

About well-posedness of optimal segmentation for Blake & Zisserman functional

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SOMMARIO: Si discute la buona posizione di un problema di minimo con discontinuità libera nel gradiente: sono provate varie condizioni di estremalità e sono esibite varie tipologie di non unicità dei minimi.

ABSTRACT: We focus well-posedness in the minimization of a second order free discontinuity problem. Several extremality conditions are proven. Various examples of multiplicity for minimizers are shown.

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1 Introduction

The interest in image segmentation arises in image analysis and computer vision theory. The first variational model for image segmentation was suggested by Mumford and Shah in [18]: they introduced the following functional

$$\int_{\Omega \setminus K} (|Du(x)|^2 + |u(x) - g(x)|^2) dx + \gamma \mathbf{H}^{n-1}(K \cap \Omega) \quad (1.1)$$

where Ω is an open subset of \mathbb{R}^n , $n \geq 1$, u is a scalar function, $K \subset \mathbb{R}^n$, D denotes the distributional gradient, $g \in L^2(\Omega)$ is a function representing the

grey levels of the image, γ is a given positive real number related to scale and contrast threshold and \mathbf{H}^{n-1} is the $n - 1$ dimensional Hausdorff measure. According to this model ([18], [17], [8]) the segmentation of the given image is achieved through the minimization of (1.1) over u and K where K is a closed subset of \mathbb{R}^n and $u \in C^1(\Omega \setminus K)$.

The existence of minimizers for the functional (1.1) was proven in [16] starting from the functional framework introduced in [15]. The existence of minimizers was proven also in [14] by a different approach in the case $n = 2$. The uniqueness of such these minimizers may fail ([2]).

Blake and Zisserman showed some limitations of the Mumford-Shah functional and introduced an alternative way to translate the image segmentation problem into a variational problem in [3]. The strong formulation of the Blake-Zisserman functional is the following functional ([6]) to be minimized among triplets u , K_0 and K_1 where K_0 and K_1 are closed sets in \mathbb{R}^n and $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ and is approximately continuous on $\Omega \setminus K_0$:

$$\int_{\Omega \setminus (K_0 \cup K_1)} \left(|D^2 u(x)|^2 + |u(x) - g(x)|^2 \right) dx + \alpha \mathbf{H}^{n-1}(K_0) + \beta \mathbf{H}^{n-1}(K_1 \setminus K_0). \quad (1.2)$$

In (1.2) Ω is an open set of \mathbb{R}^n , $n \geq 1$, $g \in L^2(\Omega)$ is a function representing the grey levels of the given image, α and β are given positive real numbers related to scale and contrast threshold, D^2 denotes the distributional hessian and \mathbf{H}^{n-1} is the $(n - 1)$ dimensional Hausdorff measure.

According to this model ([3], [8]) an optimal segmentation of the given image is achieved through the minimization of functional (1.2) over u , K_0 and K_1 . The existence of minimizers for (1.2) was proven in [13] for $n = 1$ and then in [6] for $n = 2$ starting from the weak formulation framework introduced in [5] for any dimension $n \geq 2$.

The non convex functionals (1.1) and (1.2) depend on functions and sets: in fact Mumford-Shah functional involves the two unknowns u and K , while Blake-Zisserman functional involves the three unknowns u , K_0 and K_1 .

De Giorgi introduced the basic idea to deal with problems with free discontinuity: formulate and study a relaxed version in the unknown u alone, then prove regularity results for optimal u and eventually recover the discontinuity as the singular set of such optimal u .

This program was achieved for Mumford-Shah functional in [15] by introducing a weak formulation of (1.1) where u belongs to $SBV(\Omega)$, S_u replaces K and $\int_{\Omega} |\nabla u(x)|^2 dx$ replaces $\int_{\Omega \setminus K} |Du(x)|^2 dx$ where ∇u is the absolutely continuous part of Du .

This program was achieved for the Blake-Zisserman functional in [5] by introducing a weak formulation of (1.2) where u belongs to $GSBV(\Omega)$ with ∇u in $GSBV(\Omega)^n$, S_u replaces K_0 , $S_{\nabla u} \setminus S_u$ replaces K_1 and $\int_{\Omega} |\nabla^2 u(x)|^2 dx$ replaces $\int_{\Omega \setminus (K_0 \cup K_1)} |D^2 u(x)|^2 dx$.

Concerning free discontinuity problems in image segmentation the only available result about uniqueness of the minimizer is given in [2] for the 1 dimensional Mumford-Shah functional (1.1).

In this paper we face the question of uniqueness for minimizer of 1-dimensional Blake-Zisserman functional $F_{\alpha,\beta}^g$ below. Given $g \in L^2(0,1)$, $\alpha, \beta \in \mathbb{R}$ and $u \in \mathcal{H}^2$ we define $F_{\alpha,\beta}^g : \mathcal{H}^2 \rightarrow [0, +\infty)$ as follows

$$F_{\alpha,\beta}^g(u) = \int_0^1 |\ddot{u}(x)|^2 dx + \int_0^1 |u(x) - g(x)|^2 dx + \alpha \#(S_u) + \beta \#(S_{\dot{u}} \setminus S_u). \quad (1.3)$$

Here and in the sequel for all $u \in L^2(0,1)$, \dot{u} denotes the absolutely continuous part of the distributional derivative u' of u , \ddot{u} denotes the absolutely continuous part of $(\dot{u})'$, $S_u \subseteq (0,1)$ denotes the approximate discontinuity set ([1]) of u and $S_{\dot{u}} \subseteq (0,1)$ the approximate discontinuity set of \dot{u} , $\#$ denotes the counting measure and

$$H^2(I) = \{v \in L^2(I) : v', v'' \in L^2(I)\} \quad \text{for an interval } I \subseteq \mathbb{R}$$

$$\mathcal{H}^2 = \{v \in L^2(0,1) : \#(S_v \cup S_{\dot{v}}) < +\infty, v \in H^2(I) \forall \text{ interval } I \subseteq (0,1) \setminus (S_v \cup S_{\dot{v}})\}.$$

We will call *singular set* of u the set $S_u \cup S_{\dot{u}}$ and we denote

$$m^g(\alpha, \beta) = \inf \{F_{\alpha,\beta}^g(u) \quad \forall u \in \mathcal{H}^2\},$$

$$\operatorname{argmin} F_{\alpha,\beta}^g = \{u \in \mathcal{H}^2 : F_{\alpha,\beta}^g(u) = m^g(\alpha, \beta)\},$$

the absolutely continuous part of functional $F_{\alpha,\beta}^g$ is denoted by

$$\mathcal{F}^g(u) = \int_0^1 |\ddot{u}(x)|^2 dx + \int_0^1 |u(x) - g(x)|^2 dx. \quad (1.4)$$

We emphasize that, in the 1-d case the strong and the weak version of Blake-Zisserman functional coincide: in fact if $u \in L^2(0,1)$ with $F_{\alpha,\beta}^g(u) < +\infty$ then $\#(S_{\dot{u}} \cup S_u) < +\infty$, hence $u \in C^1((0,1) \setminus (S_u \cup S_{\dot{u}})) \cap C^0((0,1) \setminus S_u) \cap \mathcal{H}^2$ and

$$\int_{(0,1)} |\ddot{u}|^2 dx = \int_{(0,1) \setminus (S_u \cup S_{\dot{u}})} |u''|^2 dx$$

so that minimizers of $F_{\alpha,\beta}^g$ automatically belong to $C^2((0,1) \setminus (S_u \cup S_{\bar{u}}))$. The complete set of Euler equations for minimizers, a compliance identity formula for functional $F_{\alpha,\beta}^g$, a priori estimates and continuous dependence of $m^g(\alpha, \beta)$ with respect to g, α, β are proven in Section 2: Theorems 2.1, 2.2, 2.3 (and 2.4 about n -d case).

It is known that $F_{\alpha,\beta}^g$ achieves a finite minimum (say $\operatorname{argmin} F_{\alpha,\beta}^g \neq \emptyset$) whenever the two following conditions are satisfied ([13]):

$$0 < \beta \leq \alpha \leq 2\beta < +\infty \quad (1.5)$$

$$g \in L^2(0,1). \quad (1.6)$$

Nevertheless minimizers are not unique in general, due to non convexity of functional (1.3). In Section 3 we show some examples of multiplicity: we exhibit $\alpha > 0$ such that $F_{\alpha,\alpha}^g$ has exactly two minimizers if $g = \chi_{[\frac{1}{2},1]}$ (see Counterexample 3.1); there are $\alpha > 0$ and $g \in L^2(0,1)$ such that uniqueness fails for any β belonging to a non empty interval $(\alpha - \varepsilon, \alpha]$ (see Counterexample 3.2); for any α and β satisfying $0 < \beta \leq \alpha < 2\beta$ there is $g \in L^2(0,1)$ with $\sharp(\operatorname{argmin} F_{\alpha,\beta}^g) > 1$ (see Counterexample 3.3). Moreover we give an example of a non empty open subset $\mathcal{N} \subseteq L^2(0,1)$ such that for any $g \in \mathcal{N}$ there are α and β satisfying (1.5) and $\sharp(\operatorname{argmin} F_{\alpha,\beta}^g) \geq 2$ (see Counterexample 3.4). The resulting picture is coherent with the appearance of instable patterns and bifurcation of optimal segmentation upon variation of parameters α and β related contrast threshold and luminance sensitivity.

In a forthcoming paper (see [4]) we will show generic uniqueness of minimizers starting from the properties shown in the present paper. We emphasize that, even for continuous piecewise affine functions g , jump and crease points of minimizers are not necessarily localized among those of g (see Section 4): hence the techniques used in [2] to prove generic uniqueness for Mumford-Shah functional cannot be directly applied here. For this reason we will follow a different strategy in [4], by carefully exploiting some intersection properties between real analytic varieties.

2 Euler equations

In this section we deduce Euler equations for minimizers of the functional $F_{\alpha,\beta}^g$. For the multidimensional situation ($n \geq 2$) we refer to [7], [10] and [12].

Theorem 2.1 (Euler equations) *If (1.5) and (1.6) hold true then every u which minimizes (1.3) in \mathcal{H}^2 is also a solution of the following system:*

$$\left\{ \begin{array}{ll} (i) & u'''' + u = g \quad \text{on } (0, 1) \setminus (S_{\tilde{u}} \cup S_u) \\ (ii) & \ddot{u}_+ = \ddot{u}_- = 0 \quad \text{on } S_{\tilde{u}} \cup S_u \cup \{0, 1\} \\ (iii) & \ddot{u}_+ = \ddot{u}_- = 0 \quad \text{on } S_u \cup \{0, 1\} \\ (iv) & \ddot{u}_+ = \ddot{u}_- \quad \text{on } S_{\tilde{u}} \\ (v) & \frac{1}{2}(u_+ + u_-) = g \quad \text{on } S_u \cap \{\text{continuity points of } g\} \end{array} \right.$$

In (ii) and (iii) we conventionally set $\ddot{u}_-(0) = \ddot{u}_+(1) = 0 = \ddot{u}_+(1) = \ddot{u}_-(0)$. If, in addition to (1.5) and (1.6), $\alpha = \beta$ then (iii),(iv) improve as follows

$$\ddot{u}_+ = \ddot{u}_- = 0 \quad \text{on } S_u \cup S_{\tilde{u}} \cup \{0, 1\}. \quad (2.1)$$

By summarizing:

$$\ddot{u} \in H^2(0, 1) \quad \text{and} \quad (\ddot{u})'' + u = g \quad \text{in } \mathcal{D}'(0, 1). \quad (2.2)$$

Proof. Let u be a minimizer in \mathcal{H}^2 of $F_{\alpha, \beta}^g$. For any $v \in BV$ we set $[v] = v_+ - v_-$ where v_- , v_+ denote respectively the left and right values of v on S_v .

We introduce the localized version of functional $F_{\alpha, \beta}^g$: once fixed g , α , β , we set, for any v in $\mathcal{H}^2(0, 1)$ and any Borel set $A \subset [0, 1]$,

$$F(v, A) = \int_A (|\ddot{v}|^2 + |v - g|^2) dx + \alpha \mathcal{H}^{n-1}(S_v \cap A) + \beta \mathcal{H}^{n-1}((S_{\tilde{v}} \setminus S_v) \cap A) \quad (2.3)$$

Step 1 - (Green formula) Assume $u \in \mathcal{H}^2 \cap H^4((0, 1) \setminus \{S_u \cup S_{\tilde{u}}\})$ then, by labelling t_l , $l = 1, \dots, \mathbb{T}$, the ordered finite set $S_u \cup S_{\tilde{u}}$, and $t_0 = 0$, $t_{\mathbb{T}+1} = 1$, for any $\varphi \in \mathcal{H}^2$ the following identity holds true

$$\begin{aligned} \sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} u'' \varphi'' dx &= \sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} u'''' \varphi dx + \\ &\sum_{l=0}^{\mathbb{T}} \left((-u'''(t_{l+1})\varphi_-(t_{l+1}) + u'''(t_l)\varphi_+(t_l)) + \right. \\ &\left. (u''_-(t_{l+1})\varphi'_-(t_{l+1}) - u''_+(t_l)\varphi'_+(t_l)) \right) \end{aligned} \quad (2.4)$$

Step 2 - At first we show that each minimizer u solves the fourth order elliptic equation (i) in the interior of $(0, 1) \setminus (S_u \cup S_{\tilde{u}})$, by performing smooth

variations. For every open set $A \subset\subset I \setminus (S_u \cup S_{\dot{u}})$, for every $\varepsilon \in \mathbb{R}$ and for every $\varphi \in C_0^\infty(A)$ we have

$$0 \leq F(u + \varepsilon\varphi, A) - F(u, A) = 2\varepsilon \left(\int_A u''\varphi'' dx + \int_A (u - g)\varphi dx \right) + o(\varepsilon)$$

where $o(\varepsilon)$ is an infinitesimal of order greater than ε . Hence

$$\int_A u''\varphi'' dx = - \int_A (u - g)\varphi dx$$

for every $\varphi \in C_0^\infty(A)$. Then (i) follows integrating by parts with (2.4). Now we seek the Euler conditions on the discontinuity set.

Step 3 - We prove the necessary conditions for extremality on S_u :

$$\ddot{u}_\pm = 0 \quad \text{on } S_u \cup \{0, 1\} \quad (2.5)$$

$$\ddot{u}_\pm = 0 \quad \text{on } S_u \cup \{0, 1\} \quad (2.6)$$

In fact, let $\varphi \in \mathcal{H}^2(0, 1) \cap C^2([t_l, t_{l+1}])$, $l = 0, \dots, \mathbb{T}$, $\text{spt}(\varphi) \subset A$, where A is a Borel set with $(S_{\dot{u}} \setminus S_u) \cap A = \emptyset$. Then for every $\varepsilon \in \mathbb{R}$ we have

$$(S_{u+\varepsilon\varphi} \cup S_{\dot{u}+\varepsilon\dot{\varphi}}) \cap A \subset S_u \cap A$$

By (2.4) we have:

$$\begin{aligned} 0 &\leq F(u + \varepsilon\varphi, A) - F(u, A) \\ &= \alpha (\#(S_{u+\varepsilon\varphi} \cap A) - \#(S_u \cap A)) + \beta \#((S_{\dot{\varphi}} \setminus S_{u+\varepsilon\varphi}) \cap A) + \\ &\quad 2\varepsilon \left(\sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} (u''\varphi'' + (u - g)\varphi) dx \right) + o(\varepsilon) \\ &= \alpha (\#(S_{u+\varepsilon\varphi} \cap A) - \#(S_u \cap A)) + \beta \#((S_{\dot{\varphi}} \setminus S_{u+\varepsilon\varphi}) \cap A) + \\ &\quad 2\varepsilon \left(\sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} (u'''\varphi + (u - g)\varphi) dx \right. \\ &\quad \left. + \ddot{u}_+(0)\varphi_+(0) - \ddot{u}_+(0)\dot{\varphi}_+(0) - \ddot{u}_-(1)\varphi_-(1) + \ddot{u}_-(1)\dot{\varphi}_-(1) \right. \\ &\quad \left. \sum_{S_u \cap A} \left([+ \ddot{u}\varphi] - [\ddot{u}\varphi] \right) \right) + o(\varepsilon) \end{aligned}$$

Up to a finite set of values of ε , we have $S_{u+\varepsilon\varphi} \cap A = S_u \cap A$ so that we can choose arbitrarily small ε satisfying

$$\#((S_{\dot{\varphi}} \setminus S_{u+\varepsilon\varphi}) \cap A) = \#((S_{\dot{\varphi}} \setminus S_u) \cap A) = 0$$

By taking into account (i) and the arbitrariness of the two traces of φ and $\dot{\varphi}$ on the two sides of points in S_u , for small ε , we can choose φ with $\varphi_{\pm} = 0$, $\dot{\varphi}_+ = 0$ and $\dot{\varphi}_-$ arbitrary or viceversa to get (2.5). Similarly, we obtain (2.6) by choosing $\dot{\varphi}_{\pm} = 0$, $\varphi_+ = 0$ and φ_- arbitrary or vice-versa.

Step 4 - We prove the necessary conditions for extremality on $S_{\dot{u}}$:

$$\ddot{u}_{\pm} = 0 \quad \text{on } S_{\dot{u}} \quad (2.7)$$

$$[\ddot{u}] = 0 \quad \text{on } S_{\dot{u}} \setminus S_u \quad (2.8)$$

Let $\varphi \in \mathcal{H}^2(0, 1) \cap C^2([t_l, t_{l+1}])$, $l = 0, \dots, \mathbb{T}$, $\text{spt}(\varphi) \subset A$, and $S_{\varphi} = \emptyset = (S_u \setminus S_{\dot{u}}) \cap A$. Up to a finite set of values of ε , so that we can choose ε arbitrarily small, we have:

$$(S_{u+\varepsilon\varphi} \cup S_{\dot{u}+\varepsilon\dot{\varphi}}) \cap A = S_{\dot{u}+\varepsilon\dot{\varphi}} \cap A = S_{\dot{u}}$$

Moreover, by (2.4):

$$\begin{aligned} 0 &\leq F(u + \varepsilon\varphi, A) - F(u, A) \\ &\leq \beta(\#(S_{\dot{u}+\varepsilon\dot{\varphi}} \cap A) - \#(S_{\dot{u}} \cap A)) + \\ &2\varepsilon \left(\sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} (u''\varphi'' + (u-g)\varphi) dx \right) + o(\varepsilon) \\ &= 2\varepsilon \left(\sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} u''''\varphi dx + (u-g)\varphi dx + \right. \\ &\quad \left. + \ddot{u}_+(0)\varphi_+(0) - \ddot{u}_+(0)\dot{\varphi}_+(0) - \ddot{u}_-(1)\varphi_-(1) + \ddot{u}_-(1)\dot{\varphi}_-(1) \right. \\ &\quad \left. \sum_{S_{\dot{u}} \cap A} \left([+ \ddot{u}\varphi] - [\ddot{u}\dot{\varphi}] \right) \right) + o(\varepsilon) \end{aligned}$$

By taking into account (i), for small ε and by the arbitrariness of φ and of the two traces of $\dot{\varphi}$ on the two sides of $S_{\dot{u}}$, we can choose φ with $\varphi_{\pm} = 0$, and arbitrary $\dot{\varphi}_+ = \dot{\varphi}_-$, to get (2.7). Analogous by choosing $\dot{\varphi}_{\pm} = 0$ and $[\dot{\varphi}] = 0$ or viceversa, we obtain (2.8).

Then (ii), (iii) and (iv) follows from (2.5)-(2.8) of steps 3 and 4.

Step 5 - We prove (v).

Assume $t \in S_u$ and g continuous at t . If $s = \frac{1}{2}(u_+(t) + u_-(t)) \neq g(t)$ then only one of the following eight cases occurs:

- | | |
|---------------------------------|---------------------------------|
| 1) $u_-(t) > u_+(t) \geq g(t)$ | 5) $u_+(t) > u_-(t) \geq g(t)$ |
| 2) $u_-(t) > s > g(t) > u_+(t)$ | 6) $u_+(t) > s > g(t) > u_-(t)$ |
| 3) $u_-(t) > g(t) > s > u_+(t)$ | 7) $u_+(t) > g(t) > s > u_-(t)$ |
| 4) $g(t) \geq u_-(t) > u_+(t)$ | 8) $g(t) \geq u_+(t) > u_-(t)$ |

To deal with 1), 2), 6), 7) choose $0 < \varepsilon \ll \text{dist}(t, (S_u \cup S_{\dot{u}} \cup \{0, 1\}) \setminus \{t\})$ and explicit the minimality of u by comparison with a variation v in a small interval:

$$v(x) = \begin{cases} u(x) & \text{if } x \in [0, t - \varepsilon) \cup (t, 1] \\ \gamma(x) = u_+(t) + \dot{u}_+(t)(x - t) & \text{if } x \in [t - \varepsilon, t] \end{cases}$$

Since $\ddot{v} \equiv 0$ in $(t - \varepsilon, t)$ and g is continuous at t then

$$\begin{aligned} F_{\alpha, \beta}^g(v) - F_{\alpha, \beta}^g(u) &= \int_0^1 |\ddot{v}(x)|^2 dx + \int_0^1 |v(x) - g(x)|^2 dx \\ &\quad - \int_0^1 |\ddot{u}(x)|^2 dx - \int_0^1 |u(x) - g(x)|^2 dx \\ &\leq \int_{t-\varepsilon}^t |v(x) - g(x)|^2 dx - \int_{t-\varepsilon}^t |u(x) - g(x)|^2 dx \\ &= \int_{t-\varepsilon}^t ((\gamma(x) - g(x))^2 - (u(x) - g(x))^2) dx \\ &= \int_{t-\varepsilon}^t (\gamma(x) - u(x))(\gamma(x) + u(x) - 2g(x)) dx \\ &\sim \int_{t-\varepsilon}^t (u_+(t) - u_-(t))(u_+(t) + u_-(t) - 2g(t)) dx < 0 \end{aligned}$$

This contradicts the minimality of u .

To deal with 3), 4), 5), 6) choose $0 < \varepsilon \ll \text{dist}(t, (S_u \cup S_{\dot{u}} \cup \{0, 1\}) \setminus \{t\})$ and explicit the minimality of u by comparison with a variation w in a small interval:

$$w(x) = \begin{cases} u(x) & \text{if } x \in [0, t) \cup (t + \varepsilon, 1] \\ \delta(x) = u_-(t) + \dot{u}_-(t)(x - t) & \text{if } x \in [t, t + \varepsilon] \end{cases}$$

which leads to the contradiction:

$$F_{\alpha, \beta}^g(w) - F_{\alpha, \beta}^g(u) \sim \int_t^{t+\varepsilon} (u_-(t) - u_+(t))(u_+(t) + u_-(t) - 2g(t)) dx < 0.$$

Step 6 - Eventually we prove (2.1): due to (iii) we have only to show

$$\ddot{u}_{\pm} = 0 \quad \text{on } (S_{\dot{u}} \setminus S_u) \text{ if } \alpha = \beta$$

Fix a Borel set A s.t. $A \subset\subset (0, 1)$, $S_u \cap A = \emptyset \neq S_{\dot{u}} \cap A$.

Let $\varphi \in \mathcal{H}^2(0, 1) \cap C^2([t_l, t_{l+1}])$, $l = 0, \dots, \mathbb{T}$ and

$$S_{\dot{\varphi}} \cap A = S_u \cap A = \emptyset \neq S_{\varphi} \cap A = S_{\dot{u}} \cap A$$

Then for every $\varepsilon \in \mathbb{R}$ we have $S_{u+\varepsilon\varphi} \cap A = S_\varphi \cap A$ and

$$(S_{u+\varepsilon\varphi} \cup S_{(\dot{u}+\varepsilon\dot{\varphi})}) \cap A = S_{\dot{u}} \cap A$$

By (2.4), (i) and (ii) we have

$$\begin{aligned} 0 &\leq F(u + \varepsilon\varphi, A) - F(u, A) \\ &= \alpha \#(S_{u+\varepsilon\varphi} \cap A) + \beta \#((S_{(\dot{u}+\varepsilon\dot{\varphi})} \setminus S_{u+\varepsilon\varphi}) \cap A) - \beta \#(S_{\dot{u}} \cap A) \\ &\quad + 2\varepsilon \left(\sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} (u''\varphi'' + (u-g)\varphi) dx \right) + o(\varepsilon) \\ &= \alpha \#(S_\varphi \cap A) + \beta \#((S_{\dot{u}} \setminus S_\varphi) \cap A) - \beta \#(S_{\dot{u}} \cap A) \\ &\quad + 2\varepsilon \left(\sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} (u'''\varphi + (u-g)\varphi) dx + \sum_{S_{\dot{u}}} ([\ddot{u}\varphi] - [\dot{u}\dot{\varphi}]) \right) + o(\varepsilon) \\ &= \alpha \#(S_\varphi \cap A) - \beta \#(S_{\dot{u}} \cap A) + 2\varepsilon \sum_{S_{\dot{u}} \cap A} [\ddot{u}\varphi] + o(\varepsilon) \end{aligned}$$

Since $S_\varphi \cap A = S_{\dot{u}} \cap A$, when $\alpha > \beta$ then the inequality is fulfilled for ε small enough, hence we do not obtain further information. On the other hand, when $\alpha = \beta$, we get

$$0 \leq F(u + \varepsilon\varphi, A) - F(u, A) = 2\varepsilon \sum_{S_{\dot{u}} \cap A} [\ddot{u}\varphi] + o(\varepsilon)$$

Then the coefficient of 2ε must vanish, hence by the arbitrariness of the two traces of φ we get (2.1).

Step 7 - The proof of (2.2) is a straightforward consequence of (i)-(iv). \square

Theorem 2.2 (Compliance identity) *Assume (1.5) and (1.6). Then any $u \in \mathcal{H}^2$ fulfilling the Euler equations (i)-(iv) of Theorem 2.1 verifies also*

$$\mathcal{F}^g(u) = \int_0^1 (gu - u^2) dx \quad (2.9)$$

and

$$F_{\alpha,\beta}^g(u) = \int_0^1 (gu - u^2) dx + \alpha \#(S_u) + \beta \#(S_{\dot{u}} \setminus S_u). \quad (2.10)$$

In particular any u minimizing $F_{\alpha,\beta}^g$ over \mathcal{H}^2 fulfills (2.9) and (2.10).

Proof. Label t_l , $l = 1, \dots, \mathbb{T}$, the ordered finite set $S_u \cup S_{\dot{u}}$ and $t_0 = 0$, $t_{\mathbb{T}+1} = 1$. Integration by parts in $\int_0^1 |\ddot{u}|^2 dx$ and (i)-(iv) of Theorem 2.1 entail

$$\begin{aligned} \int_0^1 |\ddot{u}|^2 dx &= \\ & \sum_{l=0}^{\mathbb{T}} \int_{t_l}^{t_{l+1}} (\ddot{u})'' u dx + \sum_{l=0}^{\mathbb{T}} \left((-u'''(t_{l+1})u_-(t_{l+1}) + u'''(t_l)u_+(t_l)) + \right. \\ & \quad \left. (u''_-(t_{l+1})u'_-(t_{l+1}) - u''_+(t_l)u'_+(t_l)) \right) = \int_0^1 (\ddot{u})'' u dx \\ & = \int_0^1 (g - u)u dx = \int_0^1 (gu - u^2) dx \end{aligned}$$

and the theorem follows. \square

We show a priori estimates for minima, minimizers, singular set of minimizers of $F_{\alpha, \beta}^g$ and continuous dependence of minimum value $m^g(\alpha, \beta)$ with respect to α, β in $\{(\alpha, \beta) \in \mathbb{R}^2: 0 < \beta \leq \alpha \leq 2\beta\}$ and g in $L^2(0, 1)$.

Theorem 2.3 Assume $f, g \in L^2(0, 1)$ and

$$0 < \beta \leq \alpha \leq 2\beta < +\infty, \quad 0 < b \leq a \leq 2b < +\infty. \quad (2.11)$$

Then

$$\|u\|_{L^2} \leq 2\|g\|_{L^2} \quad \forall u \in \operatorname{argmin} F_{\alpha, \beta}^g, \quad (2.12)$$

$$0 \leq m^g(\alpha, \beta) \leq \|g\|_{L^2}^2, \quad (2.13)$$

$$\begin{aligned} |m^g(\alpha, \beta) - m^h(a, b)| &\leq 5(\|g\|_{L^2} + \|h\|_{L^2})\|g - h\|_{L^2} + \\ & + \frac{\min\{\|g\|_{L^2}^2, \|h\|_{L^2}^2\}}{\min\{\alpha, a\}} |\alpha - a| + \frac{\min\{\|g\|_{L^2}^2, \|h\|_{L^2}^2\}}{\min\{\beta, b\}} |\beta - b|, \end{aligned} \quad (2.14)$$

$$\left. \begin{aligned} \sharp(S_u) &\leq \max\{j \in \mathbb{N} : j \leq 2(\|g\|_{L^2}^2 + \eta^2)/\alpha\}, \\ \sharp(S_{\dot{u}} \setminus S_u) &\leq \max\{j \in \mathbb{N} : j \leq 2(\|g\|_{L^2}^2 + \eta^2)/\beta\}, \\ &\forall u \in \operatorname{argmin} F_{\alpha, \beta}^h \text{ with } \|h - g\|_{L^2} < \eta. \end{aligned} \right\} \quad (2.15)$$

Proof. Estimate (2.13) follows from $0 \leq m^g(\alpha, \beta) \leq F_{\alpha, \beta}^g(0) = \|g\|_{L^2}^2$. By (2.13) we get the following inequality equivalent to (2.12)

$$\|u\|_{L^2}^2 \leq 2(\|u - g\|_{L^2}^2 + \|g\|_{L^2}^2) \leq 2(m^g(\alpha, \beta) + \|g\|_{L^2}^2) \leq 4\|g\|_{L^2}^2.$$

Fix $u_g \in \operatorname{argmin} F_{\alpha,\beta}^g$, $u_h \in \operatorname{argmin} F_{\alpha,\beta}^h$; then by Schwarz inequality and (2.12)

$$\begin{aligned} m^g(\alpha, \beta) &= F_{\alpha,\beta}^g(u_g) \leq F_{\alpha,\beta}^g(u_h) = F_{\alpha,\beta}^h(u_h) - \|u_h - h\|_{L^2}^2 + \|u_h - g\|_{L^2}^2 = \\ &= m^h(\alpha, \beta) - \|u_h - h\|_{L^2}^2 + \|u_h - g\|_{L^2}^2 \leq \\ &\leq m^h(\alpha, \beta) + \langle g - h, g + h - 2u_h \rangle_{L^2} \leq \\ &\leq m^h(\alpha, \beta) + (\|g\|_{L^2} + 5 \|h\|_{L^2}) \|g - h\|_{L^2}, \end{aligned}$$

similarly $m^h(\alpha, \beta) \leq m^g(\alpha, \beta) + (\|h\|_{L^2} + 5 \|g\|_{L^2}) \|g - h\|_{L^2}$. Then

$$|m^g(\alpha, \beta) - m^h(\alpha, \beta)| \leq 5(\|g\|_{L^2} + \|h\|_{L^2}) \|g - h\|_{L^2}. \quad (2.16)$$

Fix $u_{\alpha,\beta} \in \operatorname{argmin} F_{\alpha,\beta}^g$, $u_{a,b} \in \operatorname{argmin} F_{a,b}^g$; then by (1.5) and (2.13)

$$\begin{aligned} m^g(a, b) &\leq F_{a,b}^g(u_{\alpha,\beta}) = F_{\alpha,\beta}^g(u_{\alpha,\beta}) + (a - \alpha) \#(S_{u_{\alpha,\beta}}) + (b - \beta) \#(S_{\dot{u}_{\alpha,\beta}} \setminus S_{u_{\alpha,\beta}}) \\ &= m^g(\alpha, \beta) + \frac{a-\alpha}{\alpha} \alpha \#(S_{u_{\alpha,\beta}}) + \frac{b-\beta}{\beta} \beta \#(S_{\dot{u}_{\alpha,\beta}} \setminus S_{u_{\alpha,\beta}}) \leq \\ &\leq m^g(\alpha, \beta) + \frac{|\alpha-a|}{\alpha} m^g(\alpha, \beta) + \frac{|\beta-b|}{\beta} m^g(\alpha, \beta) \leq \\ &\leq m^g(\alpha, \beta) + \frac{\|g\|_{L^2}^2}{\alpha} |\alpha - a| + \frac{\|g\|_{L^2}^2}{\beta} |\beta - b|, \end{aligned}$$

similarly $m^g(\alpha, \beta) \leq m^g(a, b) + \frac{\|g\|_{L^2}^2}{a} |\alpha - a| + \frac{\|g\|_{L^2}^2}{b} |\beta - b|$. Then

$$|m^g(\alpha, \beta) - m^g(a, b)| \leq \frac{\|g\|_{L^2}^2}{\min\{\alpha, a\}} |\alpha - a| + \frac{\|g\|_{L^2}^2}{\min\{\beta, b\}} |\beta - b|. \quad (2.17)$$

Eventually inequality (2.14) follows by (2.16), (2.17) and

$$\begin{aligned} |m^g(\alpha, \beta) - m^h(a, b)| &\leq \min \left\{ |m^g(\alpha, \beta) - m^h(\alpha, \beta)| + |m^h(\alpha, \beta) - m^h(a, b)|, \right. \\ &\quad \left. |m^g(\alpha, \beta) - m^g(a, b)| + |m^g(a, b) - m^h(a, b)| \right\}. \end{aligned}$$

To prove (2.15) choose $h \in L^2(0, 1)$ with $\|h - g\|_{L^2} < \eta$ and $u \in \operatorname{argmin} F_{\alpha,\beta}^h$, then (2.13) entails

$$\alpha \#(S_u) + \beta \#(S_{\dot{u}} \setminus S_u) \leq m^h(\alpha, \beta) \leq \|h\|_{L^2}^2 \leq 2 \|g\|_{L^2}^2 + 2\eta^2. \quad \square$$

Analogous properties hold true for n -dimensional Blake-Zisserman functional.

Theorem 2.4 Fix an open set $\Omega \subseteq \mathbb{R}^n$, denote by $\mathbf{F}_{\alpha,\beta}^g$ the functional (1.2), by $\operatorname{argmin} \mathbf{F}_{\alpha,\beta}^g$ the set of minimizers of $\mathbf{F}_{\alpha,\beta}^g$, by $m^g(\alpha, \beta)$ the minimum value of $\mathbf{F}_{\alpha,\beta}^g$. Assume $f, g \in L^2(\Omega)$ and α, β and a, b fulfill (2.11). Then

$$\|u\|_{L^2} \leq 2 \|g\|_{L^2} \quad \forall u \in \operatorname{argmin} \mathbf{F}_{\alpha,\beta}^g, \quad (2.18)$$

$$0 \leq m^g(\alpha, \beta) \leq \|g\|_{L^2}^2, \quad (2.19)$$

$$\begin{aligned} |m^g(\alpha, \beta) - m^h(a, b)| &\leq 5(\|g\|_{L^2} + \|h\|_{L^2}) \|g - h\|_{L^2} + \\ &\quad \frac{\min\{\|g\|_{L^2}^2, \|h\|_{L^2}^2\}}{\min\{\alpha, a\}} |\alpha - a| + \frac{\min\{\|g\|_{L^2}^2, \|h\|_{L^2}^2\}}{\min\{\beta, b\}} |\beta - b|, \end{aligned} \quad (2.20)$$

$$\left. \begin{aligned} \mathbf{H}^{n-1}(S_u) &\leq \frac{2}{\alpha} (\|g\|^2 + \eta^2), \quad \mathbf{H}^{n-1}(S_{\nabla u} \setminus S_u) \leq \frac{2}{\beta} (\|g\|^2 + \eta^2) \\ \forall u \in \operatorname{argmin} \mathbf{F}_{\alpha, \beta}^h &\text{ with } \|h - g\|_{L^2} < \eta. \end{aligned} \right\} \quad (2.21)$$

Proof. Repeat the proof of the 1-d case (Theorem 2.3) by substituting \mathbf{H}^{n-1} to \sharp . \square

In the following Lemma we summarize and restate in a form suitable for our purposes Theorems 2.1, 3.1 and Lemma 3.6 of [13].

Theorem 2.5 *Assume $g \in L^2(0, 1)$, α, β fulfilling (1.2), $(u_l)_{l \in \mathbb{N}} \subseteq \mathcal{H}^2(0, 1)$ and $\{F_{\alpha, \beta}^g(u_l)\}_{l \in \mathbb{N}}$ is bounded.*

1. Compactness

Then there are $u \in \mathcal{H}^2(0, 1)$ and a subsequence $(u_{l_n})_{n \in \mathbb{N}}$ such that

$$\left\{ \begin{array}{l} (u_{l_n})_{n \in \mathbb{N}} \text{ converges to } u \text{ in the strong topology of } L^1(0, 1), \\ (\dot{u}_{l_n})_{n \in \mathbb{N}} \text{ converges almost everywhere to } \dot{u}, \\ (\ddot{u}_{l_n})_{n \in \mathbb{N}} \text{ converges to } \ddot{u} \text{ in the weak topology of } L^2(0, 1). \end{array} \right.$$

2. Lower semicontinuity

If $(u_l)_{l \in \mathbb{N}}$ converges strongly in L^1 to $u \in \mathcal{H}^2(0, 1)$, then

$$F_{\alpha, \beta}^g(u) \leq \liminf_{l \rightarrow +\infty} F_{\alpha, \beta}^g(u_l).$$

3. A confined single crease sequence (of a minimizing sequence) cannot converge to a jump

If $(u_l)_{l \in \mathbb{N}}$ converges strongly in L^1 to $u \in \mathcal{H}^2(0, 1)$, $(a, b) \subseteq (0, 1)$ and

$$x_l \in S_{\dot{u}_l} \setminus S_{u_l}, \quad (S_{u_l} \cup S_{\dot{u}_l}) \cap (a, b) = \{x_l\}, \quad x_l \rightarrow \bar{x} \in (a, b)$$

then $\bar{x} \notin S_u$, more precisely

$$S_u \cap (a, b) = \emptyset, \quad S_{\dot{u}} \cap (a, b) \subseteq \{\bar{x}\}. \quad \square$$

We show that \mathcal{F}^g has strictly positive infimum over suitable subsets of \mathcal{H}^2 .

Theorem 2.6 For any possibly discontinuous piecewise affine function g with $S_g \cup S_{\dot{g}} \neq \emptyset$ we introduce the subset $\mathcal{S}[g]$ of \mathcal{H}^2 as follows:
 $v \in \mathcal{S}[g]$ if and only if, either

$$(i) \begin{cases} \#(S_{\dot{v}} \setminus S_v) < \#(S_{\dot{g}} \setminus S_g) \\ \#(S_v) < \#(S_g) + \#(S_{\dot{g}} \setminus S_g) - \#(S_{\dot{v}} \setminus S_v), \end{cases}$$

or

$$(ii) \begin{cases} \#(S_v) < \#(S_g) \\ \#(S_{\dot{v}} \setminus S_v) < \#(S_{\dot{g}} \setminus S_g) + 2(\#(S_g) - \#(S_v)). \end{cases}$$

Then $\mathcal{S}[g] \neq \emptyset$ and

$$\inf_{v \in \mathcal{S}[g]} \mathcal{F}^g(v) > 0. \quad (2.22)$$

Proof. $\mathcal{S}[g]$ is not empty since $H^2(0, 1) \subseteq \mathcal{S}[g]$.

In order to show (2.22) we argue by contradiction: suppose that there is a sequence $\{v_n\}_n$ in $\mathcal{S}[g]$ with $\lim_{n \rightarrow +\infty} \mathcal{F}^g(v_n) = 0$.

Condition (i), (ii) and $\mathcal{S}[g] \subseteq \mathcal{H}^2$ entail

$$\mathcal{F}^g(v_n) + \alpha \#(S_{v_n}) + \beta \#(S_{\dot{v}_n} \setminus S_{v_n}) \leq C < +\infty \quad \forall n.$$

By Theorem 2.5(1), up to subsequences, v_n converges strongly in $L^1(0, 1)$ to a function $w \in \mathcal{H}^2$, $\dot{v}_n \rightarrow \dot{v}$ a.e, and $\ddot{v}_n \rightarrow \ddot{v}$ weakly in $L^2(0, 1)$. Lower semicontinuity of \mathcal{F}^g (Theorem 2.5(2)) implies $\mathcal{F}^g(w) = 0$ then $w = g$ a.e. in $(0, 1)$ and, by $g, w \in \mathcal{H}^2$, we have $g = w$.

Let $\mathbf{s}_n = \#(S_{v_n})$ and $\mathbf{p}_n = \#(S_{\dot{v}_n} \setminus S_{v_n})$. Up to subsequences we can assume the existence of non negative integers \mathbf{s}, \mathbf{p} such that, for any n , $\mathbf{s}_n = \mathbf{s}$, $\mathbf{p}_n = \mathbf{p}$ and the ordering of jumps and creases is independent of n . By introducing the sets $\{y_n^a\}_{a=1}^{\mathbf{s}} = S_{v_n}$ and $\{y_n^b\}_{b=1}^{\mathbf{p}} = S_{\dot{v}_n} \setminus S_{v_n}$ with $y_n^a < y_{n+1}^a$ and $y_n^b < y_{n+1}^b$ we can also assume

$$\lim_{n \rightarrow +\infty} y_n^a = y^a \quad \text{and} \quad \lim_{n \rightarrow +\infty} y_n^b = y^b.$$

The assumptions read, either (i) $\begin{cases} \mathbf{p} < \mathbf{c} \\ \mathbf{s} < \mathbf{j} + \mathbf{c} - \mathbf{p} \end{cases}$, or (ii) $\begin{cases} \mathbf{s} < \mathbf{j} \\ \mathbf{p} < \mathbf{c} + 2(\mathbf{j} - \mathbf{s}). \end{cases}$

In case (i) there is $\bar{x} \in S_{\dot{g}} \cup S_g$ such that $\bar{x} \notin \{y^a\}_{a=1}^{\mathbf{s}} \cup \{y^b\}_{b=1}^{\mathbf{p}}$, then the term $\int |\ddot{v}_n|^2$ blows up around \bar{x} , hence the contradiction $\lim_{n \rightarrow +\infty} \mathcal{F}^g(v_n) = +\infty$.

In case (ii) there is $\bar{x} \in S_g$ such that $\bar{x} \notin \{y^a\}_{a=1}^{\mathbf{s}}$ by the first condition in (ii) and, at the same time, by the second condition in (ii)

$$\lim_{n \rightarrow +\infty} y_n^{b_1} \neq \bar{x} \quad \text{or} \quad \lim_{n \rightarrow +\infty} y_n^{b_2} \neq \bar{x} \quad \forall b_1, b_2 \in \{1, \dots, \mathbf{p}\}, \quad b_1 \neq b_2;$$

then by Theorem 2.5(3) we get the contradiction $\lim_{n \rightarrow +\infty} \mathcal{F}^g(v_n) = +\infty$ as in the previous case. \square

We introduce and study the family Φ_λ of affine transformations of $L^2(0, 1)$ which are useful in exhibiting examples without uniqueness of minimizers.

Lemma 2.7 *Given $\alpha, \beta, \lambda \in \mathbb{R}$ with (1.5), we set*

$$\Phi_\lambda[v](x) = \lambda - v(1 - x), \quad \forall v \in L^2(0, 1). \quad (2.23)$$

Then for any $g \in L^2(0, 1)$ and $u \in \operatorname{argmin} F_{\alpha, \beta}^g$ we have $\Phi_\lambda[u] \in \operatorname{argmin} F_{\alpha, \beta}^{\Phi_\lambda[g]}$. In particular if $\Phi_\lambda[g] = g$ then also $\Phi_\lambda[u] \in \operatorname{argmin} F_{\alpha, \beta}^g$. Hence there is no uniqueness of minimizers for $F_{\alpha, \beta}^g$ whenever $\Phi_\lambda[g] = g$ and one can prove that a minimizer u fulfills $\Phi_\lambda[u] \neq u$.

If $g \in L^2(0, 1)$ fulfills $\sharp(\operatorname{argmin} F_{\alpha, \beta}^g) = 1$ then $\sharp(\operatorname{argmin} F_{\alpha, \beta}^{\Phi_\lambda[g]}) = 1$.

If $g \in L^2(0, 1)$ fulfills $\Phi_\lambda[g] = g$ and $\sharp(\operatorname{argmin} F_{\alpha, \beta}^g) = 1$, then $\Phi_\lambda[u] = u$.

The set $E_{\alpha, \beta}^n = \{g \in L^2(0, 1) : \sharp(\operatorname{argmin} F_{\alpha, \beta}^g) = n\}$ fulfills

$$\Phi_\lambda[E_{\alpha, \beta}^n] = E_{\alpha, \beta}^n \quad \forall \lambda \in \mathbb{R}.$$

Proof. For any $v, w \in \mathcal{H}^2(0, 1)$ we have

$$\|\Phi_\lambda[v] - \Phi_\lambda[w]\|_{L^2}^2 = \|v - w\|_{L^2}^2, \quad \|(\Phi_\lambda[v])^\cdot\|_{L^2}^2 = \|\ddot{v}\|_{L^2}^2$$

$$\sharp(S_v) = \sharp(S_{\Phi_\lambda[v]}), \quad \sharp(S_v \setminus S_v) = \sharp(S_{(\Phi_\lambda[v])^\cdot} \setminus S_{\Phi_\lambda[v]}),$$

then

$$F_{\alpha, \beta}^g(v) = F_{\alpha, \beta}^{\Phi_\lambda[g]}(\Phi_\lambda[v]) \quad \forall v \in \mathcal{H}^2(0, 1)$$

hence

$$m^{\Phi_\lambda[g]}(\alpha, \beta) \leq m^g(\alpha, \beta).$$

Since $\Phi_\lambda[\Phi_\lambda[v]] = v$ the above argument is symmetric hence

$$m^{\Phi_\lambda[g]}(\alpha, \beta) = m^g(\alpha, \beta) = F_{\alpha, \beta}^g(u) = F_{\alpha, \beta}^{\Phi_\lambda[g]}(\Phi_\lambda[u])$$

where u belongs to $\operatorname{argmin} F_{\alpha, \beta}^g$. \square

3 Counterexamples to uniqueness

In this section we show that uniqueness for the minimizer of $F_{\alpha, \beta}^g$ cannot be proven for generic data α, β and g .

The first example is given in the case $\alpha = \beta$.

Counterexample 3.1 Set $\chi = \chi_{[\frac{1}{2}, 1]}$. Let w be the unique minimizer of \mathcal{F}^χ in $H^2(0, 1)$. Observe that $F_{\alpha, \alpha}^\chi(\chi) = \alpha$ and, since $\chi \notin H^2(0, 1)$, there is $\mu = \mu(\chi) > 0$ with $\mu := \mathcal{F}^\chi(w) = F_{\alpha, \alpha}^\chi(w)$. Such μ is independent of α . Then the functional $F_{\mu, \mu}^\chi$ has at least two minimizers: χ and w , with $\chi \neq w$ since $\chi \notin H^2(0, 1)$.

Actually $F_{\mu, \mu}^\chi$ has exactly two minimizers.

To prove the last claim observe first that $F_{\mu, \mu}^\chi(u) > \mu$ if $\#(S_u \cup S_{\dot{u}}) \geq 2$. Set $\mathcal{B} = \{u \in \mathcal{H}^2 : \#(S_u) = 0, \#(S_{\dot{u}}) \leq 1\}$ and $\rho = \rho(\chi) = \inf_{u \in \mathcal{B}} \mathcal{F}^\chi(z)$.

Referring to Theorem 2.6 case (i), $\mathcal{B} \subseteq \mathcal{S}[\chi]$ hence $\rho > 0$, in any case $\rho(\chi) \leq \mu(\chi)$ since $H^2(0, 1) \subseteq \mathcal{B}$.

If $u \in \mathcal{B}$, we have either $S_u = S_{\dot{u}} = \emptyset$ then $F_{\mu, \mu}^\chi(u) \geq \mu$ with equality if and only if $u = w$; or $S_u = \emptyset$ and $\#(S_{\dot{u}}) = 1$, hence $F_{\mu, \mu}^\chi(u) \geq \rho + \mu > \mu$.

Eventually if $S_{\dot{u}} = \emptyset$ and $\#(S_u) = 1$ then either $u = \chi$ or $F_{\mu, \mu}^\chi(u) > \mu$. \square

The previous example proves that there are α and g such that $F_{\alpha, \alpha}^g$ has exactly two minimizers. Now we show that $F_{\alpha, \beta}^\chi$ may have more than one minimizer for suitable α and a continuum of choices of β , say even if (1.5) holds true and $\frac{\alpha}{\beta} \notin \mathbb{Q}$. About irrational quotient of data α, β we refer to generic uniqueness statement in Theorem 1.1 of [4].

Counterexample 3.2 Define $\chi = \chi_{[\frac{1}{2}, 1]}$, w , $\mu = \mu(\chi)$, ρ and \mathcal{B} as in Counterexample 3.1: say $F_{\alpha, \beta}^\chi(\chi) = \alpha$ and $F_{\alpha, \beta}^\chi(w) = \mu \geq \rho > 0$ with μ and ρ independent of α and β , so that $F_{\mu, \mu}^\chi$ has exactly two minimizers (χ, w) .

We claim that for any $\beta \in (\mu - \varepsilon, \mu]$, $\varepsilon = \min\{\frac{\mu}{2}, \rho\} > 0$, the functional $F_{\mu, \beta}^\chi$ has the same two minimizers χ and w and none more.

In fact $\beta > \mu/2$, $\beta > \mu - \rho$ and

$$\beta \in (\mu - \varepsilon, \mu] \subseteq \left(\frac{\mu}{2}, \mu\right] \Rightarrow \begin{cases} F_{\mu, \beta}^\chi(\chi) = \mu \\ F_{\mu, \beta}^\chi(u) > \mu \quad \text{if } \#(S_u \cup S_{\dot{u}}) \geq 2. \end{cases}$$

Moreover $0 < \beta \leq \mu < 2\beta$ hence inequality (1.5) is fulfilled by the pair μ, β . If $u \in \mathcal{B}$ we have: either $S_u = S_{\dot{u}} = \emptyset$ hence $F_{\mu, \beta}^\chi(u) \geq \mu$ with equality if and only if $u = w$, or $S_u = \emptyset$ and $\#(S_{\dot{u}}) = 1$ hence $F_{\mu, \beta}^\chi(u) \geq \rho + \beta > \rho + (\mu - \rho) = \mu$.

Eventually if $S_{\dot{u}} = \emptyset$ and $\#(S_u) = 1$ then either $u = \chi$ or $F_{\mu, \beta}^\chi(u) > \mu$. \square

Counterexample 3.3 Here we show that for any α, β satisfying the inequality $0 < \beta \leq \alpha < 2\beta$ (say a stronger constraint than (1.5)), there is $g \in L^2(0, 1)$, for instance a multiple of χ , such that $\#(\operatorname{argmin} F_{\alpha, \beta}^g) \geq 2$.

To prove the claim we exploit the homogeneity of $F_{\alpha, \beta}^g$:

$$F_{\lambda^2 \alpha, \lambda^2 \beta}^{\lambda g}(\lambda v) = \lambda^2 F_{\alpha, \beta}^g(v) \quad \forall \lambda \in \mathbb{R}, \quad \forall v \in \mathcal{H}^2, \quad \forall \alpha, \beta \text{ s.t. (1.5)}.$$

Then $F_{\lambda^2\alpha, \lambda^2\beta}^{\lambda g}$ has the same qualitative behaviour (with respect to uniqueness or non uniqueness of minimizers) of $F_{\alpha, \beta}^g$ for any $g \in L^2(0, 1)$ and α, β satisfying (1.5).

Minimizers and minima of $F_{\lambda^2\alpha, \lambda^2\beta}^{\lambda g}$ are respectively λ and λ^2 times the minimizers and minima of $F_{\alpha, \beta}^g$.

We set $\lambda = \sqrt{\frac{\alpha}{\mu(\chi)}}$ where $\mu(\chi) = \min_{H^2} \{\mathcal{F}^\chi\} = \mathcal{F}^\chi(w)$.

If $u \in H^2(0, 1)$, then either $F_{\alpha, \beta}^{\lambda\chi}(u) > F_{\alpha, \beta}^{\lambda\chi}(\lambda w) = \lambda^2\mu = \alpha$, or $u = \lambda w$ and $F_{\alpha, \beta}^{\lambda\chi}(u) = \alpha$.

If $\#(S_u) = 1$ and $\#(S_{\dot{u}}) = 0$, then either $F_{\alpha, \beta}^{\lambda\chi}(u) > F_{\alpha, \beta}^{\lambda\chi}(\lambda\chi) = \alpha$, or $u = \lambda\chi$.

If $\#(S_u \cup S_{\dot{u}}) \geq 2$, then $F_{\alpha, \beta}^{\lambda\chi}(u) > 2\beta \geq \alpha$, since $\int_0^1 |u - \lambda\chi|^2 dx > 0$.

We are left to analyze the behaviour of functional $F_{\alpha, \beta}^{\lambda\chi}$ only in the set

$\{u \in \mathcal{H}^2: \#(S_u) = 0, \#(S_{\dot{u}}) = 1\} \subset \mathcal{B}$.

Suppose first that $\rho(\chi) \geq \mu(\chi)/2$.

Since $1/2 < \beta/\alpha \leq 1$, Counterexample 3.2 implies that $F_{\mu, \mu\frac{\beta}{\alpha}}^\chi = F_{\lambda^{-2}\alpha, \lambda^{-2}\beta}^\chi$

admits exactly χ and w as minimizers. By scaling $F_{\lambda^{-2}\alpha, \lambda^{-2}\beta}^\chi$ behaves as $F_{\alpha, \beta}^{\lambda\chi}$.

Then $F_{\alpha, \beta}^{\lambda\chi}$ admits exactly $\lambda\chi$ and λw as minimizers and no more.

On the other hand suppose $\rho(\chi) < \mu(\chi)/2$.

Then, either we have the two minimizers $\lambda\chi$ and λw of $F_{\alpha, \beta}^{\lambda\chi}$, or there is a minimizer u of $F_{\alpha, \beta}^{\lambda\chi}$ with $S_u = \emptyset$ and $\#(S_{\dot{u}}) = 1$. In this last case consider the transformation Φ_λ defined by (2.23): since $\Phi_\lambda(\lambda\chi) = \lambda\chi$ Proposition 2.7 entails that $F_{\alpha, \beta}^{\lambda\chi}$ has at least two minimizers u and $\Phi_\lambda(u)$ which must be different since they have exactly one crease point. \square

Counterexample 3.4 *Here we show the existence of $\mathcal{N} \subseteq L^2(0, 1)$ with non empty interior in the strong topology of $L^2(0, 1)$ and such that for any $g \in \mathcal{N}$ there is $\beta = \beta(g)$ with $0 < \beta \leq \min_{H^2(0,1)} \mathcal{F}^g < 2\beta$ and $\#(\operatorname{argmin}_{H^2(0,1)} F_{\alpha, \beta}^g) \geq 2$ for any α satisfying*

$$\beta \leq \min_{H^2(0,1)} \mathcal{F}^g < \alpha < 2\beta. \quad (3.1)$$

Notice that (3.1) entails (1.5).

To prove the above claim we choose \mathcal{N} as a suitable L^2 neighborhood of a fixed function. Precisely we set

$$h(x) \stackrel{\text{def}}{=} \left| x - \frac{1}{2} \right|, \quad \mu(g) \stackrel{\text{def}}{=} \min_{H^2(0,1)} \mathcal{F}^g, \quad \mathcal{B} \stackrel{\text{def}}{=} \{u \in \mathcal{H}^2: \#(S_u) = 0, \#(S_{\dot{u}}) \leq 1\}.$$

We claim that

$$\exists L^2(0, 1) \text{ open neighborhood } \mathcal{N} \text{ of } h: \quad \inf_{\mathcal{B}} \mathcal{F}^g < \frac{1}{2}\mu(g) \quad \forall g \in \mathcal{N}, \quad (3.2)$$

and this will be the choice of \mathcal{N} leading to the counterexample.

To prove (3.2) we argue as follows. Consider $b = b[g](\cdot) \in \mathcal{H}^2(0, 1)$ fulfilling

$$\left. \begin{aligned} b''''(x) + b(x) &= g(x) && \text{on } (0, 1) \setminus \{1/2\}, \\ b''_+(1/2) &= b''_-(1/2) = 0, \\ b''_+(0) &= b''_-(1) = 0, \\ b'''_+(1/2) &= b'''_-(1/2), \\ b_+(1/2) &= b_-(1/2). \end{aligned} \right\} \quad (3.3)$$

By direct inspection problem (3.3) has a unique solution. Moreover $\mathcal{F}^h(b[g])$ depends continuously in L^2 with respect to g . Also $\mu(g)$ has continuous dependence on g by elliptic regularity and Theorem 2.2.

Since $h \notin H^2$, we have

$$\mathcal{F}^h(b[h]) = \mathcal{F}^h(h) = 0 < \mu(h). \quad (3.4)$$

Then (3.4) entails $\exists \mathcal{N} : 0 \leq \mathcal{F}^g(b[g]) < \frac{1}{3}\mu(h) < \frac{2}{3}\mu(h) < \mu(g) \quad \forall g \in \mathcal{N}$, say

$\exists L^2(0, 1)$ open neighborhood \mathcal{N} of h :

$$0 \leq \mathcal{F}^g(b[g]) < \frac{1}{2}\mu(g) \quad \forall g \in \mathcal{N}. \quad (3.5)$$

For any $g \in \mathcal{N}$, \mathcal{F}^g admits a minimizer over \mathcal{B} . In fact given $g \in \mathcal{N}$ and a minimizing sequence of \mathcal{F}^g over \mathcal{B} , by Theorem 2.5(1,3) we can extract a subsequence w_n strongly convergent in L^1 to a function $w \in \mathcal{B}$ with $\dot{w}_n \rightarrow \dot{w}$ a.e. and $\ddot{w}_n \rightarrow \ddot{w}$ weakly in $L^2(0, 1)$. By lower semicontinuity of \mathcal{F}^g we have that w minimizes \mathcal{F}^g over \mathcal{B} . By (3.5) w cannot belong to H^2 , hence $S_w \neq \emptyset$. By the same argument used in the proof of Theorem 2.1, w fulfills (i)-(iii) of Theorem 2.1. Then

$$\min_{\mathcal{B}} \mathcal{F}^g(u) = \mathcal{F}^g(w) \quad \forall g \in \mathcal{N}.$$

Then claim (3.2) follows by (3.5) since

$$\min_{\mathcal{B}} \mathcal{F}^g \leq \mathcal{F}^g(b[g]) < \frac{1}{2}\mu(g) \quad \forall g \in \mathcal{N}. \quad (3.6)$$

For any $g \in \mathcal{N}$ we set

$$\beta = \beta(g) \stackrel{\text{def}}{=} \mu(g) - \min_{\mathcal{B}} \mathcal{F}^g > \frac{1}{2}\mu(g) > 0. \quad (3.7)$$

Then $\beta < \mu(g) < 2\beta$ and we can choose any α such that

$$0 < \beta \leq \mu(g) < \alpha < 2\beta. \quad (3.8)$$

With the above choices for α , β and \mathcal{N} by (3.2)-(3.8) we get:

- $F_{\alpha,\beta}^g(u) \geq 2\beta > \mu(g)$ for any $u \in \mathcal{H}^2$ with $\sharp(S_{\dot{u}} \setminus S_u) > 1$,
- $F_{\alpha,\beta}^g(u) \geq \alpha > \mu(g)$ for any $u \in \mathcal{H}^2$ with $\sharp(S_u) > 0$,
- $\min_{\mathcal{H}^2} F_{\alpha,\beta}^g = \min_{\mathcal{B}} F_{\alpha,\beta}^g = F_{\alpha,\beta}^g(w) = \min_{\mathcal{B}} \mathcal{F}^g + \beta = \mu = \min_{H^2} F_{\alpha,\beta}^g$.

Since $w \notin H^2$, the minimizers of $F_{\alpha,\beta}^g$ over \mathcal{H}^2 are at least two: the minimizers of $F_{\alpha,\beta}^g$ over \mathcal{B} and the unique minimizer of $F_{\alpha,\beta}^g$ over H^2 . \square

4 Free discontinuity set of a minimizer may live outside $S_g \cup S_{\dot{g}}$

Besides the non convexity of $F_{\alpha,\beta}^g$ the following issue is among the main difficulties in the proof of generic uniqueness of minimizers: jump and crease points of a minimizer are not necessarily contained in $S_g \cup S_{\dot{g}}$. Moreover a minimizer u with $S_u \cup S_{\dot{u}} \not\subseteq S_g \cup S_{\dot{g}}$ may occur even with continuous piecewise affine datum g . This issue and the presence of the two parameters α and β instead of one prevents straightforward adaptation of methods used in [2], therefore we will employ different technical arguments in the proof of generic uniqueness of minimizers (see [4]). In this section we give an example of piecewise affine continuous functions exhibiting such phenomenon.

Theorem 4.1 *Define the following family of functions $g \in L^2(0,1)$ dependent on the parameter $a \in \mathbb{R}$*

$$g[a](x) = \left(\left| x - \frac{1}{2} \right| - a \right) \vee 0, \quad x \in [0,1]. \quad (4.1)$$

Then:

$$S_{g[a]} = \emptyset \text{ and } S_{\dot{g}[a]} = \left\{ \frac{1}{2} - a, \frac{1}{2} + a \right\} \quad \forall a \in [0, \frac{1}{2}),$$

$$\exists \alpha, \beta \text{ fulfilling (1.5), } \tilde{a} > 0 \text{ s.t.} \quad (4.2)$$

$$S_u = \emptyset, \quad S_{\dot{u}} \neq \emptyset, \quad S_{\dot{u}} \cap S_{\dot{g}[a]} = \emptyset \quad \forall u \in \operatorname{argmin} F_{\alpha,\beta}^{g[a]} \quad \forall a \in (0, \tilde{a}),$$

so that $\emptyset \neq S_{\dot{u}} \not\subseteq S_g \cup S_{\dot{g}}$ for any $a \in (0, \tilde{a})$.

Moreover either $S_{\dot{u}} = \{\frac{1}{2}\}$ or there is non uniqueness of minimizers for $F_{\alpha,\beta}^{g[a]}$.

Proof. Define $\mathcal{H}^{2,j,c} = \{u \in \mathcal{H}^2 \text{ such that } \sharp(S_u) = j \text{ and } \sharp(S_{\dot{u}} \setminus S_u) = c\}$.

Step 1 - We claim

$$\exists \bar{a} > 0, \alpha, \beta \text{ with (1.5) s.t. } \min_{\mathcal{H}^2} F_{\alpha,\beta}^{g[a]} = \min_{\mathcal{H}^{2,0,1}} F_{\alpha,\beta}^{g[a]} \quad \forall a \in (0, \bar{a}). \quad (4.3)$$

To prove (4.3), we set

$$\mu_1 = \mu_1(a) = \min_{u \in H^2(0,1)} \mathcal{F}^{g[a]}(u),$$

$$\mu_2 = \mu_2(a) = \inf_{u \in \mathcal{H}^{2,0,1}} \mathcal{F}^{g[a]}(u), \quad \mu_3 = \mu_3(a) = \inf_{u \in \mathcal{H}^{2,1,0}} \mathcal{F}^{g[a]}(u),$$

then μ_1 depends continuously on a since the map $a \mapsto g[a]$ is continuous from \mathbb{R} to $L^2([0, 1])$, $m = \mu_1(0) > 0$ since $g[0] = |x - \frac{1}{2}| \in \mathcal{H}^{2,0,1} \setminus H^2(0, 1)$. Moreover

$$0 < \mu_3(a) \leq \mu_2(a) \leq \mu_1(a), \quad (4.4)$$

in fact the first inequality in (4.4) holds true since $g[a]$ does not belong to $H^2(0, 1) \cup \mathcal{H}^{2,0,1} \cup \mathcal{H}^{2,1,0}$, the second inequality holds true by semicontinuity and the fact that for any $u \in \mathcal{H}^{2,0,1}$ there is a sequence $\{u_n\} \subseteq \mathcal{H}^{2,1,0}$ with $S_{u_n} = S_u$ for any n such that $u_n \rightarrow u$ strongly in $H^2((0, 1) \setminus S_u)$, and the last inequality follows from the embedding $H^2 \subseteq \mathcal{H}^{2,0,1}$. Then

$$\lim_{a \rightarrow 0^+} \mu_2(a) = \mu_2(0) = 0 \quad \lim_{a \rightarrow 0^+} \mu_3(a) = \mu_3(0) = 0 \quad (4.5)$$

For any $\eta \in [1, 2)$ we choose $\delta = \delta(a) > (\mu_3 - \frac{\mu_1}{2}) \vee \frac{\mu_3}{\eta} > 0$ and define

$$\alpha = \alpha(a, \delta) = \mu_1 - \mu_3 + \delta, \quad \beta = \beta(a, \eta, \delta) = \frac{\mu_1 - \mu_3 + \eta\delta}{2} \quad (4.6)$$

which will be briefly denoted α and β whenever there is no risk of confusion. Then

$$0 < \beta < \alpha \leq 2\beta, \quad (4.7)$$

$$F_{\alpha, \beta}^{g[a]}(u) \geq \mu_3 + \alpha > \mu_1 \quad \text{for any } u \in \mathcal{H}^{2,1,0}, \quad (4.8)$$

$$F_{\alpha, \beta}^{g[a]}(u) \geq 2\alpha > \mu_1 \quad \text{for any } u \in \mathcal{H}^{2,j,c} \quad \text{with } j > 1, \quad (4.9)$$

$$F_{\alpha, \beta}^{g[a]}(u) \geq 2\beta > \mu_1 \quad \text{for any } u \in \mathcal{H}^{2,j,c} \quad \text{with } c > 1 \text{ or } (j, c) = (1, 1). \quad (4.10)$$

By summarizing (4.7)-(4.10)

$$\left\{ \operatorname{argmin} F_{\alpha, \beta}^{g[a]} \right\} \subseteq H^2(0, 1) \cup \mathcal{H}^{2,0,1}. \quad (4.11)$$

Since $\mu_1 \rightarrow m$ and $\mu_2, \mu_3 \rightarrow 0$ as $a \rightarrow 0$ we can fix η and δ as before and such that $m > \eta\delta$ and choose $\varepsilon \in (0, \frac{1}{6}(m - \eta\delta))$ and \bar{a} such that

$$0 < \mu_3 \leq \mu_2 < \varepsilon, \quad \left| \beta - \frac{1}{2}(m + \eta\delta) \right| < \varepsilon, \quad |m - \mu_1| < \varepsilon \quad \forall a \in (0, \bar{a}). \quad (4.12)$$

Hence inequalities (4.12) entail

$$\mu_2 + \beta - \mu_1 \leq \varepsilon + \frac{1}{2}(m + \eta\delta) + \varepsilon - m + \varepsilon = 3\varepsilon - \frac{1}{2}(m - \eta\delta) < 0,$$

say $\mu_2 + \beta < \mu_1$ for any $a \in (0, \bar{a})$, hence (4.3) follows by (4.11).

Step 2 - We deduce the thesis starting by (4.3) and solving the Euler system of Theorem 2.1 related to one crease point at $x = t \in (0, 1)$ and no jump point.

Consider $b = b[a, t](\cdot) \in \mathcal{H}^2(0, 1)$ fulfilling

$$\left. \begin{aligned} b''''(x) + b(x) &= g[a](x) \quad \text{on } (0, 1) \setminus \{t\}, \\ b''_+(0) &= b''_-(1) = 0, \\ b''_+(t) &= b''_-(t) = 0, \\ b'''_+(0) &= b'''_-(1) = 0, \\ b'''_+(t) &= b'''_-(t), \\ b_+(t) &= b_-(t). \end{aligned} \right\} \quad (4.13)$$

By direct inspection problem (4.13) has a unique solution. We emphasize that problem (4.13) is a particular case of a general differential problem related to multiple jump points and crease points which will be discussed in [4], Theorem 2.8. Then we can define

$$\psi(a, t) = \mathcal{F}^{g[a]}(b[a, t]).$$

Symmetry of $g[a]$ with respect to $\frac{1}{2}$ (say $g[a](x) = g[a](1 - x)$) entails analogous symmetry for the solution of differential problem (4.13):

$$b[a, t](x) = b[a, 1 - t](1 - x) \quad \forall a \in (0, \bar{a}), \quad (4.14)$$

$$\psi(a, t) = \psi(a, 1 - t) \quad \forall a \in (0, \bar{a}), \quad (4.15)$$

$$\psi(a, \frac{1}{2} - a) = \psi(a, \frac{1}{2} + a) \quad \forall a \in (0, \bar{a}). \quad (4.16)$$

Eventually we set $\varphi(a) = \psi(a, \frac{1}{2} - a) - \psi(a, \frac{1}{2})$. Since $\varphi(0) = 0$, if we prove $\varphi'_+(0) > 0$ then for suitable $\tilde{a} \in (0, \bar{a})$ the thesis (4.2) follows.

To establish inequality $\varphi'_+(0) > 0$ we exploit Euler equations and compliance identity and we employ the software Maple[©] as follows (the coded instruction is contained in the appendix): first we use the symbolic computation to find the exact formula for $\psi(a, \frac{1}{2} - a)$, $\psi(a, \frac{1}{2})$ and $\varphi(a)$, then we compute exactly the right total derivative $\varphi'_+(0)$ of φ at $a = 0$, eventually we numerically

compute the value of $\varphi'_+(0)$ with error estimates and get $\varphi'_+(0) > 0$.
The above proof shows only that

$$F_{\alpha,\beta}^{g[a]}(b[a, 1/2]) < F_{\alpha,\beta}^{g[a]}(b[a, 1/2 \pm a]) \quad \forall a \in (0, \tilde{a})$$

but does not entail $b[a, 1/2] \in \operatorname{argmin} F_{\alpha,\beta}^{g[a]}$. Nevertheless, if $b[a, 1/2] \notin \operatorname{argmin} F_{\alpha,\beta}^{g[a]}$, then $u(x)$ and $u(1-x)$ are both minimizers and they do not coincide, since any minimizer must have exactly one crease point. \square

5 Appendix: Symbolic and numeric computations

In this section we provide the Maple[©] procedure used to show that $\varphi'_+(0) > 0$ in the proof of Theorem 4.1.

1. Canonical base of $\ker\left(\frac{d^4}{dt^4} + I\right)$.

```
> w_1(x) := exp(-1/2*sqrt(2)*x)*cos(1/2*sqrt(2)*x);
> w_2(x) := exp(1/2*sqrt(2)*x)*cos(1/2*sqrt(2)*x);
> w_3(x) := exp(-1/2*sqrt(2)*x)*sin(1/2*sqrt(2)*x);
> w_4(x) := exp(1/2*sqrt(2)*x)*sin(1/2*sqrt(2)*x);
```

2. A solution of the homogeneous equation in $[1/2 - a, 1/2 + a]$.

```
> dsolve({diff(d(x),x,x,x,x)+d(x)=0,
> d(1/2-a)=0,D(d)(1/2-a)=-1,
> D(D(d))(1/2-a)=0,D(D(D(d)))(1/2-a)=0});
```

3. Solution of differential system (4.13) with $t = 1/2$, and compliance identity.

```
> w(C_1,C_2,C_3,C_4,x) :=
> C_1*w_1(x)+C_2*w_2(x)+C_3*w_3(x)+C_4*w_4(x);
> dsolve(
> {diff(d(x),x,x,x,x)+d(x)=0,
> d(1/2-a)=0,
> D(d)(1/2-a)=-1,
> D(D(d))(1/2-a)=0,
> D(D(D(d)))(1/2-a)=0});
> solve(
> {eval(diff(w(C_1,C_2,C_3,C_4,x),x,x),x=0)=0,
> eval(diff(w(C_1,C_2,C_3,C_4,x),x,x,x),x=0)=0,
> eval(diff(d(x),x,x),x=1/2)+
```

```

> eval(diff(w(C_1,C_2,C_3,C_4,x),x,x),x=1/2)=0,
> eval(diff(d(x),x,x,x),x=1/2)+
> eval(diff(w(C_1,C_2,C_3,C_4,x),x,x,x),x=1/2)=0},
> {C_1,C_2,C_3,C_4});

> v(x) :=
> C_1*w_1(x)+C_2*w_2(x)+C_3*w_3(x)+C_4*w_4(x);

> ComplianceInTheMiddle(a) :=
> 2*(int((-x+1/2-a)^2,x=0..1/2-a)-
> int((-x+1/2-a)*(-x+1/2-a+v(x)),x=0..1/2-a));

> FirstDerivativeComplianceInTheMiddle :=
> simplify(coeftayl(ComplianceInTheMiddle(a),a=0,1));

```

4. Solution of differential system (4.13) with $t = 1/2 + a$, and compliance identity.

```

> w_0(C_01,C_02,C_03,C_04,x) :=
> C_01*w_1(x)+C_02*w_2(x)+C_03*w_3(x)+C_04*w_4(x);

> w_1(C_11,C_12,C_13,C_14,x) :=
> C_11*w_1(x)+C_12*w_2(x)+C_13*w_3(x)+C_14*w_4(x);

> solve(
> {eval(diff(w_0(C_01,C_02,C_03,C_04,x),x,x),x=0)=0,
> eval(diff(w_0(C_01,C_02,C_03,C_04,x),x,x,x),x=0)=0,
> eval(diff(d(x),x,x),x=1/2+a)+
> eval(diff(w_0(C_01,C_02,C_03,C_04,x),x,x),x=1/2+a)=0,
> eval(diff(d(x),x,x,x),x=1/2+a)+
> eval(diff(w_0(C_01,C_02,C_03,C_04,x),x,x,x),x=1/2+a)=
> eval(diff(w_1(C_11,C_12,C_13,C_14,x),x,x,x),x=1/2+a),
> eval(d(x),x=1/2+a)+eval(w_0(C_01,C_02,C_03,C_04,x),x=1/2+a)=
> eval(w_1(C_11,C_12,C_13,C_14,x),x=1/2+a),
> eval(diff(w_1(C_11,C_12,C_13,C_14,x),x,x),x=1)=0,
> eval(diff(w_1(C_11,C_12,C_13,C_14,x),x,x),x=1/2+a)=0,
> eval(diff(w_1(C_11,C_12,C_13,C_14,x),x,x,x),x=1)=0},
> {C_01,C_02,C_03,C_04,C_11,C_12,C_13,C_14});

> u_0(x) :=
> C_01*w_1(x)+C_02*w_2(x)+C_03*w_3(x)+C_04*w_4(x);
> u_1(x) :=

```

```

> C_11*w_1(x)+C_12*w_2(x)+C_13*w_3(x)+C_14*w_4(x);
> ComplianceRight(a) :=
> 2*int((-x+1/2-a)^2,x=0..1/2-a)-
> int((-x+1/2-a)*(-x+1/2-a+u_0(x)),x=0..1/2-a)-
> int((x-1/2-a)*(x-1/2-a+u_1(x)),x=1/2+a..1);
> FirstDerivativeComplianceRight :=
> coeftayl(ComplianceRight(a),a=0,1);
5. Evaluation of the first derivative for  $a = 0$ .
> FinalEvaluation :=
> evalf(FirstDerivativeComplianceRight-
> FirstDerivativeComplianceInTheMiddle);

```

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