and A. Garroni: *On the non-local approximation of free-discontinuity problems.* Comm. Partial Differential Equations, **23**(5-6) (1998), 817-829..
[12] A. Chambolle and G. Dal Maso: *Discrete approximation of the Mumford-Shah functional in dimension two.* ESAIM Math. Model. Numer. Anal., **33**(4) (1999), 651-672.
[13] G. Cortesani: *Sequence of non-local functionals which approximate free-discontinuity problems.* Arch. Rational Mech. Anal., **144** (1998), 357-402.
[14] G. Cortesani: *A finite element approximation of an image segmentation problem.* Math. Models Methods Appl. Sci., **9**(2) (1999), 243-259.
[15] G. Cortesani and R. Toader: *Finite element approximation of non-isotropic free-discontinuity problems.* Numer. Funct. Anal. Optim., **18**(9-10) (1997), 921-940.
[16] G. Cortesani and R. Toader: *Nonlocal approximation of nonisotropic free-discontinuity problems.* SIAM J. Appl. Math., **59**(4) (1999), 1507-1519.
[17] G. Cortesani and R. Toader: *A density result in SBV with respect to non-isotropic energies.* Nonlinear Anal., **38**(5) (1999) 585-604.
[18] G. Dal Maso: "An Introduction to Γ-Convergence". Birkhäuser, Boston, 1993.

[19] E. De Giorgi: *Free discontinuity problems in calculus of variations.* In Robert Dautray, editor, Frontiers in pure and applied mathematics. A collection of papers dedicated to Jacques-Louis Lions on the occasion of his sixtieth birthday. June 6–10, Paris 1988, pages 55–62, Amsterdam, 1991. North-Holland Publishing Co.

[20] L. Lussardi: *An approximation result for free discontinuity functionals by means of non-local energies.* Math. Meth. Appl. Sci., **31**(3) (2008) 2133-2146.

[21] L. Lussardi and E. Vitali: *Non local approximation of free-discontinuity functionals with linear growth: the one dimensional case.* Ann. Mat. Pura Appl., **186**(4) (2007) 722-744.

[22] L. Lussardi and E. Vitali: *Non local approximation of free-discontinuity problems with linear growth.* ESAIM Contr. Optim. Calc. Var., **13**(1) (2007) 135-162.

[23] M. Morini: *Sequences of singularly perturbed functionals generating free-discontinuity problems.* SIAM J. Math. Anal., **35**(3) (2003), 759-805.

[24] M. Negri: *The anisotropy introduced by the mesh in the finite element approximation of the Mumford-Shah functional.* Numer. Funct. Anal. Optim., **20**(9-10) (1999), 957-982.

[25] M. Negri: *A non-local approximation of free discontinuity problems in $SBV$ and $SBD$.* Calc. Var., **25**(1) (2006) 33-62.

(L. Lussardi) Dipartimento di Matematica e Fisica "N. Tartaglia", Università Cattolica del Sacro Cuore, via dei Musei 41, I-25121, Brescia (Italy)
*E-mail address*: l.lussardi@dmf.unicatt.it

(A. Magni) Mathematisches Institut Abt. für Reine Mathematik, Albert-Ludwigs Universität Freiburg, Eckerstrasse 1, D-79104 Freiburg im Breisgau (Germany)
*E-mail address*: annibale.magni@math.uni-freiburg.de