UNIQUENESS OF SIGNED MEASURES SOLVING THE CONTINUITY EQUATION FOR OSGOOD VECTOR FIELDS

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ABSTRACT. Nonnegative measure-valued solutions of the continuity equation are uniquely determined by their initial condition, if the characteristic ODE associated to the velocity field has a unique solution. In this paper we give a partial extension of this result to signed measure-valued solutions, under a quantitative two-sided Osgood condition on the velocity field. Our results extend those obtained for log-Lipschitz vector fields in [6].

1. INTRODUCTION

Let $T > 0$ and let

$$V(t,x) : (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

be a Borel vectorfield. We associate to V the equations

$$\dot{\gamma}(t) = V(t, \gamma(t)) \tag{ODE}$$

and (with the notation $V_t(x) = V(t,x)$)

$$\partial_t \mu_t + \mathrm{d}iv(V_t \mu_t) = 0. \tag{PDE}$$

A solution of (ODE) is an absolutely continuous curve $\gamma(t)$ such that $\dot{\gamma}(t) = V(t, \gamma(t))$ almost everywhere on $[0,T]$. We shall also consider generalized solutions in the sense of Filippov, see more details below. The so-called *continuity* or *Liouville* equation (PDE) is considered in the sense of distributions. We shall work with solutions in the class of measures. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of signed Borel measures with finite total variation on $\mathbb{R}^d$, by $\mathcal{M}^+(\mathbb{R}^d)$ the subset of non-negative finite measures, and by $|\mu| \in \mathcal{M}^+(\mathbb{R}^d)$ the total variation of a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$. We shall consider only solutions μ_t satisfying $|\mu_t|(\mathbb{R}^d) \in L^\infty(0,T)$; this is not a very restrictive assumption, because many approximation schemes do provide solutions μ_t with this property. We shall also assume that

$$\int_0^T \int_{\mathbb{R}^d} \|V_t\| d|\mu_t| dt < \infty, \tag{I}$$

a property surely satisfied if $\|V\|$ is uniformly bounded. Under these assumptions the notion of distributional solution is well defined, and it is equivalent to the requirement that, for all $\phi \in C_c^1(\mathbb{R}^d)$, $t \mapsto \int \phi \, d\mu_t$ belongs to the Sobolev space $W^{1,1}(0,T)$, with distributional derivative given by

$$\int_{\mathbb{R}^d} \langle V_t(x), \nabla\phi(x)\rangle \, d\mu_t(x).$$

Using this fact, and the uniform continuity properties of Sobolev functions on the real line, it is easy to check (see for instance [4, Proposition 8.1.7]) that we can restrict ourselves (possibly modifying μ_t in a negligible set of times) to weakly continuous solutions $t \mapsto \mu_t$, in the duality with $C_c(\mathbb{R}^d)$. Moreover, the initial condition μ_0 for (PDE) is defined in a weak sense:

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} \phi \, d\mu_t = \int_{\mathbb{R}^d} \phi \, d\mu_0 \qquad \forall \phi \in C_c(\mathbb{R}^d).$$

So, from now on only weakly continuous solutions μ_t will be considered. Reversing the time variable, also the final condition μ_T is well defined, still in the weak sense.

Our goal in the present paper is to study the relations between uniqueness for (ODE) and uniqueness for (PDE). It is known that uniqueness for (ODE) implies, via the so-called superposition principle, that nonnegative solutions of (PDE) are uniquely determined by the initial condition μ_0, see [4, 2, 3, 10, 5]. The question turns out to be much more subtle if we work in the class of signed measures. Of course, if μ_t is a solution, we can write it as the difference of the two non-negative measures μ_t^+ and μ_t^-. However, these measures need not solve the equation. This remark is reminiscent of the notion of renormalized solutions, see [7, 1]: we may call renormalized a solution μ_t such that μ_t^+ and μ_t^- are both solutions (or equivalently such that $|\mu_t|$ is a solution). It is clear that there is uniqueness if all distributional solutions are renormalized, and if there is uniqueness for (ODE), but renormalized solutions have been studied only under weak differentiability assumptions on V_t, and only in the class of absolutely continuous measures μ_t (see [2] for a survey on this topic). In this paper, we leave aside the question of the general relations between (ODE) uniqueness and (PDE) uniqueness, and we focus on a particular class of vector-fields for which (ODE) uniqueness is well-known, and derive some consequences at the (PDE) level (and, in particular, that all solutions are renormalized).

We recall that a modulus of continuity is a continuous non-decreasing function $\rho : [0,1) \longrightarrow [0,\infty)$, such that $\rho(0) = 0$. A modulus of continuity ρ is said to be Osgood if

$$\int_0^1 \frac{1}{\rho(s)} ds = +\infty.$$

We will always extend the moduli of continuity to $[1, \infty)$ by $\rho = \infty$. Typical examples of Osgood moduli of continuity are $\rho(s) = s$ and $\rho(s) = s(1 - \ln(s))$. Note that the moduli $\rho(s) = s^\alpha$, $\alpha \in (0, 1)$, are not Osgood.

It is known that uniqueness holds for (ODE) if there exist a Osgood modulus of continuity ρ and $C \in L^1(0, T)$ such that

$$\text{(O)} \qquad |\langle V(t, x) - V(t, y), x - y\rangle| \leqslant C(t)\|x - y\|\rho(\|x - y\|)$$

for all x, $y \in \mathbb{R}^d$, and all $t \in (0, T)$. Condition (O) does not seem to imply continuity of V_t: in the case when the modulus ρ is linear, (O) implies that the symmetric part of the distributional derivative is bounded, hence Korn's inequality gives that V_t is equivalent, up to Lebesgue negligible sets, to a continuous function. Since we consider measures μ_t that are possibly singular, even in the case when ρ is linear we can not apply this result to reduce ourselves to a continuous vector field; therefore we will not investigate the continuity question here (also because adding the continuity assumption would not lead to a great simplification of the uniqueness proof).

In order to prevent blow-up of solutions, the following bound is useful:

$$\text{(B)} \quad |V(t, x)| \leqslant D(t) \quad \forall x \in \mathbb{R}^d, \ \forall t \in (0, T), \text{ for some } D \in L^1(0, T).$$

The equation (ODE) is well understood under (O) and (B): There exists a unique flow map

$$X : [0, T] \times [0, T] \times \mathbb{R}^d \longmapsto \mathbb{R}^d$$

which is such that $X(s, t, \cdot)$ is a homeomorphism of $\mathbb{R}^d$ for each s and t; $X(t, t, \cdot) = Id$ for each t. In addition

$$t \mapsto X(s, t, x)$$

is a generalized solution of (ODE) in the sense of Filippov (the definition is recalled below) for each s and x. In the case when V_t is continuous, then generalized solutions in the sense of Filippov are just ordinary solutions of (ODE). Uniqueness implies that X satisfies the semigroup property

$$(1) \quad X(t_3, t_2, X(t_1, t_3, x)) = X(t_1, t_2, x) \quad \forall x \in \mathbb{R}^d, \ \forall t_1, \, t_2, \, t_3 \in [0, T].$$

The main result of this paper is the following uniqueness result:

Theorem 1. *If the vectorfield V satisfies (O) and (B), then there is uniqueness for (PDE) in the class of bounded signed measures. More precisely, if μ_t is a solution of (PDE) such that $|\mu_t|(\mathbb{R}^d) \in L^\infty(0,T)$ then*

$$\mu_t = X(0,t,\cdot)_\# \mu_0 \quad \textit{for all} \quad t \in (0,T). \tag{2}$$

In the particular case when V_t is continuous, (2) defines a solution of (PDE) with initial condition μ_0, so that Theorem 1 can also be read as an *existence* result.

The same proof would give uniqueness in the larger class of measures μ_t satisfying $|\mu_t|(\mathbb{R}^d) \in L^1(0,T)$ if conditions (O) and (B) are given in a stronger form with $C \in L^\infty(0,T)$, we leave the (easy) details to the reader.

If $\rho(s) = s$, the result is well-known. It has been proved by Bahouri and Chemin in [6] in the case $\rho(s) = s(1 - \ln(s))$ (see also [10] for related results), under the additional assumption that V has zero divergence. The proof in [6] uses Fourier analysis and Littlewood-Paley decompositions, and it is not clear to us whether it can be adapted to our more general statement.

It might be tempting to think that uniqueness for (PDE) holds in the presence of a flow of homeomorphisms solving (ODE), but we do not know whether such a result is true without an explicit bound like (O).

Let us now return to the definition of the flow X associated to V. Since V is possibly discontinuous, we consider its Filippov regularization (actually a multivalued function), namely

$$\mathbb{V}(t,x) := \bigcap_{r>0} \overline{\mathrm{co}}\left(\{V(t,y) : \|y-x\| < r\}\right),$$

where $\overline{\mathrm{co}}$ denotes closed convex hull. By definition, a generalized solution of (ODE) in the sense of Filippov is an absolutely continuous curve $X(t)$ such that the *inclusion* $\dot{X}(t) \in \mathbb{V}(t, X(t))$ holds for almost every t. Since $x \mapsto \mathbb{V}(t,x)$ is upper semicontinuous (i.e. $x_n \to x$, $v_n \in \mathbb{V}(t,x_n)$ and $v_n \to v$ imply $v \in \mathbb{V}(t,x)$), and $\mathbb{V}(t,x) \neq \emptyset$, by Filippov's theorem, see [8] or [9, Theorem 1.4.1], for all $t_1 \in [0,T]$ and $x \in \mathbb{R}^d$ there exists a Filippov solution $X(t) : [0,T] \longrightarrow \mathbb{R}^d$ satisfying $X(t_1) = x$. Furthermore, $\mathbb{V}$ inherits (O) in the form

$$|\langle v - w, x - y\rangle| \leqslant C(t)\|x-y\|\rho(\|x-y\|) \quad \forall v \in \mathbb{V}(t,x),\ \forall w \in \mathbb{V}(t,y) \tag{3}$$

and this can be used to show, by the standard argument, that X is unique. The flow $X(s,t,x)$ can now be defined by requiring that $t \mapsto X(s,t,x)$ is the only Filippov solution which satisfies $X(s) = x$.

The following strong form of uniqueness is essential for the proof :

Lemma 2. *Let $V(t,x)$ be a vector-field satisfying (O) and (B). Let $\gamma(s) = (t(s), x(s)) : [0,L] \longrightarrow [0,T] \times \mathbb{R}^d$ be a Lipschitz curve such that*

$$\dot{x}(s) = \dot{t}(s) V(t(s), x(s))$$

for almost every s. If $\int_0^L |\dot{t}(s)| C(t(s)) ds < \infty$, where $C(t)$ is the function appearing in (O), then

$$x(s) = X(t(0), t(s), x(0)).$$

Proof. Let us notice that the curve $y(s) := X(t(0), t(s), x(0))$ satisfies $\dot{y}(s) \in \dot{t}(s)\mathbb{V}(t(s), y(s))$ for almost all s, and therefore (3) gives

$$|\langle \dot{y}(s) - \dot{t}(s)V(t(s), x), y(s) - x\rangle| \leqslant |\dot{t}(s)| C(t(s)) \|y(s) - x\| \rho(\|y(s) - x\|)$$

for all $x \in \mathbb{R}^d$. In particular, we have

$$|\langle \dot{y}(s) - \dot{x}(s), y(s) - x(s)\rangle| \leqslant |\dot{t}(s)| C(t(s)) \|y(s) - x(s)\| \rho(\|y(s) - x(s)\|).$$

Denoting by d the quantity $d(s) = \|x(s) - y(s)\|$, we get (taking into account that $\dot{d}(s) = 0$ a.e. on $\{d = 0\}$)

$$|\dot{d}(s)| \leqslant |\dot{t}(s)| C(t(s)) \rho(d(s)) \qquad \text{for a.e. } s \in [0, L].$$

Since the function $|\dot{t}(s)| C(t(s))$ is integrable, and since $d(0) = 0$, we conclude that $d(s) = 0$ for all s. □

The proof of the Theorem is now based on Smirnov's decomposition of normal currents, see [11]. We expose this theory in Section 2, and then conclude the proof of Theorem 1 in Section 3.

2. Decomposition of vector fields

Let us consider the metric space $\mathcal{L} := \mathrm{Lip}([0,1]; \mathbb{R}^k)$ of Lipschitz curves $\gamma : [0,1] \longrightarrow \mathbb{R}^k$, endowed with the uniform distance and the associated Borel σ-algebra. Note that the set $\mathcal{L}$ is a Borel subset of $C([0,1]; \mathbb{R}^k)$, being a countable union of compact sets. To each curve $\gamma \in \mathcal{L}$, we associate its length $L_\gamma = \int_0^1 \|\dot{\gamma}(s)\| ds$ and the $\mathbb{R}^k$-valued measure $T^\gamma = (T_1^\gamma, \ldots, T_k^\gamma)$ on $\mathbb{R}^k$ defined by

$$\int g dT_i^\gamma = \int_0^1 g(\gamma(s)) \dot{\gamma}_i(s) ds \qquad i = 1, \ldots, k$$

for each bounded Borel function $g : \mathbb{R}^k \to \mathbb{R}$. Making the supremum among all Borel functions with $\|g\| \leqslant 1$ we get

$$|T^\gamma|(\mathbb{R}^k) \leqslant L_\gamma. \tag{4}$$

Furthermore, it is easy to check that, if γ is simple, equality in (4) holds and $|T^\gamma|$ is the image of $\|\dot\gamma\|ds$ under γ.

Let now $T=(T_1,\dots,T_k)\in[\mathcal{M}(\mathbb{R}^k)]^k$. By polar decomposition we can write $T=W\eta$, with $W:\mathbb{R}^k\to\mathbb{R}^k$ Borel unit vectorfield and $\eta\in\mathcal{M}^+(\mathbb{R}^k)$ (η is the total variation of T and W is the orienting vectorfield, uniquely determined up to η-negligible sets); we also assume that $\mathrm{div}(T)$ is (representable by) a measure $\theta\in\mathcal{M}(\mathbb{R}^k)$, namely

$$\int\langle W,\nabla\phi\rangle d\eta=-\int\phi d\theta\qquad\forall\phi\in C^1_c(\mathbb{R}^k).$$

Notice that this assumption is fulfilled by T^γ, and

$$\mathrm{div}(T^\gamma)=\delta_{\gamma(1)}-\delta_{\gamma(0)}.$$

We say that a measure $\nu\in\mathcal{M}^+(\mathcal{L})$ is a decomposition of $T=W\eta$ by simple curves if:

(i) We have

$$T=\int_{\mathcal{L}}T^\gamma d\nu(\gamma),\tag{5}$$

which explicitly means that

$$\int\langle W,f\rangle\,d\eta=\int_{\mathcal{L}}\left(\int_0^1\langle f(\gamma(s)),\dot\gamma(s)\rangle ds\right)d\nu(\gamma)$$

for each bounded Borel function $f:\mathbb{R}^k\longrightarrow\mathbb{R}^k$.

(ii) We have

$$\eta=\int_{\mathcal{L}}|T^\gamma|d\nu(\gamma)\tag{6}$$

and

$$|\theta|=\int_{\mathcal{L}}\big(\delta_{\gamma(1)}+\delta_{\gamma(0)}\big)d\nu(\gamma).\tag{7}$$

(iii) ν-almost every curve $\gamma(t)$ is simple.

Notice that condition (6) can be interpreted by saying that no cancellation occurs in (5). Analogously, by applying (5) to a gradient vectorfield f, we get

$$\theta=\mathrm{div}(W\eta)=\int_{\mathcal{L}}\mathrm{div}(T^\gamma)d\nu(\gamma)=\int_{\mathcal{L}}\big(\delta_{\gamma(1)}-\delta_{\gamma(0)}\big)d\nu(\gamma).\tag{8}$$

So, also (7) implies that no cancellation occurs in the integrals in (8).

Proposition 3. *Let $\nu\in\mathcal{M}^+(\mathcal{L})$ be a decomposition of $W\eta$ by simple curves. Then, for ν-a.e. curve γ, we have $|T^\gamma|=\gamma_\#(\|\dot\gamma\|ds)$ and*

$$\dot\gamma(s)=\|\dot\gamma(s)\|W(\gamma(s))\qquad\textit{for a.e. } s\in[0,1].\tag{9}$$

Proof. The equality $|T^\gamma| = \gamma_\#(\|\dot\gamma\| ds)$ follows from the fact that ν-almost every curve is simple. Inserting $f = W$ in (5) and taking (6) into account we get

$$\int_{\mathcal{L}} \left(|T^\gamma|(\mathbb{R}^k) - \int_0^1 \langle W(\gamma(s)), \dot\gamma(s)\rangle ds \right) d\nu(\gamma) = 0.$$

Since ν-almost every curve is simple, we have equality in (4), and we get

$$\int_{\mathcal{L}} \left(L_\gamma - \int_0^1 \langle W(\gamma(s)), \dot\gamma(s)\rangle ds \right) d\nu(\gamma) = 0,$$

so that

$$\int_{\mathcal{L}} \left(\int_0^1 \|\dot\gamma(s)\| - \langle W(\gamma(s)), \dot\gamma(s)\rangle ds \right) d\nu(\gamma) = 0.$$

The integrand being nonnegative, we get (9). □

We can now state Theorem C of [11]:

Theorem 4. *Any $T = W\eta$ as above can be decomposed as $\eta = \eta^0 + \tilde\eta$, where $div(W\eta^0) = 0$ and $W\tilde\eta$ admits a decomposition $\nu \in \mathcal{M}^+(\mathcal{L})$ by simple curves.*

It turns out that also the divergence-free part $W\eta^0$ admits a decomposition in "elementary" vector fields T^γ, but the underlying curves γ need not be in $\mathcal{L}$: in order to obtain the decomposition, also curves associated to Bohr quasiperiodic maps $\gamma : \mathbb{R} \to \mathbb{R}^k$ should be considered, see [11] for a precise discussion.

3. Proof of Theorem 1

Let μ_t be a solution of (PDE) with initial condition μ_0 and let $S \in (0, T]$. We want to prove that $\mu_S = X(0, S, .)_\sharp \mu_0$.

Let $\sigma(t, x) : (0, T) \times \mathbb{R}^d \longrightarrow \{-1, 1\}$ be the sign of μ_t. By this we mean a Borel map such that $\sigma\mu = |\mu|$. Note that we really consider here a functions σ defined at each point, and not only a class of functions up to $|\mu|$-almost everywhere equality. There is not a unique choice for the function σ, but we pick one once and for all. Let us define the vectorfield

$$W(t, x) = \frac{\sigma(t, x)}{\|(1, V(t, x))\|}(1, V(t, x))$$

and the Borel non-negative measure

$$\eta(t, x) = \chi_{(0,S)\times\mathbb{R}^d} \|(1, V(t, x))\|(dt \otimes |\mu_t|)$$

on $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$. Note that (PDE) with initial condition μ_0 at $t = 0$ and final condition μ_S at $t = S$ can be rephrased as

$$\mathrm{div}(W\eta) = \theta$$

in the sense of distributions in $\mathbb{R}^{d+1}$, where $\theta = \delta_S \otimes \mu_S - \delta_0 \otimes \mu_0$.

Let now $\eta = \eta_0 + \tilde{\eta}$ be the decomposition provided by Theorem 4, and let $\nu \in \mathcal{M}^+(\mathcal{L})$ be a decomposition of $W\tilde{\eta}$, with $k = 1 + d$. By Proposition 3, ν-a.e. curve $\gamma = (t, x)$ satisfies the ODE

$$\dot{t}(s) = \|\dot{\gamma}(s)\| \frac{\sigma(t(s), x(s))}{\|(1, V(t(s), x(s)))\|} \tag{10}$$

$$\dot{x}(s) = \|\dot{\gamma}(s)\| \frac{\sigma(t(s), x(s))}{\|(1, V(t(s), x(s)))\|} V(t(s), x(s)). \tag{11}$$

Let us prove that for ν-a.e. curve $\gamma = (t, x)$ the integrability property

$$\int_0^1 |\dot{t}(s)||C(t(s))|ds < \infty$$

holds. Indeed, take $f(t, x) = C(t)/\|(1, V(t, x))\|$ and observe that (6) gives

$$\begin{aligned} \int_{\mathcal{L}} \int_0^1 |\dot{t}(s)||C(t(s))|ds d\nu(\gamma) &= \int_{\mathcal{L}} \int f d|T^\gamma| d\nu(\gamma) = \int f d\tilde{\eta} \\ &\leqslant \int f d\eta = \int_0^T C(t)|\mu_t|(\mathbb{R}^d) dt < \infty. \end{aligned}$$

As a consequence, the estimate $\int_0^1 |\dot{t}(s)||C(t(s))|ds < \infty$ is satisfied by ν-almost every curve (t, x) in $\mathcal{L}$. In view of Lemma 2, we conclude that ν-almost every curve (t, x) in $\mathcal{L}$ satisfies

$$x(s) = X(t(0), t(s), x(0)). \tag{12}$$

Since ν-almost every curve is one to one, we conclude that $t(s)$ is one to one for ν-almost every curve. By (7) we know that $t(0) \in \{0, S\}$ and $t(1) \in \{0, S\}$ for ν-almost every curve $\gamma = (t, x)$, and therefore, either $t(0) = 0$ and $t(1) = S$, or $t(0) = S$ and $t(1) = 0$.

Denoting by $\mathcal{L}^+$ the Borel subset of $\mathcal{L}$ formed by curves $\gamma = (t, x)$ such that t is increasing on $[0, 1]$ and satisfies $t(0) = 0$ and $t(1) = S$, and by $\mathcal{L}^-$ the Borel subset of $\mathcal{L}$ formed by curves $\gamma = (t, x)$ such that t is decreasing on $[0, 1]$ and satisfies $t(0) = S$ and $t(1) = 0$, we conclude that $\nu(\mathcal{L}^+ \cup \mathcal{L}^-) = 1$. We denote by $\nu^\pm$ the restrictions of ν to $\mathcal{L}^\pm$. The measures $\nu^\pm$ are mutually singular, non-negative, and $\nu = \nu^+ + \nu^-$. Let

$$B_i : \mathcal{L}^+ \cup \mathcal{L}^- \longrightarrow \mathbb{R}^d$$

be the Borel map defined by $B_i(\gamma) = x(0)$ if $\gamma \in \mathcal{L}^+$ and $B_i(\gamma) = x(1)$ if $\gamma \in \mathcal{L}^-$. Similarly, we define

$$B_f : \mathcal{L}^+ \cup \mathcal{L}^- \longrightarrow \mathbb{R}^d$$

by $B_i(\gamma) = x(0)$ if $\gamma \in \mathcal{L}^-$ and $B_i(\gamma) = x(1)$ if $\gamma \in \mathcal{L}^+$. Note that

$$B_f = B_i \circ X(0, S, \cdot)$$

ν-almost everywhere by (12). Since $\theta = \delta_S \otimes \mu_S - \delta_0 \otimes \mu_0$, it follows from (8) that $\mu_0 = (B_i)_\sharp(\nu^- - \nu^+)$ and $\mu_S = (B_f)_\sharp(\nu^- - \nu^+)$. As a consequence, we have

$$\mu_S = X(0, S, \cdot)_\sharp \mu_0.$$

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