Mathematical Models and Methods in Applied Sciences

PHASE TRANSITIONS WITH THE LINE TENSION EFFECT: THE SUPER-QUADRATIC CASE

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Let Ω be an open bounded set of $\mathbb{R}^3$ and let W and V be two non-negative continuous functions vanishing at α, β and α', β', respectively. We analyze the asymptotic behavior as $\varepsilon \to 0$, in terms of Γ-convergence, of the following functional

$$F_\varepsilon(u) := \varepsilon^{p-2}\int_\Omega |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_\Omega W(u)dx + \frac{1}{\varepsilon}\int_{\partial\Omega} V(Tu)d\mathcal{H}^2 \quad (p>2),$$

where u is a scalar density function and Tu denotes its trace on $\partial\Omega$. We show that the singular limit of the energies F_ε leads to a coupled problem of bulk and surface phase transitions.

Keywords: Phase transitions; Line tension; Γ-convergence; Functions of bounded variation; Nonlocal variational problems

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1. Introduction

In this paper, we study a problem which concerns the analysis of liquid-liquid phase transitions, involving capillarity and line tension contributions.

According to the Van Der Waals-Cahn-Hilliard two-phases model, we consider a fluid, under isothermal conditions, confined to a bounded container $\Omega \subset \mathbb{R}^3$ and described by a mass density u on Ω, which varies continuously from the values α to β, corresponding to the phases $A = \{x \in \Omega : u(x) = \alpha\}$ and $B = \{x \in \Omega : u(x) = \beta\}$. The energy associated with u is the sum of a bulk term $\displaystyle\int_\Omega W(u)dx$, a singular perturbation $\varepsilon^2\displaystyle\int_\Omega |Du|^2dx$, which penalizes the spatial non-homogeneity of the fluid, and a boundary contribution $\lambda\displaystyle\int_{\partial\Omega} V(Tu)d\mathcal{H}^2$, where Tu denotes the trace of u

on $\partial\Omega$, ε is a small parameter giving the characteristic length of the thickness of the interface which separates the two phases, λ is the order of magnitude of the interactions of the fluid with the container wall. This model is also known as "diffuse interface model" for phase transitions. The total energy is given by

$$E^0_\varepsilon(u) = \varepsilon\int_\Omega |Du|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u)dx + \lambda\int_{\partial\Omega} V(Tu)d\mathcal{H}^2, \tag{1.1}$$

where W and V are double-well potentials, with zeroes at $\{\alpha,\beta\}$ and $\{\alpha',\beta'\}$ respectively, and $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

The asymptotic behavior of the rescaled energy (1.1) has been analyzed by several authors, mainly without taking into account the interactions between the fluid and the wall of the container; i.e., the case ($\lambda = 0$): see for instance Refs. 20, 18 and 16, but also Refs. 6 and 8 for more general perturbation terms in (1.1). The case $\lambda = 1$ was treated by Modica[19], while a different behavior for λ was studied by Alberti, Bouchitté and Seppecher in late 90's. Precisely, in Ref. 3, the authors analyzed the asymptotic behavior as ε goes to 0, via De Giorgi's Γ-convergence, of the following family of energies

$$E_\varepsilon(u) := \varepsilon\int_\Omega |Du|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u)dx + \lambda_\varepsilon\int_{\partial\Omega} V(Tu)d\mathcal{H}^2 \quad (u\in H^1(\Omega)), \tag{1.2}$$

where λ_ε satisfies

$$\varepsilon\log\lambda_\varepsilon \to k \in (0,+\infty) \quad \text{as } \varepsilon\to 0. \tag{1.3}$$

This logarithmic scaling provides a uniform control on the oscillation of the traces of minimizing sequences u_ε, and ensures that the transition of Tu_ε from α' to β' takes place in a thin layer. In fact, Alberti, Bouchitté and Seppecher proved that, under the assumption (1.3), the traces Tu_ε converge (up to a subsequence) to a function v in $BV(\partial\Omega,\{\alpha',\beta'\})$ and then the boundary phases $\{v=\alpha'\}$ and $\{v=\beta'\}$ are divided by the jump set Sv; i.e., the complement of the set of Lebesgue points of v[4]. Sv is usually referred to as "dividing line" and in general does not agree with the "contact line", that is the line where the interface between the bulk phases $\{u=\alpha\}$ and $\{u=\beta\}$ meets the boundary of Ω (see Ref. 3, Example 5.2; see also Fig.1).

Namely, the asymptotic behavior of E_ε is described by a functional Ψ which depends on the two variables u and v:

$$\Psi(u,v){:=}\sigma\mathcal{H}^2(Su) + \int_{\partial\Omega}|\tilde H(Tu)-\tilde H(v)|d\mathcal{H}^2 + \tau\mathcal{H}^1(Sv),$$
$$\forall\ (u,v)\in BV(\Omega,\{\alpha,\beta\})\times BV(\partial\Omega,\{\alpha',\beta'\}), \tag{1.4}$$

where $\sigma := |\tilde H(\beta)-\tilde H(\alpha)|$, being $\tilde H$ a primitive of $2W^{1/2}$ and $\tau := \dfrac{k}{\pi}(\beta'-\alpha')^2$.

The essence of this paper is the extension of the analysis by Alberti, Bouchitté and Seppecher, considering singular perturbation terms with super-quadratic growth. For $p>2$, we consider the family

$$F_\varepsilon(u) := \varepsilon^{p-2}\int_\Omega |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_\Omega W(u)dx + \frac{1}{\varepsilon}\int_{\partial\Omega} V(Tu)d\mathcal{H}^2, \tag{1.5}$$

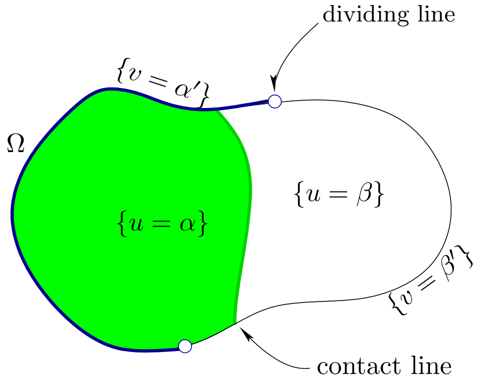

Fig. 1. Dissociation of the contact line and the dividing line.

where ε is chosen to denote the length of the boundary transition, and the other powers follow naturally by standard scaling analysis (see for instance Ref. 21, Section 2.3).

We note that the variation of the power of the gradient in the perturbation term is not a simple generalization with respect to the quadratic case. The reason is that, in the quadratic case, the natural logarithmic scaling of the energy makes the profile of the transition on the boundary irrelevant; it does not produce the so-called "equi-partition of the energy". On the contrary, such profile becomes crucial in the super-quadratic case, when equi-partition of energy occurs. This new feature will be a double-edged sword in the proof of the Γ-convergence result: some arguments will be simplified by the presence of an optimal profile problem, some other will require more care. For a detailed discussion in this direction, see Section 3 and Section 4.

Note that this article is purely variational, but some properties of the (interior) Euler-Lagrange equations of the functional have already been studied (see Refs. 23 and 24). Moreover, for the case $p = 2$, the optimal profile for the boundary transition has been studied in Ref. 10; while it is probably difficult to adapt this analysis to the more nonlinear case of the p-Laplacian.

Finally, it is worth pointing out that several open questions arise. First, when $p \in (1, 2)$, we still expect some results of Γ-convergence. Nevertheless, it is not clear what the form of the Γ-limit could be as some non-local geometric quantities will appear. Second, it could be interesting to analyze the asymptotic behavior of our functionals taking into account of the presence of an additional singular weight in the perturbation term. In this direction, there are the works in preparation by Gonzalez[15] for the case $p = 2$ and by Sire and the author[22] for the case $p > 2$. Third, not much is known in the anisotropic case.

The outline of this paper is the following. In Section 2 we fix the notation and discuss the asymptotic behavior of the functional F_ε. Since the proof of the Γ-convergence result for the functional F_ε requires several steps, in Section 3 we describe the adopted strategy and state some preliminary results. Section 4 and 5 are devoted to the proofs of the mathematical results stated in Section 2.

2. Description of the results

First, we fix the notation, also recalling some standard mathematical results used throughout the paper. Then, we analyze the asymptotic behavior of the functional F_ε defined in (1.5) stating the related main convergence result.

2.1. *Notation*

In this work, we consider different domains A with dimensions $N = 1, 2, 3$; more precisely, A will always be a bounded open set either of $\mathbb{R}^N$. We denote by ∂A the boundary of A relative to the ambient manifold; ∂A is always assumed to be Lipschitz regular. Unless otherwise stated, A is endowed with the corresponding N-dimensional Hausdorff measure, $\mathcal{H}^N$ (see Ref. 11, Chapter 2). We write $\int_A f dx$ instead of $\int_A f d\mathcal{H}^N$, and $|A|$ instead of $\mathcal{H}^N(A)$.

The N-dimensional density of A at x is the limit (if it exists) of $\mathcal{H}^N(A \cap B_r(x))/\omega_N r^N$, where $B_r(x)$ is the ball centered at x with radius r and ω_N is the measure of the unit ball in $\mathbb{R}^N$.

The *essential boundary* of A is the set of all points where A has neither density 0 nor 1 and where the density does not exist. Since the essential boundary agrees with the topological boundary when the latter is Lipschitz regular, we also denote the essential boundary by ∂A.

For every $u \in L^1_{\rm loc}(A)$, we denote by Du the derivative of u in the sense of distributions. As usual, for every $p \geq 1$, $W^{1,p}(A)$ is the Sobolev space of all $u \in L^p(A)$ such that $Du \in L^p(A)$; $BV(A)$ is the space of all $u \in L^1(A)$ with bounded variation; i.e., such that Du is a bounded Borel measure on A. We denote by Su the *jump set*; i.e., the complement of the set of Lebesgue points of u.

For every $s \in (0,1)$ and every $p \geq 1$, $W^{s,p}$ is the space of all $u \in L^p(A)$ such that the fractional semi-norm $\displaystyle\iint_{A\times A} \frac{|u(x)-u(x')|^p}{|x-x'|^{sp+N}} dx dx'$ is finite.

We denote by T the trace operator which maps $W^{1,p}(A)$ onto $W^{1-1/p,p}(\partial A)$ and $BV(A)$ onto $L^1(\partial A)$. For details and results about the theory of BV functions and Sobolev spaces we refer to Refs. 11, 4 and 1.

2.2. *The* Γ*-convergence result*

Let Ω be a bounded open subset of $\mathbb{R}^3$ with smooth boundary; let W and V be non-negative continuous functions on $\mathbb{R}$ with growth at least linear at infinity and vanishing respectively only in the "double well" $\{\alpha,\beta\}$, with $\alpha<\beta$, and $\{\alpha',\beta'\}$, with $\alpha'<\beta'$. We assume in addition that the potential V is convex near its wells. Let $p>2$ be a real number, for every $\varepsilon>0$, we consider the functional F_ε defined in $W^{1,p}(\Omega)$, given by

$$F_\varepsilon(u):=\varepsilon^{p-2}\int_\Omega |Du|^p dx+\frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_\Omega W(u)dx+\frac{1}{\varepsilon}\int_{\partial\Omega}V(Tu)d\mathcal{H}^2. \tag{2.1}$$

We analyze the asymptotic behavior of the functional F_ε in terms of Γ-convergence. Let (u_ε) be an equi-bounded sequence for F_ε; i.e., there exists a constant C such that $F(u_\varepsilon)\leq C$. We observe that the "boundedness" of the term $\frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_\Omega W(u_\varepsilon)dx$ implies that, when ε tends to 0, the sequence (u_ε) converges (up to a subsequence) to a function u which takes only the values α and β. The limit function u belongs to $BV(\Omega;\{\alpha,\beta\})$, since the term $\varepsilon^{p-2}\int_\Omega |Du_\varepsilon|^p dx$ penalizes the oscillations of u_ε, forbidding u to oscillate insanely between α and β. Moreover each u_ε has a transition from the value α to the value β in a thin layer close to the surface Su, which separates the bulk phases $\{u=\alpha\}$ and $\{u=\beta\}$. Similarly, the boundary term of F_ε forces the traces Tu_ε to take values close to α' and β', and the oscillations of the traces Tu_ε are again penalized by the integral $\varepsilon^{p-2}\int_\Omega |Du_\varepsilon|^p dx$. Then, we expect that the sequence (Tu_ε) converges to a function v in $BV(\partial\Omega)$ which takes only the values α' and β', and that a concentration of energy occurs along the line Sv, which separates the boundary phases $\{v=\alpha'\}$ and $\{v=\beta'\}$.

In view of possible "dissociation of the contact line and the dividing line", we recall that Tu may differ from v. Since the total energy $F_\varepsilon(u_\varepsilon)$ is partly concentrated in a thin layer close to Su (where u_ε has a transition from α to β), partly in a thin layer close to the boundary (where u_ε has a transition from Tu to v), and partly in the vicinity of Sv (where Tu_ε has a transition from α' to β'), we expect that the limit energy is the sum of a surface energy concentrated on Su, a boundary energy on $\partial\Omega$ (with density depending on the gap between Tu and v), and a line energy concentrated along Sv.

The asymptotic behavior of the functional F_ε is described by a functional Φ which depends on the two variables u and v. Let $\mathcal{W}$ be a primitive of $W^{(p-1)/p}$. For every $(u,v)\in BV(\Omega,\{\alpha,\beta\})\times BV(\partial\Omega,\{\alpha',\beta'\})$, we will prove that

$$\Phi(u,v):=\sigma_p\mathcal{H}^2(Su)+c_p\int_{\partial\Omega}|\mathcal{W}(Tu)-\mathcal{W}(v)|d\mathcal{H}^2+\gamma_p\mathcal{H}^1(Sv), \tag{2.2}$$

where as usual the jump sets Su and Sv are the complement of the set of Lebesgue points of u and v, respectively; c_p and σ_p are the constants defined by $c_p:=\dfrac{p}{(p-1)^{p/(p-1)}}$; $\sigma_p:=c_p|\mathcal{W}(\beta)-\mathcal{W}(\alpha)|$;

γ_p is given by the optimal profile problem

$$\gamma_p := \inf \left\{ \int_{\mathbb{R}^2_+} |Du|^p dx + \int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 : u \in L^1_{\rm loc}(\mathbb{R}^2_+) : \right. \\ \left. \lim_{t\to-\infty} Tu(t) = \alpha', \ \lim_{t\to+\infty} Tu(t) = \beta' \right\}. \tag{2.3}$$

The main convergence result is precisely stated in the following theorem.

Theorem 2.1. *Let $F_\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ and $\Phi : BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\}) \to \mathbb{R}$ defined by (2.1) and (2.2).*
Then

(i) [Compactness] *If $(u_\varepsilon) \subset W^{1,p}(\Omega)$ is a sequence such that $F_\varepsilon(u_\varepsilon)$ is bounded, then $(u_\varepsilon, Tu_\varepsilon)$ is pre-compact in $L^1(\Omega) \times L^1(\partial\Omega)$ and every cluster point belongs to $BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$.*

(ii) [Lower Bound Inequality] *For every $(u, v) \in BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$,*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v).$$

(iii) [Upper Bound Inequality] *For every $(u, v) \in BV(\Omega, \{\alpha,\beta\}) \times BV(\partial\Omega, \{\alpha',\beta'\})$ there exists a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$ and*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \Phi(u, v).$$

We can easily rewrite this theorem in term of Γ-convergence. To this aim, we extend each F_ε to $+\infty$ on $L^1(\Omega)\backslash W^{1,p}(\Omega)$ and, from Theorem 2.1, we briefly deduce the following remark.

Remark 2.1. F_ε Γ-converges on $L^1(\Omega)$ to F, given by

$$F(u) := \begin{cases} \inf \{\Phi(u, v) : v \in BV(\partial\Omega, \{\alpha',\beta'\})\} & \text{if } u \in BV(\Omega, \{\alpha,\beta\}), \\ +\infty & \text{elsewhere in } L^1(\Omega). \end{cases}$$

3. Strategy of the proof and some preliminary results

In this section, we briefly describe the adopted strategy in the proof of Theorem 2.1. We also state some convergence results (see Theorem 3.1 and Proposition 3.1) without providing proofs, but only short comments and references.

The proof of Theorem 2.1 requires several steps in which we have to analyze different effects. Then, we can deduce the terms of the limit energy Ψ, localizing three effects: the bulk effect, the wall effect and the boundary effect.

In the bulk term, the limit energy can be evaluated like in Ref. 18. Of course, we will use the super-quadratic version of the Modica-Mortola functional.

For every open set $A \subset \mathbb{R}^3$ and every real function $u \in W^{1,p}(A)$, we consider the functional

$$G_\varepsilon(u, A) := \varepsilon^{p-2} \int_A |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u) dx. \tag{3.1}$$

The energies G_ε Γ-converge in $L^1(A)$ to the functional $\sigma_p \mathcal{H}^2(Su)$, as stated in the following theorem, which can be proved thanks to simple modifications to the proof of the Modica-Mortola Theorem by Modica[18] (see also Ref. 19, Theorem 3.10).

Theorem 3.1. *For every domain $A \subset \mathbb{R}^3$ the following statements hold.*

(i) If $(u_\varepsilon) \subset W^{1,p}(A)$ is a sequence with uniformly bounded energies $G_\varepsilon(u_\varepsilon, A)$. Then (u_ε) is pre-compact in $L^1(A)$ and every cluster point belongs to $BV(A, \{\alpha, \beta\})$.

(ii) For every $u \in BV(A, \{\alpha, \beta\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$,

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) \geq \sigma_p \mathcal{H}^2(Su),$$

(iii) For every $u \in BV(A, \{\alpha, \beta\})$ there exists a sequence $(u_\varepsilon) \subset W^{1,p}(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ and

$$\limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) \leq \sigma_p \mathcal{H}^2(Su).$$

Moreover, when Su is a closed Lipschitz surface in A, the functions u_ε may be required to be $(C_W/\varepsilon^{\frac{p-2}{p-1}})$-Lipschitz continuous, and to converge to u uniformly on every set with positive distance from Su (here C_W is the supremum of $W^{1/p}$ in $[\alpha, \beta]$).

The second term of Φ can be obtained by the following proposition, adapting the approach by Modica in Ref. 19.

Proposition 3.1. *For every domain $A \subset \mathbb{R}^3$ with boundary piecewise of class C^1 and for every $A' \subset \partial A$ with Lipschitz boundary, the following statements hold.*

(i) For every $(u, v) \in BV(A, \{\alpha, \beta\}) \times BV(A', \{\alpha', \beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ and $Tu_\varepsilon \to v$ in $L^1(A')$,

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) \geq c_p \int_{A'} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2.$$

(ii) Let a function v, constant on A', and a function u, constant on A, such that $u \equiv \alpha$ or $u \equiv \beta$, be given. Then there exists a sequence (u_ε) such that $Tu_\varepsilon = v$ on A', u_ε converges uniformly to u on every set with positive distance from A' and

$$\limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) \leq c_p \int_{A'} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2.$$

Moreover, the function u_ε may be required to be $\frac{C'_W}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous (where C'_W is the maximum of $W^{1/p}$ over any interval which contains the values of u and v).

We recall that the results by Modica[19] concern a Cahn-Hilliard functional with quadratic growth in the singular perturbation term and with a boundary contribution of the form $\lambda \int_{\partial\Omega} g(Tu) d\mathcal{H}^2$, with λ not depending on ε and g a positive continuous function. Hence, we needed to adapt part of Proposition 1.2 and Proposition 1.4 in Ref. 19 to our goal (see Ref. 21, Proposition 4.3, for details).

Finally, the boundary effect. This is a delicate step, that requires a deeper analysis. The main strategy is the following: we reduce to the case in which the boundary is flat; hence we study the behavior of the energy in the three-dimensional half ball; then we reduce the problem to two dimension via a slicing argument.

Thus, the main problem becomes the analysis of the asymptotic behavior of the following two-dimensional functional, combining the interior p-energy with the boundary term:

$$H_\varepsilon(u) := \varepsilon^{p-2} \int_{D_1} |Du|^p dx + \frac{1}{\varepsilon} \int_{E_1} V(Tu) d\mathcal{H}^1, \ \forall u \in W^{1,p}(D_1), \tag{3.2}$$

where D_1 and E_1 are defined by

$$D_r := \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2, x_2 > 0 \right\},$$
$$E_r := \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2, x_2 = 0 \right\} \equiv (-r, r). \tag{3.3}$$

Note that for the case $p = 2$ the two-dimensional Dirichlet energy (3.2) can be replaced on the half-disk D_r by the $H^{1/2}$ intrinsic norm on the "diameter" E_r. This is possible thanks to the existence of an optimal constant for the trace inequality involving the L^2 norm of the gradient of a function defined on a two-dimensional domain and the $H^{1/2}$ norm of its trace on a line (see Ref. 3, Lemma 6.2 and Corollary 6.4). Hence, when $p = 2$, the analysis of the line tension effect is reduced to the one of a one-dimensional perturbation problem involving a non-local term:

$$E^1_\varepsilon(v) = \varepsilon \iint_{I \times I} \left| \frac{v(t) - v(t')}{t - t'} \right|^2 dt' dt + \lambda_\varepsilon \int_I V(v) dt,$$

where I is an open interval of $\mathbb{R}$ and λ_ε satisfies the condition (1.3).

Here, we have to analyze the asymptotic behavior of the functional H_ε, that is one of the main contribute of this paper; it will be the subject of Section 4.

3.1. *Some remarks about the structure of F_ε*

The methods used in the proof strongly require the "localization" of the functional F_ε; i.e., looking at F_ε as a function of sets. By fixing u we will be able to characterize the various effects of the problem, in the spirit of the classical "blow-up" method, developed by Fonseca and Müller[12]. In this sense, for every open set $A \subset \mathbb{R}^3$, every set $A' \subseteq \partial A$ and every function $u \in W^{1,p}(A)$, we will denote

$$F_\varepsilon(u, A, A') := \varepsilon^{p-2} \int_A |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u)dx + \frac{1}{\varepsilon} \int_{A'} V(Tu) d\mathcal{H}^2. \qquad (3.4)$$

Clearly, $F_\varepsilon(u) = F_\varepsilon(u, \Omega, \partial\Omega)$ for every $u \in W^{1,p}(\Omega)$.

Let us observe that, thanks to the growth hypothesis on the potentials W and V, we may assume that there exists a constant m such that:

$$\begin{gathered} -m \leq \alpha, \alpha', \beta, \beta' \leq m, \\ W(t) \geq W(m) \text{ and } V(t) \geq V(m) \text{ for } t \geq m, \\ W(t) \geq W(-m) \text{ and } V(t) \geq V(-m) \text{ for } t \leq -m. \end{gathered} \qquad (3.5)$$

In particular, assumption (3.5) will allow us to use the truncation argument given by the following Lemma.

Lemma 3.1. *Let a domain $A \subset \mathbb{R}^3$, a set $A' \subseteq \partial A$, and a sequence $(u_\varepsilon) \subset W^{1,p}(A)$ with uniformly bounded energies $F_\varepsilon(u_\varepsilon, A, A')$ be given. If we set $\bar{u}_\varepsilon(x) := \max\{\min\{u_\varepsilon(x), m\}, -m\}$, then*

(i) $F_\varepsilon(\bar{u}_\varepsilon, A, A') \leq F_\varepsilon(u_\varepsilon, A, A')$,
(ii) $\|\bar{u}_\varepsilon - u_\varepsilon\|_{L^1(A)}$ *and* $\|T\bar{u}_\varepsilon - Tu_\varepsilon\|_{L^1(A')}$ *vanish as* $\varepsilon \to 0$.

Proof. The inequality $F_\varepsilon(\bar{u}_\varepsilon, A, A') \leq F_\varepsilon(u_\varepsilon, A, A')$ follows immediately from (3.5). Statement (ii) follows from the fact that both W and V have growth at least linear at infinity and the integrals $\int W(u_\varepsilon)dx$ and $\int V(Tu_\varepsilon)d\mathcal{H}^2$ vanish as ε goes to 0. This is a standard argument; see, for instance, Lemma 4.4 in Ref. 21. □

4. Recovering the "contribution of the wall": the flat case

We will obtain "the contribution of the wall" to the limit energy Φ, defined by (2.2), namely $\gamma_p \mathcal{H}^1(Sv)$, by estimating the asymptotic behavior of the functional

$$F_\varepsilon(u, B\cap\Omega, B\cap\partial\Omega) = \varepsilon^{p-2} \int_{B\cap\Omega} |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_{B\cap\Omega} W(u)dx + \int_{B\cap\partial\Omega} V(Tu)d\mathcal{H}^2,$$

when B is a small ball centered on $\partial\Omega$ and $B \cap \partial\Omega$ is a flat disk. We will follow the idea of Alberti, Bouchitté and Seppecher[3], using a suitable slicing argument. Later on we will show that the flatness assumption on $B \cap \partial\Omega$ can be dropped when B is sufficiently small. Hence, we need to prove a compactness result and a lower bound inequality for the following two-dimensional functional

$$H_\varepsilon(u) := \varepsilon^{p-2} \int_{D_1} |Du|^p dx + \frac{1}{\varepsilon} \int_{E_1} V(Tu) d\mathcal{H}^1, \ \forall u \in W^{1,p}(D_1; [-m, m]), \qquad (4.1)$$

where E_r and D_r are defined by (3.3). We recall that we will always study H_ε like a reduction of F_ε. Hence there will be some hypotheses inherited by this reduction. In particular, the hypothesis $u \in [-m, m]$ in (4.1) is justified by Lemma 3.1.

Let us introduce the "localization" of the functional H_ε. For every open set $A \subset \mathbb{R}^2$, every set $A' \subset \partial A$ and every function $u \in W^{1,p}(A)$, we will denote

$$H_\varepsilon(u, A, A') := \varepsilon^{p-2} \int_A |Du|^p dx + \frac{1}{\varepsilon} \int_{A'} V(Tu) d\mathcal{H}^1. \tag{4.2}$$

If we set $u^{(\varepsilon)}(x) := u(x/\varepsilon)$ and $A/\varepsilon := \{x : \varepsilon x \in A\}$, by scaling it is immediately seen that

$$H_\varepsilon(u^{(\varepsilon)}, A, A') = H_1(u, A/\varepsilon, A'/\varepsilon). \tag{4.3}$$

In view of this scaling property, we consider the optimal profile problem, introduced in the Section 2.2; that is,

$$\gamma_p = \inf \left\{ \int_{\mathbb{R}^2_+} |Du|^p dx + \int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 : u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+) : \lim_{t \to -\infty} Tu(t) = \alpha', \ \lim_{t \to +\infty} Tu(t) = \beta' \right\} \tag{4.4}$$

and determines the line tension on the limit energy Φ.

Note that γ_p is the infimum of the "non-scaled" energy H_1 among every function with the trace that has a transition from α' to β' on the whole real line. We will show that this set of functions is non-empty (see the construction of good competitors for γ_p in Section 4.2). Moreover, in Section 4.4 we will prove that the minimum for γ_p is achieved.

4.1. *Compactness of the traces*

We prove the pre-compactness of the traces of the sequences equi-bounded for H_ε, using the trace imbedding of $W^{1,p}(D_1)$ in $W^{1-1/p,p}(\partial D_1)$ and the following lemma, which is an adaptation of Lemma 1 in Ref. 2, using the estimations in Lemma 4.1 in Ref. 13.

Lemma 4.1. *Let $(u_\varepsilon) \subset W^{1,p}(D_1)$ and let $J \subset E_1$ be an open interval. For every δ such that $0 < \delta < (\beta' - \alpha')/2$, let us define*

$$A_\varepsilon := \{x \in E_1 : Tu_\varepsilon(x) \le \alpha' + \delta\} \text{ and } B_\varepsilon := \{x \in E_1 : Tu_\varepsilon(x) \ge \beta' - \delta\}.$$

Let us set

$$a_\varepsilon := \frac{|A_\varepsilon \cap J|}{|J|} \text{ and } b_\varepsilon := \frac{|B_\varepsilon \cap J|}{|J|}. \tag{4.5}$$

Then

$$H_\varepsilon(u_\varepsilon, D_1, J) \ge \frac{2C_p(\beta - \alpha - 2\delta)^p}{(p-1)(p-2)|J|^{p-2}} \left(1 - \frac{1}{(1-a_\varepsilon)^{p-2}} - \frac{1}{(1-b_\varepsilon)^{p-2}}\right) \varepsilon^{p-2} - \frac{\pi}{2} m^p \varepsilon^{p-2} + C_\delta, \tag{4.6}$$

where C_p and C_δ are positive constants do not depending on ε.

Proof. By the Sobolev imbedding of $W^{1,p}(D_1)$ in $W^{1-1/p,p}(\partial D_1)$, there exists a constant C_p such that for every $u \in W^{1,p}(D_1)$

$$\|Tu\|_{W^{1-1/p,p}(\partial D_1)} \leq C_p \|u\|_{W^{1,p}(D_1)}.$$

It follows that there exists a constant (still denoted by C_p) such that

$$\int_{D_1} |Du|^p dx \geq C_p \iint_{J\times J} \frac{|Tu(t)-Tu(t')|^p}{|t-t'|^p} dt'dt - \int_{D_1} |u|^p dx, \quad \forall u \in W^{1,p}(D_1).$$

Hence

$$\begin{aligned} H_\varepsilon(u_\varepsilon, D_1, J) &= \varepsilon^{p-2} \int_{D_1} |Du_\varepsilon|^p dx + \frac{1}{\varepsilon}\int_J V(Tu_\varepsilon) d\mathcal{H}^1 \\ &\geq C_p \varepsilon^{p-2} \iint_{J\times J} \frac{|Tu_\varepsilon(t)-Tu_\varepsilon(t')|^p}{|t-t'|^p} dt'dt - \frac{\pi}{2} m^p \varepsilon^{p-2} \\ &\quad + \frac{1}{\varepsilon}\int_J V(Tu_\varepsilon) d\mathcal{H}^1. \end{aligned} \tag{4.7}$$

Now, let $a_0, b_0 \in \mathbb{R}$ be such that $J = (a_0, b_0)$; we denotes by $(Tu_\varepsilon)^*$ the increasing rearrangement of Tu_ε in $[a_0, b_0]$

$$Tu_\varepsilon^*(a_0 + x) := \sup\{\lambda : |\{t : Tu_\varepsilon(t) < \lambda\}| \leq x\}, \quad \forall x \in [0, b_0 - a_0]. \tag{4.8}$$

Since the non-local energy decreases under monotone rearrangements (see Ref. 14, Theorem I.1), we obtain

$$\begin{aligned} \varepsilon^{p-2} \iint_{J\times J} \frac{|Tu_\varepsilon(t)-Tu_\varepsilon(t')|^p}{|t-t'|^p} dt'dt &\geq \varepsilon^{p-2} \iint_{J\times J} \frac{|(Tu_\varepsilon)^*(t)-(Tu_\varepsilon)^*(t')|^p}{|t-t'|^p} dt'dt \\ &\geq 2\varepsilon^{p-2}(\beta'-\alpha'-2\delta)^p \int_{a_0}^{a_0+a_\varepsilon|J|} \int_{b_0-b_\varepsilon|J|}^{b_0} \frac{dt'dt}{|t-t'|^p} \\ &= \frac{2\varepsilon^{p-2}(\beta'-\alpha'-2\delta)^p}{(p-1)(p-2)|J|^{p-2}} \Bigg(1 - \frac{1}{(1-a_\varepsilon)^{p-2}} \\ &\quad - \frac{1}{(1-b_\varepsilon)^{p-2}} + \frac{1}{(1-a_\varepsilon-b_\varepsilon)^{p-2}}\Bigg). \end{aligned} \tag{4.9}$$

Denoting by ω_δ the minimum of V on $[\alpha'+\delta, \beta'-\delta]$, by (4.7) and (4.9), we obtain

$$\begin{aligned} H_\varepsilon(u_\varepsilon, D_1, J) \geq \frac{2C_p(\beta'-\alpha'-2\delta)^p}{(p-1)(p-2)|J|^{p-2}} &\Bigg(1 - \frac{1}{(1-a_\varepsilon)^{p-2}} - \frac{1}{(1-b_\varepsilon)^{p-2}} \\ &+ \frac{1}{(1-a_\varepsilon-b_\varepsilon)^{p-2}}\Bigg) \varepsilon^{p-2} - \frac{\pi}{2} m^p \varepsilon^{p-2} + \frac{C_p \omega_\delta}{\varepsilon} |J|(1-a_\varepsilon-b_\varepsilon). \end{aligned}$$

Minimizing with respect to $|J|(1-a_\varepsilon-b_\varepsilon)$, we get

$$H_\varepsilon(u_\varepsilon,D_1,J)\geq \frac{2C_p(\beta'-\alpha'-2\delta)^p}{(p-1)(p-2)|J|^{p-2}}\left(1-\frac{1}{(1-a_\varepsilon)^{p-2}}-\frac{1}{(1-b_\varepsilon)^{p-2}}\right)\varepsilon^{p-2}$$
$$-\frac{\pi}{2}m^p\varepsilon^{p-2}+2^{\frac{1}{p-1}}\frac{(p-1)^{\frac{p-2}{p-1}}}{p-2}(\beta'-\alpha'-2\delta)^{\frac{p}{p-1}}C_p\omega^{\frac{p-2}{p-1}},$$

for every $0<\delta<(\beta'-\alpha')/2$, and hence (4.6) is proved. □

We are now in position to prove the compactness result stated in the following proposition.

Proposition 4.1. *If* $(u_\varepsilon)\subset W^{1,p}(D_1;[-m,m])$ *is a sequence such that* $H_\varepsilon(u_\varepsilon)$ *is bounded then* (Tu_ε) *is pre-compact in* $L^1(E_1)$ *and every cluster point belongs to* $BV(E_1,\{\alpha',\beta'\})$.

Proof. By hypothesis, there exists a constant C such that $H_\varepsilon(u_\varepsilon)\leq C$. In particular

$$\int_{E_1}V(Tu_\varepsilon)d\mathcal{H}^1\leq C\varepsilon$$

and this implies that

$$V(Tu_\varepsilon)\to 0 \text{ in } L^1(E_1). \tag{4.10}$$

Thanks to the growth assumptions on V, (Tu_ε) is equi-integrable. Hence, by Dunford-Pettis' Theorem, (Tu_ε) is weakly relatively compact in $L^1(E_1)$; i.e., there exists $v\in L^1(E_1)$ such that (up to subsequences) $Tu_\varepsilon\rightharpoonup v$ in $L^1(E_1)$.

We have to prove that this convergence is strong in $L^1(E_1)$ and that $v\in BV(E_1,\{\alpha',\beta'\})$. This proof is standard, involving Young measures associated with sequences (see also Ref. 2, Théorème 1-(i)).

Let ν_x be the Young measure associated with (Tu_ε). Since V is a non negative continuous function in $\mathbb{R}$, we have

$$\int_{E_1}\int_{\mathbb{R}}V(t)d\nu_x(t)\leq\liminf_{\varepsilon\to0}\int_{E_1}V(Tu_\varepsilon)dx.$$

(see Ref. 25, Theorem I.16)

Hence, by (4.10), it follows that

$$\int_{\mathbb{R}}V(t)d\nu_x(t)=0,\quad \text{a.e. } x\in E_1,$$

which implies the existence of a function θ on $[0,1]$ such that

$$\nu_x(dt)=\theta(x)\delta_{\alpha'}(dt)+(1-\theta(x))\delta_{\beta'}(dt),\quad x\in E_1$$

and

$$v(x)=\theta(x)\alpha'+(1-\theta(x))\beta',\quad x\in E_1.$$

It remains to prove that θ belongs to $BV(E_1, \{0,1\})$. Let us consider the set S of the points where approximate limits of θ is neither 0 nor 1. For every $N \leq \mathcal{H}^0(S)$ we can find N disjoint intervals $\{J_n\}_{n=1,\dots,N}$ such that $J_n \cap S \neq \emptyset$ and such that the quantities a^n_ε and b^n_ε, defined by (4.5) replacing J by J_n, satisfy

$$a^n_\varepsilon \to a^n \in (0,1) \quad \text{and} \quad b^n_\varepsilon \to b^n \in (0,1) \quad \text{as } \varepsilon \text{ goes to zero.}$$

We can now apply Lemma 4.1 in the interval J_n and, taking the limit as $\varepsilon \to 0$ in the inequality (4.6), we obtain

$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon, D_1, J_n) \geq c_\delta.$$

Finally, we use the sub-additivity of the non-local functional and we get

$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon, D_1, E_1) \geq \sum_{n=1}^{N} \liminf H_\varepsilon(u_\varepsilon, D_1, J_n) \geq N c_\delta.$$

Since (u_ε) has equi-bounded energy, this implies that S is a finite set. Hence, $\theta \in BV(E_1, \{0,1\})$ and the proof of the compactness for H_ε is complete. □

4.2. *Lower bound inequality*

We will prove an optimal lower bound for H_ε.

Proposition 4.2. *For every (u,v) in $BV(D_1, \{\alpha,\beta\}) \times BV(E_1, \{\alpha',\beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(D_1; [-m,m])$ such that $u_\varepsilon \to u$ in $L^1(D_1)$ and $Tu_\varepsilon \to v$ in $L^1(E_1)$*

$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon) \geq \gamma_p \mathcal{H}^0(Sv). \tag{4.11}$$

Proof. We will prove the lower bound inequality (4.11) for v such that

$$v(t) = \begin{cases} \alpha', & \text{if } t \in (-1,0], \\ \beta', & \text{if } t \in (0,1). \end{cases}$$

Let us consider the natural extension of v to the whole real line $\mathbb{R}$ (still denoted by v); i.e.,

$$v(t) = \begin{cases} \alpha', & \text{if } t \leq 0, \\ \beta', & \text{if } t > 0. \end{cases}$$

Step 0: Strategy of the proof. We are looking for an extension of u_ε to the whole $\mathbb{R}^2_+$, namely w_ε, such that w_ε is a competitor for (4.4) and $H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+, \mathbb{R}) \simeq H_\varepsilon(u_\varepsilon, D_1, E_1)$ as $\varepsilon \to 0$ in a precise sense. More exactly, we will able to find

$s<1$ and we will construct a competitor w_ε such that, for any given $\delta>0$ there exists $\varepsilon_\delta>0$ such that

$$\begin{aligned} H_\varepsilon(u_\varepsilon) &\geq H_\varepsilon(u_\varepsilon, D_s, E_s) \\ &= H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+, \mathbb{R}) - H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+\setminus D_s, \mathbb{R}\setminus E_s) \\ &\geq \gamma_p - \delta, \quad \forall \varepsilon \leq \varepsilon_\delta. \end{aligned}$$

Step 1: Construction of the competitor. For every $s>0$, we define the harmonic extension of v from $\mathbb{R}\setminus E_s$ to $\mathbb{R}^2_+\setminus D_s$, namely $\bar{u}$, defined in polar coordinates by

$$\bar{u}(\rho,\theta) := \frac{\theta}{\pi}\alpha' + \left(1-\frac{\theta}{\pi}\right)\beta', \quad \forall \theta\in[0,\pi),\ \forall \rho \geq s.$$

We will construct the competitor w_ε simply gluing the function $\bar{u}$ and the function u_ε. Hence, for every $\varepsilon>0$, we consider the cut-off function φ in $C^\infty(\mathbb{R}^2_+)$, such that

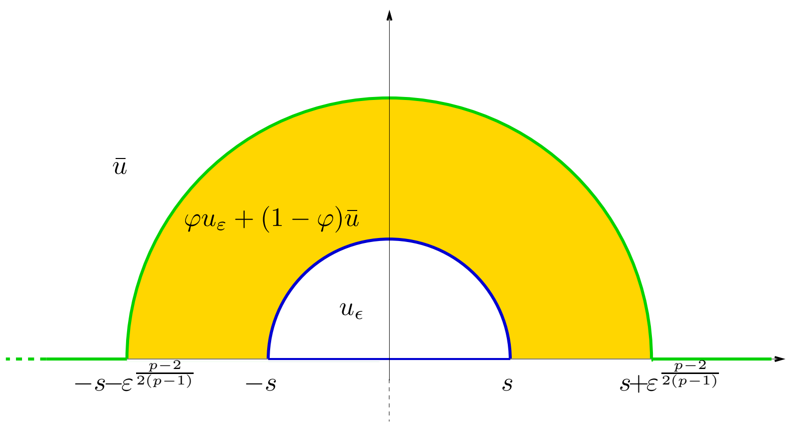

Fig. 2. The competitor w_ε.

$\varphi\equiv 1$ in D_s, $\varphi\equiv 0$ in $\mathbb{R}^2_+\setminus D_{s(\varepsilon)}$ and $|D\varphi|\leq 1/\varepsilon^{\frac{p-2}{2(p-1)}}$, where we denote by

$$s(\varepsilon) := s+\varepsilon^{\frac{(p-2)}{2(p-1)}}.$$

Thus the function w_ε can be defined as

$$w_\varepsilon := \begin{cases} u_\varepsilon & \text{in } D_s, \\ \varphi u_\varepsilon + (1-\varphi)\bar{u} & \text{in } D_{s(\varepsilon)}\setminus\ D_s, \\ \bar{u} & \text{in } \mathbb{R}^2_+\setminus D_{s(\varepsilon)}. \end{cases}$$

Note that w_ε belongs to $W^{1,p}_{\text{loc}}(\mathbb{R}^2_+)$, $\lim\limits_{t\to-\infty} Tw_\varepsilon(t) = \alpha'$ and $\lim\limits_{t\to+\infty} Tw_\varepsilon(t) = \beta'$. Clearly, w_ε is a good competitor for (4.4).

Step 2: Choice of the annulus. We need to choose an annulus in the half-disk, in which we can recover a suitable quantity of energy of u_ε. Since u_ε has equi-bounded energy $H_\varepsilon(u_\varepsilon)$ in D_1, there exists $L>0$ such that $\forall\varepsilon>0$ $\exists s\in\left(\frac{1}{2},1-\varepsilon^{\frac{p-2}{2(p-1)}}\right)$ such that

$$H_\varepsilon(u_\varepsilon, D_{s(\varepsilon)}\setminus D_s, E_{s(\varepsilon)}\setminus E_s)\leq L\varepsilon^{\frac{p-2}{2(p-1)}}. \tag{4.12}$$

Step 3: Estimates. By the scaling property of H_ε (see (4.3)), we have

$$\begin{aligned}
\gamma_p \leq H_1(w_\varepsilon^{(\varepsilon)},\mathbb{R}^2_+,\mathbb{R}) \;&=\; H_\varepsilon(w_\varepsilon,\mathbb{R}^2_+/\varepsilon,\mathbb{R}/\varepsilon)\\
\leq H_\varepsilon(w_\varepsilon,\mathbb{R}^2_+,\mathbb{R}) \;&=\; H_\varepsilon(w_\varepsilon,D_s,E_s)+H_\varepsilon(w_\varepsilon,\mathbb{R}^2_+\setminus D_s,\mathbb{R}\setminus E_s)\\
&\leq H_\varepsilon(u_\varepsilon)+\varepsilon^{p-2}\int_{\mathbb{R}^2_+\setminus D_s}|Dw_\varepsilon|^p dx+\frac{1}{\varepsilon}\int_{\mathbb{R}\setminus E_s}V(Tw_\varepsilon)d\mathcal{H}^1\\
&=: H_\varepsilon(u_\varepsilon)+I_1+I_2.
\end{aligned} \tag{4.13}$$

By definition of w_ε, the first integral in the right hand side of (4.13) can be easily estimated as follows

$$\begin{aligned}
I_1 &= \varepsilon^{p-2}\int_{\mathbb{R}^2_+\setminus D_{s(\varepsilon)}}|D\bar{u}|^p dx+\varepsilon^{p-2}\int_{D_{s(\varepsilon)}\setminus D_s}|D(\varphi u_\varepsilon+(1-\varphi)\bar{u})|^p dx\\
&\leq 3^{p-1}\varepsilon^{p-2}\int_{\mathbb{R}^2_+\setminus D_s}|D\bar{u}|^p dx+3^{p-1}\varepsilon^{p-2}\int_{D_{s(\varepsilon)}\setminus D_s}|Du_\varepsilon|^p dx\\
&\quad +6^{p-1}m^p\pi\varepsilon^{\frac{p(p-2)}{2(p-1)}}+3^{p-1}(2m)^p\pi s\varepsilon^{\frac{p-2}{2}},
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\int_{D_{s(\varepsilon)}\setminus D_s}|D\varphi|^p|u_\varepsilon-\bar{u}|^p dx &\leq \frac{1}{\varepsilon^{\frac{p(p-2)}{2(p-1)}}}\int_{D_{s(\varepsilon)}\setminus D_s}|u_\varepsilon-\bar{u}|^p\\
&\leq 2^{p-1}m^p\pi\left(\frac{1}{\varepsilon^{\frac{(p-2)^2}{2(p-1)}}}+\frac{2s}{\varepsilon^{\frac{p-2}{2}}}\right)
\end{aligned}$$

and that $|u_\varepsilon|<m$ and $|\bar{u}|<m$.

By definition of $\bar{u}$, we have

$$\int_{\mathbb{R}^2_+\setminus D_s}|D\bar{u}|^p dx=\frac{|\beta'-\alpha'|^p}{\pi^{p-1}(p-2)s^{p-2}}.$$

Hence,

$$\begin{aligned}
I_1 &\leq 3^{p-1}\varepsilon^{p-2}\int_{D_{s(\varepsilon)}\setminus D_s}|Du_\varepsilon|^p dx+\frac{3^{p-1}|\beta'-\alpha'|^p}{(p-2)\pi^{p-2}s^{p-2}}\varepsilon^{p-2}\\
&\quad +6^{p-1}m^p\pi\varepsilon^{\frac{p(p-2)}{2(p-1)}}+3^{p-1}(2m)^p\pi s\varepsilon^{\frac{p-2}{2}}.
\end{aligned} \tag{4.14}$$

Let us estimate the second integral in the right hand side of (4.13). Since $Tw_\varepsilon = \alpha'$ and $Tw_\varepsilon = \beta'$ on $\mathbb{R} \setminus E_{s(\varepsilon)}$, we have

$$I_2 = \frac{1}{\varepsilon}\int_{\mathbb{R}\setminus E_{s(\varepsilon)}} V(T\bar{u})d\mathcal{H}^1 + \frac{1}{\varepsilon}\int_{E_{s(\varepsilon)}\setminus E_s} V(Tw_\varepsilon)d\mathcal{H}^1 \equiv \frac{1}{\varepsilon}\int_{E_{s(\varepsilon)}\setminus E_s} V(Tw_\varepsilon)d\mathcal{H}^1.$$

For every $\delta > 0$, let us define

$$E_\delta := \left\{x \in E_{s(\varepsilon)} \setminus E_s : |Tu_\varepsilon - \beta'| > \delta \text{ and } |Tu_\varepsilon - \alpha'| > \delta\right\}.$$

Thanks to Step 2, there exists $N > \dfrac{L}{\omega_\delta}$ $\Big($where we denote by $\omega_\delta := \min\limits_{\substack{|t-\alpha'|\geq\delta \\ |t-\beta'|\geq\delta}} V(t)\Big)$ such that for every $\delta > 0$ there exists ε_δ such that

$$|E_\delta| \leq N\varepsilon^{\frac{p-2}{2(p-1)}}\varepsilon, \quad \forall\varepsilon \leq \varepsilon_\delta, \tag{4.15}$$

In particular, choosing δ small, the convexity of V near its wells provides

$$\begin{aligned} I_2 &= \frac{1}{\varepsilon}\int_{(E_{s(\varepsilon)}\setminus E_s)\setminus E_\delta} V(Tw_\varepsilon)d\mathcal{H}^1 + \frac{1}{\varepsilon}\int_{E_\delta} V(Tw_\varepsilon)d\mathcal{H}^1 \\ &\leq \frac{1}{\varepsilon}\int_{(E_{s(\varepsilon)}\setminus E_s)\setminus E_\delta} V(Tu_\varepsilon)d\mathcal{H}^1 + \omega_m N\varepsilon^{\frac{p-2}{2(p-1)}}, \end{aligned} \tag{4.16}$$

where $\omega_m := \max\limits_{|t|<m} V(t)$ and we used the inequality (4.15).

Finally, by (4.13), (4.14) and (4.16), we obtain, for every $\delta > 0$

$$\begin{aligned} H_\varepsilon(u_\varepsilon) &\geq \gamma_p - \Bigg(3^{p-1}H_\varepsilon(w_\varepsilon, D_{s(\varepsilon)} \setminus D_s, E_{s(\varepsilon)} \setminus E_s) + \frac{3^{p-1}|\beta' - \alpha'|^p}{(p-2)\pi^{p-1}s^{p-2}}\varepsilon^{p-2} \\ &\qquad + 6^{p-1}m^p\pi\varepsilon^{\frac{p(p-2)}{2(p-1)}} + 3^{p-1}(2m)^p\pi s\varepsilon^{\frac{p-2}{2}} + \omega_m N\varepsilon^{\frac{p-2}{2(p-1)}}\Bigg) \\ &\geq \gamma_p - \Bigg(3^{p-1}L\varepsilon + \frac{3^{p-1}|\beta' - \alpha'|^p}{(p-2)\pi^{p-1}s^{p-2}}\varepsilon^{p-2} + 6^{p-1}m^p\pi\varepsilon^{\frac{p(p-2)}{2(p-1)}} \\ &\qquad + 3^{p-1}(2m)^p\pi s\varepsilon^{\frac{p-2}{2}} + \omega_m N\varepsilon^{\frac{p-2}{2(p-1)}}\Bigg), \quad \forall\varepsilon \leq \varepsilon_\delta. \end{aligned} \tag{4.17}$$

Notice that for every $\varepsilon > 0$, $s \in \left(1/2, 1 - \varepsilon^{\frac{p-2}{2(p-1)}}\right)$. Hence, taking the limit as $\varepsilon \to 0$, we get $\liminf\limits_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon) \geq \gamma_p$, which concludes the proof. □

4.3. *Reduction to the flat case*

We prove compactness and a lower bound inequality for the following energies

$$F_\varepsilon(u_\varepsilon, \mathcal{D}, \mathcal{E}) = \varepsilon^{p-2}\int_{\mathcal{D}} |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_{\mathcal{D}} W(u)dx + \frac{1}{\varepsilon}\int_{\mathcal{E}} V(Tu)d\mathcal{H}^2,$$

where $\mathcal{D} \subset \mathbb{R}^3$ is the open half-ball centered in 0 with radius $r > 0$ and $E \subset \mathbb{R}^2$ is defined by

$$\mathcal{E} := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^2 : |x| \leq r, x_3 = 0 \right\}.$$

We will reduce to Proposition 4.1 and Proposition 4.2 via a suitable slicing argument. We use the following notation: e is an unit vector in the plane $P := \{x_3 = 0\}$;

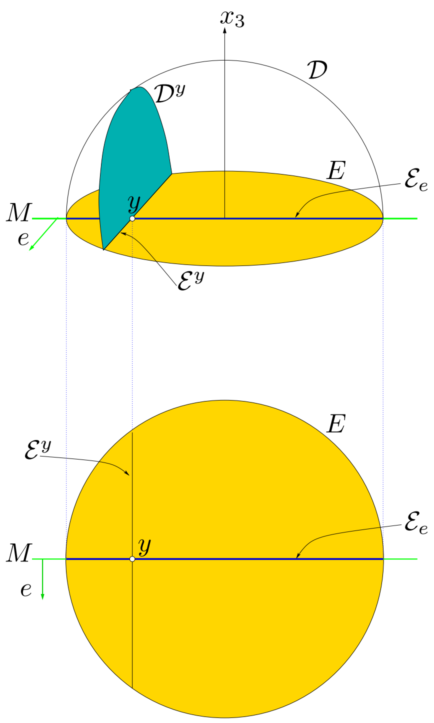

Fig. 3. The sets $\mathcal{D}, \mathcal{E}, \mathcal{E}_e, \mathcal{E}^y$ and $\mathcal{D}^y$ (see also Ref. 3, Fig. 4).

M is the orthogonal complement of e in P; π is the projection of $\mathbb{R}^3$ onto M; for every $y \in \mathcal{E}_e := \pi(\mathcal{E})$, we denote by $\mathcal{E}^y := \pi^{-1}(y) \cap \mathcal{E}$, $\mathcal{D}^y := \pi^{-1}(y) \cap \mathcal{D}$ (see Fig. 3); for every function u defined on $\mathcal{D}$ we consider the trace of u on $\mathcal{E}^y$, i.e., the one-dimensional function

$$u_e^y(t) := u(y + te).$$

Proposition 4.3. *Let $(u_\varepsilon) \subset W^{1,p}(\mathcal{D}; [-m, m])$ be a sequence with uniformly bounded energies $F_\varepsilon(u_\varepsilon, \mathcal{D}, \mathcal{E})$. Then the traces Tu_ε are pre-compact in $L^1(\mathcal{E})$ and every cluster point belongs to $BV(\mathcal{E}, \{\alpha', \beta'\})$. Moreover, if $Tu_\varepsilon \to v$ in $L^1(\mathcal{E})$, then*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \mathcal{D}, \mathcal{E}) \geq \gamma_p \left| \int_{\mathcal{E} \cap Sv} \nu_v \right| d\mathcal{H}^1. \tag{4.18}$$

Proof. By Fubini's Theorem, for every $\varepsilon > 0$, we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon, \mathcal{D}, \mathcal{E}) &\geq \varepsilon^{p-2} \int_{\mathcal{D}} |\mathcal{D}u_\varepsilon|^p dx + \frac{1}{\varepsilon} \int_{\mathcal{E}} V(Tu_\varepsilon) d\mathcal{H}^2 \\ &\geq \int_{\mathcal{E}_e} \left[\varepsilon^{p-2} \int_{D^y} |\mathcal{D}u_\varepsilon^y|^p dx + \frac{1}{\varepsilon} \int_{\mathcal{E}^y} V(Tu_\varepsilon^y) d\mathcal{H}^1 \right] dy \\ &= \int_{\mathcal{E}_e} H_\varepsilon(u_\varepsilon^y, \mathcal{D}^y, \mathcal{E}^y) dy \end{aligned} \tag{4.19}$$

and the 2D functional in the integration above has been studied in the last section. The remaining part of the proof follows exactly as in Proposition 4.7 in Ref. 3. □

4.4. *Existence of an optimal profile*

We conclude this section with the proof of the existence of a minimum for the optimal profile problem (4.4). We will use a rearrangement result in one direction to show that the minimum for γ_p is achieved by a function with non-decreasing trace.

Proposition 4.4. *The minimum for γ_p defined by (4.4) is achieved by a function u such that Tu is a non-decreasing function in $\mathbb{R}$.*

Proof. Note that, since the energy H_1 is decreasing under truncation by α' and β', it is not restrictive to minimize the problem (4.4) with the additional condition $\alpha' \leq u \leq \beta'$.

We denote by

$$X := \left\{ w : \mathbb{R} \to [\alpha', \beta'] : w \in L^1_{loc}(\mathbb{R}^2_+), \lim_{t \to -\infty} Tw(t) = \alpha', \lim_{t \to +\infty} Tw(t) = \beta' \right\}$$

$$X^* := \left\{ w \in X : Tw \text{ is non-decreasing, } Tw(t) \geq \frac{\alpha' + \beta'}{2} \text{ for } t > 0, \right. \\ \left. Tw(t) \leq \frac{\alpha' + \beta'}{2} \text{ for } t < 0 \right\}.$$

Step 1: The infimum of H_1 on X is equal to the infimum of H_1 on X^*.
Since $X^* \subset X$ we have

$$\inf_{w \in X^*} H_1(w, \mathbb{R}^2_+, \mathbb{R}) \geq \inf_{w \in X} H_1(w, \mathbb{R}^2_+, \mathbb{R}). \tag{4.20}$$

Fix u in X, we claim that for every $\delta > 0$ there exists a function u_δ in X^* such that

$$H_1(u_\delta, \mathbb{R}^2_+, \mathbb{R}) \leq H_1(u, \mathbb{R}^2_+, \mathbb{R}) + o(\delta). \tag{4.21}$$

Once we have (4.21), for every $\delta > 0$, we get

$$\inf_{w\in X^*} H_1(w,\mathbb{R}^2_+,\mathbb{R}) \le H_1(u,\mathbb{R}^2_+,\mathbb{R}) + o(\delta),\ \forall u \in X.$$

Taking the limit for $\delta \to 0$ and then the infimum on $u \in X$, we obtain

$$\inf_{w\in X^*} H_1(w,\mathbb{R}^2_+,\mathbb{R}) \le \inf_{u\in X} H_1(u,\mathbb{R}^2_+,\mathbb{R})$$

and this together with (4.20) will conclude the proof of the step. It remains to prove (4.21).

For every $S > 0$, we denote by

$$Q_S := [-S,S] \times [0,S]$$

and by $u^\star$ the monotone increasing rearrangement in direction x_1 of u in Q_S, i.e., the function $u^\star : Q_S \to \mathbb{R}$ which is increasing in $[-S,S]$ with respect to x_1 (for almost all $x' \in [0,S]$), and such that for every $\lambda \in \mathbb{R}$ and for every $x' \in [0,S]$:

$$|\,\{x_1 \in [-S,S] : u(x_1,x') \ge \lambda\}\,| = |\,\{x_1 \in [-S,S] : u^\star(x_1,x') \ge \lambda\}\,|.$$

For every $R > 0$, we define the harmonic extension of the function

$$\alpha'\chi_{(-\infty,0)}(t) + \beta'\big(1 - \chi_{[0,+\infty)})(t)\big)$$

from $\mathbb{R} \setminus E_R$ to $\mathbb{R}^2_+ \setminus D_R$; i.e., the function $\bar{u}$ that expressed in polar coordinates is given by

$$\bar{u}(\rho,\theta) := \frac{\theta}{\pi}\alpha' + \left(1 - \frac{\theta}{\pi}\right)\beta',\ \ \forall \theta \in [0,\pi],\ \forall \rho \ge R.$$

We will construct a function $\tilde{u} \in X^*$ gluing the function $\bar{u}$ and the function $u^\star$. Hence, for every $S \ge R > 0$, we consider the cut-off function φ such that

$$\varphi = 0 \text{ in } D_R,\ \ \varphi = 1 \text{ in } \mathbb{R}^2_+ \setminus D_S \text{ and } |D\varphi| \le 1/(S-R) \text{ in } D_S \setminus D_R.$$

The function $\tilde{u}$ can be defined as

$$\tilde{u} := \begin{cases} u^\star & \text{in } D_R, \\ \varphi\bar{u} + (1-\varphi)u^\star & \text{in } D_S \setminus\ D_R, \\ \bar{u} & \text{in } \mathbb{R}^2_+ \setminus D_S. \end{cases}$$

Note that $\tilde{u}$ belongs to $W^{1,p}_{\rm loc}(\mathbb{R}^2_+)$, $\lim\limits_{t\to-\infty} T\tilde{u}(t) = \alpha'$, $\lim\limits_{t\to+\infty} T\tilde{u}(t) = \beta'$ and $T\tilde{u}$ is non-decreasing in $\mathbb{R}$. Let us compute its energy.

$$\int_{\mathbb{R}^2_+} |D\tilde{u}|^p dx = \int_{D_R} |Du^\star|^p dx + \int_{D_S\setminus D_R} |D(\varphi\bar{u} + (1-\varphi)u^\star)|^p dx + \int_{\mathbb{R}^2_+\setminus D_S} |D\bar{u}|^p dx. \tag{4.22}$$

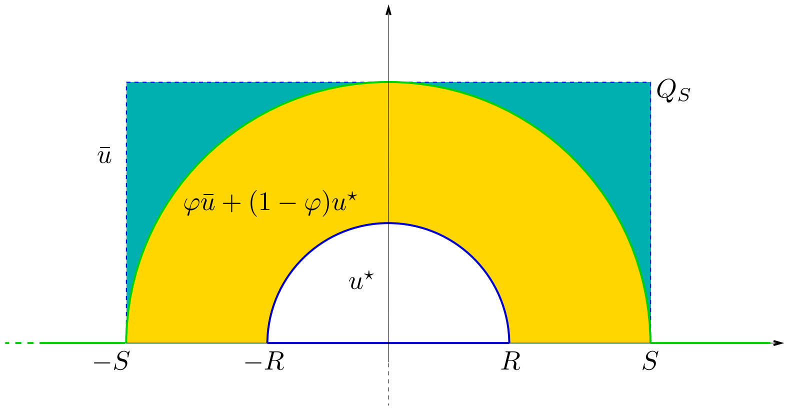

Fig. 4. The competitor $\tilde{u}$.

We estimate the integral in the set $D_S \setminus D_R$, using the fact that for every $\delta \in (0,1)$ there exists $a(\delta) \to +\infty$ as $\delta \to 0$, such that

$$(A+B+C)^p \leq (1+\delta)A^p + a(\delta)B^p + a(\delta)C^p,$$

for every non negative A, B, C.
Hence, for every $\delta \in (0,1)$, we have

$$\int_{D_S\setminus D_R} |D(\varphi\bar{u} + (1-\varphi)u^\star)|^p dx \leq (1+\delta)\int_{D_S\setminus D_R} |Du^\star|^p dx + a(\delta)\int_{D_S\setminus D_R} |D\bar{u}|^p dx$$
$$+a(\delta)\frac{|\beta' - \alpha'|^p \pi^2 (S^2 - R^2)}{(S-R)^p}, \tag{4.23}$$

where we used that $|D\varphi| \leq 1/(S-R)$ and $\bar{u}, u^\star \in [\alpha', \beta']$. By (4.22) and (4.23), we obtain

$$\int_{\mathbb{R}^2_+} |D\tilde{u}|^p dx \leq (1+\delta)\int_{D_S} |Du^\star|^p dx + a(\delta)\int_{\mathbb{R}^2_+\setminus D_R} |D\bar{u}|^p dx$$
$$+\frac{a(\delta)|\beta' - \alpha'|^p \pi^2 (S^2 - R^2)}{(S-R)^p}. \tag{4.24}$$

We can estimates the first term in the right hand side of (4.24) using the fact that monotone increasing rearrangement in one direction decreases the L^p-norm of the gradient (see Ref. 7, Theorem 3); we get

$$\int_{D_S} |Du^\star|^p dx \leq \int_{Q_S} |Du^\star|^p dx \leq \int_{Q_S} |Du|^p dx. \tag{4.25}$$

The second term in the right hand side of (4.24) can be explicitly computed

$$\int_{\mathbb{R}^2_+\setminus D_R} |D\bar{u}|^p dx = \frac{\pi^{p-1}|\beta' - \alpha'|^p}{(p-2)R^{p-2}}. \tag{4.26}$$

Finally, putting together (4.24), (4.25) and (4.26), we obtain that, for every $S \geq R > 0$ and every $\delta \in (0,1)$, the following estimate holds

$$\int_{\mathbb{R}^2_+} |D\tilde{u}|^p dx \leq (1+\delta) \int_{\mathbb{R}^2_+} |Du|^p dx + \frac{a(\delta)\pi^{p-1}|\beta' - \alpha'|^p}{(p-2)R^{p-2}} + \frac{a(\delta)|\beta' - \alpha'|^p \pi^2 (S^2 - R^2)}{(S-R)^p}, \tag{4.27}$$

with $a(\delta) \to +\infty$ as $\delta \to 0$. Now, we evaluate the second term of $H_1(\tilde{u}, \mathbb{R}^2_+, \mathbb{R})$.

$$\int_{\mathbb{R}} V(T\tilde{u}) d\mathcal{H}^1 = \int_{E_S \setminus E_R} V\big(\varphi \bar{u} + (1-\varphi) T u^\star\big) d\mathcal{H}^1 + \int_{E_R} V(Tu^\star) d\mathcal{H}^1, \tag{4.28}$$

where we used that, by definition, $\bar{u}(t) = \alpha'$ if $t < -S$ and $\bar{u}(t) = \beta'$ if $t > S$.

Since $Tu(t)$ tends to α' and β' as $t \to -\infty$ and $+\infty$ respectively and V is convex near its wells, there exists $R_0 > 0$ such that $Tu(t)$ lies in the convex wells of V for all $t \in \mathbb{R} \setminus E_R$, for every $R > R_0$, and the same does $Tu^\star(t)$. This implies that for any $S > R > R_0$ we have

$$\int_{E_S \setminus E_R} V\big(\varphi \bar{u} + (1-\varphi) T u^\star\big) d\mathcal{H}^1 \leq \int_{E_S \setminus E_R} \varphi V(\bar{u}) d\mathcal{H}^1 + \int_{E_S \setminus E_R} (1-\varphi) V(Tu^\star) d\mathcal{H}^1 \leq \int_{E_S \setminus E_R} V(Tu^\star) d\mathcal{H}^1. \tag{4.29}$$

By (4.28) and (4.29), we have

$$\int_{\mathbb{R}} V(T\tilde{u}) d\mathcal{H}^1 \leq \int_{E_S} V(Tu^\star) d\mathcal{H}^1 = \int_{E_S} V(Tu) d\mathcal{H}^1 \leq \int_{\mathbb{R}} V(Tu) d\mathcal{H}^1. \tag{4.30}$$

Finally, by (4.27) and (4.30), we have the following estimate, for every $S \geq R > R_0 > 0$ and every $\delta \in (0,1)$

$$H_1(\tilde{u}, \mathbb{R}^2_+, \mathbb{R}) \leq (1+\delta) H_1(u) + \frac{a(\delta)\pi^{p-1}|\beta' - \alpha'|^p}{(p-2)R^{p-2}} + \frac{a(\delta)|\beta' - \alpha'|^p \pi^2 (S^2 - R^2)}{(S-R)^p}. \tag{4.31}$$

This proves the claim, taking R and $S - R$ large enough.

Step 2: The infimum of H_1 on X^* is achieved.
We use the Direct Method. Take a minimizing sequence $(u_n) \subset X^*$. In particular, $H_1(u_n, \mathbb{R}^2_+, \mathbb{R}) \leq C$, Du_n converges weakly to Du in $L^p(\mathbb{R}^2_+)$ and u_n converges

to u weakly in $W^{1,p}_{\rm loc}(\mathbb{R}^2_+)$. Since $\int_{\mathbb{R}^2_+} |Du_n|^p dx$ is bounded, we can find a function $u \in L^1_{\rm loc}(\mathbb{R}^2_+)$ such that (up to a subsequence)

$$Du_n \rightharpoonup Du \text{ in } L^p(\mathbb{R}^2_+) \text{ and } u_n \rightharpoonup u \text{ in } L^p_{\rm loc}(\mathbb{R}^2_+).$$

By the trace imbedding of $W^{1,p}(\mathbb{R}^2_+)$ in $W^{1-1/p,p}(\mathbb{R})$, we have

$$Tu_n \rightharpoonup Tu \text{ in } W^{1-1/p,p}_{\rm loc}(\mathbb{R}).$$

By the compact embedding of $C^0_{\rm loc}(\mathbb{R})$ in $W^{1-1/p,p}_{\rm loc}(\mathbb{R})$ (see Ref. 1, Theorem 7.34), we have that, up to a subsequence, Tu_n locally uniformly converges to Tu. Thus Tu is non-decreasing and satisfies

$$Tu(t) \geq \frac{\alpha' + \beta'}{2} \text{ for } t > 0 \quad \text{and} \quad Tu(t) \leq \frac{\alpha' + \beta'}{2} \text{ for } t < 0.$$

Let us show that $\lim_{t \to -\infty} Tu(t) = \alpha'$ and $\lim_{t \to +\infty} Tu(t) = \beta'$. Since Tu is non-decreasing in $[\alpha', \beta']$, there exist $a \leq \dfrac{\alpha' + \beta'}{2}$ and $b \geq \dfrac{\alpha' + \beta'}{2}$ such that

$$a := \lim_{t \to -\infty} Tu(t) \quad \text{and} \quad b := \lim_{t \to +\infty} Tu(t).$$

By contradiction, we assume that either $a \neq \alpha'$ or $b \neq \beta'$. Then, since V is continuous and strictly positive in (α', β'), we obtain

$$\int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 = +\infty,$$

This is impossible, because, by Fatou's Lemma, we have

$$\int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 \leq \liminf_{n \to +\infty} \int_{\mathbb{R}} V(Tu_n) d\mathcal{H}^1 < \liminf_{n \to +\infty} H_1(u_n, \mathbb{R}^2_+, \mathbb{R}) < +\infty.$$

Hence, u is in X^*. Since H_1 is clearly lower semicontinuous on sequences such that $Du_n \rightharpoonup Du$ in L^p and $Tu_n \to Tu$ pointwise, this concludes the proof. □

5. Proof of the main result

We will prove the main result of this paper, namely the compactness, the lower bound inequality and the upper bound inequality stated in Theorem 2.1. Since we work with slicing, we may also want to evaluate "the error we make" when we perturb a three-dimensional domain to get a two-dimensional one. To this aim, we define the "isometry defect", introduced by Alberti, Bouchitté and Seppecher[3].

As usual, we denote by $O(3)$ the set of linear isometries on $\mathbb{R}^3$.

Definition 5.1. Let $A_1, A_2 \subset \mathbb{R}^3$ and let $\Psi : \overline{A_1} \to \overline{A_2}$ bi-Lipschitz homeomorphism. Then the *"isometry defect $\delta(\Psi)$ of Ψ"* is the smallest constant δ such that

$$dist(D\Psi(x), O(3)) \leq \delta, \quad \text{for a.e. } x \in A_1. \tag{5.1}$$

Here $D\Psi(x)$ is regarded as a linear mapping of $\mathbb{R}^3$ into $\mathbb{R}^3$. The distance between linear mappings is induced by the norm $\|\cdot\|$, which, for every L, is defined as the supremum of $|Lv|$ over all v such that $|v| \leq 1$. Hence, for every $L_1, L_2 : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\mathrm{dist}(L_1, L_2) := \sup_{x:|x|\leq 1} |L_1(x) - L_2(x)|.$$

By (5.1), we get

$$\|D\Psi(x)\| \leq 1 + \delta(\Psi) \text{ for a.e. } x \in A_1, \tag{5.2}$$

and then Ψ is $(1+\delta(\Psi))$-Lipschitz continuous on every convex subset of A_1. Similarly, Ψ^{-1} is $(1-\delta(\Psi))^{-1}$-Lipschitz continuous on every convex subset of A_2.

Proposition 5.1. *Let $A_1, A_2 \subset \mathbb{R}^3, \Psi : \overline{A_1} \to \overline{A_2}$ a bi-Lipschitz homeomorphism, $A_1' \subset \partial A_1, A_2' \subset \partial A_2$, be given such that $\Psi(A_1') = A_2'$ and $\delta(\Psi) < 1$. Then for every $u \in W^{1,p}(A_2)$*

$$F_\varepsilon(u, A_2, A_2') \geq (1-\delta(\Psi))^{p+3} F_\varepsilon(u \circ \Psi, A_1, A_1').$$

The proof is a simple modification of the one by Alberti Bouchitté and Seppecher in Ref. 3, Proposition 4.9, where they treat the case $p = 2$.

Proposition 5.2. *(Ref. 3, Proposition 4.10). For every $x \in \partial\Omega$ and every positive r smaller than a certain critical value $r_x > 0$, there exists a bi-Lipschitz map $\Psi_r : \overline{D_r} \to \overline{\Omega \cap B_r(x)}$ such that*

(a) Ψ_r takes D_r onto $\Omega \cap B_r(x)$ and E_r onto $\partial\Omega \cap B_r(x)$;
(b) Ψ_r is of class C^1 on D_r and $\|D\Psi_r - I\| \leq \delta_r$ everywhere in D_r, where $\delta_r \to 0$ as $r \to 0$.

Note that, in particular, the isometry defect of Ψ_r vanishes as $r \to 0$.

5.1. *Compactness*

Let a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ be given such that $F_\varepsilon(u_\varepsilon)$ is bounded. Since $F_\varepsilon(u_\varepsilon) \geq F_\varepsilon(u_\varepsilon, \Omega, \emptyset) \equiv G_\varepsilon(u_\varepsilon, \Omega)$, by the statement (i) of Theorem 3.1, the sequence (u_ε) is pre-compact in $L^1(\Omega)$ and there exists $u \in BV(\Omega, \{\alpha, \beta\})$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$.

It remains to prove that (Tu_ε) is pre-compact in $L^1(\partial\Omega)$ and that its cluster points are in $BV(\partial\Omega, \{\alpha', \beta'\})$. Thanks to Proposition 5.2 we can cover $\partial\Omega$ with finitely many balls $(B_i)_{i\in I}$ centered on $\partial\Omega$, of radius r_i such that for every $i \in I$ there exists a bi-Lipschitz map Ψ_i, with isometry defect $\delta(\Psi_i) < 1$, which satisfies $\Psi_i(D_{r_i} \cap B_i) = \Omega \cap B_i$ and $\Psi_i(E_{r_i} \cap B_i) = \partial\Omega \cap B_i$. We show that (Tu_ε) is pre-compact in $L^1(\partial\Omega \cap B_i)$ for every $i \in I$.

For every fixed i, let us set

$$u_\varepsilon^i := u_\varepsilon \circ \Psi_i.$$

Since the isometry defect of Ψ_i is smaller than 1, Proposition 5.1 implies

$$F_\varepsilon(u_\varepsilon, \Omega \cap B_i, \partial\Omega \cap B_i) \geq (1 - \delta(\Psi_i))^{p+3} F_\varepsilon(u_\varepsilon^i, D_{r_i} \cap B_i, E_{r_i} \cap B_i),$$

so $F_\varepsilon(u_\varepsilon^i, D_{r_i} \cap B_i, E_{r_i} \cap B_i)$ is bounded. Hence, the compactness of the traces Tu_ε^i in $L^1(E_{r_i})$ follows from Proposition 4.3. Finally, using the invertibility of Ψ_i, we have that (Tu_ε) is pre-compact in $L^1(\partial\Omega)$ and that its cluster points are in $BV(\partial\Omega, \{\alpha', \beta'\})$. □

5.2. *Lower bound inequality*

The proof of the lower bound inequality of Theorem 2.1 follows the lines of the proof of Theorem 2.6-(ii) by Alberti, Bouchitté and Seppecher[3], but for the estimate of the boundary effect we will use the optimal profile problem (4.4) in connection with the results proved in the previous section.

Let a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ be given such that $u_\varepsilon \to u \in BV(\Omega, \{\alpha, \beta\})$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v \in BV(\partial\Omega, \{\alpha'\beta'\})$ in $L^1(\partial\Omega)$. We have to prove that

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v), \tag{5.3}$$

where Φ is given by (2.2).
Clearly, we can assume that $\liminf\limits_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) < +\infty$.

For every $\varepsilon > 0$, let μ_ε be the energy distribution associated with F_ε with configuration u_ε; i.e., μ_ε is the positive measure given by

$$\mu_\varepsilon(B) := \varepsilon^{p-2} \int_{\Omega \cap B} |Du_\varepsilon|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_{\Omega \cap B} W(u_\varepsilon) dx + \frac{1}{\varepsilon} \int_{\partial\Omega \cap B} V(Tu_\varepsilon) d\mathcal{H}^2, \tag{5.4}$$

for every $B \subset \mathbb{R}^3$ Borel set.

Similarly, let us define

$$\begin{aligned} \mu^1(B) &:= \sigma_p \mathcal{H}^2(Su \cap B), \\ \mu^2(B) &:= c_p \int_{\partial\Omega \cap B} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2, \\ \mu^3(B) &:= \gamma_p \mathcal{H}^1(Sv \cap B). \end{aligned}$$

The total variation $\|\mu_\varepsilon\|$ of the measure μ_ε is equal to $F_\varepsilon(u_\varepsilon)$, and $\|\mu^1\| + \|\mu^2\| + \|\mu^3\|$ is equal to $\Phi(u, v)$. $\|\mu_\varepsilon\|$ is bounded and we can assume that μ_ε converges in the sense of measure to some finite measure μ in $\mathbb{R}^3$. Then, by the lower semicontinuity of the total variation, we have

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \equiv \liminf_{\varepsilon \to 0} \|\mu_\varepsilon\| \geq \|\mu\|.$$

Since the measures μ^i are mutually singular, we obtain the lower bound inequality (5.3) if we prove that

$$\mu \geq \mu^i, \quad \text{for } i = 1, 2, 3. \tag{5.5}$$

We prove that $\mu \geq \mu^i$ by showing that $\mu(B) \geq \mu^i(B)$ for all sets $B \subset \mathbb{R}^3$ such that $B \cap \Omega$ is a Lipschitz domain and $\mu(\partial B) = 0$. This class is large enough to imply the inequality (5.5) for all Borel sets B.

We have

$$\mu(B) = \lim_{\varepsilon \to 0} \mu_\varepsilon(B) \geq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega \cap B, \emptyset) \geq \sigma_p \mathcal{H}^2(Su \cap B) \equiv \mu^1(B),$$

where the last inequality follows from statement (ii) of Theorem 3.1.

Similarly, we can prove that $\mu \geq \mu^2$. We have

$$\begin{aligned} \mu(B) = \lim_{\varepsilon \to 0} \mu_\varepsilon(B) \ &\geq \ \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega \cap B, \emptyset) \\ \geq c_p \int_{\partial\Omega \cap B} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2 \ &\equiv \ \mu^2(B), \end{aligned}$$

where we used Proposition 3.1(i) with $A := B \cap \Omega$ and $A' := B \cap \partial\Omega$.

The inequality $\mu \geq \mu^3$ requires a different argument. Notice that μ^3 is the restriction of $\mathcal{H}^1$ to the set Sv, multiplied by the factor γ_p. Thus, if we prove that

$$\liminf_{r \to 0} \frac{\mu(B_r(x))}{2r} \geq \gamma_p, \quad \mathcal{H}^1\text{-a.e. } x \in Sv, \tag{5.6}$$

we obtain the required inequality.

Let us fix $x \in Sv$ such that there exists $\lim_{r \to 0} \frac{\mu(B_r(x))}{2r}$ and Sv has one-dimensional density equal to 1. We denote by ν_v the unit normal at x.

For r small enough, we choose a map Ψ_r such as in Proposition 5.2. Thus we have $\Psi_r(\overline{D_r}) = \Omega \cap B_r(x)$, $\Psi_r(E_r) = \partial\Omega \cap B_r(x)$ and $\delta(\Psi_r) \to 0$ as $r \to 0$.

Let us set

$$\bar{u}_\varepsilon := u_\varepsilon \circ \Psi_r \text{ and } \bar{v} := v \circ \Psi_r.$$

Hence, $T\bar{u}_\varepsilon \to \bar{v}$ in $L^1(E_r)$ and $\bar{v} \in BV(E_r, \{\alpha', \beta'\})$. So, thanks to Proposition 5.1, we obtain

$$\begin{aligned} \mu(B_r(x)) &= \lim_{\varepsilon \to 0} \mu_\varepsilon(B_r(x)) \\ &= \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega \cap B_r(x), \partial\Omega \cap B_r(x)) \\ &\geq \liminf_{\varepsilon \to 0} (1 - \delta(\Psi_r))^{p+3} F_\varepsilon(\bar{u}_\varepsilon, D_r, E_r). \end{aligned} \tag{5.7}$$

Moreover, by Proposition 4.3, we have

$$\liminf_{\varepsilon \to 0} F_\varepsilon(\bar{u}_\varepsilon, D_r, E_r) \geq \gamma_p \left| \int_{S\bar{v} \cap E_r} \nu_v d\mathcal{H}^1 \right|. \tag{5.8}$$

Finally, we notice that $\delta(\Psi_r)$ vanishes and $\left| \int_{S\bar{v} \cap E_r} \nu_v d\mathcal{H}^1 \right| = 2r + o(r)$ as r goes to 0. So (5.7) and (5.8) give the following inequality

$$\frac{\mu(B_r(x))}{2r} \geq \gamma_p \left(1 + \frac{o(r)}{2r}\right) \text{ as } r \to 0,$$

that implies $\mu \geq \mu^3$. This concludes the proof of the lower bound inequality. □

5.3. *Upper bound inequality*

We will construct an optimal sequence u_ε according to Theorem 2.1(iii) in a suitable partition of Ω. To this aim, and in order to use the preliminary convergence results stated in the previous sections, we need the following lemma

Lemma 5.1. *Let A be a domain in $\mathbb{R}^3, A' \subset \partial A, v : A' \to [-m, m]$ a Lipschitz function (where m is given by (3.5)) and G_ε defined by (3.1).*
Then, for every $\varepsilon > 0$, there exists an extension $u : \overline{A} \to [-m, m]$ such that

$$Lip(u) \leq \varepsilon^{-\frac{p-2}{p-1}} + Lip(v)$$

and

$$G_\varepsilon(u, A) \leq \left((\varepsilon^{\frac{p-2}{p-1}} Lip(v) + 1)^p + C_m \right) \left(\mathcal{H}^2(\partial A) + o(1) \right) \omega, \quad as\ \varepsilon \to 0, \tag{5.9}$$

where $C_m := \max_{t \in [-m,m]} W(t)$, $\omega := \min\{\|v - \alpha\|_{L^\infty}, \|v - \beta\|_{L^\infty}\}$.

Proof. It is not restrictive to assume that $A' = \partial A$; in fact, we can extend v to ∂A without increasing its Lipschitz constant. We additionally suppose that $\omega = \|v - \alpha\|_\infty$ (the case $\omega = \|v - \beta\|_\infty$ being similar).

Let us set

$$u(x) := \begin{cases} v(x) & \text{on } \partial A, \\ \alpha & \text{on } A \setminus A_{\omega\varepsilon^{(p-2)/(p-1)}}, \end{cases}$$

where A_t is the set of all x in A such that $0 < \operatorname{dist}(x, \partial A) < t$.
Then, u is $\left(\varepsilon^{-\frac{p-2}{p-1}} + \operatorname{Lip}(v)\right)$-Lipschitz continuous on $\overline{A} \setminus A_{\omega\varepsilon^{(p-2)/(p-1)}}$. Finally u can be extended to $\overline{A}$, without increasing its Lipschitz constant.
We have

$$\begin{aligned} G_\varepsilon(u, A) &= \varepsilon^{p-2} \int_{A_{\omega\varepsilon^{(p-2)/(p-1)}}} |Du|^p dx + \frac{1}{\varepsilon^{\frac{p-1}{p-2}}} \int_{A_{\omega\varepsilon^{(p-2)/(p-1)}}} W(u) dx \\ &\leq |A_{\omega\varepsilon^{(p-2)/(p-1)}}| \left(\varepsilon^{p-2} \big(\operatorname{Lip}(v) + \frac{1}{\varepsilon^{(p-1)/(p-2)}}\big)^p + \frac{1}{\varepsilon^{(p-1)/(p-2)}} C_m \right) \\ &= \left((\mathcal{H}^2(\partial A) + o(1))(\varepsilon^{\frac{p-1}{p-2}} \operatorname{Lip}(v) + 1)^p + C_m \right) \omega, \quad \text{as } \varepsilon \to 0, \end{aligned}$$

where we used that $|A_t| = (\mathcal{H}^2(\partial A) + o(1))t$ as $t \to 0$. □

Proof of the upper bound inequality. We assume that u and v (up to modifications on negligible sets) are constant in each connected component of $\Omega \setminus Su$ and

$\partial\Omega \setminus Sv$ respectively. Moreover, we can assume that the singular sets of u and v, Su and Sv respectively, are closed manifolds of class C^2 without boundary. This is so because every pair $(u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$ can be approximated in $L^1(\Omega) \times L^1(\partial\Omega)$ by pairs that fulfills those regularity assumptions (see Ref. 17, Theorem 1.24).

The idea is to construct a *partition* of Ω in four subsets, and to use the preliminary convergence results of previous sections to obtain the upper bound inequality.

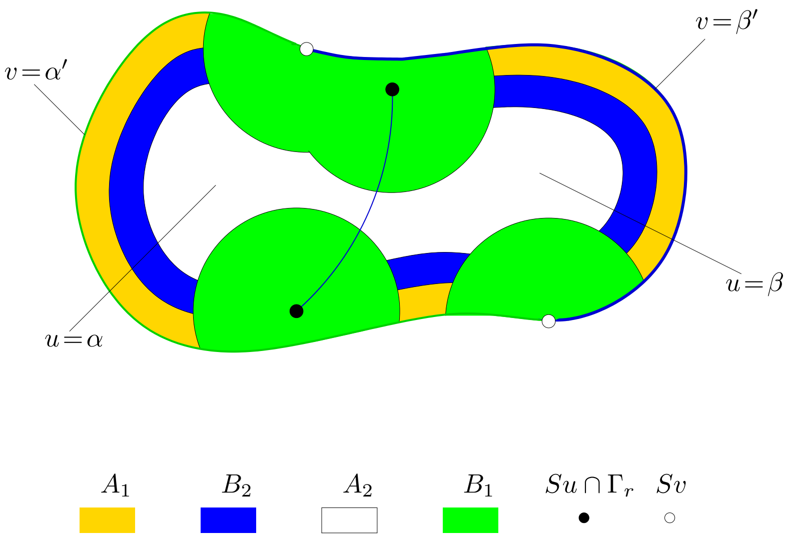

Fig. 5. Upper bound inequality - partition of Ω (see also Ref. 3, Fig. 6).

For every $r > 0$, we set

$$\Gamma_r := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) = r\}.$$

Step 1: Partition of Ω. Fix $r > 0$ such that Γ_r and Γ_{2r} are Lipschitz surfaces and $Su \cap \Gamma_r$ is a Lipschitz curve.

Now, we are ready to construct the following partition of Ω:

$$\begin{aligned}
B_1 &:= \{x \in \Omega : \mathrm{dist}(x, Sv \cup (Su \cap \Gamma_r)) < 3r\}, \\
A_1 &:= \big\{x \in \Omega \setminus \overline{B}_1 : \mathrm{dist}(x, \partial\Omega) < r\big\}, \\
B_2 &:= \big\{x \in \Omega \setminus \overline{B}_1 : r < \mathrm{dist}(x, \partial\Omega) < 2r\big\}, \\
A_2 &:= \big\{x \in \Omega \setminus \overline{B}_1 : \mathrm{dist}(x, \partial\Omega) > 2r\big\}.
\end{aligned}$$

(See Fig. 5)

For every $r > 0$ and every $\varepsilon < r^{\frac{p-1}{p-2}}$ we construct a Lipschitz function $u_{\varepsilon,r}$ in each subset.

Step 2: Construction of $u_{\varepsilon,r}$ in A_2.
We take $u_{\varepsilon,r}$ being the optimal sequence for the Modica-Mortola functional G_ε in the set A_2 (see Theorem 3.1-(iii)) and we extend it to ∂A_2 by continuity. Hence, $u_{\varepsilon,r}$ is $\dfrac{C_W}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz in $\overline{A}_2$ (here C_W is the maximum of $W^{p/(p-1)}$ in $[\alpha,\beta]$), $u_{\varepsilon,r}$ converges to u pointwise in A_2 and uniformly on $\partial A_2 \cap \partial B_2$, and

$$F_\varepsilon(u_{\varepsilon,r}, A_2, \emptyset) \equiv G_\varepsilon(u_{\varepsilon,r}, A_2) \le \sigma_p \mathcal{H}^2(Su \cap A_2) + o(1)$$
$$\le \sigma_p \mathcal{H}^2(Su) + o(1), \quad \text{as } \varepsilon \to 0. \tag{5.10}$$

Step 3: Construction of $u_{\varepsilon,r}$ in A_1.
The function u is constant (equal to α or β) on every connected component A of A_1, and the function v is constant (equal to α' or β') on $\partial A \cap \partial\Omega$. So, we can use Proposition 3.1 to get a function $u_{\varepsilon,r}$ such that $Tu_{\varepsilon,r} = v$ on $\partial A \cap \partial\Omega$ and $u_{\varepsilon,r}$ converges to u pointwise in A_1 and uniformly on every subset with positive distance from $\partial A \cap \partial\Omega$.

By Proposition 3.1(ii), $u_{\varepsilon,r}$ is $\dfrac{C'_W}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous on $\overline{A}_1$ and we can extend it to ∂A_1 with continuity. Since the distance between two different connected components of A_1 is larger than r and $\dfrac{1}{\varepsilon^{(p-2)/(p-1)}} > \dfrac{1}{r}$, choosing $C \ge 2m \vee C'_W$ it follows that $u_{\varepsilon,r}$ is $\dfrac{C}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous on $\overline{A}_1$ and agrees with v on $\partial A_1 \cap \partial\Omega$. Moreover, the function $u_{\varepsilon,r}$ satisfies

$$F_\varepsilon(u_{\varepsilon,r}, A_1, \partial A_1 \cap \partial\Omega) \equiv G_\varepsilon(u_{\varepsilon,r}, A_1) \le c_p \int_{\partial A_1 \cap \partial\Omega} |\mathcal{W}(Tu(x)) - \mathcal{W}(v(x))| d\mathcal{H}^2$$
$$+ o(1), \quad \text{as } \varepsilon \to 0. \tag{5.11}$$

Step 4: Construction of $u_{\varepsilon,r}$ in B_2.
Note that in the previous steps we have constructed an optimal sequence in $\overline{A}_1 \cup \overline{A}_2$ that is $\dfrac{C}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous; in particular it is defined and Lipschitz on $((\partial A_1 \cup \partial A_2) \cap \partial B)$, for every connected component B of B_2. By virtue of Lemma 5.1 we can extend $u_{\varepsilon,r}$ to every B, obtaining a $\dfrac{C+1}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz function that

satisfies

$$
\begin{aligned}
F_\varepsilon(u_{\varepsilon,r}, B_2, \emptyset) &\equiv G_\varepsilon(u_{\varepsilon,r}, B_2) \\
&\leq \big(((C+2)^p + C_m)(\mathcal{H}^2(\partial B_2) + o(1))\big)\omega_\varepsilon \\
&= o(1) \;\; \text{as } \varepsilon \to 0,
\end{aligned} \tag{5.12}
$$

where we used that $\omega_\varepsilon := \inf_{(\partial A_1 \cup \partial A_2)\cap \partial B_2} |u_{\varepsilon,r} - u| = o(1)$ as $\varepsilon \to 0$ (since $u_{\varepsilon,r}$ is constant on each connected components of B_2).

Step 5: Construction of $u_{\varepsilon,r}$ in B_1.
We will use an optimal profile for the minimum problem (2.3). By Proposition 4.4, there exists $\psi \in L^1_{\text{loc}}(\mathbb{R}^2_+)$ such that $T\psi(t) \to \alpha'$ as $t \to -\infty$, $T\psi(t) \to \beta'$ as $t \to +\infty$ and $H_1(\psi, \mathbb{R}^2_+, \mathbb{R}) = \gamma_p$. Now, we construct a function $w_\varepsilon : \mathbb{R}^2_+ \to \mathbb{R}$ via the same method used to provide a good competitor u_δ in the proof of Proposition 4.4(Step 1).

For every $\varepsilon > 0$, $\rho_\varepsilon, \sigma_\varepsilon \in \mathbb{R}$, we take a cut-off function $\xi \in C^\infty(\mathbb{R}^2_+)$ such that $\xi \equiv 1$ on $(\mathbb{R}^2_+) \setminus D_{\rho_\varepsilon}$ and $\xi \equiv 0$ on D_{σ_ε} such that $|D\xi| \leq \frac{1}{|\rho_\varepsilon - \sigma_\varepsilon|}$. We denote by $\bar{u}$ the function expresses in polar coordinates $\theta \in [0,\pi)$, $\rho \in [0,+\infty)$, as follows:

$$
\bar{u}(\theta,\rho) := \frac{\theta}{\pi}\alpha' + \left(1 - \frac{\theta}{\pi}\right)\beta'.
$$

We define w_ε as

$$
w_\varepsilon(x) := \begin{cases}
\psi(\frac{x}{\varepsilon}) & \text{if } x \in D_{\sigma_\varepsilon}, \\
\xi(x)\bar{u}(x) + (1-\xi(x))\psi(\frac{x}{\varepsilon}) & \text{if } x \in D_{\rho_\varepsilon} \setminus D_{\sigma_\varepsilon}, \\
\bar{u}(x) & \text{if } x \in \mathbb{R}^2_+ \setminus D_{\rho_\varepsilon},
\end{cases}
$$

Let us show that we can choose ρ_ε and σ_ε such that w_ε satisfies the following inequality

$$
H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) \leq \gamma_p + o(1), \;\text{ as } \varepsilon \to 0. \tag{5.13}
$$

We have

$$
\begin{aligned}
H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) \;=\; & H_\varepsilon(\psi^{(\varepsilon)}, D_{\sigma_\varepsilon}, E_{\sigma_\varepsilon}) + \varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}} |Dw_\varepsilon|^p dx \\
& + \frac{1}{\varepsilon}\int_{E_{\rho_\varepsilon}\setminus E_{\sigma_\varepsilon}} V(Tw_\varepsilon)d\mathcal{H}^1 \\
=: \; & H_\varepsilon(\psi^{(\varepsilon)}, D_{\sigma_\varepsilon}, E_{\sigma_\varepsilon}) + I_1 + I_2,
\end{aligned} \tag{5.14}
$$

where $\psi^{(\varepsilon)}(x) := \psi(\frac{x}{\varepsilon})$.

The first integral in the right hand side of (5.14) can be easily estimated as follows

$$\begin{aligned} I_1 &\le 3^{p-1}\varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}}|D\psi^{(\varepsilon)}|^p dx + 3^{p-1}\varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}}|D\xi|^p|\psi^{(\varepsilon)}-\bar u(x)|^p dx \\ &\quad +3^{p-1}\varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}}|D\bar u|^p dx \\ &\le 3^{p-1}\int_{D_{\rho_\varepsilon/\varepsilon}\setminus D_{\sigma_\varepsilon/\varepsilon}}|D\psi|^p dx + 3^{p-1}C^p\frac{\varepsilon^{p-2}(\rho_\varepsilon^2-\sigma_\varepsilon^2)}{(\rho_\varepsilon-\sigma_\varepsilon)^p} \\ &\quad +\frac{3^{p-1}|\beta'-\alpha'|^p}{(p-2)\pi^{p-1}}\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^{p-2}. \end{aligned} \tag{5.15}$$

While using the convexity of V near its wells and the asymptotic behavior of $T\psi(\frac{x}{\varepsilon})$, for ε small, we have

$$I_2 \le \frac{1}{\varepsilon}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}} V(T\psi^{(\varepsilon)})d\mathcal{H}^1. \tag{5.16}$$

Thus, by (5.14), (5.15), (5.16), we get

$$\begin{aligned} H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) &\le H_\varepsilon(\psi^{(\varepsilon)}, D_{\sigma_\varepsilon}, E_{\sigma_\varepsilon}) + 3^{p-1}\int_{D_{\rho_\varepsilon/\varepsilon}\setminus D_{\sigma_\varepsilon/\varepsilon}}|D\psi|^p dx \\ &\quad +3^{p-1}C^p\frac{\varepsilon^{p-2}(\rho_\varepsilon^2-\sigma_\varepsilon^2)}{(\rho_\varepsilon-\sigma_\varepsilon)^p} + \frac{3^{p-1}|\beta'-\alpha'|^p}{(p-2)\pi^{p-1}}\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^{p-2}. \end{aligned} \tag{5.17}$$

We now define the transplanted function $\bar w_\varepsilon$ on $Sv\times\mathbb{R}^2_+$ by

$$\bar w_\varepsilon(x,y) := w_\varepsilon(x), \quad \text{for every } x\in Sv \text{ and every } y\in\mathbb{R}^2_+. \tag{5.18}$$

By Fubini's Theorem, we obtain

$$\begin{aligned} F_\varepsilon(\bar w_\varepsilon, Sv\times D_{\rho_\varepsilon}, Sv\times E_{\rho_\varepsilon}) &= \mathcal{H}^1(Sv)\left(H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_{D_{\rho_\varepsilon}} W(w_\varepsilon)dx\right) \\ &\le \mathcal{H}^1(Sv)\left(H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) + C\pi\frac{\rho_\varepsilon^2}{\varepsilon^{\frac{p-2}{p-1}}}\right). \end{aligned} \tag{5.19}$$

Hence, by suitably choosing ρ_ε and σ_ε (i.e., such that $(\rho_\varepsilon^2-\sigma_\varepsilon^2)/(\rho_\varepsilon-\sigma_\varepsilon)^p \to 0$ and $\varepsilon/\sigma_\varepsilon \to 0$ as $\varepsilon\to 0$. For instance, we can take $\rho_\varepsilon = \varepsilon^{\frac{p-2}{p-1}}$ and $\sigma_\varepsilon = \varepsilon^{\frac{p-2}{2(p-1)}}$), by (5.17) and (5.19), we get

$$F_\varepsilon(\bar w_\varepsilon, Sv\times D_{\rho_\varepsilon}, Sv\times E_{\rho_\varepsilon}) \le \mathcal{H}^1(Sv)\,(\gamma_p + o(1)) \quad \text{as } \varepsilon\to 0. \tag{5.20}$$

Since Sv is a boundary in $\partial\Omega$, we can construct a diffeomorphism between the intersection of a tubular neighborhood of Sv and Ω and the product of Sv with a half-disk.

For every x in Ω, let us define the oriented distance from Sv as

$$d'(x) := \begin{cases} \mathrm{dist}(x, Sv) & \text{if } x \in \{v = \beta'\}, \\ \\ -\mathrm{dist}(x, Sv) & \text{if } x \in \{v = \alpha'\}. \end{cases}$$

For every $r > 0$, we set

$$\mathcal{S}_r := \{x \in \Omega : 0 < \mathrm{dist}(x, Sv) < r\}. \tag{5.21}$$

For every $x \in \overline{\Omega}$, we define

$$\Psi(x) := (x'', d'(x'), \mathrm{dist}(x, \partial\Omega)), \tag{5.22}$$

where x' is a projection of x on $\partial\Omega$ and x'' is a projection of x' on Sv. The function Ψ is well-defined and is a diffeomorphism of class C^2 on $\overline{\Omega} \cap U$ for some neighborhood U of Sv; and satisfies the following properties: $\Psi(\Omega \cap U) = Sv \times \mathbb{R}^2_+$; $\Psi(\partial\Omega \cap U) = Sv \times \mathbb{R} \times \{0\}$; $\Psi(x) = x,$ for every $x \in \partial\Omega$; $D\Psi(x)$ is an isometry.

We have that

$$\lim_{r \to 0} \delta_r = 0,$$

where δ_r is the isometry defect of the restriction of Ψ to $\mathcal{S}_r$.

We construct $u_{\varepsilon,r}$ on $\overline{\mathcal{S}}_{\rho_\varepsilon/2}$ as

$$u_{\varepsilon,r} := \bar{w}_\varepsilon \circ \Psi,$$

where $\bar{w}_\varepsilon$, $\mathcal{S}_r$ and Ψ are defined by (5.18), (5.21) and (5.22) respectively. For ε small, the function Ψ maps $\mathcal{S}_{\rho_\varepsilon/2}$ into $Sv \times D_{\rho_\varepsilon}$ and $\partial\mathcal{S}_{\rho_\varepsilon/2} \cap \partial\Omega$ into $Sv \times E_{\rho_\varepsilon}$, so we can use Proposition 5.1 and, by (5.20), we obtain

$$\begin{aligned} F_\varepsilon(u_{\varepsilon,r}, \mathcal{S}_{\rho_\varepsilon/2}, \partial\mathcal{S}_{\rho_\varepsilon/2} \cap \partial\Omega) &\leq (1 - \delta_\varepsilon)^{-(p+3)} F_\varepsilon(\bar{w}_\varepsilon, Sv \times D_{\rho_\varepsilon}, Sv \times E_{\rho_\varepsilon}) \\ &\leq \mathcal{H}^1(Sv)\big(\gamma_p + o(1)\big) \text{ as } \varepsilon \to 0, \end{aligned} \tag{5.23}$$

where we also used that $\delta_\varepsilon := \delta(\Psi\lfloor\mathcal{S}_{\rho_\varepsilon})$ tends to 0 as $\varepsilon \to 0$.

Notice that for ε small enough, Ψ is 2-Lipschitz continuous. Using again Lemma 5.1, we can extend $u_{\varepsilon,r}$ by setting $u_{\varepsilon,r} := v$ on the remaining part of $\partial B_1 \cap \partial\Omega$; we have that $u_{\varepsilon,r}$ is equal to v on $\partial\Omega \setminus \partial\mathcal{S}_{\rho_\varepsilon/2}$. Thus, we can extend $u_{\varepsilon,r}$ on the whole $B_1 \setminus \mathcal{S}_{\rho_\varepsilon/2}$ to a $\dfrac{2C+1}{\varepsilon^{(p-2)/(p-1)}}$-Lipschitz continuous function, which satisfies

$$\begin{aligned} F_\varepsilon(u_{\varepsilon,r}, B_1 \setminus \overline{\mathcal{S}}_{\rho_\varepsilon/2}, \partial(B_1 \setminus \overline{\mathcal{S}}_{\rho_\varepsilon/2}) \cap \partial\Omega) &= G_\varepsilon(u_{\varepsilon,r}, B_1 \setminus \overline{\mathcal{S}}_{\rho_\varepsilon/2}) \\ &\leq \big((2C+2)^p + C_m\big)\big(\mathcal{H}^2(\partial B_1) + o(1)\big) 2m \\ &\quad \text{as } \varepsilon \to 0, \end{aligned} \tag{5.24}$$

where we used $\|u_{\varepsilon,r} - \alpha\|_\infty \wedge \|u_{\varepsilon,r} - \beta\|_\infty \leq 2m$.

Step 6: Upper bound inequality. We recall that for every $r > 0$ and every $\varepsilon < r^{\frac{p-2}{p-1}}$ we have constructed a function $u_{\varepsilon,r}$ defined on the whole Ω such that

$$\limsup_{\varepsilon\to 0} \|u_{\varepsilon,r} - u\|_{L^1(\Omega)} \leq 2m(|B_1| + |B_2|) \text{ and } \limsup_{\varepsilon\to 0} \|Tu_{\varepsilon,r} - v\|_{L^1(\partial\Omega)} = 0.$$

Since $|B_1|$ and $|B_2|$ have order r^2 and r respectively, we get that $u_{\varepsilon,r} \to u$ in $L^1(\Omega)$, first taking $\varepsilon \to 0$ and then $r \to 0$.

Combining (5.10), (5.11), (5.12), (5.23) and (5.24), we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon(u_{\varepsilon,r}) \leq \sigma_p \mathcal{H}^2(Su) + c_p \int_{\partial\Omega} |\mathcal{W}(Tu(x)) - \mathcal{W}(v(x))| d\mathcal{H}^2 \\ + \gamma_p \mathcal{H}^1(Sv) - \sigma_p \mathcal{H}^2(Su \setminus A_2) \\ + \big((2C+2)^p + C_m\big)\big(\mathcal{H}^2(\partial B_1) + o(1)\big)2m. \qquad (5.25)\end{aligned}$$

Since $\mathcal{H}^2(\partial B_1)$ has order r, taking r to 0 in (5.25), we deduce the upper bound inequality (iii). Finally, applying a suitable diagonalization argument (see for instance Ref. 5, Corollary 1.18) to the sequence $u_{\varepsilon,r}$, we obtain the desired recovery sequence u_ε. This concludes the proof. □

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