Differentiability from Sobolev-Poincaré inequality

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Abstract

We provide various differentiability theorems for both functions of bounded variation and Sobolev functions in stratified groups. The proofs are based on a systematic application of different Sobolev-Poincaré inequalities.

Contents

Introduction

The central role of Poincaré inequality in Analysis and Geometry is currently the subject of several works. In particular, its connection with regularity theory for PDE's and with theory of Sobolev spaces have been widely studied, highlighting connections with different areas of Analysis. Among these, we mention the geometry of Carnot-Carathéodory spaces, the theory of quasiconformal mappings on Carnot groups, analysis on topological manifolds and on fractals. A thorough account of these developments can be found in [18], [29] and in the reference therein. In the present paper we attempt to follow the same principles, namely, we explore the connection between Poincaré inequality and fine differentiability properties.

We set our study in stratified groups, [12], which include Euclidean spaces and several types of different geometries. For instance, the Heisenberg group is a well known example of noncommutative stratified group, [28]. The classical notions of Sobolev functions and of functions with bounded variation naturally extend to the general context of stratified groups. We consider a set of left invariant vector fields $X_1, \ldots, X_m$ and we say that a locally summable map u on an open subset Ω of $\mathbb{G}$ has H-bounded variation if its distributional derivative $X_j u$ is a finite Radon measure for every $j = 1, \ldots, m$. If all distributional derivatives $X_{j_1} \cdots X_{j_p} u$ are measures for every $p \leq k$ and $j_s = 1, 2, \ldots, m$ for every $s = 1, 2, \ldots, p$, then we say that u has H-bounded k-variation, namely, $u \in BV_H^k(\Omega)$. The H-Sobolev space $W_H^{k,p}(\Omega)$ is defined in analogous fashion, it is then clear the central role played by the vector fields $\{X_j\}$. In fact, under the well known Hörmander condition, they also yield the geometry of the group, defining the so-called Carnot-Carathéodory distance, see Section 1.

In order to state our results we introduce the notion of "L^p differentiability of order k". In the Euclidean context this definition goes back to Calderón and Zygmund [5], [6]. A function $u \in L^p_{loc}(\Omega)$ is said to be L^p differentiable of order k at $x \in \Omega$ if there exists a polynomial $P_{[x]}$ of homogeneous degree less than or equal to k such that

$$\left(\fint_{B_{x,r}} |u(y) - P_{[x]}(y)|^p dy \right)^{1/p} = o(r^k) \quad \text{as} \quad r \to 0^+. \tag{1}$$

The anisotropy of stratified groups requires the notion of homogeneous degree of a polynomial, which amounts to take into account the "weight" of every coordinate, see Subsection 1.2. The polynomial $P_{[x]}$ is uniquely defined and it is called "L^p differential of order k".

It is well known that L^p differentiability turns out to be the natural concept to study fine differentiability of functions with bounded variation and Sobolev functions in Euclidean spaces. Concerning Euclidean results on L^p differentiability and approximation theorems for Sobolev functions we quote some relevant papers, [2], [3], [4], [10], [31], see also [27] and [32]. In stratified groups these questions are not still systematically studied and a complete description of fine differentiability properties of H-Sobolev functions and of functions with H-bounded variation presently lacks. The first results in this direction can be found in [1], where it is proved that functions of H-bounded k-variation are a.e. L^1 differentiable of order k.

The present paper can be thought of as a continuation of [1], where we improve the previous results using a simplified approach based on the Sobolev-Poincaré inequality as a main tool. For instance, we do not use maximal function and Rademacher theorem. In this way we also provide a new proof of Rademacher theorem on stratified groups, following Lemma 3.1.5 of [11].

Now we present our first order differentiability results. Let $u : \Omega \longrightarrow \mathbb{R}$ be a function of H-bounded variation and let $\beta = Q/(Q-1)$, where Q is the Hausdorff dimension of the group. Then u is a.e. L^β differentiable and the differential corresponds to the density of $D_H u^a$, which is the absolutely continuous part of the the measure $D_H u$, see Section 1 for precise definitions. The Euclidean version of this result has been first proved by Calderón and Zygmund, [5], see also the Federer's proof in Theorem 4.5.9(26) of [11]. We also prove that every function $u \in W^{1,p}_H(\Omega)$ is a.e. L^p differentiable and that in the case $p < Q$, defining $\gamma = Qp/(Q-p)$, every $u \in W^{1,p}_H(\Omega)$ is a.e. L^γ differentiable.

In order to highlight the common feature of all previous results we give their more concise presentation. Let us consider a (q,p)-Sobolev-Poincaré inequality of the following form

$$\left(\fint_{B_{x,r}} |u(y) - u_{B_{x,r}}|^q \, dy \right)^{1/q} \leq C \, r \, \left(\frac{\mu_p(B_{x,r})}{|B_{x,r}|} \right)^{1/p}, \tag{2}$$

where $1 \leq p \leq q$, $u \in W^{1,p}_H(\Omega)$ if $p > 1$ and $u \in BV_H(\Omega)$ if $p = 1$. The measure μ_p is defined as follows

$$\mu_p(A) = \begin{cases} \int_A |\nabla_H u(y)|^p \, dy & \text{if } p > 1 \\ |D_H u|(A) & \text{if } p = 1 \end{cases}, \tag{3}$$

where $|D_H u|$ is the total variation of u, see Section 1.

All first order differentiability results are resumed saying that inequality (2) gives the a.e. L^q differentiability of functions either in $W^{1,p}_H(\Omega)$ if $p > 1$ or in $BV_H(\Omega)$ if $p = 1$. These results turn out to be a first step in order to reach a more precise characterization of H-Sobolev functions by L^p differentiability. Such improvement would require the use of capacity in place of Lebesgue measure, [2], and certainly represents a possible new line of investigation.

Let us explain the elementary idea which allows us to obtain in a rather simple way first order differentiability results by Sobolev-Poincaré inequality. Actually we also utilize an important consequence of Poincaré inequality, namely, the so-called "representation formula": there exists a dimensional constant C such that for every $u \in C^\infty(\Omega)$ and every open ball $B_{x,2r}$ compactly contained in Ω we have the pointwise estimate

$$|u(x) - u_{B_{x,r}}| \leq C \int_{B_{x,r}} \frac{|\nabla_H u(y)|}{\rho(x,y)^{Q-1}} \, dy, \tag{4}$$

where $u_{B_{x,r}} = |B_{x,r}|^{-1} \int_{B_{x,r}} u$ and $|\nabla_H u(y)| = \left(\sum_{j=1}^m X_j u(y)^2 \right)^{1/2}$. This formula on stratified groups has been proved by Lu, [22]. Through a general method developed in spaces of homogeneous type, [14], it is possible to show that (4) is a direct consequence of Poincaré inequality.

By Fubini's theorem, inequality (4) gives the crucial estimate

$$|u(x) - u_{B_{x,r}}| \leq C\,r \left\{ \int_0^1 \left(\fint_{B_{x,tr}} |\nabla_H u(y)|^p \, dy \right)^{1/p} dt + \left(\fint_{B_{x,r}} |\nabla_H u(y)|^p \, dy \right)^{1/p} \right\} \tag{5}$$

for every $1 \leq p < \infty$, where we have used the standard notation $\fint_A u = |A|^{-1} \int_A u$. Assume for simplicity that u is smooth and consider $v(y) = u(y) - u(x) - \nabla_H u(x)^*(x^{-1}y)$. This gives the conditions $v(x) = 0$ and

$$\left(\fint_{B_{x,r}} |\nabla_H v(y)|^p \, dy \right)^{1/p} \longrightarrow 0 \quad \text{as} \quad r \to 0^+. \tag{6}$$

By p-Poincaré inequality (12) we have

$$\left(\fint_{B_{x,r}} |v(y)|^p \, dy \right)^{1/p} \leq C\,r \left(\fint_{B_{x,r}} |\nabla_H u(y)|^p \, dy \right)^{1/p} + |v_{B_{x,r}}|,$$

then applying (5) to v and using (6) it follows that $\left(\fint_{B_{x,r}} |v|^p \right)^{1/p} = o(r)$ as $r \to 0^+$. This proves that u is L^p differentiable of order one at x. Notice that (5) immediately gives us the required rate of convergence and that Sobolev-Poincaré inequality with the right exponent of summability gives the same rate of convergence. For instance, in the case $p < Q$ the Sobolev-Poincaré inequality (13) gives the estimate

$$\left(\fint_{B_{x,r}} |v(y)|^\gamma \, dy \right)^{1/\gamma} \leq C\,r \left(\fint_{B_{x,r}} |\nabla_H u(y)|^p \, dy \right)^{1/p} + |v_{B_{x,r}}|,$$

where $\gamma = Qp/(Q-p)$, then we achieve the L^γ differentiability at x. The same argument works for functions of H-bounded variation using the estimate

$$|u(x) - u_{B_{x,r}}| \leq C\,r \left\{ \int_0^1 \frac{|D_H u|(B_{x,tr})}{(tr)^Q} dt + \frac{|D_H u|(B_{x,r})}{r^Q} \right\} \tag{7}$$

in place of (5). The higher order L^p differentiability of functions in $W^{k,p}_H(\Omega)$ and $BV^k_H(\Omega)$ hides further difficulties due to the non commutativity of vector fields $\{X_j\}$. In fact, the characterization of the differential $P_{[x]}$ in (1) using distributional derivatives of u requires a basis of left invariant differential operators on $\mathbb{G}$. By the Poincaré-Birkhoff-Witt theorem this basis can be constructed using a basis of left invariant vector fields, hence Proposition 3.1 provides the characterization of the differential $P_{[x]}$. This proposition plays a crucial role in the argument by induction which connects first order differentiability results to higher order differentiability.

Finally, we give a brief overview of the paper. Section 1 is devoted to preliminary notions and the basic results which will be used in the subsequent sections. We show the simple argument that leads us to the pointwise estimates (5) and (7). We also prove Lemma 1.14, which allows to strengthen our differentiability results, as we explain below.

In Section 2 we prove first order differentiability, which have been already described along the first part of this introduction. We also show another proof of Rademacher theorem on stratified groups relying on the a.e. L^1 differentiability of functions of H-bounded variation.

Section 3 deals with higher order differentiability results. Following the approach of this paper, we add a simplified proof of the main result in [1]: every function u of H-bounded k-variation is a.e. L^1 differentiable of order k. The same technique works for H-Sobolev functions, namely, we prove that every $u \in W_H^{k,p}(\Omega)$ is a.e. L^p differentiable of order k.

In Section 4 we improve higher order differentiability in some special situations. Let $k < Q$ and let $u \in BV_H^k(\Omega)$. Then u is a.e. L^γ differentiable of order k, where $\gamma = Q/(Q-k)$. This result in Euclidean spaces fits in the case (i) of Theorem 1 in [10]. Note that the Euclidean proof of [10] uses minimizing polynomials, which suitably replace the Taylor expansion of the function, hence following a completely different approach. In the case $kp < Q$ we prove that every function $u \in W^{k,p}(\Omega)$ is a.e. L^η differentiable, where $\eta = Qp/(Q-kp)$.

In closing we add an elementary remark which applies to every of our differentiability theorems. Using the Hedberg technique, [19], we obtain Lemma 1.14, due to which every function $u : \Omega \longrightarrow \mathbb{R}$, which is L^p differentiable of order k at $x \in \Omega$ satisfies the slightly stronger differentiability condition

$$\lim_{r \to 0^+} \fint_{B_{x,r}} \left(\frac{|u(y) - P_{[x]}(y)|}{\rho(x,y)^k} \right)^p dy = 0. \tag{8}$$

This strengthened version of differentiability settles a question of [1] on the possibility of obtaining (8) by a suitable integral estimate of the difference quotient.

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1 Preliminaries

A *stratified group* is a simply connected nilpotent Lie group $\mathbb{G}$ endowed with a graded Lie algebra $\mathcal{G}$, which is decomposed into a direct sum of subspaces V_j subject to the condition $V_{j+1} = [V_j, V_1]$ for every $j \in \mathbb{N} \setminus \{0\}$, and $V_j = \{0\}$ whenever j is greater than a positive integer. We denote by ι the maximum integer such that $V_\iota \neq \{0\}$ and we call it the *nilpotence degree* of the group or the *step* of the group. Recall that for arbitrary subspaces $V, W \subset \mathcal{G}$ we have defined $[V,W] = \mathrm{span}\{[X,Y] \mid X \in V,\ Y \in W\}$.

The assumption that $\mathbb{G}$ is simply connected and nilpotent ensures that the exponential map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism. The grading of $\mathcal{G}$ allows us to define dilations on the group, i.e. maps $\delta_t : \mathcal{G} \longrightarrow \mathcal{G}$ with $\delta_t(\sum_{j=1}^{\iota} v_j) = \sum_{j=1}^{\iota} t^j v_j$, where $t > 0$ and $\sum_{j=1}^{\iota} v_j \in \mathcal{G}$ is the unique representation of a vector of $\mathcal{G}$, provided that $v_j \in V_j$ for every $j = 1, \ldots, \iota$. This notion of dilation is motivated by the fact that the composition $\exp \circ \delta_t \circ \ln : \mathbb{G} \longrightarrow \mathbb{G}$ is a group homomorphism, where we have denoted $\ln = \exp^{-1}$. We will use the same symbol to denote dilations which are read on the group. The underlying metric of the group is a left invariant Riemannian metric g such that the subspaces V_j are orthogonal each other. We will always refer to these metrics, called *graded metrics*. The Riemannian volume on $\mathbb{G}$ given by a graded metric will be denoted by v_g. It is clear that v_g is left invariant, hence it is the Haar measure of the group. For ease of notation, we will also write $v_g(A) = |A|$, where $A \subset \mathbb{G}$ is a measurable set and we will use the symbol dx when the integration is considered with respect to the Riemannian volume measure v_g. The averaged integral of a summable map $u : A \longrightarrow \mathbb{R}$ is defined as $u_A = \fint_A u = |A|^{-1} \int_A u$.

The Carnot-Carathéodory distance between two points x and x' is obtained by taking the infimum among lengths of absolutely continuous curves a.e. tangent to the horizontal subbundle and which connect x with x'. The length of connecting curves is computed by a graded metric g, hence the Carnot-Carathéodory distance ρ is left invariant and it has the important scaling property $\rho(\delta_t x, \delta_t x') = t\rho(x, x')$, where $x, x' \in \mathbb{G}$, see for instance [17]. We denote by $B_{x,r}$ the open ball of center x and radius $r > 0$ with respect to the Carnot-Carathéodory distance. We will frequently use the following scaling property

$$|B_{x,r}| = |B_1|\, r^Q \tag{9}$$

for every $x \in \mathbb{G}$ and any $r > 0$. Balls of radius r centered at the unit element e of the group will be denoted simply by B_r. The integer Q is the Hausdorff dimension of $\mathbb{G}$ with respect to the Carnot-Carathéodory distance and it is strictly greater than the topological dimension q of the group whenever $\mathbb{G}$ is not Abelian. More information on stratified groups can be found for instance in [9], [12] and [28].

The space V_1 defines the *horizontal subbundle* of the group. We introduce the *horizontal space* $H_p\mathbb{G} = \{X(p) \mid X \in V_1\}$ at $p \in \mathbb{G}$ and consider the disjoint union of all these subspaces with the relevant vector bundle topology. We denote by $H\mathbb{G}$ the horizontal subbundle. Smooth sections of the horizontal subbundle are called *horizontal vector fields*. We assume throughout the paper that Ω is an open subset of a stratified group $\mathbb{G}$. The space of smooth horizontal vector fields of Ω is denoted by $\Gamma(H\Omega)$ and the one of compactly supported horizontal vector fields by $\Gamma_c(H\Omega)$. Note that a horizontal vector field φ can be written as $\sum_{j=1}^m \varphi_j X_j$, where $(X_1, X_2, \ldots, X_m)$ is a basis of the first layer $V_1 \subset \mathcal{G}$. The *H-divergence* of φ is defined as $\sum_{j=1}^m X_j \varphi_j$ and it is denoted by $\mathrm{div}_H \varphi$. Note that the definition of H-divergence does not depend on the choice of the basis of V_1. These notions allow us to introduce functions of bounded variations naturally associated to the sub-Riemannian structure of the group, see [7], [15] for more information.

Definition 1.1 (H-BV functions) We say that a function $u \in L^1(\Omega)$ is a function of H-*bounded variation* (in short, an *H-BV function*) if

$$|D_H u|(\Omega) := \sup \left\{ \int_\Omega u(x)\ \mathrm{div}_H \varphi(x)\ dx \,\middle|\, \varphi \in \Gamma_c(H\Omega),\ |\varphi| \leq 1 \right\} < \infty \,,$$

We denote by $BV_H(\Omega)$ and $BV_{H,loc}(\Omega)$ the space of functions of H-bounded variation and of locally H-bounded variation, respectively.

By Riesz representation theorem we get the existence of a nonnegative Radon measure $|D_H u|$ and of a Borel section ν of $H\Omega$ such that $|\nu(x)| = 1$ for $|D_H u|$-a.e. $x \in \Omega$. Moreover for every $\phi \in \Gamma_c(H\Omega)$ the following integration by parts formula holds

$$\int_\Omega u(x)\, \mathrm{div}_H \phi(x)\, dx = - \int_\Omega \langle \phi, \nu \rangle\, d\, |D_H u| \,. \tag{10}$$

The vector valued measure $\nu\, |D_H u|$ is denoted by $D_H u$. The symbols $|D_H u|^a$ and $|D_H u|^s$ stand for the absolutely continuous part and the singular part of $|D_H u|$, respectively. We also define the vector measures $D_H^a u = \nu\, |D_H u|^a$ and $D_H^s u = \nu\, |D_H u|^s$.

The density of $D_H^a u$ with respect to the Haar measure of the group is denoted by $\nabla_H u$. In our arguments we will consider $\nabla_H u$ as a measurable vector functions with values in $\mathbb{R}^m$. This is possible whenever we refer to an orthonormal basis $(X_1, X_2, \ldots, X_m)$ of V_1. Note that if u is of class C^1 then $\nabla_H u = (X_1 u, \ldots, X_m u)$.

Definition 1.2 (H-BV^k functions) By induction on $k \geq 2$ we say that a measurable map $u : \Omega \longrightarrow \mathbb{R}$ has *H-bounded k-variation* (in short, H-BV^k) if for any $i = 1, \ldots, m$ the distributional derivatives $X_i u$ are representable by functions with H-bounded $(k-1)$-variation. We denote by $BV_H^k(\Omega)$ the space of all H-BV^k functions.

Definition 1.3 (H-Sobolev functions) A function $u \in L^p(\Omega)$ belongs to $W_H^{k,p}(\Omega)$ if for all $j_s \in \{1, \ldots, m\}$ and $s = 1, 2, \ldots, k$ there exists a function $v_{j_1,\ldots,j_k} \in L^p(\Omega)$ such that

$$\int_\Omega u(y)\ (X_{j_1} \cdots X_{j_k} \phi)(y)\, dy = (-1)^k \int_\Omega v_{j_1,\ldots,j_k}(y)\, \phi(y)\, dy \,, \tag{11}$$

for every $\phi \in C_c^\infty(\Omega)$, where $(X_1, \ldots, X_m)$ is an orthonormal basis of V_1.

Note that the Definitions 1.2 and 1.3 do not depend on the choice of the orthonormal basis $(X_1, \ldots, X_m)$.

1.1 Poincaré and Sobolev-Poincaré inequalities

Important tools for our study are the Poincaré inequality and the Sobolev-Poincaré inequality in stratified groups. These inequalities hold for smooth functions in the more general Carnot-Carathéodory spaces endowed with a system of Hörmander vector fields. As pointed out in [23], the Jerison result [21] includes the Poincaré inequality with respect to the L^p norm for every $p \in [1, +\infty[$.

Theorem 1.4 (Poincaré inequality) *Let Ω be an open bounded subset of $\mathbb{G}$ and let $1 \leq p < \infty$. Then there exists a constant $C > 0$ such that for every $u \in C^\infty(\Omega)$ and any ball $B_{x,2r}$ compactly contained in Ω we have*

$$\int_{B_{x,r}} |u(y) - u_{B_{x,r}}|^p \, dy \leq C r^p \int_{B_{x,r}} |\nabla_H u(y)|^p \, dy. \tag{12}$$

The following Sobolev-Poincaré inequality has been proved in [13], [22] and [30].

Theorem 1.5 (Sobolev-Poincaré inequality) *Let Ω be an open bounded subset of $\mathbb{G}$ and let $1 \leq p < Q$. Then there exists a constant $C > 0$ such that for every $u \in C^\infty(\Omega)$, setting $\gamma = Qp/(Q-p)$, we have*

$$\left(\fint_{B_{x,r}} |u(y) - u_{B_{x,r}}|^\gamma \, dy \right)^{1/\gamma} \leq C \, r \left(\fint_{B_{x,r}} |\nabla_H u(y)|^p \, dy \right)^{1/p}. \tag{13}$$

The smooth approximation of functions of bounded variation is available also in general Carnot-Carathéodory spaces, [15], [16], then the previous Sobolev-Poincaré inequality in the case $p = 1$ can be extended to H-BV functions.

Theorem 1.6 *Let Ω be an open bounded subset of $\mathbb{G}$ and let $\beta = Q/(Q-1)$. Then there exists a constant $C > 0$ such that for every $u \in BV_{H,loc}(\Omega)$ and any ball $B_{x,2r}$ compactly contained in Ω we have*

$$\left(\fint_{B_{x,r}} |u(z) - u_{B_{x,r}}|^\beta \, dz \right)^{1/\beta} \leq C \, r \, \frac{|D_H u|(B_{x,r})}{|B_{x,r}|}. \tag{14}$$

Note that as a consequence of (14) we have the following Poincaré inequality for H-BV functions

$$\int_{B_{x,r}} |u(z) - u_{B_{x,r}}| \, dz \leq C \, r \, |D_H u|(B_{x,r}). \tag{15}$$

1.2 Approximate regularity and L^p differentiability of order k

In this subsection we introduce several notions of weak regularity for measurable functions in stratified groups. We begin recalling the notion of approximate continuity.

Definition 1.7 (Approximate continuity) We say that $u \in L^1_{loc}(\Omega, \mathbb{R}^m)$ has an *approximate limit* $\lambda \in \mathbb{R}^m$ at $x \in \Omega$ if

$$\lim_{r \to 0^+} \fint_{B_{x,r}} |u(y) - \lambda| \, dy = 0 \, .$$

In the case $\lambda = u(x)$ we say that x is an *approximate continuity point* and we denote by A_u the subset of all *approximate continuity points.*

Remark 1.8 Notice that the approximate limit is uniquely defined and it does not depend on the representative element of u. In the next proposition we recall the p-approximate continuity of functions in $L^p(\Omega)$ using the in doubling property of $\mathbb{G}$. As a consequence of this result for $p = 1$ we have that $|\Omega \setminus A_u| = 0$.

Proposition 1.9 *Let $u \in L^p_{loc}(\Omega, \mathbb{R}^m)$ and let $1 \le p < \infty$. Then for a.e. $x \in \Omega$ we have*

$$\fint_{B_{x,r}} |u(y) - u(x)|^p \, dy = 0 \tag{16}$$

PROOF. Stratified groups endowed with their left invariant volume measure are in particular doubling spaces. Therefore Theorem 2.9.8 of [11] is available and it implies that for a.e. $x \in \Omega$ and for every $q \in \mathbb{Q}^m$ there exists the limit

$$\lim_{r \to 0^+} \fint_{B_{x,r}} |u(y) - q|^p \, dy = |u(x) - q|^p ,$$

then we get

$$\limsup_{r \to 0^+} \left(\fint_{B_{x,r}} |u(y) - u(x)|^p \right)^{1/p} \le 2 \, |u(x) - q| \to 0 \quad \text{as} \quad q \to u(x),$$

this completes the proof. $\square$

Definition 1.10 We define $n_j = \dim V_j$ for any $j = 1, \ldots, \iota$, $m_0 = 0$ and $m_i = \sum_{j=1}^{i} n_j$ for any $i = 1, \ldots, \iota$. We say that $(W_1, \ldots, W_q)$ of $\mathcal{G}$ is an *adapted basis*, if

$$(W_{m_{j-1}+1}, \ldots, W_{m_j}) \tag{17}$$

is a basis of V_j for any $j = 1, \ldots, \iota$.

In order to tackle the higher order approximate differentiability of H-BV^k functions we will recall some well known facts about polynomials on stratified groups. We will refer to Chapter 1.C of [12]. A function $P : \mathbb{G} \longrightarrow \mathbb{R}$ is a *polynomial on* $\mathbb{G}$ if the composition $P \circ \exp$ is a polynomial on $\mathcal{G}$. Let us fix adapted basis $(W_1, \ldots, W_q)$ and its dual basis of 1-forms $(\eta_1, \ldots, \eta_q)$. Then every polynomial $P : \mathbb{G} \longrightarrow \mathbb{R}$ can be represented as follows

$$P = \sum_{\alpha} c_\alpha \, x^\alpha, \tag{18}$$

where we have fixed a coordinate system $(x_1, \ldots, x_q)$ on $\mathbb{G}$ given by $x_i = \eta_i \circ \exp^{-1}$, where $i = 1, \ldots, q$ and for every $\alpha \in \mathbb{N}^q$ we have defined $x^\alpha = x_1^{\alpha_1} \cdots x_q^{\alpha_q}$ and only a finite number of coefficients $c_\alpha \in \mathbb{R}$ do not vanish.

The degree d_i of W_i is determined by the unique relation $W_i \in V_{d_i}$. Then we define the *homogeneous degree* of a polynomial $P : \mathbb{G} \longrightarrow \mathbb{R}$ with expression (18) as

$$\text{h-deg}(P) = \max\{d(\alpha) \mid c_\alpha \neq 0\},$$

where $d(\alpha) = \sum_{k=1}^q d_k \alpha_k$. The space of polynomials of homogeneous degree less than or equal to k will be denoted by $\mathcal{P}_{H,k}(\mathbb{G})$.

Next, we extend the definition of L^p differentiability of order k to stratified groups. This notion is due to Calderón and Zygmund, see [5] and [6].

Definition 1.11 (L^p differentiability of order k) A function $u \in L^1_{loc}(\Omega)$ is said to be *L^p differentiable of order k at $x \in \Omega$* if there exists a polynomial $T_{[x]} \in \mathcal{P}_{H,k}(\mathbb{G})$ such that,

$$\left(\fint_{B_{x,r}} |u(y) - T_{[x]}(y)|^p \, dy \right)^{1/p} = o(r^k) \quad \text{as} \quad r \to 0^+. \tag{19}$$

In the original terminology of Calderon and Zygmund, functions of previous definition were said to have a *derivative of order k in the L^p sense*. Notice that the L^p differential of order k is always uniquely defined. This follows by the next lemma and taking into account the inequality

$$\left(\fint_{B_{x,r}} |u(y) - T_{[x]}(y)|^r \, dy \right)^{1/r} \leq \left(\fint_{B_{x,r}} |u(y) - T_{[x]}(y)|^p \, dy \right)^{1/p}$$

for every $1 \leq r < p < \infty$. The same inequality ensures that L^p differentiability of order k implies L^r differentiability of the same order for all $1 \leq r \leq p$.

Lemma 1.12 (Uniqueness) *Let $P \in \mathcal{P}_{H,k}(\mathbb{G})$ be such that*

$$\lim_{r \to 0^+} \fint_{B_{x,r}} \frac{|P(y)|}{r^k} \, dy = 0.$$

Then P is the null polynomial.

PROOF. By a left translation at the unit element of the group we can assume that $x = e$. We fix a coordinate system $(x_1, \dots, x_q)$ on $\mathbb{G}$, hence we can write $P(y) = \sum_{d(\alpha)\le k} c_\alpha y^\alpha$. Suppose by contradiction that P is not vanishing. Then it is well defined the integer $p = \min\{d(\alpha) \mid c_\alpha \neq 0\}$ and $P_1(y) = \sum_{d(\alpha)=p} c_\alpha y^\alpha$. By the homogeneity of P_1 it follows that

$$P(\delta_r y) = r^p \Big(P_1(y) + \sum_{d(\alpha)>p} c_\alpha\, r^{d(\alpha)-p}\, y^\alpha \Big).$$

Defining $Q_r(y) = \sum_{d(\alpha)>p} c_\alpha\, r^{d(\alpha)-p}\, y^\alpha$, we note that

$$\sup_{y\in B_1} |Q_r(y)| \longrightarrow 0 \quad \text{as} \quad r \to 0^+.$$

By a change of variable we have

$$\fint_{B_r} \frac{|P(y)|}{r^k}\, dy = \fint_{B_1} \frac{|P(\delta_r y)|}{r^k}\, dy \longrightarrow 0 \quad \text{as} \quad r \to 0^+. \tag{20}$$

Then for a sufficiently small $r \in]0,1[$ we obtain that

$$\fint_{B_1} \frac{|P(\delta_r y)|}{r^k}\, dy \ge r^{p-k} \Big(\fint_{B_1} |P_1(y)|\, dy - \fint_{B_1} |Q_r(y)|\, dy \Big) \ge \frac{1}{2} \fint_{B_1} |P_1(y)|\, dy$$

Observing that P_1 is a nonvanishing polynomial the previous inequality and limit (20) lead us to a contradiction, then $P = 0$. □

Remark 1.13 The next lemma proves that condition (19) is equivalent to the following limit

$$\lim_{r\to 0^+} \fint_{B_{x,r}} \left(\frac{|u(y) - T_{[x]}(y)|}{\rho(x,y)^k} \right)^p dy = 0. \tag{21}$$

Lemma 1.14 *Let $g \in L^1_{loc}(\Omega)$ such that $g \ge 0$ a.e. in Ω and let $\alpha > 0$. Then the condition $\fint_{B_{x,r}} g = o(r^\alpha)$ as $r \to 0^+$ implies that*

$$\lim_{r\to 0^+} \fint_{B_{x,r}} \frac{g(y)}{\rho(x,y)^\alpha}\, dy = 0. \tag{22}$$

PROOF. Let $x \in \Omega$ and let $r > 0$ be sufficiently small. We define $r_k = r2^{-k}$ for every integer $k \ge 0$. Then we have

$$\begin{aligned} \fint_{B_{x,r}} \frac{g(y)}{\rho(x,y)^\alpha}\, dy &\le \frac{2^\alpha}{|B_{x,r}|} \sum_{k=0}^{\infty} \frac{1}{r_k^\alpha} \int_{B_{x,r_k}} g(y)\, dy = 2^\alpha \sum_{k=0}^{\infty} \frac{2^{-kQ}}{r_k^\alpha} \fint_{B_{x,r_k}} g(y)\, dy \\ &\le \frac{2^\alpha}{1-2^{-Q}} \sup_{0<s\le r} s^{-\alpha} \fint_{B_{x,s}} g(y)\, dy \longrightarrow 0 \end{aligned}$$

as $r \to 0^+$. □

1.3 Representation formula

Another crucial fact to obtain our differentiability results is the "representation formula" stated in the next theorem. In stratified groups this formula has been first obtained by Lu, [22]. For the reader's convenience we give a short proof of this result adapting the method of [14] to H-BV functions in stratified groups.

Theorem 1.15 (Representation formula) *Let $\Omega \subset \mathbb{G}$ be a bounded open set. Then there exists a constant $C > 0$ such that for every ball $B_{x,2r}$ compactly contained in Ω we have*

$$|u(x) - u_{B_{x,r}}| \leq C \int_{B_{x,r}} \frac{1}{\rho(x,z)^{Q-1}} \; d|D_H u|(z). \tag{23}$$

for every $u \in BV_{H,loc}(B_{x,r})$ whenever $x \in B_{x,r} \cap A_u$.

Proof. Let us define $r_j = r2^{-j}$ and $u_{B_{x,r_j}} = \fint_{U_{x,r_j}} u(y)dy$ for every $j \in \mathbb{N}$. By the fact that $x \in A_u$ and the Poincaré inequality (15) we obtain the following chain of estimates

$$\begin{aligned}
&|u(x) - u_{B_{x,r}}| \leq \sum_{j\in\mathbb{N}} |u_{B_{x,r_{j+1}}} - u_{B_{x,r_j}}| \leq \sum_{j\in\mathbb{N}} \fint_{B_{x,r_{j+1}}} |u - u_{B_{x,r_j}}| \\
&\leq 2^Q \sum_{j\in\mathbb{N}} \fint_{B_{x,r_j}} |u - u_{B_{x,r_j}}| \leq \frac{2^Q\, C}{|B_1|} \sum_{j\in\mathbb{N}} r_j^{1-Q} \int_{B_{x,r_j}} d|D_H u| \\
&= \frac{2^Q\, C}{|B_1|} \int_{B_{x,r}} \sum_{j\in\mathbb{N}} r_j^{1-Q} \mathbf{1}_{B_{x,r_j}} \, d|D_H u| = \frac{2^Q\, C}{|U_1|)} \int_{B_{x,r}} \sum_{2^j < r/\rho(x,y)} r_j^{1-Q} d|D_H u| \,.
\end{aligned}$$

Now we fix $\varepsilon = (Q-1)/2 > 0$ and observe that for any $y \in B_{x,r_j}$ we have

$$r_j^{1-Q} \leq \left(\frac{\rho(x,y)}{r_j}\right)^{\varepsilon} \rho(x,y)^{1-Q} \,,$$

hence the previous inequalities yield

$$|u(x) - u_{B_{x,r}}| \leq \frac{2^Q\, C}{|B_1|} \int_{B_{x,r}} \rho(x,y)^{1-Q} \sum_{2^j < r/\rho(x,y)} \left(\frac{\rho(x,y)}{r_j}\right)^{\varepsilon} d|D_H u| \,. \tag{24}$$

Now fix N_y as the maximum integer $j \in \mathbb{N}$ such that $2^j < r/\rho(x,y)$, obtaining

$$\begin{aligned}
&\sum_{2^j < r/\rho(x,y)} \left(\frac{\rho(x,y)}{r_j}\right)^{\varepsilon} = \left(\frac{\rho(x,y)}{r}\right)^{\varepsilon} \sum_{j=0}^{N_y} 2^{\varepsilon\, j} = \left(\frac{\rho(x,y)}{r}\right)^{\varepsilon} \frac{2^{\varepsilon\, N_y + \varepsilon} - 1}{2^{\varepsilon} - 1} \\
&\leq \left(\frac{\rho(x,y)}{r}\right)^{\varepsilon} 2^{\varepsilon\, N_y} \, \frac{2^{\varepsilon}}{2^{\varepsilon} - 1} \leq \frac{2^{\varepsilon}}{2^{\varepsilon} - 1} = c_Q \,.
\end{aligned}$$

Then the estimate (24) becomes

$$|u(x) - u_{B_{x,r}}| \le \frac{2^Q\, C\, c_Q}{|B_1|} \int_{B_{x,r}} \rho(x,y)^{1-Q} d|D_H u|$$

and the thesis follows. $\square$

Our method relies on a convenient formula for the integral of (23). This formula is obtained by Fubini's theorem as follows. For every $\alpha > 1$ we have

$$\int_{B_{x,r}} \frac{1}{\rho(x,z)^{\alpha-1}}\, d|D_H u|(z) = (\alpha-1) \int_{B_{x,r}} \left(\int_{\rho(x,z)}^{\infty} \frac{1}{t^\alpha}\, dt \right) d|D_H u|(z)$$
$$= (\alpha-1) \int_0^{+\infty} \frac{|D_H u|(B_{x,r} \cap B_{x,t})}{t^\alpha}\, dt = (\alpha-1) \int_0^r \frac{|D_H u|(B_{x,t})}{t^\alpha} dt + \frac{|D_H u|(B_{x,r})}{r^{\alpha-1}}\,,$$

that implies by a change of variable

$$\int_{B_{x,r}} \frac{1}{\rho(x,z)^{\alpha-1}}\, d|D_H u|(z) = \frac{(\alpha-1)}{r^{\alpha-1}} \int_0^1 \frac{|D_H u|(B_{x,tr})}{t^\alpha} dt + \frac{|D_H u|(B_{x,r})}{r^{\alpha-1}}\,. \tag{25}$$

Representation formula and the previous equality give the following theorem.

Theorem 1.16 *Let $\Omega \subset \mathbb{G}$ be a bounded open set. Then there exists a constant $C > 0$ such that for every ball $B_{x,2r}$ compactly contained in Ω we have*

$$|u(x) - u_{B_{x,r}}| \le C\, r \left\{ \int_0^1 \frac{|D_H u|(B_{x,tr})}{(tr)^Q} dt + \frac{|D_H u|(B_{x,r})}{r^Q} \right\}, \tag{26}$$

for every $u \in BV_{H,loc}(\Omega)$ whenever $x \in B_{x,r} \cap A_u$.

As an immediate consequence of the previous result we have the following corollary.

Corollary 1.17 *Let $\Omega \subset \mathbb{G}$ be a bounded open set and let $1 \le p < \infty$. Then there exists a constant $C > 0$ such that for every ball $B_{x,2r}$ compactly contained in Ω we have*

$$|u(x) - u_{B_{x,r}}| \le C\, r \left\{ \int_0^1 \left(\fint_{B_{x,tr}} |\nabla_H u(y)|^p \right)^{1/p} dt + \left(\fint_{B_{x,r}} |\nabla_H u(y)|^p \right)^{1/p} \right\}. \tag{27}$$

for every $u \in W^{1,p}_{H,loc}(\Omega)$ whenever $x \in B_{x,r} \cap A_u$.

2 Sobolev-Poincaré inequality and first order differentiability

We begin this section with some preliminary lemmas that will be used in the sequel. Throughout the paper we will use the notation v^* to indicate the map $x \longrightarrow \langle v, \ln x \rangle$ where $x \in \mathbb{G}$ and $v \in V_1$. Note that $v^*(\delta_r x) = r v^*(x)$ and $v^* : \mathbb{G} \longrightarrow \mathbb{R}$ is a group homomorphism.

Lemma 2.1 *Let μ be a nonnegative Radon measure on Ω such that there exists a negligible set $N \subset \Omega$ with $\mu(\Omega \setminus N) = 0$. Then for a.e. $x \in \Omega$ we have $\lim_{r \to 0^+} \mu(B_{x,r}) r^{-Q} = 0$.*

Proof. By contradiction, if we had a measurable subset $A \subset \Omega$, with $|A| > 0$ such that

$$\limsup_{r \to 0^+} \frac{\mu(B_{x,r})}{r^Q} > 0 \,,$$

for every $x \in A$, then we would get a measurable subset $A' \subset A \setminus N$ with positive measure and $\lambda > 0$ such that $|D_H^s u|(A') \geq \lambda |A'| > 0$. This standard fact can be checked for instance by Theorem 2.10.17 and Theorem 2.10.18 of [11]. Then inequalities

$$0 < \mu(A') \leq \mu(\Omega \setminus N)$$

lead us to a contradiction. □

Lemma 2.2 *Let $u : \Omega \longrightarrow \mathbb{R}$ be an H-BV function. Then for a.e. $x \in A_{\nabla_H u}$ and every $c \in \mathbb{R}$ the following limit holds*

$$\lim_{r \to 0^+} \frac{|D_H(u - \nabla_H u(x)^* - c)|(B_{x,r})}{r^Q} = 0.$$

Proof. Using both definitions of absolutely continuous part and singular part of $|D_H u|$ one can check without difficulty that

$$|D_H \left(u - \nabla_H u(x)^* - c\right)| = |\nabla_H u - \nabla_H u(x)| \, v_g + |D_H u|^s. \tag{28}$$

Let us first check that for a.e. $x \in \Omega$ we have

$$\lim_{r \to 0^+} \frac{|D_H u|^s(B_{x,r})}{r^Q} = 0 \,. \tag{29}$$

Radon-Nikodým theorem gives us a measurable subset $N \subset \Omega$ such that

$$|N| = 0 \qquad \text{and} \qquad |D_H u|^s(N^c) = 0.$$

Then by Lemma 2.1 the limit (29) holds a.e. in Ω. As a consequence, the assumption that x is an approximate continuity point of $\nabla_H u$ concludes the proof. □

2.1 Calderón-Zygmund theorem and L^p differentiability

The next theorem has been proved by Calderón and Zygmund for functions of bounded variation in Euclidean spaces, [5], see also [11].

Theorem 2.3 (Calderón-Zygmund) *Let $u:\Omega\longrightarrow\mathbb{R}$ be an H-BV function and define $\beta = Q/(Q-1)$. Then u is L^β differentiable of order 1 at a.e. $x\in\Omega$ and the differential is representable as $\nabla_H u(x)^*$, where $\nabla_H u$ is the density of $D_H^a u$.*

PROOF. Due to Lemma 2.2, for a.e. $x\in A_u\cap A_{\nabla_H u}$ we have

$$\lim_{r\to 0^+}\frac{|D_H v|(B_{x,r})}{r^Q}=0, \tag{30}$$

where we have defined $v(y)=u(y)-\nabla_H u(x)^*(x^{-1}y)-u(x)$ and $x\in A_v$ with $v(x)=0$. By triangle inequality we have

$$\left(\fint_{B_{x,r}}|v(y)|^\beta\,dy\right)^{1/\beta}\le\left(\fint_{B_{x,r}}|v(y)-v_{B_{x,r}}|^\beta\,dy\right)^{1/\beta}+|v_{B_{x,r}}|.$$

The Sobolev-Poincaré inequality (14) and the estimate (26) give a constant C_1 such that

$$\left(\fint_{B_{x,r}}|v(y)|^\beta\,dy\right)^{1/\beta}\le C_1\,r\left\{\int_0^1\frac{|D_H v|(B_{x,tr})}{(tr)^Q}dt+\frac{|D_H v|(B_{x,r})}{r^Q}\right\}=o(r) \tag{31}$$

where the last equality follows from condition (30). □

The simple approach used in the previous theorem can be applied to L^p differentiability, but replacing the Sobolev-Poincaré inequality (14) with the Poincaré inequality (12).

Theorem 2.4 (L^p differentiability) *Let $u\in W^{1,p}_H(\Omega)$, with $1\le p<\infty$. Then u has an L^p differential of order 1 at a.e. $x\in\Omega$ and the differential is representable as $\nabla_H u(x)^*$, where $\nabla_H u$ is the density of $D_H^a u$.*

PROOF. By Proposition 1.9, for a.e. $x\in\Omega\cap A_u$ we have

$$\lim_{r\to 0^+}\fint_{B_{x,r}}|\nabla_H u(y)-\nabla_H u(x)|^p\,dy=0.$$

Let us define $v(y)=u(y)-\nabla_H u(x)^*(x^{-1}y)-u(x)$. Notice that $x\in A_v$ with $v(x)=0$ and

$$\lim_{r\to 0^+}\fint_{B_{x,r}}|\nabla_H v(y)|^p\,dy=0. \tag{32}$$

By triangle inequality we have

$$\left(\fint_{B_{x,r}} |v(y)|^p \, dy\right)^{1/p} \le \left(\fint_{B_{x,r}} |v(y) - v_{B_{x,r}}|^p \, dy\right)^{1/p} + |v_{B_{x,r}}|. \tag{33}$$

The estimate (27) and the condition (32) imply that $|v_{B_{x,r}}| = o(r)$ as $r \to 0^+$. Finally, we apply Poincaré inequality (12) to the right-hand side of (33), hence (32) ends the proof. □

In the case $1 \le p < Q$ Theorem 2.4 can be improved using the Sobolev-Poincaré inequality (13)

Theorem 2.5 *Let $u \in W^{1,p}_H(\Omega)$ with $1 \le p < Q$. Then u has an L^γ differential of order 1 at a.e. $x \in \Omega$ and the differential is representable as $\nabla_H u(x)^*$, where $\nabla_H u$ is the density of $D^a_H u$ and $\gamma = Qp/(Q-p)$.*

PROOF. Proposition 1.9 ensures that for a.e. $x \in \Omega \cap A_u$ we have

$$\lim_{r \to 0^+} \fint_{B_{x,r}} |\nabla_H u(y) - \nabla_H u(x)|^p \, dy = 0.$$

Let us define $v(y) = u(y) - \nabla_H u(x)^*(x^{-1}y) - u(x)$, observing that

$$\lim_{r \to 0^+} \fint_{B_{x,r}} |\nabla_H v(y)|^p \, dy = 0. \tag{34}$$

The triangle inequality gives

$$\left(\fint_{B_{x,r}} |v(y)|^\gamma \, dy\right)^{1/\gamma} \le \left(\fint_{B_{x,r}} |v(y) - v_{B_{x,r}}|^\gamma \, dy\right)^{1/\gamma} + |v_{B_{x,r}}|. \tag{35}$$

From estimate (27) and condition (34) we obtain $|v_{B_{x,r}}| = o(r)$ as $r \to 0^+$. Finally, we apply the Sobolev-Poincaré inequality (13) to the right-hand side of (35), hence (34) ends the proof. □

2.2 Another proof of Rademacher theorem

As a consequence of the a.e. L^1 differentiability of H-BV functions we obtain another proof of the Rademacher theorem for real-valued Lipschitz maps on stratified groups. This result for group-valued maps is much more complicated and it is due to Pansu [26]. In the case of real-valued Lipschitz maps on Carnot-Carathéodory spaces, this theorem has been proved by Monti and Serra Cassano, [25], using a Morrey-type estimate in Carnot-Carathéodory spaces due to Lu, [24].

Here we present a different approach which follows Lemma 3.1.5 of [11]. After this lemma Rademacher theorem follows by the a.e. approximate differentiability. Our proof extends this lemma on stratified groups and uses the a.e. L^1 differentiability of functions of H-bounded variation. Notice that the part concerning the a.e. approximate differentiability in Theorem 3.1.6 of [11] extends to the setting of stratified groups with nontrivial technical difficulties.

Theorem 2.6 (Rademacher) *Let $u : \Omega \longrightarrow \mathbb{R}$ be a Lipschitz map. Then for a.e. $x \in \Omega$ there exists a unique vector $\nabla_H u(x) \in V_1$ such that*

$$\lim_{y\to x} \frac{|u(y) - u(x) - \nabla_H u(x)^*(x^{-1}y)|}{\rho(x,y)} = 0.$$

Proof. It is not difficult to check that u has distributional derivative $X_j u$ in $L^\infty(\Omega)$ for every $j = 1, \dots, m$. In particular $u \in BV_H(\Omega)$ and Theorem 2.3 largely ensures that u is L^1 differentiable at a.e. $x \in \Omega$. Let us pick $x \in \Omega$ at which u is L^1 differentiable with differential $\nabla_H u(x)^*$. By Lemma 1.14 with $\alpha = 1$ we obtain the limit

$$\lim_{r\to 0^+} \fint_{B_{x,r}} \frac{|u(y) - u(x) - \nabla_H u(x)^*(x^{-1}y)|}{\rho(x,y)}\, dy = 0.$$

In particular u is approximately differentiable at x in the Federer sense, [11], namely, for every $\varepsilon > 0$ the set

$$E_\varepsilon = \Big\{ y \in \Omega \mid |u(y) - u(x) - \nabla_H u(x)^*(x^{-1}y)| \le \varepsilon\, \rho(x,y) \Big\}$$

has the property $|E_\varepsilon \cap B_{x,r}|\, |B_{x,r}|^{-1} \longrightarrow 1$ as $r \to 0^+$. Then x is a density point of E_ε hence the doubling property gives that $\operatorname{dist}(y, E_\varepsilon) = o\,(\rho(y,x))$ as $\rho(y,x) \to 0^+$. Fix $r_0 > 0$ such that $\operatorname{dist}(y, E_\varepsilon) \le \varepsilon\, \rho(y,x)$ for every $\rho(y,x) \le r_0$. We choose $y \in \overline{B}_{x,r_0}$, assuming that $\overline{B}_{x,r_0} \subset \Omega$. If $y \notin E_\varepsilon \cap \overline{B}_{x,r_0} \subset \Omega$, then $\operatorname{dist}(y, E_\varepsilon) > 0$ and there exists $a \in E_\varepsilon$ such that $\rho(y,a) < 2\operatorname{dist}(y, E_\varepsilon)$. We denote by L the Lipschitz constant of u. We have

$$\begin{aligned}
&|u(y) - u(x) - \nabla_H u(x)^*(x^{-1}y)| \le |\nabla_H u(x)^*(a^{-1}y)| + |u(a) - u(x) - \nabla_H u(x)^*(x^{-1}a)| \\
&+|u(y) - u(a)| \le \|\nabla_H u(x)^*\|\, 2\varepsilon\rho(y,x) + \varepsilon\, \rho(y,x) + 2\, L\, \varepsilon\, \rho(y,x).
\end{aligned}$$

In the case $y \in E_\varepsilon \cap \overline{B}_{x,r_0}$, by definition of E_ε, we have $|u(y) - u(x) - \nabla_H u(x)^*(x^{-1}y)| \le \varepsilon\rho(y,x)$. The arbitrary choice of ε concludes the proof. $\square$

3 Poincaré inequality and higher order differentiability

In the first part of this section we review some basic facts about left invariant differential operators on groups. In particular we recall the important Poincaré-Birkhoff-Witt theorem, which will be of great importance to our differentiability results of higher order.

As in $\mathbb{R}^n$ for every $k \in \mathbb{N} \setminus \{0\}$ there is a natural correspondence between differential operators ∂_x^α with $|\alpha| = \sum_{j=1}^n \alpha_j \le k$ and polynomials of degree less than or equal to k. The same analogy holds for stratified groups. For convenience of notation, we will identify the Lie algebra $\mathcal{G}$ of left invariant vector fields with the isomorphic vector space of first order left invariant differential operators. Let $(W_1, W_2, \dots W_q)$ be a graded basis of $\mathcal{G}$, regarded as a basis of first order left invariant differential operators. Recall that a homogeneous left invariant differential operator Z has degree d if $\delta_r^* Z = r^d Z$, namely,

$$\delta_r^* Z(\varphi) = Z(\varphi \circ \delta_r) = d\varphi(\delta_r Z) = r^d d\varphi(Z) = r^d Z\varphi$$

for every $r > 0$ and $\varphi \in C^\infty(\mathbb{G})$. By this definition one easily checks that the left invariant vector field W_i of degree d_i has also homogeneous degree d_i as left invariant differential operator. With this notion we can define the homogeneous degree of a left invariant differential operator. We consider a left invariant differential operator

$$Z = \sum_\alpha c_\alpha W^\alpha,$$

where only a finite number of coefficients $c_\alpha \in \mathbb{R}$ do not vanish and W^α is defined by the following ordered terms

$$W^\alpha = W_1^{\alpha_1} \cdot \cdots \cdot W_q^{\alpha_q},$$

for every $\alpha = (\alpha_1, \ldots, \alpha_q) \in \mathbb{N}^q$. We define the *homogeneous degree* of Z as

$$\text{h-deg}(Z) = \max\{d(\alpha) \mid c_\alpha \neq 0\},$$

where $d(\alpha) = \sum_{k=1}^q d_k \alpha_k$. The finite dimensional space of left invariant differential operators of homogeneous degree less than or equal to k will be denoted by $\mathcal{A}_k(\mathbb{G})$. The Poincaré-Birkhoff-Witt theorem states that the set of differential operators

$$\left\{W_1^{\alpha_1} \cdot \cdots \cdot W_q^{\alpha_q} \mid (\alpha_1, \ldots, \alpha_q) \in \mathbb{N}^q\right\}$$

is a basis of the algebra of left invariant differential operators on $\mathbb{G}$, see p.21 of [12]. The next proposition states a precise relation between polynomials and left invariant differential operators on groups, see Proposition 1.30 of [12].

Proposition 3.1 *There exists an isomorphism $L : \mathcal{P}_{H,k}(\mathbb{G}) \longrightarrow \mathcal{A}_k(\mathbb{G})$, given by*

$$L(P) = \sum_{d(\alpha) \le k} (W^\alpha P)(0)\ W^\alpha.$$

Definition 3.2 (Mixed derivatives) Let $u \in BV_H^k(\Omega)$ and let $Y_1, \ldots, Y_p \in \mathcal{G}$ be first order left invariant differential operators such that $\sum_{j=1}^p \text{h-deg}(Y_j) \le k$. We define the Radon measure $D_{Y_1 \cdots Y_p} u$ as follows

$$\int_\Omega \phi\ dD_{Y_1 \cdots Y_p} u = (-1)^p \int_\Omega u(x)\ \ (Y_p \cdots Y_1 \phi)(x)\, dx \qquad \forall \phi \in C_c^\infty(\Omega). \tag{36}$$

with respect to the Haar measure is denoted by $W^\alpha u$.

Some remarks here are in order. By the fact that the flow associated to $Y \in \mathcal{G}$ preserves the measure, we have that $\operatorname{div}(Y) = 0$, hence by the classical divergence theorem the relation (36) holds for smooth functions u with $D_{Y_1 \cdots Y_p} u = (Y_1 \cdots Y_p u)\ v_g$.

Note also that the existence of distributional derivative $D_{Y_1 \cdots Y_p} u$ as measure follows by the Lie bracket generating condition on horizontal vector fields $(X_1, \ldots, X_m)$ that are now regarded as differential operators. It suffices to observe that every left invariant differential operator Z with h-deg$(Z)=k$ can be written as a linear combination of iterated commutators of $(X_1,\dots,X_m)$ up to k times. Then one uses the definition of H-BV^k function, which exactly requires that every iterated distributional derivative $X_{j_1}\cdots X_{j_k}u$ is a measure, where $1\le j_1,\dots,j_k\le m$.

By Radon-Nikodým theorem we have $D_{Y_1,\dots,Y_p}u=\big(D_{Y_1,\dots,Y_p}u\big)^a+\big(D_{Y_1,\dots,Y_p}u\big)^s$, where the terms of the sum are the absolutely continuous part and the singular part of $D_{Y_1\cdots Y_p}u$, respectively. The density of $\big(D_{Y_1\cdots Y_p}u\big)^a$ will be denoted simply by $Y_1\cdots Y_pu$. Recall that for every $i=1,\dots,m$ the differential operators $W_i\in\mathcal{G}$ have homogeneous degree equal to one. We will denoted these operators by X_i, where $i=1,\dots,m$.

Due to the Poincaré-Birkhoff-Witt theorem for every $j_1,\dots,j_p\in\{1,2,\dots,m\}$, there exist coefficients $\{c^{\alpha_1,\dots,\alpha_q}_{j_1,\dots,j_p}\}$ such that

$$X_{j_1}\cdots X_{j_p}=\sum_{d(\alpha)\le p}c^{\alpha_1,\dots,\alpha_q}_{j_1,\dots,j_p}\,W_1^{\alpha_1}\cdots W_q^{\alpha_q}, \tag{37}$$

where $\alpha=(\alpha_1,\dots,\alpha_q)$ and X_{j_s} is a horizontal differential operator for every $s=1,\dots,p$, namely, it has homogeneous degree equal to one. By a direct computation, it is not difficult to see that Definition 3.2 and the linearity of (37) give the relation

$$D_{X_{j_1}\cdots X_{j_p}}u=(-1)^p\sum_{d(\alpha)\le p}(-1)^{|\alpha|}c^{\alpha_1,\dots,\alpha_q}_{j_p,\dots,j_1}\,D_{W_1^{\alpha_1}\cdots W_q^{\alpha_q}}u \tag{38}$$

foe every $u\in BV^k_H(\Omega)$ and $p\le k$.

3.1 L^1 differentiability of H-BV^k functions

This subsection deals with the L^1 differentiability of order k of H-BV^k functions. We start with a preliminary lemma.

Lemma 3.3 *Let $u\in BV^k_H(\Omega)$. Then for a.e. $x\in\bigcap_{d(\alpha)\le k}A_{W^\alpha u}$ there exists a unique polynomial $P_{[x]}\in\mathcal{P}_{H,k}(\mathbb{G})$ such that $W^\alpha P_{[x]}(x)=W^\alpha u(x)$ and*

$$\lim_{r\to0^+}|D_{W^\alpha}(u-P_{[x]})|(B_{x,r})r^{-Q}=0,$$

for every $d(\alpha)\le k$.

Proof. By Proposition 3.1 for every $x\in\bigcap_{d(\alpha)\le k}A_{W^\alpha u}$ we have a unique polynomial $P_{[x]}\in\mathcal{P}_{H,k}(\mathbb{G})$ such that $W^\alpha P_{[x]}(x)=W^\alpha u(x)$ for every $d(\alpha)\le k$. By the regularity of $P_{[x]}$ we have the decomposition

$$|D_{W^\alpha}(u-P_{[x]})|=|W^\alpha u-W^\alpha P_{[x]}|\,v_g+|D_{W^\alpha}u|^s.$$

Then Lemma 2.1 concludes the proof. □

Theorem 3.4 *Every H-BV^k function $u : \Omega \longrightarrow \mathbb{R}$ is a.e. L^1 differentiable of order k.*

Proof. By Lemma 3.3 for a.e. $x \in \bigcap_{d(\alpha)\le k} A_{W^\alpha u}$ we have a unique polynomial $P_{[x]} \in \mathcal{P}_{H,k}(\mathbb{G})$ such that $W^\alpha P_{[x]}(x) = W^\alpha u(x)$ and

$$\lim_{r\to 0^+} |D_{W^\alpha}(u - P_{[x]})|(B_{x,r})r^{-Q} = 0.$$

for every $d(\alpha) \le k$. By induction on k, suppose that our claim is satisfied for $k-1 \ge 1$. Theorem 2.3 in particular ensures that our claim holds for $k-1 = 1$. For every $j = 1,\dots,m$ the summable function $X_j u$ is H-BV^{k-1} and there exists a unique polynomial $R_j \in \mathcal{P}_{H,k-1}(\mathbb{G})$ such that

$$⨍_{B_{x,r}} |X_j u(y) - R_j(y)|\, dy = o(r^{k-1}) \quad \text{as} \quad r \to 0^+ \tag{39}$$

and which satisfies the condition $W^\beta X_j u(x) = W^\beta R_j(x)$ for every $d(\beta) \le k-1$. By Poincaré-Birkhoff-Witt theorem we have a finite number of coefficients $\{c^\alpha_{\beta,j}\}$ such that

$$W^\beta X_j u(x) = \sum_{d(\alpha)\le k} c^\alpha_{\beta,j} W^\alpha u(x) = \sum_{d(\alpha)\le k} c^\alpha_{\beta,j} W^\alpha P_{[x]} = W^\beta X_j P_{[x]}(x)\,, \tag{40}$$

then $W^\beta R_j(x) = W^\beta X_j P_{[x]}(x)$ for every $d(\beta) \le k-1$. By virtue of Proposition 3.1 and observing that $X_j P_{[x]} \in \mathcal{P}_{H,k-1}(\mathbb{G})$ we have $R_j = X_j P_{[x]}$ for very $j = 1,\dots,m$. Then, by (39) for every $j = 1,\dots,m$ we have

$$⨍_{B_{x,r}} |X_j \left(u - P_{[x]}\right)(y)|\, dy = o(r^{k-1}) \quad \text{as} \quad r \to 0^+. \tag{41}$$

Let us define $v = u - P_{[x]}$ and observe that (41) yields the condition

$$\int_{B_{x,r}} |X_j v(y)|\, dy = o(r^{Q+k-1}) \quad \text{as} \quad r \to 0^+. \tag{42}$$

The Poincaré inequality (15) and the estimate (26) give us a constant C_1 such that

$$⨍_{B_{x,r}} |v(y)|\, dy \le C_1\, r \left\{ \int_0^1 \frac{|D_H v|(B_{x,tr})}{(tr)^Q} dt + \frac{|D_H v|(B_{x,r})}{r^Q} \right\}.$$

It follows that

$$⨍_{B_{x,r}} |v(y)|\, dy \le C_1\, r^{1-Q} \sum_{j=1}^m \int_{B_{x,r}} |X_j v(y)|\, dy \; + C_1\, r^{1-Q} \sum_{j=1}^m \int_0^1 t^{-Q} \int_{B_{x,rt}} |X_j v(y)|\, dt.$$

This inequality together with condition (42) finish the proof. □

3.2 L^p differentiability of H-Sobolev functions

We first observe that $W^{k,p}_H(\Omega) \subset BV^k_H(\Omega)$, then all previous considerations for H-BV^k functions hold for Sobolev functions of the same order. We have in addition that $D_{W^\alpha} u = (W^\alpha u)\ v_g$ where $W^\alpha u \in L^p(\Omega)$ for every $d(\alpha) \le k$.

Lemma 3.5 *Let $u \in W^{k,p}_H(\Omega)$. Then for a.e. $x \in \bigcap_{d(\alpha)\le k} A_{W^\alpha u}$ there exists a unique polynomial $P_{[x]} \in \mathcal{P}_{H,k}(\mathbb{G})$ such that $W^\alpha P_{[x]}(x) = W^\alpha u(x)$ and*

$$\lim_{r\to 0^+} \fint_{B_{x,r}} |W^\alpha u(y) - W^\alpha P_{[x]}(y)|^p\, dy = 0,$$

for every $d(\alpha) \le k$.

Proof. By Proposition 1.9 and Proposition 3.1 for a.e. $x \in \Omega$ we have a unique polynomial $P_{[x]} \in \mathcal{P}_{H,k}(\mathbb{G})$ such that $W^\alpha P_{[x]}(x) = W^\alpha u(x)$ for every $d(\alpha) \le k$, where the number $W^\alpha u(x)$ satisfies the property

$$\lim_{r\to 0^+} \fint_{B_{x,r}} |W^\alpha u(y) - W^\alpha P_{[x]}(x)|^p\, dy = 0,$$

for every $d(\alpha) \le k$. Then by continuity of $W^\alpha P_{[x]}$ the thesis follows. □

Theorem 3.6 *Every function $u \in W^{k,p}_H(\Omega)$ is a.e. L^p differentiable of order k.*

Proof. By Lemma 3.5 for a.e. $x \in \Omega$ we have a unique polynomial $P_{[x]} \in \mathcal{P}_{H,k}(\mathbb{G})$ such that $W^\alpha P_{[x]}(x) = W^\alpha u(x)$ and

$$\lim_{r\to 0^+} \fint_{B_{x,r}} |W^\alpha u(y) - W^\alpha P_{[x]}(y)|^p\, dy = 0,$$

for every $d(\alpha) \le k$. By induction on k, suppose that our claim is satisfied for $k-1 \ge 1$. Theorem 2.4 in particular ensures that our claim holds for $k-1=1$. For every $j = 1, \ldots, m$ we have $X_j u \in W^{k-1,p}_H(\Omega)$, then there exists a unique polynomial $R_j \in \mathcal{P}_{H,k-1}(\mathbb{G})$ such that

$$\left(\fint_{B_{x,r}} |X_j u(y) - R_j(y)|^p\, dy \right)^{1/p} = o(r^{k-1}) \quad \text{as} \quad r \to 0^+ \tag{43}$$

and which satisfies the condition $W^\beta X_j u(x) = W^\beta R_j(x)$ for every $d(\beta) \le k-1$. By Poincaré-Birkhoff-Witt theorem we have a finite number of coefficients $\{c^\alpha_{\beta,j}\}$ such that

$$W^\beta X_j u(x) = \sum_{d(\alpha)\le k} c^\alpha_{\beta,j} W^\alpha u(x) = \sum_{d(\alpha)\le k} c^\alpha_{\beta,j} W^\alpha P_{[x]} = W^\beta X_j P_{[x]}(x)\,,$$

then $W^\beta R_j(x) = W^\beta X_j P_{[x]}(x)$ for every $d(\beta) \leq k-1$. By virtue of Proposition 3.1 and observing that $X_j P_{[x]} \in \mathcal{P}_{H,k-1}(\mathbb{G})$ we have $R_j = X_j P_{[x]}$ for very $j = 1, \ldots, m$. Then, by (43) for every $j = 1, \ldots, m$ we have

$$\left(\fint_{B_{x,r}} |X_j \left(u - P_{[x]}\right)(y)|^p \, dy \right)^{1/p} = o(r^{k-1}) \quad \text{as} \quad r \to 0^+. \tag{44}$$

Let us define $v = u - P_{[x]}$ and observe that (44) yields

$$\left(\fint_{B_{x,r}} |X_j v(y)|^p \, dy \right)^{1/p} = o(r^{k-1}) \quad \text{as} \quad r \to 0^+.$$

for every $j = 1, \ldots, m$. Then we achieve the condition

$$\left(\fint_{B_{x,r}} |\nabla_H v(y)|^p \, dy \right)^{1/p} = o(r^{k-1}) \quad \text{as} \quad r \to 0^+. \tag{45}$$

By triangle inequality and Poincaré inequality (12) we obtain

$$\left(\fint_{B_{x,r}} |v(y)|^p \, dy \right)^{1/p} \leq C\, r \left(\fint_{B_{x,r}} |\nabla_H v(y)|^p \, dy \right)^{1/p} + |v_{B_{x,r}}|.$$

In view of (27) we have the estimate

$$|v_{B_{x,r}}| \leq C\, r \left\{ \int_0^1 \left(\fint_{B_{x,tr}} |\nabla_H u(y)|^p \right)^{1/p} dt + \left(\fint_{B_{x,r}} |\nabla_H u(y)|^p \right)^{1/p} \right\}$$

for functions in $W^{1,p}_{H,loc}(\Omega)$. Then there exists a constant C_1 such that

$$\left(\fint_{B_{x,r}} |v(y)|^p \, dy \right)^{1/p} \leq C_1\, r \left\{ \int_0^1 \left(\fint_{B_{x,tr}} |\nabla_H u(y)|^p \right)^{1/p} dt + \left(\fint_{B_{x,r}} |\nabla_H u(y)|^p \right)^{1/p} \right\}.$$

As a result, the condition (45) and the previous inequality conclude the proof. $\square$

4 Sobolev-Poincaré inequality and higher order differentiability

Here we study some special cases where it is possible to improve the differentiability results of the previous section.

4.1 Differentiability of H-BV^k functions in the case $k < Q$

Theorem 4.1 *Let $u \in BV_H^k(\Omega)$ with $1 \leq k < Q$ and let $\gamma = Q/(Q-k)$. Then u is a.e. L^γ differentiable of order k.*

PROOF. By Lemma 3.3 for a.e. $x \in \bigcap_{d(\alpha)\leq k} A_{W^\alpha u}$ we have a unique polynomial $P_{[x]} \in \mathcal{P}_{H,k}(\mathbb{G})$ such that $W^\alpha P_{[x]}(x) = W^\alpha u(x)$ and

$$\lim_{r\to 0^+} |D_{W^\alpha}(u - P_{[x]})|(B_{x,r}) r^{-Q} = 0.$$

for every $d(\alpha) \leq k$. By induction on k, suppose that our claim is satisfied for $k-1 \geq 1$ and that $u \in L^\eta_{loc}(\Omega)$, where $\eta = Q/(Q-k+1)$. Theorem 2.3 and Sobolev-Poincaré inequality (14) ensure that our claim holds for $k-1 = 1$. For every $j = 1, \dots, m$ the functions $X_j u$ and u are H-BV^{k-1}, then the induction hypothesis implies that $u, X_j u \in L^\eta_{loc}(\Omega)$, that there exists a unique polynomial $R_j \in \mathcal{P}_{H,k-1}(\mathbb{G})$ such that

$$\left(\fint_{B_{x,r}} |X_j u(y) - R_j(y)|^\eta \, dy \right)^{1/\eta} = o(r^{k-1}) \quad \text{as} \quad r \to 0^+, \tag{46}$$

and which satisfies the condition $W^\beta X_j u(x) = W^\beta R_j(x)$ for every $d(\beta) \leq k-1$. By Poincaré-Birkhoff-Witt theorem we have a finite number of coefficients $\{c^\alpha_{\beta,j}\}$ such that

$$W^\beta X_j u(x) = \sum_{d(\alpha)\leq k} c^\alpha_{\beta,j} W^\alpha u(x) = \sum_{d(\alpha)\leq k} c^\alpha_{\beta,j} W^\alpha P_{[x]} = W^\beta X_j P_{[x]}(x)\,, \tag{47}$$

then $W^\beta R_j(x) = W^\beta X_j P_{[x]}(x)$ for every $d(\beta) \leq k-1$. By virtue of Proposition 3.1 and observing that $X_j P_{[x]} \in \mathcal{P}_{H,k-1}(\mathbb{G})$ we have $R_j = X_j P_{[x]}$ for very $j = 1, \dots, m$. Then every $j = 1, \dots, m$ we have $X_j \left(u - P_{[x]}\right) = X_j u - R_j$. As a result, defining $v = u - P_{[x]}$ and using condition (46) for every $j = 1, \dots, m$ we achieve

$$\left(\fint_{B_{x,r}} |X_j v(y)|^\eta \, dy \right)^{1/\eta} = o(r^{k-1}). \tag{48}$$

The condition $k < Q$ implies the inequality $\eta = Q\,(Q-k+1)^{-1} < Q$, hence the Sobolev-Poincaré inequality (13) yields

$$\left(\fint_{B_{x,r}} |v(y) - v_{B_{x,r}}|^\gamma \, dy \right)^{1/\gamma} \leq C\, r \left(\fint_{B_{x,r}} |\nabla_H v(y)|^\eta \, dy \right)^{1/\eta},$$

where $\gamma = Q\eta/(Q-\eta) = Q/(Q-k)$. From condition (48) for every $j = 1, \ldots, m$ we obtain

$$\left(\fint_{B_{x,r}} |v(y) - v_{B_{x,r}}|^\gamma \, dy\right)^{1/\gamma} = o(r^k). \tag{49}$$

Notice that $v \in W^{1,\eta}_{H,loc}(\Omega)$, then we can apply (27) getting the inequality

$$|v_{B_{x,r}}| \le C\, r \left\{ \int_0^1 \left(\fint_{B_{x,tr}} |\nabla_H v(y)|^\eta\right)^{1/\eta} dt + \left(\fint_{B_{x,r}} |\nabla_H u(y)|^\eta\right)^{1/\eta} \right\}, \tag{50}$$

hence condition (48) gives $|v_{B_{x,r}}| = o(r^k)$ as $r \to 0^+$. Finally, by triangle inequality we have

$$\left(\fint_{B_{x,r}} |v(y)|^\gamma \, dy\right)^{1/\gamma} \le \left(\fint_{B_{x,r}} |v(y) - v_{B_{x,r}}|^\gamma \, dy\right)^{1/\gamma} + |v_{B_{x,r}}| = o(r^k),$$

hence the thesis follows. □

4.2 Differentiability of H-Sobolev functions in the case $kp < Q$

Theorem 4.2 *Let $u \in W^{k,p}_H(\Omega)$ with $1 \le kp < Q$ and let $\gamma = Qp/(Q-kp)$. Then u is a.e. L^γ differentiable of order k.*

PROOF. By Lemma 3.5 for a.e. $x \in \Omega$ we have a unique polynomial $P_{[x]} \in \mathcal{P}_{H,k}(\mathbb{G})$ such that $W^\alpha P_{[x]}(x) = W^\alpha u(x)$ and

$$\lim_{r \to 0^+} \fint_{B_{x,r}} |W^\alpha u(y) - W^\alpha P_{[x]}(y)|^p \, dy = 0,$$

for every $d(\alpha) \le k$. By induction on k, suppose that our claim is satisfied for $k-1 \ge 1$ and that $u \in L^\eta_{loc}(\Omega)$, where $\eta = Qp\,(Q-(k-1)p)^{-1}$. Theorem 2.5 and Sobolev-Poincaré inequality (13) ensure that our claim holds for $k-1 = 1$. We have $u, X_j u \in W^{k-1,p}_H(\Omega)$ for every $j = 1, \ldots, m$. By the induction hypothesis, $u, X_j u \in L^\eta_{loc}(\Omega)$ and for every $j = 1, \ldots, m$ there exists a unique polynomial $R_j \in \mathcal{P}_{H,k-1}(\mathbb{G})$ such that

$$\left(\fint_{B_{x,r}} |X_j u(y) - R_j(y)|^\eta \, dy\right)^{1/\eta} = o(r^{k-1}), \tag{51}$$

which satisfies the condition $W^\beta X_j u(x) = W^\beta R_j(x)$ for every $d(\beta) \le k-1$. By Poincaré-Birkhoff-Witt theorem we have a finite number of coefficients $\{c^\alpha_{\beta,j}\}$ such that

$$W^\beta X_j u(x) = \sum_{d(\alpha) \le k} c^\alpha_{\beta,j} W^\alpha u(x) = \sum_{d(\alpha) \le k} c^\alpha_{\beta,j} W^\alpha P_{[x]} = W^\beta X_j P_{[x]}(x)\,,$$

then $W^\beta R_j(x) = W^\beta X_j P_{[x]}(x)$ for every $d(\beta) \le k-1$. In view of Proposition 3.1 and observing that $X_j P_{[x]} \in \mathcal{P}_{H,k-1}(\mathbb{G})$ we have $R_j = X_j P_{[x]}$ for every $j = 1, \dots, m$. Then, defining $v = u - P_{[x]}$ we obtain $X_j v = X_j u - R_j$ for every $j = 1, \dots, m$. Thus, relation (51) yields

$$\left(\fint_{B_{x,r}} |X_j v(y)|^\eta \, dy \right)^{1/\eta} = o(r^{k-1}).$$

Then we arrive at the condition

$$\left(\fint_{B_{x,r}} |\nabla_H v(y)|^\eta \, dy \right)^{1/\eta} = o(r^{k-1}). \tag{52}$$

Recall that by induction hypothesis $v \in W^{1,\eta}_H(\Omega)$, furthermore the condition $kp < Q$ gives $\eta < Q$, therefore we can apply Sobolev-Poincaré inequality (13), getting

$$\left(\fint_{B_{x,r}} |v(y)|^\gamma \, dy \right)^{1/\gamma} \le C\, r \left(\fint_{B_{x,r}} |\nabla_H v(y)|^\eta \, dy \right)^{1/\eta} + |v_{B_{x,r}}|,$$

where $\gamma = Q\eta/(Q-\eta) = Qp/(Q-kp)$. Moreover, by virtue of (27) we have the estimate

$$|v_{B_{x,r}}| \le C\, r \left\{ \int_0^1 \left(\fint_{B_{x,tr}} |\nabla_H u(y)|^\eta \right)^{1/\eta} dt + \left(\fint_{B_{x,r}} |\nabla_H u(y)|^\eta \right)^{1/\eta} \right\}$$

Then there exists a constant C_1 such that

$$\left(\fint_{B_{x,r}} |v(y)|^\gamma \, dy \right)^{1/\gamma} \le C_1\, r \left\{ \int_0^1 \left(\fint_{B_{x,tr}} |\nabla_H u(y)|^\eta \right)^{1/\eta} dt + \left(\fint_{B_{x,r}} |\nabla_H u(y)|^\eta \right)^{1/\eta} \right\}.$$

Finally, relation (52) and the previous inequality conclude the proof. □

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