Approximation by Γ-convergence of a curvature-depending functional in Visual Reconstruction

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1 Introduction

In recent years a number of variational models related to reconstruction problems in Computer Vision have been proposed (for a survey see e.g. the monographs [7, 32, 37]). For the image segmentation problem, Mumford and Shah [33] proposed to minimize the functional

$$\mathcal{G}^{MS}(u,C)=\int_{\Omega\setminus C}|\nabla u|^2dx+\int_{\Omega}|u-g|^2dx+\mathcal{H}^1(C),$$

where $\Omega\subset\mathbb{R}^2$ is a bounded open set (the image domain), $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, and $g\in L^\infty(\Omega)$ is the input image. The functional has to be minimized over all closed sets $C\subset\overline{\Omega}$ and all $u\in\mathcal{C}^1(\Omega\setminus C)$. The function u represents a denoised approximation of the input image g, and C represents the set of boundaries of the segmentation. The Mumford and Shah variational model can be extended to several visual reconstruction problems (see March [29]): computation of depth from stereo images (Shah [39]), computation of optical flow (Nesi [34]), shape from shading (Shah [40]).

The existence of minimizers of $\mathcal{G}^{MS}$ has been proved independently by Dal Maso, Morel and Solimini [21] and De Giorgi, Carriero and Leaci [23] using the compactness and lower semicontinuity theorems of Ambrosio [3]. Mumford and Shah [33] studied the properties of minimizers (u,C) of $\mathcal{G}^{MS}$ assuming that C is a finite union of simple $\mathcal{C}^{1,1}$ curves meeting $\partial\Omega$ and meeting each other only at their endpoints. They proved that the vertices of C may only be: (i) triple points where three curves meet with equal angles; (ii) points on the boundary of Ω where one curve meets $\partial\Omega$ perpendicularly; (iii) 'crack-tips' where a curve ends and meets nothing.

In Computer Vision, the constraints imposed on the segmentations highlighted by such results constitute a drawback of the variational model. In particular, corners and T-junctions, which are relevant features for pattern recognition, are distorted. Since the length measure is not sensitive to corners and junctions, to allow for such singularities in the segmentations it is necessary to consider curvature-depending energies. Functionals that include both the integral of squared curvature $\int_C \kappa^2 d\mathcal{H}^1$ and a cost associated with singularities along the curves themselves have been proposed by Anzellotti [6]. Mantegazza [28], Shah [41], Terzopoulos [43, 44], functionals based on curvature have been proposed by Nitzberg and Mumford [35] and Nitzberg, Mumford and Shiota [36] for the problem of segmentation with depth.

We consider the functional

$$\mathcal{G}(u, C, P) = \#(P) + \int_C (1+\kappa^2)\, d\mathcal{H}^1 + \int_{\Omega\setminus(C\cup P)} |\nabla u|^2\, dx + \int_\Omega |u-g|^2\, dx,$$

where C is a family of curves, P is the set of the endpoints of the curves in C, and $\#(P)$ is the number of points in P. It should be noted that the functional $\mathcal{G}$ involves the recursive application of a one-dimensional version of the Mumford-Shah functional along the curves of the family C. The embedded structure of the resulting variational model reflects the recursive embedding of visual singularities, from surfaces to contours, to points. This is an instance of a general variational formulation of computer vision problems proposed by Terzopoulos [43]. The functional $\mathcal{G}$ has been proposed in this form by Anzellotti; existence results for minimizers of $\mathcal{G}$ can be found in [19].

In order to recover corners and junctions, in Blake and Zisserman [13] the contours obtained by minimizing the Mumford-Shah energy are subject to a subsequent process, in which the contours are regarded as fixed. The singularities along the curves are then computed by minimizing the energy $\#(P) + \int_C \kappa^2 d\mathcal{H}^1$. The drawback of such an approach is that the one-dimensional process can no longer feed back to the contour-detecting energy $\int_{\Omega\setminus C} |\nabla u|^2 dx$.

Existence results for minimizers of $\mathcal{G}$ are relatively simple, since energy bounds imply a bound of the number of components of C and on their norm as $W^{2,2}$-functions. On the contrary, the numerical minimization of the overall functional $\mathcal{G}$ is a difficult task, which reflects the difficulties of the challenging problem of recovering geometrical properties of the visible surfaces from two-dimensional image functions. An algorithm for minimizing a curvature-depending functional, based on ideas of Γ-convergence, has been proposed by Shah [41] when C is a family of closed curves with corner points, but without junctions. In the present paper we propose the approximation of the full functional $\mathcal{G}$ by means of a family of functionals which are, at least in principle, numerically more tractable, and we prove the Γ-convergence of the approximating functionals to $\mathcal{G}$.

From the analytical viewpoint, one of the main novelties of the approach is the idea of an iteration of a gradient-theory approach to construct an approximation by energies of 'elliptic type' of functionals as $\mathcal{G}$ defined on triplets (u, C, P) of functions, curves and points (i.e., objects of 'dimension' 2, 1 and 0). Another remarkable feature is that the approximation result is subdivided in two parts: a first approximation is performed by means of a new type of energies where points and curves are substituted by sets, while the final approximation by functionals defined on smooth functions is reduced to the first by means of the coarea formula and a 'mollification' argument where sharp interfaces are substituted by 'optimal profiles'.

The idea of an intermediate approach by functionals defined on sets is already contained in the proofs of many approximation results for free-discontinuity problems (see e.g. [15]) and is partially formalized in a paper by Bourdin and Chambolle in the case of the Mumford-Shah functional (see [14] Lemma 2). In our case the first idea is to construct a variational approximation of the functional $\#(P)$ that simply counts the number of the points of a set P by another functional whose minimizers are discs of small radius ε (additional conditions will force these discs to contain the target set of points); another requirement is that this functional should be subsequently transformed to an energy defined on functions by means of the coarea formula. Following some suggestions from a paper by Braides and Malchiodi [17] such a functional is given by

$$\mathcal{E}_\varepsilon^{(1)}(D) = \frac{1}{4\pi}\int_{\partial D}\Big(\frac{1}{\varepsilon} + \varepsilon\kappa^2(x)\Big)\, d\mathcal{H}^1(x),$$

where κ denotes the curvature of ∂D. The number $1/4\pi$ is a normalization factor that derives from the fact that minimizers of $\mathcal{E}^{(1)}(D)$ are given by balls of radius ε. This functional may be interpreted, upon scaling, as a singular perturbation of the perimeter functional by a curvature term.

The next step is then to construct another energy defined on sets, that approximates the functional $\int_C (1+\kappa^2) d\mathcal{H}^1$, where C is a (finite) union of $W^{2,2}$-curves with endpoints contained in P. To this end we approximate C away from D by sets A, whose energy is defined as

$$\mathcal{E}_\varepsilon^{(2)}(A, D) = \frac{1}{2}\int_{(\partial A)\setminus D} (1+\kappa^2)\, d\mathcal{H}^1.$$

Since no dependence on ε is present in this energy, the condition that A shrinks to C must be imposed by requiring in addition that $\mathrm{meas}(A) \le a_\varepsilon = o(1)$ as $\varepsilon \to 0$. The factor $1/2$ depends on the fact that, as A tends to C, each curve of C is the limit of two arcs of ∂A.

The intermediate approximation is thus constructed by assembling the pieces above and the simpler terms that account for u:

$$\mathcal{E}_\varepsilon(u, A, D) = \mathcal{E}_\varepsilon^{(1)}(D) + \mathcal{E}_\varepsilon^{(2)}(A, D) + \int_{\Omega\setminus(A\cup D)} |\nabla u|^2\, dx + \int_\Omega |u-g|^2\, dx$$

defined for A and D compactly contained in Ω (with the condition that $\mathrm{meas}(A) \le a_\varepsilon$). Note that $A\cup D$ contains the singularities of u. In this way a triplet (u, C, P) is approximated by means of a triplet $(u, A, D) = (u_\varepsilon, A_\varepsilon, D_\varepsilon)$ as in Fig. 1.

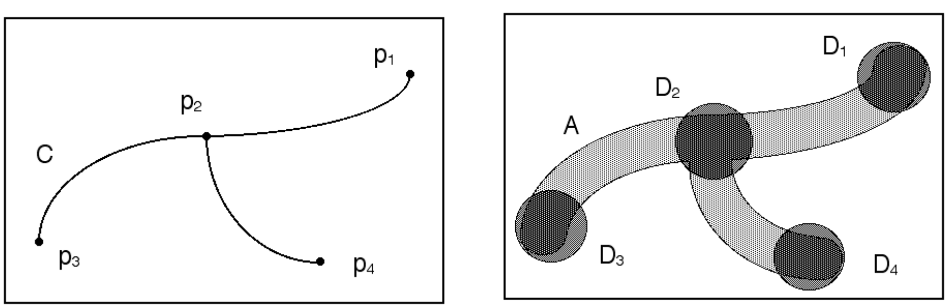

Figure 1: curves C and points $P = \{p_i\}$, and approximating sets A and $D = \bigcup_i D_i$

To obtain an energy defined on functions, we again use a gradient-theory approach as by Modica and Mortola [31] (see also e.g. [15]), where it is shown that the perimeter measures $\mathcal{H}^1 \llcorner \partial A$ and $\mathcal{H}^1 \llcorner \partial D$ are approximated by the measures $\mathcal{H}_\varepsilon^1(s, \nabla s)\, dx$ and $\mathcal{H}_\varepsilon^1(w, \nabla w)\, dx$ where

$$\mathcal{H}_\varepsilon^1(s, \nabla s) = \zeta_\varepsilon |\nabla s|^2 + \frac{s^2(1-s)^2}{\zeta_\varepsilon}, \qquad \mathcal{H}_\varepsilon^1(w, \nabla w) = \zeta_\varepsilon |\nabla w|^2 + \frac{w^2(1-w)^2}{\zeta_\varepsilon},$$

$\zeta_\varepsilon \to 0$ as $\varepsilon \to 0$, and s and w are optimal-profile functions approximating $1-\chi_A$ and $1-\chi_D$, respectively. We define the curvature of s and w as

$$\kappa(\nabla s) = \begin{cases} \operatorname{div}\left(\frac{\nabla s}{|\nabla s|}\right) & \text{if } \nabla s \neq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad \kappa(\nabla w) = \begin{cases} \operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) & \text{if } \nabla w \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

respectively. The next step is formally to replace the characteristic functions $1-\chi_A$ and $1-\chi_D$ by functions s and w. The term $\mathcal{E}^{(1)}_\varepsilon(D)$ is then substituted by

$$\mathcal{G}^{(1)}_\varepsilon(w) = \int_\Omega \Big(\frac{1}{\varepsilon} + \varepsilon\kappa^2(\nabla w)\Big)\mathcal{H}^1_\varepsilon(w,\nabla w)\,dx;$$

the term $\mathcal{E}^{(2)}_\varepsilon(A,D)$ by

$$\mathcal{G}^{(2)}_\varepsilon(s,w) = \int_\Omega w^2(1+\kappa^2(\nabla s))\mathcal{H}^1_\varepsilon(s,\nabla s)\,dx,$$

and the constraint that $\operatorname{meas}(A) \le a_\varepsilon$ by an integral penalization

$$\mathcal{I}_\varepsilon(s,w) = \frac{1}{\mu_\varepsilon}\int_\Omega ((1-s)^2 + (1-w)^2)\,dx,$$

(where $\mu_\varepsilon \to 0$) that forces s and w to be equal to 1 almost everywhere in the limit as $\varepsilon \to 0$, so that we construct a candidate functional

$$\mathcal{G}_\varepsilon(u,s,w) = \frac{1}{4\pi b_0}\mathcal{G}^{(1)}_\varepsilon(w) + \frac{1}{2b_0}\mathcal{G}^{(2)}_\varepsilon(s,w) + \int_\Omega s^2|\nabla u|^2\,dx + \int_\Omega |u-g|^2\,dx + \mathcal{I}_\varepsilon(s,w),$$

where b_0 is a normalization constant.

The main result of the paper is showing that these elliptic energies are indeed variational approximations in the sense of De Giorgi's Γ-convergence of the energy $\mathcal{G}$, for a suitable choice of ζ_ε and μ_ε (see Theorem 3.9)

The construction of $\mathcal{G}_\varepsilon$ is close in spirit to the Ambrosio and Tortorelli approach ([5], [15]). A technical but important difference is that in [5] the double-well potential $s^2(1-s)^2$ in the approximation of the perimeter is replaced by the single-well potential $(1-s)^2$. This modification breaks the symmetry between 0 and 1 and forces automatically s to tend to 1 as $\varepsilon \to 0^+$. Unfortunately, it also forbids recovery sequences to be bounded in $W^{2,2}$. With this substitution the curvatures terms in $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}_\varepsilon$ would necessarily be unbounded. In our case the necessary symmetry breaking is obtained by adding the 'lower order' term $\mathcal{I}_\varepsilon$.

Another issue raised by this approach is the proper definition of convergence of functions or sets to curves and points. While in the results by Modica and Mortola [31] or Ambrosio and Tortorelli [5] it is possible, and convenient, to identify sets of finite perimeter with characteristic functions in BV, and segmentation curves with free-discontinuity sets of SBV functions, respectively, in our case we have chosen not to place our objects in a common functional framework. On the contrary we have chosen a 'lighter' approach by keeping the domains of the approximating functionals and of the limit disjoint. We use the Hausdorff convergence to define the convergence of sets to curves and points, and an *ad hoc* definition of convergence of functions, that takes into account the convergence of sub-level sets. This approach is in line with recent works as those by Alberti, Baldo and Orlandi [1] on the convergence of Ginzburg-Landau energies (see also Sandier and Serfaty [38]), or of Friesecke, James and Muller [27] on the limits of thin structures, where the identification of some lower-dimensional geometric objects as limits of full-dimensional functions is described using Γ-convergence.

We note that the complex form of the functionals $\mathcal{G}_\varepsilon$, in particular of $\mathcal{G}_\varepsilon^{(1)}$, seems necessary despite the simple form of the target energy. Indeed, in order to describe an energy defined on points in the limit, it seems necessary to consider degenerate functionals. Our approach may be compared with that giving vortices in the Ginzburg-Landau theory (see e.g. the book by Bethuel, Brezis and Hélein [12]) or concentration of energies for functionals with critical growth (see e.g. the book by Flucher [26], and a recent approach by Γ-convergence by Amar and Garroni [2]). In both cases energies on points are recovered as limits of complex energies.

Finally we remark that the Γ-convergence result still holds, with minor changes in our proof, by replacing the lower order term $\int_\Omega |u-g|^2 dx$ with a wide class of perturbations that allow the application of the variational method to other visual reconstruction problems, such as for instance the ones considered in [29, 34], and that numerical approximations may be simplified by the use of a conjecture by De Giorgi on the approximation of curvature functionals (see Section 7).

2 Notation and preliminary definitions

We denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the usual euclidean norm and scalar product in $\mathbb{R}^2$, by dist the euclidean distance in $\mathbb{R}^2$, and by $B_\rho(x_0)$ the open ball centered at x_0 with radius ρ. For any set A, χ_A will be the *characteristic function* of A; that is, $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \notin A$. We say that A is of class $\mathcal{C}^\infty$ if A is open, and its restriction to some neighbourhood of any $x \in \partial A$ is the subgraph of a function of class $\mathcal{C}^\infty$ with respect to a suitable orthogonal coordinate system. If $\Omega \subseteq \mathbb{R}^2$ is a bounded open set then we write $A \in \mathcal{C}_c^\infty(\Omega)$ if A is of class $\mathcal{C}^\infty$ and $A \subset\subset \Omega$. The *Hausdorff distance* between two closed sets C and K is defined as $\mathrm{d}_{\mathcal{H}}(C,K) = \inf\{r > 0 : C \subset (K)_r, K \subset (C)_r\}$, where $(A)_r = \{x \in \mathbb{R}^2 : \mathrm{dist}(x,A) < r\}$, for a generic set $A \subseteq \mathbb{R}^2$.

We denote by $\mathrm{meas}(B)$ the Lebesgue measure of the set $B \subseteq \mathbb{R}^2$, by $\mathcal{H}^1$ the one-dimensional Hausdorff measure and by $\#$ the counting measure. We will use standard notation for the Lebesgue and Sobolev spaces L^p and $W^{k,p}$.

2.1 Curves. Length and curvature energies

We call a *curve* any function $\gamma : [a,b] \to \mathbb{R}^2$ in $W^{2,2}(a,b)$ such that $|\dot\gamma| \neq 0$ in $[a,b]$. The points $\gamma(a)$ and $\gamma(b)$ are the *endpoints* of γ, the set $[\gamma] = \{\gamma(t) : t \in [a,b]\}$ is the *trace* of γ. A curve will be identified with its representative in $C^1([a,b])$. A curve is *simple* if $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$ or $\{t_1,t_2\} = \{a,b\}$. A *regular closed curve* is a curve on some interval $[a,b]$ that may be extended to a $(b-a)$-periodic $W^{2,2}_{\mathrm{loc}}$ function on $\mathbb{R}$ (i.e., its endpoints join smoothly).

Let $C = \{\gamma^i\}_i$ be a family of curves. If the curves γ_i are parameterized on disjoint intervals $[a_i,b_i]$ then with an abuse of notation we will write $C : S \to \mathbb{R}^2$, where $S = \bigcup_i [a_i,b_i]$. We denote by $[C]$ the *trace* of C, defined as the union of all the traces of the curves in C; we say that C is *disjoint* if $[\gamma^i] \cap [\gamma^j] = \emptyset$ for any i, j with $i \neq j$.

Let γ be a curve defined on $[a,b]$; we define the *tangent unit vector* at the point $t \in [a,b]$ as $\tau(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$, and the *curvature* $\kappa(t)$ by $\kappa(t) = |\dot{\tau}(t)|/|\dot{\gamma}(t)|$. Note that the functionals

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| dt, \qquad K(\gamma) = \int_a^b \kappa^2(t) |\dot{\gamma}(t)| dt$$

are independent of the particular parametrization chosen. $L(\gamma)$ is the *length* of γ and K is the *integral of the square of the curvature* along γ. Furthermore, given a subset $I \subseteq [a,b]$, we define the localized versions of L and K by

$$L(\gamma, I) = \int_I |\dot{\gamma}(t)| dt, \qquad K(\gamma, I) = \int_I \kappa^2(t) |\dot{\gamma}(t)| dt,$$

respectively. We denote by s the *arc-length parameter*; then we have $|\dot{\gamma}(s)| = 1$, $\tau(s) = \dot{\gamma}(s)$, $\kappa(s) = |\ddot{\gamma}(s)|$, and

$$K(\gamma) = \int_0^{L(\gamma)} \kappa^2(s) ds = \int_0^{L(\gamma)} |\ddot{\gamma}(s)|^2 ds. \tag{2.1}$$

We will need the following simple geometric lemma (for a proof see [8] Lemma 3.1 and [19] Section 3). The angle between two vectors will be assumed to be in $[0, \pi]$.

Lemma 2.1. *Let γ be a curve defined on $[a,b]$ and let ω denote the maximum angle between its tangent vectors at any two points in $[a,b]$. The following inequalities hold:*

(i) *if $\omega > \pi/3$ then $L(\gamma)K(\gamma) \geq 1$;*

(ii) *if γ is a closed curve then $\omega \geq \pi/2$;*

(iii) *if $\omega \leq \pi/3$ then $L(\gamma) \leq 2|\gamma(b) - \gamma(a)|$;*

(iv) *if γ is a simple regular closed curve then $L(\gamma)K(\gamma) \geq 4\pi^2$.*

If A is a bounded subset of $\mathbb{R}^2$ of class $\mathcal{C}^\infty$, then ∂A is locally the graph of a function f of class $\mathcal{C}^\infty$. Hence the curvature $\kappa(x)$ of ∂A can be defined locally by means of the classical formulas involving the second derivatives of f. The function $\kappa(x)$ does not depend on the choice of the coordinate system used to describe ∂A as a graph, and belongs to $L^2(\partial A, \mathcal{H}^1)$. Let A be a bounded subset of $\mathbb{R}^2$ of class $\mathcal{C}^\infty$; a finite family $C = \{\gamma^i\}_i$ of regular closed curves is a *parametrization* of ∂A if C is disjoint, each curve of the family is simple, and $[C] = \partial A$. Then, using (2.1), we have

$$\mathcal{H}^1(\partial A) = \sum_{\gamma \in C} L(\gamma), \qquad \int_{\partial A} \kappa^2 d\mathcal{H}^1 = \sum_{\gamma \in C} K(\gamma). \tag{2.2}$$

Note that each bounded subset A of $\mathbb{R}^2$ of class $\mathcal{C}^\infty$ admits a parametrization.

2.2 Admissible families of curves

Let $C = \{\gamma^i\}_i$ be a family of curves; we denote by $P(C)$ the set of the endpoints of all the curves in C with the exception of those regular and closed. Then we define the functional

$$\mathcal{A}(C) = \sum_{\gamma \in C} \big(K(\gamma) + L(\gamma)\big) + \#P(C).$$

Following [19] we give the definition of an admissible family of curves.

Definition 2.2. *We say that a family $C = \{\gamma^i\}_i$ of curves is* admissible *if the following conditions are satisfied:*

(i) $\mathcal{A}(C) < +\infty$*;*

(ii) $\dot\gamma^i(t_1)$ *and* $\dot\gamma^j(t_2)$ *are parallel whenever* $\gamma^i(t_1) = \gamma^j(t_2)$ *with* t_1 *and* t_2 *interior to the domains of* γ^i *and* γ^j*, respectively, and with possibly* $i = j$*.*

We say that C is an admissible family of curves in Ω *if in addition* $[C] \subset \overline{\Omega}$*.*

Condition (ii) asserts that two (possibly coinciding) curves may meet at a point different from an endpoint only if they have the same tangent at that point. In [19] it has been proved that if C is admissible, then the total number of curves in C is finite, so that $[C]$ is a closed set. Note that this does not follow simply from $\#P(C) < +\infty$ since the curves in C may have common endpoints.

A notion of convergence of a sequence of traces of admissible families of curves, that takes into account possible reparameterizations and that curves may converge to points, has been introduced in [19] as follows.

Definition 2.3. *We say that a sequence of traces of admissible families $\{[C_h]\}_h$ of curves in Ω* converges *to the trace of an admissible family of curves $[C]$ in Ω* up to the finite set of points $P \subset \overline{\Omega}$ *if for any h there exists an admissible family, whose trace is $[C_h]$ (and which is still denoted by C_h), such that the following conditions are satisfied:*

(i) *each of the families C_h contains a number m of curves $\{\gamma_h^1, \dots, \gamma_h^m\}$, with m independent of h, such that, for any $i = 1, \dots, m$, the sequence $\{\gamma_h^i\}_h$, reparametrized on a fixed interval, converges weakly in $W^{2,2}$ to a curve γ^i;*

(ii) *the maximum distance of the trace of the remaining curves of C_h from the set P goes to zero (i.e.,* $\operatorname{dist}([C_h \setminus \{\gamma_h^1, \dots, \gamma_h^m\}], P) \to 0$*) ;*

(iii) *if we set $C' = \{\gamma^1, \dots, \gamma^m\}$, then $[C'] = [C]$.*

We need a further notion of convergence for sequences of compact sets.

Definition 2.4. *We say that a sequence of compact sets $\{\mathcal{K}_h\}_h$* converges in the Hausdorff metric *to the compact set $\mathcal{K}$* up to the finite set of points P *if there exists a sequence of compact sets $\{\widehat{\mathcal{K}}_h\}_h$ such that $\widehat{\mathcal{K}}_h \subseteq \mathcal{K}_h$ for any h, $\{\widehat{\mathcal{K}}_h\}_h$ converges to $\mathcal{K}$ in the Hausdorff metric, and the maximum distance of $\mathcal{K}_h \setminus \widehat{\mathcal{K}}_h$ from the set P goes to zero.*

A 'dense' class of families of curves with respect to the convergence in Definition 2.3 are those satisfying the finiteness property as defined in [8] (see Lemma 6.1).

Definition 2.5. *We say that a family of curves C satisfies the* finiteness property *if C is finite and there exists a finite set of points F such that $[C] \setminus F$ can be written locally as the graph of a function of class $W^{2,2}$.*

The following notion of equivalent families of curves is useful when dealing with different parameterizations of $[C]$.

Definition 2.6. *Let C and C' be two admissible families of curves in Ω. We say that C and C' are* equivalent *if $[C] = [C']$, $P(C) = P(C')$ and*

$$\sum_{\gamma \in C} L(\gamma) = \sum_{\gamma \in C'} L(\gamma), \qquad \sum_{\gamma \in C} K(\gamma) = \sum_{\gamma \in C'} K(\gamma).$$

3 The functional framework and the main results

In this section we define the curvature-depending functional introduced by Coscia in [19]. We then define two families of approximating functionals defined on boundaries of smooth sets, and on smooth functions, respectively.

3.1 The energy functional

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set; let $g \in L^2(\Omega)$ and let α_K, α_L and α_P be positive numbers. For any admissible family C of curves in Ω, and any finite set of points $P \subset \overline{\Omega}$ such that $P(C) \subseteq P$, we define the functionals:

$$\mathcal{F}(C, P) = \sum_{\gamma \in C} \big(\alpha_K K(\gamma) + \alpha_L L(\gamma)\big) + \alpha_P \# P,$$

$$\mathcal{F}_0(C, P) = \inf\{\mathcal{F}(\widehat{C}, \widehat{P}) : [\widehat{C}] = [C],\ \widehat{P} \setminus [\widehat{C}] = P \setminus [C]\}.$$

The functional $\mathcal{F}_0$ allows us to deal with the trace of of a family of curves independently of the parametrization of the curves themselves.

Remark 3.1. The infimum in the definition of $\mathcal{F}_0(C, P)$ is a minimum; namely, there exist an admissible family C^* of curves in Ω and a finite set of points $P^* \subset \overline{\Omega}$ such that

$$[C^*] = [C], \quad P^* \setminus [C^*] = P \setminus [C], \qquad \mathcal{F}(C^*, P^*) = \mathcal{F}_0(C, P). \tag{3.1}$$

This can be easily proved by reasoning similarly as in the proof of [19] Theorem 4.2.

We denote by $X(\Omega)$ the family of all triplets (u, C, P) such that $P \subset \overline{\Omega}$ is a finite set of points, C is an admissible family of curves in Ω such that $P(C) \subseteq P$ and $u \in W^{1,2}(\Omega \setminus [C])$, and we introduce the functional $\mathcal{G} : X(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{G}(u, C, P) = \int_{\Omega \setminus [C]} |\nabla u|^2 \, dx + \mathcal{F}_0(C, P) + \int_\Omega |u - g|^2 \, dx.$$

The compactness and lower semicontinuity result below for the functional $\mathcal{G}$ follows from [19] Theorem 4.2. We say that a sequence of sets of points $\{P_h\}_h \subset \overline{\Omega}$ *converges to the set* P if each of the sets P_h contains a number n of points $\{x_h^1, \dots, x_h^n\}$, with n independent of h, such that $x_h^i \to x^i$ for any $i = 1, \dots, n$, and $\bigcup_{i=1}^n \{x^i\} = P$.

Theorem 3.2 (coerciveness and lower semicontinuity of $\mathcal{G}$). *Let $\{(u_h, C_h, P_h)\}_h \subset X(\Omega)$ be a sequence such that $\sup_h \mathcal{G}(u_h, C_h, P_h) < +\infty$; then there exist a subsequence $\{(u_{h_k}, C_{h_k}, P_{h_k})\}_k$ and a triplet $(u, C, P) \in X(\Omega)$, such that $\{[C_{h_k}]\}_k$ converges to $[C]$ up to the set P, P_{h_k} converges to P, $u_{h_k} \rightharpoonup u$ weakly in $W^{1,2}(\Omega')$ for every $\Omega' \subset\subset \Omega \setminus ([C] \cup P)$ (in particular, $u_{h_k} \to u$ almost everywhere in Ω), and*

$$\liminf_{h \to +\infty} \mathcal{G}(u_h, C_h, P_h) \geq \mathcal{G}(u, C, P).$$

Remark 3.3. By the results above, taking u_h constant (e.g. $u_h = 0$), we deduce that the corresponding coerciveness property holds for $\mathcal{F}_0(C, P)$ and, consequently, for $\mathcal{F}(C, P)$.

The object of this work is to obtain an approximation of the functional $\mathcal{G}$ in a suitable sense by functionals defined on (smooth) functions. This approximation will be described below, first introducing an intermediate approximation where curves and points are seen as limits of (smooth) sets.

3.2 The approximating functionals $\mathcal{E}_\varepsilon$ defined on sets

We first introduce the functionals $\mathcal{E}_\varepsilon$ defined on (functions and) boundaries of smooth sets and approximating $\mathcal{G}$. In the following, without losing generality, we assume $\alpha_K = \alpha_L = \alpha_P = 1$. We set

$$Y(\Omega) = \{(u, A, D) : u \in W^{1,2}(\Omega);\ A, D \in \mathcal{C}_c^\infty(\Omega)\},$$

and, for every $\varepsilon > 0$, we denote by $E_\varepsilon : Y(\Omega) \to [0, +\infty]$ the functional defined by

$$\begin{aligned} E_\varepsilon(u, A, D) \quad &= \quad \int_\Omega (1 - \chi_{A\cup D})|\nabla u|^2\, dx + \frac{1}{2}\int_{\partial A} (1 - \chi_D)(1 + \kappa^2)\, d\mathcal{H}^1 \\ &\quad + \frac{1}{4\pi}\int_{\partial D} \Big(\frac{1}{\varepsilon} + \varepsilon\kappa^2\Big) d\mathcal{H}^1 + \int_\Omega |u - g|^2\, dx, \end{aligned}$$

where $\kappa(x)$ denotes either the curvature of ∂A at $x \in \partial A$, or the curvature of ∂D at $x \in \partial D$, respectively.

Let a_ε be a positive infinitesimal as $\varepsilon \to 0^+$; we denote by $\mathcal{E}_\varepsilon : Y(\Omega) \to [0, +\infty]$ the functional defined by

$$\mathcal{E}_\varepsilon(u, A, D) = \begin{cases} E_\varepsilon(u, A, D) & \text{if } \operatorname{meas}(A \cup D) \le a_\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

We define the following convergence for sequences $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$.

Definition 3.4. *We say that a sequence $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$* converges weakly *to the triplet $(u, C, P) \in X(\Omega)$, if* $\operatorname{meas}(A_h \cup D_h) \to 0$ *and the following properties hold:*

(i) *$\{\partial D_h\}_h$ converges in the Hausdorff metric to the set P;*

(ii) *$\{\partial A_h\}_h$ converges in the Hausdorff metric to $[C]$ up to the set P;*

(iii) *$u_h \to u$ in $L^1(\Omega)$.*

The above definition describes the concentration of the smooth sets D_h and A_h on sets of points and traces of curves, respectively. We now define the Γ-convergence of the functionals $\mathcal{E}_\varepsilon$ to the functional $\mathcal{G}$ with respect to the convergence above.

Definition 3.5. *We say that $\mathcal{E}_\varepsilon$* Γ-converge *to $\mathcal{G}$ as $\varepsilon \to 0^+$ if for every sequence $\{\varepsilon_h\}_h$ of positive numbers converging to zero and for every triplet $(u, C, P) \in X(\Omega)$ the following two conditions are fulfilled :*

(i) (liminf inequality) *for every sequence $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ converging weakly to (u, C, P), we have*

$$\liminf_{h\to+\infty} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) \ge \mathcal{G}(u, C, P); \tag{3.2}$$

(ii) (limsup inequality) *there exists a sequence $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ converging weakly to (u, C^*, P^*) such that*

$$\limsup_{h\to+\infty} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) \le \mathcal{G}(u, C, P), \tag{3.3}$$

where C^*, $P^* \subset \overline{\Omega}$ are as in (3.1).

We will prove the following theorem which states that a sequence of triplets in $Y(\Omega)$, asymptotically minimizing the functional $\mathcal{E}_\varepsilon$, admits a subsequence converging weakly to a minimizer of $\mathcal{G}$.

Theorem 3.6 (approximation by energies defined on sets). *The functionals $\mathcal{E}_\varepsilon$ Γ-converge to $\mathcal{G}$ as $\varepsilon \to 0^+$. Moreover, if $\{\varepsilon_h\}_h$ is a sequence of positive numbers converging to zero, and $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ is a sequence such that*

$$\lim_{h\to+\infty}\Big(\mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) - \inf_{Y(\Omega)} \mathcal{E}_{\varepsilon_h}\Big) = 0,$$

then there exist a subsequence $\{(u_{h_k}, A_{h_k}, D_{h_k})\}_k$ and a minimizer (u, C, P) of $\mathcal{G}$ such that $\{(u_{h_k}, A_{h_k}, D_{h_k})\}_k$ converges weakly to (u, C, P).

The proof of the liminf inequality will be given by Theorem 4.4, that of the limsup inequality by Theorem 6.3. The convergence of minimum problems is a direct consequence of Γ-convergence (see [16] Section 1.5) and of the equi-coerciveness of $\mathcal{E}_\varepsilon$ proved in Theorem 4.1.

3.3 The functionals $\mathcal{G}_\varepsilon$ defined on smooth functions

We now introduce the functionals $\mathcal{G}_\varepsilon$ approximating $\mathcal{G}$ and defined on smooth functions. We set

$$W(\Omega) = \{(u, s, w) : u \in W^{1,2}(\Omega);\ 1-s, 1-w \in \mathcal{C}_0^\infty(\Omega; [0,1])\}.$$

If $1-s \in \mathcal{C}_0^\infty(\Omega; [0,1])$, using Sard's theorem (see e.g. [4]), for a.e. $\lambda \in (0,1)$ we have

$$\{s = \lambda\} = \partial\{s < \lambda\}, \quad \{s < \lambda\} \in \mathcal{C}_c^\infty(\Omega), \quad |\nabla s| \neq 0 \text{ on } \{s = \lambda\}.$$

Then we set

$$\kappa(\nabla s) = \operatorname{div}\left(\frac{\nabla s}{|\nabla s|}\right) \qquad \text{on } \{s=\lambda\} \quad \text{for a.e. } \lambda \in (0,1)\ ,$$

and, $V : \mathbb{R} \to [0, +\infty)$ being defined by $V(t) = t^2(1-t)^2$,

$$\mathcal{H}^1_\varepsilon(s, \nabla s) = \varepsilon|\nabla s|^2 + \frac{V(s)}{\varepsilon}. \tag{3.4}$$

The quantity $\kappa(\nabla s)$ is the curvature of the level set $\{s = \lambda\}$, and $\mathcal{H}^1_\varepsilon(s, \nabla s)$ is the *Modica-Mortola density* of elliptic functionals approximating the perimeter functional (see [31], [15]). If $1-w \in \mathcal{C}_0^\infty(\Omega; [0,1])$ the quantities $\kappa(\nabla w)$ and $\mathcal{H}^1_\varepsilon(w, \nabla w)$ are defined analogously.

Let now β_ε, μ_ε be positive infinitesimals as $\varepsilon \to 0^+$ such that

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon|\log\varepsilon|}{\beta_\varepsilon} = 0, \qquad \lim_{\varepsilon\to 0^+}\frac{\beta_\varepsilon}{\mu_\varepsilon} = 0; \tag{3.5}$$

for every $\varepsilon > 0$ we define

$$\mathcal{G}^{(1)}_\varepsilon(w) = \int_{\Omega\setminus\{|\nabla w|=0\}} \Big(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\kappa^2(\nabla w)\Big)\mathcal{H}^1_\varepsilon(w, \nabla w)\, dx,$$

and

$$\mathcal{G}^{(2)}_\varepsilon(s,w)=\int_{\Omega\setminus\{|\nabla s|=0\}} w^2(1+\kappa^2(\nabla s))\mathcal{H}^1_\varepsilon(s,\nabla s)\,dx.$$

We denote by $\mathcal{G}_\varepsilon : W(\Omega)\to[0,+\infty]$ the functional defined by

$$\begin{aligned}\mathcal{G}_\varepsilon(u,s,w) \quad &= \quad \int_\Omega s^2|\nabla u|^2\,dx+\frac{1}{4\pi b_0}\mathcal{G}^{(1)}_\varepsilon(w)+\frac{1}{2b_0}\mathcal{G}^{(2)}_\varepsilon(s,w)+\int_\Omega |u-g|^2\,dx\\ &\qquad +\frac{1}{\mu_\varepsilon}\int_\Omega(1-s)^2\,dx+\frac{1}{\mu_\varepsilon}\int_\Omega(1-w)^2\,dx,\end{aligned}\tag{3.6}$$

where $b_0=2\int_0^1\sqrt{V(t)}dt$.

We define the following convergence for sequences $\{(u_h,s_h,w_h)\}_h\subset W(\Omega)$ that describes the concentration of the level sets of the smooth functions w_h and s_h on sets of points and traces of curves, respectively. The definition is particularly complex in order to ensure the compactness of the convergence under the hypothesis of uniform boundedness of the energies $\mathcal{G}_\varepsilon$, since such an assumption does not guarantee analogous bounds for all level sets.

Definition 3.7. *We say that a sequence $\{(u_h,s_h,w_h)\}_h\subset W(\Omega)$* converges weakly *to the triplet $(u,C,P)\in X(\Omega)$, if, after setting*

$$\{x\in\Omega: s_h(x)<\lambda\}=A^\lambda_h,\qquad \{x\in\Omega: w_h(x)<\theta\}=D^\theta_h,$$

the following properties hold:

(i) *for any $\theta,\lambda\in(0,1)$ there exist a finite set of points $P^\theta\subset\overline{\Omega}$ and an admissible family C^λ of curves in Ω such that the sequence $\{(u_h,A^\lambda_h,D^\theta_h)\}_h$ converges weakly to (u,C^λ,P^θ);*

(ii) *we have $[C]=\bigcap\{[C^\lambda]:0<\lambda<1\}$ and $P=\bigcap\{P^\theta:0<\theta<1\}$.*

We now can define the Γ-convergence of the functionals $\mathcal{G}_\varepsilon$ to the functional $\mathcal{G}$ with respect to the convergence above.

Definition 3.8. *We say that $\mathcal{G}_\varepsilon$* Γ-converge *to $\mathcal{G}$ as $\varepsilon\to0^+$ if for every sequence $\{\varepsilon_h\}_h$ of positive numbers converging to zero and for every triplet $(u,C,P)\in X(\Omega)$ the following two conditions are fulfilled:*

(i) (liminf inequality) *for every sequence $\{(u_h,s_h,w_h)\}_h\subset W(\Omega)$ converging weakly to (u,C,P), we have*

$$\liminf_{h\to+\infty}\mathcal{G}_{\varepsilon_h}(u_h,s_h,w_h)\geq\mathcal{G}(u,C,P);\tag{3.7}$$

(ii) (limsup inequality) *there exists a sequence $\{(u_h,s_h,w_h)\}_h\subset W(\Omega)$ converging weakly to (u,C^*,P^*) such that*

$$\limsup_{h\to+\infty}\mathcal{G}_{\varepsilon_h}(u_h,s_h,w_h)\leq\mathcal{G}(u,C,P),\tag{3.8}$$

where C^* and P^* are as in (3.1).

We will prove the following theorem which states that a sequence of triplets in $W(\Omega)$, asymptotically minimizing the functional $\mathcal{G}_\varepsilon$, admits a subsequence converging weakly to a minimizer of $\mathcal{G}$.

Theorem 3.9 (approximation by functionals defined on smooth functions). *The functionals $\mathcal{G}_\varepsilon$ Γ-converge to $\mathcal{G}$ as $\varepsilon \to 0^+$. Moreover, if $\{\varepsilon_h\}_h$ is a sequence of positive numbers converging to zero, and $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ is a sequence such that*

$$\lim_{h\to+\infty}\Big(\mathcal{G}_{\varepsilon_h}(u_h, s_h, w_h) - \inf_{W(\Omega)} \mathcal{G}_{\varepsilon_h}\Big) = 0,$$

then there exist a subsequence $\{(u_{h_k}, s_{h_k}, w_{h_k})\}_k$ and a minimizer (u, C, P) of $\mathcal{G}$ such that $\{(u_{h_k}, s_{h_k}, w_{h_k})\}_k$ converges weakly to (u, C, P).

The proof of the liminf inequality will be given by Theorem 5.3, that of the limsup inequality by Theorem 6.4. The convergence of minimum problems is a direct consequence of Γ-convergence (see [16] Section 1.5) and of the equi-coerciveness of $\mathcal{G}_\varepsilon$ proved in Theorem 5.1.

4 Approximation by functionals defined on sets

4.1 Equicoerciveness

We now prove the equicoerciveness of the family of functionals $\mathcal{E}_\varepsilon$.

Theorem 4.1. *Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. Let $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ be a sequence such that*

$$\sup_{h\in\mathbb{N}} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) < +\infty. \tag{4.1}$$

Then there exist a subsequence $\{(u_{h_k}, A_{h_k}, D_{h_k})\}_k$ and a triplet $(u, C, P) \in X(\Omega)$ such that $\{(u_{h_k}, A_{h_k}, D_{h_k})\}_k$ converges weakly to (u, C, P). Moreover, the following properties hold:

(a) *there exists a sequence $\{C_k\}_k$ of disjoint families of simple curves in Ω such that $[C_k] \subseteq \partial A_{h_k}$ for any k, $\{[C_k]\}_k$ converges to $[C]$ up to the set P (see Definition 2.3), and the maximum distance of $\partial A_{h_k} \setminus [C_k]$ from the set P goes to zero;*

(b) *$u_{h_k} \rightharpoonup u$ weakly in $W^{1,2}(\Omega')$ for every $\Omega' \subset\subset \Omega \setminus ([C] \cup P)$.*

Proof. Step 1. *We prove property* (i) *of the convergence in Definition* 3.4.
Since $\{D_h\}_h \subset \mathcal{C}^\infty_c(\Omega)$, for any $h \in \mathbb{N}$ there exists a parametrization of ∂D_h by means of a finite family $\widetilde{C}_h$ of regular closed curves such that $\widetilde{C}_h$ is disjoint, each curve of the family is simple, and $[\widetilde{C}_h] = \partial D_h$ for any h.

Using (2.2), (4.1) and Lemma 2.1 (iv), for h large enough we have

$$\begin{aligned} M_1 &\geq \frac{1}{4\pi}\int_{\partial D_h}\Big(\varepsilon_h\kappa^2 + \frac{1}{\varepsilon_h}\Big)d\mathcal{H}^1 = \frac{1}{4\pi}\sum_{\gamma\in\widetilde{C}_h}\Big(\varepsilon_h K(\gamma) + \frac{1}{\varepsilon_h}L(\gamma)\Big) \\ &\geq \frac{1}{4\pi}\sum_{\gamma\in\widetilde{C}_h} 2(L(\gamma)K(\gamma))^{1/2} \geq \#\widetilde{C}_h, \end{aligned} \tag{4.2}$$

where M_1 is a positive constant independent of h. Since $\#\widetilde{C}_h$ is uniformly bounded with respect to h, there exists a subsequence $\{\widetilde{C}_{h_k}\}_k$ such that $\#\widetilde{C}_{h_k} = n$ independent of k. Let $\widetilde{C}_{h_k} = \{\widetilde{\gamma}^1_{h_k}, \dots, \widetilde{\gamma}^n_{h_k}\}$.

From (4.2) it follows in particular that for any $\gamma \in \widetilde{C}_h$ we have $L(\gamma) \le 4\pi M_1 \varepsilon_h$, so that $\lim_h \max\{L(\gamma) : \gamma \in \widetilde{C}_h\} = 0$. Then, possibly passing to a subsequence (not relabelled), there exists a finite set of points $P = \{x^1, \dots, x^n\} \subset \overline{\Omega}$ (not necessarily all distinct) such that for any $\rho > 0$ we can find $k_\rho \in \mathbb{N}$ with

$$[\widetilde{\gamma}^i_{h_k}] \subseteq B_\rho(x^i) \qquad \text{for all } k > k_\rho \text{ and } i \in \{1, \dots, n\}. \tag{4.3}$$

Since $[\widetilde{C}_{h_k}] = \partial D_{h_k}$ property (i) then follows from (4.3).

Step 2: *we prove property* (a)*, which implies condition* (ii) *in Definition* 3.4*.*
We set $\delta = \min\{|x^i - x^j| : x^i, x^j \in P,\ x^i \ne x^j\}$, and, for $\eta > 0$ with $\eta \le \delta/4$, define

$$Q_\eta = \Big(\bigcup_{i=1}^n B_\eta(x^i)\Big) \cap \Omega,$$

so that there exists $k_\eta \in \mathbb{N}$ such that $\partial D_{h_k} \subset Q_\eta$ for any $k > k_\eta$. Since $A_h \in \mathcal{C}^\infty_c(\Omega)$, then, for $k > k_\eta$ we may write

$$\partial A_{h_k} \setminus Q_\eta = [\widehat{C}_{\eta,k}] \cup N,$$

where $\widehat{C}_{\eta,k}$ is a family of simple curves (not necessarily finite) having all the endpoints on ∂Q_η, and $N \subseteq \partial Q_\eta$. We denote by $\omega(\gamma)$ the maximum angle between the tangent vectors to γ at any two points. We denote by $\gamma \in C^1_{\eta,k}$ the subset of $\widehat{C}_{\eta,k}$ of the curves γ such that $\omega(\gamma) > \pi/3$. Using Lemma 2.1(ii) it follows that all closed curves in $\widehat{C}_{\eta,k}$ belong to $C^1_{\eta,k}$. If $\gamma \in \widehat{C}_{\eta,k} \setminus C^1_{\eta,k}$, then γ joins two distinct points $P^i_k \ne P^j_k$ such that $P^i_k, P^j_k \in \partial Q_\eta$. We define $\gamma \in C^2_{\eta,k}$ as the subset of $\widehat{C}_{\eta,k}$ of those γ such that

$$\omega(\gamma) \le \pi/3 \quad \text{and } P^i_k \in \partial B_\eta(x^i),\ P^j_k \in \partial B_\eta(x^j), \quad \text{with } x^i \ne x^j.$$

Then we set $C_{\eta,k} = C^1_{\eta,k} \cup C^2_{\eta,k}$. Using (2.2) and (4.1) we have for $k > k_\eta$

$$\begin{aligned} M_2 &\ge \int_{\partial A_{h_k}} (1 - \chi_{D_{h_k}}(x))(1+\kappa^2)\, d\mathcal{H}^1 \\ &\ge \int_{\partial A_{h_k} \setminus Q_\eta} (1+\kappa^2)\, d\mathcal{H}^1 \ge \sum_{\gamma \in C_{\eta,k}} \big(K(\gamma) + L(\gamma)\big), \end{aligned} \tag{4.4}$$

where M_2 is a positive constant independent of k. Using (4.4), the inequality $(K+L)^2 \ge 4KL$, and Lemma 2.1(i), we find

$$\begin{aligned} M_2 &\ge \sum_{\gamma \in C^1_{\eta,k}} \big(K(\gamma) + L(\gamma)\big) \ge 2\#C^1_{\eta,k}\ , \\ M_2 &\ge \sum_{\gamma \in C^2_{\eta,k}} L(\gamma) \ge (\delta - 2\eta)\#C^2_{\eta,k}\ . \end{aligned}$$

Since $K(\gamma) \leq M_2$ for any curve $\gamma \in C^1_{\eta,k}$, again using Lemma 2.1(i), it follows that $L(\gamma) \geq 1/M_2$ and

$$L(\gamma) \geq \min\Big\{\frac{1}{M_2}, \delta - 2\eta\Big\} \qquad \text{for all } \gamma \in C_{\eta,k}. \tag{4.5}$$

Then, since $\eta \leq \delta/4$ and all closed curves in $C_{\eta,k}$ are regular, we have

$$\#P(C_{\eta,k}) \leq 2\#C_{\eta,k}, \qquad \#C_{\eta,k} \leq \Big(\frac{1}{2} + \frac{2}{\delta}\Big)M_2. \tag{4.6}$$

It follows that $C_{\eta,k}$ is a finite family and, since the traces of the curves in $C_{\eta,k}$ are pairwise disjoint, $C_{\eta,k}$ is an admissible family of curves in Ω for any $k > k_\eta$. Using (4.4) and (4.6) we have

$$\mathcal{F}(C_{\eta,k}, P(C_{\eta,k})) \leq 2\Big(1 + \frac{2}{\delta}\Big)M_2 \qquad \text{for all } k > k_\eta. \tag{4.7}$$

Then, by the coerciveness of $\mathcal{F}$ (Remark 3.3) and possibly passing to a subsequence, there exists an admissible family C_η of curves in Ω such that $\{[C_{\eta,k}]\}_k$ converges to $[C_\eta]$, up to a finite set of points contained in ∂Q_η.

If $\gamma \in \widehat{C}_{\eta,k} \setminus C_{\eta,k}$, then $\omega(\gamma) \leq \pi/3$ and γ joins two distinct points $P^i_k \neq P^j_k$ such that $P^i_k, P^j_k \in \partial B_\eta(x^q)$, for some $q \in \{1, \ldots, n\}$. Then, by Lemma 2.1(iii) we get

$$L(\gamma) \leq 2|P^i_k - P^j_k| \leq 4\eta, \qquad \text{if } \gamma \in \widehat{C}_{\eta,k} \setminus C_{\eta,k},$$

from which it follows that

$$\partial A_{h_k} \setminus [C_{\eta,k}] \subseteq \bigcup_{i=1}^{n} B_{3\eta}(x^i). \tag{4.8}$$

Let now $\{\eta_l\}_l$ be a sequence of positive numbers converging to zero as $l \to +\infty$. By Remark 3.3 and (4.7), we have for l large enough

$$2\Big(1 + \frac{2}{\delta}\Big)M_2 \geq \liminf_{k\to+\infty} \mathcal{F}(C_{\eta_l,k}, P(C_{\eta_l,k})) \geq \mathcal{F}(C_{\eta_l}, P(C_{\eta_l})),$$

so that $\mathcal{F}(C_{\eta_l}, P(C_{\eta_l}))$ is uniformly bounded with respect to l. Since the endpoints of the curves C_{η_l} converge to a set of points contained in P, again using Remark 3.3, up to a subsequence, there exists an admissible family C of curves in Ω such that $\{[C_{\eta_l}]\}_l$ converges to $[C]$ up to the set P, as $l \to +\infty$. The proof of (a) is concluded by a diagonal argument producing the family C_k, upon showing that $P(C) \subseteq P$. By construction, the set of the limits of the endpoints of all the curves in C_k is contained in P. It remains to show that no new endpoints arise. This may happen in two cases: i) if a sequence of closed curves in C_k converges to a nonclosed curve; ii) if a sequence of regular closed curves in C_k converges to a nonregular curve. Case (i) is ruled out by the uniform convergence, and case (ii) is ruled out by the weak $W^{2,2}_{\mathrm{loc}}$ convergence of the curves in C_k (e.g. as in Step 1 of the proof of [19] Theorem 4.2). It follows that $P(C) \subseteq P$; moreover, by the construction of the families of curves $C_{\eta,k}$, we have

$$(P \setminus P(C)) \cap [C] = \emptyset. \tag{4.9}$$

The proof of (a) is then completed.

STEP 3: *we prove property* (b)*, that implies condition* (ii) *in Definition* 3.4.

Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset\subset \Omega \setminus ([C] \cup P)$ invading $\Omega \setminus ([C] \cup P)$. The distance $\mathrm{dist}(\Omega_j, [C] \cup P)$ is positive for any j, so that by (4.3), (4.8) and the convergence of $\{[C_k]\}_k$ to $[C]$, there exists $\eta_j > 0$ such that

$$\Omega_j \cap \Big(\bigcup_{i=1}^{n} B_{3\eta_j}(x^i) \Big) = \emptyset;$$

hence there exists $k_j \in \mathbb{N}$ such that for all $k \geq k_j$

$$\Omega_j \cap [\widetilde{C}_{h_k}] = \Omega_j \cap \partial D_{h_k} = \emptyset, \qquad \Omega_j \cap [\widehat{C}_{\eta_j,k}] = \Omega_j \cap \partial A_{h_k} = \emptyset.$$

Then for any $x \in \Omega_j$ there exists $\rho > 0$ such that either $B_\rho(x) \subseteq A_{h_k}$ or $B_\rho(x) \subseteq \Omega \setminus A_{h_k}$, for any $k \geq k_j$. Since $\mathrm{meas}(A_{h_k}) \to 0$ it follows that $\Omega_j \cap A_{h_k} = \emptyset$ for any $k \geq k_j$. Analogously $\Omega_j \cap D_{h_k} = \emptyset$.

Hence $u_{h_k} \in W^{1,2}(\Omega_j)$ for any $k \geq k_j$, and using (4.1) we have

$$\begin{aligned} \sup_j \sup_k \int_{\Omega_j} (|u_{h_k}|^2 + |\nabla u_{h_k}|^2)\, dx &\leq \sup_k \int_{\Omega \setminus (A_{h_k} \cup D_{h_k})} (|u_{h_k}|^2 + |\nabla u_{h_k}|^2)\, dx \\ &\leq \sup_k \int_{\Omega \setminus (A_{h_k} \cup D_{h_k})} (2|u_{h_k} - g|^2 + 2|g|^2 + |\nabla u_{h_k}|^2)\, dx < +\infty. \end{aligned} \tag{4.10}$$

Hence, possibly passing to a subsequence, $\{u_{h_k}\}_k$ converges weakly in $W^{1,2}(\Omega_j)$ and a.e. in Ω_j to a function $u \in W^{1,2}(\Omega_j)$ for any j. By using a diagonal argument we obtain a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus ([C] \cup P))$ and a subsequence $\{u_{h_k}\}_k$ converging to u weakly in $W^{1,2}(\Omega')$ for every $\Omega' \subset\subset \Omega \setminus ([C] \cup P)$.

Using (4.10), the weak lower semicontinuity of the $W^{1,2}$ norm and the fact that $\{\Omega_j\}_j$ invades $\Omega \setminus ([C] \cup P)$, we deduce that $u \in W^{1,2}(\Omega \setminus ([C] \cup P))$. Since P is a finite set of points we have $u \in W^{1,2}(\Omega \setminus [C])$ and (b) is proved. ■

Remark 4.2. (i) In the following we will use the construction in Step 3 above; in particular that possibly passing to a subsequence, there exists a sequence $\{\eta_h\}_h$ of positive numbers converging to zero such that

$$\partial D_h \subset Q_{\eta_h} \quad \text{for all } h, \text{ where } Q_{\eta_h} = \big(\bigcup \{ B_{\eta_h}(x) : \ x \in P \} \big) \cap \Omega, \tag{4.11}$$

and there exists a sequence $\{C_h\}_h$ of disjoint families of simple curves in Ω that satisfies condition (a) in Theorem 4.1 and such that $P(C_h) \in \partial Q_{\eta_h}$ for any h.

(ii) Note that in the proof of (a) (Step 2 above) we only need an estimate on the part of the energy $E_{\varepsilon_h}(u_h, A_h, D_h)$ not depending on u_h.

4.2 Γ-convergence: lower inequality

We now prove the lower inequality (3.2) of the Γ-convergence of the family of functionals $\mathcal{E}_\varepsilon$ to the functional $\mathcal{G}$. We need the following technical lemma. It asserts that the weak convergence of (u_h, A_h, D_h) to (u, C, P) under an equi-boundedness assumption on the energies $\mathcal{E}_\varepsilon$ occurs in such a way that each curve is obtained as a limit of two curves belonging to ∂A_h.

Lemma 4.3. *Let $\{\varepsilon_h\}_h$ be an infinitesimal sequence of positive numbers and let the sequence $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ converge weakly to $(u, C, P) \in X(\Omega)$ and*

$$\sup_{h\in\mathbb{N}} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) < +\infty.$$

Then, there exists a sequence $\{C_h\}_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ of disjoint families of simple curves in Ω such that $[C_h] \subseteq \partial A_h$ for any h, $\{[C_h]\}_h$ converges to $[C]$ up to the set P, m is an even number independent of h, $\{\gamma_h^i\}_h$ converges weakly in $W^{2,2}$ to a curve γ^i for any $i \in \{1, \dots, m\}$, and

$$K(\gamma^{2i-1}) = K(\gamma^{2i}), \qquad L(\gamma^{2i-1}) = L(\gamma^{2i}), \tag{4.12}$$

for $i = 1, \dots, \frac{m}{2}$.

Proof. We denote by $C_h = \{\gamma_h^1, \dots, \gamma_h^{m_h}\}$ the families of curves obtained in Theorem 4.1. We also suppose that η_h and C_h are as in Remark 4.2(i), that $m_h = m$ is independent of h, and the sequence $\{\gamma_h^i\}_h$ converges weakly in $W^{2,2}$ to a curve γ^i for any $i \in \{1, \dots, m\}$. Let $C' = \{\gamma^1, \dots, \gamma^m\}$; then, by Definition 2.3(iii) and the construction of $\{C_h\}_h$, we have $[C'] = [C]$ and $P(C') \subseteq P$. Moreover C' is an admissible family of curves in Ω.

Let $p \in P$; by (4.9), for any $\gamma \in C'$ such that p is not an endpoint of γ we have $\mathrm{dist}(p, [\gamma]) > 0$. Then we define $d^{(1)} > 0$ by

$$d^{(1)} = \min\{\mathrm{dist}(p, [\gamma]) : \ \gamma \in C', \ p \in P \text{ and } p \text{ is not an endpoint of } \gamma\}.$$

Let $p \in P(C')$ and let $\gamma \in C'$ be a curve having p as an endpoint; since $\gamma : [a, b] \to \mathbb{R}^2$ for some interval $[a, b]$ and $\gamma \in W^{2,2}(a, b)$, we may select $\nu = \nu(p, \gamma) > 0$ such that:

(i) if γ is not a closed curve, then $\gamma^{-1}(B_\nu(p)) = I_\nu$, where $I_\nu \subset [a, b]$ is an interval such that the set $\{\gamma(t) : t \in I_\nu\}$ intersects $\partial B_r(p)$ in only one point for any $r < \nu$;

(ii) if γ is a closed curve, then $\gamma^{-1}(B_\nu(p)) = I_\nu \cup I'_\nu$, where $I_\nu, I'_\nu \subset [a, b]$ are disjoint intervals such that both the sets $\{\gamma(t) : t \in I_\nu\}$ and $\{\gamma(t) : t \in I'_\nu\}$ intersect $\partial B_r(p)$ in only one point for any $r < \nu$.

We then set

$$d^{(2)} = \min\{\nu(p, \gamma) : \ \gamma \in C', \ p \in P(C') \text{ and } p \text{ is an endpoint of } \gamma\}.$$

Note that $d^{(2)} > 0$. We also set

$$\delta = \min\{|x - y| : x, y \in P, \ x \neq y\},$$

and we choose $\rho > 0$ such that $\rho < \min\{d^{(1)}/2, d^{(2)}, \delta/2\}$. Then, let $0 < \sigma < \rho$ and let

$$Q_\rho = \bigcup \{B_\rho(x) : x \in P\}, \qquad Q_\sigma = \bigcup \{B_\sigma(x) : x \in P\}. \tag{4.13}$$

As in the proof of Theorem 4.1, we have for h large enough $\partial A_h \cap (\Omega \setminus Q_\sigma) = [C_h] \cap (\Omega \setminus Q_\sigma)$; moreover the endpoints of all the curves in C_h are contained in Q_σ.

Step 1: *we cover $[C] \setminus Q_\rho$ by open rectangles in which $[C]$ is the union of graphs.*

For any point $q \in [C] \setminus Q_\rho$ let $R(q)$ be an open rectangle centered at q, having two sides parallel to the tangent line $T_{[C]}(q)$ of $[C]$ at q, and such that each curve $\gamma \in C'$ meeting $R(q)$ has the following properties:

(a) $\gamma^{-1}(R(q)) = \bigcup_{j=1}^{H} I_j$, where H is a finite integer and the I_j are disjoint open intervals;

(b) $\gamma(I_j)$ is a cartesian graph in $R(q)$ of a function of class $W^{2,2}$ with respect to $T_{[C]}(q)$ for any $j = 1, \ldots, H$;

(c) γ does not meet the two sides of $R(q)$ which are parallel to $T_{[C]}(q)$.

Such a rectangle exists, since $\{\gamma^1, \ldots, \gamma^m\}$ is an admissible family of curves.

We cover the set $[C] \setminus Q_\rho$ with the family of all such open rectangles $\{R(q)\}_q$ and such that the diameter of $R(q)$ is strictly smaller than $\rho - \sigma$ for any q. As $[C] \setminus Q_\rho$ is a compact set we get

(d) there exists a finite set of points $M \subset [C] \setminus Q_\rho$ such that the finite subfamily of open rectangles $\{R(q)\}_{q \in M}$ with diameters strictly smaller than $\rho - \sigma$ covers $[C] \setminus Q_\rho$.

Note that $Q_\sigma \cap R(q) = \emptyset$ for any $q \in M$ and the set $(\bigcup_{q \in M} R(q)) \cup Q_\rho$ covers $[C]$.

Thanks to the $W^{2,2}$ weak convergence of γ_h^i to γ^i, for any $q \in M$ and for h large enough, we have that each curve $\gamma_h \in C_h$ meeting $R(q)$ satisfies properties (a), (b) and (c) with the same integer H and the same intervals I_j.

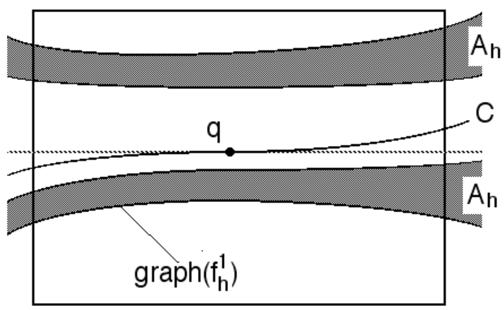

Figure 2: description of A_h inside $R(q)$

Step 2: *we locally parameterize $\partial A_h \setminus Q_\rho$.*

Let $q \in M$ and let (ξ, η) denote the coordinates of points in a local coordinate system in $R(q)$, with the origin in q, such that $T_{[C]}(q)$ coincides with the ξ-axis and $R(q) = (-\xi_0, \xi_0) \times (-\eta_0, \eta_0)$. Then we have

$$[C_h] \cap R(q) = \bigcup \{\mathrm{graph}(f_h^j) : 1 \leq j \leq r\},$$

where, the curves $\gamma_h \in C_h$ being simple and disjoint, for all $\xi \in (-\xi_0, \xi_0)$

$$f_h^j \in W^{2,2}(-\xi_0, \xi_0) \ \text{ for all } j \in \{1, \ldots, r\}, \qquad f_h^1(\xi) < f_h^2(\xi) < \cdots < f_h^r(\xi).$$

Since $\partial A_h \cap R(q) = [C_h] \cap R(q)$ and $\mathrm{meas}(A_h) \leq a_{\varepsilon_h}$ converges to zero, using the $W^{2,2}$ weak convergence of γ_h^i to γ^i and property (c) above, we deduce that the following properties are satisfied for h large enough:

$$\{(\xi,\eta) \in R(q) : \eta < f_h^1(\xi)\} \cap A_h = \emptyset, \qquad \{(\xi,\eta) \in R(q) : \eta > f_h^r(\xi)\} \cap A_h = \emptyset,$$

r is an even integer,

$$\{(\xi,\eta) \in R(q) : f_h^{2k-1}(\xi) < \eta < f_h^{2k}(\xi)\} \subset A_h \quad \text{for } k = 1, \ldots, \frac{r}{2}, \tag{4.14}$$

$$\{(\xi,\eta)\in R(q): f_h^{2k}(\xi)<\eta<f_h^{2k+1}(\xi)\}\cap A_h=\emptyset \quad \text{for } k=1,\ldots,\frac{r}{2}-1, \tag{4.15}$$

and, for any $k=1,\ldots,\frac{r}{2}$, the functions f_h^{2k-1}, f_h^{2k} converge weakly in $W^{2,2}(-\xi_0,\xi_0)$ to the same function $f^k\in W^{2,2}(-\xi_0,\xi_0)$ such that $\mathrm{graph}(f^k)\subseteq[\gamma]\cap R(q)$ for some $\gamma\in C'$.

Step 3: *We show that $A_h\setminus Q_\rho$ may be decomposed into a finite number of connected components converging to $[C]$ and that the number of such connected components is $\#C'/2$.*

Fix $i\in\{1,\ldots,m\}$ and suppose that the curve γ^i is parametrized on the interval $[a,b]$. We construct a finite sequence of open rectangles $\{R(q_n)\}_{n=1}^N$, not necessarily all distinct, with $R(q_n)\in\{R(q)\}_{q\in M}$ for any $n=1,\ldots,N$, and having the following properties: for any $n\in\{1,\ldots,N\}$ there exists an open interval $I^i(n)=(a_n,b_n)\subset[a,b]$ such that

$$I^i(n)\subseteq {\gamma^i}^{-1}(R(q_n)), \qquad [\gamma^i]\setminus Q_\rho\subset\gamma^i\Big(\bigcup_{n=1}^N I^i(n)\Big), \tag{4.16}$$

$$I^i(n)\cap I^i(n+1)\neq\emptyset, \qquad b_n<b_{n+1}, \qquad \text{for all } n\in\{1,\ldots,N-1\}. \tag{4.17}$$

We show that we can construct $\{R(q_n)\}_n$ in such a way that there exist $l\in\{1,\ldots,m\}$, with $l\neq i$ and open intervals $I^l(n)\subseteq{\gamma^l}^{-1}(R(q_n))$, $n=1,\ldots,N$, such that

$$[\gamma^l]\setminus Q_\rho\subset\gamma^l\Big(\bigcup_{n=1}^N I^l(n)\Big), \qquad \{\gamma^i(t):t\in I^i(n)\}=\{\gamma^l(t):t\in I^l(n)\}, \tag{4.18}$$

for any $n=1,\ldots,N$.

Assume that γ^i has two endpoints p_1, p_2 with $p_1\neq p_2$. Hence, if $q\in M\cap[\gamma^i]$ we have $R(q)\cap B_\rho(p)=\emptyset$ for any $p\in P$ such that $p\notin\{p_1,p_2\}$. We will construct the sequence $\{R(q_n)\}_{n=1}^N$ recursively. Choose $q_1\in M$ in such a way that $R(q_1)$ contains $[\gamma^i]\cap\partial B_\rho(p_1)$, that is a single point because of property (i) above and since $\rho<d^{(2)}$. Suppose that the rectangle $R(q_n)$ has been chosen; then we choose the rectangle $R(q_{n+1})$ as follows.

Let (ξ,η) denote a local coordinate system in $R(q_n)$ such that $T_{[C]}(q_n)$ coincides with the ξ-axis and $R(q_n)=(-\xi_0,\xi_0)\times(-\eta_0,\eta_0)$. Then, for h large enough, $[C_h]\cap R(q_n)$ is the union of the graphs of r functions f_h^j, $j=1,\ldots,r$. Let $k\in\{1,\ldots,r/2\}$; using properties (a), (b) and (4.14), (4.15), there exist an index $l\in\{1,\ldots,m\}$, possibly depending on n, and open intervals

$$I_h^i(n)\subseteq{\gamma_h^i}^{-1}(R(q_n)), \qquad I_h^l(n)\subseteq{\gamma_h^l}^{-1}(R(q_n)),$$

such that

$$\mathrm{graph}(f_h^{2k-1})=\{\gamma_h^i(t):t\in I_h^i(n)\}, \qquad \mathrm{graph}(f_h^{2k})=\{\gamma_h^l(t):t\in I_h^l(n)\}.$$

Since the functions f_h^{2k-1}, f_h^{2k} converge weakly in $W^{2,2}(-\xi_0,\xi_0)$ to the same function f^k, and γ_h^i, γ_h^l converge weakly in $W^{2,2}$ to the curves γ^i and γ^l, respectively, it follows that there exist open intervals $I^i(n)=(a_n,b_n)\subseteq{\gamma^i}^{-1}(R(q_n))$ and $I^l(n)\subseteq{\gamma^l}^{-1}(R(q_n))$, such that

$$\mathrm{graph}(f^k)=\{\gamma^i(t):t\in I^i(n)\}=\{\gamma^l(t):t\in I^l(n)\}.$$

Assume that $p_1=\gamma^i(a)$ and set

$$x_h=(-\xi_0,f_h^{2k-1}(-\xi_0)), \quad y_h=(\xi_0,f_h^{2k-1}(\xi_0)), \quad z_h=(\xi_0,f_h^{2k}(\xi_0)).$$

If $n = 1$, since $R(q_1)$ contains $[\gamma^i] \cap \partial B_\rho(p_1)$, the diameter of $R(q)$ is strictly smaller than $\rho - \sigma$, and $\rho < d^{(2)}$, we can choose the coordinate system (ξ, η) in such a way that $\gamma^i(a_1) \in B_\rho(p_1)$ and $\gamma^i(b_1) \notin B_\rho(p_1)$. For the same reason, if $n = 1$ we have $l \neq i$. Since $x_h \to \gamma^i(a_1)$ and $y_h \to \gamma^i(b_1)$, we have $x_h \in B_\rho(p_1)$ and $y_h \notin B_\rho(p_1)$ for h large enough. If $n > 1$, using the property (i) of the function γ^i, $\rho < d^{(2)}$, and (4.17) we have $\gamma^i(b_n) \notin B_\rho(p_1)$. Since $y_h \to \gamma^i(b_n)$, it follows that $y_h \notin B_\rho(p_1)$ for h large enough. Since $R(q_n)$ is open, either $y_h \in B_\rho(p_2)$ or we can find $q_{n+1} \in M$, with $R(q_{n+1}) \neq R(q_n)$, such that $y_h \in R(q_{n+1})$. Then, since f_h^{2k-1} and f_h^{2k} converge weakly to the same function in $W^{2,2}(-\xi_0, \xi_0)$ as $h \to +\infty$, we have either $z_h \in B_\rho(p_2)$ or $z_h \in R(q_{n+1})$ for h large enough.

If $y_h, z_h \in B_\rho(p_2)$ we set $n = N$ and the procedure stops. Otherwise, if $y_h, z_h \in R(q_{n+1})$, we can find a local coordinate system in $R(q_{n+1})$, with coordinates $(\widetilde{\xi}, \widetilde{\eta})$, such that $[C_h] \cap R(q_{n+1})$ is the union of the graphs of $\widetilde{r}$ functions $\widetilde{f}_h^j$, $j = 1, \ldots, \widetilde{r}$, and there exists $s \in \{1, \ldots, \widetilde{r} - 1\}$ such that

$$\{(\widetilde{\xi}, \widetilde{\eta}) \in R(q_{n+1}) : \widetilde{f}_h^s(\widetilde{\xi}) < \widetilde{\eta} < \widetilde{f}_h^{s+1}(\widetilde{\xi})\} \subset A_h,$$

$y_h \in \mathrm{graph}(\widetilde{f}_h^s)$, $z_h \in \mathrm{graph}(\widetilde{f}_h^{s+1})$. Moreover there exist open intervals

$$I_h^i(n+1) \subseteq {\gamma_h^i}^{-1}(R(q_{n+1})), \qquad I_h^l(n+1) \subseteq {\gamma_h^l}^{-1}(R(q_{n+1})),$$

such that $\mathrm{graph}(\widetilde{f}_h^s) = \{\gamma_h^i(t) : t \in I_h^i(n+1)\}$, and $\mathrm{graph}(\widetilde{f}_h^{s+1}) = \{\gamma_h^l(t) : t \in I_h^l(n+1)\}$. Then, from the convergence properties of γ_h^i and γ_h^l it follows that there exist open intervals $I^i(n+1)$, satisfying (4.17), and $I^l(n+1)$ such that

$$\{\gamma^i(t) : t \in I^i(n+1)\} = \{\gamma^l(t) : t \in I^l(n+1)\}.$$

Hence we can choose the index l independent of n and, since $l \neq i$ if $n = 1$, we have $l \neq i$ for any n. Then, using the property (a) above of γ^i, after a finite number of steps conditions (4.16), (4.17) and (4.18) are satisfied and the procedure stops. The same argument also holds, with slight changes, if γ^i is a closed curve (either regular or not), by using property (ii) and again the condition $\rho < d^{(2)}$.

STEP 4: *proof of the energy equalities in* (4.12).

By (4.16) and (4.18) there exist closed intervals J_ρ^i and J_ρ^l, contained in the intervals of parametrization of γ^i and γ^l, respectively, such that $[\gamma^i] \setminus Q_\rho = \gamma^i(J_\rho^i)$, $[\gamma^l] \setminus Q_\rho = \gamma^l(J_\rho^l)$, and

$$K(\gamma^i, J_\rho^i) = K(\gamma^l, J_\rho^l), \qquad L(\gamma^i, J_\rho^i) = L(\gamma^l, J_\rho^l),$$

for any $\rho > 0$. Then, letting $\rho \to 0^+$, the intervals J_ρ^i and J_ρ^l invade the intervals of parametrization of γ^i and γ^l, so that we have

$$K(\gamma^i) = K(\gamma^l), \qquad L(\gamma^i) = L(\gamma^l).$$

Let now $i' \neq i, l$; since the curves $\gamma_h \in C_h$ are simple and disjoint, using (4.14), (4.15) and arguing as before, we find an index $l' \neq i'$, with $l' \neq i, l$, such that $K(\gamma^{i'}) = K(\gamma^{l'})$ and $L(\gamma^{i'}) = L(\gamma^{l'})$. By repeating this argument it follows that m is an even number and the curves $\{\gamma^1, \ldots, \gamma^m\}$ can be ordered in such a way that the relation (4.12) holds. This concludes the proof of the lemma. ■

Theorem 4.4 (lower bound). *Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every triplet $(u, C, P) \in X(\Omega)$ and for every sequence $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ converging weakly to (u, C, P), we have*

$$\liminf_{h\to+\infty} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) \geq \mathcal{G}(u, C, P).$$

Proof. Let $(u, C, P) \in X(\Omega)$ and let $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ converge weakly to (u, C, P). Possibly extracting a subsequence we may assume that

$$\lim_{h\to+\infty} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) = \liminf_{h\to+\infty} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) < +\infty,$$

otherwise the result is trivial. Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset\subset \Omega \setminus ([C] \cup P)$ invading $\Omega \setminus ([C] \cup P)$.

Using Theorem 4.1(b) and possibly passing to a subsequence, we have $u_h \rightharpoonup u$ weakly in $W^{1,2}(\Omega_j)$, so that

$$\int_{\Omega_j} |u-g|^2 \, dx = \lim_{h\to+\infty} \int_{\Omega_j} |u_h - g|^2 \, dx \leq \liminf_{h\to+\infty} \int_{\Omega} |u_h - g|^2 \, dx$$

for any j, from which it follows

$$\liminf_{h\to+\infty} \int_{\Omega} |u_h - g|^2 \, dx \geq \int_{\Omega} |u-g|^2 \, dx. \tag{4.19}$$

By Definition 3.4(i) and (ii), and again using the weak $W^{1,2}(\Omega_j)$ convergence of u_h to u, we have

$$\liminf_{h\to+\infty} \int_{\Omega\setminus(A_h\cup D_h)} |\nabla u_h|^2 \, dx \geq \liminf_{h\to+\infty} \int_{\Omega_j} |\nabla u_h|^2 \, dx \geq \int_{\Omega_j} |\nabla u|^2 \, dx,$$

for any j. Since $u \in W^{1,2}(\Omega \setminus [C])$, it follows

$$\liminf_{h\to+\infty} \int_{\Omega\setminus(A_h\cup D_h)} |\nabla u_h|^2 \, dx \geq \int_{\Omega\setminus[C]} |\nabla u|^2 \, dx. \tag{4.20}$$

Using (4.2), (4.3) and Definition 3.4(i), we find

$$\liminf_{h\to+\infty} \frac{1}{4\pi} \int_{\partial D_h} \left(\varepsilon_h \kappa^2 + \frac{1}{\varepsilon_h}\right) d\mathcal{H}^1 \geq \# P. \tag{4.21}$$

To prove the statement of the theorem it will be enough to show that

$$\liminf_{h\to+\infty} \frac{1}{2} \int_{\partial A_h} (1 - \chi_{D_h})(1+\kappa^2) \, d\mathcal{H}^1 \geq \sum_{\gamma\in\widehat{C}} \big(K(\gamma) + L(\gamma)\big), \tag{4.22}$$

where $\widehat{C}$ is an admissible family of curves in Ω such that $[\widehat{C}] = [C]$ and $P(\widehat{C}) \subseteq P$. Indeed, collecting inequalities (4.19–4.22) we find

$$\begin{aligned}
\liminf_{h\to+\infty} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) \quad &\geq \quad \int_{\Omega\setminus[C]} |\nabla u|^2 \, dx + \mathcal{F}(\widehat{C}, P) + \int_{\Omega} |u-g|^2 \, dx \\
&\geq \quad \int_{\Omega\setminus[C]} |\nabla u|^2 \, dx + \mathcal{F}_0(C, P) + \int_{\Omega} |u-g|^2 \, dx = \mathcal{G}(u, C, P),
\end{aligned}$$

which is the desired result.

Let $\{C_h\}_h$ be disjoint families of simple curves in Ω satisfying Theorem 4.1(a), $\{\eta_h\}_h$ be as in Remark 4.2(i), and

$$\begin{aligned}\liminf_{h\to+\infty}\int_{\partial A_h}(1-\chi_{D_h})(1+\kappa^2)\,d\mathcal{H}^1 &\geq \liminf_{h\to+\infty}\int_{\partial A_h\setminus Q_{\eta_h}}(1+\kappa^2)\,d\mathcal{H}^1\\ &\geq \liminf_{h\to+\infty}\sum_{\gamma\in C_h}\big(K(\gamma)+L(\gamma)\big). \qquad (4.23)\end{aligned}$$

Let $C_h=\{\gamma_h^1,\ldots,\gamma_h^m\}$; then $P(C_h)\in\partial Q_{\eta_h}$ for any h, and the sequence $\{\gamma_h^i\}_h$ converges weakly in $W^{2,2}$ to a curve γ^i for any $i\in\{1,\ldots,m\}$. Let $C'=\{\gamma^1,\ldots,\gamma^m\}$; then C' is an admissible family of curves in Ω such that $[C']=[C]$ and $P(C')\subseteq P$. Moreover, by Lemma 4.3, the curves $\gamma^1,\ldots,\gamma^m$ satisfy (4.12).

Arguing as in the proof of (4.5) we have $L(\gamma)\geq c>0$ for any $\gamma\in C_h$, where c is a constant independent of h. From the weak convergence in $W^{2,2}$ of γ_h^i we easily deduce (as in Step 1 in the proof of [19] Theorem 4.2)

$$\liminf_{h\to+\infty}\sum_{\gamma\in C_h}\big(K(\gamma)+L(\gamma)\big)\geq\sum_{\gamma\in C'}\big(K(\gamma)+L(\gamma)\big).$$

Then it follows that

$$\liminf_{h\to+\infty}\frac{1}{2}\int_{\partial A_h}(1-\chi_{D_h})(1+\kappa^2)\,d\mathcal{H}^1\geq\frac{1}{2}\sum_{\gamma\in C'}\big(K(\gamma)+L(\gamma)\big).$$

Using Lemma 4.3 we set

$$\widehat{C}=\{\gamma^{2i}\}_{i=1,\ldots,\frac{m}{2}},$$

so that we have $[\widehat{C}]=[C']=[C]$, $P(\widehat{C})=P(C')\subseteq P$, and $\widehat{C}$ is an admissible family of curves in Ω. Then, using (4.12), it follows

$$\frac{1}{2}\sum_{\gamma\in C'}\big(K(\gamma)+L(\gamma)\big)=\sum_{\gamma\in\widehat{C}}\big(K(\gamma)+L(\gamma)\big),$$

which implies (4.22) and concludes the proof of the theorem. ■

5 Approximation by energies defined on smooth functions

In this section we prove the equicoerciveness of the family $\mathcal{G}_\varepsilon$ and inequality (3.7).

5.1 Equicoerciveness

The equicoerciveness of $\mathcal{G}_\varepsilon$ will be proved by applying the previous results on energies defined on sets to the sub-level sets of the functions s_h and w_h.

Theorem 5.1. *Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. Let $\{(u_h,s_h,w_h)\}_h\subset W(\Omega)$ be a sequence such that*

$$\sup_{h\in\mathbb{N}}\mathcal{G}_{\varepsilon_h}(u_h,s_h,w_h)<+\infty. \qquad (5.1)$$

Then there exist a subsequence $\{(u_{h_k}, s_{h_k}, w_{h_k})\}_k$ and a triplet $(u, C, P) \in X(\Omega)$ such that $\{(u_{h_k}, s_{h_k}, w_{h_k})\}_k$ converges weakly to (u, C, P). Moreover, $u_{h_k} \rightharpoonup u$ weakly in $W^{1,2}(\Omega')$ for every $\Omega' \subset\subset \Omega \setminus ([C] \cup P)$.

Proof. We first check Definition 3.7(i) in the following Steps 1–5. We set $\beta_{\varepsilon_h} = \beta_h$ and $\mu_{\varepsilon_h} = \mu_h$; we also denote the value in (5.1) by M.

STEP 1: *Energy estimates for $D_h^\theta = \{w_h < \theta\}$.*
For any $\theta \in (0, 1)$ we have

$$\operatorname{meas}(D_h^\theta) \leq \frac{1}{(1-\theta)^2} \int_\Omega (1 - w_h)^2 \, dx \leq \frac{\mu_h M}{(1-\theta)^2}. \tag{5.2}$$

Using the coarea formula (see e.g. [4]) and the algebraic inequality $\varepsilon_h |\nabla w_h|^2 + \varepsilon_h^{-1} V(w_h) \geq 2|\nabla w_h| \sqrt{V(w_h)}$, we have

$$\begin{aligned} \mathcal{G}^{(1)}_{\varepsilon_h}(w_h) &\geq 2 \int_{\Omega \setminus \{|\nabla w_h| = 0\}} |\nabla w_h| \sqrt{V(w_h)} \Big(\frac{1}{\beta_h} + \beta_h \kappa^2(\nabla w_h) \Big) \, dx \\ &\geq 2 \int_0^1 \sqrt{V(\theta)} \int_{\{w_h = \theta\} \cap \{|\nabla w_h| \neq 0\}} \Big(\frac{1}{\beta_h} + \beta_h \kappa^2(\nabla w_h) \Big) d\mathcal{H}^1 d\theta. \end{aligned} \tag{5.3}$$

By Sard's theorem, for any h there exists a negligible set $T_{w_h} \subseteq (0, 1)$ such that

$$\{w_h = \theta\} = \partial\{w_h < \theta\}, \quad \{w_h < \theta\} \in \mathcal{C}^\infty_c(\Omega) \quad \text{for } \theta \in (0, 1) \setminus T_{w_h} \ ,$$

and, denoting by κ the curvature of $\{w_h = \theta\}$,

$$|\nabla w_h| \neq 0 \text{ on } \{w_h = \theta\}, \quad |\operatorname{div}(\nabla w_h / |\nabla w_h|)| = |\kappa| \text{ on } \{w_h = \theta\} \qquad \text{for } \theta \in (0, 1) \setminus T_{w_h},$$

(for a similar argument see the proof of [9] Theorem 4.2). Using (5.3) and Fatou's Lemma it follows

$$M \geq 2 \int_{(0,1) \setminus T_w} \sqrt{V(\theta)} \liminf_{h \to +\infty} \int_{\partial\{w_h < \theta\}} \Big(\frac{1}{\beta_h} + \beta_h \kappa^2 \Big) \, d\mathcal{H}^1 \, d\theta, \tag{5.4}$$

where $T_w = \bigcup_{h \in \mathbb{N}} T_{w_h}$. Hence, there exists a negligible set $F_w \supseteq T_w$ such that, for any $\theta \in (0, 1) \setminus F_w$, we have

$$\liminf_{h \to +\infty} \int_{\partial D_h^\theta} \Big(\frac{1}{\beta_h} + \beta_h \kappa^2 \Big) \, d\mathcal{H}^1 \leq M_\theta < +\infty, \tag{5.5}$$

where M_θ is a positive constant depending on θ, but not on h.

STEP 2: *Compactness properties of D_h^θ.*
Let $\theta \in (0, 1) \setminus F_w$; as in Step 1 of the proof of Theorem 4.1, using (5.5), up to the extraction of a subsequence possibly depending on θ, there exists a finite set of points $P^\theta \subset \overline{\Omega}$ such that Definition 3.7(i) is satisfied. Hence, we can find a dense countable subset $\mathcal{N} = \{\theta_i\}_i$ of $(0, 1)$, a sequence of finite sets of points $\{P^{\theta_i}\}_i \subset \overline{\Omega}$ and, by using a diagonal argument, a subsequence $\{w_{h_k}\}_k$ such that, for any $\theta_i \in \mathcal{N}$, Definition 3.7(i) is satisfied.

Fix $\theta \in (0, 1)$ and let $\theta_i \in \mathcal{N}$ be such that $\theta_i > \theta$, and, consequently, $D_{h_k}^\theta \subseteq D_{h_k}^{\theta_i}$. Using Definition 3.7(i), for any $\delta > 0$ there exists $k_0 = k_0(\delta)$ such that, for any $k > k_0$, we have $D_{h_k}^{\theta_i} \cap B_\delta(x) \neq \emptyset$ for any $x \in P^{\theta_i}$. Since $D_{h_k}^\theta$ is open, for any $\delta > 0$ and any $x \in P^{\theta_i}$

such that $D_{h_k}^{\theta} \cap B_\delta(x) \neq \emptyset$, we may choose $\theta_n = \theta_n(x)$ such that $\theta_n \in \mathcal{N}$, $\theta_n < \theta$ and, for k large enough, $D_{h_k}^{\theta_n} \cap B_\delta(x) \neq \emptyset$. We set

$$\widehat{\theta} = \max\{\theta_n(x) : x \in P^{\theta_i}\}.$$

Then, for any $x \in P^{\theta_i}$, the inclusion $D_{h_k}^{\widehat{\theta}} \subseteq D_{h_k}^{\theta}$ implies

$$D_{h_k}^{\widehat{\theta}} \cap B_\delta(x) \subseteq D_{h_k}^{\theta} \cap B_\delta(x) \subseteq D_{h_k}^{\theta_i} \cap B_\delta(x).$$

Since $\widehat{\theta}, \theta_i \in \mathcal{N}$, by letting $\delta \to 0^+$, it follows that condition (i) in Definition 3.7 is satisfied for any $\theta \in (0,1)$ by setting $P^\theta = P^{\widehat{\theta}}$.

STEP 3: *Energy estimates for $A_h^\lambda = \{s_h < \lambda\}$.*
For any $\lambda \in (0,1)$ we have

$$\operatorname{meas}(A_h^\lambda) \leq \frac{1}{(1-\lambda)^2} \int_\Omega (1-s_h)^2 \, dx \leq \frac{\mu_h M}{(1-\lambda)^2}. \tag{5.6}$$

Again using the coarea formula we have

$$\mathcal{G}_{\varepsilon_h}^{(2)}(s_h, w_h) \geq 2 \int_0^1 \sqrt{V(\lambda)} \int_{\{s_h = \lambda\} \cap \{|\nabla s_h| \neq 0\}} w_h^2 \left(1 + \kappa^2(\nabla s_h)\right) \, d\mathcal{H}^1 d\lambda. \tag{5.7}$$

Arguing as before, for any h there exists a set $T_{s_h} \subseteq (0,1)$ of zero Lebesgue measure such that $\{s_h = \lambda\} = \partial\{s_h < \lambda\}$, and $\{s_h < \lambda\} \in \mathcal{C}_c^\infty(\Omega)$ for any $\lambda \in (0,1) \setminus T_{s_h}$. Using (5.7) we find

$$M \geq 2 \int_{(0,1)\setminus T_s} \sqrt{V(\lambda)} \int_{\partial\{s_h<\lambda\}} w_h^2 (1+\kappa^2) \, d\mathcal{H}^1 d\lambda,$$

where $T_s = \bigcup_{h\in\mathbb{N}} T_{s_h}$, from which, for any $\theta \in (0,1)$, it follows

$$M \geq 2 \int_{(0,1)\setminus T_s} \sqrt{V(\lambda)} \int_{\partial A_h^\lambda} (1 - \chi_{D_h^\theta}) \theta^2 (1+\kappa^2) \, d\mathcal{H}^1 d\lambda, \tag{5.8}$$

since $\theta^2 \leq w_h^2$ for any $x \notin D_h^\theta$. Then, using Fatou's Lemma we have

$$M \geq 2\theta^2 \int_{(0,1)\setminus T_s} \sqrt{V(\lambda)} \liminf_{h\to+\infty} \int_{\partial A_h^\lambda} (1 - \chi_{D_h^\theta})(1+\kappa^2) \, d\mathcal{H}^1 d\lambda.$$

Hence, for any $\theta \in (0,1)$ there exists a set $F_s^\theta \supseteq T_s$ of zero Lebesgue measure such that, for any $\lambda \in (0,1) \setminus F_s^\theta$, we have

$$\liminf_{h\to+\infty} \int_{\partial A_h^\lambda} (1 - \chi_{D_h^\theta})(1+\kappa^2) \, d\mathcal{H}^1 \leq \frac{M_\lambda}{\theta^2} < +\infty, \tag{5.9}$$

where M_λ is a positive constant depending on λ, but not on h.

STEP 4: *$W^{2,2}$-weak compactness properties of A_h^λ for a dense set of λ.*
Let P^θ be as in Step 2 and $P = \bigcap\{P^\theta : 0 < \theta < 1\}$. Since $\#P^\theta < +\infty$ for all $\theta \in (0,1)$ and $P^{\theta_1} \subseteq P^{\theta_2}$ when $\theta_1 \leq \theta_2$, then there exists $\theta_0 \in (0,1)$ such that $P = P^{\theta_0}$.

Let $\theta = \theta_0$ and $\lambda \in (0,1) \setminus F_s^{\theta_0}$. By Remark 4.2(i), using (5.6) and (5.9), up to the extraction of a subsequence possibly depending on λ, it follows that there exist a sequence $\{C_h^\lambda\}_h$ of disjoint families of simple curves and an admissible family C^λ of curves in Ω with $P(C^\lambda) \subseteq P$, such that $[C_h^\lambda] \subseteq \partial A_h^\lambda$, $\{[C_h^\lambda]\}_h$ converges to $[C^\lambda]$ up to the set P, and the maximum distance of $\partial A_h^\lambda \setminus [C_h^\lambda]$ from P goes to zero as $h \to +\infty$. Then we can find a dense countable subset $\mathcal{D} = \{\lambda_i\}_i$ of $(0,1)$, a sequence $\{C^{\lambda_i}\}_i$ of admissible families of curves in Ω with $P(C^{\lambda_i}) \subseteq P$ and, by using a diagonal argument, a subsequence $\{s_{h_k}\}_k$, a sequence $\{C_k^{\lambda_i}\}_k$ of disjoint families of simple curves for any $i \in \mathbb{N}$, such that $[C_k^{\lambda_i}] \subseteq \partial A_{h_k}^{\lambda_i}$, $\{[C_k^{\lambda_i}]\}_k$ converges to $[C^{\lambda_i}]$ up to the set P, and the maximum distance of $\partial A_{h_k}^{\lambda_i} \setminus [C_k^{\lambda_i}]$ from P goes to zero.

Choose $\sigma > 0$ and $\rho > \sigma$, and let Q_ρ, Q_σ be defined as in (4.13). Let $\lambda_i \in \mathcal{D}$; we cover $[C^{\lambda_i}] \setminus Q_\rho$ by a finite family $\{R(q)\}_{q \in M}$ of open rectangles with diameters strictly smaller than $\rho - \sigma$ satisfying properties (a)–(d) as in the proof of Lemma 4.3. Using (4.14), for any $q \in M$ we have for large enough k:

$$A_{h_k}^{\lambda_i} \cap R(q) = \bigcup_{j=1}^{r/2} B_k^{\lambda_i}(j,q), \qquad B_k^{\lambda_i}(j,q) \cap B_k^{\lambda_i}(l,q) = \emptyset \quad \text{if } j \neq l,$$

$$B_k^{\lambda_i}(j,q) = \{(\xi,\eta) \in R(q) : f_k^{2j-1}(\xi) < \eta < f_k^{2j}(\xi)\} \quad \text{for } j = 1, \ldots, \frac{r}{2}, \tag{5.10}$$

where $r = r(\lambda_i, q)$ is a finite integer, (ξ,η) denotes a local coordinate system in $R(q)$ and f_k^{2j-1}, f_k^{2j} are functions of class $W^{2,2}$.

By Step 3 in the proof of Lemma 4.3 there exists $\rho_0 > 0$ such that, for any $\rho < \rho_0$ there exists $k_0 = k_0(\rho)$ such that, for any $k > k_0$, the set $A_{h_k}^{\lambda_i} \setminus Q_\rho$ has a finite number of connected components equal to $\#C^{\lambda_i}/2$. Moreover, if $\mathcal{A}_k^{\lambda_i}$ is a connected component of $A_{h_k}^{\lambda_i} \setminus Q_\rho$, we have $\lim_k d_{\mathcal{H}}(\partial\mathcal{A}_k^{\lambda_i}, [\gamma] \setminus Q_\rho) = 0$, where $\gamma \in C^{\lambda_i}$. Then, for any $q \in M$ such that $[\gamma] \cap R(q) \neq \emptyset$ and for any $k > k_0$ we have $\mathcal{A}_k^{\lambda_i} \cap R(q) \neq \emptyset$. Let now $\lambda_m \in \mathcal{D}$ be such that $\lambda_m < \lambda_i$: from the inclusion $A_{h_k}^{\lambda_m} \subseteq A_{h_k}^{\lambda_i}$ it follows that $[C^{\lambda_m}] \subseteq [C^{\lambda_i}]$. Let $\mathcal{A}_k^{\lambda_m}$ be a connected component of $A_{h_k}^{\lambda_m} \setminus Q_\rho$: if $\mathcal{A}_k^{\lambda_m} \cap \mathcal{A}_k^{\lambda_i} \neq \emptyset$, then, arguing as in Step 3 in the proof of Lemma 4.3 , for any $k > k_0$ it follows that

$$\mathcal{A}_k^{\lambda_i} \cap R(q) \neq \emptyset \quad \Longrightarrow \quad \mathcal{A}_k^{\lambda_m} \cap R(q) \neq \emptyset. \tag{5.11}$$

Step 5: *compactness properties of A_h^λ with respect to the Haurdorff distance for all λ.* Fix $\lambda \in (0,1)$ and let $\lambda_i \in \mathcal{D}$ be such that $\lambda_i > \lambda$. Let $\{\sigma_m\}_m$, $\{\rho_m\}_m$ be sequences of positive numbers converging to zero as $m \to +\infty$, and let $\{R(q)\}_{q \in \widetilde{M}(m)}$ be finite families of open rectangles covering $[C^{\lambda_i}] \setminus Q_{\rho_m}$ in such a way that

$$\{R(q)\}_{q \in \widetilde{M}(m_1)} \subseteq \{R(q)\}_{q \in \widetilde{M}(m_2)} \tag{5.12}$$

for any m_1, m_2 with $m_1 < m_2$. Such families are easily constructed by a recursive procedure: set $\widetilde{M}(0) = \emptyset$ and $\{R(q)\}_{q \in \widetilde{M}(m)}$ is obtained by adding to the covering related to $m-1$ a finite covering of $([C^{\lambda_i}] \setminus Q_{\rho_m}) \setminus \bigcup_{q \in \widetilde{M}(m-1)} R(q)$ by open rectangles satisfying (a)–(d) as in Step 1 of Lemma 4.3 with $\rho = \rho_m$ and $\sigma = \sigma_m$.

Then for any $m \in \mathbb{N}$ and any $q \in \widetilde{M}(m)$ we have

$$A_{h_k}^{\lambda} \cap R(q) \subseteq \bigcup_{j=1}^{r/2} B_k^{\lambda_i}(j,q).$$

Since $A^\lambda_{h_k}$ is open, for any $q \in \widetilde{M}(m)$ and any $j \in \{1, \ldots, \frac{r}{2}(\lambda_i, q)\}$ such that $A^\lambda_{h_k} \cap B^{\lambda_i}_k(j,q) \neq \emptyset$, we may choose $\lambda_n = \lambda_n(j,q)$ such that $\lambda_n \in \mathcal{D}$, $\lambda_n < \lambda$ and, for k large enough, $A^{\lambda_n}_{h_k} \cap B^{\lambda_i}_k(j,q) \neq \emptyset$. Let

$$\lambda_m = \max\Big\{\lambda_n(j,q) : q \in \widetilde{M}(m),\ j = 1, \ldots, \frac{r}{2}(\lambda_i, q)\Big\}.$$

Then, for any $q \in \widetilde{M}(m)$ and any $l \in \{1, \ldots, \frac{r}{2}(\lambda_m, q)\}$, the inclusion $A^{\lambda_m}_{h_k} \subseteq A^\lambda_{h_k}$ implies

$$B^{\lambda_m}_k(l,q) \subseteq A^\lambda_{h_k} \cap B^{\lambda_i}_k(j,q) \subset B^{\lambda_i}_k(j,q) \tag{5.13}$$

for some $j \in \{1, \ldots, \frac{r}{2}(\lambda_i, q)\}$.

As in Step 3 of the proof of Lemma 4.3, the convergence of $[C^{\lambda_m}_k]$ to $[C^{\lambda_m}]$ up to the set P implies that, for any m and any $q \in \widetilde{M}(m)$,

$$\lim_{k \to +\infty} d_{\mathcal{H}}\Big(\bigcup_{l=1}^{r/2} \partial B^{\lambda_m}_k(l,q), [C^{\lambda_m}] \cap \overline{R(q)}\Big) = 0, \tag{5.14}$$

where $r = r(\lambda_m, q)$. Moreover, using (5.10), if $B^{\lambda_m}_k(l,q) \subseteq B^{\lambda_i}_k(j,q)$, it follows that $\partial B^{\lambda_m}_k(l,q)$, $\partial B^{\lambda_i}_k(j,q)$ converge in the Hausdorff metric to the same compact set, that is the graph of a function of class $W^{2,2}$. Hence, using (5.13) and (5.14) we find that

$$\lim_{k \to +\infty} d_{\mathcal{H}}\Big(\partial A^\lambda_{h_k} \cap \overline{R(q)}, [C^{\lambda_m}] \cap \overline{R(q)}\Big) = 0$$

for any m and any $q \in \widetilde{M}(m)$. Then

$$\lim_{k \to +\infty} d_{\mathcal{H}}(\partial A^\lambda_{h_k} \setminus Q_{\rho_m}, [C^{\lambda_m}] \setminus Q_{\rho_m}) = 0 \qquad \text{for any } m \in \mathbb{N}. \tag{5.15}$$

As $m \to +\infty$, λ_m increases; we show that there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$ we have $\lambda_m = \lambda_{m_0} \in \mathcal{D}$. Let $m_0 \in \mathbb{N}$ be such that for any $m \geq m_0$ there exists $k_0 = k_0(\rho_m)$ such that, for any $k > k_0$, the set $A^{\lambda_i}_{h_k} \setminus Q_{\rho_m}$ has $\#C^{\lambda_i}/2$ connected components. If $m > m_0$ then $\lambda_m \geq \lambda_{m_0}$; let now $q \in \widetilde{M}(m)$ and $j \in \{1, \ldots, \frac{r}{2}(\lambda_i, q)\}$ be such that

$$R(q) \notin \{R(q)\}_{q \in \widetilde{M}(m_0)}, \qquad R(q) \supset A^\lambda_{h_k} \cap B^{\lambda_i}_k(j,q) \neq \emptyset.$$

Using (5.13) there exist $\mathcal{A}^{\lambda_i}_k$, $\mathcal{A}^{\lambda_m}_k$, connected components of $A^{\lambda_i}_{h_k} \setminus Q_{\rho_m}$, $A^{\lambda_m}_{h_k} \setminus Q_{\rho_m}$, respectively, such that

$$\mathcal{A}^{\lambda_m}_k \cap \mathcal{A}^{\lambda_i}_k \cap R(q) \neq \emptyset. \tag{5.16}$$

Then, using (5.11) and (5.12), there exists $q \in \widetilde{M}(m_0)$ such that (5.16) holds, from which it follows that there exists a connected component $\mathcal{A}^{\lambda_{m_0}}_k$ of $A^{\lambda_{m_0}}_{h_k} \setminus Q_{\rho_{m_0}}$ such that $\mathcal{A}^{\lambda_{m_0}}_k \cap \mathcal{A}^{\lambda_m}_k \neq \emptyset$. Again using (5.11) and (5.12), for any $q \in \widetilde{M}(m)$ such that $\mathcal{A}^{\lambda_m}_k \cap R(q) \neq \emptyset$ we have $\mathcal{A}^{\lambda_{m_0}}_k \cap R(q) \neq \emptyset$. Hence we may choose $\lambda_m = \lambda_{m_0}$, with $\lambda_{m_0} < \lambda$. Then, by using (5.15) and letting $\rho_m \to 0^+$, there exists a sequence of compact sets $\{\mathcal{K}^\lambda_k\}_k$ such that $\mathcal{K}^\lambda_k \subseteq \partial A^\lambda_{h_k}$, $\lim_k d_{\mathcal{H}}(\mathcal{K}^\lambda_k, [C^{\lambda_{m_0}}]) = 0$, and the maximum distance of $\partial A^\lambda_{h_k} \setminus \mathcal{K}^\lambda_k$ from the set P goes to zero. The proof of the compactness property of the level sets A^λ_h is then completed setting $C^\lambda = C^{\lambda_{m_0}}$.

STEP 6: *proof of condition* (ii) *in Definition* 3.7.

Let $\lambda_i, \lambda_m \in \mathcal{D}$ be such that $\lambda_m < \lambda_i$; then $[C^{\lambda_m}] \subseteq [C^{\lambda_i}]$. For a given $\rho > 0$ let $\mathcal{A}_k^{\lambda_i}$, $\mathcal{A}_k^{\lambda_m}$ be connected components respectively of $A_{h_k}^{\lambda_i} \setminus Q_\rho$, $A_{h_k}^{\lambda_m} \setminus Q_\rho$ such that $\mathcal{A}_k^{\lambda_i} \cap \mathcal{A}_k^{\lambda_m} \neq \emptyset$. If ρ is small enough, using (5.11) it follows that the sets $\partial\mathcal{A}_k^{\lambda_i}$ and $\partial\mathcal{A}_k^{\lambda_m}$ converge in the Hausdorff metric, as $k \to +\infty$, to the same compact set $[\gamma] \setminus Q_\rho$, where $[\gamma] = [\gamma']$ and $\gamma \in C^{\lambda_i}$, $\gamma' \in C^{\lambda_m}$. Then, by letting $\rho \to 0^+$, we have

$$[C^{\lambda_i}] \cap [C^{\lambda_m}] = \bigcup\{[\gamma^j] : \gamma^j \in \widehat{C}^{\lambda_i}\}, \tag{5.17}$$

where $\widehat{C}^{\lambda_i}$ is a subfamily of C^{λ_i}. By Step 5 above, for any $\lambda \in (0,1)$ there exists $\lambda_p \in \mathcal{D}$ such that the sequence of compact sets $\mathcal{K}_k^\lambda \subseteq \partial A_{h_k}^\lambda$ converges to $[C^{\lambda_p}]$ in the Hausdorff metric. Then, using (5.17), there exists $\lambda_0 \in \mathcal{D}$ such that

$$[C^{\lambda_0}] = \bigcap\{[C^\lambda] : 0 < \lambda < 1\}.$$

Since $P(C^{\lambda_0}) \subseteq P$, the proof is completed by setting $C = C^{\lambda_0}$.

STEP 7: *compactness properties for u_h.*

For any $\lambda \in (0,1)$ we have

$$M \geq \int_\Omega s_h^2 |\nabla u_h|^2 \, dx \geq \lambda^2 \int_\Omega (1 - \chi_{A_h^\lambda}) |\nabla u_h|^2 \, dx$$

since $\lambda^2 \leq s_h^2$ for any $x \notin A_h^\lambda$. Hence, for any $\theta \in (0,1)$ and for any $\lambda \in (0,1)$, we get

$$\int_\Omega (1 - \chi_{A_h^\lambda \cup D_h^\theta}) |\nabla u_h|^2 \, dx \leq \frac{M}{\lambda^2} < +\infty, \qquad \text{for all } h \in \mathbb{N}. \tag{5.18}$$

Moreover, using (5.2) and (5.6), we find

$$\operatorname{meas}(A_h^\lambda \cup D_h^\theta) \leq \Big(\frac{1}{(1-\theta)^2} + \frac{1}{(1-\lambda)^2} \Big) \mu_h M = o(1)$$

as $h \to +\infty$, for any $\theta, \lambda \in (0,1)$.

Since $P = P^{\theta_0}$ and $C = C^{\lambda_0}$, using the results previously obtained, there exist subsequences $\{s_{h_k}\}_k$ and $\{w_{h_k}\}_k$ with the following properties: the maximum distance of $\partial D_{h_k}^{\theta_0}$ from the set P goes to zero, there exists a sequence $\{C_k^{\lambda_0}\}_k$ of disjoint families of simple curves, such that $[C_k^{\lambda_0}] \subseteq \partial A_{h_k}^{\lambda_0}$, $\{[C_k^{\lambda_0}]\}_k$ converges to $[C]$ up to the set P, and the maximum distance of $\partial A_{h_k}^{\lambda_0} \setminus [C_k^{\lambda_0}]$ from P goes to zero. Then, by using (5.18) with $\lambda = \lambda_0$ and $\theta = \theta_0$, and by repeating the same arguments of the proof of Theorem 4.1, the compactness property of the functions u_h follows. This yields condition (i) in Definition 3.7 and concludes the proof of the theorem. ■

Remark 5.2. By using the method of proof of Theorem 5.1, an analogous equicoercivity result can be obtained in the two-dimensional case for the families of functionals considered in [9].

5.2 Γ-convergence: lower bound

We now prove the liminf inequality (3.7).

Theorem 5.3. *Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every triplet $(u, C, P) \in X(\Omega)$ and for every sequence $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ converging weakly to (u, C, P), we have*

$$\liminf_{h\to+\infty} \mathcal{G}_{\varepsilon_h}(u_h, s_h, w_h) \geq \mathcal{G}(u, C, P).$$

Proof. Let $(u, C, P) \in X(\Omega)$ and let $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ converge weakly to (u, C, P). Possibly extracting a subsequence we may assume that

$$\lim_{h\to+\infty} \mathcal{G}_{\varepsilon_h}(u_h, s_h, w_h) = \liminf_{h\to+\infty} \mathcal{G}_{\varepsilon_h}(u_h, s_h, w_h) < +\infty.$$

Using Definition 3.7(i), the inequality

$$\liminf_{h\to+\infty} \int_\Omega |u_h - g|^2\, dx \geq \int_\Omega |u - g|^2\, dx \tag{5.19}$$

follows as in the proof of Theorem 4.4. Let $\lambda \in (0, 1)$ and $\theta \in (0, 1)$; we have

$$\int_\Omega s_h^2 |\nabla u_h|^2\, dx \geq \lambda^2 \int_\Omega (1 - \chi_{A_h^\lambda \cup D_h^\theta}) |\nabla u_h|^2\, dx. \tag{5.20}$$

Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset\subset \Omega \setminus ([C^\lambda] \cup P^\theta)$ invading $\Omega \setminus ([C^\lambda] \cup P^\theta)$. By Theorem 5.1, possibly passing to a subsequence, $u_h \rightharpoonup u$ weakly in $W^{1,2}(\Omega_j)$. Using Definition 3.7(i), we have

$$\liminf_{h\to+\infty} \lambda^2 \int_\Omega (1 - \chi_{A_h^\lambda \cup D_h^\theta}) |\nabla u_h|^2\, dx \geq \lambda^2 \liminf_{h\to+\infty} \int_{\Omega_j} |\nabla u_h|^2\, dx \geq \lambda^2 \int_{\Omega_j} |\nabla u|^2\, dx,$$

for any j. Since $u \in W^{1,2}(\Omega \setminus [C])$ and $[C] \subseteq [C^\lambda]$, using (5.20) it follows

$$\liminf_{h\to+\infty} \int_\Omega s_h^2 |\nabla u_h|^2\, dx \geq \lambda^2 \int_{\Omega\setminus[C^\lambda]} |\nabla u|^2\, dx = \lambda^2 \int_{\Omega\setminus[C]} |\nabla u|^2\, dx.$$

By letting $\lambda \to 1$ we obtain

$$\liminf_{h\to+\infty} \int_\Omega s_h^2 |\nabla u_h|^2\, dx \geq \int_{\Omega\setminus[C]} |\nabla u|^2\, dx. \tag{5.21}$$

Using the coarea formula and Sard's Theorem as in (5.3) and (5.4), we get

$$\liminf_{h\to+\infty} \mathcal{G}^{(1)}_{\varepsilon_h}(w_h) \geq 2 \liminf_{h\to+\infty} \int_{(0,1)\setminus T_w} \sqrt{V(\theta)} \int_{\partial\{w_h<\theta\}} \Big(\frac{1}{\beta_h} + \beta_h \kappa^2\Big)\, d\mathcal{H}^1 d\theta.$$

Using Fatou's Lemma, Definition 3.7(i) and (4.21), we obtain

$$\begin{aligned}\liminf_{h\to+\infty} \mathcal{G}^{(1)}_{\varepsilon_h}(w_h) \;&\geq\; 2 \int_{(0,1)\setminus T_w} \sqrt{V(\theta)} \liminf_{h\to+\infty} \int_{\partial D_h^\theta} \Big(\frac{1}{\beta_h} + \beta_h \kappa^2\Big)\, d\mathcal{H}^1 d\theta \\ &\geq\; 2 \int_{(0,1)\setminus T_w} 4\pi \# P^\theta \sqrt{V(\theta)} d\theta,\end{aligned}$$

from which, since $P \subseteq P^\theta$, we get

$$\liminf_{h\to+\infty} \mathcal{G}^{(1)}_{\varepsilon_h}(w_h) \geq 4\pi b_0 \# P. \tag{5.22}$$

Let $\theta \in (0,1)$; again using the coarea formula and Sard's Theorem as in (5.7) and (5.8), we obtain

$$\liminf_{h\to+\infty} \mathcal{G}^{(2)}_{\varepsilon_h}(s_h, w_h) \geq 2 \liminf_{h\to+\infty} \int_{(0,1)\setminus T_s} \sqrt{V(\lambda)} \int_{\partial A_h^\lambda} (1-\chi_{D_h^\theta})\theta^2(1+\kappa^2)\, d\mathcal{H}^1 d\lambda,$$

from which, using Fatou's Lemma, we get

$$\liminf_{h\to+\infty} \mathcal{G}^{(2)}_{\varepsilon_h}(s_h, w_h) \geq 2\theta^2 \int_{(0,1)\setminus T_s} \sqrt{V(\lambda)} \liminf_{h\to+\infty} \int_{\partial A_h^\lambda} (1-\chi_{D_h^\theta})(1+\kappa^2)\, d\mathcal{H}^1 d\lambda. \tag{5.23}$$

Hence, there exists a negligible set $F_s^\theta \supseteq T_s$ such that estimate (5.9) holds. Let $\lambda \in (0,1)\setminus F_s^\theta$; by Remark 4.2, using (5.6) and (5.9), up to the extraction of a subsequence possibly depending on λ and θ, it follows that there exist a sequence $\{C_h^{\lambda,\theta}\}_h$ of disjoint families of simple curves and an admissible family $C^{\lambda,\theta}$ of curves in Ω with $P(C^{\lambda,\theta}) \subseteq P^\theta$, such that $[C_h^{\lambda,\theta}] \subseteq \partial A_h^\lambda$, $\{[C_h^{\lambda,\theta}]\}_h$ converges to $[C^{\lambda,\theta}]$ up to the set P^θ, and the maximum distance of $\partial A_h^\lambda \setminus [C_h^{\lambda,\theta}]$ from P^θ goes to zero as $h \to +\infty$.

Arguing as in (4.23) we then obtain

$$\liminf_{h\to+\infty} \int_{\partial A_h^\lambda} (1-\chi_{D_h^\theta})(1+\kappa^2)\, d\mathcal{H}^1 \geq \sum_{\gamma \in C^{\lambda,\theta}} \big(K(\gamma)+L(\gamma)\big). \tag{5.24}$$

Let $\rho > 0$, $Q_{\rho,\theta} = \bigcup\{B_\rho(x) : x \in P^\theta\}$, and let $\mathcal{A}_h^{\lambda,\theta}$ be a connected component of $A_h^\lambda \setminus Q_{\rho,\theta}$. Then, by Step 3 of Lemma 4.3, there exist pairs of curves $\gamma_h^i, \gamma_h^j \in C_h^{\lambda,\theta}$ and $\gamma^i, \gamma^j \in C^{\lambda,\theta}$ such that, possibly passing to a subsequence, the following properties hold for h large enough:

$$\partial\mathcal{A}_h^{\lambda,\theta} \setminus \overline{Q}_{\rho,\theta} = ([\gamma_h^i] \cup [\gamma_h^j]) \setminus \overline{Q}_{\rho,\theta} \text{ for all } \rho > 0,\ \gamma_h^i \rightharpoonup \gamma^i,\ \gamma_h^j \rightharpoonup \gamma^j \text{ weakly in } W^{2,2}, \tag{5.25}$$

the endpoints of γ_h^i, γ_h^j belong to $Q_{\rho,\theta}$, and

$$[\gamma^i] = [\gamma^j], \qquad K(\gamma^i) + L(\gamma^i) = K(\gamma^j) + L(\gamma^j). \tag{5.26}$$

If $\theta = \theta_0$ we may set $C^{\lambda,\theta_0} = C^\lambda$. Hence, by Definition 3.7(i), it follows that $[C^{\lambda,\theta}] = [C^\lambda]$ for any $\theta \in (0,1)$. Since $P = P^{\theta_0} \subseteq P^\theta$, if $\theta = \theta_0$ we set $Q_{\rho,\theta_0} = Q_\rho$. Then, for each connected component $\mathcal{A}_h^\lambda$ of $A_h^\lambda \setminus Q_\rho$, by (5.25) and (5.26), for h large enough we may write

$$\partial\mathcal{A}_h^\lambda \setminus \overline{Q}_{\rho,\theta} = \bigcup \left\{\partial\mathcal{A}_h^{\lambda,\theta} \setminus \overline{Q}_{\rho,\theta} : \mathcal{A}_h^{\lambda,\theta} \subseteq \mathcal{A}_h^\lambda\right\} \text{ for all } \rho > 0,\ \theta \in (0,1),$$

from which, again using (5.25) and letting $\rho \to 0^+$, it follows

$$\sum_{\gamma \in C^{\lambda,\theta}} \big(K(\gamma)+L(\gamma)\big) = \sum_{\gamma \in C^\lambda} \big(K(\gamma)+L(\gamma)\big). \tag{5.27}$$

Hence, using (4.12), there exists an admissible family $\widetilde{C}^\lambda$ of curves in Ω such that $[\widetilde{C}^\lambda] = [C^\lambda]$, $P(\widetilde{C}^\lambda) \subseteq P$, and

$$\sum_{\gamma \in C^\lambda} \big(K(\gamma) + L(\gamma)\big) = 2 \sum_{\gamma \in \widetilde{C}^\lambda} \big(K(\gamma) + L(\gamma)\big). \tag{5.28}$$

As in Step 6 of Theorem 5.1 there exists $\lambda_0 \in \mathcal{D}$ such that $C = C^{\lambda_0}$. Let now $\lambda > \lambda_0$ and let $\gamma \in \widetilde{C}^\lambda$ be the weak limit in $W^{2,2}$ of $\gamma_h \in C_h^\lambda$, up to the extraction of a subsequence. Then, using (5.25), for any h there exists a connected component $\mathcal{A}_h^\lambda$ of $A_h^\lambda \setminus Q_\rho$ such that $[\gamma_h] \setminus Q_\rho \subseteq \partial \mathcal{A}_h^\lambda$. Let $\mathcal{A}_h^{\lambda_0}$ be a connected component of $A_h^{\lambda_0} \setminus Q_\rho$ such that $\mathcal{A}_h^{\lambda_0} \cap \mathcal{A}_h^\lambda \neq \emptyset$. Since $\mathcal{A}_h^{\lambda_0} \subseteq \mathcal{A}_h^\lambda$, using (5.11) and arguing as in the proof of Lemma 4.3, there exist $\gamma' \in C^{\lambda_0}$ and a sequence of curves $\{\gamma'_h\}_h$ with $\gamma'_h \in C_h^{\lambda_0}$ for any h such that, up to a subsequence, $\gamma'_h \rightharpoonup \gamma'$ in $W^{2,2}$ and

$$[\gamma] = [\gamma'], \qquad K(\gamma) + L(\gamma) = K(\gamma') + L(\gamma').$$

Hence, since $C = C^{\lambda_0}$, for any $\lambda > \lambda_0$ there exist a subfamily $\widehat{C}^\lambda$ of $\widetilde{C}^\lambda$ and a subfamily $\widehat{C}$ of C such that $[\widehat{C}] = [C]$, $P(\widehat{C}) \subseteq P$, and

$$\sum_{\gamma \in \widetilde{C}^\lambda} \big(K(\gamma) + L(\gamma)\big) \geq \sum_{\gamma \in \widehat{C}^\lambda} \big(K(\gamma) + L(\gamma)\big) = \sum_{\gamma \in \widehat{C}} \big(K(\gamma) + L(\gamma)\big). \tag{5.29}$$

Let $\lambda \leq \lambda_0$; since the inclusion $A_h^\lambda \subseteq A_h^{\lambda_0}$ implies $[C^\lambda] \subseteq [C^{\lambda_0}]$ and Definition 3.7(ii) implies $[C^{\lambda_0}] = [C] \subseteq [C^\lambda]$, we have $[C^\lambda] = [C]$. Then, arguing as before, for any $\lambda < \lambda_0$ there exist subfamilies of curves $\widehat{C}^\lambda$ and $\widehat{C}$ such that $[\widehat{C}] = [C]$, $P(\widehat{C}) \subseteq P$, and inequality (5.29) holds. Collecting (5.23), (5.24) and (5.27) we obtain for any $\theta \in (0,1)$:

$$\liminf_{h \to +\infty} \mathcal{G}^{(2)}_{\varepsilon_h}(s_h, w_h) \geq 2\theta^2 \int_{(0,1) \setminus T_s} \sqrt{V(\lambda)} \sum_{\gamma \in C^\lambda} \big(K(\gamma) + L(\gamma)\big) d\lambda,$$

from which, using (5.28) and (5.29), it follows

$$\liminf_{h \to +\infty} \mathcal{G}^{(2)}_{\varepsilon_h}(s_h, w_h) \geq 2 b_0 \theta^2 \sum_{\gamma \in \widehat{C}} \big(K(\gamma) + L(\gamma)\big).$$

By letting $\theta \to 1$ we get

$$\liminf_{h \to +\infty} \mathcal{G}^{(2)}_{\varepsilon_h}(s_h, w_h) \geq 2 b_0 \sum_{\gamma \in \widehat{C}} \big(K(\gamma) + L(\gamma)\big). \tag{5.30}$$

Since $[\widehat{C}] = [C]$ and $P(\widehat{C}) \subseteq P$, using (3.6) and collecting the inequalities (5.19), (5.21-5.22) and (5.30), we find

$$\begin{aligned} \liminf_{h \to +\infty} \mathcal{G}_{\varepsilon_h}(u_h, s_h, w_h) &\geq \int_{\Omega \setminus [C]} |\nabla u|^2 \, dx + \mathcal{F}(\widehat{C}, P) + \int_\Omega |u - g|^2 \, dx \\ &\geq \int_{\Omega \setminus [C]} |\nabla u|^2 \, dx + \mathcal{F}_0(C, P) + \int_\Omega |u - g|^2 \, dx = \mathcal{G}(u, C, P), \end{aligned}$$

which is the desired result. ■

6 Γ-convergence: upper bounds

In this section we prove the limsup inequalities (3.3) and (3.8) of the Γ-convergence of the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{G}_\varepsilon$ to the functional $\mathcal{G}$, respectively. We recall that a set Ω is star-shaped if there exists $x_0 \in \Omega$ such that $\xi(x - x_0) + x_0 \in \Omega$ for any $x \in \Omega$ and any $\xi \in [0, 1]$.

6.1 Upper bound for functionals defined on sets

First we prove in the following lemma that the families of curves satisfying the finiteness property are dense in the class of the admissible families of curves in Ω. Such a density result has been proved by Bellettini and Mugnai in [10] for admissible families of regular closed curves. In this case they also show that the approximating families C_h have traces contained in the trace of the target family C. In the present paper we give a different proof, without such a requirement, but that takes into account the presence of the endpoints $P(C)$. In the proof we will make use of some of the results in [10].

Lemma 6.1. *Let Ω be a star-shaped bounded open set in $\mathbb{R}^2$ and let $C = \{\gamma^1, \dots, \gamma^m\}$ be an admissible family of curves in Ω. Then there exist an admissible family $\widehat{C} = \{\widehat{\gamma}^1, \dots, \widehat{\gamma}^m\}$ of curves in Ω such that $[\widehat{C}] = [C]$, $P(\widehat{C}) = P(C)$,*

$$\sum_{\widehat{\gamma} \in \widehat{C}} \big(K(\widehat{\gamma}) + L(\widehat{\gamma})\big) = \sum_{\gamma \in C} \big(K(\gamma) + L(\gamma)\big), \tag{6.1}$$

and a sequence $\{C_h\}_h$ of admissible families of curves in Ω such that $C_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ and the following properties hold:

(i) *C_h satisfies the finiteness property for any h;*

(ii) *$[C_h] \subset \Omega$ and $\#P(C_h) = \#P(C)$ for any h;*

(iii) *$P(C_h)$ converges to $P(C)$;*

(iv) *the sequence $\{\gamma_h^i\}_h$, parameterized on a fixed interval, converges strongly in $W^{2,2}$ to $\widehat{\gamma}^i$ for any $i = 1, \dots, m$.*

Proof. We first assume that $[C] \subset \Omega$.

STEP 1: *In order to construct locally the family C_h we cover $[C] \setminus P(C)$ with a finite number of open sets. We use open rectangles to cover $[C] \setminus P(C)$ away from $P(C)$ and conical sets close to $P(C)$.*

For any point $q \in [C] \setminus P(C)$ let $R(q)$ be an open rectangle centered at q, having two sides parallel to the tangent line $T_{[C]}(q)$ of $[C]$ at q, and such that each curve $\gamma \in C$ meeting $R(q)$ satisfies properties **(a)–(c)** in Step 1 of the proof of Lemma 4.3 (that will be used also in the present proof with the same notation) and

(d) if I_j are as in (a), then $q \in \gamma(I_j)$ for any j (in particular, if $H = 1$ in (a) this reduces to $q \in [\gamma]$).

Such a rectangle exists, since C is an admissible family of curves.

Let $v, \tau \in \mathbb{R}^2$ with $|\tau| = 1$, $\theta \in (0, \pi/2)$ and $a > 0$; we define the *open finite half-cone* $\Lambda \subset \mathbb{R}^2$, with vertex at v, radius a and axis in the direction of τ, as the set

$$\Lambda = \{x \in \mathbb{R}^2 : |x - v| \cos\theta < \langle x - v, \tau\rangle < a\}.$$

Let now $p \in P(C)$ and let $\gamma \in C$ be a curve having p as an endpoint. If γ is not closed let $c = \gamma^{-1}(p)$ and let $\tau(p)$ be the tangent unit vector of γ at p, defined as $\lim_{t\to c} \dot\gamma(t)/|\dot\gamma(t)|$. If γ is closed the two tangent unit vectors of γ at p are defined analogously. Then the family C has a finite number of tangent vectors at p. For any of such vectors $\tau(p)$ let $\Lambda(p) \subset \mathbb{R}^2$ be an open finite half-cone, with vertex at p and axis in the direction of $\tau(p)$, such that each curve $\gamma \in C$ meeting $\Lambda(p)$ has the following properties:

(e) p is an endpoint of γ and $\tau(p)$ is a tangent unit vector of γ at p;

(f) γ is a cartesian graph in $\Lambda(p)$ of a function of class $W^{2,2}$ with respect to the axis of the half-cone;

(g) γ does not meet the lateral sides of $\Lambda(p)$.

Such a finite half-cone exists, since C is an admissible family of curves.

We cover the set $[C]$ with a family of open rectangles $\{R(q)\}_q$ and a finite family of half-cones $\{\Lambda(p_j)\}_{j=1}^N$ having the above properties. Since $[C]$ is a compact set, there exists a finite set of points $\{q_1, \ldots, q_M\} \subset [C] \setminus P(C)$ such that the finite subfamily of open rectangles $\{R(q_i)\}_{i=1}^M$ and the half-cones cover $[C]$.

Step 2: *The construction of $\{C_h\}$ by finite induction on curves $\{C_h^i\}$ $i \in \{1, \ldots, M\}$ will be obtained by reparameterizing locally the curves in $\{C_h^{i-1}\}$ as ordered graphs and subsequently splitting these graphs. The parameter h will enter in this splitting in such a way that C_h converges to C. In this step we describe the inductive hypotheses.*

For any $h \in \mathbb{N}$ let $C_h^0 = C$, let $1 \le i \le M$, and suppose that C_h^{i-1} has been defined. Then C_h^i is obtained by modifying C_h^{i-1} only on $R(q_i)$, i.e. $[C_h^i] \setminus R(q_i) = [C_h^{i-1}] \setminus R(q_i)$, in such a way that the following properties are satisfied:

(i′) C_h^i is an admissible family consisting of m curves;

(ii′) for any $j = 1, \ldots, M$, each curve $\gamma_h \in C_h^i$ such that $[\gamma_h] \cap R(q_j) \neq \emptyset$ satisfies properties (a)-(d);

(iii′) there exists a finite set of points F_i such that the set $[C_h^i] \cap R(q_i) \setminus F_i$ can be written locally as the graph of a function of class $W^{2,2}$;

Besides these geometric conditions, in which h enters only as a parameter, we will also have the following convergence property:

(iv′) the curves of C_h^i converge strongly in $W^{2,2}$ to the curves of a family $\widehat{C}^i$ such that $[\widehat{C}^i] = [C]$ and

$$\sum_{\widehat\gamma \in \widehat{C}^i} \big(K(\widehat\gamma) + L(\widehat\gamma)\big) = \sum_{\gamma \in C} \big(K(\gamma) + L(\gamma)\big). \tag{6.2}$$

Step 3: *Ordered local reparameterization of* C_h^{i-1}

With fixed $q = q_i$ let (ξ, η) denote the coordinates of points in a local coordinate system in $R(q)$, with the origin in q, such that the tangent line $T_{[C]}(q)$ of $[C]$ at q coincides with the ξ-axis and $R(q) = (-\xi_0, \xi_0) \times (-\eta_0, \eta_0)$. We denote

$$R^+(q) = \{(\xi, \eta) \in R(q) : \xi > 0\} \quad \text{and} \quad R^-(q) = \{(\xi, \eta) \in R(q) : \xi < 0\}. \tag{6.3}$$

We will work on $R^+(q_i)$, since the modification of C_h^{i-1} on $R^-(q_i)$ is analogous.

By the inductive properties of C_h^{i-1} we have

$$(C_h^{i-1})^{-1}(R^+(q_i)) = \bigcup\big\{(\gamma_h)^{-1}(R^+(q_i)) : \gamma_h \in C_h^{i-1}\big\} = \bigcup\{I_{kl} : 1 \le l \le n(k),\ 1 \le k \le m\}, \tag{6.4}$$

where $n(k) \in \mathbb{N} \cup \{0\}$ is the number of times that the k-th curve of C_h^{i-1} crosses $R^+(q_i)$ for $k = 1, \dots, m$, and I_{kl} are open pairwise disjoint intervals. Note that (6.4) holds also for $C = C_h^0$, so that the induction process may start. Then there exist r functions f_h^j, $j = 1, \dots, r$, of class $W^{2,2}(0, \xi_0)$, with $r = \sum_{k=1}^m n(k)$, such that each function f_h^j corresponds in a one-to-one way to an interval I_{kl} such that the graph of f_h^j is the image $\gamma_h(I_{kl})$ for some curve $\gamma_h \in C_h^{i-1}$. In particular we have

$$[C_h^{i-1}] \cap R^+(q_i) = \bigcup\{\mathrm{graph}(f_h^j) : 1 \le j \le r\},$$

and, the family C_h^{i-1} being admissible,

$$f_h^{j\prime}(\xi) = f_h^{k\prime}(\xi) \quad \text{whenever} \quad f_h^j(\xi) = f_h^k(\xi). \tag{6.5}$$

For any $\xi \in [0, \xi_0]$ let $\{j_1, \dots, j_r\}$ be a permutation dependent on ξ of $\{1, \dots, r\}$ such that

$$f_h^{j_1}(\xi) \le f_h^{j_2}(\xi) \le \cdots \le f_h^{j_r}(\xi).$$

Then we define r functions $g_h^j : [0, \xi_0] \to (-\eta_0, \eta_0)$, $j = 1, \dots, r$, by means of

$$g_h^1(\xi) = f_h^{j_1}(\xi), \dots, g_h^r(\xi) = f_h^{j_r}(\xi) \quad \text{for any } \xi \in [0, \xi_0]. \tag{6.6}$$

Using (6.5) it follows that the function g_h^j is continuous for any $j = 1, \dots, r$. Then we have

$$g_h^1(\xi) \le g_h^2(\xi) \le \cdots \le g_h^r(\xi) \quad \text{for any } \xi \in [0, \xi_0], \tag{6.7}$$

and

$$[C_h^{i-1}] \cap R(q_i) = \bigcup\{\mathrm{graph}(g_h^j) : 1 \le j \le r\}.$$

Using (6.5) it follows that $g_h^j \in W^{2,2}[0, \xi_0]$ for any $j = 1, \dots, r$ (see also [10] Lemma 4.3).

We now construct an admissible family $\widehat{C}_h^i$ of curves, having the same trace and the same number of curves as C_h^{i-1}, and such that the images in $R^+(q_i)$ of the curves of $\widehat{C}_h^i$ are given by the graphs of the functions g_h^j. By (6.6), for any $j \in \{1, \dots, r\}$, there exists an index $s_j \in \{1, \dots, r\}$ such that $f_h^j(\xi_0) = g_h^{s_j}(\xi_0)$ and $s_j \notin \{s_1, \dots, s_{j-1}\}$. Fix now $j \in \{1, \dots, r\}$ and let I_{kl} be such that the graph of f_h^j is the image $\gamma_h(I_{kl})$ for some curve $\gamma_h \in C_h^{i-1}$. Using (6.6) and property (d) we have $f_h^j(0) = g_h^{s_j}(0)$. Then we modify the curve γ_h only in the interval I_{kl} in order to obtain a new curve $\widehat{\gamma}_h$ whose image $\widehat{\gamma}_h(I_{kl})$ is given by graph$(g_h^{s_j})$. We repeat this construction for every $j \in \{1, \dots, r\}$.

The family C_h^{i-1} is modified in the rectangle $R^-(q_i)$ in an analogous way. We then obtain an admissible family $\widehat{C}_h^i$ of m curves such that

$$[\widehat{C}_h^i] = [C_h^{i-1}], \qquad P(\widehat{C}_h^i) = P(C_h^{i-1}) = P(C).$$

Moreover, using (6.6) and the locality of the weak derivatives of Sobolev functions, we have

$$\sum_{\widehat{\gamma}\in\widehat{C}_h^i} \big(K(\widehat{\gamma}) + L(\widehat{\gamma})\big) = \sum_{\gamma\in C_h^{i-1}} \big(K(\gamma) + L(\gamma)\big) \tag{6.8}$$

(see also the proof of [10] Lemma 3.9).

STEP 4: *Compatibility of the reparameterization of C_h^{i-1} with* **(iv′)**.

We now prove that the curves of $\widehat{C}_h^i$ converge strongly in $W^{2,2}$ to the curves of a family $\widehat{C}^i$ such that $[\widehat{C}^i] = [C]$ and equality (6.2) holds. If $i = 1$ the family $\widehat{C}_h^1$ does not depend on h, we have $[\widehat{C}_h^1] = [C]$ and the equality of the energies is given by (6.8). Then, for any $i = 2, \dots, M$, we set

$$Q_i = \bigcup_{j=1}^{i} R(q_j).$$

By using the inductive property (iii′) of C_h^{i-1} there exists a finite set of points Z_{i-1} such that the set $[C_h^{i-1}] \cap Q_{i-1} \setminus Z_{i-1}$ can be written locally as the graph of a function of class $W^{2,2}$. It follows that we may write

$$(C_h^{i-1})^{-1}(R^+(q_i) \cap Q_{i-1}) = \bigcup_{l=1}^{L} J_l,$$

where J_l are open pairwise disjoint intervals which satisfy the following properties (see also the Appendix):

(v′) for any $l, l' \in \{1, \dots, L\}$ either $C_h^{i-1}(J_l) \cap C_h^{i-1}(J_{l'}) = \emptyset$ or $C_h^{i-1}(J_l) = C_h^{i-1}(J_{l'})$;

(vi′) $x \notin C_h^{i-1}(J_l)$ for every $l \in \{1, \dots, L\}$ and for every point $x \in Z_{i-1}$.

For any $l \in \{1, \dots, L\}$ there exists an index $j_l \in \{1, \dots, r\}$ and an interval $(\xi_{l,1}, \xi_{l,2}) \subset [0, \xi_0]$ such that

$$C_h^{i-1}(J_l) = \{(\xi, \eta) \in R^+(q_i) \cap Q_{i-1} : \xi_{l,1} < \xi < \xi_{l,2}, \eta = f_h^{j_l}(\xi)\},$$

and the intervals J_l correspond to the intervals $(\xi_{l,1}, \xi_{l,2})$ in a one-to-one way. Using (6.6) and properties (v′), (vi′) it follows that the functions g_h^j can be chosen in such a way that, for any interval $(\xi_{l,1}, \xi_{l,2})$, the index j_l corresponds in a one-to-one way to an index $j'_l \in \{1, \dots, r\}$ such that

$$f_h^{j_l}(\xi) = g_h^{j'_l}(\xi) \qquad \text{for any } \xi \in (\xi_{l,1}, \xi_{l,2}). \tag{6.9}$$

An analogous result holds in $R^-(q_i)$. Moreover, by induction we have

$$[C_h^{i-1}] \setminus Q_{i-1} = [C] \setminus Q_{i-1},$$

so that

$$[\widehat{C}^i_h] \cap R(q_i) \setminus Q_{i-1} = [C] \cap R(q_i) \setminus Q_{i-1}. \tag{6.10}$$

Using the property (iv′) of the family C^{i-1}_h it follows that, up to a subsequence, the functions f^j_h converge strongly in $W^{2,2}$. Since the family C^{i-1}_h has been constructed by modifying the family C only on the set Q_{i-1}, using (6.9), (6.10) and the property (iv′) of C^{i-1}_h, it follows that the family $\widehat{C}^i_h$ converges strongly in $W^{2,2}$ to a family of curves $\widehat{C}^i$ with $[\widehat{C}^i] = [C]$. Equality (6.2) then follows from (6.8), the same equality for $\widehat{C}^{i-1}$, and the $W^{2,2}$ convergence of $\widehat{C}^i_h$.

STEP 5: *Definition of C^i_h by splitting the curves of $\widehat{C}^i_h$ in $R(q_i)$.*
We now define the family C^i_h in $R^+(q_i)$. Let $\nu \le M$ be an integer, let $\{q_{i_1}, \dots, q_{i_\nu}\}$ denote the subset of the points of $\{q_1, \dots, q_M\}$ that lie in $R^+(q_i)$, and let $\{\xi_{i_1}, \dots, \xi_{i_\nu}\}$ denote the respective ξ coordinates. Let $\phi : [0, \xi_0] \to [0, 1]$ denote a $\mathcal{C}^\infty$ function such that $\phi(\xi) > 0$ for any $\xi \notin \{0, \xi_{i_1}, \dots, \xi_{i_\nu}, \xi_0\}$ and ϕ vanishes at the points $\{0, \xi_{i_1}, \dots, \xi_{i_\nu}, \xi_0\}$ with all its derivatives. Then, for any $h, h' \in \mathbb{N}$ we define

$$\tilde{g}^j_{h,h'}(\xi) = g^j_h(\xi) + \frac{j}{h'r}\phi(\xi), \quad \text{for any } j = 1, \dots, r, \tag{6.11}$$

for any $\xi \in [0, \xi_0]$. For h' large enough we have $-\eta < \tilde{g}^j_{h,h'}(\xi) < \eta$ for any $\xi \in [0, \xi_0]$.

Fix $j \in \{1, \dots, r\}$ and let I_{kl} be such that the graph of g^j_h is the image $\widehat{\gamma}_h(I_{kl})$ of a curve $\widehat{\gamma}_h \in \widehat{C}^i_h$. We then modify the curve $\widehat{\gamma}_h$ only in the interval I_{kl} in order to obtain a new curve $\widehat{\gamma}_{h,h'}$ whose image $\widehat{\gamma}_{h,h'}(I_{kl})$ is given by graph$(\tilde{g}^j_{h,h'})$. We repeat this construction for every $j \in \{1, \dots, r\}$. Set

$$F^+_i = \{q_{i_1}, \dots, q_{i_\nu}\} \cup \big([\widehat{C}^i_h] \cap \partial R^+(q_i)\big), \tag{6.12}$$

which is a finite set of points.

We modify the family $\widehat{C}^i_h$ in the rectangle $R^-(q_i)$ in an analogous way, and we define a finite set of points F^-_i. We have thus obtained a family of m curves $C^i_{h,h'}$ by modifiying C^{i-1}_h only on $R(q_i)$ and such that, for any h, $C^i_{h,h'}$ converges strongly to $\widehat{C}^i_h$ in $W^{2,2}$ as $h' \to +\infty$. By using a diagonal argument we obtain a family C^i_h of curves which satisfies properties (i′) and (iv′). Using (6.7) and (6.11) property (ii′) is satisfied for h large enough; in particular, property (d) of the curves of C^i_h in the rectangles follows from (6.12). Moreover, again using (6.7) and (6.11) property (iii′) is satisfied by setting $F_i = F^+_i \cup F^-_i$.

By induction there exists a finite set of points $Z = Z_M$ such that the set $[C^M_h] \cap Q_M \setminus Z$ can be written locally as the graph of a function of class $W^{2,2}$. Hence the set

$$[C^M_h] \setminus \Big(Z \cup \bigcup_{j=1}^N \Lambda(p_j)\Big)$$

can also be written locally as the graph of a function of class $W^{2,2}$. Moreover, we can choose the half-cones $\Lambda(p_j)$ pairwise disjoint.

STEP 6: *Extension of the construction to the half-cones.*
We now construct the family of curves C_h by modifying C^M_h only on the half-cones, i.e.

$$[C_h] \setminus \Big(\bigcup_{j=1}^N \Lambda(p_j)\Big) = [C^M_h] \setminus \Big(\bigcup_{j=1}^N \Lambda(p_j)\Big),$$

in such a way that for any $j = 1, \dots, N$, there exists a finite set of points G_j such that the set $[C_h] \cap \Lambda(p_j) \setminus G_j$ can be written locally as the graph of a function of class $W^{2,2}$. Moreover $P(C_h) = P(C)$. The modification of C_h^M on the half-cones $\Lambda(p_j)$ is performed in the same way as the previous modification on the half-rectangles $R^+(q_i)$. Arguing as before, we then obtain a sequence of admissible families C_h of m curves converging strongly in $W^{2,2}$ to a family $\widehat{C}$ such that $[\widehat{C}] = [C]$, $P(\widehat{C}) = P(C)$ and equality (6.1) holds.

By construction, there exists a finite set of points F such that $[C_h] \setminus F$ can be written locally as the graph of a function of class $W^{2,2}$ for any h large enough, so that C_h satisfies the finiteness property for any h large enough. Since $[C] \subset \Omega$ we have also $P(C_h) = P(C)$ and $[C_h] \subset \Omega$ for large enough h. Hence, if $[C] \subset \Omega$ the lemma is proved.

Step 7: *We remove the assumption* $[C] \subset \Omega$.

Let Ω be star-shaped with respect to the point x_0, and let $\{c_k\}_k \subset (0,1)$ be a sequence of points such that $\lim_{k\to+\infty} c_k = 1$. We define a sequence C'_k of families of curves in the following way. For any curve $\gamma \in C$, with $\gamma : [a,b] \to \mathbb{R}^2$, we define

$$\gamma'_k(t) = x_0 + c_k(\gamma(t) - x_0) \qquad \text{for any } t \in [a,b].$$

Since Ω is star-shaped with respect to x_0 and $0 < c_k < 1$ for any k we have

$$[C'_k] = \bigcup\{[\gamma'_k] : \ \gamma'_k \in C'_k\} \subset \Omega \qquad \text{for any } k.$$

Since $c_k \to 1$ one can check that C'_k converges strongly in $W^{2,2}$ to C. The proof of the lemma then follows from the case $[C] \subset \Omega$ and a diagonal argument. ■

The lemma below follows from a result by Chambolle and Doveri ([18], Proposition 1 of Appendix).

Lemma 6.2. *Let $\mathcal{K} \subset \overline{\Omega}$ be a compact set, and let $\{\mathcal{K}_h\}_h \subset \overline{\Omega}$ be a sequence of compact sets converging to $\mathcal{K}$ in the Hausdorff metric such that $\sup_{h\in\mathbb{N}} \mathcal{H}^1(\mathcal{K}_h) < +\infty$. Assume that the sets $\mathcal{K}_h$ have for any h a fixed finite number of closed connected components. Let $u \in W^{1,2}(\Omega \setminus \mathcal{K})$. Then there exists a sequence $\{u_h\}_h$ with $u_h \in W^{1,2}(\Omega \setminus \mathcal{K}_h)$ for any h, such that $u_h \to u$ strongly in $L^2(\Omega)$ and $\nabla u_h \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^2)$.*

We now may prove the limsup inequality of Γ-convergence. Given a target triplet (u, C, P), with the aid of Lemma 6.1 (and some results in the Appendix) we may suppose that C is composed by disjoint curves. The approximating sets D_h, A_h are then constructed as suitable neighbourhoods of P and C, respectively. The functions u_h are then constructed with the help of Lemma 6.2 above.

Theorem 6.3 (upper bound). *Let Ω be a bounded, open and star-shaped set, and let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every triplet $(u, C, P) \in X(\Omega)$ there exists a sequence $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ converging weakly to (u, C^*, P^*) such that*

$$\limsup_{h\to+\infty} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) \leq \mathcal{G}(u, C, P), \tag{6.13}$$

where C^* and P^* are as in (3.1).

Proof. First assume that C^* satisfies the finiteness property and that $[C^*] \subset \Omega$.

STEP 1: *reduction to families of disjoint simple curves.*

By adapting a method introduced by Bellettini, Dal Maso and Paolini in [8] we may construct a sequence $C'_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ of families of simple curves of class $\mathcal{C}^\infty$ such that $\{\gamma_h^i\}_h$ converges strongly in $W^{2,2}$ to a curve γ^i for any $i = 1, \dots, m$, the family $C' = \{\gamma^1, \dots, \gamma^m\}$ is admissible and satisfies the finiteness property, C' and C^* are equivalent and the following properties hold for any h:

$$P(C'_h) = P(C'), \qquad \big([\gamma_h^i] \cap [\gamma_h^j]\big) \setminus P(C') = \emptyset \quad \text{for all } i, j,\ i \neq j. \tag{6.14}$$

The proof of the construction of this family is obtained by a careful modification of results in [8]. We have included a complete proof in the Appendix for the reader's convenience.

STEP 2: *Construction of the sequences of sets $\{A_h\}_h$ and $\{D_h\}_h$.*

Let $C'_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ with m independent of h. Since the curves γ_h^i are of class $\mathcal{C}^\infty$ and are converging strongly in $W^{2,2}$, for any $p \in P(C')$ and any curve $\gamma_h^i \in C'_h$ having p as an endpoint, the following properties hold for any h large enough:

(i) if γ_h^i is not a closed curve, then $[\gamma_h^i]$ intersects $\partial B_r(p)$ in only one point for any $r \leq \varepsilon_h$;

(ii) if γ_h^i is a closed curve, then $[\gamma_h^i]$ intersects $\partial B_r(p)$ in only two points for any $r \leq \varepsilon_h$.

Then we define

$$D_h = \bigcup \{B_{\varepsilon_h}(p) :\ p \in P^*\}. \tag{6.15}$$

Moreover, for h large enough and any regular closed curve $\gamma_h^i \in C'_h$ we have $[\gamma_h^i] \cap D_h = \emptyset$.

Since $\gamma_h^i \to \gamma^i$ strongly in $W^{2,2}$ for any $i = 1, \dots, m$, using (6.14) and properties (i) and (ii), we may find m sequences of sets $\{A_h^i\}_h \subset \mathcal{C}_c^\infty(\Omega)$ such that $\text{meas}(A_h^i) \to 0$ for any i, and the following properties hold for any $i = 1, \dots, m$ and for any h:

$$\begin{aligned} [\gamma_h^i] \setminus D_h \subset A_h^i, \qquad & \overline{A}_h^i \cap \overline{A}_h^j = \emptyset \quad \text{for all } i \neq j, \\ \partial A_h^i \setminus D_h \quad = \quad & [\gamma_h^{i+}] \cup [\gamma_h^{i-}], \end{aligned} \tag{6.16}$$

where γ_h^{i+} and γ_h^{i-} are simple curves of class $\mathcal{C}^\infty$ such that $[\gamma_h^{i+}] \cap [\gamma_h^{i-}] = \emptyset$, and $\gamma_h^{i+} \to \gamma^i$ and $\gamma_h^{i-} \to \gamma^i$ strongly in $W^{2,2}$. Then we set

$$A_h = \bigcup_{i=1}^m A_h^i,$$

and we have $A_h, D_h \in \mathcal{C}_c^\infty(\Omega)$ for h large enough.

STEP 3: *Construction of the sequence of functions $\{u_h\}_h$.*

Let $\mathcal{K}_h = \bigcup_{i=1}^m [\gamma_h^i] \setminus D_h$. The sequence of compact sets $\{\mathcal{K}_h\}_h$ converges in the Hausdorff metric to the set $[C'] = [C^*]$; moreover the number of connected components of $\mathcal{K}_h$ is m for all h and $\sup_{h \in \mathbb{N}} \mathcal{H}^1(\mathcal{K}_h) < +\infty$. Then, since $u \in W^{1,2}(\Omega \setminus [C^*])$, by Lemma 6.2 there exists a sequence $\{\widehat{u}_h\}_h$ with $\widehat{u}_h \in W^{1,2}(\Omega \setminus \mathcal{K}_h)$ for any h, such that $\widehat{u}_h \to u$ strongly in $L^2(\Omega)$ and $\nabla \widehat{u}_h \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^2)$.

Let $\{\rho_h\}_h$ be a sequence of positive numbers converging to zero such that

$$\{x \in \Omega : \text{dist}(x, \mathcal{K}_h) < \rho_h\} \subset A_h \quad \text{for any } h.$$

Let then $\{g_h\}_h \subset \mathcal{C}^\infty(\mathbb{R}^2)$ be a sequence of functions such that $0 \le g_h \le 1$ for any h and

$$g_h(x) = 1 \text{ on } \{x \in \Omega : \operatorname{dist}(x, \mathcal{K}_h) < \rho_h/2\}, \qquad g_h(x) = 0 \text{ on } \{x \in \Omega : \operatorname{dist}(x, \mathcal{K}_h) \ge \rho_h\}, \tag{6.17}$$

for any h. Then we define $u_h = (1 - g_h)\widehat{u}_h$ for any h so that $u_h \in W^{1,2}(\Omega)$.

Step 4: *Estimate of the upper bound.*

By construction we have $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ and $\{(u_h, A_h, D_h)\}_h$ converges weakly to (u, C^*, P^*).

Since $\widehat{u}_h \to u$ in $L^2(\Omega)$ and $\operatorname{meas}(A_h) \to 0$, using the definition of the function g_h, we have $\|u_h - \widehat{u}_h\|_{L^2} \to 0$ and

$$\lim_{h\to+\infty} \int_\Omega |u_h - g|^2\, dx = \int_\Omega |u - g|^2\, dx. \tag{6.18}$$

Since $\nabla\widehat{u}_h \to \nabla u$ in $L^2(\Omega;\mathbb{R}^2)$, $\operatorname{meas}(A_h \cup D_h) \to 0$, and $u_h(x) = \widehat{u}_h(x)$ if $x \notin A_h$, we have

$$\lim_{h\to+\infty} \int_\Omega (1 - \chi_{A_h\cup D_h})|\nabla u_h|^2\, dx = \lim_{h\to+\infty} \int_\Omega (1 - \chi_{A_h\cup D_h})|\nabla \widehat{u}_h|^2\, dx = \int_{\Omega\setminus[C^*]} |\nabla u|^2\, dx. \tag{6.19}$$

Using (6.15) we get

$$\lim_{h\to+\infty} \frac{1}{4\pi} \int_{\partial D_h} \Big(\frac{1}{\varepsilon_h} + \varepsilon_h \kappa^2\Big)\, d\mathcal{H}^1 = \lim_{h\to+\infty} \frac{1}{4\pi} \sum_{p\in P^*} \int_{\partial B_{\varepsilon_h}(p)} \Big(\frac{1}{\varepsilon_h} + \varepsilon_h\kappa^2\Big) d\mathcal{H}^1 = \#P^*. \tag{6.20}$$

Then, using (6.16), we have

$$\begin{aligned}
&\limsup_{h\to+\infty} \frac{1}{2} \int_{\partial A_h} (1 - \chi_{D_h})(1 + \kappa^2)\, d\mathcal{H}^1 \le \lim_{h\to+\infty} \frac{1}{2} \int_{\partial A_h \setminus D_h} (1 + \kappa^2)\, d\mathcal{H}^1 \\
&= \frac{1}{2} \lim_{h\to+\infty} \sum_{i=1}^m \big(K(\gamma_h^{i+}) + L(\gamma_h^{i+}) + K(\gamma_h^{i-}) + L(\gamma_h^{i-})\big) = \sum_{\gamma\in C'} \big(K(\gamma) + L(\gamma)\big).
\end{aligned} \tag{6.21}$$

Since C' and C^* are equivalent, collecting (6.18)–(6.21), we obtain

$$\limsup_{h\to+\infty} \mathcal{E}_{\varepsilon_h}(u_h, A_h, D_h) \le \int_{\Omega\setminus[C^*]} |\nabla u|^2\, dx + \mathcal{F}(C^*, P^*) + \int_\Omega |u - g|^2\, dx = \mathcal{G}(u, C, P). \tag{6.22}$$

Step 5: *We remove the assumptions that C^* satisfies the finiteness property and $[C^*] \subset \Omega$.*

By Lemma 6.1 there exist an admissible family C^0 of curves in Ω such that $[C^0] = [C^*]$, $P(C^0) = P(C^*)$,

$$\sum_{\gamma\in C^0} \big(K(\gamma) + L(\gamma)\big) = \sum_{\gamma\in C^*} \big(K(\gamma) + L(\gamma)\big),$$

and a sequence $\{C^0_k\}_k$ of admissible families of curves in Ω satisfying the properties (i)-(iv) of Lemma 6.1. Then we have

$$\lim_{k\to+\infty} \mathcal{F}(C^0_k, P^0_k) = \mathcal{F}(C^*, P^*), \tag{6.23}$$

where $P_k^0 = P(C_k^0) \cup (P^* \setminus P(C^*))$.

Since the sequence of compact sets $\{[C_k^0]\}_k$ converges in the Hausdorff metric to the set $[C^0] = [C^*]$, the number of connected components of $[C_k^0]$ and $\mathcal{H}^1([C_k^0])$ are uniformly bounded with respect to k, and $u \in W^{1,2}(\Omega \setminus [C^*])$, by Lemma 6.2 there exists a sequence $\{u_k^0\}_k$ with $u_k^0 \in W^{1,2}(\Omega \setminus [C_k^0])$ for any k, such that

$$\lim_{k\to+\infty} \int_\Omega |u_k^0 - g|^2\, dx = \int_\Omega |u-g|^2\, dx, \qquad \lim_{k\to+\infty} \int_{\Omega\setminus[C_k^0]} |\nabla u_k^0|^2\, dx = \int_{\Omega\setminus[C^*]} |\nabla u|^2\, dx. \tag{6.24}$$

Using (6.23) and (6.24) we get

$$\lim_{k\to+\infty} \Big(\int_{\Omega\setminus[C_k^0]} |\nabla u_k^0|^2\, dx + \mathcal{F}(C_k^0, P_k^0) + \int_\Omega |u_k^0 - g|^2\, dx \Big) = \mathcal{G}(u, C, P),$$

from which, since C_k^0 satisfies the finiteness property and $[C_k^0] \subset \Omega$, using (6.22) and a diagonal argument, the statement of the theorem follows. ■

6.2 Upper bound for functionals defined on smooth functions

Theorem 6.4. *Let Ω be a star-shaped bounded open set, and let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every triplet $(u, C, P) \in X(\Omega)$ there exists a sequence $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ converging weakly to (u, C^*, P^*) such that*

$$\limsup_{h\to+\infty} \mathcal{G}_{\varepsilon_h}(u_h, s_h, w_h) \le \mathcal{G}(u, C, P),$$

where C^* and P^* are as in (3.1).

Proof. Step 1: *We use Theorem* 6.3 *to find recovery sets by means of which to construct sequences s_h, w_h.*

We set $\beta_{\varepsilon_h} = \beta_h$ and $\mu_{\varepsilon_h} = \mu_h$. Let $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ be the sequence converging weakly to (u, C^*, P^*) and satisfying (6.13) constructed in the proof of Theorem 6.3. In particular, we have

$$D_h = \bigcup \{B_{\beta_h}(p_h) : p_h \in P_h\}, \tag{6.25}$$

where $\{P_h\}_h$ is a sequence of finite sets of points converging to P^*, and β_h satisfies (3.5). Since $A_h, D_h \in \mathcal{C}_c^\infty(\Omega)$ for any h, we may choose the sequence $\{P_h\}_h$ in such a way that

$$\{x \in \Omega : \mathrm{dist}(x, A_h) < 2\varepsilon_h |\log \varepsilon_h|\} \subset\subset \Omega, \quad \{x \in \Omega : \mathrm{dist}(x, D_h) < 2\varepsilon_h |\log \varepsilon_h|\} \subset\subset \Omega,$$

for h large enough. Moreover, as in (6.16), for any h we may write

$$\partial A_h \setminus D_h^0 = \bigcup_{i=1}^m [\gamma_h^i], \qquad D_h^0 = \{x \in \Omega : \delta_{D_h}(x) < -2\varepsilon_h |\log \varepsilon_h|\}, \tag{6.26}$$

where γ_h^i, $i = 1, \ldots, m$, are simple curves such that $[\gamma_h^i] \cap [\gamma_h^j] = \emptyset$ for any $i \ne j$, and the family $\{\gamma_h^1, \ldots, \gamma_h^m\}$ converges strongly in $W^{2,2}$ to a family of curves having the same trace of C^*. The curves γ_h^i can be chosen in such a way that

$$\{x \in \Omega : \mathrm{dist}(x, [\gamma_h^i]) \le 2\varepsilon_h |\log \varepsilon_h|\} \cap \{x \in \Omega : \mathrm{dist}(x, [\gamma_h^j]) \le 2\varepsilon_h |\log \varepsilon_h|\} = \emptyset$$

for any $i \neq j$, $i,j \in \{1,\ldots,m\}$, and for large enough h. Furthermore, we choose the sequence $\{A_h\}_h$ in such a way that

$$\lim_{h\to+\infty} \frac{\mathrm{meas}(A_h)}{\beta_h} = 0. \tag{6.27}$$

STEP 2: *Upper bound by 'mollification' of the sets A_h, D_h.*
Let $\eta \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ be defined by $\eta(t) = \frac{1}{2}\left(1+\tanh\frac{t}{2}\right)$. Note that

$$\eta'(t) = \sqrt{V(\eta(t))}. \tag{6.28}$$

It is well known that η is an 'optimal profile' for the transition energy in the Modica and Mortola perimeter approximation (see also [15] Section 3.2). For any $h \in \mathbb{N}$ let $\zeta_h : [0,+\infty) \to [0,1]$ be a $\mathcal{C}^\infty$ function such that

$$\zeta_h = 1 \quad \text{on} \quad [0, |\log\varepsilon_h|], \qquad \zeta_h = 0 \quad \text{on} \quad [2|\log\varepsilon_h|, +\infty),$$

$$\zeta_h' < 0 \quad \text{in} \quad (|\log\varepsilon_h|, 2|\log\varepsilon_h|), \qquad \|\zeta_h'\|_{L^\infty([|\log\varepsilon_h|,2|\log\varepsilon_h|])} = O(1/|\log\varepsilon_h|).$$

As in [9], we set

$$\eta_h(t) = \begin{cases} \eta\Big(\dfrac{t}{\varepsilon_h}\Big)\zeta_h\Big(\dfrac{t}{\varepsilon_h}\Big) + 1 - \zeta_h\Big(\dfrac{t}{\varepsilon_h}\Big) & \text{if } t \geq 0 \\ 1 - \eta_h(-t) & \text{if } t < 0. \end{cases}$$

Let $A \subset \mathbb{R}^2$ and let δ_A denote the *signed distance function* from ∂A negative inside A:

$$\delta_A(x) = \mathrm{dist}(x, A) - \mathrm{dist}(x, \mathbb{R}^2 \setminus A).$$

Moreover we recall the notation $(A)_r = \{x \in \mathbb{R}^2 : \mathrm{dist}(x,A) < r\}$.

We now construct the sequences of functions $\{s_h\}_h$ and $\{w_h\}_h$. We set

$$s_h(x) = \eta_h(\delta_{A_h}(x)), \qquad w_h(x) = \eta_h(\delta_{D_h}(x)) \qquad \text{for all } x \in \Omega. \tag{6.29}$$

Since $A_h, D_h \in \mathcal{C}^\infty_c(\Omega)$ we have $(u_h, s_h, w_h) \in W(\Omega)$ for h large enough. Moreover, one can check that the sequence $\{(u_h, s_h, w_h)\}_h$ converges weakly to (u, C^*, P^*). Set now

$$D_h^1 = \{x \in \Omega : |\delta_{D_h}(x)| < \varepsilon_h|\log\varepsilon_h|\},\ D_h^2 = \{x \in \Omega : \varepsilon_h|\log\varepsilon_h| < |\delta_{D_h}(x)| < 2\varepsilon_h|\log\varepsilon_h|\}.$$

Using the definition of $\mathcal{G}^{(1)}_\varepsilon$ we have

$$\begin{aligned} \mathcal{G}^{(1)}_{\varepsilon_h}(w_h) &= \int_{D_h^1}\Big(\frac{1}{\beta_h} + \beta_h\kappa^2(\nabla w_h)\Big)\mathcal{H}^1_{\varepsilon_h}(w_h, \nabla w_h)\,dx \\ &\quad + \int_{D_h^2}\Big(\frac{1}{\beta_h} + \beta_h\kappa^2(\nabla w_h)\Big)\mathcal{H}^1_{\varepsilon_h}(w_h, \nabla w_h)\,dx = I_h + II_h. \end{aligned} \tag{6.30}$$

Using (3.4), (6.28), the equality $|\nabla w_h| = |\eta_h'(\delta_{D_h})|$, and the coarea formula we get for h large enough

$$\begin{aligned} I_h &= 2\int_{D_h^1} |\nabla w_h|\sqrt{V(w_h)}\Big(\frac{1}{\beta_h} + \beta_h\kappa^2(\nabla w_h)\Big)dx \\ &= 2\int_{\theta_h^1}^{\theta_h^2} \sqrt{V(\theta)}\int_{\{w_h=\theta\}}\Big(\frac{1}{\beta_h} + \beta_h\kappa^2\Big)d\mathcal{H}^1 d\theta, \end{aligned}$$

where $\theta_h^1 = \varepsilon_h/(1+\varepsilon_h)$ and $\theta_h^2 = 1/(1+\varepsilon_h)$. Then, we easily get

$$I_h = 2\int_{\theta_h^1}^{\theta_h^2} \sqrt{V(\theta)}d\theta \int_{\partial D_h} \Big(\frac{1}{\beta_h} + \beta_h \kappa^2\Big) d\mathcal{H}^1 + O(\varepsilon_h|\log\varepsilon_h|) \int_{\theta_h^1}^{\theta_h^2} \sqrt{V(\theta)}d\theta$$

(see also the proof of [9] Theorem 4.3), from which, using (6.20), we obtain

$$\lim_{h\to+\infty} I_h = 8\pi \# P^* \int_0^1 \sqrt{V(\theta)}d\theta. \tag{6.31}$$

Again using the coarea formula and arguing as above we get

$$II_h = O(1)\int_{D_h^2} \mathcal{H}^1_{\varepsilon_h}(w_h, \nabla w_h)\, dx,$$

from which, using the explicit form of w_h (e.g. as in [31, 42]), it follows that $\lim_h II_h = 0$. Then, using (6.30) and (6.31) we have

$$\lim_{h\to+\infty} \mathcal{G}^{(1)}_{\varepsilon_h}(w_h) = 4\pi b_0 \# P^*. \tag{6.32}$$

We now set

$$A_h^1 = \{x\in\Omega : |\delta_{A_h}(x)| < \varepsilon_h|\log\varepsilon_h|\},\ A_h^2 = \{x\in\Omega : \varepsilon_h|\log\varepsilon_h| < |\delta_{A_h}(x)| < 2\varepsilon_h|\log\varepsilon_h|\}.$$

Using the definition of $\mathcal{G}^{(2)}_\varepsilon$ we have

$$\begin{aligned}\mathcal{G}^{(2)}_{\varepsilon_h}(s_h, w_h) \quad &= \quad \int_{A_h^1} w_h^2 \left(1+\kappa^2(\nabla s_h)\right) \mathcal{H}^1_{\varepsilon_h}(s_h, \nabla s_h)\, dx \\ &\quad + \int_{A_h^2} w_h^2 \left(1+\kappa^2(\nabla s_h)\right) \mathcal{H}^1_{\varepsilon_h}(s_h, \nabla s_h)\, dx = \widetilde{I}_h + \widetilde{II}_h. \end{aligned} \tag{6.33}$$

Using the coarea formula and arguing as before we get for h large enough

$$\widetilde{I}_h \le 2\int_{A_h^1\setminus D_h^0} |\nabla s_h|\sqrt{V(s_h)}(1+\kappa^2(\nabla s_h))dx = 2\int_{\theta_h^1}^{\theta_h^2} \sqrt{V(\theta)} \int_{\{s_h=\theta\}\setminus D_h^0} (1+\kappa^2)\, d\mathcal{H}^1 d\theta.$$

Then it follows that

$$\widetilde{I}_h \le 2\int_{\theta_h^1}^{\theta_h^2} \sqrt{V(\theta)}d\theta \int_{\partial A_h\setminus D_h^0} (1+\kappa^2)\, d\mathcal{H}^1 + O(\varepsilon_h|\log\varepsilon_h|), \tag{6.34}$$

from which, using (6.21) and (6.26), we obtain

$$\limsup_{h\to+\infty} \widetilde{I}_h \le 4\sum_{\gamma\in C^*} \big(K(\gamma)+L(\gamma)\big) \int_0^1 \sqrt{V(\theta)}d\theta. \tag{6.35}$$

Analogously we have that $\lim_h \widetilde{II}_h = 0$. Using (6.33) and (6.35) we have

$$\limsup_{h\to+\infty} \mathcal{G}^{(2)}_{\varepsilon_h}(s_h, w_h) \le 2b_0 \sum_{\gamma\in C^*} \big(K(\gamma)+L(\gamma)\big). \tag{6.36}$$

Then, from Steps 3 and 4 in the proof of Theorem 6.3, by taking ρ_h in (6.17) small enough we have $u_h(x) = \widehat{u}_h(x)$ if $x \notin A_h^0$, with $A_h^0 = \{x \in \Omega : \delta_{A_h}(x) < -2\varepsilon_h|\log \varepsilon_h|\}$. Moreover, $\nabla\widehat{u}_h \to \nabla u$ in $L^2(\Omega;\mathbb{R}^2)$, from which it follows

$$\limsup_{h\to+\infty} \int_\Omega s_h^2 |\nabla u_h|^2\, dx \leq \lim_{h\to+\infty} \int_{\Omega\setminus A_h^0} |\nabla \widehat{u}_h|^2\, dx = \int_{\Omega\setminus[C^*]} |\nabla u|^2\, dx. \tag{6.37}$$

Moreover we have

$$\lim_{h\to+\infty} \int_\Omega |u_h - g|^2\, dx = \int_\Omega |u-g|^2\, dx. \tag{6.38}$$

Finally, using (3.5), (6.25) and (6.27) we get

$$\limsup_{h\to+\infty} \frac{1}{\mu_h} \int_\Omega (1-w_h)^2\, dx \leq \lim_{h\to+\infty} \frac{\operatorname{meas}((D_h)_{2\varepsilon_h|\log\varepsilon_h|})}{\mu_h} = 0, \tag{6.39}$$

and

$$\limsup_{h\to+\infty} \frac{1}{\mu_h} \int_\Omega (1-s_h)^2\, dx \leq \lim_{h\to+\infty} \frac{\operatorname{meas}((A_h)_{2\varepsilon_h|\log\varepsilon_h|})}{\mu_h} = 0. \tag{6.40}$$

The statement of the theorem then follows collecting (6.32) and (6.36)-(6.40). ■

7 Connection with a conjecture by E. De Giorgi

In this section we prove the upper inequality of Γ-convergence for a family of functionals based on a conjecture by E. De Giorgi [22]. According to that conjecture the Γ-convergence for curvature-depending functionals can be obtained by replacing

$$\operatorname{div}\Big(\frac{\nabla s}{|\nabla s|}\Big) \qquad \text{by} \qquad 2\varepsilon\Delta s - \frac{V'(s)}{\varepsilon}$$

in the approximating functionals. An analogous replacement takes place for the term involving $\kappa(\nabla w)$. The resulting approximating functionals are more convenient for the purpose of numerical computations. However, the validity of the lower inequality is an open problem.

For every $\varepsilon > 0$ we define

$$\begin{aligned}
\widehat{\mathcal{G}}_\varepsilon^{(1)}(w) &= \int_\Omega \Big(\frac{1}{\beta_\varepsilon} + \frac{\beta_\varepsilon}{b_1}\Big(2\varepsilon\Delta w - \frac{V'(w)}{\varepsilon}\Big)^2\Big)\mathcal{H}_\varepsilon^1(w,\nabla w)\, dx,\\
\widehat{\mathcal{G}}_\varepsilon^{(2)}(s,w) &= \int_\Omega w^2\Big(1 + \frac{1}{b_1}\Big(2\varepsilon\Delta s - \frac{V'(s)}{\varepsilon}\Big)^2\Big)\mathcal{H}_\varepsilon^1(s,\nabla s)\, dx,
\end{aligned}$$

where $b_1 = (8/b_0)\int_0^1 V(t)^{3/2}dt$. We denote by $\widehat{\mathcal{G}}_\varepsilon : W(\Omega) \to [0,+\infty]$ the functional defined by

$$\begin{aligned}
\widehat{\mathcal{G}}_\varepsilon(u,s,w) &= \int_\Omega s^2|\nabla u|^2\, dx + \frac{1}{4\pi b_0}\widehat{\mathcal{G}}_\varepsilon^{(1)}(w) + \frac{1}{2b_0}\widehat{\mathcal{G}}_\varepsilon^{(2)}(s,w) + \int_\Omega |u-g|^2\, dx\\
&\quad + \frac{1}{\mu_\varepsilon}\int_\Omega (1-s)^2\, dx + \frac{1}{\mu_\varepsilon}\int_\Omega (1-w)^2\, dx.
\end{aligned}$$

We prove the following proposition.

Proposition 7.1. *Let Ω be a star-shaped bounded open set, and let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every triplet $(u, C, P) \in X(\Omega)$ there exists a sequence $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ converging weakly to (u, C^*, P^*) such that*

$$\limsup_{h \to +\infty} \widehat{\mathcal{G}}_{\varepsilon_h}(u_h, s_h, w_h) \leq \mathcal{G}(u, C, P),$$

where C^* and P^* are as in (3.1).

Proof. The proof is similar to that of Theorem 6.4. Hence we sketch the estimate for the term $\widehat{\mathcal{G}}^{(2)}_{\varepsilon_h}$, the estimates for the other terms being similar.

The optimal sequence $\{(u_h, s_h, w_h)\}_h$ is constructed in the same way as in the proof of Theorem 6.4. By using the same notations, we decompose the term $\widehat{\mathcal{G}}^{(2)}_{\varepsilon_h}(s_h, w_h)$ into the integrals over the sets A_h^1 and A_h^2, respectively denoted by I_h and II_h. For any $x \in A_h^1$ we have for h large enough

$$\Delta \delta_{A_h}(x) = \kappa_h^t(x),$$

where $\kappa_h^t(x)$ is the curvature of the level set $\{x : \delta_{A_h}(x) = t\}$. Then, using (6.28) and (6.29), we have $2\varepsilon_h \eta_h''(t) = V'(\eta_h(t))/\varepsilon_h$, and

$$2\varepsilon_h \Delta s_h - \frac{V'(s_h)}{\varepsilon_h} = 2\varepsilon_h \eta_h' \Delta \delta_{A_h} + 2\varepsilon_h \eta_h'' - \frac{V'(\eta_h)}{\varepsilon_h} = 2\varepsilon_h \eta_h' \kappa(\nabla s_h).$$

Using the coarea formula and arguing as in the proof of Theorem 6.4 we get for h large enough

$$\begin{aligned}
I_h &\leq 2 \int_{A_h^1 \setminus D_h^0} |\nabla s_h| \sqrt{V(s_h)} \Big(1 + \frac{4}{b_1} \varepsilon_h^2 (\eta_h')^2 \kappa^2(\nabla s_h)\Big) dx \\
&= 2 \int_{\theta_h^1}^{\theta_h^2} \sqrt{V(\theta)} \int_{\{s_h = \theta\} \setminus D_h^0} \Big(1 + \frac{4}{b_1} \varepsilon_h^2 (\eta_h')^2 \kappa^2\Big) d\mathcal{H}^1 d\theta \\
&= 2 \int_{\theta_h^1}^{\theta_h^2} \sqrt{V(\theta)} d\theta \int_{\partial A_h \setminus D_h^0} d\mathcal{H}^1 \\
&\qquad + \frac{8}{b_1} \int_{\theta_h^1}^{\theta_h^2} V(\theta)^{3/2} d\theta \int_{\partial A_h \setminus D_h^0} \kappa^2 d\mathcal{H}^1 + O(\varepsilon_h |\log \varepsilon_h|) \\
&= 2 \int_{\theta_h^1}^{\theta_h^2} \sqrt{V(\theta)} d\theta \int_{\partial A_h \setminus D_h^0} (1 + \kappa^2)\, d\mathcal{H}^1 + O(\varepsilon_h |\log \varepsilon_h|),
\end{aligned}$$

which coincides with the estimate (6.34). The other terms are estimated in an analogous way. ■

Remark 7.2. The result of Proposition 7.1 holds essentially with the same proof even if the terms $\widehat{\mathcal{G}}^{(1)}_\varepsilon$ and $\widehat{\mathcal{G}}^{(2)}_\varepsilon$ are replaced with the following functionals

$$\widehat{\mathcal{G}}^{(1)}_\varepsilon(w) = \frac{\beta_\varepsilon}{2\varepsilon} \int_\Omega \Big(2\varepsilon \Delta w - \frac{V'(w)}{\varepsilon}\Big)^2 dx + \frac{1}{\beta_\varepsilon} \int_\Omega \mathcal{H}^1_\varepsilon(w, \nabla w)\, dx,$$

and

$$\widehat{\mathcal{G}}^{(2)}_\varepsilon(s, w) = \int_\Omega \mathcal{H}^1_\varepsilon(s, \nabla s)\, dx + \frac{1}{2\varepsilon} \int_\Omega w^2 \Big(2\varepsilon \Delta s - \frac{V'(s)}{\varepsilon}\Big)^2 dx,$$

which are simpler for numerical computations (see for instance [24], [25], [30]).

8 Appendix

By adapting a method introduced by Bellettini, Dal Maso and Paolini in [8] we will prove the result in Step 1 of Theorem 6.3.

Proposition 8.1. *Let C^* satisfy the finiteness property and $[C^*] \subset \Omega$. Then there exists a sequence $C'_h = \{\gamma_h^1, \ldots, \gamma_h^m\}$ of families of simple curves of class $\mathcal{C}^\infty$ such that $\{\gamma_h^i\}_h$ converges strongly in $W^{2,2}$ to a curve γ^i for any $i = 1, \ldots, m$, the family $C' = \{\gamma^1, \ldots, \gamma^m\}$ is admissible and satisfies the finiteness property, C' and C^* are equivalent, and* (6.14) *holds for all h.*

Before proving this proposition we introduce some notation following [8]. Let S be the union of m closed and pairwise disjoint intervals $[a_i, b_i]$, $i = 1, \ldots, m$, and let $C : S \to \mathbb{R}^2$ be a family of curves that satisfies the finiteness property (Definition 2.5). Then there exists a finite number of points $\{t_1, \ldots, t_p\} \in S$ such that $\{a_1, b_1, \ldots, a_m, b_m\} \subseteq \{t_1, \ldots, t_p\}$ and the unique finite partition $\mathcal{P}$ of $S \setminus \{t_1, \ldots, t_p\}$ composed of open intervals having endpoints in $\{t_1, \ldots, t_p\}$ satisfies the following properties:

(i) for any $I, H \in \mathcal{P}$ either $C(I) \cap C(H) = \emptyset$ or $C(I) = C(H)$;

(ii) C is injective on $\overline{I}$ for every $I \in \mathcal{P}$;

(iii) $C(t_i) \notin C(H)$ for every $H \in \mathcal{P}$ and for every $i = 1, \ldots, p$.

For every $I \in \mathcal{P}$ the set $C(I)$ will be called a *branch* of C, and for every $i = 1, \ldots, p$ the point $C(t_i)$ is called a *node* of C. Observe that $P(C)$ is contained in the set of all the nodes of C. Let B be a branch of C, and let $x \in B$; then the number of elements of $C^{-1}(x)$ depends only on B and does not depend on x. This number is called the *multiplicity of the branch* B. If T is an open subinterval of $S \setminus \{t_1, \ldots, t_p\}$, then the set $C(T)$ will be called an *arc* of C. Let J be an arc of C; since J is contained in a branch B, the *multiplicity of* J is the multiplicity of B.

Let C be an admissible family of curves in Ω that satisfies the finiteness property, and let q be a node of C such that $q \notin P(C)$. Then there exists a tangent unit vector $\tau(q)$ of $[C]$ at q. Let B_1 and B_2 be two branches of C having endpoint q. We say that B_1 and B_2 *lie on the same side* with respect to q if there exists a neighbourhood U of q such that $(x - q) \cdot \tau(q) > 0$ for every $x \in (B_1 \cup B_2) \cap U$. Otherwise, we say that B_1 and B_2 *lie on opposite sides* with respect to q.

We say that two open intervals $I, H \in \mathcal{P}$ are *consecutive* if their closures $\overline{I}, \overline{H}$ have one point belonging to $\{t_1, \ldots, t_p\}$ in common. If B, B' are two branches of C having multiplicity one, then they are *consecutive* if there exist two consecutive open intervals $I, H \in \mathcal{P}$ such that $B = C(I)$ and $B' = C(H)$. Two consecutive branches have a common endpoint and if the family C is admissible they lie on opposite sides with respect to it.

The following property can be proved (see [8], Section 6):

(iv) let C be an admissible family of curves in Ω that satisfies the finiteness property and let all branches of C have multiplicity one. Then, given a branch B of C having endpoint $q \notin P(C)$, the number of branches of C having endpoint q and lying on the same side of B with respect to q is the same of the number of the branches having endpoint q and lying on the opposite side.

Remark 8.2. Let C be as in (iv) and let q be a node of C such that $q \notin P(C)$. Let $R(q)$ be an open rectangle centered at q and having two sides parallel to the tangent line of $[C]$ at q. We use the local coordinates (ξ, η) and the notation introduced for $R^{\pm}(q)$ in (6.3). If $R(q)$ is sufficiently small, by the Implicit Function Theorem it follows that the set $R^{+}(q) \cap ([C] \setminus \{q\})$ is composed of r different arcs $J_1, \ldots, J_r$ having q as an endpoint. For any $k = 1, \ldots, r$ the arc J_k is the cartesian graph in $R^{+}(q)$ of a function f_k of class $W^{2,2}$ with respect to the ξ-axis. Following [8] we order the arcs $J_1, \ldots, J_r$ in such a way that $f_1(\xi) < f_2(\xi) < \cdots < f_r(\xi)$ for any $\xi \in (0, \xi_0)$ and we call the index k the *ordering number* of J_k with respect to q. Let now B be a branch of C having endpoint q and such that $B \cap R^{+}(q) \neq \emptyset$. Each branch B' having endpoint q and lying on the same side of B with respect to q contains one and only one arc J_k. We say that the index k is the *ordering number* of B' with respect to q.

By using property (iv) above, the set $R^{-}(q) \cap ([C] \setminus \{q\})$ is composed of r different arcs which are the cartesian graphs in $R^{-}(q)$ of functions of class $W^{2,2}$. Then the branches having endpoint q and lying on the opposite side of B with respect to q can be ordered in an analogous way using such functions.

We can now prove Proposition 8.1

Proof. We may assume (up to a reparametrization which does not change the energy $\mathcal{F}(C^*, P^*)$) that C^* is a map $C^* : S \to \mathbb{R}^2$, where the set S is a finite union of closed and pairwise disjoint intervals, and $\left|\frac{dC^*}{dt}(t)\right| = 1$ for any $t \in S$. Let $\{t_1, \ldots, t_p\}$ denote the finite subset of S as above, and let $\mathcal{P}$ denote the corresponding partition of $S \setminus \{t_1, \ldots, t_p\}$ into open intervals. First we construct a sequence $\{\widetilde{C}_h\}_h$ of admissible families of curves in Ω such that $\widetilde{C}_h : S \to \mathbb{R}^2$, $\widetilde{C}_h$ converges strongly to C^* in $W^{2,2}(S; \mathbb{R}^2)$, and the following properties are satisfied for any h: (a) $P(\widetilde{C}_h) = P(C^*)$; (b) the family $\widetilde{C}_h$ satisfies the finiteness property; (c) the nodes of $\widetilde{C}_h$ coincide with the nodes of C^*; (d) all branches of $\widetilde{C}_h$ have multiplicity one.

We follow the construction of [8] Lemma 6.1. Let (q_1, q_2) be a pair of consecutive nodes of C^*, and let B be a branch joining q_1 and q_2 having multiplicity r. Then $C^{*-1}(B) = \bigcup_{j=1}^{r} I_j$, with $I_j \in \mathcal{P}$ for any $j = 1, \ldots, r$. The branch B is approximated by r branches $B_h^1, \ldots, B_h^r$. Let $\varphi : \mathbb{R}^2 \to [0, +\infty)$ of class $\mathcal{C}^\infty$ which vanishes at the points q_1, q_2 with all its derivatives, a function $v \in \mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R}^2)$, and $h_0 \in \mathbf{N}$ such that, if we set for any $j = 1, \ldots, r$,

$$B_h^j = \widetilde{C}_h(I_j), \qquad \widetilde{C}_h(t) = C^*(t) + \frac{j}{hr}\varphi(C^*(t))v(C^*(t)), \quad t \in \overline{I}_j, \tag{8.1}$$

then, for any $h > h_0$, $\widetilde{C}_h$ is injective on $\overline{I}_j$, and $B_h^j \cap B_h^{j'} = \emptyset$ for any $j \neq j'$ (to construct such functions we may use the method of proof of Lemma 6.1 of [8]). We repeat this construction for all pairs of consecutive nodes of C^* and for all branches. It follows that for any $h > h_0$ the approximating branches corresponding to different branches of C^* are disjoint [8]. Moreover, since $[C^*] \subset \Omega$ we have also $[\widetilde{C}_h] \subset \Omega$ for large enough h. Then we obtain a sequence $\{\widetilde{C}_h\}_h$ of families of curves with the required properties.

Now we show that for any $h > h_0$ there exists a family of curves $\widehat{C}_h$ satisfying the finiteness property such that $\widehat{C}_h$ and $\widetilde{C}_h$ are equivalent, all branches of $\widehat{C}_h$ have multiplicity one, and consecutive branches have the same ordering number. We adapt the construction of [8] Theorem 6.1 to our purposes. For any $h > h_0$ we associate to the family of curves $\widetilde{C}_h$ a finite undirected graph G_h whose vertices are the nodes of $\widetilde{C}_h$, and whose edges are the branches of $\widetilde{C}_h$ [11]. A chain of edges of the graph G_h is called simple if it does not use the same edge twice; a cycle is a simple chain whose endpoints are the same vertex.

Fix $h > h_0$; to construct the family $\widehat{C}_h$ we first show that there exists a finite number of chains of edges of the graph G_h with the following properties:

(i) every edge of G_h is contained in the family of chains;

(ii) the endpoints of every chain which is not a cycle belong to $P(\widetilde{C}_h)$; every cycle contains at most one vertex belonging to $P(\widetilde{C}_h)$;

(iii) the chains are all simple and pairwise disjoint;

(iv) if E_1 and E_2 are two edges of the same chain with a common vertex v, and E_2 is the successor of E_1 in the chain, then $v \notin P(\widetilde{C}_h)$, E_1 and E_2 lie on opposite sides with respect to v and have the same ordering number.

We construct the chains of edges $\{E_i\}_i$ inductively. Let v_i and v_{i+1} denote the vertices of E_i and let v_1 be a vertex of G_h such that $v_1 \in P(\widetilde{C}_h)$. We choose the first edge E_1 of the chain as an arbitrary edge having vertex v_1. Now suppose that the edge E_{i-1} has been defined. If $v_i \in P(\widetilde{C}_h)$ then E_{i-1} is the last edge of the chain, and the construction is concluded. Otherwise, we define E_i as the unique branch of $\widetilde{C}_h$ lying on the opposite side of E_{i-1} with respect to v_i and having the same ordering number. By using the same method of proof of [8] Theorem 6.1 it follows that the chain $\{E_1, \ldots, E_i\}$ is simple. By induction we then get a simple chain $\{E_i\}_i$ which satisfies the properties (ii) and (iv).

Then we repeat the algorithm starting from an edge not contained in the previous chain and having a vertex belonging to $P(\widetilde{C}_h)$. After a finite number of implementations of the algorithm all edges having a vertex belonging to $P(\widetilde{C}_h)$ are reached, since one can prove, arguing as in [8], that all the chains obtained are simple and pairwise disjoint. We repeat the algorithm starting from an edge not contained in the previous chains: arguing again as above we obtain cycles which do not contain vertices belonging to $P(\widetilde{C}_h)$. It follows that after a finite number of implementations of the algorithm all edges of the graph G_h are reached, so that all the properties (i)-(iv) are satisfied.

We now reparametrize the curves of the family $\widetilde{C}_h$ on a finite union S_h of closed and pairwise disjoint intervals in such a way that $|\frac{d\widetilde{C}_h}{dt}(t)| = 1$ for any $t \in S_h$. We still denote by $\widetilde{C}_h : S_h \to \mathbb{R}^2$ this parametrization. Let $\{t_1^h, \ldots, t_p^h\}$ denote the finite subset of S_h such that, for any $l = 1, \ldots, p$, we have $t_l^h = \widetilde{C}_h^{-1}(q)$ for some node q of $\widetilde{C}_h$. Let $\mathcal{P}$ still denote the partition of $S_h \setminus \{t_1^h, \ldots, t_p^h\}$ into open intervals. Since each branch B of the family $\widetilde{C}_h$ corresponds to an edge of the graph G_h in a one-to-one way, there exists a bijection between the set of all edges of G_h and the partition $\mathcal{P}$.

We now construct the family of curves $\widehat{C}_h$ using the family of chains obtained above. We reparametrize $[\widetilde{C}_h]$ by means of the following surgery operations on the set S_h as in the proof of [8] Theorem 6.1. Let $\{E_i\}_i$ be a chain of edges, and let $\{I_i\}_i \subseteq \mathcal{P}$ be the sequence of the corresponding intervals according to the bijection. For any i let $I_i = (a_i, b_i)$. If $\widetilde{C}_h(a_i) \neq v_i$, we reverse the orientation of the branch $\widetilde{C}_h(I_i)$ in such a way that v_i becomes the image of a_i and v_{i+1} becomes the image of b_i. Next, for any i, we glue together b_i with a_{i+1} by means of translations of the intervals I_i. We then get a curve γ defined on a single interval $I = (a, b)$ such that $\gamma(a)$ is the initial endpoint of the chain $\{E_i\}_i$ and $\gamma(b)$

is the terminal endpoint. Since the tangent vector of $[\widetilde{C}_h]$ exists at any node $q \notin P(\widetilde{C}_h)$ and $|\frac{d\widetilde{C}_h}{dt}(t)| = 1$, the property (iv) of the chain ensures that $\dot{\gamma}$ is continuous in (a, b).

Repeating this procedure for all the chains we obtain an admissible family of curves which we denote by $\widehat{C}_h$. Then the properties (i)-(iv) of the chains imply that $\widehat{C}_h$ and $\widetilde{C}_h$ are equivalent. Moreover, by construction, the family $\widehat{C}_h$ has all branches with multiplicity one and consecutive branches have the same ordering number.

Since (8.1) implies that $t_l^h \to t_l$ for any $l = 1, \dots, p$, we may repeat on $S \setminus \{t_1, \dots, t_p\}$ the same surgery operations made on $S_h \setminus \{t_1^h, \dots, t_p^h\}$. We then obtain a set S' and a family of curves $C' : S' \to \mathbb{R}^2$ such that C' and C^* are equivalent. Since the graphs G_h are all isomorphic for any $h > h_0$, by identifying isomorphic graphs, the chains of edges constructed above do not depend on h and we may reparametrize the families of curves $\widehat{C}_h$ on the set S' independent of h. It follows that $\widehat{C}_h$ converges strongly to C' in $W^{2,2}(S'; \mathbb{R}^2)$.

We now construct sequences $\{\widehat{C}_{h,k}\}_k$ of families of simple curves such that, for any $h > h_0$, $\widehat{C}_{h,k}$ converges strongly to $\widehat{C}_h$ in $W^{2,2}(S'; \mathbb{R}^2)$, and there exists $k_0 \in \mathbf{N}$ such that for any $k > k_0$

$$P(\widehat{C}_{h,k}) = P(\widehat{C}_h) = P(C'), \quad ([\gamma^i_{h,k}] \cap [\gamma^j_{h,k}]) \setminus P(C') = \emptyset \text{ for all } \gamma^i_{h,k}, \gamma^j_{h,k} \in \widehat{C}_{h,k},\ i \neq j.$$

We follow the construction of [8] Lemma 6.2. Fix $h > h_0$ and let q be a node of $\widehat{C}_h$ such that $q \notin P(\widehat{C}_h)$. Let $\tau(q)$ be a tangent unit vector of $[\widehat{C}_h]$ at q, and let $\nu(q)$ be a normal unit vector of $[\widehat{C}_h]$ at q. Let $R(q)$ be an open rectangle centered at q and having two sides parallel to $\tau(q)$. Let (ξ, η) denote the coordinates of points in a local coordinate system in $R(q)$, with the origin in q, such that the tangent line of $[\widehat{C}_h]$ at q coincides with the ξ-axis and $R(q) = (-\xi_0, \xi_0) \times (-\eta_0, \eta_0)$. Let $R^\pm(q)$ as in (6.3). If $R(q)$ is sufficiently small, using Remark 8.2 we may write:

$$R^+(q) \cap ([\widehat{C}_h] \setminus \{q\}) = \bigcup \{J_h^{i+} : 1 \le i \le n\},\ R^-(q) \cap ([\widehat{C}_h] \setminus \{q\}) = \bigcup \{J_h^{i-} : 1 \le i \le n\},$$

where the index i denotes the ordering number of the arcs J_h^{i+}, J_h^{i-} with respect to q.

Let $\widehat{C}_h^{-1}(q) = \{t_1^h, \dots, t_n^h\}$. For any $i = 1, \dots, n$, let $I_h^{i+} = \widehat{C}_h^{-1}(J_h^{i+})$, $I_h^{i-} = \widehat{C}_h^{-1}(J_h^{i-})$, and $I_h^i = I_h^{i+} \cup I_h^{i-} \cup \{t_i^h\}$. Since consecutive branches of $\widehat{C}_h$ have the same ordering number, I_h^{i+} and I_h^{i-} are consecutive open intervals of S', hence the set I_h^i is an open interval. We choose the normal unit vector $\nu(q)$ in such a way that $(0, 1)$ are the components of $\nu(q)$ with respect to the system of coordinates (ξ, η). We now define (see the proof of [8] Lemma 6.2):

$$\widehat{C}_{h,k}(t) = \widehat{C}_h(t) + \nu(q) \frac{i}{kn} \psi(\pi(\widehat{C}_h(t))), \tag{8.2}$$

if $t \in I_h^i$ for some $i = 1, \dots, n$, and $\widehat{C}_{h,k}(t) = \widehat{C}_h(t)$ elsewhere on S'. In (8.2) $\pi : R(q) \to (-\xi_0, \xi_0)$ denotes the projection of $R(q)$ onto $(-\xi_0, \xi_0)$, and $\psi : (-\xi_0, \xi_0) \to [0, 1]$ denotes a $\mathcal{C}^\infty$ function with the following properties: $\psi(0) = 1$, $\psi(\xi) = 0$ for any $\frac{\xi_0}{2} \le \xi < \xi_0$, $\psi'(\xi) < 0$ for any $0 < \xi < \frac{\xi_0}{2}$, and $\psi(-\xi) = \psi(\xi)$ for any $\xi \in [0, \xi_0)$.

Using (8.2) we have $\widehat{C}_{h,k}(I_h^i) \cap \widehat{C}_{h,k}(I_h^j) = \emptyset$ for any $i \neq j$. We repeat this construction for any node q of $\widehat{C}_h$ such that $q \notin P(\widehat{C}_h)$, by taking the rectangles $R(q)$ pairwise disjoint. Then we obtain, for any $h > h_0$, a sequence $\{\widehat{C}_{h,k}\}_k$ of families of curves with the required properties for large enough k.

Hence, for any $h > h_0$, by using a diagonal argument we obtain a sequence of families of simple curves satisfying (6.14) and converging strongly in $W^{2,2}$ to the curves γ^i of C'.

Since any family of curves can be approximated in the $W^{2,2}$ norm by a family of curves of class $\mathcal{C}^\infty$, the sequence $\{C'_h\}_h$ of families of simple curves is then obtained by means of a diagonal argument. Finally, since $[C^*] \subset \Omega$ we have also $[C'_h] \subset \Omega$ for large enough h. ■

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