# CURVATURE AND DISTANCE FUNCTION FROM A MANIFOLD

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ABSTRACT. This paper is concerned with the relations between the differential invariants of a smooth manifold embedded in the Euclidean space and the square of the distance function from the manifold. In particular, we are interested in curvature invariants like the mean curvature vector and the second fundamental form. We find that these invariants can be computed in a very simple way using the third order derivatives of the squared distance function.
Moreover, we study a general class of functionals depending on the derivatives up to a given order $\gamma$ of the squared distance function and we find an algorithm for the computation of the Euler equation. Our class of functionals includes as particular cases the well known Area functional ($\gamma = 2$), the integral of the square of the quadratic norm of the second fundamental form ($\gamma = 3$) and the Willmore functional.

## CONTENTS



## 1. INTRODUCTION

In the last years, a large interest has grown in connection with geometric evolution problems, also with motivations coming from the Mathematical Physics (phase transitions, Stefan problem). A model problem is the evolution of surfaces by mean curvature, which can be considered as the gradient flow of the Area functional. Indeed, if $M$ is a compact $n$–manifold embedded in $\mathbb{R}^N$ without boundary, and if $\Phi_t$ is a family of diffeomorphisms of $\mathbb{R}^N$ such that $\Phi_0$ is the identity, then

$$\frac{d}{dt}\left[\mathcal{H}^n\big(\Phi_t(M)\big)\right]_{t=0} = -\int_M \langle \mathbf{H}, X\rangle \, d\mathcal{H}^n$$

where $X = [\Phi_t]'_{t=0}$ is the infinitesimal generator of $\Phi_t$, $\mathcal{H}^n$ is the $n$–dimensional Hausdorff measure and $\mathbf{H}$ is the mean curvature vector of $M$.
This mathematical problem is intriguing because the appearance of singularities

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during the flow makes it necessary (with the exceptions of planar Jordan curves, convex shapes, codimension 1 graphs) a weak approach to obtain a global in time solution of the evolution problem. Starting from the pioneering work of Brakke [2], a large literature is by now available on this subject (see for instance Chen–Giga–Goto [4], Evans–Spruck [12], Huisken [19], Ilmanen [21] and the references therein). The weak formulations are mainly based either on Geometric Measure Theory (currents, varifolds), or on the theory of viscosity solutions (the level set approach of Chen–Giga–Goto [4], Evans–Spruck [12]). In the latter approach, a crucial rôle (see for instance Ambrosio–Soner [1], Evans–Soner-Souganidis [11] and Soner [30]) is played by the analytical properties of the distance function $d^M(x)$ from the manifold (see also Delfour–Zolésio [8, 9]). For instance, in the codimension 1 case $n = (N-1)$, it turns out that $M_t = \partial U_t$ flows by mean curvature if and only if

$$d_t(x,t) = \Delta d(x,t) \qquad \text{for } x \in M_t$$

where $d(x,t)$ is equal to the signed distance function from $M_t$, i.e. $d(x,t) = -d^{M_t}(x)$ if $x \in U_t$ and $d(x,t) = d^{M_t}(x)$ if $x \notin U_t$. Since the signed distance function makes no sense in higher codimension problems, De Giorgi suggested in [5], [6] and [7] to work with the squared distance function $\eta^M(x) = [d^M(x)]^2/2$. Setting $\eta(x,t) = \eta^{M_t}(x)$, it turns out that (see Ambrosio–Soner [1]) the mean curvature flow is characterized by the equation

$$(\nabla\eta)_t(x,t) = \Delta(\nabla\eta)(x,t) \qquad \text{for } x \in M_t$$

because $-\nabla\eta(x,t)$ represents the displacement of $x \in M_t$ and $-\Delta(\nabla\eta)(x,t)$ is the mean curvature vector of $M_t$ at $x$.

One of the goals of this paper is a systematic study of the connections between the analytical properties of $\eta^M$ and the geometric invariants of the manifold $M$. In particular, in Section 2 we will prove that $d^3\eta^M(x)$ and the second fundamental form $\mathbf{B}_x$ of $M$ are mutually connected for any $x \in M$ by simple linear relations; in addition, for any normal vector $p$ the eigenvalues of $\langle \mathbf{B}_x, p\rangle$ on the tangent space (in some sense, the "principal curvatures" in the direction $p$) are linked to the eigenvalues of $\nabla^2\eta^M(x_s)$ for points $x_s$ on the normal line $x + sp$.

Our motivations are also related to the analysis of general classes of geometric functionals, including the Area functional and the Willmore functional (see Chen [3], Simon [29], Weiner [32], Willmore [34])

$$\mathcal{F}(M) = \int_M |\mathbf{H}|^2 \, d\mathcal{H}^n.$$

More generally, functionals depending on the second fundamental form of $M$ have been widely investigated in the literature (see Langer [23], Reilly [26], Rund [27] and Voss [31]). By our preceding analysis we see that in principle any autonomous "geometric" functional can be written as

$$\mathcal{F}(M) = \int_M f\left(\eta^M_{i_1 i_2}, \dots, \eta^M_{j_1 \dots j_\gamma}\right) d\mathcal{H}^n \tag{1.1}$$

for some function $f$ depending on the derivatives of $\eta^M$ up to a given order $\gamma$. In this setting, the Area functional and the Willmore functional respectively correspond to

$$\frac{1}{N-n}\int_M \sum_{i,j} |\eta^M_{ij}|^2 \, d\mathcal{H}^n, \qquad \int_M \sum_{i,k} |\eta^M_{ikk}|^2 \, d\mathcal{H}^n. \tag{1.2}$$

One of the main result of this paper is a constructive algorithm for the computation of the first variation of the functional $\mathcal{F}$ in (1.1). Specifically, under smoothness assumptions on $f$ we prove that there exists a unique normal vector field $E_{\mathcal{F}}$ such that

$$\frac{d}{dt}\mathcal{F}\big(\Phi_t(M)\big)\bigg|_{t=0} = \int\limits_M \langle E_{\mathcal{F}}|X\rangle \, d\mathcal{H}^n$$

for any family of diffeomorphisms $\Phi_t$ whose infinitesimal generator is $X$. In general, $E_{\mathcal{F}}$ depends on the derivatives of $\eta^M$ up to the order $(2\gamma - 1)$, and if $f$ is a polynomial the same is true for $E_{\mathcal{F}}$.
We carry out an explicit computation for the generalization of the Willmore functional

$$\mathcal{F}_p(M) = \int\limits_M |\mathbf{H}|^p \, d\mathcal{H}^n.$$

In particular, in the codimension one case, we recover some of the results found by the above mentioned authors (see Reilly [26], Voss [31]).

The advantages of our approach are its full generality and its independence by the dimension and the codimension. However, it should be said that assumptions like $n = 1$ or $n = (N - 1)$ are very often important to get a manageable expression for $E_{\mathcal{F}}$. Another difficulty is related to the fact that, in the codimension 1 case, any symmetric functions of the principal curvatures is in principle representable as in (1.1), but this representation is in practice not easy, with the notable exceptions of $|\mathbf{H}|^p$ and $|\mathbf{B}|^p$.

Another advantage of this approach is the possibility to consider also functionals depending on the derivatives of $\mathbf{B}$ up to any order. To this purpose, in the final part of the paper we consider the functionals

$$\mathcal{G}_\gamma(M) = \int\limits_M \sum_{i_1,\dots,i_\gamma} |\eta^M_{i_1,\dots,i_\gamma}|^2 \, d\mathcal{H}^n. \tag{1.3}$$

We already noticed that $\mathcal{G}_2$ is a multiple of the Area functional; recently, De Giorgi suggested in [6], [7] a parametric approximation of the mean curvature flow problem based on the gradient flow of the functionals $\mathcal{G}_2 + \epsilon\mathcal{G}_\gamma$ with $\gamma > n+1$ (in order to gain a Sobolev embedding for the tangent spaces). For the evolution of curves, this problem has been studied by Wen [33] with $\gamma = 3$ and with a constraint on length.

We notice that the function inside the integral in (1.3) is equal to $3|\mathbf{B}|^2$ for $\gamma = 3$; assuming in addition $n = 2$, the functional $\mathcal{G}_3$ coincides up to multiplicative and additive constants depending on the genus of $M$ (see the discussion at the beginning of Section 5) with the Willmore functional. For $\gamma > 2$, computing the leading term in the first variation of $E_{\mathcal{G}_\gamma}$ we find that this term is equal, up to a multiplicative constant, to the normal component of $\Delta^M \circ \ldots \circ \Delta^M \mathbf{H}$, where the Laplace–Beltrami operator $\Delta^M$ on $M$ acts $(\gamma - 2)$ times on $\mathbf{H}$.

We now give a brief explanation of the main idea behind our proof of the first variation in (1.1). Let us assume, for the sake of simplicity, that the function $f$ in (1.1) is smooth and depends only on $\nabla^3\eta^M$, as the second functional in (1.2). Let $M_t := \Phi_t(M)$ be the deformed manifolds; the classical parametric method consists in the computation of

$$\frac{d}{dt}\mathbf{B}(x,t)\bigg|_{t=0}$$

where $\mathbf{B}(x,t)$ is the second fundamental form of $M_t$ at $\Phi_t(x)$. Using the Area Formula to carry the integrals from $M_t$ to $M$, also the derivative at $t = 0$ of the tangential Jacobian $J\Phi_t(x)$ on $M$ is needed.

Still using the Area Formula, we proceed in a slightly different way: by the relations between $\nabla^3\eta^M$ and $\mathbf{B}$ we basically need to compute the derivative

$$\frac{d}{dt}\nabla^3\eta(x,t)\bigg|_{t=0} \tag{1.4}$$

where $\eta(\cdot,t)$ is the squared distance function from $M_t$. By the smoothness of $\eta$ we can reverse the order of differentiation to get

$$\nabla^3\left(\frac{d}{dt}\eta(x,t)\bigg|_{t=0}\right).$$

Computing $\eta_t(x,0)$ in Lemma 4.5 we find that (1.4) is given by

$$-\nabla^3\left\langle\nabla\eta^M(x)\,|\,X(x-\nabla\eta^M(x))\right\rangle$$

where $X$ is the infinitesimal generator of $\Phi_t$. The same argument works with higher order derivatives of $\eta$, which correspond to derivatives of $\mathbf{B}$.
We conclude noticing that, in principle, the method works also for second or higher order variations: one needs to compute the derivatives

$$\frac{\partial^k}{\partial t^k}\eta(x,t)\bigg|_{t=0}, \qquad \frac{\partial^k}{\partial t^k}J^M\Phi_t(x)\bigg|_{t=0}.$$

However, in this paper we confine our attention only to first variations.

**Acknowledgement .** This paper is the fruit of many stimulating conversations with Ennio De Giorgi, whose recent death has been a great loss for the whole mathematical community. We would like to dedicate this paper to him.

## 2. Squared Distance Function from a Manifold

In all this paper, $e_1,\dots,e_N$ is the canonical basis of $\mathbb{R}^N$, $M$ is a smooth, compact $n$–manifold without boundary embedded in $\mathbb{R}^N$ and $T_xM$, $N_xM$ are respectively the tangent space and the normal space to $M$ at $x\in M$.

The *distance function* $d^M(x)$ and the *squared distance function* $\eta^M(x)$ are respectively defined by

$$d^M(x)=\mathrm{dist}(x,M), \qquad \eta^M(x)=\frac{1}{2}[d^M(x)]^2$$

for any $x\in\mathbb{R}^N$ (we will drop $M$ when no ambiguity is possible). In this section we analyse the differentiability properties of $d$ and $\eta$ and the connection between the derivatives of these functions and some geometric properties of $M$.

It is well known that $d$ is a Lipschitz function with Lipschitz constant 1; moreover, for any differentiability point $x$ of $d$ there exists $y\in M$ such that

$$\nabla d(x)=\frac{x-y}{|x-y|} \tag{2.1}$$

provided $x\notin M$. In particular, at any differentiability point $x$ the minimizer $y$ in $M$ is uniquely determined and we will denote it by $\pi^M(x)$. By the chain rule, we have also

$$|\nabla\eta(x)|^2=2\eta(x) \tag{2.2}$$

at any differentiability point of $\eta$ (the identity is trivial if $\eta(x)=0$).

The above mentioned properties are true even if $M$ is merely a closed set (additional regularity properties of $d$ are studied in [14, 17]). In general, only one sided estimates on the second derivatives of $d$, $\eta$ are available, based on the convexity of the function $A(x)=|x|^2/2-\eta(x)$, which can be represented by

$$\max_{y\in M}\ \langle x|y\rangle-\frac{1}{2}|y|^2.$$

The function $A$ which will be extensively considered in Section 4.

However, it is natural to expect higher regularity of $M$ to lead to higher regularity of $d$ and $\eta$ (see also, in the case $n = (N-1)$, [18, 8, 9, 16, 22]). Assuming, as we do, that $M$ is a smooth manifold, the following simple result can be proved (see for instance [1], Theorem 3.1):

**Theorem 2.1.** *There exists a constant $\sigma > 0$ such that $\eta$ is smooth in the region*

$$\Omega = \{x \in \mathbb{R}^N : d(x) < \sigma\}.$$

*Moreover, for any $x \in M$ the Hessian matrix $\nabla^2\eta(x)$ is the (matrix of) orthogonal projection onto the normal space $N_xM$.*

It should be remarked that $d(x) = \sqrt{2\eta(x)}$ is smooth on $\Omega \setminus M$ but it is not smooth up to $M$. In the codimension one case $n = (N-1)$ this difficulty can be amended writing $M = \partial E$ and using the *signed distance function*

$$d^*(x) = \begin{cases} d(x) & \text{if } x \notin E; \\ -d(x) & \text{if } x \in E. \end{cases}$$

As shown in [1], in higher codimension problems the function $\eta$ is a good substitute of $d^*(x)$ in the analysis of mean curvature flow problems.

The following result is concerned with the Hessian matrix of $\eta$ out of $M$.

**Theorem 2.2.** *Let $x \in \Omega$, let $y$ be its projection on $M$ and let*

$$B(s) = \nabla^2\eta\big(y + s(x-y)\big)$$

*for any $s \in [0, d(x)]$. Then, the matrices $B(s)$ are diagonal in a common basis, and denoting by $\lambda_1(s), \dots, \lambda_N(s)$ their eigenvalues in increasing order, we have*

$$\lambda_{N-n+1}(s) = \lambda_{N-n+2}(s) = \cdots = \lambda_N(s) = 1 \qquad \forall s \in [0, d(x)].$$

*The remaining eigenvalues are strictly less than 1 and satisfy the ODE*

$$\lambda_i'(s) = \frac{\lambda_i(s)(1-\lambda_i(s))}{s} \qquad \forall s \in (0, d(x)]$$

*for $i = 1, \dots, N$. Finally, the quotients $\lambda_i(s)/s$ are bounded in $(0, d(x)]$.*

The proof of Theorem 2.2 (see [1], Theorem 3.2) is mainly based on the identities

$$\eta_i\eta_{ij} = \eta_j, \qquad \eta_{ik}\eta_{ij} + \eta_i\eta_{ijk} = \eta_{jk},$$

obtained by differentiation of (2.2). Using the identity

$$\nabla^2\eta = d\nabla^2 d + \nabla d \otimes \nabla d$$

and using (2.1), it follows that also $\nabla^2 d\big(y + s(x-y)\big)$ is diagonal in the same basis; in addition, one eigenvalue, corresponding to the eigenvector $\nabla d(x)$, is 0, $(N-n-1)$ eigenvalues are equal to $1/s$ and the $n$ remaining ones $\beta_1(s), \dots, \beta_n(s)$ are bounded and satisfy

$$\beta_i'(s) = -\beta_i^2(s) \qquad \forall s \in (0, d(x)]. \tag{2.3}$$

A straightforward consequence of Theorem 2.2 is the following result.

**Corollary 2.3.** *Let $x \in \Omega$, $y = \pi^M(x)$ and let $\mathbf{K}_x : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbf{R}$ be the symmetric 3–linear form induced by $d^3\eta(x)$. Then*

$$\mathbf{K}_x(u, v, w) = 0$$

*if at least two of the vectors $u$, $v$, $w$ belong to $N_yM$.*

## 3. Second Fundamental Form and $d^3\eta$

In this section we analyse the connection between the second fundamental form of $M$ and $d^3\eta$. We begin with the definition of the *tangential gradient* $\nabla^M f$.

**Definition 3.1.** Let $x \in M$, and let $f$ be a $C^1$ real valued function defined in a neighborhood $U$ of $x$. The tangential gradient $\nabla^M f(x)$ is the projection of $\nabla f(x)$ on $T_xM$.

It is easy to check that $\nabla^M f$ depends only on the restriction of $f$ to $M \cap U$; moreover, an extension argument shows that $\nabla^M f$ can also be defined for functions initially defined only on $M \cap U$. In a similar way we can define the *tangential divergence* of a vector field and the *tangential Laplacian* of a function $g$:

$$\operatorname{div}^M X(x) = \sum_{i=1}^{k} \nabla_i^M X^i(x), \qquad \Delta^M g(x) = \operatorname{div}^M \nabla^M g(x). \tag{3.1}$$

By Theorem 2.1 we infer

$$\nabla_i^M f(x) = P_{ij}(x)\nabla_j f(x) \quad \text{with} \quad P_{ij}(x) = (\delta_{ij} - \eta_{ij}(x)), \tag{3.2}$$

because $\eta_{ij}(x)$ is the matrix of orthogonal projection on $N_xM$. Notice that the formula defining $P_{ij}(x)$ makes sense also on $\Omega \setminus M$: in this case, Theorem 2.2 implies

$$P(x) : T_yM \to T_yM, \qquad \operatorname{Ker} P(x) \supset N_yM \tag{3.3}$$

where $y = x - \nabla\eta(x)$ is the projection of $x$ on $M$. However, in general $P(x)$ is not the identity on $T_yM$. Finally, we notice that $P_{ij}(x) = \nabla_i^M x_j$ for any $x \in M$.

Now we introduce the *second fundamental form* $\mathbf{B}$ and the *mean curvature vector* $\mathbf{H}$ of $M$.

**Definition 3.2.** Let $x \in M$, $u$, $v \in T_xM$ and let $\phi(y)$ be a smooth vector field defined in a neighborhood $U$ of $x$ such that $\phi(y) \in T_yM$ for any $y \in M \cap U$ and $\phi(x) = u$. The second fundamental form

$$\mathbf{B}_x : T_xM \times T_xM \to N_xM$$

is defined by

$$\mathbf{B}_x(u,v) = \big[\frac{\partial \phi}{\partial v}(x)\big]^{\perp}$$

where $^{\perp}$ stands for the projection on $N_xM$. We extend $\mathbf{B}_x$ to $\mathbb{R}^N \times \mathbb{R}^N$ by setting

$$\mathbf{B}_x(v,w) = \mathbf{B}_x\big(P(x)v, P(x)w\big) \qquad \forall v,\, w \in \mathbb{R}^N$$

and we denote by $\mathbf{B}_{ij}^k(x) = \langle \mathbf{B}_x(e_i,e_j) | e_k \rangle$ the components on $\mathbf{B}_x$ in the canonical basis. Finally, the mean curvature vector $\mathbf{H}(x)$ is the trace of the second fundamental form:

$$\mathbf{H}_i(x) = \sum_j \mathbf{B}_{jj}^i(x).$$

The 3-tensor $\mathbf{B}_x$ embodies all information on the curvature properties of $M$ at $x$, while $\mathbf{H}(x)$ is important in connection with the tangential divergence theorem (see Theorem 4.2). It is well known that $\mathbf{B}_x(u,v)$ is symmetric with respect to $(u,v)$ and that it can be computed by

$$\mathbf{B}_x(u,v) = -\sum_{\alpha=1}^{N-n} \langle u | \nabla_v^M \nu^\alpha(x) \rangle \nu^\alpha \qquad \forall u,\, v \in T_xM, \tag{3.4}$$

where $\nu^1, \ldots, \nu^{N-n}$ is any smooth orthonormal basis of the normal space to $M$ in a neighborhood of $x$.

For an extensive discussion of these results see for instance [10], Chapter 6.

We now define a new 3-tensor $\mathbf{C}_x$ with components (in the canonical basis)

$$C_{ijk}(x) = \nabla_i^M P_{jk}(x) = \nabla_i^M \nabla_j^M x_k. \tag{3.5}$$

Since for any $x \in M$ the matrix $P(x)$ is the orthogonal projection on $T_xM$, also $\mathbf{C}_x$ is expected to contain all information on the curvature of $M$ (see [20, 24]). In the following two theorems we find that $d^3\eta(x)$, $\mathbf{C}_x$ and $\mathbf{B}_x$ are mutually connected by simple linear relations. The proof of the first theorem is taken from [20].

**Theorem 3.3.** *The components of $\mathbf{B}_x$ and the components of $\mathbf{C}_x$ are related for any $x \in M$ by the identities*

$$\mathbf{B}_{ij}^k(x) = P_{mj}(x)C_{ikm}(x) = P_{li}(x)C_{jkl}(x), \qquad C_{ijk} = \mathbf{B}_{ij}^k(x) + \mathbf{B}_{ik}^j(x). \tag{3.6}$$

*In addition, the mean curvature vector $\mathbf{H}(x)$ of $M$ is given by*

$$\mathbf{H}_i(x) = \sum_l C_{lil}(x). \tag{3.7}$$

*Proof.* Let $x \in M$, $u = e_i$, $v = e_j$ and let $u' = P(x)e_i$, $v' = P(x)e_j$ be the projections of $u$, $v$ on $T_xM$. Taking $\phi(y) = P(y)e_i$, we have

$$\big[\frac{\partial \phi^r}{\partial v'}(x)\big] = P_{jm}(x)(P_{ir})_m(x) \qquad r = 1, \dots, N$$

so that, using the identity $P^2 = P$ on $M$ we get

$$\begin{aligned} \mathbf{B}_{ij}^k(x) &= P_{jm}(x)\big[(P_{ik})_m(x) - P_{kl}(x)(P_{il})_m(x)\big] \\ &= \nabla_j^M P_{ik}(x) - P_{kl}(x)\nabla_j^M P_{il}(x) \\ &= P_{il}(x)\nabla_j^M P_{kl}(x) = P_{il}(x)C_{jkl}(x). \end{aligned}$$

Now we prove the second identity in (3.6). Using the first identity in (3.6) and the symmetry of $P$ we get

$$\begin{aligned} \mathbf{B}_{ij}^k(x) + \mathbf{B}_{ik}^j(x) &= P_{lj}(x)C_{ikl}(x) + P_{lk}(x)C_{ijl}(x) \\ &= P_{lj}(x)\nabla_i^M P_{kl}(x) + P_{lk}(x)\nabla_i^M P_{jl}(x) \\ &= \nabla_i^M\big(P_{jl}(x)P_{lk}(x)\big) = \nabla_i^M P_{jk}(x) = C_{ijk}(x). \end{aligned}$$

Finally, we prove (3.7):

$$\mathbf{H}_i(x) = \mathbf{B}_{kk}^i(x) = P_{lk}(x)C_{kil}(x) = \sum_l C_{lil}(x).$$

□

**Theorem 3.4.** *The tensor $\mathbf{C}_x$ and $d^3\eta(x)$ are related for any $x \in M$ by the identities*

$$C_{ijk}(x) = -P_{il}(x)\eta_{ljk}(x), \qquad \eta_{ijk}(x) = -\frac{1}{2}\big\{C_{ijk}(x) + C_{jki}(x) + C_{kij}(x)\big\}. \tag{3.8}$$

*Proof.* The first identity is an easy consequence of the fact that $d^2\eta(x)$ is the orthogonal projection on $N_xM$. To prove the second one, we write (omitting the dependence on $x$)

$$\begin{aligned} \eta_{ijk} &= -C_{ijk} + (\delta_{is} - P_{is})\eta_{sjk} = -C_{ijk} + (\delta_{is} - P_{is})(-C_{jsk} + (\delta_{jt} - P_{jt})\eta_{stk}) \\ &= -C_{ijk} + (\delta_{is} - P_{is})(-C_{jsk} + (\delta_{jt} - P_{jt})(-C_{kst} + (\delta_{kl} - P_{kl})\eta_{stl})) \\ &= -C_{ijk} - C_{jsk}(\delta_{is} - P_{is}) - C_{kst}(\delta_{is} - P_{is})(\delta_{jt} - P_{jt}) \\ &\quad +(\delta_{is} - P_{is})(\delta_{jt} - P_{jt})(\delta_{kl} - P_{kl})\eta_{stl}. \end{aligned}$$

By Corollary 2.3, the last term is zero, so that (3.6) yields

$$\begin{aligned} \eta_{ijk} &= -C_{ijk} - C_{jki} + C_{jsk}P_{is} - C_{kij} + C_{kit}P_{jt} + C_{ksj}P_{si} - C_{kst}P_{is}P_{jt} \\ &= -C_{ijk} - C_{jki} - C_{kij} + \mathbf{B}_{ij}^k + \mathbf{B}_{ki}^j + \mathbf{B}_{jk}^i - P_{jt}\mathbf{B}_{ik}^t. \end{aligned}$$

Since $\mathbf{B}(e_i, e_k) \in N_x M$, $P_{jt}\mathbf{B}^t_{ik} = 0$; exchanging $i$ with $j$ in the above formula, averaging and using the second identities in (3.6) we eventually get

$$\begin{aligned}\eta_{ijk} &= -C_{ijk} - C_{jki} - C_{kij} + \frac{1}{2}\big\{\mathbf{B}^k_{ij} + \mathbf{B}^j_{ki} + \mathbf{B}^i_{jk} + \mathbf{B}^k_{ji} + \mathbf{B}^i_{kj} + \mathbf{B}^j_{ik}\big\} \\ &= -\frac{1}{2}\big\{C_{ijk} + C_{jki} + C_{kij}\big\}.\end{aligned}$$

□

We are particularly interested in the expression of $\mathbf{H}(x)$ as a function of $d^3\eta(x)$. To begin with, we notice that (3.7) implies

$$\begin{aligned}\mathbf{H}_i(x) &= \sum_l C_{lil}(x) = -P_{lk}(x)\eta_{kil}(x) = -\eta_{kik}(x) + \eta_{lk}(x)\eta_{kil}(x) \\ &= -\eta_{kik}(x) + \frac{1}{2}(\eta_{lk}\eta_{lk})_i(x)\end{aligned}$$

for $i = 1, \ldots, N$. Since $\nabla^2\eta(x)$ is symmetric, $\eta_{lk}(x)\eta_{lk}(x)$ coincides with the sum of the squares of the eigenvalues of $\nabla^2\eta(x)$. By Theorem 2.2, this quantity is equal to $n + o(|x - x^0|)$ near every point $x^0 \in M$, hence $(\eta_{lk}\eta_{lk})_i(x)$ vanishes on $M$. As conjectured in [5] and proved in [1], it follows that

$$\mathbf{H}(x) = -\Delta(\nabla\eta)(x) \qquad \forall x \in M. \tag{3.9}$$

More generally, using (3.6) and (3.8) we can write each component $\mathbf{B}^k_{ij}(x)$ of the second fundamental form as a function of $\nabla^3\eta(x)$:

$$\begin{aligned}\mathbf{B}^k_{ij}(x) &= P_{jm}(x)C_{ikm}(x) = -P_{jm}(x)P_{il}(x)\eta_{lkm}(x) \\ &= -(\delta_{jm} - \eta_{jm}(x))(\delta_{il} - \eta_{il}(x))\eta_{lkm}(x) \\ &= -\eta_{ijk}(x) + \eta_{mj}(x)\eta_{kim}(x) + \eta_{li}(x)\eta_{kjl}(x) - \eta_{jm}(x)\eta_{il}(x)\eta_{lkm}(x) \\ &= -\eta_{ijk}(x) + \eta_{mj}(x)\eta_{kim}(x) + \eta_{mi}(x)\eta_{kjm}(x) \\ &= (\eta_{im}\eta_{mj}(x) - \eta_{ij})_k(x). \qquad (3.10)\end{aligned}$$

Conversely, using the second identities in (3.8), (3.6) we get

$$\begin{aligned}\eta_{ijk}(x) &= -\frac{1}{2}\big\{C_{ijk}(x) + C_{jki}(x) + C_{kij}(x)\big\} \\ &= -\frac{1}{2}\big\{\mathbf{B}^k_{ij}(x) + \mathbf{B}^j_{ik}(x) + \mathbf{B}^i_{jk}(x) + \mathbf{B}^k_{ji}(x) + \mathbf{B}^j_{ki}(x) + \mathbf{B}^i_{kj}(x)\big\} \\ &= -\mathbf{B}^k_{ij}(x) - \mathbf{B}^i_{jk}(x) - \mathbf{B}^j_{ki}(x). \qquad (3.11)\end{aligned}$$

We conclude this section with the analysis of the geometric significance of the eigenvalues $\lambda_i(s)$ in Theorem 2.2. Keeping the same notations of Theorem 2.2, let $x_s = y + s(x - y)$, $p = \nabla d(x)$; denoting by $\lambda_1(s), \ldots, \lambda_n(s)$ the eigenvalues of $\nabla^2\eta(x_s)$ strictly less than 1 and by $w_1, \ldots, w_n$ be the corresponding eigenvectors (independent of $s$) spanning $T_yM$, the following theorem holds:

**Theorem 3.5.** *For any $i = 1, \ldots, n$ we have*

$$\lim_{s\to 0^+} \frac{\lambda_i(s)}{s} = \lambda_i.$$

*Moreover, the numbers $\lambda_i$ are the eigenvalues of the symmetric bilinear form*

$$-\big\langle \mathbf{B}_y(u, v)|p\big\rangle \qquad u,\, v \in T_yM$$

*and $w_i$ are the corresponding eigenvectors.*

*Proof.* By the remarks following Theorem 2.2, $\lambda_i(s)/s$ are eigenvalues $\beta_i(s)$ of $\nabla^2 d^M(x_s)$. Let $\pi$ be the affine $(n+1)$–space generated by $T_yM$ and $p$ and passing through $y$. In addition let $\Sigma \subset \pi$ be the smooth $n$–manifold obtained projecting $U \cap M$ on $\pi$, for a suitable neighborhood $U$ of $y$, and let $\overline{\mathbf{B}}_y$ be the second fundamental form of $\Sigma$ at $y$. Viewing $\Sigma$ as a codimension 1 surface in $\pi$, we denote (see for instance [10]) by $\lambda_1, \ldots, \lambda_n$ the *principal curvatures* at $y$ of $\Sigma$ (with the orientation induced near $y$ by $p$), defined as the eigenvalues of the symmetric bilinear form

$$\langle \overline{\mathbf{B}}_y(u,v)|p\rangle \qquad u,\, v \in T_y\Sigma.$$

To prove the theorem, we first assume $n = (N-1)$. Under this assumption, $\Sigma = M$ and the property is a straightforward consequence of the well known formula (see for instance [18], Lemma 14.17)

$$\beta_i(s) = \frac{-\lambda_i}{1 - s\lambda_i} \qquad \forall s \in (0, d(x)]$$

for the eigenvalues $\beta_i(s)$ of $\nabla^2 d^\Sigma(x_s)$ corresponding to eigenvectors in $\pi$ (see also (2.3) or [13]).

In the general case, we notice that, by Theorem 2.1, $\eta^\Sigma$ is smooth near $y$ and

$$\limsup_{z\to y,\, z\in\pi} \frac{|\eta^M(z) - \eta^\Sigma(z)|}{|z-y|^4} < +\infty \tag{3.12}$$

because $\Sigma$ is obtained projecting $M$ on the space $\pi$ containing $y + T_yM$. By (3.12) we infer

$$\lim_{s\to 0^+} \frac{\nabla^2\eta^M(x_s) - \nabla^2\eta^\Sigma(x_s)}{s} = 0.$$

Since the matrices are diagonal in the same basis, denoting by $\overline{\lambda}_i(s)$ the eigenvalues of $\nabla^2\eta^\Sigma(x_s)$ corresponding to the directions $w_i$, $\lambda_i(s)/s$ converge to the same limit of $\overline{\lambda}_i(s)/s$, i.e., $\lambda_i$.

Finally, we prove the last statement. By (3.12) we infer

$$d^3\eta^M(y)(u,v,p) = d^3\eta^\Sigma(y)(u,v,p) \qquad \forall u,\, v \in T_yM = T_y\Sigma,$$

so that (3.10) yields

$$\langle \mathbf{B}_y(u,v)|p\rangle = \langle \overline{\mathbf{B}}_y(u,v)|p\rangle \qquad \forall u,\, v \in T_yM$$

because $p \in N_yM \cap N_y\Sigma$. This proves that $\lambda_i$ are the eigenvalues of $-\langle \mathbf{B}_y|p\rangle$ and that $w_i$ are the corresponding eigenvectors. □

*Remark* 3.6. In particular, the sum of the eigenvalues $\beta_i(s) = \lambda_i(s)/s$ of $\nabla^2 d(x_s)$ converges as $s \to 0^+$ to $-\langle \mathbf{H}(y)|p\rangle$. This property, already proved in [1], has been used to extend the level set approach (see [25, 12, 4]) to the evolution of surfaces of any codimension.

## 4. Functionals Depending on the Distance

In this section we use the squared distance function $\eta$ to compute the first variation of several geometric functionals. For our purposes, it is technically more convenient to work with the convex function

$$A^M(x) = \frac{1}{2}|x|^2 - \eta(x) = \frac{|x|^2 - d^2(x)}{2},$$

smooth in the "tubular neighborhood" $\Omega$ of the manifold $M$ introduced in Theorem 2.1.

The greater convenience of $A^M$ can be explained noticing that $\nabla^2 A^M(x)$ is, for $x \in M$, the projection matrix on $T_xM$, and this quantity often appears in the computation of tangential gradients. For the reader's convenience, we reformulate now the results of the preceding sections in terms of $A^M$:

**Proposition 4.1.** *(a) For any $x \in \Omega$, $\nabla A^M(x)$ is the projection $y = \pi^M(x)$ of $x$ on $M$. In addition $\nabla^2 A^M(x)$ is zero on $N_yM$ and maps $T_yM$ into $T_yM$. If $x = y \in M$, then $\nabla^2 A^M(x)$ is the matrix of orthogonal projection on $T_xM$;*
*(b) for any $x \in \Omega$, the 3–linear form $\mathbf{K}_x : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ given by*

$$\mathbf{K}_x(u, v, w) = \sum_{ijk} A^M_{ijk}(x) u_i v_j w_k$$

*is equal to zero if at least two of the 3 vectors $u$, $v$, $w$, are normal to $M$ at $\nabla A^M(x)$;*
*(c) for $x \in M$, the second fundamental form $\mathbf{B}_x$ and the mean curvature vector $\mathbf{H}(x)$ are related to the derivatives of $A^M(x)$ by*

$$\mathbf{B}^k_{ij}(x) = A^M_{jm}(x) A^M_{il}(x) A^M_{mlk}(x) = \left(\delta_{kl} - A^M_{kl}(x)\right) A^M_{ijl}, \tag{4.1}$$

$$\mathbf{H}(x) = \sum_j A^M_{jij}(x), \tag{4.2}$$

$$\nabla^M_i A^M_{jk}(x) = \mathbf{B}^k_{ij}(x) + \mathbf{B}^j_{ik}(x). \tag{4.3}$$

*Proof.* The first statement follows by Theorem 2.1 and the second one by Corollary 2.3. The first equality in (4.1) and (4.2) follow by (3.9) and (3.10). The second equality in (4.1) can be obtained multiplying in (3.11) by $(I - \nabla^2 A^M)$. Finally (4.3) is a restatement of the second equality (3.6). □

In this section we will consider functionals $\mathcal{F}$ defined on the class of smooth $n$–manifolds $M$ (compact and with no boundary) of the following kind:

$$\mathcal{F}(M) = \int_M f\left(A^M_{ij\,\ldots\,k}\right)\, d\mathcal{H}^n(x) \tag{4.4}$$

assuming that the smooth function $f$ depends only on a finite number of derivatives of $A^M$, of order at most $s$. For the sake of simplicity we assume that $\mathcal{F}$ is autonomous, i.e., $f$ does not depend on $\nabla A^M(x) = x$. Notice that in this way we can define every functional depending on the curvature, using the relations (4.2).

### 4.1. Area Formula and Divergence Theorem.

For a general smooth map $\Phi : M \to \mathbb{R}^k$ we can consider the *tangential Jacobian*,

$$J^M\Phi(x) = \left[\det\left(d^M\Phi^*_x \circ d^M\Phi_x\right)\right]^{1/2}$$

where $d^M\Phi_x : T_xM \to \mathbb{R}^k$ is the linear map induced by the the tangential gradient and $\left(d^M\Phi_x\right)^* : \mathbb{R}^k \to T_xM$ is the adjoint map.

**Theorem 4.2** (Tangential Divergence Theorem)**.** *For any smooth vector field $X$ defined on $M$ the following formula holds:*

$$\int_M \operatorname{div}^M X\, d\mathcal{H}^n = -\int_M \langle \mathbf{H} \,|\, X\rangle\, d\mathcal{H}^n.$$

A simple consequence of this theorem is the equation,

$$\int_M f\Delta^M g\, d\mathcal{H}^n = \int_M g\Delta^M f\, d\mathcal{H}^n.$$

Another fundamental result is the following Area Formula.

**Theorem 4.3** (Area Formula)**.** *If $\Phi$ is a smooth injective map from $M$ to $\mathbb{R}^k$, then we have*

$$\int_{\Phi(M)} f(y)\,d\mathcal{H}^n(y) = \int_M f(\Phi(x))\,J^M\Phi(x)\,d\mathcal{H}^n(x) \tag{4.5}$$

*for every $f \in C^0(\mathbb{R}^k)$.*

For detailed discussions and proofs of these and of related facts we refer to the books of Federer [15] and of Simon [28].

4.2. **First Variation.** We can now study the first variation of the general functional (4.4) deforming the manifold $M$ along some vector field $X$.

Let $\Phi_t(x)$ be a smooth one–parameter family of diffeomorphisms of $\Omega$ in itself, with $\Phi_0$ equal to the identity. That is, $\Phi : \mathbb{R}\times\Omega \to \Omega$ is a smooth function acting on the manifold $M$ and giving a *deformation* $M_t = \Phi_t(M)$ which is clearly again a smooth $n$–manifold. We want to compute the derivative of $\mathcal{F}(M_t)$ at $t = 0$, i.e.

$$\frac{d}{dt}\int_{M_t} f\left(A^{M_t}_{i_1i_2},\,\dots\,,A^{M_t}_{j_1\,\dots\,j_\gamma}\right)\,d\mathcal{H}^n(x)\Bigg|_{t=0}.$$

The main result of this section is the following:

**Theorem 4.4.** *There exists a unique vector field $E_{\mathcal{F}}(A^M)$ such that*

$$\frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} = \int_M \langle E_{\mathcal{F}}\left(A^M\right)\mid X\rangle\,d\mathcal{H}^n \tag{4.6}$$

*for any family $\Phi_t$ whose infinitesimal generator is $X$. Moreover $E_{\mathcal{F}}(A^M)$ is normal and:*
*(i) if $f$ depends on the derivatives of $A^M$ up to the order $\gamma$, then $E_{\mathcal{F}}(A^M)$ depends on the derivatives of $A^M$ up to the order $(2\gamma-1)$;*
*(ii) if the function $f$ in the functional (4.4) is a polynomial, then $E_{\mathcal{F}}(A^M)$ is a polynomial in the derivatives of $A^M$;*
*(iii) if $\gamma = 2$ in (i), we have*

$$E_{\mathcal{F}}\left(A^M\right) = -f\left(A^M_{ij}\right)\mathbf{H} - 2\left(\nabla^M_j\phi_{ij} + \phi_{ij}\mathbf{H}_j\right)e_i^\perp + 2\phi_{ij}\mathbf{B}^s_{ij}e_s$$

*where $\phi_{ij}(x) = \partial f/\partial A^M_{ij}(x)$ and $e_i^\perp = (I - \nabla^2A^M)e_i$ is the normal component of $e_i$.*

By the same argument leading to Theorem 2.2, choosing $\Omega$ small enough we can assume that for $t \in (-\varepsilon,\varepsilon)$ all the manifolds $M_t$ are contained in $\Omega$ and that the function $A^t(x) = A^{M_t}(x)$ is a smooth function in $t \in (-\varepsilon,\varepsilon)$ and $x \in \Omega$.

Applying the Area Formula 4.5 to the map $\Phi_t : M \to M_t$ we can rewrite the derivative as

$$\frac{d}{dt}\int_M f\left(A^t_{i_1i_2}(\Phi_t(x)),\,\dots,\,A^t_{j_1\,\dots\,j_\gamma}(\Phi_t(x))\right)\,J^M\Phi_t(x)\,d\mathcal{H}^n(x)\Bigg|_{t=0}$$

where $J^M\Phi_t(x)$ denotes the tangential Jacobian on $M$ of the map $\Phi_t$.
Hence, carrying the derivative under the integral sign, we find out

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} &= \int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha}\,\frac{d}{dt}\left[A^t_\alpha(\Phi_t(x))\right]\Big|_{t=0}\,d\mathcal{H}^n(x)\\ &\quad+\int_M f\left(A^{M_t}_{i_1i_2},\,\dots\,,A^{M_t}_{j_1\,\dots\,j_\gamma}\right)\,\frac{d}{dt}J^M\Phi_t(x)\Big|_{t=0}\,d\mathcal{H}^n(x).\end{aligned}$$

where $\gamma$ is a multiindex such that $|\alpha| \leq \gamma$.
Now, the derivative of the Jacobian is simply the tangential divergence of its infinitesimal generator field $X(x) = \frac{d\Phi_t(x)}{dt}\Big|_{t=0}$ and the derivative of the function $[A^t_\alpha(\Phi_t(x))]$ can be expressed by

$$\frac{d}{dt}\left[A^t_\alpha(\Phi_t(x))\right]\Big|_{t=0} = \frac{\partial A^t_\alpha}{\partial t}(x)\Big|_{t=0} + \langle \nabla A^M_\alpha(x) \,|\, X(x)\rangle.$$

Using the fact that the function $A^t(x)$ is smooth, we can exchange the order of differentiation in the middle term of this equation to get

$$\frac{\partial A^t_\alpha}{\partial t}(x)\Big|_{t=0} = D^\alpha \left\{ \frac{\partial}{\partial t} A^t(x)\Big|_{t=0} \right\}.$$

To go on, we need to compute the derivative of the function $A^t(x)$ at $t = 0$.

**Lemma 4.5.** *Under the above smoothness assumptions, we have*

$$\frac{\partial}{\partial t} A^t(x)\Big|_{t=0} = -\langle \nabla A^M(x) - x \,|\, X(\nabla A^M(x))\rangle \tag{4.7}$$

*where $X$ is the infinitesimal generator field of $\Phi_t$.*

*Proof.* We consider a point $x \in \Omega$ and we define $y = \pi^M(x) \in M$ and $z = \Phi_t(y) \in M_t$. We have $d^2(x, M) = \|x - y\|^2$ and $d^2(x, M_t) \leq \|x - z\|^2$, hence

$$\begin{aligned}\frac{A^{M_t}(x) - A^M(x)}{t} &= -\frac{1}{2}\frac{d^2(x, M_t) - d^2(x, M)}{t} \\ &\geq \frac{1}{2}\frac{\|x - y\|^2 - \|x - z\|^2}{t} = \frac{\langle z - y \,|\, 2x - y - z\rangle}{2t}.\end{aligned}$$

Now $z - y = \Phi_t(y) - y$ is infinitesimal as $t \to 0$, moreover

$$\Phi_t(y) = y + tX\left(\pi^M(x)\right) + o(t).$$

Then the last term of the equation above tends to

$$\langle X(\pi^M(x)) \,|\, x - \pi^M(x)\rangle = -\langle \nabla A^M(x) - x \,|\, X(\nabla A^M(x))\rangle.$$

This proves that

$$\liminf_{t\to 0} \frac{A^{M_t}(x) - A^M(x)}{t} \geq -\langle \nabla A^M(x) - x \,|\, X(\nabla A^M(x))\rangle.$$

Now, using a similar reasoning with $y = \pi^{M_t}(x)$ and $z = \Phi_t^{-1}(y)$, we obtain the opposite estimate

$$\limsup_{t\to 0} \frac{A^{M_t}(x) - A^M(x)}{t} \leq -\langle \nabla A^M(x) - x \,|\, X(\nabla A^M(x))\rangle$$

and this proves the lemma. ∎

We can now write the following general formula

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} &= \int_M f\left(A^{M_t}_{i_1 i_2}, \ldots, A^M_{j_1 \ldots j_\gamma}\right) \operatorname{div}^M X(x)\, d\mathcal{H}^n(x) \\ &+ \int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha} \langle \nabla A^M_\alpha(x) \,|\, X(x)\rangle\, d\mathcal{H}^n(x) \\ &- \int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha} D^\alpha \left[\langle \nabla A^M(x) - x \,|\, X(\nabla A^M(x))\rangle\right] d\mathcal{H}^n(x).\end{aligned}$$

Applying the tangential divergence theorem 4.2 to the first term and adding together gradient and tangential gradient of the functions $A^M$ we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} = &-\int_M f\left(A^M_{i_1 i_2},\,\dots\,,A^M_{j_1\,\dots\,j_\gamma}\right)\langle \mathbf{H}\,|\,X\rangle\,d\mathcal{H}^n\\ &+\int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha}\langle \nabla^\perp A^M_\alpha\,|\,X\rangle\,d\mathcal{H}^n\\ &-\int_M \sum_\alpha \frac{\partial f}{\partial A^M_\alpha} D^\alpha\left[\langle \nabla A^M(x)-x\,|\,X(\nabla A^M(x))\rangle\right]\,d\mathcal{H}^n(x)\end{aligned} \tag{4.8}$$

recalling that $\mathbf{H}$ is the mean curvature and that the sign "$^\perp$" denotes the projection on the normal space to the manifold $M$.

It is now clear that the only problem to get to the Euler equations for $\mathcal{F}$ relies on the computation of the last term, and in particular on the study of the derivatives

$$D^\alpha\left[\langle \nabla A^M(x)-x\,|\,X(\nabla A^M(x))\rangle\right]. \tag{4.9}$$

Before proceeding to the computation of (4.9), we want to make some remarks on the first variation:

**Proposition 4.6.** *The first variation of the functional (4.4) depends only on the values on $M$ of the infinitesimal generator $X$. Moreover if the vector field $X$ is tangent to $M$, the first variation is zero.*

*Proof.* Since $\nabla A^M(x)\in M$ for any $x\in\Omega$, (4.8) clearly implies that the first variation depends only on $X|_M$. If $X$ is tangential, the first term is zero because $\mathbf{H}(x)\in N_xM$, the second one is clearly zero and the last one vanishes because $\nabla A^M(x)-x$ is normal to $M$ at $\nabla A^M(x)$ for any $x\in\Omega$. □

Since (4.8) is linear in $X$, splitting $X(x)$ in $P(x)X(x)$ and $(I-P(x))X(x)$, we can assume in the following that $X(x)$ is normal to $M$ at $\nabla A^M(x)$ for any $x\in\Omega$.

Now we go on with the study of the equation (4.9) assuming that the multiindex $\alpha$ is described by $(i_1,\,\dots\,i_r)$ with $\gamma\geq r\geq 2$. We can distribute the derivatives on the two terms inside the scalar product. If all the derivatives act on the right term in the scalar product the result is zero, because the quantity $\nabla A^M(x)-x$ is zero on the manifold $M$. If all the derivatives go on the left term, it is simple to see that we obtain exactly the second term, with the opposite sign, in equation (4.8) which simplifies. So we study the terms with at least one derivative on $X(\nabla A^M(x))$ and at least one on $\nabla A^M(x)-x$.
Forgetting the term on the left in the scalar product, which will produce functions of kind $A^M_{j_1\,\dots\,j_t}$, we reduce ourselves to study the derivatives of functions of the type $\varphi(\nabla A^M(x))$ at points of $M$, where $\varphi: M\to\mathbb{R}$.

**Proposition 4.7.** *For every multiindex $\beta$ the derivative $D^\beta[\varphi(\nabla A^M(x))]$ can be expressed on $M$ by a sum of terms*

$$g(x)\nabla^M_{j_1}\circ\nabla^M_{j_2}\circ\,\dots\,\circ\nabla^M_{j_l}\varphi(x)$$

*with $l\leq|\beta|$ and with the functions $g$ being polynomials in the derivatives of $A^M$ up to the order $|\beta|+1$.*

*Proof.* We fix a notation, denoting by $\nabla^M f(x)$ the projection of the gradient of the function $f$ on the tangent space of $M$ at the point $\pi^M(x)$ even if $x\notin M$. This vector clearly coincides with the tangential gradient if $x\in M$.

We prove by induction on $n = |\beta|$, that every derivative can be written as a sum of terms of the following kind:

$$g\left(A^M\right) \nabla^M_{j_1} \circ \nabla^M_{j_2} \circ \ldots \circ \nabla^M_{j_l} \varphi(\nabla A^M(x)) \tag{4.10}$$

for $x \in \Omega$, $l \leq n$ and where $g\left(A^M\right)$ denotes a function of the derivatives of $A^M$ up to the order $(n+1)$ (here we tangentially differentiate $\varphi(y)$ $l$ times and we evaluate the derivatives at $\nabla A^M(x)$).
If $n = 1$ we have only one derivative, hence

$$\frac{\partial}{\partial x_i}[\varphi(\nabla A^M(x))] = \nabla_k \varphi(\nabla A^M(x))\, A^M_{ki}(x).$$

By the properties (a) of $\nabla^2 A^M$, in the equation above we can consider $\nabla^M_k \varphi$ instead of $\nabla\varphi$, and the first case of the induction is achieved.
Now, assuming the proposition true for $(n-1)$, to get the induction step we have to differentiate with respect to $x_i$ a formula like (4.10). If the additional derivative $\nabla_i$ acts on $g\left(A^M\right)$ it does not matter, while when it acts on the other factor we apply the same reasoning of the case $n = 1$ to get a term of the form

$$g\left(A^M\right) A^M_{ik}(x) \nabla^M_k \circ \nabla^M_{j_1} \circ \nabla^M_{j_2} \circ \ldots \circ \nabla^M_{j_l} \varphi(\nabla A^M(x)).$$

Finally, if $x$ belongs to $M$ we have $\nabla A^M(x) = x$ and the statement follows. □

*Proof of Theorem 4.4.* The uniqueness of $E_{\mathcal{F}}(A^M)$ easily follows by the possibility to choose $\Phi_t(x) = x + tX(x)$ where $X$ is any vector field.
We first consider the special case $\gamma = 2$, that is, functionals depending on the tangent space of $M$. In this case the Euler equation easily follows by formula (4.8), using the argument of the case $|\beta| = 1$ in Proposition 4.7:

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} &= -\int_M f\left(A^M_{ij}\right) \langle \mathbf{H} \,|\, X\rangle \, d\mathcal{H}^n \\ &\quad - \int_M \frac{\partial f}{\partial A^M_{ij}} \left(A^M_{is}(x) - \delta_{is}\right) \nabla^M_j X^s(x)\, d\mathcal{H}^n(x) \\ &\quad - \int_M \frac{\partial f}{\partial A^M_{ij}} \left(A^M_{js}(x) - \delta_{js}\right) \nabla^M_i X^s(x)\, d\mathcal{H}^n(x). \end{aligned}$$

Defining $\phi_{ij}$ as in the statement of the theorem and using the tangential divergence theorem we find out

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(M_t)\Big|_{t=0} &= -\int_M f\left(A^M_{ij}\right) \langle \mathbf{H} \,|\, X\rangle \, d\mathcal{H}^n \\ &+ \int_M \left[\phi_{ij}(x)\left(A^M_{is}(x) - \delta_{is}\right) \mathbf{H}_j(x) + \nabla^M_j(\phi_{ij}(x)(A^M_{is}(x) - \delta_{is}))\right] X^s(x)\, d\mathcal{H}^n(x) \\ &+ \int_M \left[\phi_{ij}(x)\left(A^M_{js}(x) - \delta_{js}\right) \mathbf{H}_i(x) + \nabla^M_i(\phi_{ij}(x)(A^M_{js}(x) - \delta_{js}))\right] X^s(x)\, d\mathcal{H}^n(x). \end{aligned}$$

Finally, using the orthogonality of $X$ we obtain (4.4).

In general, the existence of $E_{\mathcal{F}}(A^M)$ and its computing algorithm are described by the following steps:

- **Step 1** We distribute the derivatives on the two terms in the scalar product in the last line of (4.8), avoiding to have all the derivatives acting on one alone.
- **Step 2** Write the derivation operator on the field $X$ in terms of tangential gradients, following Proposition 4.7.

- **Step 3** Bring derivatives away from the field $X$, using the identity $f\nabla_i^M X^s = \nabla_i^M(fX^s) - X^s\nabla_i^M f$, and then the tangential divergence theorem 4.2 to exchange the integral of $\nabla_i^M(fX^s)$ with the integral of $-\mathbf{H}_i fX^s$. Iterating this procedure we get to an expression $g^s X^s$, which we are interested in.

In particular, we obtain that $E_{\mathcal{F}}(A^M)$ has a polynomial dependence on the derivatives of $A^M$ if the same is true for $f$. Applying Proposition 4.7 to expressions like

$$\frac{\partial f}{\partial A^M_\alpha}\left[\langle D^\beta\left(\nabla A^M(x)-x\right) \mid D^\tau X(\nabla A^M(x))\rangle\right]$$

with $\beta+\tau=\alpha$ and $\beta,\,\tau\neq 0$, one finds terms of the following form

$$g_\sigma(x)\nabla^M_{\sigma_1}\circ\ldots\circ\nabla^M_{\sigma_l}X(x)$$

with $g_\sigma$ depending on the derivatives of $A^M$ up to the order $|\alpha|$ and $l\leq|\alpha|-1$; integrating by parts we obtain terms depending on the derivatives of $A^M$ up to the order $l+|\alpha|$. Since $l\leq|\alpha|-1$ and $|\alpha|\leq\gamma$, we get terms with derivatives of order at most $(2\gamma-1)$. □

## 5. Euler Equations for some Particular Functionals

In this section we study and compute effectively the Euler equations $E_{\mathcal{F}}(A^M)$ in some cases. We will consider the following functionals:

$$\mathcal{F}_p(M)=\int_M |\mathbf{H}|^p\,d\mathcal{H}^n,\qquad\qquad \mathcal{G}_\gamma(M)=\int_M\sum_{|\alpha|=\gamma}|A^M_\alpha|^2\,d\mathcal{H}^n \tag{5.1}$$

defined on compact, smooth $n$–manifolds $M$ embedded in $\mathbb{R}^N$ with $\partial M=\emptyset$.
The function $\mathbf{H}$ appearing inside the first integral is the mean curvature vector of $M$ that, as we remarked, can be expressed by $\Delta\left(\nabla A^M\right)$. The Willmore functional corresponds to the case of surfaces in $\mathbb{R}^3$ with $p=2$, for further references on this topic see [34]. We also notice that for $\gamma=2$ the functional $\mathcal{G}_\gamma$ reduces to $n\mathcal{H}^n(M)$, whose first variation is $-n\mathbf{H}$.

If $\gamma=3$, by (3.11) the functional $\mathcal{G}_\gamma$ is equal to 3 times the integral of the square of the quadratic norm of $\mathbf{B}$. By the Gauss–Bonnet theorem, in the case $n=2$, $N=3$ the functionals $\mathcal{F}_2$ and $\mathcal{G}_3$ are proportional, because the deformation does not change the genus of the manifold and $|\mathbf{B}|^2$ is equal to $|\mathbf{H}|^2-2\lambda_1\lambda_2$, where $\lambda_1,\,\lambda_2$ are the principal curvatures. In particular, in this case we have $E_{\mathcal{G}_3}=3\,E_{\mathcal{F}_2}$ (see also Remark 5.3).

### 5.1. Codazzi–Mainardi's Equations.

In the computations of this section we will need the following result.

**Proposition 5.1.** *At every point of the manifold $M$, the following relation holds,*

$$\begin{aligned}\nabla_i^M\mathbf{B}^l_{jk}-\nabla_j^M\mathbf{B}^l_{ik} = \textstyle\sum_s\big\{&\mathbf{B}^l_{ks}\nabla^M_i P_{js}-\mathbf{B}^l_{ks}\nabla^M_j P_{is}\\ &+\mathbf{B}^l_{js}\nabla^M_i P_{ks}-\mathbf{B}^l_{is}\nabla^M_j P_{ks}\\ &+\mathbf{B}^s_{ik}\nabla^M_j P_{ls}-\mathbf{B}^s_{jk}\nabla^M_i P_{ls}\big\}.\end{aligned}$$

For a proof of this proposition, consult the book of Do Carmo [10] at Chapter 6, Section 2.

*Remark* 5.2. We notice that in the codimension one case this relation becomes very simple: denoting with $\nu$ a locally smooth, unit normal vector field and with $\mathbf{B}^\nu$ the symmetric bilinear form $\langle\mathbf{B}|\nu\rangle$, we have

$$\nabla_i^M\mathbf{B}^\nu_{jk}-\nabla_j^M\mathbf{B}^\nu_{ik} = \nu_j\left[\mathbf{B}^\nu\right]^2_{ik}-\nu_i\left[\mathbf{B}^\nu\right]^2_{jk}\,.$$

Moreover, setting in this formula $j = k$ and summing over the index $k$, we get the equation

$$\sum_k \nabla^M_k \mathbf{B}^\nu_{ik} \;=\; \nabla^M_i H + \nu_i B^2 \tag{5.2}$$

where $H$ is $\langle \mathbf{H} | \nu \rangle$ and $B^2$ is the square of the quadratic norm of $\mathbf{B}^\nu$.

5.2. **First Variation of $\mathcal{F}_p$.** We have seen in (4.8) that the first variation is expressed by

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_p(M_t)\Big|_{t=0} = & -\int_M |\mathbf{H}|^p \,\langle \mathbf{H} \,|\, X\rangle \, d\mathcal{H}^n \\
& + \int_M \langle \nabla^\perp |\mathbf{H}|^p \,|\, X \rangle \, d\mathcal{H}^n \\
& - p \sum_{ijl} \int_M |\mathbf{H}|^{p-2}\, A^M_{ill} D^{ijj} \left[ \langle \nabla A^M(x) - x \,|\, X(\nabla A^M(x))\rangle \right] \, d\mathcal{H}^n(x).
\end{aligned} \tag{5.3}$$

By Proposition 4.6, we can assume that $X(x)$ is normal to $M$ at $\nabla A^M(x)$ for any $x \in \Omega$. Studying the last term and distributing the derivatives in the scalar product we obtain the following:

- With 3 derivatives on the left term we get

  $$-\int_M \langle \nabla^\perp |\mathbf{H}|^p \,|\, X\rangle \, d\mathcal{H}^n$$

  that simplifies with the second term in (5.3).
- 3 derivatives on the right term give zero, because the function $\nabla A^M(x) - x$ is zero on $M$.
- 2 derivatives on the left term,

  $$\begin{aligned}
  & -p \int_M |\mathbf{H}|^{p-2}\, A^M_{ill} \langle D^{jj} \left(\nabla A^M(x) - x\right) \,|\, \nabla^M_i X(x)\rangle \, d\mathcal{H}^n(x) \\
  & -2p \int_M |\mathbf{H}|^{p-2}\, A^M_{ill} \langle D^{ij} \left(\nabla A^M(x) - x\right) \,|\, \nabla^M_j X(x)\rangle \, d\mathcal{H}^n(x).
  \end{aligned}$$

  The first term is zero because $A^M_{ill}$ is a normal vector and $\nabla^M_i X$ is a tangential gradient. The second one, using the tangential divergence theorem can be expressed as

  $$\begin{aligned}
  & 2p \int_M |\mathbf{H}|^{p-2}\, \mathbf{H}_j A^M_{ill} \langle \nabla A^M_{ij}(x) \,|\, X(x)\rangle \, d\mathcal{H}^n(x) \\
  & +2p \int_M \nabla^M_j \left\{ |\mathbf{H}|^{p-2}\, A^M_{ill} A^M_{ijs} \right\} X^s \, d\mathcal{H}^n.
  \end{aligned}$$

  Finally, by the fact that the 3–tensor $A^M_{ijk}$ gives zero when applied to the two normal vectors $A^M_{ill}$ and $\mathbf{H}_j$ (see Proposition 4.1(b)), we get

  $$2p \int_M \nabla^M_j \left\{ |\mathbf{H}|^{p-2}\, \mathbf{H}_i A^M_{ijs} \right\} X^s \, d\mathcal{H}^n.$$

- 2 derivatives on the right term,

$$-p\int_M |\mathbf{H}|^{p-2}\, A^M_{ill} D^i \left(\nabla_s A^M(x) - x_s\right)\, D^{jj}\left[X^s(\nabla A^M(x))\right]\, d\mathcal{H}^n(x)$$

$$-2p\int_M |\mathbf{H}|^{p-2}\, A^M_{ill} D^j \left(\nabla_s A^M(x) - x_s\right)\, D^{ij}\left[X^s(\nabla A^M(x))\right]\, d\mathcal{H}^n(x).$$

Using an orthogonality argument like above, we see that the second of these two terms vanishes, while the first one gives

$$p\int_M |\mathbf{H}|^{p-2}\, A^M_{sll}\frac{\partial}{\partial x_j}\left\{A^M_{jr}(x)A^M_{rt}(\nabla A^M(x))X^s_t(\nabla A^M(x))\right\}\, d\mathcal{H}^n(x)$$

$$= p\int_M |\mathbf{H}|^{p-2}\, A^M_{sll}A^M_{jr}(x)\frac{\partial}{\partial x_j}\left\{A^M_{rt}(\nabla A^M(x))X^s_t(\nabla A^M(x))\right\}\, d\mathcal{H}^n(x)$$

$$+ p\int_M |\mathbf{H}|^{p-2}\, A^M_{sll}A^M_{jjr}(x)A^M_{rt}(\nabla A^M(x))X^s_t(\nabla A^M(x))\, d\mathcal{H}^n(x)$$

$$= p\int_M |\mathbf{H}|^{p-2}\, A^M_{sll}\nabla^M_r \circ \nabla^M_r X^s(x)\, d\mathcal{H}^n(x) = p\int_M |\mathbf{H}|^{p-2}\, A^M_{sll}\Delta^M X^s\, d\mathcal{H}^n(x)$$

where we used extensively Proposition 4.1(a) and in particular the identity $A^M_{jr}(x)A^M_{rt}(\nabla A^M(x)) = A^M_{jt}(x)$. Substituting $A^M_{sll}$ with $\mathbf{H}_s$ and using the properties of the tangential Laplacian, this final term is equal to

$$p\int_M \Delta^M\left(|\mathbf{H}|^{p-2}\,\mathbf{H}_s\right) X^s\, d\mathcal{H}^n.$$

Finally, adding all these results together, we get

$$\frac{d}{dt}\mathcal{F}_p(M_t)\Big|_{t=0} = -\int_M |\mathbf{H}|^p\,\langle \mathbf{H}\,|\,X\rangle\, d\mathcal{H}^n + p\int_M \Delta^M\left(|\mathbf{H}|^{p-2}\,\mathbf{H}_i\right)X^i\, d\mathcal{H}^n$$
$$+2p\int_M \nabla^M_j\left\{|\mathbf{H}|^{p-2}\,\mathbf{H}_s A^M_{ijs}\right\}X^i\, d\mathcal{H}^n.$$

Using the orthogonality of $X$ and Proposition 4.1(b) we can simplify again the last term to get

$$2p\, e^\perp_i \nabla^M_j\left\{|\mathbf{H}|^{p-2}\mathbf{H}_s A^M_{ijs}\right\} = 2p\, e^\perp_i |\mathbf{H}|^{p-2}\nabla^M_j\left\{\mathbf{H}_s A^M_{ijs}\right\}.$$

Now we have,

$$e^\perp_i \nabla^M_j\left\{\mathbf{H}_s A^M_{ijs}\right\} = e^\perp_i \nabla^M_j\left\{\mathbf{H}_s \mathbf{B}^s_{ij}\right\} = e^\perp_i \mathbf{H}_s \nabla^M_j \mathbf{B}^s_{ij}$$

and using the relation of Proposition 5.1,

$$e^\perp_i \nabla^M_j \mathbf{B}^s_{ij} = \mathbf{B}^s_{jt}\nabla^M_j P_{it}$$

hence, substituting this quantity in the equation above, the term we are dealing with becomes

$$\mathbf{H}_s\mathbf{B}^s_{jt}\nabla^M_j P_{it} = \mathbf{H}_s\mathbf{B}^s_{jt}A^M_{jti} = \mathbf{H}^s\mathbf{B}^s_{jt}\mathbf{B}^i_{jt}.$$

Then we get the Euler equation of $\mathcal{F}_p$,

$$E_{\mathcal{F}_p} = -|\mathbf{H}|^p\,\mathbf{H} + 2p\,|\mathbf{H}|^{p-2}\mathbf{H}^s\mathbf{B}^s_{jt}\mathbf{B}^i_{jt}e_i + p\,\Delta^M\left(|\mathbf{H}|^{p-2}\,\mathbf{H}_i\right)e^\perp_i, \tag{5.4}$$

where we denoted by $e^\perp_i = (I - \nabla^2 A^M)e_i$ the normal projections of the vectors of the canonical basis of $\mathbb{R}^N$.

In the codimension 1 case $n = (N-1)$, we have a scalar form of the Euler equations. Indeed, considering the classical second fundamental form $\mathbf{B}^\nu = \langle \mathbf{B} | \nu \rangle$, locally we can write $X(x) = \varphi(x)\nu(x)$ and $\mathbf{B}^l_{jt} = \nu_l \mathbf{B}^\nu_{jt}$, hence $\mathbf{H}(x) = H(x)\nu(x)$, with $\nu$ smooth unit normal vector field and $\varphi$ in $C^\infty(M)$.
The equation (5.4) becomes

$$\begin{aligned}\langle E_{\mathcal{F}_p} \,|\, X \rangle = &- |H|^p H\varphi + 2p\varphi |H|^{p-2} H \,\text{Trace}\, [\mathbf{B}^\nu]^2 + p\varphi \Delta^M \left( |H|^{p-2} H \right)\\ &+ p\varphi \, |H|^{p-2} \, H \nu_i \Delta^M \nu_i\end{aligned}$$

where we used the fact that $\nu_i \nabla^M \nu_i$ is equal to zero because $\nu$ is a unit vector field. By the same reason $\nu_i \Delta^M \nu_i = -\langle \nabla^M \nu_i | \nabla^M \nu_i \rangle = -\sum_i \left| \nabla^M \nu_i \right|^2$, which is the square of the quadratic norm of the bilinear form $\mathbf{B}^\nu$, indeed, by (3.4) we have

$$\mathbf{B}^\nu_{ij} = -\nabla^M_i \nu_j = -\nabla^M_j \nu_i. \tag{5.5}$$

The term Trace $[\mathbf{B}^\nu]^2$ is clearly also equal to $|\mathbf{B}^\nu|^2$; if we denote such quantity with $B^2$, then we can write

$$E_{\mathcal{F}_p} = \left[ -|H|^p H + p \, |H|^{p-2} \, H B^2 + p \, \Delta^M \left( |H|^{p-2} \, H \right) \right] \nu.$$

In particular, setting $p = 2$ we have the nice equation (see [34])

$$E_{\mathcal{F}_2} = \left[ 2 \, \Delta^M H + 2HB^2 - H^3 \right] \nu$$

corresponding to the Willmore functional.

5.3. **First Variation of $\mathcal{G}_\gamma$.** The other functional we are interested in is

$$\mathcal{G}_\gamma(M) = \int_M \sum_{|\alpha|=\gamma} |A^M_\alpha|^2 \, d\mathcal{H}^n, \tag{5.6}$$

as usual defined on compact $n$–manifolds in $\mathbb{R}^N$ with $\partial M = \emptyset$. By the remarks at the beginning of this section, we can assume $\gamma > 2$. We perform the full computation of $E_{\mathcal{G}_\gamma}$ only for the case $\gamma = 3$ in codimension 1, in the general case we only study the part of $E_{\mathcal{G}_\gamma}$ containing the greatest number of derivatives of the function $A^M$.

Reasoning like in the previous example, the first variation is given by

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = &- \int_M |A^M_{ijk}|^2 \langle \mathbf{H} | X \rangle \, d\mathcal{H}^n\\ &- 2 \int_M A^M_{ijk} \, A^M_{ijs} \nabla^M_k X^s \, d\mathcal{H}^n\\ &- 2 \int_M A^M_{ijk} \left( A^M_{is} - \delta_{is} \right) \nabla^M_j \nabla^M_k X^s \, d\mathcal{H}^n\end{aligned}$$

and permuting cyclically the indexes $i$, $j$ and $k$ in the last two integrals. Hence this gives,

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = &- 3 \int_M B^2 \langle \mathbf{H} | X \rangle \, d\mathcal{H}^n\\ &+ 6 \int_M \nabla^M_k \left( \nabla^M_k A^M_{ij} \, A^M_{ijs} \right) X^s \, d\mathcal{H}^n\\ &+ 3 \int_M A^M_{ijk} \left( \delta_{is} - A^M_{is} \right) \left( \nabla^M_j \circ \nabla^M_k + \nabla^M_k \circ \nabla^M_j \right) X^s \, d\mathcal{H}^n.\end{aligned}$$

Now we use the fact that $A^M_{is} = \delta_{is} - \nu_i\nu_s$, moreover we set $\mathbf{H} = H\nu$ and $X = \varphi\nu$. Substituting these quantities in the formula above and simplifying the terms equal to zero by orthogonality, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = &- 3\int_M \varphi H B^2 \, d\mathcal{H}^n \\ &+ 6\int_M \varphi \Delta^M \left(\nu_i\nu_j\right) \nabla^M_i \nu_j \, d\mathcal{H}^n \\ &- 6\int_M \varphi \nabla^M_k \left(\nu_i\nu_j\right) \nabla^M_k A^M_{ijs}\nu_s \, d\mathcal{H}^n \\ &- 3\int_M \nabla^M_j \nu_k \left(\nabla^M_j\nabla^M_k \varphi + \nabla^M_k\nabla^M_j \varphi\right) \, d\mathcal{H}^n \\ &- 3\int_M \varphi\nabla^M_j \nu_k\nu_s \left(\nabla^M_j\nabla^M_k \nu_s + \nabla^M_k\nabla^M_j \nu_s\right) \, d\mathcal{H}^n.\end{aligned}$$

Indeed, using the properties stated in Proposition 4.1, we can compute

$$\begin{aligned}\varphi\nabla^M_k \left(\nabla^M_k A^M_{ij} A^M_{ijs}\right)\nu_s &= -\varphi\nabla^M_k \left(\nabla^M_k \nu_i\nu_j\nabla^M_i A^M_{js} + \nabla^M_k \nu_j\nu_i\nabla^M_j A^M_{is}\right)\nu_s \\ &= \varphi\nabla^M_k \left\{\left(\nabla^M_k \nu_i\right)\left(\nabla^M_i \nu_s\right) + \left(\nabla^M_k \nu_j\right)\left(\nabla^M_j \nu_s\right)\right\}\nu_s \\ &= -2\left(\nabla^M_k \nu_i\right)\left(\nabla^M_i \nu_s\right)\left(\nabla^M_k \nu_s\right).\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = &- 3\int_M \varphi H B^2 \, d\mathcal{H}^n - 12\int_M \varphi\left(\nabla^M_k \nu_i\right)\left(\nabla^M_k \nu_j\right)\left(\nabla^M_i \nu_j\right) \, d\mathcal{H}^n \\ &- 3\int_M \nabla^M_j \nu_k \left(\nabla^M_j\nabla^M_k \varphi + \nabla^M_k\nabla^M_j \varphi\right) \, d\mathcal{H}^n \\ &+ 6\int_M \varphi\left(\nabla^M_j \nu_k\right)\left(\nabla^M_s \nu_j\right)\left(\nabla^M_k \nu_s\right) \, d\mathcal{H}^n \\ = &- 3\int_M \varphi H B^2 \, d\mathcal{H}^n \\ &- 3\int_M \nabla^M_j \nu_k \left(\nabla^M_j\nabla^M_k \varphi + \nabla^M_k\nabla^M_j \varphi\right) \, d\mathcal{H}^n \\ &- 6\int_M \varphi\left(\nabla^M_k \nu_i\right)\left(\nabla^M_j \nu_k\right)\left(\nabla^M_i \nu_j\right) \, d\mathcal{H}^n.\end{aligned}$$

We introduce now the following elementary symmetric functions of the eigenvalues $\lambda_i$ of $\mathbf{B}^\nu = \langle \mathbf{B} | \nu\rangle$,

$$\mathcal{S}_t = \sum_{i_1 < i_2 \, \dots \, < i_t} \lambda_{i_1}\lambda_{i_2} \, \dots \, \lambda_{i_t}, \qquad \text{for } t \leq n$$

and we define $\mathcal{S}_t = 0$ for $t > n$. The last term in the equation above can be written as

$$\left(\nabla^M_k \nu_i\right)\left(\nabla^M_k \nu_j\right)\left(\nabla^M_i \nu_j\right) = -\,\text{Trace}\,\left[\mathbf{B}^\nu\right]^3 = -\sum_i \lambda_i^3.$$

Using the formula

$$\mathcal{S}_1^3 = -2(\lambda_1^3 + \lambda_2^3 + \, \dots \, + \lambda_n^3) + 3\mathcal{S}_1[\mathcal{S}_1^2 - 2\mathcal{S}_2] + 6\mathcal{S}_3$$

and recalling that $H = \mathcal{S}_1$ and $\mathcal{S}_1^2 - 2\mathcal{S}_2 = B^2$, we have

$$2(\lambda_1^3 + \lambda_2^3 + \ \dots \ + \lambda_n^3) = H^3 - 3HB^2 + 6\mathcal{S}_3.$$

Substituting this term in the equation above, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_3(M_t)\Big|_{t=0} = &-3\int_M \varphi HB^2\, d\mathcal{H}^n + 3\int_M \mathbf{B}^\nu_{jk}\left(\nabla^M_j\nabla^M_k\varphi + \nabla^M_k\nabla^M_j\varphi\right)\, d\mathcal{H}^n\\ &+ 18\int_M \varphi\mathcal{S}_3\, d\mathcal{H}^n - 3\int_M \varphi H^3\, d\mathcal{H}^n + 9\int_M \varphi HB^2\, d\mathcal{H}^n,\end{aligned}$$

and finally

$$\begin{aligned}\frac{d}{dt}\int_M B^2\, d\mathcal{H}^n\Bigg|_{t=0} = &\, 2\int_M \varphi HB^2\, d\mathcal{H}^n + 6\int_M \varphi\mathcal{S}_3\, d\mathcal{H}^n - \int_M \varphi H^3\, d\mathcal{H}^n\\ &+ \int_M \mathbf{B}^\nu_{jk}\left(\nabla^M_j\nabla^M_k\varphi + \nabla^M_k\nabla^M_j\varphi\right)\, d\mathcal{H}^n.\end{aligned}$$

Now, to conclude it is sufficient to show that the last term of the formula above is equal to

$$2\int_M \varphi\,\Delta^M H\, d\mathcal{H}^n.$$

This can be done with the help of Codazzi–Mainardi's equations, in particular using the relation (5.2)

$$\begin{aligned}\int_M \mathbf{B}^\nu_{jk}\,\nabla^M_j\nabla^M_k\varphi\, d\mathcal{H}^n &= -\int_M \nabla^M_j\mathbf{B}^\nu_{jk}\nabla^M_k\varphi\, d\mathcal{H}^n\\ &= -\int_M \left(\nabla^M_k H + \nu_k B^2\right)\nabla^M_k\varphi\, d\mathcal{H}^n = -\int_M \nabla^M_k H\,\nabla^M_k\varphi\, d\mathcal{H}^n\\ &= \int_M \varphi\,\nabla^M_k\nabla^M_k H\, d\mathcal{H}^n = \int_M \varphi\,\Delta^M H\, d\mathcal{H}^n.\end{aligned}$$

Hence the Euler equation of the functional $\mathcal{G}_3$ (three times the integral of $B^2$) is given by

$$E_{\mathcal{G}_3} = 3\,\left[2\Delta^M H + 2HB^2 - H^3 + 6\mathcal{S}_3\right]\nu.$$

*Remark* 5.3. As we noticed that $B^2 = H^2 - 2\mathcal{S}_2$, we have also that the Euler equation of the functional

$$\mathcal{L}_2(M) = \int_M \mathcal{S}_2\, d\mathcal{H}^n$$

in codimension 1, is given by

$$E_{\mathcal{L}_2} = -3\mathcal{S}_3\nu.$$

For a complete discussion of Euler equations of functionals depending on the elementary symmetric functions of the eigenvalues of the second fundamental form, see [31].

**Theorem 5.4.** *For any $\gamma > 2$ the Euler equation of the functional $\mathcal{G}_\gamma$ is given by*

$$E_{\mathcal{G}_\gamma} = 2\gamma(-1)^{\gamma-1} \sum_{j,i_2,k_2 \dots i_\gamma,k_\gamma} \left( A^M_{i_2k_2} \dots A^M_{i_\gamma k_\gamma} A^M_{j\, i_2k_2 \dots i_\gamma k_\gamma} \right) e_j^\perp + g(A^M)$$

$$= 2\gamma(-1)^{\gamma-1} \sum_{j=1}^{N} \left( \overbrace{\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M}^{(\gamma-2)\text{–times}} \mathbf{H}_j \right) e_j^\perp + h(A^M) \tag{5.7}$$

*where the vector fields $g(A^M)$, $h(A^M)$ are polynomials in the derivatives of $A^M$ up to the order $(2\gamma - 2)$.*

*Proof.* We follow the line of proof of Proposition 4.7 and we notice that, by Proposition 4.6, we can assume that the infinitesimal generator $X$ is a normal vector field. We have seen (see (4.8)) that the term with the highest number of derivatives arises from the integral

$$-2 \int_M A^M_{i_1 \dots i_\gamma} D^{i_1 \dots i_\gamma} \langle \nabla A^M(x) - x \,|\, X(\nabla A^M(x)) \rangle \, d\mathcal{H}^n(x)$$

when all but one of the derivatives $D^{i_j}$ act on the field $X$. We suppose that the only derivative going on the left is $D^{i_1}$. Hence, we have to study

$$-2 \int_M A^M_{i_1 \dots i_\gamma} \left( A^M_{i_1 k}(x) - \delta_{i_1 k} \right) D^{i_2 \dots i_\gamma} \left[ X^k(\nabla A^M(x)) \right] d\mathcal{H}^n(x).$$

After doing the first derivative on $X(\nabla A^M(x))$ we get $A^M_{i_\gamma j}(x)(\nabla^M_j X^k)(\nabla A^M(x))$; it is clear that if we are only interested in the term containing the highest derivative, we can avoid to distribute derivatives on $A^M_{i_\gamma j}(x)$ and then consider only the term containing the derivatives of the field. Iterating this argument we get

$$-2 \int_M A^M_{i_1 \dots i_\gamma} \left( A^M_{i_1 k}(x) - \delta_{i_1 k} \right) A^M_{i_\gamma j_\gamma} \dots A^M_{i_2 j_2} \nabla^M_{j_2} \circ \dots \circ \nabla^M_{j_\gamma} X^k(x) \, d\mathcal{H}^n(x).$$

Now we have to apply the tangential divergence theorem 4.2, noticing again that if we are interested only in the highest derivative term, so we can limit ourselves to differentiate the term $A^M_{i_1 \dots i_\gamma}$. Moreover, since we apply the theorem with tangential fields, no term containing $\mathbf{H}$ appears. After doing this we obtain

$$(-1)^\gamma 2 \int_M \left[ \nabla^M_{j_\gamma} \circ \dots \circ \nabla^M_{j_2} A^M_{i_1 \dots i_\gamma} \right] \left( A^M_{i_1 k} - \delta_{i_1 k} \right) A^M_{i_\gamma j_\gamma} \dots A^M_{i_2 j_2} X^k(x) \, d\mathcal{H}^n(x).$$

Using the orthogonality of $X$ we get

$$-2(-1)^\gamma \int_M \left[ \nabla^M_{j_\gamma} \circ \dots \circ \nabla^M_{j_2} \left( \nabla_{i_1} A^M \right)_{i_2 \dots i_\gamma} \right] A^M_{i_\gamma j_\gamma} \dots A^M_{i_2 j_2} X^{i_1}(x) \, d\mathcal{H}^n(x).$$

Hence, performing the tangential derivatives and adding on all indexes we get the first equality in (5.7).

To get the second equality, we apply in the inverse direction the derivative of a product formula to *carry inside* the components of the projection $A^M_{i_t k_t}$, in order to obtain the tangential Laplacians. Notice that, with a reasoning similar to the one above, in doing this we only introduce terms with an order of differentiation at most $(2\gamma - 2)$. In this way we obtain

$$E_{\mathcal{G}_\gamma} = 2\gamma(-1)^{\gamma-1} \sum_{j=1}^{N} \left( \overbrace{\Delta^M \circ \Delta^M \circ \dots \circ \Delta^M}^{(\gamma-1)\text{–times}} \left( \nabla_j A^M \right) \right) e_j^\perp + h(A^M).$$

The last step in proving this representation formula for $\mathcal{G}_\gamma$ is to show that $\mathbf{H}(x) = \Delta^M\left(\nabla A^M\right)(x)$, at every $x \in M$. We have

$$\Delta^M\left(A^M_i\right) = \nabla^M_k\left(A^M_{ki}\right) = \mathbf{B}^i_{kk} + \mathbf{B}^k_{ki} = \mathbf{H}_i + \mathbf{B}^k_{ki}$$

where we used the relation (4.3). To conclude it is hence sufficient to notice that $\mathbf{B}^k_{ki} = 0$ by orthogonality. □

*Remark* 5.5. We remark that in the two expressions above for the leading term, we cannot substitute $e_i^\perp$ with $e_i$, because of the fact that neither the first nor the second are in general normal vectors. This can be seen considering a torus $T$ in $\mathbb{R}^3$ with the biggest radius equal to 2 and the smallest one equal to 1, for instance that one defined by

$$T \equiv \left\{(2-\cos\nu)\cos\theta, (2-\cos\nu)\sin\theta, \sin\nu) \in \mathbb{R}^3 \mid (\nu,\theta) \in \mathbb{R}^2\right\}$$

and computing these two vectors in the first meaningful case $\gamma = 3$ at the point $(2,0,1)$ on the top of $T$.

The function $\eta^T$ for the torus is given by

$$\eta^T(x,y,z) = \frac{1}{2}\left[\sqrt{\left(\sqrt{x^2+y^2}-2\right)^2 + z^2} - 1\right]^2.$$

Setting

$$A(x,y,z) = A^T(x,y,z) = \frac{\|(x,y,z)\|^2}{2} - \eta^T(x,y,z)$$

and using the *Mathematica*[1] package we computed

$$\sum_{i_2,k_2,i_3,k_3} A_{i_2k_2}A_{i_3k_3}A_{j\,i_2k_2i_3k_3} = A_{j1111} + 2A_{j2211} + A_{j2222}$$

with $j = 1$ at $(2,0,1)$ and we found the value $-3$, hence there is a tangential component in the leading term of the first representation in (5.7).

For the second term we show the computation explicitly. We have that $\Delta^M\mathbf{H}_i = \Delta^M(H\nu_i)$, hence

$$\begin{aligned}\Delta^M\mathbf{H}_i &= \nu_i\Delta^M H + 2\langle\nabla^M\nu_i \,|\, \nabla^M H\rangle + H\Delta\nu_i\\ &= \nu_i\Delta^M H + 2\nabla^M_k\nu_i\nabla^M_k H + H\nabla^M_k\nabla^M_k\nu_i\\ &= \nu_i\Delta^M H - 2\mathbf{B}^\nu_{ik}\nabla^M_k H - H\nabla^M_k\mathbf{B}^\nu_{ik}.\end{aligned}$$

Now we apply the relation (5.2) to the last term in the equation above to get

$$\begin{aligned}\Delta^M\mathbf{H}_i &= \nu_i\Delta^M H - 2\mathbf{B}^\nu_{ik}\nabla^M_k H - H\nabla^M_i H - \nu_i H B^2\\ &= \nu_i\left(\Delta^M H - HB^2\right) - \left(H\delta_{ik} + 2\mathbf{B}^\nu_{ik}\right)\nabla^M_k H.\end{aligned}$$

Since at the point $(2,0,1)$ of the torus $T$ we have $\mathbf{B}^\nu_{11} = 1$ and $\mathbf{B}^\nu_{2j} = 0$, hence $H = 1$, the vector $e_i\Delta^M\mathbf{H}_i$ has a tangential part given by

$$-3(\nabla^M_1 H)e_1 - H(\nabla^M_2 H)e_2. \tag{5.8}$$

The quantity $H = \langle\mathbf{H} \,|\, \nu\rangle$ in a neighborhood of the point $(2,0,1)$ is

$$H(x,y,z) = 2 - \frac{2}{\sqrt{x^2+y^2}}$$

then

$$\nabla H(x,y,z) = 2\frac{xe_1 + ye_2}{\left(x^2+y^2\right)^{3/2}}.$$

[1]*Mathematica* is a registered trademark of Wolfram Research.

At the point $(2,0,1)$ we have

$$\nabla H(2,0,1) \;=\; \frac{1}{2}e_1 \;=\; \nabla^M H(2,0,1)$$

because the gradient is a tangent vector.
This, with the (5.8) shows that $e_i\Delta\mathbf{H}_i$ can have a non zero tangential component.

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