Variational Approximation of a Second Order Free Discontinuity Problem in Computer Vision

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Abstract

We consider a functional, proposed by Blake and Zisserman for computer vision problems, which depends on free discontinuities, free gradient discontinuities and second order derivatives. We show how this functional can be approximated by elliptic functionals defined on Sobolev spaces. The approximation takes place in a variational sense, the De Giorgi Γ–convergence, and extends to this second order model an approximation of the Mumford–Shah functional obtained by Ambrosio and Tortorelli. For the purpose of illustration an algorithm based on the Γ–convergent approximation is applied to the problem of computing depth from stereo images and some numerical examples are presented.

Key words: theory and algorithms for computer vision, variational problems, Γ–convergence, functions of bounded variation.

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1 Introduction

In recent years, variational principles with a free discontinuity set have been introduced to solve reconstruction problems in computer vision theory (see for instance [4, 26, 31]). The variational approach to the image segmentation problem proposed by Mumford and Shah [27] consists in minimizing the functional

$$E(u,K) = \int_{\Omega\setminus K} \left(|\nabla u|^2 + \mu \, |u-g|^2 \right) dx + \alpha \mathcal{H}^{n-1}(K\cap\Omega) , \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open set, $\mathcal{H}^{n-1}$ is the Hausdorff $(n-1)$–dimensional measure, $g \in L^\infty(\Omega)$, and α, $\mu > 0$ are fixed positive parameters. The functional has to be minimized over all closed sets $K \subset \overline{\Omega}$ and all $u \in C^1(\Omega \backslash K)$. In the case $n = 2$ the function g represents the image to be segmented. By minimizing the functional one tries to detect the discontinuities of g due to the edges of the objects in the image, and to cancel the discontinuities due to noise and small irregularities. The set K contains the jump points of u and represents the edges of the objects. The functional penalizes large sets K, and outside K the function u is required to be close to g and C^1.

The Mumford and Shah variational principle can be extended to several reconstruction problems of Computer Vision [25]: stereo reconstruction [32], computation of optical flow [28], shape from shading [33]. Variational problems involving functionals of this form are usually called free discontinuity problems, after a terminology introduced by De Giorgi [18].

The Mumford and Shah model has some drawbacks: it is unable to reconstruct crease discontinuities and yields the over-segmentation of steep gradients (the so-called ramp effect). To overcome these defects of the first order model, Blake and Zisserman [9] introduced a second order functional which can be written in the form

$$\begin{aligned} F(u,K_0,K_1) \;&=\; \int_{\Omega\backslash(K_0\cup K_1)} \Big(\mid \nabla^2 u \mid^2 + \Phi(x,u) \Big)\, dx \\ &+\; \alpha \mathcal{H}^{n-1}(K_0 \cap \Omega) + \beta \mathcal{H}^{n-1}((K_1 \backslash K_0) \cap \Omega) \;, \end{aligned} \tag{1.2}$$

with α, $\beta > 0$ positive parameters. The functional has to be minimized over the unknown sets K_0, K_1, with $K_0 \cup K_1$ closed and $u \in C^2(\Omega \backslash (K_0 \cup K_1))$ approximately continuous on $\Omega \backslash K_0$. If $\Phi(x,u) = \mu \mid u - g \mid^2$ and $n = 2$, the functional (1.2) is just that one introduced in [9] (the thin plate surface under tension). In the second order model, K_0 represents the set of jump points for u, and $K_1 \backslash K_0$ is the set of crease points. Since the reconstruction of crease discontinuities is particularly relevant in those Computer Vision problems which require the reconstruction of visible surfaces from two–dimensional images, we have then introduced in (1.2) the function $\Phi(x,u)$. A suitable choice of this function will allow us to apply this variational method to computer vision problems as for instance the computation of depth from pairs of stereo images (see [25]).

If the conditions (see [9])

$$\beta \le \alpha \le 2\beta \tag{1.3}$$

are satisfied, the existence of minimizers for the functional $F(u, K_0, K_1)$ has been proved, in the case $n = 2$ and $\Phi(x,u) = \mu \mid u - g \mid^2$, by Carriero, Leaci and Tomarelli [13] (notice that (1.3) are necessary and sufficient for the lower semicontinuity of $\overline{F}$ with respect to the L^1 convergence). The proof is based on a weak formulation of the problem by setting

$$\overline{F}(u) = \int_\Omega \Big(\mid \nabla^2 u \mid^2 + \Phi(x,u) \Big)\, dx + \alpha \mathcal{H}^{n-1}(S_u) + \beta \mathcal{H}^{n-1}(S_{\nabla u} \backslash S_u) \;, \tag{1.4}$$

where ∇u denotes an approximate differential, S_u is the discontinuity set of u in an approximate sense, and $S_{\nabla u}$ is the discontinuity set of ∇u. In [12] the existence of minimizers for the functional $\overline{F}$ over the space

$$\Big\{ u : \Omega \to \mathbf{R} : \; u \in L^2(\Omega), \; u \in GSBV(\Omega), \; \nabla u \in [GSBV(\Omega)]^n \Big\} \;, \tag{1.5}$$

has been proved in any space dimension n, $GSBV(\Omega)$ being the space of generalized special functions of bounded variation introduced in [17]. A regularity theorem in [13] then shows that, for $n=2$, any weak minimizer actually provides a minimizing triplet (u, K_0, K_1) of F by taking a suitable representative of the function and the closure of S_u and $S_{\nabla u}$.

Ambrosio and Tortorelli [5, 6] approximated the Mumford and Shah functional (1.1) by a family of elliptic functionals defined on Sobolev spaces. The approximation takes place in a variational sense, the De Giorgi Γ–convergence. The approximating elliptic functionals proposed in [6] are defined by

$$E_\epsilon(u,s) = \int_\Omega (s^2+\lambda_\epsilon) \mid \nabla u \mid^2 \, dx + \mu \int_\Omega \mid u-g \mid^2 \, dx + \alpha \mathcal{G}_\epsilon(s) \ , \tag{1.6}$$

where the approximation takes place as $\epsilon \to 0^+$, $\lambda_\epsilon \to 0^+$, and

$$\mathcal{G}_\epsilon(s) = \int_\Omega \left[\epsilon \mid \nabla s \mid^2 + \frac{(s-1)^2}{4\epsilon} \right] dx \ . \tag{1.7}$$

The variable $s \in [0,1]$ is related to the set of jumps K. The minimizing s_ϵ are near to 0 in a neighbourhood of the set K, and far from the neighbourhood they are close to 1. The neighbourhood shrinks as $\epsilon \to 0$. The Ambrosio and Tortorelli's approximation can be used to find an effective algorithm for computing the minimizers of E [25, 29, 30]. The approximation has been applied to several computer vision problems in [28, 32, 33], and further improvements have been proposed and experimented in [34].

In the present paper we consider the following family of functionals

$$\begin{aligned} F_\epsilon(u,s,\sigma) &= \int_\Omega (\sigma^2+\kappa_\epsilon) \mid \nabla^2 u \mid^2 \, dx + \int_\Omega \Phi(x,u)\, dx + (\alpha-\beta)\mathcal{G}_\epsilon(s) \\ &+ \beta \mathcal{G}_\epsilon(\sigma) + \xi_\epsilon \int_\Omega (s^2+\zeta_\epsilon) \mid \nabla u \mid^\gamma \, dx \ , \end{aligned} \tag{1.8}$$

for suitable infinitesimals κ_ϵ, ξ_ϵ, ζ_ϵ. A slight variant of these functionals has been proposed by Bellettini and Coscia [7] in the case $n=1$ and in that case the Γ–convergence of F_ϵ to $\overline{F}$ has been proved (see also the discussion in the beginning of §6). We extend their Γ–convergence result in the following way: we prove the lower inequality of Γ–convergence in any space dimension n, and we prove the upper inequality when u is bounded and $\mid \nabla u \mid \in L^\gamma(\Omega)$, under a very mild regularity assumption on the sets S_u and $S_{\nabla u}$, which is fulfilled in computer vision applications. In the particular case when $\alpha=\beta$ and $n=2$ we obtain a full Γ-convergence theorem.

The extension of the Ambrosio and Tortorelli's approximation to the second order problem presents several difficulties. The lower inequality cannot be obtained by means of the slicing technique and consequent reduction to a one-dimensional problem used in [5, 6]. Such a reduction yields the operator norm of the Hessian matrix in the Γ–limit instead of the euclidean norm. The second derivatives are then estimated by adapting a global technique proposed by Ambrosio in [3] and relying on a compactness theorem in the space (1.5) due to Carriero, Leaci and Tomarelli [12]. Conversely, the jump part of the functional is estimated by using a slicing argument, taking into account that the space $GSBV$ is a vector space under a suitable energy condition (Proposition 4.3).

The major difficulty in the proof of the upper inequality consists in obtaining a suitable estimate on $\int \mid \nabla u \mid^\gamma dx$ from the finiteness of (1.4). Such an estimate would permit us to adapt the constructive part of Ambrosio and Tortorelli's proof [6] to the second order problem. In the case $\alpha = \beta$, $n = \gamma = 2$, an estimate which yields a full Γ–convergence result is obtained by means of a suitable interpolation inequality in $W^{2,2}$ (Proposition 4.6). If $\alpha \neq \beta$ we prove the upper inequality under the assumption that u is bounded and $\mid \nabla u \mid \in L^{\gamma}(\Omega)$, which yields only a partial result.

The discretization of the functional (1.2) is not straightforward and it is difficult to apply gradient descent with respect to the unknown sets K_0 and K_1. Conversely, the Γ–convergent approximation yields a sequence of functionals (1.8) which are numerically much more tractable, so that discretization and gradient descent may be applied in a straightforward way. In particular a simple discretization method, commonly used for computer vision problems [35], may be applied to the functionals (1.8). We then apply the Γ–convergence result to the problem of computation of depth from stereo images, and we present some computer experiments on synthetic images to illustrate the feasibility of the approximation.

2 Notations and Preliminary Results

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set. We denote by $\mathcal{B}(\Omega)$ the σ-algebra of all the Borel subsets of Ω; for any $C \in \mathcal{B}(\Omega)$ we denote by meas(C) the Lebesgue n–dimensional measure of C and by $\mathcal{H}^{n-1}(C)$ the Hausdorff $(n-1)$–dimensional measure of C. We denote by $B_\rho(x)$ the open ball $\{y \in \mathbf{R}^n :\mid y - x \mid < \rho\}$. We denote by $\mathbf{M}^{n\times n}$ the space of $n \times n$ matrices endowed with the euclidean norm. We introduce the following notations: $s \wedge t = \min\{s,t\}$, $s \vee t = \max\{s,t\}$ for every $s, t \in \mathbf{R}$; given two vectors a, b, we set $\langle a, b\rangle = a \cdot b = \sum_i a_i b_i$ and $(a \otimes b)_{ij} = a_i b_j$.

For any Borel function $u : \Omega \to \mathbf{R}$ we define the approximate upper and lower limits $u^+(x)$, $u^-(x)$ by

$$u^+(x) = \inf\left\{t \in [-\infty, +\infty] : \lim_{\rho\to 0^+} \frac{\text{meas}(\{y \in B_\rho(x) : u(y) > t\})}{\rho^n} = 0\right\} ,$$

$$u^-(x) = \sup\left\{t \in [-\infty, +\infty] : \lim_{\rho\to 0^+} \frac{\text{meas}(\{y \in B_\rho(x) : u(y) < t\})}{\rho^n} = 0\right\} .$$

The set

$$S_u = \{x \in \Omega :\ u^-(x) < u^+(x)\}$$

is the discontinuity set of u in an approximate sense and it is negligible with respect to Lebesgue measure (see [20], Section 2.9.13). Let $z = u^+(x) = u^-(x) \in \mathbf{R}$; we say that $\nabla u(x) \in \mathbf{R}^n$ is the approximate differential of u at x if $v^+(x) = 0$, where

$$v(y) = \frac{\mid u(y) - z - \langle \nabla u(x), y - x\rangle \mid}{\mid y - x \mid} \qquad \forall y \in \Omega \setminus \{x\} .$$

If u is differentiable at x then $\nabla u(x)$ is the classical gradient. In the one–dimensional case we shall use the notation u' in place of ∇u. An important property of the approximate differential is the fact that

$$\nabla u(x) = 0 \qquad \text{a.e. on} \quad \{y \in \Omega : u(y) = c\} \qquad \forall c \in \mathbf{R} . \tag{2.9}$$

We denote by $BV(\Omega)$ the space of functions of bounded variation in Ω, i.e. the functions $u \in L^1(\Omega)$ such that the distributional derivative of u is representable by means of a vector measure $Du = (D_1 u, \cdots, D_n u)$ with finite total variation. We denote by $| Du |$ the measure total variation of Du. If $u \in BV(\Omega)$ then ∇u exists almost everywhere in Ω and coincides almost everywhere with the Radon–Nikodym derivative of Du with respect to the Lebesgue measure [11]. Moreover, the set S_u is countably $(n-1)$–rectifiable, i.e. representable as a disjoint union $\cup_{i=1}^{\infty} K_i \cup N$, where $\mathcal{H}^{n-1}(N) = 0$ and K_i are compact sets, each contained in a C^1 hypersurface $\Gamma_i \subset \mathbf{R}^n$ [16].

Let $E \subset \mathcal{B}(\Omega)$; we define

$$P(E,\Omega) = \sup\left\{ \int_E div\ \phi\, dx : \phi \in C_0^1(\Omega;\mathbf{R}^n), | \phi | \leq 1 \right\}.$$

We say that E is a set of finite perimeter in Ω if $P(E,\Omega) < +\infty$. By Riesz's theorem (see [21]), E is a set of finite perimeter if and only if $1_E \in BV(\Omega)$, and $P(E,\Omega) = | D1_E | (\Omega)$.

The following Fleming-Rishel co-area formula (see [21]) establishes an important connection between BV functions and sets of finite perimeter:

$$| Du | (\Omega) = \int_{-\infty}^{+\infty} P(\{x \in \Omega : u(x) > t\}, \Omega)\, dt. \tag{2.10}$$

We say that $u \in BV(\Omega)$ belongs to the space of special functions of bounded variation $SBV(\Omega)$ if

$$| Du | (\Omega) = \int_\Omega | \nabla u |\ dx + \int_{S_u} | u^+ - u^- |\ d\mathcal{H}^{n-1}\ .$$

Functions like the Cantor–Vitali function, whose derivative is concentrated on Cantor's middle third set, are then excluded by $SBV(\Omega)$ (see [1, 17]).

Given a Borel function $u : \Omega \to \mathbf{R}$ we say that $u \in GSBV(\Omega)$ if (see [2, 17])

$$-N \vee u \wedge N \in SBV_{\mathrm{loc}}(\Omega) \qquad \forall N \in \mathbf{N}. \tag{2.11}$$

The jump set of u is given by

$$S_u = \bigcup_{N=1}^{\infty} S_{-N\vee u\wedge N}.$$

Furthermore, if $u \in GSBV(\Omega)$ then S_u is countably $(n-1)$–rectifiable, ∇u exists almost everywhere in Ω and is given by (see [2])

$$\nabla u = \nabla(-N \vee u \wedge N) \quad \text{a.e. on} \quad \{x \in \Omega :| u |\leq N\} \qquad \forall N \in \mathbf{N}.$$

We also set

$$GSBV^2(\Omega) = \{u \in GSBV(\Omega) :\ \nabla u \in [GSBV(\Omega)]^n\}\ .$$

Given $u \in GSBV^2(\Omega)$, we use the notation $\nabla^2_{i,j} u = \nabla_j(\nabla_i u)$ and, in the one–dimensional case, $u'' = (u')'$. Moreover we set

$$S_{\nabla u} = \bigcup_{i=1}^{n} S_{\nabla_i u}.$$

The following compactness result has been proved by Carriero, Leaci and Tomarelli in [12].

Theorem 2.1 *Let $(u_h) \subset GSBV^2(\Omega)$ be a sequence such that*

$$\|u_h\|_{L^2} \ , \qquad \mathcal{H}^{n-1}(S_{u_h} \cup S_{\nabla u_h}) \ , \qquad \int_\Omega |\ \nabla^2 u_h\ |^2\ dx$$

are uniformly bounded in h. Then there exist a subsequence (u_{h_k}) and $u \in GSBV^2(\Omega) \cap L^2(\Omega)$ such that, as $k \to +\infty$,

$$\begin{aligned} &u_{h_k} \to u \qquad \text{strongly in } L^1(\Omega) \\ &\nabla u_{h_k} \to \nabla u \qquad a.e. \text{ in } \Omega \\ &\nabla^2 u_{h_k} \rightharpoonup \nabla^2 u \qquad \text{weakly in } L^2(\Omega; \mathbf{M}^{n\times n})\ . \end{aligned}$$

Finally, we recall the following lemma (see [10]).

Lemma 2.2 *Let $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ be a σ-finite measure, and let $(f_h) \subset L^1(\Omega)$ be a sequence of non-negative functions. Then,*

$$\int_\Omega \sup_{i\in\mathbf{N}} f_i(x)\, d\mu(x) = \sup\left\{ \sum_{i=1}^k \int_{A_i} f_i(x)\, d\mu(x) : A_i \subset \Omega \text{ open and mutually disjoint, } k \in \mathbf{N} \right\}.$$

We now recall the definition and some properties of Γ–convergence (see [15]). Let X be a metric space, and let $f_\epsilon : X \to [0, +\infty]$ be a family of functions indexed by $\epsilon > 0$. We say that f_ϵ Γ–converge as $\epsilon \to 0^+$ to $f : X \to [0, +\infty]$ if the following two conditions

$$\forall x_\epsilon \to x \quad \liminf_{\epsilon\to 0^+} f_\epsilon(x_\epsilon) \geq f(x), \tag{2.12}$$

and

$$\exists x_\epsilon \to x \quad \limsup_{\epsilon\to 0^+} f_\epsilon(x_\epsilon) \leq f(x), \tag{2.13}$$

are fulfilled for every $x \in X$. The Γ–limit, if it exists, is unique and lower semicontinuous. The Γ–convergence is stable under continuous perturbations, that is, $f_\epsilon + g$ Γ–converge to $f + g$ if f_ϵ Γ–converge to f and g is continuous. The most important property of Γ–convergence is the following: if (x_ϵ) is asymptotically minimizing, i.e.

$$\lim_{\epsilon\to 0^+} \left(f_\epsilon(x_\epsilon) - \inf_X f_\epsilon \right) = 0, \tag{2.14}$$

and if x_{ϵ_h} converge to x for some sequence $\epsilon_h \to 0$, then x minimizes f.

3 Statement of Main Results

Let $\Phi(x, u) = \mu \mid u - g \mid^2$, $\mu > 0$, and $0 < \beta \leq \alpha \leq 2\beta$. For every $u \in GSBV^2(\Omega) \cap L^2(\Omega)$ and every $g \in L^\infty(\Omega)$, we write (1.4) as

$$\overline{F}(u) = \int_\Omega \left(\mid \nabla^2 u \mid^2 + \mu \mid u - g \mid^2 \right) dx + (\alpha - \beta)\mathcal{H}^{n-1}(S_u) + \beta\mathcal{H}^{n-1}(S_u \cup S_{\nabla u}).$$

In [12], using Theorem 2.1 and a suitable lower semicontinuity theorem in $GSBV^2(\Omega)$, Carriero, Leaci and Tomarelli proved that the problem

$$(\mathcal{P}) \qquad \min\left\{\overline{F}(u):\ u \in GSBV^2(\Omega)\cap L^2(\Omega)\right\}$$

has at least one solution.
For every $\epsilon > 0$ and any function $v \in W^{1,2}(\Omega;[0,1])$, let us define

$$\mathcal{G}_\epsilon(v) = \int_\Omega \left(\epsilon \mid \nabla v \mid^2 + \frac{(v-1)^2}{4\epsilon}\right) dx.$$

Our aim is to approximate $\overline{F}$, in the sense of Γ–convergence, by a family of elliptic functionals F_ϵ which are formally defined by

$$\begin{aligned} F_\epsilon(u,s,\sigma) &= \int_\Omega (\sigma^2+\kappa_\epsilon) \mid \nabla^2 u \mid^2\ dx + \mu \int_\Omega \mid u-g \mid^2\ dx + (\alpha-\beta)\mathcal{G}_\epsilon(s) \\ &+ \beta\mathcal{G}_\epsilon(\sigma) + \xi_\epsilon \int_\Omega (s^2+\zeta_\epsilon) \mid \nabla u \mid^\gamma\ dx \end{aligned} \tag{3.15}$$

for suitable nonnegative infinitesimals κ_ϵ, ξ_ϵ, ζ_ϵ (in some cases they are allowed to vanish, see the statements below). This formula makes sense if $u \in W^{2,2}(\Omega)$ and $s,\ \sigma \in W^{1,2}(\Omega)$; however, in the case $\kappa_\epsilon = 0$, because of the coefficient σ^2 multiplying the second derivatives, the functionals F_ϵ are not coercive in these spaces. In Section 5 we identify a domain $\mathcal{D}(\Omega)$ of the functionals F_ϵ such that the problem

$$(\mathcal{P}_\epsilon) \qquad \min\left\{F_\epsilon(u,s,\sigma):\ (u,s,\sigma)\in\mathcal{D}(\Omega)\right\}$$

has at least one solution, provided $\gamma > 2$.

We define

$$X(\Omega) = L^2(\Omega)\times L^\infty(\Omega;[0,1])\times L^\infty(\Omega;[0,1]) \supset \mathcal{D}(\Omega)$$

and we denote by $\mathcal{F} : X(\Omega) \to [0,+\infty]$ the functional defined by

$$\mathcal{F}(u,s,\sigma) = \begin{cases} \overline{F}(u) & \text{if } u \in GSBV^2(\Omega),\ s\equiv 1,\ \sigma\equiv 1, \\ +\infty & \text{otherwise.}\end{cases}$$

Analogously, we denote by $\mathcal{F}_\epsilon : X(\Omega) \to [0,+\infty]$ the functional defined by

$$\mathcal{F}_\epsilon(u,s,\sigma) = \begin{cases} F_\epsilon(u,s,\sigma) & \text{if } (u,s,\sigma)\in\mathcal{D}(\Omega), \\ +\infty & \text{otherwise.}\end{cases}$$

We first prove the lower inequality of Γ–convergence.

Theorem 3.1 *Assume that $\gamma \geq 2$, that*

$$\lim_{\epsilon\to 0^+} \frac{\xi_\epsilon}{\epsilon^{\gamma-1}} = +\infty \tag{3.16}$$

and that either $\kappa_\epsilon > 0$ for ϵ small enough or $\zeta_\epsilon > 0$ for ϵ small enough. Then, for every triple $(u,s,\sigma)\in X(\Omega)$ and for every family $(u_\epsilon,s_\epsilon,\sigma_\epsilon)\in\mathcal{D}(\Omega)$ converging to (u,s,σ) in $[L^1(\Omega)]^3$ as $\epsilon\to 0^+$ we have

$$\liminf_{\epsilon\to 0^+} \mathcal{F}_\epsilon(u_\epsilon,s_\epsilon,\sigma_\epsilon) \geq \mathcal{F}(u,s,\sigma)\ .$$

Moreover, (3.16) can be replaced by the condition $\xi_\epsilon \geq 0$ in the case $\alpha = \beta$.

Then we prove the equi-coercivity of the family $(\mathcal{F}_\epsilon)$ under the same assumptions on γ and on the infinitesimals κ_ϵ, ξ_ϵ, ζ_ϵ made in Theorem 3.1.

Theorem 3.2 *Let $(u_\epsilon, s_\epsilon, \sigma_\epsilon) \in \mathcal{D}(\Omega)$ be such that*

$$\sup_{\epsilon>0} \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon, \sigma_\epsilon) < +\infty .$$

Then the family $(u_\epsilon, s_\epsilon, \sigma_\epsilon)$ is relatively compact in the $[L^1(\Omega)]^3$ topology as $\epsilon \to 0^+$ and any limit point is of the form $(u, 1, 1)$ with $u \in GSBV^2(\Omega) \cap L^2(\Omega)$.

We now consider the upper inequality of Γ–convergence.

We first state our full Γ-convergence result in the special case when $n = 2$, $\gamma = 2$ and $\alpha = \beta$. We recall that a domain Ω is strictly star-shaped if there exists $x_0 \in \Omega$ such that $t(\Omega - x_0) + x_0 \subset\subset \Omega$ for any $t \in [0, 1)$.

Theorem 3.3 *Assume that $n = \gamma = 2$, $\alpha = \beta$ and Ω is strictly star-shaped. Assume that $\kappa_\epsilon > 0$ and $\kappa_\epsilon = o(\epsilon^4)$, while $\xi_\epsilon = \zeta_\epsilon = 0$. Then the family $(\mathcal{F}_\epsilon)$ Γ-converges to $\mathcal{F}$ as $\epsilon \to 0^+$.*

Then from the properties of Γ–convergence and Theorem 3.2, if $(\bar{u}_\epsilon, \bar{s}_\epsilon, \bar{\sigma}_\epsilon)$ minimizes $\mathcal{F}_\epsilon$, then the family $(\bar{u}_\epsilon, \bar{s}_\epsilon, \bar{\sigma}_\epsilon)$ is relatively compact in $[L^1(\Omega)]^3$ as $\epsilon \to 0^+$ and any limit point corresponds to a triple $(u, 1, 1)$ with u minimizer of $\overline{F}$.
Notice that in the case $\alpha = \beta$, $\xi_\epsilon = 0$, the functionals $\mathcal{F}_\epsilon$ do not depend on s, hence we can write them in the much simpler form

$$F_\epsilon(u, \sigma) = \int_\Omega (\sigma^2 + \kappa_\epsilon) \mid \nabla^2 u \mid^2 \, dx + \mu \int_\Omega \mid u - g \mid^2 \, dx + \beta \mathcal{G}_\epsilon(\sigma) \ .$$

Now we consider a more general situation. For every set $A \subset \mathbf{R}^n$ and every positive real number ρ, we denote by $(A)_\rho$ the open tubular neighbourhood of A with radius ρ, that is $(A)_\rho = \{x \in \mathbf{R}^n : \ \mathrm{dist}(x, A) < \rho\}$. We define the Minkowski $(n-1)$–dimensional upper and lower content of the set A respectively by

$$\mathcal{M}^*(A) = \limsup_{\rho\to 0^+} \frac{\mathrm{meas}((A)_\rho)}{2\rho} \ , \qquad \mathcal{M}_*(A) = \liminf_{\rho\to 0^+} \frac{\mathrm{meas}((A)_\rho)}{2\rho} \ .$$

It can be shown (see [20], Section 3.2.39) that $\mathrm{meas}((A)_\rho)/\rho$ converges to $2\mathcal{H}^{n-1}(A)$ as $\rho \to 0^+$ for any compact subset A of a C^1 hypersurface. In particular, by inner approximation this implies

$$\mathcal{M}_*(A) \geq \mathcal{H}^{n-1}(A)$$

for any $u \in BV(\Omega)$ and any Borel set $A \subset S_u$, because $\mathcal{H}^{n-1}$-almost all of S_u can be covered by C^1 hypersurfaces. The inequality $\mathcal{M}^*(A) \leq \mathcal{H}^{n-1}(A)$, which implies

$$\lim_{\rho\to 0^+} \frac{\mathrm{meas}((A)_\rho)}{2\rho} = \mathcal{H}^{n-1}(A) \ ,$$

holds under very mild regularity assumptions on the set A [5].

We are able to prove the upper inequality of Γ–convergence under the assumption that $u \in L^\infty(\Omega)$, $|\nabla u| \in L^\gamma(\Omega)$ and that, for the sets S_u and $S_u \cup S_{\nabla u}$, Hausdorff measure and Minkowski content coincide.

Theorem 3.4 *Assume that $\gamma \geq 2$, $\kappa_\epsilon = 0$, $\zeta_\epsilon > 0$, ξ_ϵ satisfies (3.16) and $\xi_\epsilon \zeta_\epsilon = o(\epsilon^{\gamma-1})$. Then, for every triple $(u, s, \sigma) \in X(\Omega)$ such that $u \in L^\infty(\Omega)$, $|\nabla u| \in L^\gamma(\Omega)$ and*

$$\mathcal{M}^*(S_u) \leq \mathcal{H}^{n-1}(S_u) \ , \qquad \mathcal{M}^*(S_u \cup S_{\nabla u}) \leq \mathcal{H}^{n-1}(S_u \cup S_{\nabla u}) \ ,$$

there exist $(u_\epsilon, s_\epsilon, \sigma_\epsilon) \in \mathcal{D}(\Omega)$ converging to (u, s, σ) in $[L^1(\Omega)]^3$ as $\epsilon \to 0^+$ such that

$$\limsup_{\epsilon \to 0^+} \mathcal{F}_\epsilon(u_\epsilon, s_\epsilon, \sigma_\epsilon) \leq \mathcal{F}(u, s, \sigma). \tag{3.17}$$

Remark 1. By the stability property of Γ–convergence with respect to continuous perturbations, the Γ–convergence result still holds if in the functional $\mathcal{F}$ the term $\mu \mid u - g \mid^2$ is replaced by $\Phi(x, u)$ in such a way that the functional $u \to \int_\Omega \Phi(x, u)\, dx$ is continuous with respect to the strong $L^1(\Omega)$ topology. Let Φ be a Carathéodory function on $\Omega \times \mathbf{R}$, i.e. $\Phi(\cdot, p)$ is measurable for any $p \in \mathbf{R}$ and $\Phi(x, \cdot)$ is continuous for almost every $x \in \Omega$. Then a sufficient condition for the continuity is the following [19]:

$$\begin{cases} \Phi : \Omega \times \mathbf{R} \to \mathbf{R} \quad \textit{is Carathéodory} \ , \\ 0 \leq \Phi(x, u) \leq a(x) + b \mid u \mid \ , \end{cases}$$

with $a \in L^1(\Omega)$ and $b \geq 0$.

4 Basic properties of $GSBV^2$ functions

In this section we give some technical results concerning the one–dimensional sections of functions $u \in GSBV(\Omega)$. Let $\nu \in \mathbf{S}^{n-1} = \{x \in \mathbf{R}^n :\mid x \mid = 1\}$ be a fixed direction. We set

$$\begin{aligned} \Pi_\nu &= \{x \in \mathbf{R}^n : \langle x, \nu \rangle = 0\}, \\ \Omega_x &= \{t \in \mathbf{R} : x + t\nu \in \Omega\} \quad (x \in \Pi_\nu), \\ \Omega_\nu &= \{x \in \Pi_\nu : \Omega_x \neq \emptyset\}. \end{aligned}$$

The sets Ω_x are the 1-dimensional slices of Ω indexed by $x \in \Pi_\nu$, and Ω_ν is the projection of Ω on Π_ν. Given $u \in GSBV(\Omega)$, we define for $\mathcal{H}^{n-1}$–a.e. $x \in \Omega_\nu$ the restriction

$$u_x(t) = u(x + t\nu) \qquad \text{for a.e. } t \in \Omega_x.$$

The following slicing result can be obtained from [1], Theorem 3.3, and [2], section 1.

Lemma 4.1 *Let $u : \Omega \to \mathbf{R}$ be a measurable function. Then $u \in GSBV(\Omega)$ if and only if, for any $\nu \in \mathbf{S}^{n-1}$, $u_x \in GSBV(\Omega_x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$ and*

$$\int_{A_\nu} |D(-N \vee u_x \wedge N)|(A_x)\, d\mathcal{H}^{n-1} < +\infty \tag{4.18}$$

for any open set $A \subset\subset \Omega$ and any $N \in \mathbf{N}$.
Moreover, if $u \in GSBV(\Omega)$ and $\nu \in \mathbf{S}^{n-1}$, then for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$ we have:
(a) $u'_x(t) = \langle \nabla u(x + t\nu), \nu \rangle$ *for a.e. $t \in \Omega_x$;*
(b) $S_{u_x} = (S_u)_x$.

The proof of the following lemma can be found in Federer [20], Section 3.2.22.

Lemma 4.2 *For every countably $\mathcal{H}^{n-1}$-rectifiable set $E \subset \mathbf{R}^n$ there exists a Borel function $\nu_E : E \to \mathbf{S}^{n-1}$ such that*

$$\int_E \mid \langle \nu, \nu_E(x) \rangle \mid \, d\mathcal{H}^{n-1}(x) = \int_{E_\nu} \mathcal{H}^0(E_x) \, d\mathcal{H}^{n-1}(x) \qquad \forall \nu \in \mathbf{S}^{n-1}.$$

The function $\nu_E(x)$ is a normal unit vector to E at x in an approximate sense (see [20], Section 3.2.16).

Although $GSBV(\Omega)$ is not a vector space, we can prove that the natural energy spaces for our problems do have a vector structure.

Proposition 4.3 *The set*

$$Y = \left\{ u \in GSBV(\Omega) : \int_\Omega \mid \nabla u \mid \, dx + \mathcal{H}^{n-1}(S_u) < +\infty \right\}$$

is a vector space.

Proof. Let u_1, $u_2 \in Y$ and $\nu \in \mathbf{S}^{n-1}$ be fixed. By Lemma 4.1(a,b) and Lemma 4.2 we have

$$\int_{\Omega_x} |u'_{ix}| \, dt + \mathcal{H}^0(S_{u_{ix}}) < +\infty \qquad \text{for } i = 1,\, 2$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$, because

$$\int_{\Omega_\nu} \left[\int_{\Omega_x} |u'_{ix}| \, dt + \mathcal{H}^0(S_{u_{ix}}) \right] d\mathcal{H}^{n-1} \le \int_\Omega |\nabla u_i| \, dx + \mathcal{H}^{n-1}(S_{u_i}) < +\infty.$$

In particular $u_{ix} \in L^\infty_{\mathrm{loc}}(\Omega_x)$ and since $SBV_{\mathrm{loc}}(\Omega_x)$ is a vector space $u_{1x} + u_{2x}$ belongs to $SBV_{\mathrm{loc}}(\Omega_x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$. Since the condition (4.18) is easily verified the conclusion follows by using Lemma 4.1. ■

Finally, we show how in $GSBV^2(\Omega)$ second order derivatives and jump set of the derivative can be recovered as well by a slicing method.

Lemma 4.4 *Let $u \in GSBV^2(\Omega)$ be such that*

$$\int_\Omega \mid \nabla^2 u \mid \, dx + \mathcal{H}^{n-1}(S_{\nabla u}) < +\infty.$$

Then, for any $\nu \in \mathbf{S}^{n-1}$ the function $\langle \nabla u, \nu \rangle$ belongs to $GSBV(\Omega)$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$ we have:

(a) $u'_x \in GSBV(\Omega_x)$;

(b) $u''_x(t) = \langle \nabla \langle \nabla u, \nu \rangle, \nu(x + t\nu) \rangle$ *for a.e.* $t \in \Omega_x$;

(c) $S_{u'_x} = (S_{\nabla u \cdot \nu})_x$.

Proof. By Proposition 4.3 it follows that $\langle \nabla u, \nu \rangle \in GSBV(\Omega)$ whenever $\nabla_i u \in GSBV(\Omega)$ for $i = 1, \ldots, n$. By Lemma 4.1(a) it follows that $u'_x = \langle \nabla u, \nu \rangle_x$ a.e. in Ω_x for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$; in particular $u'_x \in GSBV(\Omega_x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$. Then, statements (b), (c) follow by applying Lemma 4.1(a,b) to $\langle \nabla u, \nu \rangle$. ■

Corollary 4.5 *The set*

$$\left\{u \in GSBV^2(\Omega) : \int_\Omega \mid \nabla^2 u \mid \, dx + \mathcal{H}^{n-1}(S_u \cup S_{\nabla u}) < +\infty\right\}$$

is a vector space.

We conclude this section with an interpolation inequality in $W^{2,2}$ which provides a mild estimate of $\int |\nabla u|^2 \, dx$ with the Blake–Zisserman energy (see also [12]).

Proposition 4.6 *Let A, $B \subset \mathbf{R}^n$ be open sets with $(A)_{2r} \subset\subset B$. Then*

$$\int_A |\nabla u|^2 \, dx \leq 16n \left[r^{-2} \int_B u^2 \, dx + 2r^2 \int_B |\nabla^2 u|^2 \, dx\right] \qquad \forall u \in W^{2,2}_{\rm loc}(B) \ . \tag{4.19}$$

Proof. We prove the inequality only in the case $n = 1$; the general case can be achieved by a slicing argument, taking into account Lemma 4.4(b).

Let $x \in A$ and choose $x_1 \in [x + r, x + 2r]$, $x_2 \in [x - 2r, x - r]$ such that

$$ru(x_1) = \int_{x+r}^{x+2r} u(s) \, ds, \qquad ru(x_2) = \int_{x-2r}^{x-r} u(s) \, ds$$

and $x_3 \in [x_2, x_1]$ such that $u'(x_3) = [u(x_1) - u(x_2)]/(x_1 - x_2)$. Then, for any $y \in [x - 2r, x + 2r]$, using twice Hölder inequality we estimate

$$\begin{aligned} |u'(y)|^2 &\leq 2|u'(x_3)|^2 + 2\left(\int_{x_3}^y u''(s) \, ds\right)^2 \leq \frac{4(u^2(x_1) + u^2(x_2))}{r^2} + 2|x_3 - y| \left|\int_{x_3}^y |u''(s)|^2 \, ds\right| \\ &\leq \frac{4}{r^3} \int_{x-2r}^{x+2r} u^2(s) \, ds + 8r \int_{x-2r}^{x+2r} |u''(s)|^2 \, ds. \end{aligned}$$

By integration we obtain

$$\int_{x-2r}^{x+2r} |u'|^2 \, dy \leq \frac{16}{r^2} \int_{x-2r}^{x+2r} u^2 \, dy + 32r^2 \int_{x-2r}^{x+2r} |u''|^2 \, dy \ .$$

By a standard covering argument the conclusion follows. ■

5 The Approximation Framework

In this section we find a domain suitable for coercivity and lower semicontinuity of the functionals F_ϵ formally defined by (3.15).

We often set $w = (u, s, \sigma)$ and we always assume that $0 \leq s \leq 1$, $0 \leq \sigma \leq 1$ almost everywhere. If $\kappa_\epsilon = 0$ we define $p = 2\gamma/(\gamma + 2)$ and

$$\mathcal{D}(\Omega) = \{(u, s, \sigma) \in X(\Omega) : \ u, \, s, \, \sigma \in W^{1,2}(\Omega), \ \sigma \nabla u \in W^{1,p}(\Omega; \mathbf{R}^n)\} \ ,$$

if $\kappa_\epsilon > 0$ we define

$$\mathcal{D}(\Omega) = W^{2,2}_{\rm loc}(\Omega) \times W^{1,2}(\Omega; [0,1]) \times W^{1,2}(\Omega; [0,1]) \ .$$

If $u \in \mathcal{D}(\Omega)$ the approximate differentiability of u and of $\sigma\nabla u$ imply that $\nabla^2 u$ exists almost everywhere in $\{\sigma > 0\}$ and is given by

$$\nabla^2 u = \frac{\nabla(\sigma\nabla u) - \nabla\sigma \otimes \nabla u}{\sigma} \ . \tag{5.20}$$

We also set $\nabla^2 u = 0$ in $\{\sigma = 0\}$ (this set is empty if $\kappa_\epsilon > 0$).

We now prove a compactness theorem for the sublevels of F_ϵ.

Theorem 5.1 *Assume that $\gamma > 2$, $\kappa_\epsilon + \zeta_\epsilon > 0$ and let $(w_h) = (u_h, s_h, \sigma_h) \subset \mathcal{D}(\Omega)$ be a sequence such that*

$$\sup_h F_\epsilon(w_h) < +\infty.$$

Then there exist a subsequence (w_{h_k}) and $w = (u, s, \sigma) \in \mathcal{D}(\Omega)$ such that (w_{h_k}) converge in $[L^1(\Omega)]^3$ to w and (∇u_{h_k}) converge a.e. to ∇u in $\{\sigma > 0\}$.

Proof. The case when $\kappa_\epsilon > 0$ is a straightforward consequence of Rellich theorem and (4.19), which shows that (u_h) is bounded in $W^{2,2}(A)$ for any open set $A \subset\subset \Omega$. Hence, in the following we consider the more delicate case when $\kappa_\epsilon = 0$ and $\zeta_\epsilon > 0$.

From the definition of F_ϵ the sequences (s_h) and (σ_h) are bounded in $W^{1,2}(\Omega)$. Moreover, since $(|\nabla u_h|)$ is bounded in $L^\gamma(\Omega)$ and

$$\nabla(\sigma_h \nabla u_h) = \sigma_h \nabla^2 u_h + \nabla\sigma_h \otimes \nabla u_h$$

also $v_h = \sigma_h \nabla u_h$ are bounded in $W^{1,p}(\Omega; \mathbf{R}^n)$. Hence, possibly extracting a further subsequence we can assume that (v_{h_k}) is converging a.e. in Ω. It easily follows that $\nabla u_{h_k} = v_{h_k}/\sigma_{h_k}$ converge a.e. to ∇u in $\{\sigma > 0\}$.

In order to prove that $\sigma\nabla u \in W^{1,p}(\Omega; \mathbf{R}^n)$ (hence $w \in \mathcal{D}(\Omega)$) we notice that by Hölder inequality, we have

$$\lim_{h\to+\infty} \int_{\{\sigma=0\}} |\sigma_h \nabla u_h| \, dx = 0$$

hence (possibly extracting a subsequence) $\sigma_h \nabla u_h$ converge a.e. to $\sigma\nabla u$ in the whole of Ω. Since $(\sigma_h \nabla u_h)$ is also bounded in $W^{1,p}(\Omega; \mathbf{R}^n)$, it follows that $\sigma\nabla u \in W^{1,p}(\Omega; \mathbf{R}^n)$ and that $\sigma_{h_k} \nabla u_{h_k}$ weakly converge in $W^{1,p}(\Omega; \mathbf{R}^n)$ to $\sigma\nabla u$. ■

Now we prove the lower semicontinuity of F_ϵ.

Theorem 5.2 *Assume that $\gamma > 2$, $\kappa_\epsilon + \zeta_\epsilon > 0$ and let $(w_h) = (u_h, s_h, \sigma_h) \subset \mathcal{D}(\Omega)$ be converging in $[L^1(\Omega)]^3$ to $w = (u, s, \sigma) \in \mathcal{D}(\Omega)$. Then*

$$\liminf_{h\to+\infty} F_\epsilon(w_h) \geq F_\epsilon(w) \ .$$

Proof. Also in this case we consider only the more difficult case when $\kappa_\epsilon = 0$ and $\zeta_\epsilon > 0$. It is not restrictive to assume that $(F_\epsilon(w_h))$ is converging to a finite limit and, by Theorem 5.1 and its proof, we can also assume that ∇u_h converge to ∇u a.e. in $\{\sigma > 0\}$ and $\sigma_h \nabla u_h$ weakly converge in $W^{1,p}(\Omega; \mathbf{R}^n)$ to $\sigma\nabla u$.

Since the sequences (s_h) and (σ_h) are bounded in $W^{1,2}(\Omega)$, they weakly converge respectively to s, σ and therefore the terms $\mathcal{G}_\epsilon(s)$ and $\mathcal{G}_\epsilon(\sigma)$ are lower semicontinuous. The lower semicontinuity of $\int_\Omega (s^2 + \zeta_\epsilon)|\nabla u|^\gamma \, dx$ directly follows by Ioffe lower semicontinuity theorem (see [10], Theorem 4.1.1).

Finally, the identity

$$\nabla(\sigma_h \nabla u_h) = \sigma_h \nabla^2 u_h + \nabla \sigma_h \otimes \nabla u_h$$

and the weak convergence of $\nabla(\sigma_h \nabla u_h)$ to $\nabla(\sigma \nabla u)$ easily imply that $\nabla^2 u_h$ weakly converge to $\nabla^2 u$ in $L^2(K; \mathbf{M}^{n \times n})$ on any compact set $K \subset \Omega$ on which (σ_h) uniformly converges to σ, (∇u_h) uniformly converges to ∇u and $\inf_K \sigma > 0$. Then, Ioffe lower semicontinuity theorem again gives

$$\int_K \sigma^2 |\nabla^2 u|^2 \, dx \leq \liminf_{h \to +\infty} \int_K \sigma_h^2 |\nabla^2 u_h|^2 \, dx.$$

Let $\delta > 0$; by Egorov theorem we can cover almost all of $\{\sigma \geq \delta\}$ by an increasing sequence of compact sets on which (σ_h) and (∇u_h) are uniformly converging. As a consequence, the inequality above holds with $\{\sigma \geq \delta\}$ in place of K, and letting $\delta \downarrow 0$ we obtain the lower semicontinuity of the term $\int_\Omega \sigma^2 |\nabla^2 u|^2 \, dx$. ■

From the compactness and the lower semicontinuity properties of the functional F_ϵ it follows that for any $\epsilon > 0$ the problem

$$(\mathcal{P}_\epsilon) \qquad \min \{F_\epsilon(u, s, \sigma) : \ (u, s, \sigma) \in \mathcal{D}(\Omega)\}$$

has at least one solution, provided $\gamma > 2$.

6 The Lower Inequality

In this section we prove the lower inequality of Γ–convergence (2.12) and the equi-coercivity of the family $(\mathcal{F}_\epsilon)$. In the following it will be convenient also to consider functionals depending on the domain of integration.

The following lower bound for the jump terms in the one dimensional case has been shown by Bellettini and Coscia in [7], Theorem 3.1.

Lemma 6.1 *Assume that κ_ϵ, ξ_ϵ, ζ_ϵ are as in Theorem 3.1 and $\gamma \geq 2$. Let $I \subset \mathbf{R}$ be a bounded open set and $\epsilon_h \to 0^+$. Then, for every sequence (w_h) converging to w in $[L^1(I)]^3$ as $h \to +\infty$ such that $\mathcal{F}_{\epsilon_h}(w_h)$ is bounded we have*

$$\liminf_{h \to +\infty} [(\alpha - \beta)\mathcal{G}_{\epsilon_h}(s_h, I) + \beta \mathcal{G}_{\epsilon_h}(\sigma_h, I)] \geq (\alpha - \beta)\mathcal{H}^0(S_u \cap I) + \beta \mathcal{H}^0 \left((S_u \cup S_{u'}) \cap I\right) \ . \tag{6.21}$$

The condition (3.16) on ξ_ϵ can be dropped in the case $\alpha = \beta$.

Since our functionals are slightly different from those in [7], some remarks are necessary. Indeed, the functionals in [7] are given by

$$\begin{aligned} F_\epsilon(u, s, \sigma) \ &= \ \int_\Omega (\sigma^2 + \kappa_\epsilon) \mid \nabla^2 u \mid^2 \ dx + \int_\Omega \mid u - g \mid^2 \ dx + (\alpha - \beta)\mathcal{G}_\epsilon(s) \\ &+ \ \beta \mathcal{G}_\epsilon(\sigma) + \xi_\epsilon \int_\Omega s^2 \mid \nabla u \mid^2 \ dx \ , \end{aligned}$$

hence the differences with respect to ours are two: first they assume that $\zeta_\epsilon = 0$ and $\gamma = 2$ and then they prove the lower bound only in the case when $\kappa_\epsilon > 0$ (hence $u_h \in W^{2,2}(I)$). The assumption that $\zeta_\epsilon = 0$ is not a problem, since smaller functionals are considered, and also a general exponent γ can be considered, provided (3.16) holds. However, for technical reasons related to the proof of the Γ–limsup inequality, in particular the difficulty to estimate the second derivatives of $u_\epsilon = \psi_\epsilon u$ in the proof of Theorem 3.4 (this can be avoided in the case $n = 1$ using suitable interpolating cubic polynomials), we have preferred a different formulation of the energy in the larger class $\mathcal{D}(\Omega)$, which still provides compactness of minimizing sequences and lower semicontinuity of the energy. Moreover, the proof of the Γ–liminf inequality of Bellettini and Coscia works, essentially with no modification, also for our more general functionals. Notice also that our full Γ-convergence result Theorem 3.3 fits exactly in the Bellettini and Coscia framework.

The reason why no condition on ξ_ϵ (besides $\xi_\epsilon \geq 0$) is necessary in the case $\alpha = \beta$ is that the term $\xi_\epsilon \int s^2 |\nabla u|^\gamma \, dx$ has been added to the energy to force σ_h to tend to zero at least *twice* (paying asymptotically at least $2\beta \geq \alpha$) near to jump points of u if s_h is far away from 0 (if this does not happen and (3.16) holds, then the additional term diverges, see Lemma 3.2(i) of [7]); in the case when $\alpha = \beta$ it is not necessary to force this behaviour of σ_h, since σ_h is already forced by the other terms of F_ϵ to tend to zero at least *once* (paying asymptotically at least β) near to jump points of u or of u', regardless of the values of s_h.

Finally, we notice that we can restate (6.21) as follows

$$\begin{aligned} &\liminf_{h\to+\infty} \left[t\mathcal{F}_{\epsilon_h}(w_h) + (\alpha-\beta)\mathcal{G}_{\epsilon_h}(s_h, I) + \beta\mathcal{G}_{\epsilon_h}(\sigma_h, I)\right] \\ \geq\ & (\alpha-\beta)\mathcal{H}^0(S_u \cap I) + \beta\mathcal{H}^0\left((S_u \cup S_{u'}) \cap I\right) \qquad \forall t > 0 \ . \end{aligned} \tag{6.22}$$

The advantage of this new formulation is that the apriori assumption that $\mathcal{F}_{\epsilon_h}(w_h)$ is bounded can be dropped.

Proof of Theorem 3.1

Let $(w_\epsilon) \in \mathcal{D}(\Omega)$, $w \in X(\Omega)$, be such that $w_\epsilon \to w$ in $[L^1(\Omega)]^3$ as $\epsilon \to 0^+$. We assume that

$$+\infty > L = \liminf_{\epsilon\to 0} \mathcal{F}_\epsilon(w_\epsilon, \Omega) = \lim_{h\to+\infty} \mathcal{F}_{\epsilon_h}(w_{\epsilon_h}, \Omega) \ , \tag{6.23}$$

otherwise the result is trivial. For notational simplicity we set $w_{\epsilon_h} = (u_h, s_h, \sigma_h)$ and we assume that w_{ϵ_h} converge a.e. to (u, s, σ) as $h \to +\infty$.

We also assume that (w_h) converges to w fast enough, i.e. $\sum_h \|w_h - w\|_{L^1} < +\infty$. This assumption and Fubini theorem imply (with the notation of Section 4)

$$\lim_{h\to\infty} w_{hx} = w_x \quad \text{a.e. in } \Omega_x \qquad\qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Omega_\nu$$

for any direction $\nu \in \mathbf{S}^{n-1}$, and this will be useful in the sequel.

If either s or σ were not identically equal to 1, then by the Fatou lemma we would get

$$L \geq \liminf_{h\to+\infty} \left[(\alpha-\beta)\int_{\{s\neq 1\}} \frac{(s_h-1)^2}{4\epsilon_h}\,dx + \beta\int_{\{\sigma\neq 1\}} \frac{(\sigma_h-1)^2}{4\epsilon_h}\,dx\right] \geq +\infty,$$

which contradicts the assumption that $L < +\infty$. Therefore, we will assume that $s \equiv 1$ and $\sigma \equiv 1$.

The proof now follows by proving separately the following inequalities:

$$\liminf_{h\to+\infty}\int_\Omega {\sigma_h}^2 \mid \nabla^2 u_h \mid^2 \, dx \geq \int_\Omega \mid \nabla^2 u \mid^2 \, dx \ , \tag{6.24}$$

$$\liminf_{h\to+\infty}[(\alpha-\beta)\mathcal{G}_{\epsilon_h}(s_h,\Omega)+\beta\mathcal{G}_{\epsilon_h}(\sigma_h,\Omega)] \geq (\alpha-\beta)\mathcal{H}^{n-1}(S_u)+\beta\mathcal{H}^{n-1}(S_u\cup S_{\nabla u}) \ . \tag{6.25}$$

Possibly extracting a subsequence (this is allowed, since we are assuming that $\mathcal{F}_{\epsilon_h}(w_{\epsilon_h})$ is converging) we can assume that both lim inf in (6.24) and (6.25) are finite limits, denoted by L_1 and L_2 respectively.

We first prove (6.24). Let $\psi(t)=\int_0^t(1-\tau)\,d\tau$; using (6.23) we have

$$\int_\Omega \mid \nabla\psi(\sigma_h) \mid \, dx = \int_\Omega \mid \nabla\sigma_h \mid (1-\sigma_h)\,dx \leq \int_\Omega \left[\epsilon_h \mid \nabla\sigma_h \mid^2 + \frac{(1-\sigma_h)^2}{4\epsilon_h}\right] dx \leq L+1 \ ,$$

for h large enough. Then, by the co-area formula (2.10), we have

$$\int_0^{\psi(1)} P(\{\psi(\sigma_h)>t\},\Omega)\,dt = \int_\Omega \mid \nabla\psi(\sigma_h) \mid \, dx \leq L+1$$

for h large enough. By the Fatou lemma we then get

$$\int_{\psi(a)}^{\psi(1)} \liminf_{h\to+\infty} P(\{\psi(\sigma_h)>t\},\Omega)\,dt \leq \liminf_{h\to+\infty}\int_\Omega \mid \nabla\psi(\sigma_h) \mid \, dx \leq L+1$$

for any $a\in(0,1)$. Therefore there exists $t_0=\psi(\theta)\in(\psi(a),\psi(1))$ for some $\theta\in(a,1)$ such that

$$\liminf_{h\to+\infty} P(\{\psi(\sigma_h)>t_0\},\Omega) \leq l < +\infty \tag{6.26}$$

with $l=(L+1)/[\psi(1)-\psi(a)]$.

Then, if we set $E_h=\{\sigma_h>\theta\}$, by (6.26) we get $P(E_h,\Omega)\leq l+1$ for infinitely many h; for notational simplicity we will assume in the following that the inequality is true for any h (in the general case a further subsequence must be extracted). By the L^1 convergence of (σ_h) to 1 we obtain

$$\mathrm{meas}(\Omega\setminus E_h) \leq \frac{1}{1-\theta}\int_\Omega(1-\sigma_h)\,dx \to 0 \ .$$

Then we define

$$v_h = u_h 1_{E_h} \ . \tag{6.27}$$

By the locality property (2.9) we get

$$\nabla v_h = 1_{E_h}\nabla u_h \ , \qquad \nabla^2 v_h = 1_{E_h}\nabla^2 u_h \ ,$$

for a.e. $x\in\Omega$. Since

$$-N\vee v_h\wedge N = 1_{E_h}[-N\vee u_h\wedge N] \qquad\qquad \forall N\in\mathbf{N} \ ,$$

from [36] Ch. 4 Section 6.4 it follows that $v_h\in GSBV(\Omega)$. Analogously,

$$-N\vee(\nabla_i v_h)\wedge N = 1_{E_h}[-N\vee(\nabla_i u_h)\wedge N] \in SBV_{\mathrm{loc}}(\Omega)$$

for any $N \in \mathbf{N}$ and any $i = 1, \dots, n$. Then $v_h \in GSBV^2(\Omega)$ and we have

$$\mathcal{H}^{n-1}(S_{v_h} \cup S_{\nabla v_h}) \le P(E_h, \Omega) \le l + 1 \quad \text{for every } h \in \mathbf{N} \; .$$

Then, since $\sigma_h \ge \theta$ on E_h, the sequence (v_h) satisfies all the assumptions of the compactness Theorem 2.1, hence we can assume (again, possibly passing to a subsequence) that (v_h) converges in $L^1(\Omega)$ to some function $v \in GSBV^2(\Omega)$ with $\nabla v_h \to \nabla v$ a.e. in Ω and $\nabla^2 v_h$ weakly converging to $\nabla^2 v$ in $L^2(\Omega; \mathbf{M}^{n\times n})$. Since (u_h) converges to u in $L^1(\Omega)$ and $\mathrm{meas}(\Omega \setminus E_h) \to 0$, we obtain that $u = v \in GSBV^2(\Omega)$; moreover, by the lower semicontinuity of quadratic forms with respect to weak convergence in L^2 we get

$$L_1 \ge \liminf_{h\to+\infty} \int_{E_h} \theta^2 \mid \nabla^2 u_h \mid^2 \, dx = \liminf_{h\to+\infty} \int_{\Omega} \theta^2 \mid \nabla^2 v_h \mid^2 \, dx \ge \int_{\Omega} \theta^2 \mid \nabla^2 u \mid^2 \, dx \; .$$

By letting $a \uparrow 1$ (hence $\theta \to 1$) we obtain (6.24).

The relation (6.25) will be proved using (6.22) and a slicing argument. Let $A \subset \Omega$ open, and $\nu \in \mathbf{S}^{n-1}$ be fixed. By using the notation of Section 4 we have

$$\begin{aligned}
\mathcal{G}_{\epsilon_h}(s_h, A) \;\; &\ge \;\; \int_A \left(\epsilon_h \mid \langle \nabla s_h, \nu\rangle \mid^2 + \frac{(s_h-1)^2}{4\epsilon_h} \right) dx \\
&= \;\; \int_{A_\nu} d\mathcal{H}^{n-1}(x) \int_{A_x} \left(\epsilon_h \mid s'_{hx} \mid^2 + \frac{(s_{hx}-1)^2}{4\epsilon_h} \right) dt \\
&= \;\; \int_{A_\nu} \mathcal{G}_{\epsilon_h}(s_{hx}, A_x)\, d\mathcal{H}^{n-1}(x) \; .
\end{aligned}$$

An analogous relation holds for $\mathcal{G}_{\epsilon_h}(\sigma_h, A)$ and, taking into account Lemma 4.4, for $\mathcal{F}_{\epsilon_h}(w_h, A)$.

Since w_{hx} converge to w_x in $[L^1(\Omega_x)]^3$ for $\mathcal{H}^{n-1}$-almost every $x \in \Omega_\nu$, by using Fatou lemma, (6.22), Lemmas 4.1 and 4.4, and eventually Lemma 4.2 we get

$$\begin{aligned}
&\liminf_{h\to+\infty} [t\mathcal{F}_{\epsilon_h}(w_h) + (\alpha-\beta)\mathcal{G}_{\epsilon_h}(s_h, A) + \beta\mathcal{G}_{\epsilon_h}(\sigma_h, A)] \\
\ge \;\; & \int_{A_\nu} \liminf_{h\to+\infty} [t\mathcal{F}_{\epsilon_h}(w_{hx}) + (\alpha-\beta)\mathcal{G}_{\epsilon_h}(s_{hx}, A_x) + \beta\mathcal{G}_{\epsilon_h}(\sigma_{hx}, A_x)] \; d\mathcal{H}^{n-1}(x) \\
\ge \;\; & (\alpha-\beta)\int_{A_\nu} \mathcal{H}^0(S_{u_x} \cap A_x)\, d\mathcal{H}^{n-1}(x) + \beta \int_{A_\nu} \mathcal{H}^0((S_{u_x} \cup S_{u'_x}) \cap A_x)\, d\mathcal{H}^{n-1}(x) \\
= \;\; & (\alpha-\beta)\int_{A_\nu} \mathcal{H}^0((S_u \cap A)_x)\, d\mathcal{H}^{n-1}(x) + \beta \int_{A_\nu} \mathcal{H}^0(((S_u \cup S_{\nabla u\cdot\nu}) \cap A)_x)\, d\mathcal{H}^{n-1}(x) \\
= \;\; & \alpha \int_{S_u\cap A} \mid \langle \nu, \nu_u(x)\rangle \mid \; d\mathcal{H}^{n-1}(x) + \beta \int_{(S_{\nabla u\cdot\nu}\setminus S_u)\cap A} \mid \langle \nu, \nu_{\nabla u}(x)\rangle \mid \; d\mathcal{H}^{n-1}(x) \; ,
\end{aligned}$$

for any $t > 0$, where $\nu_u(x)$ and $\nu_{\nabla u}(x)$ are approximate unit normals to S_u and $S_{\nabla u}$ respectively. Letting $t \downarrow 0$ we obtain

$$\begin{aligned}
&\liminf_{h\to+\infty} [(\alpha-\beta)\mathcal{G}_{\epsilon_h}(s_h, A) + \beta\mathcal{G}_{\epsilon_h}(\sigma_h, A)] \\
\ge \;\; & \alpha \int_{S_u\cap A} \mid \langle \nu, \nu_u(x)\rangle \mid \; d\mathcal{H}^{n-1}(x) + \beta \int_{(S_{\nabla u\cdot\nu}\setminus S_u)\cap A} \mid \langle \nu, \nu_{\nabla u}(x)\rangle \mid \; d\mathcal{H}^{n-1}(x) \; .
\end{aligned} \tag{6.28}$$

We now apply Lemma 2.2 in the following framework:

- $f_\nu(x) = \alpha \mid \langle \nu, \nu_u(x) \rangle \mid 1_{S_u} + \beta \mid \langle \nu, \nu_{\nabla u}(x) \rangle \mid 1_{S_{\nabla u \cdot \nu} \setminus S_u}$,
- $\mu = \mathcal{H}^{n-1} \lfloor (S_u \cup S_{\nabla u})$,

with ν varying in a countable dense subset D of $\mathbf{S}^{n-1}$. Since $\sup_{\nu \in D} f_\nu = \alpha 1_{S_u} + \beta 1_{S_{\nabla u} \setminus S_u}$ (because any $x \in S_{\nabla u}$ belongs to $S_{\nabla u \cdot \nu}$ provided $\langle \nu, \nabla^+ u(x) - \nabla^- u(x) \rangle \neq 0$), by Lemma 2.2 we have that

$$(\alpha - \beta)\mathcal{H}^{n-1}(S_u) + \beta \mathcal{H}^{n-1}(S_u \cup S_{\nabla u})$$

is equal to the supremum of

$$\sum_{i=1}^{k} \left\{ \alpha \int_{S_u \cap A_i} \mid \langle \nu_i, \nu_u(x) \rangle \mid \, d\mathcal{H}^{n-1}(x) + \beta \int_{(S_{\nabla u \cdot \nu_i} \setminus S_u) \cap A_i} \mid \langle \nu_i, \nu_{\nabla u}(x) \rangle \mid \, d\mathcal{H}^{n-1}(x) \right\}$$

among all finite families (A_i, ν_i) with $\nu_i \in D$ and $A_i \subset \Omega$ open and pairwise disjoint. By (6.28) and the superadditivity of the lim inf operator, any of these sums is less than L_2, whence the inequality (6.25) follows (see also [5]). ■

Proof of Theorem 3.2
By the equiboundedness of $\mathcal{F}_\epsilon(w_\epsilon)$ it follows immediately that $(s_\epsilon, \sigma_\epsilon) \to (1,1)$ in $[L^1(\Omega)]^2$ as $\epsilon \to 0^+$.

Reasoning as in the proof of (6.24) of Theorem 3.1 we can find a sequence $\epsilon_h \to 0^+$ and measurable sets E_h such that $\mathrm{meas}(\Omega \setminus E_h) \to 0$ and $v_{\epsilon_h} = u_{\epsilon_h} 1_{E_h}$ converge in $L^1(\Omega)$ to $u \in GSBV^2(\Omega) \cap L^2(\Omega)$. Since (u_ϵ) is equibounded in $L^2(\Omega)$, by Hölder inequality $\|u_{\epsilon_h} - v_{\epsilon_h}\|_{L^1} \to 0$ as $h \to +\infty$, hence $u_{\epsilon_h} \to u$ in $L^1(\Omega)$. ■

7 The Upper Inequality

Proof of Theorem 3.3
We can assume without losing generality that $\mu = 0$, $u \in GSBV^2(\Omega) \cap L^2(\Omega)$, $\mid \nabla^2 u \mid \in L^2(\Omega)$ and $\sigma \equiv 1$. Since we are assuming that $\alpha = \beta$ we simply set $s_\epsilon \equiv 1$ for any $\epsilon > 0$.

Assuming that Ω is star-shaped with respect to the origin, we set $\Omega_t = t\Omega$ with $t \in (0,1)$ and construct a family $w_\epsilon = (u_\epsilon, s_\epsilon, \sigma_\epsilon) \in \mathcal{D}(\Omega_t)$ such that (as in the previous section we emphasize the dependence on the domain of integration)

$$\limsup_{\epsilon \to 0^+} \mathcal{F}_\epsilon(w_\epsilon, \Omega_t) \leq \mathcal{F}(w, \Omega) \ . \tag{7.29}$$

Then, the functions $w_{\epsilon,t}(x) = w_\epsilon(tx)$ belong to $\mathcal{D}(\Omega)$ and satisfy

$$\limsup_{\epsilon \to 0^+} \mathcal{F}_\epsilon(w_{\epsilon,t}, \Omega) \leq t^{-n} \mathcal{F}(w, \Omega) \ ,$$

hence the desired family of the Γ–limsup inequality can be constructed by a diagonal argument by letting $t \uparrow 1$.

In order to construct the family (w_ϵ) satisfying (7.29) we follow the outline of [6], assuming first that

$$\mathcal{M}^* \left((S_u \cup S_{\nabla u}) \cap K \right) = \mathcal{H}^{n-1} \left((S_u \cup S_{\nabla u}) \cap K \right) \qquad \text{for any } K \subset \Omega \text{ compact} \ . \tag{7.30}$$

We restrict our choice to the functions u_ϵ and σ_ϵ that, outside a tubular neighborhood of $S_u \cup S_{\nabla u}$, with radius depending on ϵ, are respectively equal to u and 1.

Setting $\tilde{S}_u = S_u \cup S_{\nabla u}$ and $\tau(x) = \mathrm{dist}(x, \tilde{S}_u)$, by the interpolation inequality (4.19) we obtain a constant C depending only on u such that

$$\int_{\Omega_t \setminus (\tilde{S}_u)_r} |\nabla u|^2 \, dx \leq C r^{-2} \tag{7.31}$$

for any r sufficiently small. In view of our assumption on κ_ϵ, we can find an infinitesimal b_ϵ faster than ϵ such that $\kappa_\epsilon = o(b_\epsilon^4)$ (for instance $b_\epsilon = (\epsilon \kappa_\epsilon^{1/4})^{1/2}$), and an infinitesimal η_ϵ faster than $\sqrt{\epsilon}$ such that $a_\epsilon = -2\epsilon \ln \eta_\epsilon$ is infinitesimal (for instance $\eta_\epsilon = \epsilon$).
With this choice of the infinitesimals, we then define

$$\sigma_\epsilon(x) \quad = \quad \begin{cases} 0 & \text{if } x \in (\tilde{S}_u)_{b_\epsilon} \\ 1 - \eta_\epsilon & \text{if } x \in \Omega_t \setminus (\tilde{S}_u)_{a_\epsilon + b_\epsilon} \ . \end{cases}$$

Let now y_ϵ be the solution of the Cauchy problem

$$\dot{y}(t) = \frac{1-y}{2\epsilon}, \qquad y(b_\epsilon) = 0,$$

that is $y_\epsilon(t) = 1 - \exp\left[(b_\epsilon - t)/(2\epsilon)\right]$. We complete the definition of σ_ϵ by setting

$$\sigma_\epsilon(x) = y_\epsilon \circ \tau(x) \qquad \text{if } x \in (\tilde{S}_u)_{a_\epsilon + b_\epsilon} \setminus (\tilde{S}_u)_{b_\epsilon} \ .$$

Now we turn to the choice of u_ϵ. To this aim, we build a smooth function $\psi_\epsilon : \Omega \to [0,1]$ such that $\psi_\epsilon = 0$ in $\{\tau \leq b_\epsilon/2\}$, $\psi_\epsilon = 1$ in $\{\tau \geq b_\epsilon\}$ and $|\nabla \psi_\epsilon| = O(1/b_\epsilon)$, $|\nabla^2 \psi_\epsilon| = O(1/b_\epsilon^2)$. Taking into account that $|\nabla \tau| = 1$ almost everywhere, a function ψ_ϵ with the required properties can be built as $(\chi \circ \tau) * \rho$, where ρ is a convolution kernel with diameter $b_\epsilon/3$ and $\chi(s) = [0 \vee (6s/b_\epsilon - 4) \wedge 1]$. The assumptions on u and the interpolation inequality (4.19) yield

$$u \in W^{2,2}_{\mathrm{loc}}(\Omega \setminus \overline{\tilde{S}_u}) \ ,$$

so that, if we set $u_\epsilon = u\psi_\epsilon$, we have $u_\epsilon \in W^{2,2}(\Omega_t)$.
With these choices, we get

$$\mathcal{F}_\epsilon(w_\epsilon, \Omega_t) = \int_{\Omega_t} (\sigma_\epsilon^2 + \kappa_\epsilon) \mid \nabla^2 u_\epsilon \mid^2 \ dx \tag{7.32}$$

$$+\beta \mathcal{G}_\epsilon(\sigma_\epsilon, \Omega_t \cap ((\tilde{S}_u)_{a_\epsilon + b_\epsilon} \setminus (\tilde{S}_u)_{b_\epsilon})) \tag{7.33}$$

$$+\beta \frac{\mathrm{meas}(\Omega_t \cap (\tilde{S}_u)_{b_\epsilon})}{4\epsilon} \tag{7.34}$$

$$+\beta \frac{\eta_\epsilon^2}{4\epsilon} \mathrm{meas}(\Omega_t \setminus (\tilde{S}_u)_{a_\epsilon + b_\epsilon}) \ . \tag{7.35}$$

Since $u_\epsilon \equiv u$ on $\{\sigma_\epsilon > 0\}$ (because $\psi_\epsilon \equiv 1$ on $\{\tau \geq b_\epsilon\} \supset \{\sigma_\epsilon > 0\}$), the upper limit of the term in (7.32) does not exceed

$$\int_\Omega |\nabla^2 u|^2 \, dx + \limsup_{\epsilon \to 0^+} \kappa_\epsilon \int_{\Omega_t \cap \{b_\epsilon/2 \leq \tau \leq b_\epsilon\}} |\nabla^2 u_\epsilon|^2 \, dx \ .$$

Taking into account (7.31), the identity

$$\nabla^2 u_\epsilon = \psi_\epsilon \nabla^2 u + 2\nabla\psi_\epsilon \otimes \nabla u + u\nabla^2\psi_\epsilon \ ,$$

and our choice of b_ϵ we obtain that the lim sup above is zero.

Concerning the term in (7.33), from the proof of Theorem 3.1 of [6] it follows that its upper limit does not exceed $\beta\mathcal{M}^*(\tilde{S}_u \cap \overline{\Omega}_t)$; by (7.30) we obtain that the upper limit is less than $\beta\mathcal{H}^{n-1}(\tilde{S}_u)$.

The term in (7.34) is infinitesimal because the Minkowski content is finite and $b_\epsilon = o(\epsilon)$ and similarly the term in (7.35) is infinitesimal because $\eta_\epsilon^2 = o(\epsilon)$.

This proves, under the additional assumption (7.30), the existence of a family (w_ϵ) satisfying (7.29). The assumption can be removed as follows. Consider for any $\lambda > 0$ the penalized problem

$$\min\left\{\int_\Omega \left(|\nabla^2 v|^2 + \lambda|v-u|^2\right) dx + \beta\mathcal{H}^{n-1}\left(S_v \cup S_{\nabla v}\right) :\ v \in GSBV^2(\Omega)\cap L^2(\Omega)\right\} \ ,$$

and let u_λ be a minimizer (see [12]). Notice that $\overline{F}(u_\lambda) \le \overline{F}(u) < +\infty$, hence $u_\lambda \to u$ as $\lambda \to +\infty$. Then, it has been proved in [14] that any function u_λ fulfils (7.30), and therefore a family (w_ϵ) satisfying (7.29) for $(u,1,1)$ can be obtained from those already constructed for $(u_\lambda, 1, 1)$ by a diagonal argument. ■

Proof of Theorem 3.4

Since the proof is similar to that of Theorem 3.3 we sketch only the relevant differences. The function σ_ϵ is defined in the same way and s_ϵ is constructed analogously in a tubular neighborhood of S_u. Let $\tau_1(x) = \mathrm{dist}(x, S_u)$. In order to construct u_ϵ we fix some smooth function ψ_ϵ such that $\psi_\epsilon = 0$ in $\{\tau_1 \le b_\epsilon/2\}$, $\psi_\epsilon = 1$ in $\{\tau_1 \ge b_\epsilon\}$ and $|\nabla\psi_\epsilon| = O(1/b_\epsilon)$. The assumptions on u yield $u \in W^{1,\gamma}(\Omega\setminus\overline{S}_u)$, so that setting $u_\epsilon = u\psi_\epsilon$ we have $u_\epsilon \in W^{1,\gamma}(\Omega)$ and $\sigma_\epsilon\nabla u_\epsilon \in W^{1,p}(\Omega;\mathbf{R}^n)$.

The cut off function ψ_ϵ is built only in the tubular neighborhood of S_u, otherwise the term $\xi_\epsilon\int(s_\epsilon^2+\zeta_\epsilon)\,|\nabla u_\epsilon|^\gamma\,dx$ can not be controlled in the neighborhood of $S_{\nabla u}\setminus S_u$. Then we must set $\kappa_\epsilon = 0$ otherwise $\mathcal{F}_\epsilon(w_\epsilon)$ is not finite.

With these choices the estimates proceed in the same way as in the proof of Theorem 3.3 taking into account that the upper limit of the term $\xi_\epsilon\int(s_\epsilon^2+\zeta_\epsilon)\,|\nabla u_\epsilon|^\gamma\,dx$ does not exceed

$$\xi_\epsilon\int_\Omega |\nabla u|^\gamma\,dx + \limsup_{\epsilon\to 0^+}\xi_\epsilon\zeta_\epsilon\int_{\Omega\cap\{b_\epsilon/2\le\tau_1\le b_\epsilon\}}|\nabla u_\epsilon|^\gamma\,dx \ .$$

In view of the assumption on $\xi_\epsilon\zeta_\epsilon$, we can find an infinitesimal b_ϵ faster than ϵ such that $\xi_\epsilon\zeta_\epsilon = o(b_\epsilon^{\gamma-1})$, for instance $b_\epsilon = \left(\epsilon(\xi_\epsilon\zeta_\epsilon)^{\frac{1}{\gamma-1}}\right)^{1/2}$, so that the lim sup above is zero. ■

8 An Application to the Computation of Depth From Stereo Images

The Γ–convergent approximation has been experimented on the problem of computation of depth from a pair of stereo images for the purpose of illustration. In the following Ω denotes the open set $(0,1)\times(0,1)$ of $\mathbf{R}^2$, and $x = (x_1, x_2)$. In the case of parallel camera geometry [22] we choose the expression of the function $\Phi(x,u)$ used in [23, 24, 25]:

$$\Phi(x,u) = \mu\left[L(x_1,x_2) - R(x_1+u,x_2)\right]^2 \ ,$$

where u is the disparity function, $\mu > 0$ is a parameter, and R, L are bounded continuous functions corresponding to the right and left image intensities. Depth is inversely proportional to disparity. The Γ–convergence theorem may be applied if the functions R and L satisfy the conditions of the Remark 1, which can be fulfilled for instance by means of a convolution of the image intensities with a smooth kernel having a suitably small diameter. For the purpose of illustration we set $\gamma = 2$.

A simple discretization method, commonly used for computer vision problems [35], may be applied to the functionals $\mathcal{F}_\epsilon$ in a straightforward way. Discrete versions of u, s and σ are defined on a square lattice of coordinates (ih, jh) where $h = 1/(N-1)$, $0 \le i \le N-1$, $0 \le j \le N-1$. We denote by $u^h_{i,j}$, $s^h_{i,j}$ and $\sigma^h_{i,j}$, an approximation of $u(ih, jh)$, $s(ih, jh)$ and $\sigma(ih, jh)$, respectively. We denote by $u^h, s^h, \sigma^h \in \mathbf{R}^{N^2}$ the vectors of the discrete variables. Then we set $\kappa_\epsilon = 0$, $\zeta_\epsilon > 0$ and we discretize

$$\mathcal{F}^1_\epsilon(u, s, \sigma) = \int_\Omega \sigma^2 \mid \nabla^2 u \mid^2 \, dx + \xi_\epsilon \int_\Omega (s^2 + \zeta_\epsilon) \mid \nabla u \mid^2 \, dx \ ,$$

by

$$\begin{aligned} \mathcal{F}^1_{\epsilon,h}(u^h, s^h, \sigma^h) \;=\; & \sum_{i,j} \Big\{ (\sigma^h_{i,j})^2 \frac{1}{h^2} \Big[(u^h_{i+1,j} - 2u^h_{i,j} + u^h_{i-1,j})^2 \\ + \; & 2(u^h_{i+1,j+1} - u^h_{i,j+1} - u^h_{i+1,j} + u^h_{i,j})^2 \\ + \; & (u^h_{i,j+1} - 2u^h_{i,j} + u^h_{i,j-1})^2 \Big] \\ + \; & \xi_\epsilon ((s^h_{i,j})^2 + \zeta_\epsilon) \Big[(u^h_{i+1,j} - u^h_{i,j})^2 + (u^h_{i,j+1} - u^h_{i,j})^2 \Big] \Big\} \ . \end{aligned} \tag{8.36}$$

We set

$$\mathcal{F}^2_\epsilon(s, \sigma) = (\alpha - \beta)\mathcal{G}_\epsilon(s) + \beta \mathcal{G}_\epsilon(\sigma) \ ,$$

and we discretize $\mathcal{G}_\epsilon(s)$ by

$$\mathcal{G}_{\epsilon,h}(s^h) = \sum_{i,j} \left\{ \epsilon \left[(s^h_{i+1,j} - s^h_{i,j})^2 + (s^h_{i,j+1} - s^h_{i,j})^2 \right] + \frac{h^2}{4\epsilon} (s^h_{i,j} - 1)^2 \right\} \ , \tag{8.37}$$

and analogously for $\mathcal{G}_\epsilon(\sigma)$. Then we set

$$\mathcal{F}^3(u) = \mu \int_\Omega [L(x_1, x_2) - R(x_1 + u(x_1, x_2), x_2)]^2 \, dx \ ,$$

which is discretized by

$$\mathcal{F}^3_h(u^h) = \mu \sum_{i,j} h^2 \left(L^h_{i,j} - R^h_{i+u^h_{i,j},j} \right)^2 \ , \tag{8.38}$$

where $R^h_{i,j}$, $L^h_{i,j}$ denote an approximation of $R(ih, jh)$, $L(ih, jh)$ and, since $u^h_{i,j}$ is generally not an integer, the discretization of R is computed by means of a linear interpolation. We set

$$\mathcal{F}_{\epsilon,h}(u^h, s^h, \sigma^h) = \mathcal{F}^1_{\epsilon,h}(u^h, s^h, \sigma^h) + \mathcal{F}^2_{\epsilon,h}(s^h, \sigma^h) + \mathcal{F}^3_h(u^h) \ .$$

In order to recover a stable solution the grid must resolve the width of the transition region of the functions s and σ. Then the discretization step should be at least $h =$

$o(\epsilon)$ as it has been shown in [8] for the discretization of the Ambrosio and Tortorelli's approximating functionals. A global solution of the discrete nonconvex variational problem could be computed by means of a stochastic optimization method. However we use a faster deterministic continuation procedure in which α and β are considered as continuation variables [35]. The functional $\mathcal{F}_\epsilon^1 + \mathcal{F}_\epsilon^2$ becomes increasingly convex for larger values of these variables. Then a solution of the system of equations

$$\nabla \mathcal{F}_{\epsilon,h}(u^h, s^h, \sigma^h) = 0 \ ,$$

is computed by using a nonlinear Gauss–Seidel iterative method, with α and β initially set to high values, then gradually lowered. The continuation procedure yields experimental good, although not globally optimal, solutions. The parameters α and β are lowered according to the rule

$$\alpha^{(k)} = \alpha_0(c)^k, \qquad \beta^{(k)} = \beta_0(c)^k, \tag{8.39}$$

where α_0, β_0 are the initial values, $c < 1$ is a real number, and each step k of the continuation procedure consists of 32 nonlinear Gauss–Seidel iterations.

The Γ–convergence theorem and the continuation algorithm have been experimented on synthetic stereo pairs of images corresponding to simple patterns. The images are discretized with N=256. The brightness patterns of all the surfaces represented in the synthetic images are linear combinations of spatially orthogonal sinusoids. The spatial frequency of the sinusoids is chosen to give a reasonably strong brightness gradient such as that usually required for binocular stereo matching (see also March [23, 24, 25]). The range of brightness values for L, R is [0,255]. Depth has to be recovered from the local geometrical distortion of the brightness pattern in the left image relative to the one in the right image. The periodicity of the brightness pattern causes further difficulties to the problem of recovering disparity because of the presence of many ambiguous corresponding points in the two images.

The algorithm was started with an initial estimate of the disparity function u equal to a constant value, and setting the functions σ, s equal to 1 everywhere. The values of the parameters in the functional were chosen on the basis of the results of a number of experiments.

Figures 1 and 2 show the two images L and R of a stereo pair representing an object shaped as a revolution surface and portrayed against a plane background. The value of disparity ranges from 14 to 32 pixels ($14h \leq u \leq 32h$). The stereo disparity u in the images of Figures 1–2 is discontinuous along the occluding boundary between the curved surface and the plane background. The function u has no creases in this example.

We set $\sqrt{\mu} = 26$ and $c = 0.8$ in (8.39). The continuation procedure is iterated for 43 steps (1376 total Gauss–Seidel iterations) and the final values of the parameters are $\alpha = 37$, $\beta = 32.7$. Figure 3 shows the function σ computed with $\epsilon = 2 \cdot 10^{-2}$, and figure 4 shows the same function computed with $\epsilon = 1.3 \cdot 10^{-2}$. Figure 5 shows the function σ computed with $\epsilon = 6.5 \cdot 10^{-3}$: in this case σ reaches values of order 10^{-5} along the discontinuity set of u. In the figures representing the functions σ and s by means of grey values, white corresponds to 0 and black corresponds to 1. The figures show the convergence of the functions σ_ϵ towards the discontinuity set of the disparity u as ϵ decreases, thus illustrating the behaviour of Γ–convergence in this specific example. Figure 6 shows the function s computed with $\epsilon = 6.5 \cdot 10^{-3}$. Because of the presence of the factor ξ_ϵ converging to zero, the level sets of the functions s_ϵ might not approximate the set S_u. For instance in the present example s takes values between 0.5 and 0.7 along some portions of the discontinuity contour. However, in the proof of the lower inequality of Γ–convergence in the one–dimensional case, Bellettini and Coscia [7] show that, if s_ϵ does not tend to vanish near the points of S_u as $\epsilon \to 0$, then the functions σ_ϵ must approach zero at least twice in a neighbourhood of S_u, if ξ_ϵ converges to zero with a rate satisfying (3.16). Because of the slicing argument used in the estimate of the terms involving the Hausdorff measures in the proof of the lower inequality, an analogous property also holds in the n–dimensional case. As a consequence of this property in the numerical experiments the creases appear as contours along which the function σ is close to zero, and the jumps appear as contours with larger thickness. This effect was observed also by Blake and Zisserman [9], and may be used to discriminate creases from jumps. Finally figure 7 shows the surfaces corresponding to the disparity map recovered from the stereo pair. The discontinuity set is correctly reconstructed along the occluding contour.

Figures 8 and 9 show the images L, R of a stereo pair representing a truncated pyramid laid upon a plane background. Disparity ranges from 12 to 24 pixels ($12h \leq u \leq 24h$) and, in this example, the function u has creases and no jumps.

We set $\sqrt{\mu} = 47$ and $c = 0.76$. The continuation procedure consists of 45 steps (1440 total Gauss–Seidel iterations) and the final values of the parameters are $\alpha = 14.6$, $\beta = 7.3$. Figure 10 shows the function σ computed with $\epsilon = 8 \cdot 10^{-3}$. Figure 11 shows σ computed with $\epsilon = 2 \cdot 10^{-3}$: in this case σ reaches values of order either 10^{-3} or 10^{-2} along the set of creases of u. The figures show the capability of Γ–convergence in the localization of the creases. Figure 12 shows the surface recovered from the stereo pair with the creases correctly reconstructed.

Figures 13 and 14 show the last stereo pair used in the computer experiments which is obtained from the previous one by introducing a jump between the truncated pyramid and the plane background. Disparity ranges from 8 to 24 pixels ($8h \leq u \leq 24h$). In this example the function u has both creases and jumps.

We set $\sqrt{\mu} = 47$ and $c = 0.76$. The continuation procedure consists of 43 steps (1376 Gauss–Seidel iterations) and the final values of the parameters are $\alpha = 14.3$, $\beta = 12.6$. Figures 15 and 16 show, respectively, the functions σ and s computed with $\epsilon = 2 \cdot 10^{-3}$. The function σ reaches values of order 10^{-5} along the jumps, and of order 10^{-2} along the creases, while s is about 0.3 along the jumps. Figure 15 shows that the thickness of the curves representing the jumps is larger than thickness of the curves representing the creases. Finally figure 17 shows the surfaces corresponding to the disparity map recovered from the stereo pair. Both discontinuities and gradient discontinuities are reconstructed.

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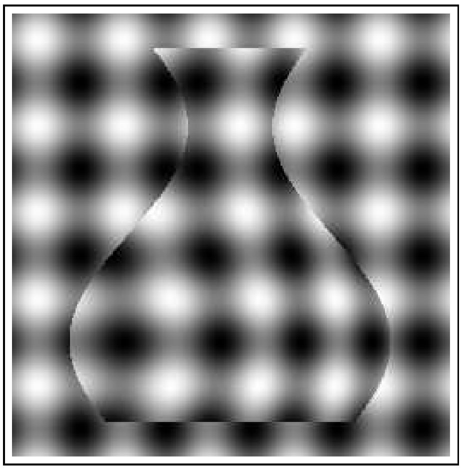

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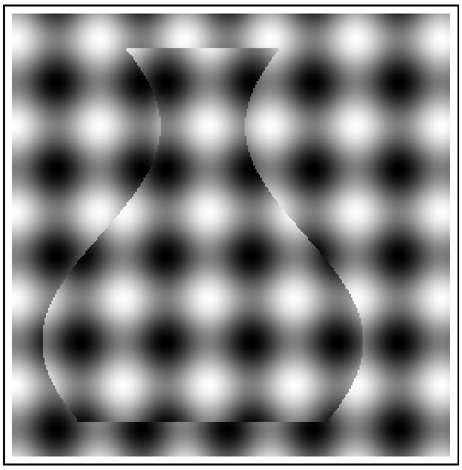
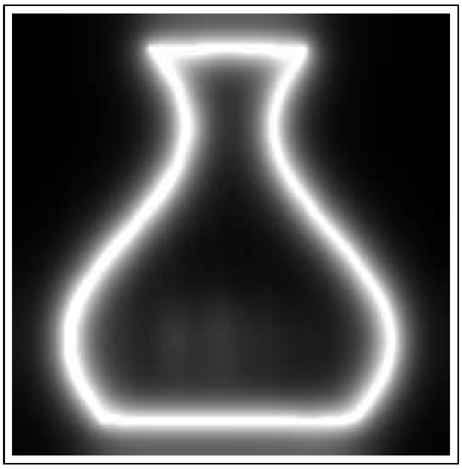


2

Figure 1: L image of a synthetic stereo pair (only jumps).
Figure 2: R image of a synthetic stereo pair (only jumps).



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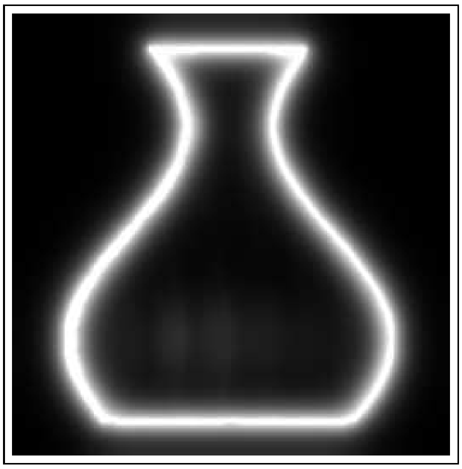
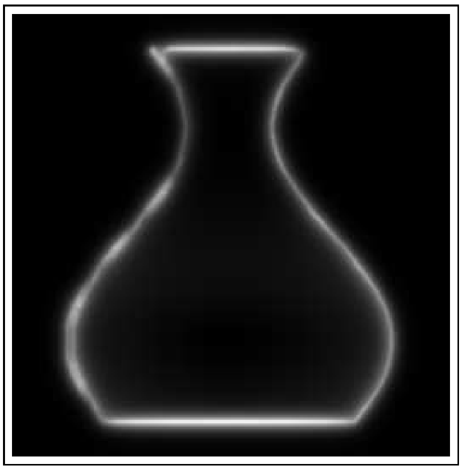


4

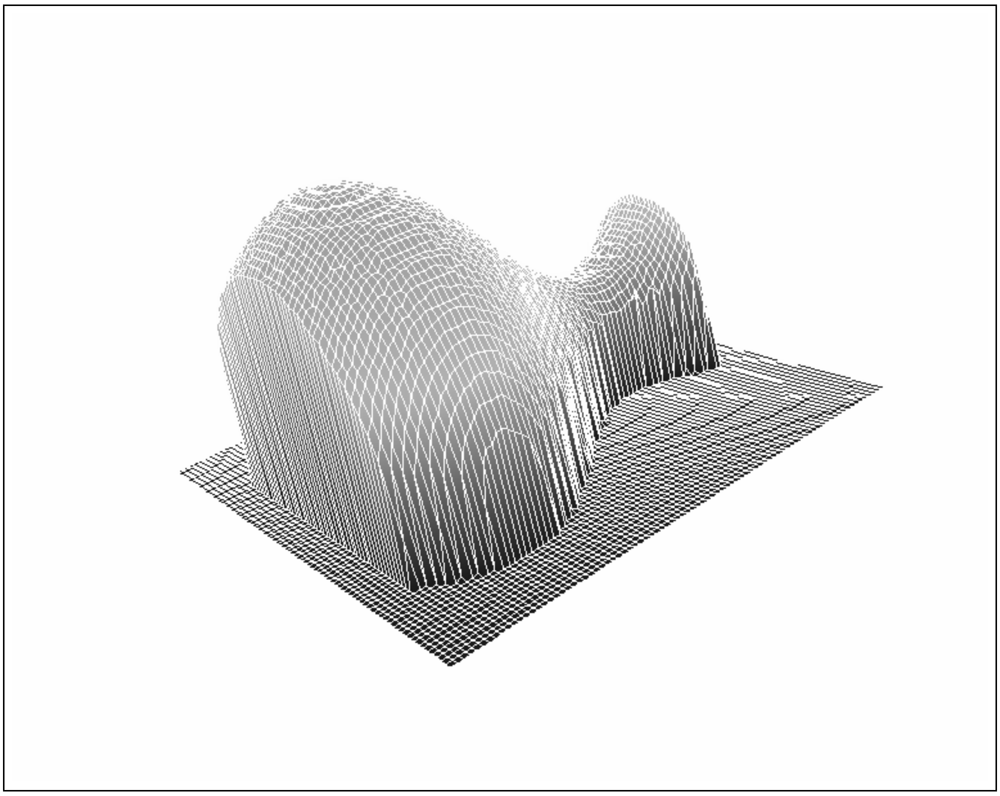
Figure 3: the function σ computed with $\epsilon = 2.0 \cdot 10^{-2}$.
Figure 4: the function σ computed with $\epsilon = 1.3 \cdot 10^{-2}$.

5



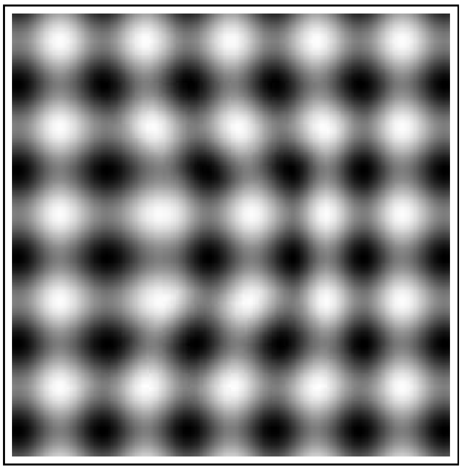
6

Figure 5: the function σ computed with $\epsilon = 6.5 \cdot 10^{-3}$.
Figure 6: the function s computed with $\epsilon = 6.5 \cdot 10^{-3}$.



7

Figure 7: surfaces with jumps recovered from the stereo pair 1–2.



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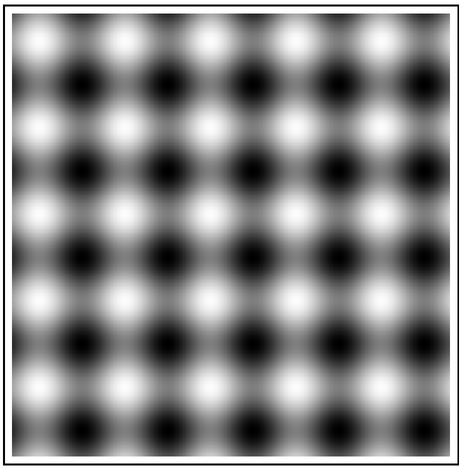
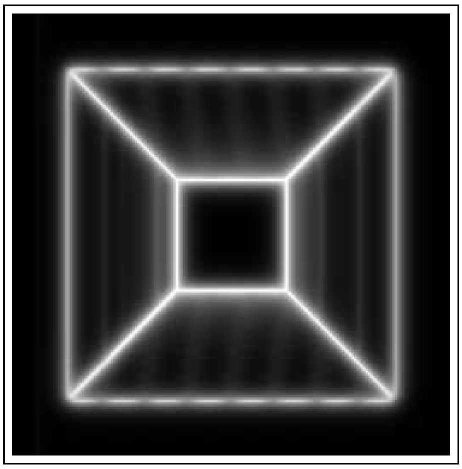


9

Figure 8: L image of a synthetic stereo pair (only creases).
Figure 9: R image of a synthetic stereo pair (only creases).



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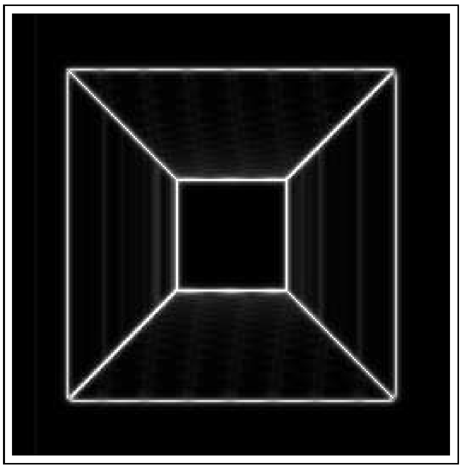
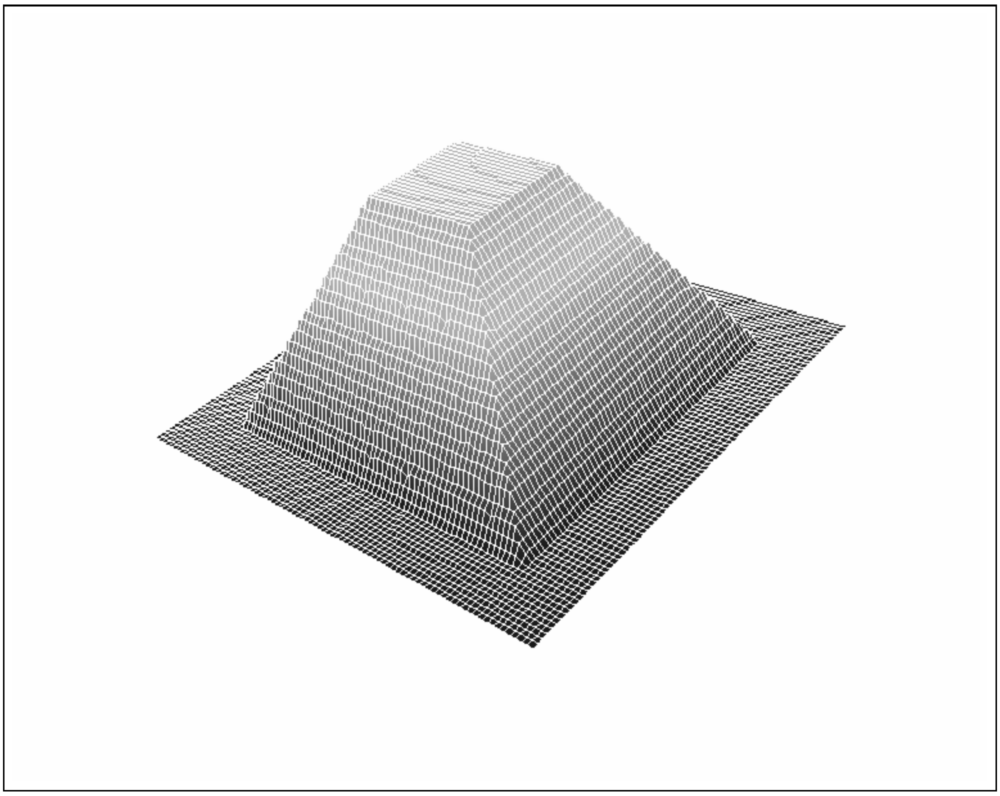


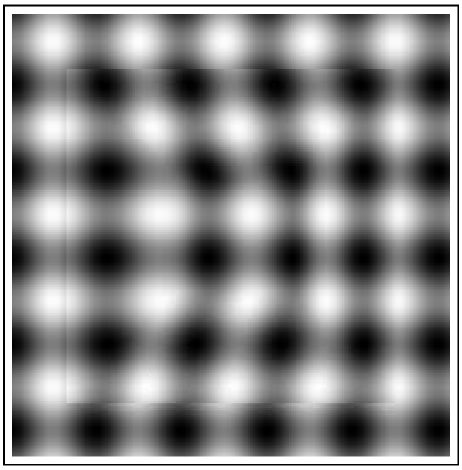
11

Figure 10: the function σ computed with $\epsilon = 8.0 \cdot 10^{-3}$.
Figure 11: the function σ computed with $\epsilon = 2.0 \cdot 10^{-3}$.



12

Figure 12: surface with creases recovered from the stereo pair 8–9.



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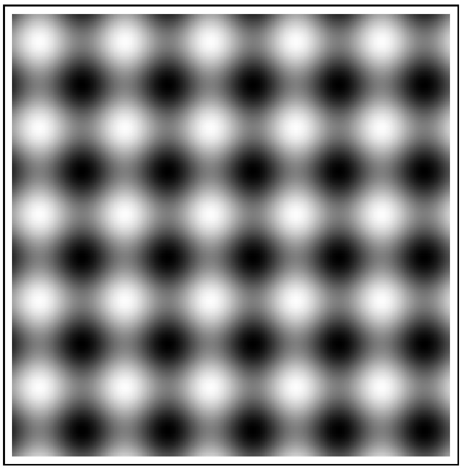
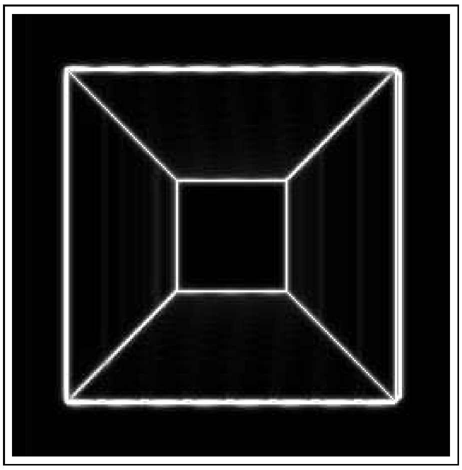


14

Figure 13: L image of a synthetic stereo pair (jumps+creases).
Figure 14: R image of a synthetic stereo pair (jumps+creases).



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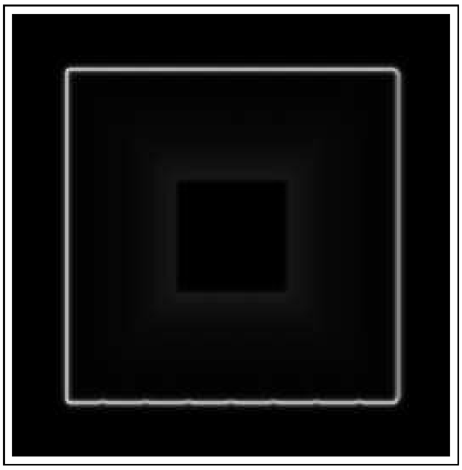
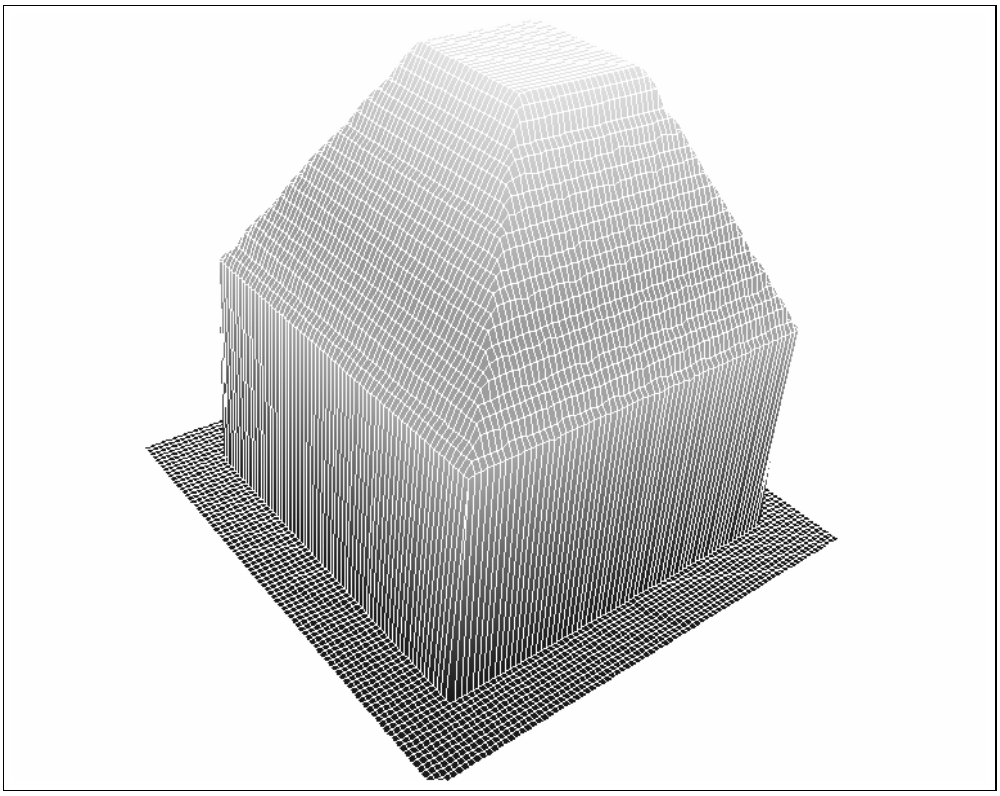


16

Figure 15: the function σ computed with $\epsilon = 2.0 \cdot 10^{-3}$.
Figure 16: the function s computed with $\epsilon = 2.0 \cdot 10^{-3}$.



17

Figure 17: surfaces with jumps and creases recovered from the stereo pair 13–14.