ON THE OPTIMAL REINFORCEMENT OF AN ELASTIC MEMBRANE

GIUSEPPE BUTTAZZO, NICOLAS VARCHON

Abstract. We consider an elastic membrane occupying a domain Ω of $\mathbb{R}^N$ under the action of a given exterior load f. The membrane can be reinforced by the addition of a suitable potential term in the energy; this is usually a boundary term but also other situations can be considered. We study the optimal configuration of the stiffeners which provide the best reinforcement of the membrane.

Keywords: Elastic membrane, boundary reinforcement, optimization problems.

AMS Classification Numbers: 49J45, 74P10, 49Q10

1. Introduction

Reinforcing an elastic structure subjected to a given load is a problem which arises in several applications. The literature on the topic is very wide; for a clear description of the problem from a mechanical point of view and a related bibliography we refer for instance to the beautiful recent paper by Villaggio [6].

We consider here the simple case of an elastic membrane occupying a domain Ω and subjected to a given exterior load $f \in L^2(\Omega)$. The shape u of the membrane in the equilibrium configuration is then characterized as the solution of the partial differential equation

$$-\Delta u = f \quad \text{in } \Omega \tag{1.1}$$

together with the corresponding boundary conditions of Neumann or Dirichlet type on $\partial\Omega$ (or on a part of it).

The reinforcement of the membrane is usually performed at its boundary, by the addition of suitable stiffeners, whose total amount is prescribed. Mathematically, this is described by a nonnegative coefficient $a(x)$ which acts on the Neumann part of the boundary $\partial\Omega$ and modifies the boundary conditions into the new ones:

$$\frac{\partial u}{\partial \nu} + a(x)u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

The problem of finding an optimal reinforcement for the membrane then consists in the determination of a coefficient $a(x)$ which optimizes a given cost functional. The optimization criterion we consider in this paper is the so called *elastic compliance*, which has to be minimized in order to provide a membrane as stiff as possible. More precisely, for every coefficient $a \in L^1(\partial\Omega)$ we consider the related energy

$$\begin{aligned} \mathcal{E}(a) \;=\; \inf\Big\{ \int_\Omega |Du|^2\,dx + \int_{\partial\Omega} a(x)u^2\,d\mathcal{H}^{N-1} \\ -2\int_\Omega f(x)u\,dx \;:\; u \in H^1(\Omega)\Big\} \end{aligned} \tag{1.3}$$

and the compliance

$$\mathcal{C}(a) = -\mathcal{E}(a), \tag{1.4}$$

so that the optimal reinforcement problem can be written as

$$\min\Big\{\mathcal{C}(a) \;:\; a \in L^1(\partial\Omega),\ \int_{\partial\Omega} a\,d\mathcal{H}^{N-1} \le m\Big\}. \tag{1.5}$$

When some part Γ_0 of the boundary is fixed, that is Dirichlet conditions are imposed on Γ_0, in (1.3) the space $H^1(\Omega)$ has to be replaced by the space $\{u \in H^1(\Omega) \;:\; u = 0 \text{ on } \Gamma_0\}$ and correspondingly in (1.5) the coefficient $a(x)$ will be searched among the ones which vanish on Γ_0.

More generally, in this paper we allow the reinforcement to take place in some prescribed compact set $K \subset \overline{\Omega}$ so that in (1.5) the constraint $\operatorname{supp} a \subset K$ has to be added. We will show that the constraint above is variationally closed; in other words, a minimizing sequence $\{a_n\}_{n\in\mathbb{N}}$ may converge to a measure μ with nonzero singular part, as it happens in several shape optimization problems (see for instance [1]). However, due to the particular form of the reinforcement problems, we will show that this does not occur and that an optimal coefficient a_{opt} actually exists in $L^1(K)$ with possibly a boundary part in $L^1(\partial K)$.

We end the paper by some numerical computations on some two dimensional examples.

2. The Mathematical Setting

Let us now precise the optimization problem we study and the notation used throughout this paper.

The capacity of a subset A is defined by

$$\mathrm{Cap}(A) = \inf\Big\{ \int_{\mathbb{R}^N} \big[|\nabla u|^2 + u^2\big]\,dx \;:\; u \in \mathcal{U}_A\Big\}.$$

where $\mathcal{U}_A$ is the set of all functions u of the Sobolev space $H^1(\mathbb{R}^N)$ such that $u \ge 1$ almost everywhere in a neighborhood of A.

If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subset E$ with $\mathrm{Cap}(Z) = 0$, we say that $P(x)$ holds quasi-everywhere on E (shortly q.e. on E). The expression almost everywhere (shortly a.e.) refers, as usual, to the Lebesgue measure.

We denote by $\mathcal{L}$ the $N-$dimensional Lebesgue measure and by $\mathcal{H}^d$ the $d-$dimensional Hausdorff measure. Let $B(x, r)$ be the ball of center x and radius r. The average of a function u in the ball $B(x,r)$ is defined by

$$\bar{u}_r(x) = \frac{1}{\omega_N r^N} \int_{B(x,r)} u \, d\mathcal{L} \,. \tag{2.1}$$

We recall the classical result about Sobolev functions: if $u \in H^1(\Omega)$, then there exists a Borel set $E \subset \Omega$ such that $\mathrm{Cap}(E) = 0$ and for all $x \in \Omega \setminus E$, the following limit exists

$$\tilde{u}(x) = \lim_{r \to 0^+} \bar{u}_r(x);$$

the function $\tilde{u}$ is called the quasi-continuous representative of u.

Let μ and ν be two Borel measures in $\mathbb{R}^N$; following the notation concerning the differentiation of measures (see [4]), we define for each point x in $\mathbb{R}^N$

$$\underline{D}_\nu \mu(x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \,. \tag{2.2}$$

The following result can be easily deduced from [4] (Lemma 1, page 36).

Lemma 2.1. *Let $0 < \alpha < +\infty$ be fixed. Suppose that $\underline{D}_\nu \mu(x) \le \alpha$ for μ-a.e. $x \in A$. Then*

$$\mu(A) \le \alpha \nu(A).$$

In particular, if there exists a constant α such that $\underline{D}_\nu \mu(x) \le \alpha$ for μ-a.e. $x \in \mathbb{R}^N$, then the measure μ is absolutely continuous with respect to the measure ν. In this case, $\underline{D}_\nu \mu \in L^\infty_\nu(\mathbb{R}^N)$ and by the differentiation theorem of Radon measures (see [4]), $\underline{D}_\nu \mu$ is then the density of μ with respect to the measure ν, that is $\mu = \underline{D}_\nu \mu \nu$.

We consider a smooth bounded connected open set Ω and two compact sets Γ_0 and K included in the closure of Ω; the sets above are supposed to have a smooth boundary, the sense of "smooth" being precised later. We denote by $\mathcal{M}_0(K \setminus \Gamma_0)$ the class of all nonnegative Borel measures (possibly $+\infty$ valued) supported in $K \setminus \Gamma_0$ which vanish on all sets of capacity zero. Given $f \in L^2(\Omega)$, for every measure $\mu \in \mathcal{M}_0(K \setminus \Gamma_0)$ we consider the related energy

$$\begin{aligned} \mathcal{E}(\mu) \;\; = \;\; \inf \Big\{ & \int_\Omega |Du|^2 \, dx + \int_{\overline{\Omega}} u^2 \, d\mu \\ & -2 \int_\Omega f(x) u \, dx \; : \; u \in H^1(\Omega, \Gamma_0) \Big\}. \end{aligned} \tag{2.3}$$

Here $H^1_0(\Omega, \Gamma_0)$ is the Hilbert space of all functions $u \in H^1(\Omega)$ which vanish on Γ_0. This last condition is intended in the sense of capacity, that is $\tilde{u} = 0$ q.e. on Γ_0, where $\tilde{u}$ denotes the quasi-continuous representative of u.

The compliance of a measure $\mu \in \mathcal{M}_0(K \setminus \Gamma_0)$ is defined by

$$\mathcal{C}(\mu) = -\mathcal{E}(\mu).$$

We choose a Lagrangian formulation to take into account the total mass of the reinforcement and consider the optimization problem

$$\min_{\mu\in\mathcal{M}_0(K\setminus\Gamma_0)}\{\mathcal{C}(\mu)+\alpha\mu(K)\},\tag{2.4}$$

where α is a nonnegative parameter. This formulation is equivalent to the following

$$\min_{\mu\in\mathcal{M}_0(K\setminus\Gamma_0)}\{\mathcal{C}(\mu):\mu(K)=C\},\tag{2.5}$$

where C is a suitable nonnegative constant. It is well known (see for instance [1], [3]) that every measure in $\mathcal{M}_0(K\setminus\Gamma_0)$ can be obtained as a γ-limit of a sequence in $L^1(K)$; therefore, if $K=\partial\Omega$, problem (2.5) is the relaxed formulation of the initial boundary reinforcement problem (1.5).

If Γ_0 is a set with positive capacity or if μ is a nonzero measure, then the expression

$$\Big(\int_\Omega|\nabla u|^2\,dx+\int_{\overline{\Omega}}u^2\,d\mu\Big)^{1/2}$$

defines a norm on the Hilbert space $X_\mu(\Omega)=H^1_0(\Omega,\Gamma_0)\cap L^2(\Omega,\mu)$. In this case, problem (2.4) has an unique solution which is the weak solution of the state equation (see for instance [1])

$$\begin{cases}-\Delta u+u\mu&=&f&\text{in}&\Omega,\\ u&=&0&\text{on}&\Gamma_0.\end{cases}\tag{2.6}$$

A weak solution of (2.6) is a function in $X_\mu(\Omega)$ solving the variational problem

$$\int_\Omega\nabla u\nabla v\,dx+\int_{\overline{\Omega}}uv\,d\mu=\int_\Omega fv\,dx\qquad\forall v\in X_\mu(\Omega).\tag{2.7}$$

3. The Results

We first deal with the problem of existence of solutions.

Theorem 3.1. *Problem (2.4) has at least one solution.*

Proof. Let $(\mu_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (2.4). Up to a subsequence, we may assume that $(\mu_n)_{n\in\mathbb{N}}$ γ-converges to a measure μ which belongs to $\mathcal{M}_0(K\setminus\Gamma_0)$. Consequently, the sequence of functionals $G_n(u)=\int_\Omega|\nabla u|^2\,dx+\int_{\overline{\Omega}}u^2\,d\mu_n$ Γ-converges to the functional $G(u)=\int_\Omega|\nabla u|^2\,dx+\int_{\overline{\Omega}}u^2\,d\mu$ in $L^2(\Omega)$ (see [3]). We deduce that

$$\mathcal{C}(\mu)=\lim_{n\to\infty}\mathcal{C}(\mu_n)$$

and since the total mass $\mu\mapsto\mu(\Omega)$ is lower semicontinuous under the γ-convergence (see [2]), the measure μ is a solution of (2.4). To conclude the proof, it is sufficient to show that $\mathcal{C}(\mu)<+\infty$. Indeed, this does not occur only if $\mathrm{Cap}(\Gamma_0)=0$, $\mu\equiv0$ and $\int_\Omega f\,dx\neq0$; in this case it is enough to compare $\mathcal{C}(\mu)$ to $\mathcal{C}(\nu)$ where ν is a nonzero admissible measure of $\mathcal{M}_0(K\setminus\Gamma_0)$. □

The next lemma gives some necessary conditions of optimality for the solutions of the optimization problem (2.4) and has the uniqueness of the solution as a direct consequence.

Lemma 3.2. *Let μ be a solution of problem (2.4) and let u be the solution of (2.6) associated to μ. Then*

$$|u| \le \sqrt{\alpha} \text{ q.e. in } K \text{ and } |u| = \sqrt{\alpha}\ \mu\text{-a.e.} \tag{3.1}$$

Moreover μ is the unique measure for which the conditions above are satisfied.

Proof. The proof of the first part is similar to the proof of the optimality conditions in [2]. We only change the first family of perturbations to take into account the fact that the admissible measures are supported in $K \setminus \Gamma_0$. The first family of perturbations we consider is $\mu_\varepsilon = \mu + \varepsilon\phi\mathcal{L}\llcorner_{K\setminus\Gamma_0}$.

Let now $(\mu_i)_{i=1,2}$ be two measures which satisfy the optimality conditions, and let $(u_i)_{i=1,2}$ be the associated solutions. Since u_1 and u_2 are in $L^\infty(K)$, their quasi-continuous representatives belong to $X_{\mu_1}(\Omega) \cap X_{\mu_2}(\Omega)$. Taking the difference between the variational equations (2.7) verified respectively by u_1 and u_2, and using $(u_2 - u_1)$ as test function, we obtain

$$\int_\Omega |\nabla(u_2 - u_1)|^2\, dx + \int_K u_2(u_2 - u_1)\, d\mu_2 - \int_K u_1(u_2 - u_1)\, d\mu_1 = 0. \tag{3.2}$$

Introducing in (3.2) the optimality conditions (3.1) previously proved, it is easy to obtain that $u_1 = u_2$. Let now u be the quasi-continuous representative of u_1; then by taking again the difference between the variational equations verified by u respectively for the measure μ_1 and μ_2, we obtain that for all $\varphi \in X_{\mu_1}(\Omega) \cap X_{\mu_2}(\Omega)$

$$\int_\Omega u\varphi\, d\mu_1 = \int_\Omega u\varphi\, d\mu_2. \tag{3.3}$$

By taking $\varphi = u\psi$ as test function in (3.3), with $\psi \in H^1_0(\Omega, \Gamma_0)$ we easily obtain the uniqueness of the optimal measure, and the proof is concluded. □

We now give and prove the main result of this paper.

Theorem 3.3. *Let $f \in L^p(\Omega)$ with $p > N$. Suppose that the boundary of K is $C^{1,1}$. Then there exists a function $a \in L^\infty(\partial K)$ such that the solution of the problem (2.4) verifies*

$$\mu = \frac{1}{\sqrt{\alpha}}\Big(f^+\mathcal{L}\llcorner_{U^+} + f^-\mathcal{L}\llcorner_{U^-}\Big) + a\mathcal{H}^{N-1}\llcorner_{\partial K} \tag{3.4}$$

where

$$U^+ = \{u = \sqrt{\alpha}\} \cap K \quad \text{and} \quad U^- = \{u = -\sqrt{\alpha}\} \cap K \tag{3.5}$$

being u the solution of the state equation (2.6) associated to μ. Moreover there exists a constant C depending on N, Ω and K such that

$$\|a\|_{L^\infty(\partial K)} \le C\frac{\|f\|_{L^p(\Omega)}}{\sqrt{\alpha}}\ .$$

Proof. Without loss of generality we can suppose that $K \cup \Gamma_0 \subset \Omega$. Indeed, consider problem (2.4) posed in the open set $\tilde{\Omega}$ such that $\overline{\Omega} \subset \tilde{\Omega}$ with the source f extended by zero outside Ω. Lemma 3.2 implies that the solution of problem (2.4) in $\tilde{\Omega}$ coincides with the extension by zero of the solution of problem (2.4) in Ω.

Consider a test function $\varphi \in \mathcal{D}(K_{\text{int}} \setminus \Gamma_0)$ where K_{int} is the interior of K. For all nonnegative integers p, the function $u^p\varphi$ belongs to $X_\mu(\Omega)$. By introducing $u^{2p+1}\varphi$ as test function in the equation (2.7) and letting $p \to +\infty$, we obtain by the same argument used in [5], that

$$\mu\llcorner_{K_{\text{int}}} = \frac{1}{\sqrt{\alpha}}\Big(f^+\mathcal{L}\llcorner_{\{u=\sqrt{\alpha}\}\cap K} + f^-\mathcal{L}\llcorner_{\{u=-\sqrt{\alpha}\}\cap K}\Big). \tag{3.6}$$

Consider $x \in \partial K \setminus \Gamma_0$ and $(\varphi_\varepsilon)_\varepsilon$ the family of functions defined by

$$\varphi_\varepsilon(y) = \begin{cases} 1 - \frac{|y-x|^2}{\varepsilon^2} & \text{if } |y-x| < \varepsilon \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Taking $u\varphi_\varepsilon$ as test function in (2.7) we obtain

$$\int_\Omega \varphi_\varepsilon |\nabla u|^2\, dx + \frac{1}{2}\int_\Omega \nabla u^2 \nabla\varphi_\varepsilon\, dx + \int_\Omega u^2\varphi_\varepsilon\, d\mu = \int_\Omega f u\varphi_\varepsilon\, dx. \tag{3.8}$$

Using the optimality condition of Lemma 3.2, we have

$$\int_\Omega \varphi_\varepsilon\, d\mu \le \frac{1}{\alpha}\Big(\int_\Omega f u\varphi_\varepsilon\, dx - \frac{1}{2}\int_\Omega \nabla u^2\nabla\varphi_\varepsilon\, dx\Big). \tag{3.9}$$

Let $w \in H^1(\Omega \setminus K)$ be the solution of

$$\begin{cases} -\Delta w = |f| & \text{in } \Omega \setminus K, \\ w = \sqrt{\alpha} & \text{on } \partial K, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

By the assumptions made on f and K the function w belongs to $W^{1,\infty}(\Omega \setminus K)$ and there exists a nonnegative constant $C_1 = C_1(\Omega, K)$ such that $\|\nabla w\|_{L^\infty(\Omega\setminus K)} \le C_1\|f\|_{L^p(\Omega)}$. Since $|u| \le \sqrt{\alpha}$ in K we have $|u| \le w$ in $\Omega \setminus K$. The function u^2 and w^2 belong to $H^1(\Omega)$. Considering x such that $|u(x)| = \sqrt{\alpha}$, by Lemma 4.3 of [2] we have

$$\liminf_{\varepsilon\to 0} -\frac{1}{\varepsilon^{N-1}}\int_\Omega \nabla(u^2 - w^2)\nabla\varphi_\varepsilon\, dx \le 0. \tag{3.11}$$

Moreover, a straightforward computation shows that there exists two constants C_2, C_3 depending only on N such that

$$\Big|\int_\Omega \nabla w^2\nabla\varphi_\varepsilon\, dx\Big| \le C_2\sqrt{\alpha}\|\nabla w\|_{L^\infty(\Omega\setminus K)}\varepsilon^{N-1} + C_3\|\nabla w\|^2_{L^\infty(\Omega\setminus K)}\varepsilon^N.$$

Then there exists $C_4 = C_4(N, \Omega, K)$ such that

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^{N-1}}\Big|\int_\Omega \nabla w^2\nabla\varphi_\varepsilon\, dx\Big| \le C_4\sqrt{\alpha}\|f\|_{L^p(\Omega)} \tag{3.12}$$

and since $u \in L^\infty(\Omega)$,

$$\frac{1}{\varepsilon^{N-1}} \int_\Omega f u \varphi_\varepsilon \, dx \le C_4 \varepsilon^{1-N/p} \|u\|_{L^\infty(\Omega)} \|f\|_{L^p(\Omega)}. \tag{3.13}$$

Passing to the limit in (3.9) and using facts (3.11), (3.12) and (3.13), we obtain that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N-1}} \int_\Omega \varphi_\varepsilon \, d\mu \le \frac{C_4}{\sqrt{\alpha}} \|f\|_{L^p(\Omega)}. \tag{3.14}$$

Thanks to the smoothness of the boundary of K, there exists $C_5 = C_5(K)$ such that

$$\underline{D}_{\mathcal{H}^{N-1}_{\llcorner \partial K}} \mu(x) \le C_5 \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^{N-1}} \int_\Omega \varphi_\varepsilon \, d\mu. \tag{3.15}$$

Combining (3.14) and (3.15), we conclude that for $\mathcal{H}^{N-1}$-a.e. $x \in \partial K \setminus \Gamma_0$

$$\underline{D}_{\mathcal{H}^{N-1}_{\llcorner \partial K}} \mu \llcorner \partial K \le \frac{C}{\sqrt{\alpha}} \|f\|_{L^p(\Omega)} \tag{3.16}$$

where $C = C(N, \Omega, K)$. This ends the proof. □

It is possible to extend the result above to the case when the boundary of K is piecewise smooth.

Proposition 3.4. *Suppose that the boundary of K is a countable union of $C^{1,1}$ curves. Then there exists $a \in L^1(\partial K)$ such that the characterization of the optimal measure (3.4)-(3.5) is still valid.*

In $\mathbb{R}^2$, the conclusion of Theorem 3.3 is still valid if all the connected components of $\Omega \setminus K$ which contain the support of f are convex polygons.

Proof. If the boundary of K is piecewise $C^{1,1}$ the constant C_1 depends on the point x. So we can only conclude that $\mu_{\llcorner \partial K}$ is absolutely continuous with respect to the Lebesgue measure. In the particular case of convex polygons in $\mathbb{R}^2$, we know that the solution w belongs to $W^{2,2+\varepsilon}$, so the constant C_1 does not depend on x and the proof is still valid. □

Remark 3.5. • Under an additional hypothesis of regularity, the singular part of the optimal measure can be expressed in terms of α and u. If we suppose that the surface density a is strictly positive in a neighborhood of $x \in \partial K \setminus \Gamma_0$, then u is a C^1 function in a neighborhood of x and a direct application of the Green formula gives that

$$a = -\frac{u}{\alpha} \Big(\frac{\partial u}{\partial n}\Big|_{\partial K_{\text{ext}}} + \frac{\partial u}{\partial n}\Big|_{\partial K_{\text{int}}} \Big) \tag{3.17}$$

where $\frac{\partial u}{\partial n}\Big|_{\partial K_{\text{ext}}}$ and $\frac{\partial u}{\partial n}\Big|_{\partial K_{\text{int}}}$ are respectively the outside and inside normal derivatives of u on the boundary of K.

• In the particular case where Γ_0 is empty, and the function f is nonnegative, the solution of the optimization problem is explicit. Indeed, by the uniqueness of the measure μ given by Lemma 3.2, the optimal measure is given by

$$\mu = -\frac{1}{\sqrt{\alpha}} \frac{\partial w}{\partial n}\Big|_{\partial K_{\rm int}} \tag{3.18}$$

where w is the solution of (3.10), which coincides with the solution of the state equation associated to μ.

4. Some Numerical Computations

In this section we give some examples of numerical solutions. At left of the three figures, we represent the data of the problem and at right the density of the optimal measure in a gray scale. In the two first examples, the design region Ω is the square. In the first one (Figure 1), the support K is the boundary of Ω, and the function f is equal to -1 in the white part, to 1 in the black one, and to zero elsewhere. In the second one (Figure 2), the support K of the measure is in gray and the function f is equal to 1 in the upper half square and to zero in the other part. In the third and last example, we consider the triangle as design region (in gray at left of Figure 3) and two Dirichlet parts Γ_0 on the boundary of the triangle (in black). The source f is constant and the support K is the whole triangle; the optimal measure is represented at right in black,

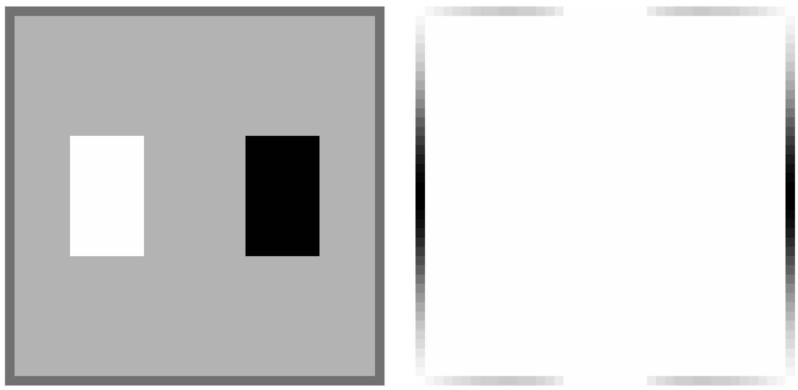

Figure 1. left: support and source, right: optimal measure

Acknowledgements. This work is part of the European Research Training Network "Homogenization and Multiple Scales" (HMS2000) under contract HPRN-2000-00109. The second author wishes to thank the Department of Mathematics of the University of Pisa for the kind and warm hospitality.

References

[1] BUTTAZZO, G.; DAL MASO, G. *Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions*, Appl. Math. Optim. **23** (1991), no. 1, 17–49.

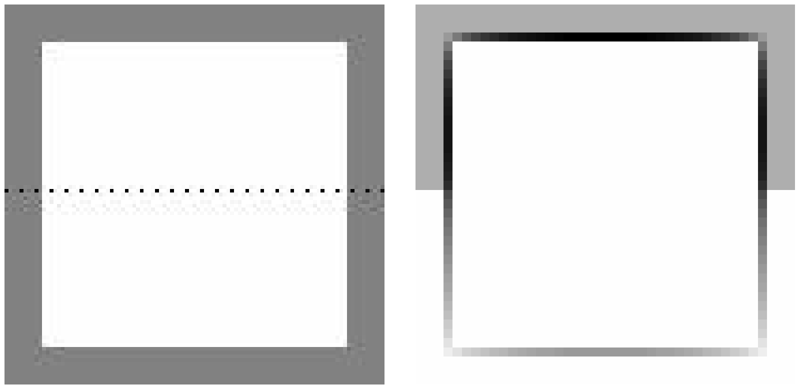

FIGURE 2. left: support and source, right: optimal measure

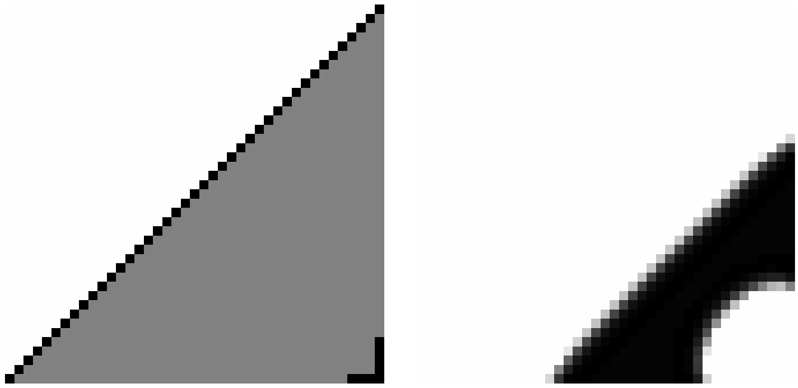

FIGURE 3. left: support and source, right: optimal measure

[2] G. BUTTAZZO, N. VARCHON, H. ZOUBAIRI: *Optimal measures for elliptic problems.* Preprint Dipartimento di Matematica Università di Pisa, Pisa (2004). Available at `http://cvgmt.sns.it/papers/butvarzou04/`

[3] DAL MASO, G.; MOSCO, U. *Wiener's criterion and* Γ*-convergence* , Appl. Math. Optim. **15** (1987), no. 1, 15–63.

[4] EVANS, L.C.; GARIEPY, R.F. *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, 1992.

[5] VARCHON, N. *Optimal measures for nonlinear cost functionals.* Preprint Dipartimento di Matematica Università di Pisa, Pisa (2004).

[6] VILLAGGIO, P. *Calcolo delle variazioni e teoria delle strutture.* Boll. Un. Mat. Ital. Ser. VIII, **VII-A** (2004), 49–76.

Giuseppe Buttazzo
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2
56127 Pisa, ITALY
`buttazzo@dm.unipi.it`

Nicolas Varchon
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2
56127 Pisa, ITALY
`varchon@mail.dm.unipi.it`