An elliptic equation with history

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Abstract

We prove the existence and uniqueness for a semilinear elliptic problem with memory, both in the weak and the classical setting. This problem describes the effective behaviour of a biological tissue under the injection of an electrical current in the radiofrequency range.

1 Introduction

Let Ω be an open bounded subset of $\boldsymbol{R}^N$ with regular boundary and let $T > 0$. We study the existence, uniqueness and regularity for the solution of the semilinear problem

$$\begin{cases} -\mathrm{div}\Big(A(x)\nabla_x u + \int_0^t B(x,t-\tau)\nabla_x u(x,\tau)\; d\tau\Big) = g(x,t,u) & \text{in } \Omega\times(0,T)\,, \\ u = f & \text{in } \partial\Omega\times(0,T)\,, \end{cases} \tag{1.1}$$

where $A(x)$ is a simmetric and positive definite matrix, $B(x,t)$ is a simmetric matrix, $g : \Omega\times(0,T)\times\boldsymbol{R}\to\boldsymbol{R}$ and $f : \overline{\Omega}\times(0,T)\to\boldsymbol{R}$ are given functions. More precisely, in Section 2 we prove the well-posedness of problem (1.1) in a weak sense, by using a fixed point technique:

Theorem 1.1 *Let $A \in L^\infty(\Omega;\boldsymbol{R}^{N^2})$ be such that $\lambda|\xi|^2 \le A(x)\xi\cdot\xi \le \Lambda|\xi|^2$, for suitable $0<\lambda<\Lambda<+\infty$, for almost every $x\in\Omega$ and every $\xi\in\boldsymbol{R}^N$; let $B\in L^2(0,T;L^\infty(\Omega;\boldsymbol{R}^{N^2}))$, and let $f\in L^2(0,T;H^1(\Omega))$. Assume that $g:\Omega\times(0,T)\times\boldsymbol{R}\to\boldsymbol{R}$ is a Carathéodory function such that*

(G1) $\quad g(\cdot,\cdot,0)\in L^2(0,T;H^{-1}(\Omega))$
(G2) $\quad |g(x,t,s)-g(x,t,s')|\le L|s-s'|$ *for a.e.* $(x,t)\in\Omega\times(0,T)$, *and every* $s,s'\in\boldsymbol{R}$,

where $L\le\frac{\lambda}{3C}$, if C is the best constant in the classical Poincaré inequality on Ω.

Then, there exists a unique function $u\in L^2(0,T;H^1(\Omega))$ satisfying in the sense of distributions problem (1.1).

In Section 3 we prove that, under further regularity assumptions on the data, existence and uniqueness of classical solutions of (1.1) hold true, by using a delay technique. This regularity is instrumental in applications (see [2]).

Theorem 1.2 *Let $m \geq 0$ be any fixed integer and let also $0 < \alpha < 1$. Let $A \in C^{1+\alpha}(\overline{\Omega}; \boldsymbol{R}^{N^2})$ satisfy the assumption of Theorem 1.1 and $B \in C^0([0,T]; C^{1+\alpha}(\overline{\Omega}; \boldsymbol{R}^{N^2}))$ be such that $B' \in L^2(0,T; W^{1,\infty}(\Omega; \boldsymbol{R}^{N^2}))$. Assume that $g \in C^0([0,T]; C^{m+\alpha}(\overline{\Omega} \times \boldsymbol{R}))$ satisfies (G2) of Theorem 1.1, with $\gamma L < 1$, where γ is a structural constant depending only on $\lambda, \Lambda, N, \Omega, \beta, A, B$, and that there exists $L_0 > 0$ such that*

$$\text{(G3)} \qquad |g(x,t,s)|, |\nabla_x g(x,t,s)|, |g_t(x,t,s)| \leq L|s| + L_0 \,.$$

Let $f \in C^0([0,T]; C^{m+2+\alpha}(\overline{\Omega}))$, with $f_t \in L^\infty(0,T; C^{m+2+\alpha}(\overline{\Omega}))$. Then there exists a unique function $u \in C^0([0,T]; C^{1+\alpha}(\overline{\Omega})) \cap L^\infty(0,T; C^{m+2+\alpha}(\overline{\Omega}))$ solving (1.1) in the classical sense.

In the linear case, our problem can be compared to the ones studied in the context of linear elasticity in [4, 5], where (1.1) is reduced to a Volterra equation and solved, under suitable hypotheses, by means of the spectral theory in $C^0([0,T]; C^{2+\alpha}(\overline{\Omega}))$. Problem (1.1), again in the linear case, is also studied in [3], in the context of weak solvability. There, the Fourier trasform technique is applied, under some assumptions on the asymptotic behaviour of the kernel B, in order to obtain the existence in the space $L^2(-\infty, +\infty; H^2(\Omega))$.

From the physical point of view, problem (1.1) describes the effective behaviour of a biological tissue under the injection of an electrical current in the radiofrequency range ([1, 2]). Here, the unknown u represents the electrical potential and the driven electrical current $-A(x)\nabla_x u - \int_0^t B(x,t-\tau)\nabla_x u(x,\tau)\, d\tau$ depends on the history of the electrical field $-\nabla_x u$, therefore it is non local in time.

2 Proof of Theorem 1.1

We note that, possibly replacing u with $v = u - f$ and g with

$$\tilde{g}(x,t,u) = g(x,t,u) - \operatorname{div}\Big(A(x)\nabla_x f + \int_0^t B(x,t-\tau)\nabla_x f(x,\tau)\,\mathrm{d}\tau\Big)\,,$$

there is no loss of generality in assuming $f \equiv 0$. Consider the Banach space $X = L^2(0,T_1; H^1_o(\Omega))$, endowed with the usual norm

$$\|u\|_{L^2(0,T_1;H^1_o(\Omega))} := \left(\int_0^{T_1}\int_\Omega |\nabla_x u|^2\, dx\, dt\right)^{1/2}\,,$$

where T_1 will be chosen later. Let us introduce an operator H acting on X by means of $H(u) = w$, where w is the solution of

$$-\operatorname{div}\Big(A(x)\nabla_x w\Big) = \operatorname{div}\Big(\int_0^t B(x,t-\tau)\nabla_x u(x,\tau)\,\mathrm{d}\tau\Big) + g(x,t,u)\,,$$

with null trace on $\partial\Omega$, and t fixed almost everywhere in $(0,T)$. Clearly, the operator H is well defined; moreover, multiplying the previous equation by w and integrating by parts, we obtain that $H(X) \subset X$.

Given $u_1, u_2 \in X$, we have that $w = H(u_1) - H(u_2)$ has null trace on the boundary $\partial\Omega$ and solves

$$-\operatorname{div}\Big(A(x)\nabla_x w\Big) = \operatorname{div}\Big(\int_0^t B(x,t-\tau)\nabla_x u(x,\tau)\,\mathrm{d}\tau\Big) + g(x,t,u_1) - g(x,t,u_2)\,,$$

where $u = u_1 - u_2$. Again multiplying the previous equation by w and integrating by parts, it follows

$$\begin{aligned} \|w\|^2_{L^2(0,T_1;H^1_o(\Omega))} &= \|H(u_1) - H(u_2)\|^2_{L^2(0,T_1;H^1_o(\Omega))} \\ &\leq \tfrac{\gamma}{\delta}\, T_1 \|u\|^2_{L^2(0,T_1;H^1_o(\Omega))} + \gamma\, \delta \|w\|^2_{L^2(0,T_1;H^1_o(\Omega))} \\ &\quad + \tfrac{CL}{2\lambda}\|u\|^2_{L^2(0,T_1;H^1_o(\Omega))} + \tfrac{CL}{2\lambda}\|w\|^2_{L^2(0,T_1;H^1_o(\Omega))}\,, \end{aligned}$$

where γ depends only on λ and B. Now, recalling that $L \le \frac{\lambda}{3C}$ and choosing $\delta = \frac{1}{6\gamma}$ and $T_1 < \frac{1}{36\gamma^2}$, we can absorb the second and the fourth term of the last inequality into the left hand side, obtaining that H is a contraction. So, it admits a unique fixed point, i.e., a solution of (1.1) exists in X. Noting that the width T_1 of the time interval is independent of the iteration step, we may conclude the proof by iterating this argument over $(0, T)$.

3 Proof of Theorem 1.2

We will prove that the unique weak solution found in Theorem 1.1 actually belongs to $C^0([0,T]; C^{1+\alpha}(\overline{\Omega})) \cap L^\infty(0,T; C^{m+2+\alpha}(\overline{\Omega}))$.

Assume $m = 0$ and set

$$\|u\|^t_{(2+\alpha)} := \operatorname*{ess\,sup}_{0\le\tau\le t} \|u(\cdot,\tau)\|_{(2+\alpha)}\,, \tag{3.1}$$

where $\|\cdot\|_{(m+\alpha)}$ is the norm in $C^{m+\alpha}(\overline{\Omega})$. Analogously, if we indicate with $\|\cdot\|_{2,q}$ and $\|\cdot\|_\infty$ the norms in $W^{2,q}(\Omega)$ and $L^\infty(\Omega)$, respectively, we can consider the corresponding norms $\|\cdot\|^t_{2,q}$ and $\|\cdot\|^t_\infty$ in the sense of (3.1).

Let us introduce a sequence of approximating problems

$$\begin{aligned} &-\operatorname{div}\Big(A(x)\nabla_x u_h\Big) = \operatorname{div}\Big(\textstyle\int_0^{t_h} B(x,t-\tau)\nabla_x u_h(x,\tau)\,\mathrm{d}\tau\Big) + g(x,t,u_h(x,t_h)) &&\text{in } \Omega\times(0,T)\\ &u_h(x,t) = f(x,t) &&\text{on } \partial\Omega\times(0,T)\end{aligned} \tag{3.2}$$

where $t_h = \max(0, t-h)$, for $0 < t < T$ and any fixed $h > 0$, and $u_h(x,0) =: u_0(x)$ is given by the unique solution of the standard elliptic nonlinear equation

$$-\operatorname{div}\Big(A(x)\nabla u_0(x)\Big) = g(x,0,u_0(x))$$

which coincides with $f(\cdot,0)$ on the boundary $\partial\Omega$.

Existence of a solution $u_h \in C^0([0,T]\times\overline{\Omega}\times\boldsymbol{R}) \cap L^\infty(0,T; C^{2+\alpha}(\overline{\Omega}))$ is elementary, moreover, by standard elliptic estimates, using also (G3), we have that, for every $t \in [0,T]$,

$$\begin{aligned}\|u_h\|^t_{2+\alpha} &\le \gamma\left(\int_0^{t_h}\|u_h\|^\tau_{2+\alpha}\,d\tau + L\|u_h\|^t_{2+\alpha} + \|\nabla_x g(u_h)\|^t_\infty + \|g(u_h)\|^t_\infty + \|f\|^t_{2+\alpha}\right)\\ &\le \gamma\left(\int_0^t \|u_h\|^\tau_{2+\alpha}\,d\tau + 3L\|u_h\|^t_{2+\alpha} + 2L_0 + \|f\|^T_{2+\alpha}\right)\,,\end{aligned} \tag{3.3}$$

where γ is a structural constant depending on $\lambda, \Lambda, N, \Omega, \beta, A, B$. Taking into account that L is small and using Gronwall's Lemma, it follows that $\|u_h\|^T_{2+\alpha} \le \gamma$, where γ now depends also on $L, L_0, \|f\|^T_{2+\alpha}$. Now let us consider the following problem

$$\begin{aligned}-\operatorname{div}\Big(A(x)\nabla_x u'_h\Big) &= \operatorname{div}\Big(B(x,h)\frac{\partial t_h}{\partial t}\nabla_x u_h(x,t) + \int_0^{t_h} B'(x,t-\tau)\nabla_x u_h(x,\tau)\,\mathrm{d}\tau\Big)\\ &\quad + g_s(x,t,u_h(x,t_h))u'_h(x,t_h)\frac{\partial t_h}{\partial t} + g_t(x,t,u_h(x,t_h))\,,\end{aligned} \tag{3.4}$$

with $u'_h = f_t$ on the boundary $\partial\Omega$ and u' stands for the temporal derivative of u.

This is a standard linear elliptic problem in u'_h (where u_h and $u'_h(x,t_h)$ are regarded as known functions), with a non zero source term. Hence (see e.g. [6], Chp. 9) for every $q \ge 2$, we have existence and uniqueness of a solution $u'_h \in L^\infty(0,T; W^{2,q}(\Omega))$; moreover,

$$\|u'_h\|^t_{2,q} \le \gamma\left(\|u_h\|^t_{2,q} + L\|u'_h\|^{t_h}_{L^q} + \|g_t(u_h)\|^t_\infty + \|f_t\|^t_{2,q}\right)\,.$$

Therefore, since L is sufficiently small, using again (G3), we obtain

$$\|u_h'\|_{2,q}^T \le \gamma(\|u_h\|_{2,q}^T + L_0 + \|f_t\|_{2,q}^T) \le \gamma \, . \tag{3.5}$$

By (3.5) and taking into account that (3.3) implies $\|u_h\|_{2+\alpha}^T \le \gamma$, it follows that

$$\begin{array}{ll} u_h, (u_h)_{x_i}, (u_h)_{x_i x_j} & \in L^\infty(\Omega_T) \\ (u_h)_t, (u_h)_{t x_i}, (u_h)_{t x_i x_j} & \in L^q(\Omega_T) \end{array}$$

uniformly with respect to h, where $\Omega_T = \Omega \times (0,T)$. This implies that, if we choose $q > N$ sufficiently large, the sequences $\{u_h\}, \{(u_h)_{x_i}\}$ are compactly embedded in $C^0(\overline{\Omega}_T)$, being uniformly Hölder continuous with exponent α. Now, up to a subsequence, we can pass to the limit for $h \to 0^+$ in the weak formulation of (3.2), obtaining that $u_h \to u \in C^0([0,T]; C^{1+\alpha}(\overline{\Omega}))$, where u is a solution of (1.1). Moreover, $u \in L^\infty(0,T; C^{2+\alpha}(\overline{\Omega}))$, as follows by applying the calculations in (3.3) to the original problem solved by u.

Now let $m \in \boldsymbol{N}$. By classical elliptic estimates, it follows

$$\|u(\cdot,t)\|_{(m+2+\alpha)} \le \gamma_1 \|g(u)\|_{(m+\alpha)}^t + \gamma_1 \|f\|_{(m+2+\alpha)}^t + \gamma_2 \int_0^t \|u(\cdot,\tau)\|_{(m+2+\alpha)} \, \mathrm{d}\tau$$

where $g(u) = g(x,t,u(x,t))$. After an application of Gronwall's lemma, we obtain

$$\|u\|_{(m+2+\alpha)}^T \le \gamma_1 \|g(u)\|_{(m+\alpha)}^T + \gamma_1 \|f\|_{(m+2+\alpha)}^T \, . \tag{3.6}$$

Hence, $u \in L^\infty(0,T; C^{m+2+\alpha}(\overline{\Omega}))$, whenever $\|g(u)\|_{(m+\alpha)}^T$ is bounded. If $m = 1$, by the first part of the proof we have that $\|g(u)\|_{(1+\alpha)}^T$ is bounded, which implies that u actually belongs to $L^\infty(0,T; C^{3+\alpha}(\overline{\Omega}))$. The proof is then concluded by induction over m in (3.6).

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