# MINIMAL MEASURES, ONE-DIMENSIONAL CURRENTS AND THE MONGE-KANTOROVICH PROBLEM

LUIGI DE PASCALE, MARIA STELLA GELLI, LUCA GRANIERI

Abstract. In recent works L.C. Evans has noticed a strong analogy between Mather's theory of minimal measures in Lagrangian dynamic and the theory developed in the last years for the optimal mass transportation (or Monge-Kantorovich) problem. In this paper we start to investigate this analogy by proving that to each minimal measure it is possible to associate, in a natural way, a family of curves on the space of probability measures. These curves are absolutely continuous with respect to the metric structure related to the optimal mass transportation problem. Some minimality properties of such curves are also addressed.

**Keywords.** Mather's minimal measures, Monge-Kantorovich problem, optimal transport problems, normal 1-currents.
**MSC 2000.** 37J50, 49Q20, 49Q15.

## 1. Introduction

In this paper we study some aspects of a problem proposed by L.C. Evans about the relationships between Mather's problem of minimal measures and the Monge-Kantorovich problem. The bridge between the two problems will be a careful analysis of a minimization problem on the space $\mathcal{N}_1^c$ of normal closed 1-currents which generalizes some results of [6].

An important point is that the Monge-Kantorovich problem requires on the Lagrangian less regularity than the Theory of Mather. We will try to deal with a less regular Lagrangian using suitable relaxed formulations when needed.

To give a more detailed description of the results of the paper we turn now our attention to the description of the two problems and of the main results. Let us start with the hypotheses which will be assumed to be true throughout the paper.

$(M, g)$ will be a compact Riemannian manifold without boundary, $L : TM \to \mathbb{R}$ is a Lagrangian which satisfies the following properties:

(1) $L$ is regular (say $C^1$),
(2) $L$ is uniformly superlinear in $v$,
(3) $L(x, \cdot)$ is strictly convex in the fiber for all $x$.
    (This means $L(x, tv + (1-t)w) \leq tL(x, v) + (1-t)L(x, w)$ for all $v, w$ and $t \in (0, 1)$, equality holding if and only if $v = w$).

$H : T^*M \to \mathbb{R}$ will denote the convex conjugate of $L$. The relationship between $L$ and $H$ defines the Legendre transform $\ell : TM \to T^*M$ by $(x, q) = \ell(x, p)$ if $H(x, q) = p \cdot q - L(x, p)$ ($q = \frac{\partial L}{\partial p}(x, p)$). If $L$ is more regular the flow associated to $L$ and the Hamiltonian flow of $H$ are conjugate through $\ell$, however in this paper we will need only that the Legendre transform is a Borel map (see [22] for a complete introduction to the Legendre transform and its properties).

### 1.1. **Mather's minimal measures.**

A measure $\mu \in \mathcal{M}(TM)$ will be said closed if and only if for all exact forms $\omega$

$$\int_{TM} \langle \omega(x), v\rangle d\mu(x, v) = 0. \tag{1.1}$$

According to this definition we set

$$\mathcal{M}^c = \left\{ \mu \in \mathcal{M}(TM) \ : \ \mu \text{ is closed}, \ \mu \geq 0 \, , \, \mu(TM) = 1, \ \int_{TM} L(x, v) d\mu < \infty \right\}.$$

To each measure $\mu \in \mathcal{M}^c$ we can associate the homology class of $\mu$ which we will denote by $[\mu] \in H_1(M, \mathbb{R})$ (by duality with $H^1$). Indeed, thanks to the fact that $L$ is superlinear and $\int_{TM} L(x, v)d\mu < \infty$, $\mu$ acts in a natural way on the set of the closed 1-forms on $M$ by

$$\omega \mapsto \int_{TM} \langle \omega(x), v\rangle d\mu(x, v)$$

and thanks to condition (1.1) this action passes to the quotient by the exact forms.

Once we fix an homology class $[h]$ Mather's variational problem amounts to:

$$\min_{\mathcal{M}^c} \left\{ \int_{TM} L(x, v) d\mu(x, v) \ : \ [\mu] = [h] \right\}. \tag{P1}$$

We will use the notation

$$\mathcal{A}(\mu) := \int_{TM} L(x, v) d\mu(x, v).$$

The relevance of Mather theory is due to the fact that it is a theory of invariant sets and measures under assumptions extremely weaker than those of KAM theory. Actually, it is strictly related to the weak KAM theory developed by Fathi. We will recall some more details throughout the paper. Some references for this are [6, 13, 23].

A remarkable property of problem (P1) is the following: we minimize an action functional which depends on $L$ on measures which are merely closed (notice that every invariant measure is also closed) and it turns out that, when $L$ is more regular, the minimal measures are invariant for the flow associated to the Lagrangian $L$ (see for example [6, 13]).

1.2. **The Monge-Kantorovich problem.**

Optimal transport problems, also known as Monge-Kantorovich problems, have been very intensively studied in the last 10 years and, due to the numerous and important applications to PDE, shape optimization and Calculus of variations, we witnessed a spectacular development of the field. Describing in details this theory is out of the aim of this paper, the interested reader may look at the book and lecture notes [1, 11, 37], the paper [25] and for some of the applications [8, 9, 30]. Our description will be restricted to the setting of a compact Riemannian manifold $M$ without boundary, however many of the concepts of this section could be formulated in general metric spaces.

Let $c : M \times M \to \mathbb{R}^+$ be a lower semicontinuous function. The Monge problem is formulated as follows: given two probability measures $\nu^+, \nu^-$ find a map $t : M \to M$ such that $t_\sharp \nu^+ = \nu^-$ ($\sharp$ denotes the push-forward of measures) and such that $t$ minimizes

$$\int_M c(x, t(x))d\mu$$

among the maps with the same property. It may happen that the set of admissible maps is empty (e.g. $\nu^+ = \delta_x$ and $\nu^- = \frac{1}{2}(\delta_y + \delta_z)$). Thus the problem is reformulated in its Kantorovich's relaxation. Find $\gamma \in \mathcal{P}(M \times M)$ such that $\pi^1_\sharp \gamma = \nu^+$ and $\pi^2_\sharp \gamma = \nu^-$ ($\pi^1$ and $\pi^2$ are the projection on factors of $M \times M$) and such that $\gamma$ minimizes

$$\int_{M\times M} c(x, y)d\gamma(x, y).$$

If $t$ is admissible for the Monge problem then the measure associated in the usual way to the graph of $t$ (i.e. $(id \times t)_\sharp \nu^+$) is admissible for the Kantorovich problem. However the class of admissible measures for the Kantorovich problem is never empty as it contains $\nu^+ \otimes \nu^-$. Moreover the Kantorovich problem is linear. Existence of minimizers for the Monge problem is difficult and may fail, while for the Kantorovich problem semicontinuity of $c$ is enough.

If $c$ is a distance then the cost

$$d_c(\nu^+, \nu^-) = \min_{\substack{\pi^1_\sharp\gamma=\nu^+ \\ \pi^2_\sharp\gamma=\nu^-}} \int_{M\times M} c(x, y)d\gamma(x, y),$$

defines a distance on $\mathcal{P}(M)$. If $c$ is the distance of the manifold then for $p \geq 1$ also

$$d_p(\nu^+, \nu^-) = \Big( \min_{\substack{\pi^1_\sharp\gamma=\nu^+ \\ \pi^2_\sharp\gamma=\nu^-}} \int_{M\times M} d^p(x, y)d\gamma(x, y)\Big)^{1/p}$$

defines a distance on $\mathcal{P}(M)$ called Wasserstein distance. Moreover $(\mathcal{P}(M), d_p)$ is complete, separable and $d_p$ metrizes the weak$*$ convergence of measures. The metric structure induced on $\mathcal{P}(M)$ is extremely rich and will play an important role in this paper. In particular $(\mathcal{P}(M), d_p)$ admits a tangent space. We will get in more details in a subsection of preliminary results.

Finally, whenever $c(x,y)$ is a "length" cost, which means that there exists a Lagrangian $\mathfrak{L} : TM \to \mathbb{R}^+$ such that

$$c(x,y) = \inf_{\substack{\gamma \in AC([0,1],M) \\ \gamma(0)=x, \gamma(1)=y}} \int \mathfrak{L}(\gamma(t), \dot{\gamma}(t))dt,$$

then the transport problem has a third formulation due to Brenier.

Minimize

$$\int_0^1 \int_M \mathfrak{L}(x, v(x,t)) d\rho_t(x) dt,$$

among the pair $(\rho, v)$ where $\rho : [0,1] \to \mathcal{P}(M)$ is a curve, and for each $t$, $v(\cdot, t)$ is a tangent vector field on $M$ defined $\rho_t$ almost everywhere, which satisfies the continuity equation:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0 & \textit{on } (0,1) \times M, \\ \\ \rho(0) = \nu^+, \quad \rho(1) = \nu^- & , \end{cases} \tag{1.2}$$

in a distributional sense.

1.3. **Description of the results of the paper.**

The paper is devoted to different aims that we briefly summarize in the following, referring to each section for more precise statements. In the first part we extend some results of [6] to the case of Lagrangians not necessarily homogeneous in $v$ (see section 3), in the second part we give a description of minimal measures in terms of curves on the space of probability measures which are solutions of equation (1.2) (see section 4). This provides in some sense an Eulerian description of the minimal measures as opposed to the Lagrangian description of [6]. Notice that in our setting the flow associated to the Lagrangian, in general, is not defined in a classical sense and that equation (1.2) is strictly related to the Cauchy problem for non-smooth vector fields ([2, 15]). More in details in section 3 we show that problem (P1) is equivalent to a variational problem on the space $\mathcal{N}_1(M)$ of normal 1-currents (thus extending some result of [6] where this equivalence was proved for $L(x,v) = g(v,v)$). Due to the non-homogeneity of $L$, the definition of the action functional on the space of normal 1-currents requires some justifications and some technical accuracy (see Lemma 3.1 and Lemma 3.3). The definition we give is mostly inspired by the work of Brenier on transport problems.

As by-product of this first equivalence we obtain that each minimal measure is supported on a graph. This last result was originally proved by Mather; here we provide an extremely simple proof.

In section 4 we prove that the problem on currents, introduced in section 3, is in turn equivalent to the problem of minimizing a convex functional on the set of periodic solutions of equation (1.2), which satisfy a suitable topological constraint. This problem (dropping the constraint) is very similar to Brenier's formulation of the Monge-Kantorovich problem. The idea of this last equivalence stems from the study of the transport density (see for example [9, 14]). In the same section we also extend this equivalence to a larger set of solutions of equation (1.2) which are in some sense quasi-periodic curves in the space of probability measures.

Each problem will be described in detail in the corresponding section.

Section 3 and 4 contain the main results of the paper. In section 4 we also present some examples and address some question of minimality as a program for future research. Indeed the equivalence proved in section 4 permits to associate to each minimal measure $\mu$ a family of solutions of (1.2). Each $\rho$ belonging to this family can be interpreted as an absolutely continuous curve in the space of probability measures equipped with the Wasserstein metric (see the previous subsection).

The natural question that arises from our analysis is the following. Does $\rho$ enjoy any intrinsic minimality property? The right minimality property, in our opinion, has been introduced in [7] using some costs studied also in [28]. We will be more precise in the last section of the paper. The problem in dealing with such question is that $\rho$ minimizes a functional on $AC([0,1],\mathcal{P}(M))$ with an additional constraint assigned on $M$ instead of $\mathcal{P}(M)$.

**Acknowledgments.** The authors wish to thanks Luigi Ambrosio and Giuseppe Buttazzo for the constant attention during the preparation of this paper.

A first version of this paper in which one more section was present circulated for a while. In that version we made use of a result of some other authors which was not completely correct. We are indebted to Albert Fathi who remarked that something was not clear and helped us to find the point.

**Note.** Some months after this paper was written another paper on this topic appeared [7]. In that paper the authors exploit the instruments of the weak KAM theory of Albert Fathi to introduce a transport problem related to a generic Lagrangian (even time depending) and to relate this transport problem to the Mather's one.

## 2. Preliminary results

### 2.1. Closed, normal 1-currents.

This section collects some definitions and technical facts about normal 1-currents and their representations which will be used explicitly in the paper or which are useful to give sense to some definitions. The exposition is adapted to the fact that the manifold $M$ is compact and then different from what would be on an open subset $\Omega$ of $\mathbb{R}^N$.

By $\Gamma^\infty(T^*M)$ we denote the space of $\mathcal{C}^\infty$ 1-dimensional forms. $\Gamma^\infty(T^*M)$ is usually equipped with the norm $\|\omega\|_\infty = \sup_M |\omega(x)|$. The space of normal, 1-dimensional currents is the space of linear, continuous functionals on $\Gamma^\infty(T^*M)$ and will be denoted by $\mathcal{N}_1(M)$. This last space is naturally endowed with a weak convergence which will be denoted by $T_n \rightharpoonup T_\infty$.

The subspace of the currents $T \in \mathcal{N}_1(M)$ such that $\langle T, \omega\rangle = 0$ whenever $\omega$ is an exact form is the space of normal, closed 1-currents and it is denoted by $\mathcal{N}_1^c(M)$.

The mass norm of a normal current (in short: "mass") is defined as follows:

$$\|T\| = \sup\{\langle T, \omega\rangle \mid \|\omega\|_\infty \leq 1\}.$$

The boundary of a current is defined by duality with the differential through the formula:

$$\langle T, d\phi\rangle = \langle \partial T, \phi\rangle \quad \text{for all} \quad \phi \in \mathcal{C}^\infty(M).$$

Then for a current $T$, being closed is equivalent to $\partial T = 0$.

Another well known fact, following from the definition, is that given a sequence $T_n \in \mathcal{N}_1^c(M)$ such that $\|T_n\| \leq C$ up to subsequences there exists $T \in \mathcal{N}_1^c(M)$ such that for all $\omega \in \Gamma^\infty(T^*M)$, $\lim_n \langle T_n, \omega\rangle = \langle T, \omega\rangle$.

A very good property of this class of currents is that they can be represented by integration. We will use two different representations for normal 1-currents. Any $T \in \mathcal{N}_1^c(M)$ can be represented by integration using a probability measure $\sigma$ on $M$ and a tangent vector field $X$ defined $\sigma$-a.e. as follows:

$$\langle T, \omega\rangle = \int_M \langle \omega(x), X\rangle d\sigma,$$

in this case we briefly write $T = \sigma \wedge X$. The vector field $X$ belongs to $L^1_\sigma(M, TM)$ to keep the notations compact we will sometime write $X \in L^1_\sigma$ instead. The same convention will be adopted for forms.

In line with [6] we choose the following canonical representation (which we denote by an underscript $s$ which stands for "special"):

$$\sigma_s(B) = \frac{1}{\|T\|} \sup\{\langle T, \omega\rangle \mid \omega \in \Gamma^\infty(T^*M),\ \|\omega\|_\infty \leq 1,\ \mathrm{spt}(\omega) \subset B\}, \tag{2.1}$$

for all open subsets $B$ of $M$, and $X_s$ is determined by the Radon-Nikodym Theorem and satisfies $g(X_s, X_s) = \|T\|^2$ $\sigma_s$-a.e. This special representation has no relevance in the general results of this paper, but it is important to relate our results to the geometric ones contained in [6].

There are many references in the literature for the representation theorem for normal currents, among them we refer to [24] or Theorem 1 in section 2.3 of [26].

Given a representation $\sigma\wedge X$ of a current $T$ we can obtain another representation (which in special cases will coincide with the first one) as follows: choose a function $c \in L^1_\sigma$ such that $\int_M c\, d\sigma = 1$ and $c(x) > 0$ $\sigma - a.e.$, then $\sigma_1 = c(x)\sigma$ is a probability measure and $X_1 = X/c$ is a vector field such that $\sigma_1 \wedge X_1 = T$.

The second representation formula for closed normal 1-current is related to the most elementary examples of such currents. Indeed the first example of closed normal 1-current is given by the integration along a closed, Lipschitz path on the manifold $M$. However periodicity is not the only way for a curve to produce a closed current.

*Example* 2.1. Let $v$ be an irrational direction on the torus $\mathbb{T}^n$ and consider the curve $\gamma(t) = x_0 + tv \,(mod.\ \mathbb{Z}^n)$ defined for $t \in \mathbb{R}$ and $x_0 \in \mathbb{T}^n$ fixed. For $k \in \mathbb{N}$, define the normal 1-current associated to the curve $\gamma_{|(-k,k)}$ as follows:

$$\langle T_k, \omega\rangle = \frac{1}{2k}\int_{-k}^{k} \langle \omega(\gamma(t)), \dot\gamma(t)\rangle dt.$$

It is well known (see for example [5]) that we can consider the limit $\langle T_\infty, \omega\rangle = \lim_{k\to+\infty}\langle T_k, \omega\rangle$ for any 1-form $\omega$ on the torus. Since

$$\langle T_\infty, df\rangle = \lim_{k\to+\infty} \frac{f(\gamma(k)) - f(\gamma(-k))}{2k} = 0,$$

$T_\infty$ is a closed current. The same construction can be carried on for a $\gamma$ which is $[0,1]$-periodic, in this case

$$\langle T_\infty, \omega\rangle = \int_0^1 \langle \omega(\gamma(t)), \dot\gamma(t)\rangle dt.$$

Indeed

$$\begin{aligned}\langle T_k, \omega\rangle &= \frac{1}{2k}\int_{-k}^{k} \langle \omega(\gamma(t)), \dot\gamma(t)\rangle dt = \frac{1}{2k}\sum_{h=1}^{2k}\int_{-k+h-1}^{-k+h} \langle \omega(\gamma(t)), \dot\gamma(t)\rangle dt \\ &= \frac{1}{2k}\sum_{h=1}^{2k}\int_0^1 \langle \omega(\gamma(t)), \dot\gamma(t)\rangle dt = \int_0^1 \langle \omega(\gamma(t)), \dot\gamma(t)\rangle dt.\end{aligned}$$

It turns out that this example contains the building block of normal closed 1-currents. Proofs of what follows here are contained in [6] and [35], although our exposition is closer to the second reference.

**Definition 2.2.** *A current $T \in \mathcal{N}_1^c$ is said an elementary solenoid if there exists a Lipschitz curve $\gamma : \mathbb{R} \to M$ enjoying the following properties:*

(1) *$Lip(\gamma) \le \|T\|$;*

(2) *$\langle T, \omega\rangle = \lim_{k\to\infty} \frac{1}{2k}\int_{-k}^{k} \langle \omega(\gamma(t)), \dot\gamma(t)\rangle dt$, for any 1-form $\omega$;*

(3) *$\gamma(\mathbb{R}) \subset \mathrm{spt}(T)$.*

*In particular the semicontinuity of the mass together with (1), (2), (3) implies that $|\dot\gamma(t)| = \|T\|$ a.e. We denote by $\mathcal{S}$ the set of all elementary solenoids.*

The following is Theorem B in [35].

**Theorem 2.3.** *Every $T \in \mathcal{N}_1^c$ can be decomposed into elements of $\mathcal{S}$ in the sense that there exists a probability measure $\nu$ on $\mathcal{S}$ such that*

$$T = \int_{\mathcal{S}} R d\nu(R) \qquad \text{and} \qquad \|T\| = \int_{\mathcal{S}} \|R\| d\nu(R).$$

2.2. **The continuity equation and the tangent space to $\mathcal{P}(M)$.**

The continuity equation (1.2) has been used in the Monge-Kantorovich theory since its beginning for many applications. The fact that it characterizes the a.c. curves on the space of probability measures equipped with the Wasserstein metric was only recently proved in [3]. Here we summarize some results from that book.

We restrict our attention to the case of $\mathcal{P}_p(M) := (\mathcal{P}(M), d_p)$ for $p > 1$. The tangent space to $\mathcal{P}_p(M)$ at a point $\mu$ is defined as the closure in $L^p(\mu)$ of the images of gradients of smooth functions via the duality map i.e.:

$$Tan_\mu \mathcal{P}_p(M) := \overline{\{j_q(\nabla\varphi) \ : \ \varphi \in C^\infty(M)\}}_{L^p(\mu)}$$

where $q$ is the dual exponent of $p$ and $j_q : L^q(\mu) \to L^p(\mu)$ denotes the map

$$j_q(v) = |v|^{q-2} v.$$

The following theorem relates absolutely continuous curves in $\mathcal{P}_p(M)$ to the continuity equation and, in some sense, justifies the definition of the tangent space.

**Theorem 2.4.** *Let $\rho : [0,1] \to \mathcal{P}_p(M)$ be a curve. If $\rho$ is a.c. and $|\rho'| \in L^1(0,1)$ is its metric derivative, then there exists a Borel vector field $v : (t,x) \mapsto v_t(x)$ such that*

$$v_t \in L^p(\rho_t) \quad \text{and} \quad \|v_t\|_{L^p(\rho_t)} \leq |\rho'|(t) \quad \text{for} \quad \mathcal{L}^1 - a.e.\ t \in [0,1] \tag{2.2}$$

*and the continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot v\rho = 0 \quad in \quad (0,1) \times M,$$

*is satisfied in the sense of distributions. Moreover for a.e. $t \in (0,1)$ $v_t$ belongs to $T_{\rho(t)}\mathcal{P}_p(M)$.*

*Conversely if $\rho$ satisfies the continuity equation for some vector fields $v_t$ such that $\|v_t\|_{L^p(\rho_t)} \in L^1(0,1)$ then $t \mapsto \rho(t)$ is a.c. and*

$$|\rho'|(t) \leq \|v_t\|_{L^p(\rho_t)} \quad for \quad \mathcal{L}^1 - a.e.\ t \in [0,1].$$

*Remark* 2.5. The minimality property (2.2) uniquely determines a tangent metric field $v_t$. Theorem 2.4 holds also for $p = 1$ (a proof is contained in [1]). In the case $p = 1$ the tangent vector field $v_t$ is not uniquely determined anymore.

Theorem 2.4 and the definition of tangent space were motivated by the work of Otto [34], where the concepts have been introduced in a different, but more formal, point of view.

In [3] there is also an infinitesimal characterization of the tangent space in terms of transport maps.

## 3. Energy on currents and equivalence with Mather's problem

In this section we will introduce an energy functional on $\mathcal{N}_1$ associated to the Lagrangian $L$. As $L$ is not homogeneous, after some considerations one is convinced that the functional should depend not only on the current but also on the parametrization we choose for it. And indeed the definition we give turns out to be the natural one from many points of view that will be clear later on. However some reader may prefer to take as the definition the formula given in Lemma 3.3.

For each $\sigma \in \mathcal{P}(M)$ and each $T \in \mathcal{N}_1^c$ we define the following functional:

$$\mathcal{L}(\sigma, T) := \sup_{\alpha,\omega} \int_M \alpha(x) d\sigma + \langle T, \omega\rangle \tag{3.1}$$

where the sup is taken for $\alpha$ continuous function and $\omega \in \Gamma^\infty(T^*M)$ which satisfy $\alpha(x) + H(x, \omega(x)) \leq 0$ pointwise. Observe that this definition is related to the natural duality between $\mathcal{C}(M) \times \Gamma^\infty(T^*M)$ and $\mathcal{M}(M) \times \mathcal{N}_1(M)$.

**Lemma 3.1.** *$\mathcal{L}(\sigma, T)$ is bounded from below and if it is finite then there exists a vector field $X \in L^1_\sigma(M)$ such that $T = \sigma \wedge X$.*

*Proof.* Since $L$ is bounded from below by a constant $C$ one easily deduce that $H(x, 0) = \sup\{-L(x, v) \mid v \in T_xM\} \leq -C$, i.e. the couple ($\alpha \equiv C$, $\omega = 0$) is admissible for the supremum in (3.1) for any $(\sigma, T)$. It follows that $\mathcal{L}(\sigma, T) \geq C$. Let us now fix $(\sigma, T)$ and assume that $\mathcal{L}(\sigma, T) < +\infty$. First we consider the special representation of $T$ introduced in equation (2.1). The measure $\sigma_s \in \mathcal{P}(M)$ and the vector field $X_s \in L^\infty_{\sigma_s}(M)$ satisfy $T = \sigma_s \wedge X_s$ and $g(X_s, X_s) = \|T\|^2$ $\sigma_s$-a.e., moreover we extend $X_s$ to all of $M$ by setting $X_s = 0$ outside the support of $\sigma_s$. The result is achieved if $\sigma_s$ is a.c. with respect to $\sigma$. Let us decompose $\sigma_s = \sigma_s^a + \sigma_s^s$ where $\sigma_s^a << \sigma$ and $\sigma_s^s$ is singular w.r.t. $\sigma$; we will prove that $|\sigma_s^s| = 0$. We can rewrite the functional $\mathcal{L}(\sigma, T)$ as

$$\mathcal{L}(\sigma, T) = \sup_{\alpha,\omega} \int_M \alpha(x)\, d\sigma + \int_M \langle \omega(x), X_s(x)\rangle\, d\sigma_s. \tag{3.2}$$

Let us now consider the 1-form $\tilde{\omega}$ such that $\|\tilde{\omega}\| = 1$ for $\sigma_s$-a.e. $x$

$$\langle \tilde{\omega}(x), X_s(x)\rangle = g(X_s(x), X_s(x))^{\frac{1}{2}} = \|T\|$$

and let $A$ be a Borel set on which $\sigma_s^s$ is concentrated and s.t. $\sigma(A) = 0$. We define $\omega(x) := \tilde{\omega}(x)\chi_A(x)$. In general this form will be not continuous but, by using Lusin's Theorem with respect to the measure $\sigma + \sigma_s^s$ on M, we can find regular forms $\omega_\varepsilon(x)$ such that $||\omega_\varepsilon||_{L^\infty(\sigma+\sigma_s^s)} \leq 1$, $\omega_\varepsilon(x) \to \omega(x)$ pointwise $(\sigma + \sigma_s^s)$-a.e. and $(\sigma + \sigma_s^s)\big(\{x \in M \,|\, \omega_\varepsilon(x) \neq \omega(x)\}\big) < \varepsilon$. Let us define $\alpha_\varepsilon(x) = -H(x, \omega_\varepsilon(x))$. By the continuity of $H$ we have that the functions $\alpha_\varepsilon$ are continuous functions on $M$ and they converge pointwise $(\sigma + \sigma_s^s)$-a.e. to $\alpha(x) := -H(x, \omega(x))$. Moreover,

$(\alpha_\varepsilon, \omega_\varepsilon)$ is admissible for (3.1). Thus we get

$$\begin{aligned} \mathcal{L}(\sigma, T) &\geq \int_M \alpha_\varepsilon(x)\, d\sigma + \int_M \langle \omega_\varepsilon(x), X_s(x)\rangle\, d\sigma_s \\ &= \int_{\{\omega_\varepsilon \neq \omega\}} \alpha_\varepsilon(x)\, d\sigma - \int_{\{\omega_\varepsilon = \omega\}} H(x, \omega(x))\, d\sigma \\ &\quad + \int_M \langle \omega_\varepsilon(x), X_s(x)\rangle\, d\sigma_s \qquad (3.3) \\ &\geq -\|\alpha_\varepsilon\|_\infty \sigma(\{\omega_\varepsilon \neq \omega\}) - \int_{\{\omega_\varepsilon = \omega\}} H(x, 0)\, d\sigma \\ &\quad + \int_M \langle \omega_\varepsilon(x), X_s(x)\rangle\, d\sigma_s \\ &\geq -C_1\varepsilon + C + \int_M \langle \omega_\varepsilon(x), X_s(x)\rangle\, d\sigma_s \end{aligned}$$

where $C_1 := \sup_{M\times\{|\omega|\leq 1\}} |H(x,\omega)|$ and we have used the bounds $\|\alpha_\varepsilon\|_\infty \leq C_1$, and $H(x, 0) \leq -C$. Passing to the limit as $\varepsilon \to 0^+$ in (3.3), taking into account that

$$\lim_{\varepsilon\to 0^+} \int_M \langle \omega_\varepsilon(x), X_s(x)\rangle\, d\sigma_s = \int_M \langle \omega(x), X_s(x)\rangle\, d\sigma_s\,,$$

we get

$$\begin{aligned} \mathcal{L}(\sigma, T) &\geq \int_M \langle \tilde\omega(x)\chi_A(x), X_s(x)\rangle\, d\sigma_s + C \qquad (3.4) \\ &= \int_A \langle \tilde\omega(x), X_s(x)\rangle\, d\sigma_s^s + C = \|T\| |\sigma_s^s| + C. \end{aligned}$$

For any positive number $K \in \mathbb{R}$, one can repeat the same argument above replacing $\omega(x)$ with $K\omega(x)$, that is, considering $(\alpha_\varepsilon^K, K\omega_\varepsilon)$ as a test couple in (3.1). The previous estimates still hold and (3.3) reads now

$$\begin{aligned} \mathcal{L}(\sigma, T) &\geq \int_M \langle \tilde\omega(x)\chi_A(x), X_s(x)\rangle\, d\sigma_s - C \\ &= -C_K\varepsilon + C + \int_M \langle K\omega_\varepsilon(x), X_s(x)\rangle\, d\sigma_s \end{aligned}$$

where $C_K := \sup_{M\times\{|\omega|\leq K\}} |H(x,\omega)|$. Passing to the limit as $\varepsilon \to 0^+$ eventually we get that for any $K > 0$

$$\mathcal{L}(\sigma, T) \geq \int_A \langle K\tilde\omega(x), X_s(x)\rangle\, d\sigma_s^s + C = K\|T\| |\sigma_s^s| + C.$$

Since $\mathcal{L}(\sigma, T) < \infty$ we than have $|\sigma_s^s| = 0$.

□

*Remark* 3.2. In the definition of $\mathcal{L}(\sigma,T)$ we can require that the constraint is satisfied with equality rather than inequality i.e., taking the supremum on pairs $(\alpha,\omega)\in C(M)\times\Gamma^\infty(T^*M)$ such that $\alpha(x)+H(x,\omega(x))=0$ pointwise. Thus $\mathcal{L}(\sigma,T)$ can be rewritten also as

$$\mathcal{L}(\sigma,T)=\sup_\omega\int_M(-H(x,\omega(x)))\,d\sigma+\langle T,\omega\rangle$$

where the sup is taken for $\omega\in\Gamma^\infty(T^*M)$. Moreover, if $\mathcal{L}(\sigma,T)$ is finite, then we can enlarge the class of admissible $\omega$ as follows. Let

$$\Gamma_\sigma:=\{\omega \text{ forms with coefficients in } L^\infty_\sigma\}$$

then

$$\mathcal{L}(\sigma,T)=\sup_{\omega\in\Gamma_\sigma}\int_M(-H(x,\omega(x)))\,d\sigma+\langle T,\omega\rangle. \tag{3.5}$$

This fact can be easily proved following the same procedure of Lemma 3.1. Indeed, given $\omega\in\Gamma_\sigma$ by Lusin's Theorem we can find $\omega_\varepsilon\in\Gamma^\infty(T^*M)$ such that $\omega_\varepsilon\to\omega$ in $L^\infty_\sigma$. Then

$$\mathcal{L}(\sigma,T)\geq\int_M(-H(x,\omega_\varepsilon(x))+\langle\omega_\varepsilon(x),X(x)\rangle)\,d\sigma \tag{3.6}$$

where we have taken $X(x)$ as in the thesis of Lemma 3.1. It suffices then to pass to the limit as $\varepsilon\to 0$ and notice that $H(x,\omega_\varepsilon(x))\to H(x,\omega(x))$ in $L^1_\sigma$ and $\lim_{\varepsilon\to0}\int_M\langle\omega_\varepsilon,X\rangle\,d\sigma=\int_M\langle\omega,X\rangle\,d\sigma$ to get (3.5), the inverse inequality being obvious.

**Lemma 3.3.** *If $\mathcal{L}(\sigma,T)$ is finite then*

$$\mathcal{L}(\sigma,T)=\int_M L(x,X(x))d\sigma,$$

*where $X$ is the vector field such that $T=\sigma\wedge X$.*

*Proof.* Let us fix $(\sigma,T)$ such that $\mathcal{L}(\sigma,T)<+\infty$, and let $X(x)\in L^1_\sigma$ be such that $T=\sigma\wedge X$ (see Lemma 3.1). We briefly recall that

$$L(x,v)+H(x,\omega)\geq\langle\omega,v\rangle \qquad \forall x\in M,\ \forall v\in T_xM,\ \forall\omega\in T^*_xM \tag{3.7}$$

with equality achieved in case $(x,\omega)=\ell(x,v)$, where $\ell$ is the Legendre Transform. Thus for any couple $(\alpha,\omega)$ such that $\alpha(x)+H(x,\omega(x))=0$ we have

$$\begin{aligned}&\int_M(\alpha(x)+\langle\omega,X(x)\rangle)\,d\sigma\\ =\ &\int_M(-H(x,\omega(x))+\langle\omega,X(x)\rangle)\,d\sigma\leq\int_M L(x,X(x))d\sigma.\end{aligned}$$

Taking the supremum with respect to $(\alpha, \omega)$ we deduce

$$\mathcal{L}(\sigma, T) \leq \int_M L(x, X(x)) d\sigma.$$

Let now $\omega(x)$ be such that $(x, \omega(x)) = \ell(x, X(x))$ and let

$$\omega_K := \begin{cases} \omega(x) & \text{if } |\omega(x)| \leq K \\ 0 & \text{otherwise.} \end{cases}$$

By the regularity of $\ell$ one obtain the right measurability properties for $\omega$ and then, in particular, $\omega_K \in L^\infty_\sigma$. Moreover, defined $A_K := \{x : |\omega(x)| \leq K\}$ we have that $A_K^c$ converges to $\emptyset$ in measure (or equivalently $\chi_{A_K} \to 1$ in $L^1_\sigma$).

Taking into account Remark 3.2 we can take $\omega_K$ as admissible form and compute

$$\begin{aligned} \mathcal{L}(\sigma, T) &\geq \int_M (-H(x, \omega_K(x)) + \langle \omega_K, X(x) \rangle)\, d\sigma \\ &= \int_{A_k} (-H(x, \omega(x)) + \langle \omega, X(x) \rangle)\, d\sigma + \int_{A_k^c} (-H(x, \omega_K(x)) + \langle \omega_K, X \rangle)\, d\sigma \\ &= \int_M L(x, X(x)) \chi_{A_K}(x)\, d\sigma + \int_M (-H(x, 0))(1 - \chi_{A_K}(x))\, d\sigma. \end{aligned}$$

Hence, since $L$ is bounded from below and from the boundedness of $H(x, 0)$, we can take the limit as $K \to +\infty$ and get

$$\mathcal{L}(\sigma, T) \geq \int_M L(x, X(x))\, d\sigma. \tag{3.8}$$

□

To each measure $\mu \in \mathcal{M}^c(TM)$ we can apply the disintegration theorem and write $\mu = \sigma \otimes \nu_x$ for a suitable family of probability measures $\nu_x$ concentrated on the fiber $T_xM$ and $\sigma = \pi_\sharp \mu$ where $\pi : TM \to M$ is the projection $(x, v) \mapsto x$. Moreover as seen in the introduction it is possible to associate to $\mu$ a closed normal 1-current $\tilde{p}(\mu)$ by the formula

$$\langle \tilde{p}(\mu), \omega \rangle := \int_{TM} \langle \omega(x), v \rangle d\mu(x, v) = \int_M \int_{T_xM} \langle \omega(x), v \rangle d\nu_x(v) d\sigma(x).$$

By the definition of $\mathcal{M}^c(TM)$ and the Jensen inequality we have, using the notation: $\mathfrak{m}(x) = \int_{T_xM} v d\nu_x$,

$$\infty > \int_{TM} L(x, v) d\mu = \int_M \int_{T_xM} L(x, v) d\nu_x(v) d\sigma(x) \geq$$
$$\int_M L(x, \mathfrak{m}(x)) d\sigma(x) \geq \mathcal{L}(\pi_\sharp \mu, \tilde{p}(\mu)).$$

Then thanks to Lemmas 3.1 and 3.3

$$\mathcal{L}(\pi_\sharp\mu, \tilde{p}(\mu)) = \int_M L(x, \mathfrak{m}(x))d\sigma. \tag{3.9}$$

To each current $T$ in $\mathcal{N}_1^c$ we can associate its homology class $[T]$. Once an homology class $[h]$ is fixed it makes sense to study the following problem:

$$\min\{\mathcal{L}(\sigma, T) \ : \ \sigma \in \mathcal{P}(M),\ T \in \mathcal{N}_1^c(M),\ [T] = [h]\}. \tag{P2}$$

In particular as the homology classes of $\mu$ and $\tilde{p}(\mu)$ coincide the digression above implies that for each homology class the minimum in problem (P2) is finite.

**Proposition 3.4.** *For each homology class $[h]$ there exists at least a minimizer for (P2).*

*Proof.* Let $T \in \mathcal{N}_1^c$ be such that $[T] = [h]$. By the superlinearity of $L$ for the special representation of $T$, $(\sigma_s, X_s)$, we have $\int_M L(x, X_s(x))d\sigma_s \leq A$ for some constant $A$ depending on $\|T\|$. Consider now a minimizing sequence $(\sigma_n, T_n)$. We can assume that $\mathcal{L}(\sigma_n, T_n) \leq A$. By Lemma 3.1 $T_n = \sigma_n \wedge X_n$, with $X_n \in L^1_{\sigma_n}$. We may also assume that $\sigma_n \overset{*}{\rightharpoonup} \sigma$. Thanks to Remark 3.2 we can take a form $\omega_n(x)$ such that $\langle \omega_n(x), X_n(x)\rangle = |X_n(x)|$ , with $\|\omega_n\|_\infty \leq 1$. The functions $\alpha_n(x) = -H(x, \omega_n(x))$ are equibounded. We have

$$\int_M \alpha_n(x)d\sigma_n + \int_M |X_n(x)|d\sigma_n = \int_M \alpha_n(x)d\sigma_n + \langle T_n, \omega_n\rangle \leq \mathcal{L}(\sigma_n, T_n) \leq A.$$

Hence $\int_M |X_n(x)|d\sigma_n \leq A$ . For any form $\omega$ such that $\|\omega\|_\infty \leq 1$ , we can evaluate:

$$\langle T_n, \omega\rangle = \int_M \langle \omega(x), X_n(x)\rangle d\sigma_n \leq \int_M |X_n(x)|d\sigma_n \leq A.$$

Then $\|T_n\| \leq A$. Compactness of 1-currents ensures that there exists $T \in \mathcal{N}_1^c(M)$ such that $T_n \rightharpoonup T$. It is easy to check that $T$ is closed and $[T] = [h]$. If a pair $(\alpha, \omega)$ is admissible for (P2) we have:

$$\int_M \alpha(x)d\sigma + \langle T, \omega\rangle = \lim_{n\to+\infty} \int_M \alpha(x)d\sigma_n + \langle T_n, \omega\rangle \leq \lim_{n\to+\infty} \mathcal{L}(\sigma_n, T_n).$$

Passing to the supremum we obtain $\mathcal{L}(\sigma, T) \leq \lim_{n\to+\infty} \mathcal{L}(\sigma_n, T_n)$, i.e. $(\sigma, T)$ is optimal. □

Let us denote by $\mathcal{PN}_1^c$ the set of pair $(\sigma, T)$ such that $\mathcal{L}(\sigma, T) < \infty$ and study the relationship between minimizers of the two problems.

We can define $i : \mathcal{PN}_1^c \to \mathcal{M}^c(TM)$ as

$$i(\sigma, T) = (id \times X)_\sharp \sigma,$$

where $T = \sigma \wedge X$ (see Lemma 3.1). Then the following theorem holds.

**Theorem 3.5.** *Problems (P1) and (P2) are equivalent in the sense that:*

(1) *the minimal values in (P1) and (P2) coincide,*
(2) *for each* $(\sigma, T) \in \mathcal{PN}_1^c$ $\tilde{p}(i(\sigma,T)) = T$ *and* $[T] = [i(\sigma,T)]$. *Moreover* $\mathcal{A}(i(\sigma,T)) = \mathcal{L}(\sigma,T)$.

*Proof.* Notice that by simple computations $\tilde{p}(i(\sigma,T)) = T$, $[T] = [i(\sigma,T)]$ and $\mathcal{A}(i(\sigma,T)) = \mathcal{L}(\sigma,T)$. Thus the minimum in (P1) is smaller then the minimum in (P2). The Jensen inequality implies the following

$$\mathcal{A}(\mu) = \int_{TM} L(x,v)d\mu = \int_M \int_{T_xM} L(x,v)d\nu_x(v)d\sigma \geq \\ \geq \int_M L(x, \int_{T_xM} v d\nu_x(v))d\sigma = \mathcal{L}(\pi_\sharp\mu, \tilde{p}(\mu)). \quad (3.10)$$

Then the two minimal values coincide. □

*Remark* 3.6. From the proof of the previous theorem it follows that each minimal measure is supported on a graph. In fact, if $\mu$ is a minimal measure, then

$$\mathcal{A}(\mu) \geq \mathcal{L}(\pi_\sharp\mu, \tilde{p}(\mu)) = \mathcal{A}(i(\pi_\sharp\mu, \tilde{p}(\mu))) \geq \mathcal{A}(\mu)$$

where the last inequality follows from the minimality of $\mu$. Going back to inequality (3.10) we obtain (still denoting by $\sigma \otimes \nu_x$ the disintegration of $\mu$) for $\sigma$-a.e. $x$ the equality

$$\int_{T_xM} L(x,v)d\nu_x = L(x, \int_{T_xM} v d\nu_x),$$

and this last equality, together with the strict convexity of $L$, implies $\nu_x = \delta_{\varphi(x)}$.

We conclude this section with a proposition which connects the equivalence just proved with a classical geometric problem in the space of currents and to the results of [6].

For each $T \in \mathcal{N}_1^c$ we denote by $\sigma_s \wedge X_s$ the special parametrization introduced in (2.1) and recall that $\sigma_s$ is a probability measure and $g(X_s, X_s) = \|T\|^2$ $\sigma_s$-a.e.

**Proposition 3.7.** *Let* $L(x,v) = g(v,v)^\alpha$ *for some* $\alpha > \frac{1}{2}$, *let* $T \in \mathcal{N}_1^c$ *and let* $\sigma \wedge X$ *be any parametrization of* $T$ *then*

$$\mathcal{L}(\sigma, T) \geq \mathcal{L}(\sigma_s, T) = \|T\|^{2\alpha}.$$

*Proof.* We have $\sigma = c(x)\sigma_s$ and $X = \frac{1}{c(x)}X_s$ for a positive function $c \in L^1_{\sigma_s}$ such that $\int_M c(x)d\sigma_s = 1$. Then

$$\mathcal{L}(\sigma,T) = \int_M g(X,X)^\alpha d\sigma = \int_M g(X_s,X_s)^\alpha \frac{c(x)}{c(x)^{2\alpha}} d\sigma_s = \\ = \|T\|^{2\alpha} \int_M c(x)^{1-2\alpha} d\sigma_s \geq \|T\|^{2\alpha} \Big( \int_M c(x) d\sigma_s \Big)^{1-2\alpha} = \|T\|^{2\alpha},$$

where the inequality above follows from the Jensen inequality and the convexity of the function $y \mapsto y^{1-2\alpha}$. □

The previous result in the case $\alpha = 1$ was proved by Bangert in [6]. The interpretation of Proposition 3.7 is that whenever $L(x, v) = g^\alpha(v, v)$ for some $\alpha > \frac{1}{2}$ our problem is equivalent to the classical problem of minimizing the mass in a given homology class. As one expects the minimizers do not depend on $\alpha$ and the minimal value is homogeneous with respect to $\alpha$.

*Remark* 3.8. The probability measure $\sigma$ in a minimal pair $(\sigma, T)$ has a counterpart in the theory of optimal transport which is the so called transport density. The transport density is a very useful tool in the optimal transport theory and in some of its applications ([8, 9, 10, 14, 18]).

The weak KAM theory of A. Fathi [20, 21, 22], suggests that for a minimal pair $(\sigma, T)$ the supremum in Remark 3.2 is achieved at a pair $\alpha(x) \equiv c$ and $\omega(x) = du(x)$ where $u$ is a viscosity solution of $H(x, du(x)) = c$. Let us discuss the case of some special Hamiltonians.

First observe that if we denote by $\omega(x)$ the Legendre transform of $X(x)$, then the following equality is satisfied

$$\mathcal{L}(\sigma, T) = \int_M [-H(x, \omega(x)) + \langle \omega(x), X(x) \rangle] d\sigma,$$

since the supremum in (3.5) is in fact attained at $\omega$. Since $X(x) \in L^1_\sigma$ in general we cannot say too much about the regularity of $\omega(x)$.

Thus, assuming that $(\alpha, \omega)$ is a maximizing pair in (3.1) we have

$$\begin{aligned}\mathcal{L}(\sigma, T) &= \int_M L(x, X) d\sigma = \int_M \alpha(x) d\sigma + \int_M \langle \omega, X \rangle d\sigma \\ &\leq \int_M [-H(x, \omega) + \langle \omega, X \rangle] d\sigma \leq \int_M L(x, X) d\sigma.\end{aligned}$$

Then, in fact, equality holds in all the previous estimates. Thus, it follows that

$$\alpha(x) + H(x, \omega(x)) = 0 \quad \sigma - a.e.$$

The weak KAM theory deals with weak solutions of the equation

$$c_0 + H(x, du(x)) = 0.$$

Let us consider the constant

$$c_0 = \min \Big\{ \int_{TM} L(x, v) d\mu \mid \mu \in \mathcal{M}^c \Big\}. \tag{3.11}$$

The constant $c_0$ (with the opposite sign) is known as Mané critical value, see [13, 22, 23]. This minimum will be attained in some homology class $[h]$. By the existence theorem $c_0 = \int_M L(x, X) d\sigma$ for a suitable optimal pairs $(\sigma, \sigma \wedge X)$ in such homology class. We will now use this optimal pair to prove a well known formula in a particular case.

**Lemma 3.9.** *For symmetric Lagrangians $L(x,v) = L(x,-v)$ one has:*

$$c_0 = \sup_{u\in\mathcal{C}^1} \min_{x\in M} -H(x,du).$$

*Proof.* Let $u \in \mathcal{C}^1$ . We have:

$$c_0 = \int_M L(x,X)d\sigma \geq \int_M \langle du, X\rangle d\sigma - \int_M H(x,du)d\sigma =$$
$$\int_M -H(x,du)d\sigma \geq \min_{x\in M} -H(x,du),$$

where the second equality depends on the fact that $\sigma \wedge X$ is closed. Taking the supremum we obtain

$$c_0 \geq \sup_{u\in\mathcal{C}^1} \min_{x\in M} -H(x,du).$$

Viceversa, consider a point $x_0 \in M$ such that $L(x_0,0) = \min_{x\in M} L(x,0)$, and the closed measure $\delta_{(x_0,0)}$ on $TM$. Observe that also the Hamiltonian $H$ is strictly convex, superlinear and symmetric. Then

$$\forall (x,\omega) \in T^*M : H(x,0) \leq H(x,\omega).$$

So we can evaluate

$$c_0 \leq \int_{TM} L(x,v)d\delta_{(x_0,0)} = L(x_0,0) \leq L(x,0) = -H(x,0) = -H(x,dc),$$

where $c$ denotes any constant function on the manifold $M$. But this holds for every $x \in M$, hence

$$c_0 \leq \min_{x\in M} -H(x,dc) \leq \sup_{u\in\mathcal{C}^1} \min_{x\in M} -H(x,du).$$

□

*Remark* 3.10. This min-max formula is well known (see [13, 27]) and also holds for non-symmetric Lagrangians. We have chosen symmetric Lagrangians in order to have a simpler proof. Observe that in the symmetric case the above supremum in the min-max formula is in fact attained, at least on constant functions.

Let $u$ be a function such that $c_0 = min_{x\in M} - H(x,du)$. Then the pair $(c_0, du)$ is admissible for $\mathcal{L}(\sigma, T)$. But we have also

$$0 = \int_M \langle du, X\rangle d\sigma \leq \int_M [L(x,X) + H(x,du)]d\sigma = \int_M [c_0 + H(x,du)]d\sigma \leq 0,$$

where the first equality follows from the fact that $T$ is closed. Hence

$$c_0 + H(x,du) = 0 \quad \sigma - a.e.$$

The above equation is true for every minimizing measure $\mu$ in (3.11), so if we consider $\mathcal{M}_0 \subset M$ as the projected Mather's set $\pi(\overline{\bigcup_\mu \mathrm{spt}(\mu)})$ (see [13, 22]) we can state the following:

**Theorem 3.11.** *For symmetric Lagrangians the minimal measures are concentrated in the set of points where the function $x \mapsto H(x,0)$ achieves its maximum.*

*Remark* 3.12. In particular, considering Lagrangians of the type $\frac{1}{2}|v|^2 - V(x)$, the previous theorem says that the minimal measures concentrate on the set of maximum points of the potential $-V$. Moreover, this result explains in a particular case the duality between the weak KAM theory and the Mather theory.

## 4. Transport equations and Eulerian representations of Mather's measures

In this section we will show that Mather's problem is equivalent to two different minimization problems on some space of curves on $\mathcal{P}(M)$ which solve the continuity equation.

The first problem we consider is the following: minimize the convex and l.s.c. functional

$$\mathcal{K}(\rho, v) = \int_0^1 \int_M L(x, v(x,t)) d\rho_t dt, \tag{4.1}$$

among all pairs $\rho : [0,1] \to \mathcal{P}(M)$ and $v : M \times [0,1] \to TM$ which solve

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0 \quad \text{in} \quad (0,1) \times M, \\ \\ \rho(0) = \rho(1) \end{cases} \tag{4.2}$$

and satisfy an homological constraint in a sense that will be explained right below. To each solution $(\rho, v)$ of (4.2) we can associate a closed, normal 1-current

$$T_{\rho,v} : \omega \mapsto \int_0^1 \int_M \langle \omega(x), v(x,t) \rangle d\rho_t dt.$$

As already noticed $T_{\rho,v}$ has a well defined homology class. Then the problem is formulated as follows

$$\min \left\{ \mathcal{K}(\rho, v) \mid (4.2) \text{ holds, and } [T_{\rho,v}] = [h] \right\}. \tag{P3}$$

**Definition 4.1.** *Each minimizing pair $(\rho, v)$ will be called an Eulerian representation of the current $T_{\rho,v}$ and of the corresponding minimal measure.*

Notice that different Eulerian representations can be associated to the same current or measure. In the following we will shortly write $\rho_t$ and $v_t$ to denote respectively the measure $\rho(t)$ and the vector field $v(\cdot, t)$.

To each pair $(\rho, v)$ solution of (4.2) we can associate a pair $(\sigma, T)$ admissible for problem (P2) by the map

$$(\rho, v) \xrightarrow{\bar{c}} \Big( \int_0^1 \rho dt, T_{\rho,v} \Big).$$

Viceversa, for each pair $(\sigma, T)$ which is admissible for problem (P2) we define the map $\underline{c}(\sigma, T) = (\rho, v)$ where $\rho_t = \sigma$ and $v(\cdot, t) = X$ (recall that $T = \sigma \wedge X$) for all $t$. It is easy to check that $\underline{c}(\sigma, T) = (\rho, v)$ is solution of (4.2).

**Theorem 4.2.** *The minimal values in (P2) and (P3) coincide. Moreover the maps $\overline{c}$ and $\underline{c}$ associate minimizers of one problem to minimizers of the other.*

*Proof.* If $(\sigma, T)$ is such that $\mathcal{L}(\sigma, T) < \infty$ then $\rho(t) \equiv \sigma$ and $v \equiv X$ is a solution of (4.2) and

$$\mathcal{K}(\rho, v) = \int_0^1 \int_M L(x, X(x)) d\sigma dt = \mathcal{L}(\sigma, T).$$

Moreover $T_{\rho,v}$ and $T$ are in the same homology class.

On the other hand, let $(\rho, v)$ be a solution of (4.2) such that $\mathcal{K}(\rho, v) < \infty$ then $\overline{c}(\rho, v)$ is such that $\mathcal{L}(\overline{c}(\rho, v)) < \infty$. We need to check that

$$|T_{\rho,v}|(B) = 0 \quad \text{whenever} \quad \int_0^1 \rho_t dt(B) = 0$$

and this follows from the inequality

$$|T_{\rho,v}|(B) \leq \int_0^1 \int_B |v_t| d\rho_t dt,$$

the absolute continuity of $v_t \rho_t$ with respect to $\rho_t$ and the positivity of $\rho_t$.

Let us compare the values of $\mathcal{K}$ and $\mathcal{L} \circ \overline{c}$. To do this we first remark that the functional $\mathcal{K}$ can be also defined by duality in a similar way as already done in equation (3.1) and Lemma 3.3, see for example [11] pag. 93. If $(\alpha, \omega)$ is an admissible test pair for (3.1) then we can use it as test also in the definition of $\mathcal{K}$. It follows that

$$\int_0^1 \int_M \alpha(x) d\rho_t dt + \int_0^1 \int_M \langle \omega(x), v \rangle d\rho_t dt = \langle (\alpha, \omega), \overline{c}(\rho, v) \rangle$$

which implies, taking the supremum over the test functions which do not depend on $t$, $\mathcal{L}(\overline{c}(\rho, v)) \leq \mathcal{K}(\rho, v)$. □

*Remark* 4.3. By the strict convexity of the Lagrangian $L$, once we fix the curve $\rho_t$ there exists essentially only one vector field $v_t$ such that $(\rho, v)$ is solution of (4.2) and $\mathcal{K}(\rho, v)$ is minimal.

It may happen that for some currents $T \in \mathcal{N}_1^c(M)$ the only periodic parametrization is the parametrization constant with respect to $t$. We now prove the equivalence of problem (P2) with another problem (P4) which includes (P3) in a suitable sense and guarantees the existence of non trivial minimal parametrizations for all minimal measures.

Here we consider the pairs $\rho : \mathbb{R} \to \mathcal{P}(M)$ and $v : M \times \mathbb{R} \to TM$ which solves

$$\frac{\partial \rho}{\partial t} + \nabla \cdot v\rho = 0 \quad \text{on } \mathbb{R} \times M, \tag{4.3}$$

and such that the sequence of currents $T_k$ given by

$$\langle T_k, \omega \rangle = \frac{1}{2k} \int_{-k}^{k} \int_M \langle \omega(x), v(x,t) \rangle d\rho_t dt \tag{4.4}$$

converges toward a closed, normal 1-current $T_{\rho,v}$. If $(\rho, v)$ is a periodic solution of (4.2) then the limit of the sequence in (4.4) coincides with $T_{\rho,v}$ already defined so that there is no abuse of notation here.

As already noticed $T_{\rho,v}$ has a well defined homology class.

The cost to be minimized will be given by

$$\mathcal{K}_1(\rho, v) = \liminf_{k \to \infty} \frac{1}{2k} \int_{-k}^{k} \int_M L(x, v(x,t)) d\rho_t dt. \tag{4.5}$$

Then the new problem is written as

$$\min \{\mathcal{K}_1(\rho, v) \mid (4.3) \text{ holds, and } [T_{\rho,v}] = [h]\}. \tag{P4}$$

The equivalence between problems (P2) and (P4) can be proved exactly as in Theorem 4.2 and then we will skip the proof. Let us now make a parallel between the probabilistic representations of normal 1-dimensional currents (Theorem 2.3 which is Theorem B in [35] and Theorem 3.5 in [6]) and solutions of equations (4.2) and of (4.3) (Theorem 8.2.1 of [3]). We will then use the structure of this general theorem to produce an example of multiplicity of Eulerian representations of a minimal measure.

**Theorem 4.4.** *Let $T \in \mathcal{N}_1^c(M) \setminus \{0\}$ then there exists a non-constant solution $(\rho, v)$ of (4.3) such that $T_{\rho,v} = T$.*

*Proof.* By theorem 2.3 (see also Theorem 3.5 in [6]) there exists a probability measure $\nu$ on the space of elementary solenoids $\mathcal{S}$ which represents $T$ in the sense explained in section 2.1, and at least one of these solenoids is non-constant. Each solenoid $\mathcal{R}$ in the support of $\nu$ is identified by a curve $\gamma_{\mathcal{R}}$. Rescaling the time variable we replace $\gamma_{\mathcal{R}}(t)$ by $\gamma_{\mathcal{R}}(\frac{\|T\|}{\|\mathcal{R}\|}t)$ so that the tangent vector has modulus $\|T\|$ for a.e. $t$. Given two solenoids $\mathcal{R}_1$ and $\mathcal{R}_2$ in the support of $\nu$, if $\gamma_{\mathcal{R}_1}(A) = \gamma_{\mathcal{R}_2}(B)$ for two sets of positive measure $A$ and $B$ in $\mathbb{R}$ then for a.e. $t$ and $s$ such that $\gamma_{\mathcal{R}_1}(t) = \gamma_{\mathcal{R}_2}(s)$ either $\dot\gamma_{\mathcal{R}_1}(t) = \dot\gamma_{\mathcal{R}_2}(s)$ or $\dot\gamma_{\mathcal{R}_1}(t) = -\dot\gamma_{\mathcal{R}_2}(s)$. The equality

$$\|T\| = \int_{\mathcal{S}} \|\mathcal{R}\| d\nu,$$

forbids cancellations so that only $\dot\gamma_{\mathcal{R}_1}(t) = \dot\gamma_{\mathcal{R}_2}(s)$ is allowed.

Then we define: $\rho(t) = \int_{\mathcal{S}} \delta_{\gamma(t)} d\nu(\gamma)$ and $v(x,t) = \dot\gamma(t)$ if $\gamma(t) = x$ for some $\gamma \in \mathrm{spt}(\nu)$ and $v(x,t) = 0$ otherwise. It is easy to check that $(\rho, v)$ is a solution of (4.3). $\square$

*Remark* 4.5. In the previous proof we could also remark that

$$\lim_{k\to\infty}\Big(\frac{1}{2k}\int_{-k}^{k}\rho_t dt, T_k\Big) = (\sigma_s, T). \tag{4.6}$$

Then by Proposition 3.7, in the case $L(x,v) = g^\alpha(v,v)$, $\alpha > 1/2$, the solution of (4.3) given by Theorem 4.4 is an Eulerian representation of the associated minimal measure.

*Example* 4.6. Using a variation of the technique we used in the previous proof it is sometimes possible to build many explicit parametrizations of a minimal measure $\mu$. In this example we deal with the flat 2-torus $\mathbb{T}^2$. The example will be better described with the help of the figure 4.1. By $\gamma_1$ and $\gamma_2$ we denote both the curves and their supports. The distinction will be clear from the context.

Let $\gamma_1$ and $\gamma_2$ be as in Figure 4.1.

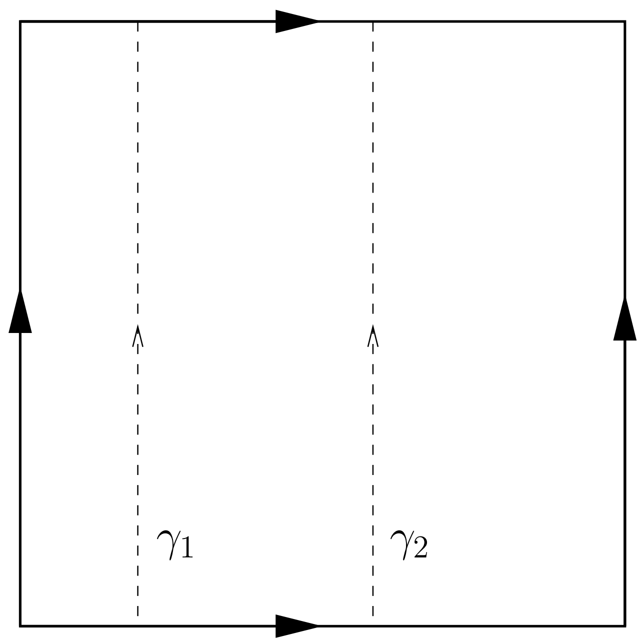


FIGURE 4.1. An example of path on the torus.

Consider $\mu \in \mathcal{P}(T\mathbb{T}^2)$ defined by $\mu = \dfrac{1}{2}((\mathcal{H}^1 \llcorner \gamma_1)\otimes\delta_{e_2} + (\mathcal{H}^1 \llcorner \gamma_2)\otimes\delta_{e_2})$. The measure $\mu$ is invariant for the geodesic flow, the homology class of $\mu$ is $[\mu] = (0,1) \in \mathbb{R}\times\mathbb{R} = H_1(\mathbb{T}^2)$, and $\mu$ is minimal being its cost equal to the length of the minimal geodesic in this class.

The following are some of the minimal parametrizations of $\mu$:

(1) $\rho_1(t) = \frac{1}{2}(\delta_{\gamma_1(t)} + \delta_{\gamma_2(t)})$, $\quad v_1(x,t) = e_2$,

(2) $\rho_2(t) = \frac{1}{2}\delta_{\gamma_1(t)} + \frac{1}{2}\mathcal{H}^1 \llcorner \gamma_2, \quad v_2(x,t) = e_2,$
(3) Constant parametrization,
(4) Parametrizations obtained by exchanging the role of $\gamma_1$ and $\gamma_2$ in the previous examples,
(5) Compositions of the previous examples with time shift.

4.1. **A notion of minimality.** In this section we discuss a notion of local minimality which we expect should be satisfied by the Eulerian representations of a minimal measure. This is also the occasion to make a better comparison with the recent results of [7].

The homology class $[h]$ can be seen as a parameter for problem (P1) and we denote by $\beta(h)$ the corresponding minimal value. The function $\beta$ is a convex function on $H_1(M)$ and its convex conjugate $\beta^* : H^1(M) \to \mathbb{R}$ plays a role in introducing the following modified Lagrangians. Notice that $\beta = \beta^{**}$. For a fixed homology class $[h]$ let us denote by $\omega$ a form in the cohomology class at which $\max\{\langle h, \omega\rangle - \beta^*(\omega)\}$ is attained. Then problem (P1) is equivalent to the unconstrained minimization of the action functional related to $L_\omega := L - \omega$.

Let us now recall some results from [28] (and from [7] for time dependent Lagrangians). Let

$$c_s^t(x,y) = \inf_{\substack{\gamma\in AC([s,t],M)\\ \gamma(s)=x,\gamma(t)=y}} \int L_\omega(\gamma(\tau),\dot\gamma(\tau))d\tau,$$

and let $\nu^+$ and $\nu^-$ in $\mathcal{P}_1(M)$. We look for $\gamma \in \mathcal{P}(M\times M)$ such that $\pi^1_\sharp\gamma = \nu^+$ and $\pi^2_\sharp\gamma = \nu^-$ ($\pi^1$ and $\pi^2$ are the projection on factors of $M\times M$) and such that $\gamma$ minimizes

$$C_s^t(\nu^+,\nu^-) = \min\Big\{\int_{M\times M} c_s^t(x,y)d\gamma(x,y) \mid \gamma\in\mathcal{P}(M\times M)\ \pi^1_\sharp\gamma=\nu^+, \pi^2_\sharp\gamma=\nu^-\Big\}.$$

By Theorem 4.5 of [28] this transport problem is equivalent to

$$C_s^t(\nu^+,\nu^-) = \min\Big\{\int_{M\times(s,t)} L(x,v(x,t))d\Sigma(x,t)\Big\}, \tag{4.7}$$

where the minimum is taken in the set of vector fields $v$ and $\Sigma\in\mathcal{M}^+(M\times[s,t])$ such that $\partial(\Sigma\wedge(v(x,t),1)) = \nu^-\otimes\delta_t - \nu^+\otimes\delta_s$.

If $\rho$ is admissible for problem (P3) (with fixed homology class $[h]$), then $v(x,t)$ and $\rho_{|_{[s,t]}}\otimes\mathcal{L}^1$ are admissible for (4.7).

Under some more regularity assumption in [7] it is proved that every Eulerian representation $\rho$ of a minimal measure $\mu$ is locally minimal for the costs associated to its homology class in the sense that for all $s\le t\le\tau$

$$C_s^\tau(\rho_s,\rho_\tau) = C_s^t(\rho_s,\rho_t) + C_t^\tau(\rho_t,\rho_\tau).$$

## References

[1] L. Ambrosio, Lecture Notes on Transport Problems, in 'Mathematical Aspects of Evolving Interfaces". Lecture Notes in Mathematics **1812**, Springer, Berlin, 2003, 1-52.

[2] L. Ambrosio, Transport equation and Cauchy problem for $BV$ vector fields, Invent. Math. **158** (2004), no. 2, 227–260.

[3] L. Ambrosio, N. Gigli, G. Savaré, Gradient Flows in Metric Spaces and in the Space of Probability Measures, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel 2005.

[4] L. Ambrosio, P. Tilli, Topics on Analysis in Metric Spaces, Oxford Lecture Series in Mathematics and Its Applications, **25**, Oxford University Press, Oxford, 2004.

[5] V. Arnold, Geometrical methods in the theory of ordinary differential equations, Springer-Verlag, Berlin & New York, 1982.

[6] V. Bangert, Minimal measures and minimizing closed normal one-currents, GAFA Geom. Funct. Anal. **9** (1999), no. 3, 413–427

[7] P. Bernard, B. Buffoni, Optimal mass transportation and Mather theory, Preprint (december 2004), to appear on J. Eur. Math. Soc. (JEMS)

[8] G. Bouchitté, G. Buttazzo, Characterization of optimal shapes and masses through Monge–Kantorovich equation, J. Eur. Math. Soc. (JEMS) **3** (2001), 139–168.

[9] G. Bouchitté, G. Buttazzo and P. Seppecher, Shape optimization solutions via Monge–Kantorovich equation, C. R. Acad. Sci. Paris **324-I** (1997), 1185–1191.

[10] G. Bouchitté, G. Buttazzo, L. De Pascale, A $p$-Laplacian approximation for some mass optimization problems, Journal of Optimization Theory and Applications **18** (2003), 1–25.

[11] Y. Brenier, Extended Monge-Kantorovich Theory, in Optimal Transportation and Applications, Lecture Notes in Mathematics **1813**, Springer, Berlin, 2003, 91-121.

[12] J. Benamou, Y. Brenier, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, Numer. Math. **84** (2000), no. 3, 375–393.

[13] G. Contreras, R. Iturriaga, Global minimizers of autonomous Lagrangians. IMPA, Rio de Janeiro, 1999.

[14] L. De Pascale, A. Pratelli, Regularity properties for Monge transport density and for solutions of some shape optimization problems, Calc. Var. Partial Differential Equations **14** (2002), 249–274.

[15] R.J. Di Perna, P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. **98** (1989), 511–547.

[16] L.C. Evans, Some new PDE methods for weak KAM theory, Calc. Var. Partial Differential Equations **17** (2003), no. 2, 159–177.

[17] L.C. Evans, Partial differential equations and Monge-Kantorovich mass transfer (survey paper), *Current Developments in Mathematics, 1997*, International Press (1999), edited by S.T. Yau.

[18] L.C. Evans, W. Gangbo, Differential Equations Methods for the Monge–Kantorovich Mass Transfer Problem, Mem. Amer. Math. Soc., Vol. **137** (1999).

[19] L.C. Evans, D. Gomes, Linear programming interpretation of Mather's variational principle, ESAIM Control Optim.Calc.Var. **8** (2002), 693-702.

[20] A. Fathi, Thèorème KAM faible et thèorie de Mather sur les systèmes lagrangiens, C. R. Acad. Sci. Paris Sèr. I Math. **324** (1997), no. 9, 1043–1046.

[21] A. Fathi, Solutions KAM faibles conjuguèes et barrières de Peierls. C. R. Acad. Sci. Paris Sèr. I Math. **325** (1997), no. 6, 649–652.

[22] A. Fathi, Weak Kam Theorem in Lagrangian Dynamics, forthcoming book, Cambridge University Press.

[23] A. Fathi, A. Siconolfi, Existence of $C^1$ critical subsolutions of the Hamilton-Jacobi equation, Invent. Math. **155** (2004), no. 2, 363-388.

[24] H. Federer, Geometric Measure Theory, Springer (Berlin), 1969.

[25] W. Gangbo, R.J. McCann, The geometry of optimal transportation, Acta Math. **177** (1996), 113-161.

[26] M. Giaquinta, G. Modica, J. Souček, Cartesian currents in the calculus of variations. I. Cartesian currents. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 37. Springer-Verlag, Berlin, 1998.

[27] D. Gomes, Viscosity solutions of Hamilton-Jacobi equations, and asymptotics for Hamiltonian systems, Calc. Var. Partial Differential Equations **14** (2002), no. 3, 345–357.

[28] L. Granieri, On action minimizing measures for the Monge-Kantorovich problem, Preprint (july 2004), to appear on NoDEA Nonlinear Differential Equations Appl.

[29] S. Hohloch, Optimale Massebewegung im Monge-Kantorovich Transportproblem, Diplomarbeit, Dezember 2002.

[30] R. Jordan, D. Kinderlehrer, F. Otto, The variational formulation of the Fokker-Plank equation, Siam J. Math. Anal. **29** (1998), 1-17.

[31] U. Lang, V. Schroeder, Kirszbraun's theorem and metric spaces of bounded curvature, GAFA Geom. Funct. Anal. **7** (1997), 535–560.

[32] J. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, Math. Z. **207** (1991), no. 2, 169–207.

[33] J. Mather, Minimal measures, Comment. Math. Helv. **64** (1989), no. 3, 375–394.

[34] F. Otto, The geometry of dissipative evolution equations: the porus medium equation, Comm. Partial Differential Equations **26** (2001) no. 1-2, 101-174.

[35] S.K. Smirnov, Decomposition of solenoidal vector charges into elementary solenoids and the structure of normal one-dimensional currents, St. Petersburg Math. J. **5** (1994), no.4, 841-867.

[36] K.T. Sturm, Stochastics and Analysis on Metric Spaces, lecture notes in preparation.

[37] C. Villani, Topics in Mass Transportation, Graduate Studies in mathematics **58**, AMS, Providence, RI.

Dipartimento di Matematica Applicata, Università di Pisa, via Bonanno Pisano 25/B, 56126 Pisa, Italy

*E-mail address*: `depascal@dm.unipi.it`

Dipartimento di Matematica, Università di Pisa, via Buonarroti 2, 56127 Pisa, Italy

*E-mail address*: `gelli@mail.dm.unipi.it,`

*E-mail address*: `granieri@mail.dm.unipi.it`