A LOWER SEMICONTINUITY RESULT FOR SOME INTEGRAL FUNCTIONALS IN THE SPACE SBD

FRANÇOIS EBOBISSE

Abstract. The purpose of this paper is to study the lower semicontinuity with respect to the strong L^1-convergence, of some integral functionals defined in the space SBD of special functions with bounded deformation. Precisely, we prove that, if $u \in SBD(\Omega)$, $(u_h) \subset SBD(\Omega)$ converges to u strongly in $L^1(\Omega, \mathbb{R}^n)$ and the measures $|E^j u_h|$ converge weakly $*$ to a measure ν singular with respect to the Lebesgue measure, then

$$\int_\Omega f(x, \mathcal{E}u)\, dx \le \liminf_{h\to\infty} \int_\Omega f(x, \mathcal{E}u_h)\, dx$$

provided the integrand f satisfies a weak convexity property and standard growth assumptions of order $p > 1$.

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Contents

1. Introduction

Our goal in this paper is to extend in the framework of functions with bounded deformation, the following theorem by Ambrosio [2] for integral functionals defined in the space SBV of special functions of bounded variation.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \times \mathbb{R}^k \times \mathbb{R}^{n\times k}$ be a Carathédory function satisfying:*

(i) *for a.e. every $x \in \Omega$, for every $(u, \xi) \in \mathbb{R}^k \times \mathbb{R}^{n\times k}$,*

$$|\xi|^p \le f(x, u, \xi) \le a(x) + \Psi(|u|)(1 + |\xi|^p),$$

where $p > 1$, $a \in L^1(\Omega)$ and the function $\Psi : [0, \infty) \to [0, \infty)$ is continuous;

(ii) *for a.e. every* $x \in \Omega$ *and every* $u \in \mathbb{R}^k$, $f(x,u,\cdot)$ *is quasi-convex.*

Then for every $u \in SBV(\Omega,\mathbb{R}^k)$ *and any sequence* $(u_h) \subset SBV(\Omega,\mathbb{R}^k)$ *converging to* u *in* $L^1_{loc}(\Omega,\mathbb{R}^k)$ *and such that*

$$\sup_h \mathcal{H}^{n-1}(S_{u_h}) < \infty \tag{1.1}$$

we have

$$\int_\Omega f(x,u,\nabla u)\,dx \le \liminf_{h\to\infty} \int_\Omega f(x,u_h,\nabla u_h)\,dx.$$

Theorem 1.1 extends in the SBV setting a classical lower semicontinuity result by Acerbi-Fusco [1] in the Sobolev space $W^{1,p}(\Omega)$.

Later Kristensen in [19] extended Theorem 1.1 under the weaker assumptions

$$\sup_h \int_{S_{u_h}} \theta(|u_h^+ - u_h^-|)d\mathcal{H}^{n-1} < \infty \tag{1.2}$$

for some function θ such that $\theta(r)/r \to \infty$ as $r \to 0^+$, and f is a normal integrand, i.e., for a.e. $x \in \Omega$, $f(x,\cdot,\cdot)$ is lower semicontinuous in $\mathbb{R}^k \times \mathbb{R}^{n\times k}$ and there exists a Borel function $\tilde{f} : \Omega \times \mathbb{R}^k \times \mathbb{R}^{n\times k} \to [0,\infty]$ such that $f(x,\cdot,\cdot) = \tilde{f}(x,\cdot,\cdot)$.

In the proof of Theorem 1.1 as well as in the Acerbi-Fusco result, the use of Lusin type approximation of functions in the given space (BV or Sobolev spaces) by Lipschitz continuous functions is crucial.

Recently, Theorem 1.1 has been extended by Fonseca-Leoni-Paroni [16] to functionals depending also on the hessian matrices.

In this paper we deal with first order variational problem, but with integral functionals depending explicitly on the symmetrized derivative $Eu := (Du + Du^T)/2$ and defined in the space SBD of special functions with bounded deformation.

The main result of the paper is the following lower semicontinuity theorem:

Theorem 1.2. *Let* $p > 1$ *and let* $f : \Omega \times \mathrm{M}^{n\times n}_{\mathrm{sym}} \to [0,\infty)$ *be a Carathéodory function satisfying:*

(i) *for a.e. every* $x \in \Omega$, *for every* $\xi \in \mathrm{M}^{n\times n}_{\mathrm{sym}}$,

$$\frac{1}{C}|\xi|^p \;\le\; f(x,\xi) \;\le\; \phi(x) + C(1+|\xi|^p),$$

for some constant $C > 0$ *and a function* $\phi \in L^1(\Omega)$;

(ii) *for a.e. every* $x_0 \in \Omega$, $f(x_0,\cdot)$ *is symmetric quasi-convex i.e.,*

$$f(x_0,\xi) \le \fint_A f(x_0,\xi + \mathcal{E}\varphi(x))dx \tag{1.3}$$

for every bounded open subset A *of* $\mathbb{R}^n$, *for every* $\varphi \in W_0^{1,\infty}(A,\mathbb{R}^n)$ *and* $\xi \in \mathrm{M}^{n\times n}_{\mathrm{sym}}$.

Then for every $u \in SBD(\Omega)$, *for any sequence* $(u_h) \subset SBD(\Omega)$ *converging to* u *strongly in* $L^1(\Omega,\mathbb{R}^n)$ *with* $|E^j u_h|$ *converging weakly* $*$ *to a positive measure* ν *singular with respect to* *the Lebesgue measure, we have*

$$\int_\Omega f(x,\mathcal{E}u)\,dx \;\le\; \liminf_{h\to\infty}\int_\Omega f(x,\mathcal{E}u_h)\,dx.$$

In the literature there are various results (see [5, 22, 23, 9]) on lower semicontinuity and relaxation of convex integral functionals in BD with linear growth in the strain tensor, in connection with mathematical problems in elasto-plasticity. Concerning non convex functionals with linear growth we mention the papers [12, 6, 13]. As far as the author knows, there is no result on lower semicontinuity of non convex volume energies with superlinear growth in the strain tensor. So, Theorem 1.2 is the first lower semicontinuity result for this class of functionals.

The proof of Theorem 1.2 follows the lines of Theorem 1.1. We use the blow-up method introduced in [17] and described as a two-steps process whose first step here is the proof of the lower semicontinuity result whenever Ω is the unit ball $B(0,1)$, the limit function is linear and $|E^j u_h|(B(0,1))$ converges to zero (see Proposition 3.1). In a second step, we use a blow-up argument through the approximate differentiability of BD functions to reduce the problem into the first step.

The use of Lusin type approximation for BD functions is crucial in the proof of Proposition 3.1. This result established in [11] and refined here in Proposition 2.8 is obtained using a "Poincaré type" inequality for BD functions (see Theorems 2.2 and 2.3) together with the maximal function of Radon measures.

This paper is organized as follows. In section 2 we collect and prove some fine properties of BD functions that will be used in the proof of our main result. Section 3 is devoted to the proof of Theorem 1.2. In section 4, we discuss the assumption (in Theorem 1.2) that the measures $|E^j u_h|$ converge weakly $*$ to a positive measure ν singular with respect to the Lebesgue measure. In Example 4.7 we consider a minimization problem in SBD with a unilateral constraint on the jump sets and we show that minimizing sequences (u_h) satisfy the assumption on $|E^j u_h|$. However, as shown in Example 4.4, this assumption is not always compatible with the SBD compactness criterion (Theorem 4.1). Precisely, we construct a sequence of functions (u_h) in SBD that satisfies the assumptions of Theorem 4.1, while the sequence of measures $|E^j u_h|$ converges weakly $*$ to a measure proportional to the Lebesgue measure.

2. Notation and Preliminary results

Let $n \ge 1$ be an integer. We denote by $M^{n\times n}$ the space of $n\times n$ matrices and by $\mathrm{M}^{n\times n}_{\mathrm{sym}}$ the subspace of symmetric matrices in $M^{n\times n}$. For any $\xi \in M^{n\times n}$, ξ^T is the transpose of ξ. Given u, $v \in \mathbb{R}^n$, $u\otimes v$ and $u \odot v := (u\otimes v + v\otimes u)/2$ denote the tensor and symmetric products of u and v, respectively. $\mathbf{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. We use the standard notation, $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ to denote respectively the Lebesgue outer measure and the $(n-1)$-dimensional Hausdorff measure. For every set $E \subset \mathbb{R}^n$, $\overline{E}$ and $|E|$ stand respectively for the closure and the Lebesgue outer measure of E, while χ_E denotes the charateristic function of E, i.e., $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. For $1 \le p \le \infty$, $\|\cdot\|_p$ will denote the norm in the L^p space.

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{B}(\Omega)$ the family of Borel subsets of Ω. For any $x \in \Omega$ and $\rho > 0$, $B(x,\rho)$ stands for the open ball of $\mathbb{R}^n$ centered at x with radius ρ and whenever $x = 0$ and $\rho = 1$ we simply write B_1. We will use the notation w_n for the Lebesgue measure of the ball B_1. If μ is a Radon measure, we denote by $|\mu|$ its total variation.

Let $u \in L^1_{loc}(\Omega, \mathbb{R}^m)$. We recall that a point $x \in \Omega$ is a *Lebesgue point* of u if there exists $\tilde{u}(x) \in \mathbb{R}^m$ such that

$$\lim_{r \to 0^+} \fint_{B(x,r)} |u(y) - \tilde{u}(x)| dx = 0;$$

the vector $\tilde{u}(x)$ is called the *approximate limit* of u at x. Ω_u denotes the set of of Lebesgue points of u and $S_u := \Omega \setminus \Omega_u$ is called the *approximate discontinuity set* of u. By Lebesgue's differentiation theorem, the set S_u is $\mathcal{L}^n$-negligible and the function $\tilde{u} : \Omega_u \to \mathbb{R}^m$ called *Lebesgue representative* of u coincides with u $\mathcal{L}^n$-almost everywhere in Ω_u.
We recall also that a point $x \in S_u$ is an *approximate jump point* of u if there exists $(u^+(x), u^-(x), \nu_u(x)) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbf{S}^{n-1}$ with $u^+(x) \neq u^-(x)$ such that

$$\lim_{r \to 0^+} \fint_{B^{\pm}(x,r,\nu_u(x))} |u(y) - u^{\pm}(x)| dx = 0$$

where $B^{\pm}(x, r, \nu_u(x)) := \{y \in B(x,r)\colon \langle y - x, \pm\nu_u(x)\rangle > 0\}$ and $u^{\pm}(x)$ are called the *one-sided Lebesgue limits* of u at x with respect to the direction $\nu_u(x)$. The triplet $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined up to a change of orientation of $\nu_u(x)$ and a simultaneously permutation of $u^+(x)$ and $u^-(x)$. The Borel subset $J_u \subset S_u$ called *Jump set* of u is the set of approximate jump points of u.

Definition 2.1. *We say that $u : \Omega \to \mathbb{R}^n$ is a function with bounded deformation in Ω if $u \in L^1(\Omega, \mathbb{R}^n)$ and $Eu := (Du + Du^T)/2 \in \mathcal{M}_b\big(\Omega, \mathrm{M}^{n\times n}_{\mathrm{sym}}\big)$, where Du is the distributional gradient of u and $\mathcal{M}_b\big(\Omega, \mathrm{M}^{n\times n}_{\mathrm{sym}}\big)$ is the space of $\mathrm{M}^{n\times n}_{\mathrm{sym}}$-valued Radon measures with finite total variation in Ω.*

The space $BD(\Omega)$ of functions with bounded deformation in Ω was introduced in [20] and studied, for instance in [5], [18], [22], [23] in relation with the static model of Hencky in perfect plasticity. $BD(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega,\mathbb{R}^n)} + |Eu|(\Omega)$$

where $|Eu|(\Omega)$ is the total variation of the measure Eu in Ω.

It is well known (see Temam [23]) that the trace operator $\mathrm{Tr} : BD(\Omega) \to L^1(\partial\Omega, \mathbb{R}^n)$ is continuous.

Whenever the open set Ω is assumed to be connected, the kernel of the operator E is the class of *infinitesimal rigid motions* denoted here by $\mathcal{R}$, and composed of affine maps of the form $Mx + b$, where M is a skew-symmetric $n \times n$ matrix and $b \in \mathbb{R}^n$. Therefore $\mathcal{R}$ is a finite-dimensional subspace.

Fine properties of BD functions were studied, for instance, in [4], [8] and [18]. The following "Poincaré type" inequality for BD functions has been proved by Kohn [18] (see also [4]).

Theorem 2.2. *Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with Lipschitz boundary. Let $R : BD(\Omega) \to \mathcal{R}$ be a continuous linear map which leaves $\mathcal{R}$ fixed.*
Then there exists a positive constant $C(\Omega, R)$ such that:

$$\int_\Omega |u - R(u)|dx \le C(\Omega, R)|Eu|(\Omega) \quad \text{for any } u \in BD(\Omega). \tag{2.1}$$

When Ω is an open ball of $\mathbb{R}^n$ there is a precise representation of the rigid motion $R(u)$, given in the following theorem.

Theorem 2.3. *Let $u \in BD(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\rho > 0$. Then there exists a vector $d_\rho(u)(x) \in \mathbb{R}^n$ and an $n \times n$ skew-symmetric matrix $A_\rho(u)(x)$ such that:*

$$\int_{B(x,\rho)} |u(y) - d_\rho(u)(x) - A_\rho(u)(x)(y - x)|dy \le C(n)\rho|Eu|(B(x, \rho)) \tag{2.2}$$

where $C(n)$ is a positive constant depending only on the dimension n.
Moreover, $d_\rho(u)(x)$ and $A_\rho(u)(x)$ are expressed as singular integrals in the following ways:

$$d^i_\rho(u)(x) := \sum_{l,m=1}^{n} \int_{|y-x|\ge\rho} \frac{\bigwedge^i_{lm}(y - x)}{nw_n|y - x|^n} dEu_{lm}(y); \tag{2.3}$$

$$A^{ij}_\rho(u)(x) := \sum_{l,m=1}^{n} \int_{|y-x|\ge\rho} -\frac{\Gamma^{lm}_{ij}(y - x)}{2w_n|y - x|^{n+2}} dEu_{lm}(y), \tag{2.4}$$

where $\bigwedge$ and Γ, respectively third and fourth-order tensor valued functions, are defined and studied in [18], [4].

We recall that if $u \in BD(\Omega)$, then the jump set J_u of u is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable Borel set and the following decomposition of the measure Eu holds

$$Eu = \mathcal{E}u\mathcal{L}^n + E^s u = \mathcal{E}u\mathcal{L}^n + E^j u + E^c u\,, \tag{2.5}$$

where $E^j u := ([u] \odot \nu_u)\mathcal{H}^{n-1} \llcorner J_u$, $[u] := u^+ - u^-$, $\mathcal{E}u$ is the density of the absolutely continuous part of Eu with respect to $\mathcal{L}^n$, $E^s u$ is the singular part, and $E^c u$ is the *Cantor part* and vanishes on Borel subsets that are σ-finite with respect to $\mathcal{H}^{n-1}$ (see [4]).

Hereinafter we will use the following proposition proved in [4, Proposition 7.8 and Remark 7.9]

Proposition 2.4. *Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a 0-homogeneous function, smooth and with mean value zero on the unit sphere $\mathbf{S}^{n-1}$. For any Radon measure μ with finite total variation in*

$\mathbb{R}^n$, *let us define the functions*

$$h_\rho(x) := \int_{|y-x|\geq\rho} \frac{K(y-x)}{|y-x|^n} d\mu(y) \qquad \rho > 0.$$

Then the function $h(x) := \sup_{\rho>0} |h_\rho(x)|$ *satisfies the following weak* L^1 *estimate*

$$\big|\{x \in \mathbb{R}^n \colon h(x) > t\}\big| \leq \frac{C(n,K)}{t} |\mu|(\mathbb{R}^n). \tag{2.6}$$

Moreover, if $\mu = f\mathcal{L}^n$ *with* $f \in L^p(\mathbb{R}^n)$*, then the following strong* L^p *estimate holds*

$$\|h\|_p \leq C(n,K)\,\|f\|_p\,. \tag{2.7}$$

Let us recall also the theorem by Ambrosio-Coscia-Dal Maso [4] on the approximate differentiability of BD functions.

Theorem 2.5. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$ *with Lipschitz boundary. Let* $u \in BD(\Omega)$*. Then for* $\mathcal{L}^n$*-almost every* $x \in \Omega$ *there exists an* $n \times n$ *matrix* $\nabla u(x)$ *such that*

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B(x,\rho)} \frac{|u(y) - u(x) - \nabla u(x)(y-x)|}{\rho} dy = 0\,, \tag{2.8}$$

and

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B(x,\rho)} \frac{|(u(y) - u(x) - \mathcal{E}u(x)(y-x), y-x)|}{|y-x|^2} dy = 0 \tag{2.9}$$

for $\mathcal{L}^n$*-almost every* $x \in \Omega$*.*

In particular, by (2.8) u is approximately differentiable $\mathcal{L}^n$-almost everywhere in Ω and the function ∇u satisfies the weak L^1 estimate

$$\mathcal{L}^n(\{x \in \Omega \,:\, |\nabla u(x)| > t\}) \leq \frac{C(n,\Omega)}{t} \,\|u\|_{BD(\Omega)} \quad \forall t > 0,$$

where $C(n,\Omega)$ is a positive constant depending only on n and Ω.
From (2.9) and (2.8) one can easily see that

$$\mathcal{E}u(x) = (\nabla u(x) + \nabla u(x)^T)/2 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega. \tag{2.10}$$

Analogously to the space SBV introduced by De Giorgi and Ambrosio (see for instance [3]), the space SBD was introduced by Bellettini and Coscia in [7] and studied in [8].

Definition 2.6. *The space* $SBD(\Omega)$ *of* special functions with bounded deformation*, is the space of functions* $u \in BD(\Omega)$ *such that the measure* $E^c u$ *in (2.5) is zero.*

We set

$$A(u)(x) := \sup_{\rho>0} |A_\rho(u)(x)|$$

with $A_\rho(u)$ the anti-symmetric matrix defined in (2.4). Note that for every $u \in SBD(\Omega)$,

$$A_\rho(u) = L_\rho(u) + J_\rho(u)$$

with

$$L_\rho^{ij}(u)(x) := \sum_{l,m=1}^{n} \int_{|y-x|\geq \rho} -\frac{\Gamma_{ij}^{lm}(y-x)}{2w_n|y-x|^{n+2}}\mathcal{E}u_{lm}(y)dy \tag{2.11}$$

and

$$J_\rho^{ij}(u)(x) := \sum_{l,m=1}^{n} \int_{|y-x|\geq \rho} -\frac{\Gamma_{ij}^{lm}(y-x)}{2w_n|y-x|^{n+2}}dE^j u_{lm}(y). \tag{2.12}$$

We set also

$$L(u)(x) := \sup_{\rho>0} |L_\rho(u)(x)| \quad \text{ and } \quad J(u)(x) := \sup_{\rho>0} |J_\rho(u)(x)|. \tag{2.13}$$

Let us recall that, given a $\mathbb{R}^m$-valued Radon Measure μ in $\mathbb{R}^n$, the *maximal function* of μ is defined by

$$M(\mu)(x) := \sup_{\rho>0} \frac{|\mu|(B(x,\rho))}{|B(x,\rho)|} \qquad \forall x \in \mathbb{R}^n.$$

Whenever $\mu = g\mathcal{L}^n$, we recover the maximal function of the function g (see [21]).

The following theorem on Lusin type approximation of BD functions is proved in [11].

Theorem 2.7. *Let Ω be either $\mathbb{R}^n$ or a Lipschitz bounded open subset of $\mathbb{R}^n$ and $u \in BD(\Omega)$. Then for any $\lambda > 0$, there exists a Lipschitz continuous function $v_\lambda : \Omega \to \mathbb{R}^n$ with $lip(v_\lambda) \leq C\lambda$ such that:*

$$\big|\{x \in \Omega \,:\, v_\lambda(x) \neq u(x)\}\big| \leq \frac{C}{\lambda} \|u\|_{BD(\Omega)}, \tag{2.14}$$

where C is a positive constant only depending on n and Ω.

In the following proposition we further refine the estimate (2.14) when the function $u \in SBD(\Omega)$ with $\mathcal{E}u \in L^p(\Omega, \mathrm{M}^{n\times n}_{\mathrm{sym}})$.

Proposition 2.8. *Let $p \in (1,\infty)$, $\lambda > 0$ and $u \in SBD_p(\mathbb{R}^n)$. Then there exists a function $v_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz continuous with $lip(v_\lambda) \leq C\lambda$, $|v_\lambda(x)| \leq C\lambda$ for every $x \in \mathbb{R}^n$, and for any Borel subset E of $\mathbb{R}^n$ we have the following estimate*

$$\begin{aligned}\big|E \cap \{x \in \mathbb{R}^n \,:\, v_\lambda(x) \neq u(x)\}\big| &\leq \frac{C}{\lambda}[||u||_{L^1(\mathbb{R}^n,\mathbb{R}^n)} + |E^j u|(E)] + \\ &+ \frac{1}{\lambda^p}\int_{E\cap\{L(u)(x)>\lambda\}} |L(u)(x)|^p dx + \frac{1}{\lambda^p}\int_{E\cap\{M(|\mathcal{E}u|)>\lambda\}} [M(|\mathcal{E}u|)]^p dx.\end{aligned} \tag{2.15}$$

where C is a positive constant only depending on n.

Proof. For $\lambda > 0$, we set

$$E_\lambda := \{x \in \mathbb{R}^n\colon\ M(|u|\mathcal{L}^n + |Eu|)(x) \leq 3\lambda \text{ and } A(u)(x) \leq 2\lambda\}.$$

It has been proved in Theorem 2.7 that $u|_{E_\lambda\setminus S_u}$ is Lipschitz continuous with Lipschitz constant less or equal to a positive constant proportional to λ. Moreover, from the Lebesgue differentiation theorem we have also

$$|u(x)| \leq 3\lambda \quad \forall x \in E_\lambda \setminus S_u.$$

The function v_λ is then obtained from $u|_{E_\lambda\setminus S_u}$ by Kirszbraun's Theorem (see Federer [15, Theorem 2.10.43]).

Now given $E \in \mathcal{B}(\mathbb{R}^n)$, since $E \cap \{x \in \mathbb{R}^n : v_\lambda(x) \neq u(x)\} \subset E \setminus E_\lambda$, it is sufficient to estimate the measure of $E \setminus E_\lambda$.

Note that

$$\big|E \cap \{x \in \mathbb{R}^n\colon\ A(u)(x) > 2\lambda\}\big| \leq \big|E \cap \{x \in \mathbb{R}^n\colon\ L(u)(x) > \lambda\}\big| + \big|E \cap \{x \in \mathbb{R}^n\colon\ J(u)(x) > \lambda\}\big|$$

where L and J are defined in (2.11) and (2.12). From Proposition 2.4 and Chebychev's inequality we get respectively

$$\big|E \cap \{x \in \mathbb{R}^n\colon\ J(u)(x) > \lambda\}\big| \leq \frac{C(n)}{\lambda}|E^j u|(E)$$

and

$$\big|E \cap \{x \in \mathbb{R}^n\colon\ L(u)(x) > \lambda\}\big| \leq \frac{1}{\lambda^p}\int_{E\cap\{L(u)(x)>\lambda\}} |L(u)(x)|^p dx.$$

Hence,

$$\big|E \cap \{x \in \mathbb{R}^n\colon\ A(u)(x) > 2\lambda\}\big| \leq \frac{C(n)}{\lambda}|E^j u|(E) + \frac{1}{\lambda^p}\int_{E\cap\{L(u)(x)>\lambda\}} |L(u)(x)|^p dx. \tag{2.16}$$

On the other hand, using covering theorems (see [3], [15]) and the properties of maximal functions of L^p functions, we obtain the estimates

$$\begin{aligned}\big|E \cap \{x \in \mathbb{R}^n\colon\ M(|u|\mathcal{L}^n + |Eu|)(x) > 3\lambda\}\big| &\leq \big|E \cap \{x \in \mathbb{R}^n\colon\ M(|u|\mathcal{L}^n)(x) > \lambda\}\big| \\ &+ \big|E \cap \{x \in \mathbb{R}^n\colon\ M(|E^j u|)(x) > \lambda\}\big| + \big|E \cap \{x \in \mathbb{R}^n\colon\ M(|\mathcal{E}u|)(x) > \lambda\}\big| \\ &\leq \frac{C(n)}{\lambda}[||u||_{L^1(\mathbb{R}^n,\mathbb{R}^n)} + |E^j u|(E)] + \frac{1}{\lambda^p}\int_{E\cap\{M(|\mathcal{E}u|)>\lambda\}} [M(|\mathcal{E}u|)]^p\, dx.\end{aligned} \tag{2.17}$$

The estimate (2.15) is then obtained by adding (2.16) to (2.17). □

Remark 2.9. Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$ and $u \in SBD(\Omega)$ with $\mathcal{E}u \in L^p(\Omega, \mathrm{M}^{n\times n}_{\mathrm{sym}})$. Let $\overline{u}$ be the extension of u by 0 outiside Ω. It is easy to see that

$$E\overline{u} := \mathcal{E}u\mathcal{L}^n \llcorner \Omega + E^j u \llcorner \Omega - \mathrm{tr}(u) \odot \nu\mathcal{H}^{n-1} \llcorner \partial\Omega$$

where $\mathrm{tr}(u)$ and ν are respectively the trace of u on $\partial\Omega$ and the outer unit normal vector to $\partial\Omega$. Therefore from the continuity of the trace operator for BD functions, we get

$$\overline{u} \in SBD(\mathbb{R}^n) \quad \text{with} \quad \mathcal{E}\overline{u} \in L^p(\mathbb{R}^n, \mathrm{M}^{n\times n}_{\mathrm{sym}}).$$

Applying then Proposition 2.8 to $\overline{u}$, we obtain the following estimate for every $E \in \mathcal{B}(\mathbb{R}^n)$

$$
\begin{aligned}
&\big|E \cap \{x \in \mathbb{R}^n : v_\lambda(x) \neq \overline{u}(x)\}\big| \\
\leq\ & \frac{C(n,\Omega)}{\lambda}\Big[||u||_{L^1(\Omega,\mathbb{R}^n)} + |E^j u|(\Omega \cap E) + \int_{\partial\Omega\cap E} |\mathrm{tr}(u)| d\mathcal{H}^{n-1}\Big] \\
& + \frac{1}{\lambda^p}\int_{E\cap\{L(u)(x) > \lambda\}} |L(u)(x)|^p dx + \frac{1}{\lambda^p}\int_{E\cap\{M(|\mathcal{E}u|)>\lambda\}} [M(|\mathcal{E}u|)]^p dx.
\end{aligned}
$$

In particular for any $E \in \mathcal{B}(\Omega)$ we get

$$
\big|E \cap \{x \in \Omega : v_\lambda(x) \neq u(x)\}\big| \leq \frac{C(n,\Omega)}{\lambda}[||u||_{L^1(\Omega,\mathbb{R}^n)} + |E^j u|(\Omega)] \tag{2.18}
$$

$$
+ \frac{1}{\lambda^p}\int_{E\cap\{L(u)(x) > \lambda\}} |L(u)(x)|^p dx + \frac{1}{\lambda^p}\int_{E\cap\{M(|\mathcal{E}u|)>\lambda\}} [M(|\mathcal{E}u|)]^p\, dx.
$$

3. The proof of our main result

This section is essentially devoted to the proof of Theorem 1.2. The following proposition will be crucial.

Proposition 3.1. *Let $f_h : \Omega \times \mathrm{M}^{n\times n}_{\mathrm{sym}} \to [0,\infty)$ be a sequence of Carathéodory functions satisfying for a.e. every $x \in \Omega$, for every $\xi \in \mathrm{M}^{n\times n}_{\mathrm{sym}}$,*

$$
\frac{1}{C}|\xi|^p \leq f_h(x,\xi) \leq \phi_h(x) + C(1+|\xi|^p),
$$

for some constant $C > 0$ and a sequence (ϕ_h) uniformly bounded in $L^1(B_1)$. Assume that there exist an $\mathcal{L}^n$-negligible set $N \subset B_1$ and a symmetric quasi-convex function $f : \mathrm{M}^{n\times n}_{\mathrm{sym}} \to [0,\infty)$ such that $\lim_{h\to\infty} f_h(y,\xi) = f(\xi)$ uniformly on compact subsets of $\mathrm{M}^{n\times n}_{\mathrm{sym}}$ and for any $y \in B_1 \setminus N$. Then, for any sequence (u_h) in $SBD(B_1)$ converging strongly in $L^1(B_1,\mathbb{R}^n)$ to a linear function u, with $\lim_{h\to\infty} |E^j u_h|(B_1) \to 0$, we have

$$
\int_{B_1} f(\mathcal{E}u)\, dx \leq \liminf_{h\to\infty} \int_{B_1} f_h(x,\mathcal{E}u_h)\, dx.
$$

Proof. Let $(u_h) \subset SBD(B_1)$ be a sequence which converges strongly in $L^1(B_1,\mathbb{R}^n)$ to a linear function u and $\lim_{h\to\infty} |E^j u_h|(B_1) \to 0$. Up to substituting u_h by $u_h - u$ and $f_h(x,z)$ by $f_h(x,z+\mathcal{E}u)$ we can assume that $u \equiv 0$. So, we have to prove that

$$
|B_1|f(0) \leq \liminf_{h\to\infty} \int_{B_1} f_h(x,\mathcal{E}u_h)\, dx. \tag{3.1}
$$

Up to a subsequence we assume that

$$
\liminf_{h\to\infty} \int_{B_1} f_h(x,\mathcal{E}u_h)\, dx = \lim_{h\to\infty} \int_{B_1} f_h(x,\mathcal{E}u_h)\, dx < \infty.
$$

So the sequence $(\mathcal{E}u_h)$ is uniformly bounded in $L^p(B_1,\mathrm{M}^{n\times n}_{\mathrm{sym}})$. We set

$$
\Psi_h := [M(\mathcal{E}u_h)]^p + |L(u_h)|^p + |\phi_h| \tag{3.2}
$$

where M is the maximal function and L is defined in (2.13). From the assumptions and from Proposition 2.4, we have that (Ψ_h) is a bounded sequence in $L^1(B_1)$. So, By Chacon Bitting Lemma (see for instance [3, Lemma 5.32]) there exist a subsequence of (Ψ_h) (still denoted (Ψ_h)) and a decreasing sequence of sets $(E_k) \subset \mathcal{B}(B_1)$ such that $|E_k| \to 0$ as $k \to \infty$ and the sequence $(\Psi_h 1_{B_1 \setminus E_k})_h$ is equiintegrable for any $k \in \mathbb{N}$. We introduce the following modulus of equiintegrability for the sequence $(\Psi_h 1_{B_1 \setminus E_k})_h$

$$W_k(\delta) := \sup \left\{ \begin{array}{l} \limsup_{h\to\infty} \int_F \Psi_h \, dx \colon \; F \in \mathcal{B}(B_1), \\ F \subset B_1 \setminus E_k \text{ and } |F| \leq \delta \end{array} \right\} \quad \forall \delta > 0, \, \forall k \in \mathbb{N}. \tag{3.3}$$

It follows that $W_k(\delta) \to 0$ as $\delta \to 0$.
Now, from Proposition 2.8 and Remark 2.9 we have for any integer $m \geq 1$, a Lipschitz continuous function $v_{h,m} : \overline{B}_1 \to \mathbb{R}^n$ and a set $E_{h,m} \in \mathcal{B}(B_1)$ such that

$$Lip(v_{h,m}) \leq C(n, B_1)m, \; |v_{h,m}(x)| \leq C(n, B_1)m \;\; \forall x \in \overline{B}_1, \quad v_{h,m} = u_h \text{ in } B_1 \setminus E_{h,m} \tag{3.4}$$

and for any $E \in \mathcal{B}(B_1)$ the following estimate holds

$$\begin{aligned} \big|E_{h,m} \setminus E\big| \leq & \frac{C(n, B_1)}{m} [||u_h||_{L^1(B_1,\mathbb{R}^n)} + |E^j u_h|(B_1)] \\ & + \frac{1}{m^p} \int_{\{L(u_h)(x)>m\}\setminus E} |L(u_h)(x)|^p dx \, + \, \frac{1}{m^p} \int_{\{M(|\mathcal{E}u_h|)>m\}\setminus E} [M(|\mathcal{E}u_h|)]^p dx. \end{aligned} \tag{3.5}$$

In particular for $E = E_k$ we get from the definition of Ψ_h that

$$\big|E_{h,m} \setminus E_k\big| \leq \frac{C(n, B_1)}{m} [||u_h||_{L^1(B_1,\mathbb{R}^n)} + |E^j u_h|(B_1)] + \frac{2}{m^p} \int_{\{\Psi_h > m^p\}\setminus E_k} \Psi_h \, dx. \tag{3.6}$$

We set $S := \sup_h ||\Psi_h||_1$. Using the fact that $\big|\{\Psi_h > m^p\}\big| \leq \frac{S}{m^p}$ together with $u_h \to 0$ strongly in $L^1(B_1, \mathbb{R}^n)$ and $|E^j u_h|(B_1) \to 0$ (by assumptions), we get from (3.6) that

$$\limsup_{h\to\infty} m^p \big|E_{h,m} \setminus E_k\big| \leq 2W_k\Big(\frac{S}{m^p}\Big). \tag{3.7}$$

From the inequality (3.7), it is easy to see (for m large enough) that

$$\limsup_{h\to\infty} \int_{E_{h,m}\setminus E_k} \phi_h \, dx \leq \limsup_{h\to\infty} \int_{E_{h,m}\setminus E_k} \Psi_h \, dx \leq 2W_k\Big(\frac{S}{m^p}\Big).$$

Now from (3.4), it follows by Ascoli-Arzelà that the sequence $(v_{h,m})_h$ is relatively compact in $C(\overline{B}_1, \mathbb{R}^n)$. Hence, using a diagonal argument, we get up to a subsequence that, for every integer $m \geq 1$, $v_{h,m}$ converges uniformly to a function $v_m \in C(\overline{B}_1, \mathbb{R}^n)$ as $h \to \infty$.
Since, $|E_k| \to 0$ as $k \to \infty$, to get (3.1), it is enough to prove that

$$|B_1 \setminus E_k| f(0) \, \leq \, \liminf_{h\to\infty} \int_{B_1} f_h(x, \mathcal{E}u_h) \, dx \qquad \forall k \in \mathbb{N}. \tag{3.8}$$

We have the following estimates

$$\begin{aligned}\int_{B_1} f_h(x,\mathcal{E}u_h)\,dx \geq \int_{B_1\setminus(E_{h,m}\cup E_k)} f_h(x,\mathcal{E}u_h)\,dx &= \int_{B_1\setminus(E_{h,m}\cup E_k)} f_h(x,\mathcal{E}v_{h,m})\,dx \\ &= \int_{B_1\setminus E_k} f_h(x,\mathcal{E}v_{h,m})\,dx - \int_{E_{h,m}\setminus E_k} f_h(x,\mathcal{E}v_{h,m})\,dx \\ &\geq \int_{B_1\setminus E_k} f_h(x,\mathcal{E}v_{h,m})\,dx - \int_{E_{h,m}\setminus E_k} \phi_h dx - Cm^p\big|E_{h,m}\setminus E_k\big|.\end{aligned}$$

So, passing to the limit as $h\to\infty$, and using (3.6) and (3.7) we get that

$$\liminf_{h\to\infty}\int_{B_1} f_h(x,\mathcal{E}u_h)\,dx \geq \liminf_{h\to\infty}\int_{B_1\setminus E_k} f_h(x,\mathcal{E}v_{h,m})\,dx - CW_k\Big(\frac{S}{m^p}\Big). \tag{3.9}$$

Now from the assumption on the convergence of $f_h(x,\xi)$ to $f(\xi)$, we get

$$\liminf_{h\to\infty}\int_{B_1\setminus E_k} f_h(x,\mathcal{E}v_{h,m})\,dx \geq \liminf_{h\to\infty}\int_{B_1\setminus E_k} f(\mathcal{E}v_{h,m})\,dx. \tag{3.10}$$

Using the symmetric quasi-convexity of the function f, we also get

$$\liminf_{h\to\infty}\int_{B_1\setminus E_k} f(\mathcal{E}v_{h,m})\,dx \geq \int_{B_1\setminus E_k} f(\mathcal{E}v_m)\,dx. \tag{3.11}$$

Indeed, f symmetric quasi-convex means that $f\circ\pi$ is quasi-convex in the classical sense, where π is the projection on symmetric matrices. Since $lip(v_{h,m})\leq C(\Omega,n)m$, it is easy to see that the $(v_{h,m})_h$ converges weakly $\star$ in $W^{1,\infty}(B_1,\mathbb{R}^n)$ to the function v_m and hence (3.11) follows from a classical lower semicontinuity theorem by Morrey (see for instance Dacorogna [10]).

Finally putting together (3.9), (3.10) and (3.11) we get

$$\liminf_{h\to\infty}\int_{B_1} f_h(x,\mathcal{E}u_h)\,dx \geq \int_{B_1\setminus E_k} f(\mathcal{E}v_m)\,dx - CW_k\Big(\frac{S}{m^p}\Big). \tag{3.12}$$

On the other hand, from (3.7) we have also that

$$m^p\big|\{x\in B_1\setminus E_k\colon\ v_m(x)\neq 0\}\big| \leq 2W_k\Big(\frac{S}{m^p}\Big). \tag{3.13}$$

In fact, from the L^1-norm lower semicontinuity of the map

$$u\to\big|\{x\in B_1\setminus E_k\colon\ |u|(x)\neq 0\}\big| = \int_{B_1\setminus E_k}\chi_{(0,\infty)}(|u|(x))dx,$$

it follows that

$$\begin{aligned} m^p\big|\{x\in B_1\setminus E_k\colon\ v_m(x)\neq 0\}\big| &\leq \liminf_{h\to\infty} m^p\big|\{x\in B_1\setminus E_k\colon\ (v_{h,m}-u_h)(x)\neq 0\}\big| \\ &= \liminf_{h\to\infty} m^p\big|E_{h,m}\setminus E_k\big| \leq 2W_k\Big(\frac{S}{m^p}\Big).\end{aligned}$$

Now, setting $A_m := \{x \in B_1 \setminus E_k \colon\ v_m(x) \neq 0\}$, we obtain from (3.12) that

$$\liminf_{h\to\infty} \int_{B_1} f_h(x, \mathcal{E}u_h)\, dx \geq \int_{B_1\setminus(E_k\cup A_m)} f(0)\, dx - CW_k\Big(\frac{S}{m^p}\Big) \tag{3.14}$$

So, passing to the limit in (3.12) as $m \to \infty$ and using (3.13) we finally obtain (3.8) and this achieves the proof of the proposition. □

Remark 3.2. We recall that the terminology *symmetric quasi-convexity* is already available in the literature. It has been introduced and properly used in [12] (see also [6, 13]).

Now we are in the position to prove the main result of this paper.

The proof of Theorem 1.2. Let (u_h) be a sequence such that u_h converges strongly to u in $L^1(\Omega, \mathbb{R}^n)$ and $|E^j u_h|$ converges weak $*$ to the measure ν singular with respect to the Lebesgue measure. We assume that

$$\liminf_{h\to\infty} \int_\Omega f(x, \mathcal{E}u_h)\, dx \;=\; \lim_{h\to\infty} \int_\Omega f(x, \mathcal{E}u_h)\, dx \;<\; \infty.$$

So, up to a subsequence, the sequence of measures $f_h(x, \mathcal{E}u_h)\mathcal{L}^n \llcorner \Omega$ converges weakly $*$ to a positive measure μ. To prove (1.4), it is enough to prove that

$$\frac{d\mu}{d\mathcal{L}^n}(x_0) \;\geq\; f(x_0, \mathcal{E}u(x_0)) \qquad \text{a.e. } x_0 \in \Omega. \tag{3.15}$$

In fact, from the lower semicontinuity of the total variations of measure with respect to weak $*$ convergence and from the inequality (3.15) it follows that

$$\liminf_{h\to\infty} \int_\Omega f(x, \mathcal{E}u_h)\, dx \geq \mu(\Omega) \geq \int_\Omega \frac{d\mu}{d\mathcal{L}^n}(x)\, dx \geq \int_\Omega f(x, \mathcal{E}u(x))\, dx.$$

So, let us prove that (3.15) holds. To this aim, we use a characterization of Carathéodory functions by Scorza-Dragoni (see e.g. [14, Page 235]), to get for every $i \in \mathbb{N}$ a compact set $K_i \subset \Omega$ such that $|\Omega \setminus K_i| < 1/i$ and $f|_{K_i\times \mathrm{M}^{n\times n}_{\mathrm{sym}}}$ is continuous in $K_i \times \mathrm{M}^{n\times n}_{\mathrm{sym}}$. Let K_i^1 be the set of Lebesgue points of the function χ_{K_i}. We set

$$F := \bigcup_{i\in\mathbb{N}} (K_i \cap K_i^1)$$

and it follows that $|\Omega \setminus F| \;\leq\; |\Omega \setminus (K_i \cap K_i^1)| \;=\; |\Omega \setminus K_i| \;\leq\; 1/i \to 0 \quad$ as $i \to \infty$. We fix $x_0 \in F$ such that:

(i) x_0 is an approximate differentiability point of u and such that $\mathcal{E}u(x_0) = \frac{\nabla u(x_0)+\nabla u(x_0)^T}{2}$;

(ii) $\displaystyle\frac{d\nu}{d\mathcal{L}^n}(x_0) \;=\; \lim_{\varepsilon\to 0} \frac{\nu(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} = 0;$

(iii) $\displaystyle\frac{d\mu}{d\mathcal{L}^n}(x_0) \;=\; \lim_{\varepsilon\to 0} \frac{\mu(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} \;<\; \infty.$

We consider the sequence $\varepsilon_k \searrow 0^+$ such that $\nu(\partial B(x_0, \varepsilon_k)) = 0$ and $\mu(\partial B(x_0, \varepsilon_k)) = 0$. Note that such a sequence exists since the set $\{\varepsilon > 0\colon\ \nu(\partial B(x_0, \varepsilon)) > 0,\ \nu(\partial B(x_0, \varepsilon)) > 0\}$ is a most countable set. From the fact that $u_h \to u$ strongly in $L^1(\Omega,\mathbb{R}^n)$ and the approximate differentiability of u at x_0 we get that

$$\lim_{k\to\infty}\lim_{h\to\infty} ||u_{k,h}-w_0||_{L^1(\mathcal{B}_1,\mathbb{R}^n)}=0 \tag{3.16}$$

where

$$u_{k,h} := \frac{u_h(x_0+\varepsilon_k y)-u(x_0)}{\varepsilon_k} \quad \text{and} \quad w_0(y):=\nabla u(x_0)y.$$

We have also that

$$\begin{aligned}
|E^j u_{k,h}|(B_1) &= \int_{B_1\cap J_{u_{k,h}}} \left|(u^+_{k,h}-u^-_{k,h})\odot \nu_{u_{k,h}}\right| d\mathcal{H}^{n-1}\\
&= \varepsilon_k^{-n}\int_{B(x_0,\varepsilon_k)\cap J_{u_h}} \left|(u^+_h-u^-_h)\odot \nu_{u_h}\right| d\mathcal{H}^{n-1}\\
&= \frac{|E^j u_h|(B(x_0,\varepsilon_k))}{\varepsilon_k^n} \le \frac{|E^j u_h|(\overline{B(x_0,\varepsilon_k)})}{\varepsilon_k^n}.
\end{aligned}$$

Hence

$$\begin{aligned}
\limsup_{k\to\infty}\limsup_{h\to\infty}|E^j u_{k,h}|(B_1) &\le \limsup_{k\to\infty}\limsup_{h\to\infty}\frac{|E^j u_h|(\overline{B(x_0,\varepsilon_k)})}{\varepsilon_k^n}\\
&\le \limsup_{k\to\infty}\frac{\nu(\overline{B(x_0,\varepsilon_k)})}{\varepsilon_k^n} = 0.
\end{aligned} \tag{3.17}$$

On the other hand, setting $f_k(y,\xi):=f(x_0+\varepsilon_k y,\xi)$ we get that

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^n}(x_0) &\ge \limsup_{k\to\infty}\frac{\mu(\overline{B(x_0,\varepsilon_k)})}{|B(x_0,\varepsilon_k)|}\\
&\ge \limsup_{k\to\infty}\limsup_{h\to\infty}\frac{1}{|B(x_0,\varepsilon_k)|}\int_{B(x_0,\varepsilon_k)} f(x,\mathcal{E}u_h)\,dx\\
&\ge \limsup_{k\to\infty}\limsup_{h\to\infty}\frac{1}{w_n}\int_{B_1} f(x_0+\varepsilon_k y,\mathcal{E}u_{k,h})\,dy\\
&= \limsup_{k\to\infty}\limsup_{h\to\infty}\frac{1}{w_n}\int_{B_1} f_k(y,\mathcal{E}u_{k,h})\,dy.
\end{aligned}$$

By a standard diagonal argument we may extract a subsequence $v_k := u_{k,h_k}$ such that

$$\lim_{k\to\infty}||v_k-w_0||_{L^1(B_1,\mathbb{R}^n)}=0, \qquad \lim_{k\to\infty}|E^j v_k|(B_1)=0$$

and

$$\frac{d\mu}{d\mathcal{L}^n}(x_0) \ge \limsup_{k\to\infty}\frac{1}{w_n}\int_{B_1} f_k(y,\mathcal{E}v_k)\,dy.$$

Now, since $x_0\in F$, there exist $i_0\in\mathbb{N}$ such that $x_0\in K_{i_0}\cap K^1_{i_0}$. So, the sequence $\chi_{\frac{K_{i_0}-x_0}{\varepsilon_k}}$ converges strongly to 1 in $L^1(B_1)$ and hence, up to a subsequence $\chi_{\frac{K_{i_0}-x_0}{\varepsilon_k}}(y)\to 1$ for a.e. $y\in B_1$. So, for k large enough we have that $x_0+\varepsilon_k y\in K_{i_0}$ for a.e. $y\in B_1$. Hence, for every

$\xi \in \mathrm{M}^{n\times n}_{\mathrm{sym}}$ we get that $\lim_{k\to\infty} f(x_0+\varepsilon_k y,\xi) = f(x_0,\xi)$ for a.e. $y \in B_1$. Therefore, we finally obtain for a.e. $y \in B_1$ that

$$\lim_{k\to\infty} f_k(y,\xi) = f(x_0,\xi) \tag{3.18}$$

locally uniformly in $\mathrm{M}^{n\times n}_{\mathrm{sym}}$. So, applying Proposition 3.1 to the sequence (v_k), we get

$$\frac{d\mu}{d\mathcal{L}^n}(x_0) \ge \liminf_{k\to\infty} \frac{1}{w_n}\int_{B_1} f_k(y,\mathcal{E}v_k)\,dy \ge \frac{1}{w_n}\int_{B_1} f(x_0,\mathcal{E}u(x_0))\,dy = f(x_0,\mathcal{E}u(x_0))$$

which gives (3.15) and achieves the proof of the theorem. □

4. Some examples and remarks

In the proof of Theorem 1.2, the assumption on $|E^j u_h|$ has played a crucial role in order to perform the blow-up argument. Note that any sequence $(u_h) \subset W^{1,p}(\Omega,\mathbb{R}^n)$ such that $u_h \to u$ strongly in $L^1(\Omega,\mathbb{R}^n)$ satisfies trivially the assumptions of the theorem. For examples of sequences that are not necessarily in $W^{1,p}(\Omega,\mathbb{R}^n)$, we consider here a variational problem with a uniform L^∞ constraint on the admissible functions and a unilateral constraint the jump sets.
Let us recall here the compactness criterion in SBD by Bellettini-Coscia-Dal Maso [8].

Theorem 4.1. *Let $\phi : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function such that*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty. \tag{4.1}$$

Let (u_h) be a sequence in $SBD(\Omega)$ such that

$$\int_\Omega |u_h|dx + |E^j u_h|(\Omega) + \int_\Omega \phi(|\mathcal{E}u_h|)dx + \mathcal{H}^{n-1}(J_{u_h}) \le C \tag{4.2}$$

for some positive constant C independent of h. Then there exists a subsequence, still denoted by (u_h) and a function $u \in SBD(\Omega)$ such that

$$u_h \to u \text{ strongly in } L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^n), \tag{4.3}$$

$$\mathcal{E}u_h \rightharpoonup \mathcal{E}u \text{ weakly in } L^1(\Omega,\mathrm{M}^{n\times n}_{\mathrm{sym}}), \tag{4.4}$$

$$E^j u_h \rightharpoonup E^j u \text{ weakly} \star \text{ in } \mathcal{M}_b(\Omega,\mathrm{M}^{n\times n}_{\mathrm{sym}}), \tag{4.5}$$

$$\mathcal{H}^{n-1}(J_u) \le \liminf_{h\to+\infty} \mathcal{H}^{n-1}(J_{u_h}). \tag{4.6}$$

In the next example we consider a variational problem for which the minimizing sequences satisfy the assumption on the measures $|E^j u_h|$ in Theorem 1.2.

Example 4.2. Let $K \neq \emptyset$ be a non closed subset of Ω such that $0 < \mathcal{H}^{n-1}(K) < \infty$ and let $\{F(x)\}_{x\in\Omega}$ be a family of uniformly bounded closed subsets of $\mathbb{R}^n$. We consider the following variational problem:

$$\min_{\substack{u\in SBD(\Omega)\\ J_u\subset K\\ u(x)\in F(x) \text{ a.e. in } \Omega}} \int_\Omega f(x,\mathcal{E}u)\,dx \tag{4.7}$$

with $f : \Omega \times \mathrm{M}^{n\times n}_{\mathrm{sym}} \to [0,\infty)$ being a Carathéodory function that satisfies the assumptions of Theorem 1.2 that we repeat here for the reader's convenience:

(i) for a.e. every $x \in \Omega$, for every $\xi \in \mathrm{M}^{n\times n}_{\mathrm{sym}}$,

$$\frac{1}{C}|\xi|^p \le f(x,\xi) \le \phi(x) + C(1+|\xi|^p),$$

for some constant $C > 0$ and a function $\phi \in L^1(\Omega)$;

(ii) for a.e. every $x_0 \in \Omega$, $f(x_0,\cdot)$ is symmetric quasi-convex i.e,

$$f(x_0,\xi) \le \fint_A f(x_0,\xi+\mathcal{E}\varphi(x))dx \tag{4.8}$$

for every bounded open subset A of $\mathbb{R}^n$, for every $\varphi \in W^{1,\infty}_0(A,\mathbb{R}^n)$ and $\xi \in \mathrm{M}^{n\times n}_{\mathrm{sym}}$.

Let us prove that Problem (4.7) admits a solution.
First of all, By the rectifiability of jump sets of BD functions, note that the inclusion $J_u \subset K$ will be intended up to a $\mathcal{H}^{n-1}$-negligible set.

Now let $(u_h) \subset SBD(\Omega)$ be a minimizing sequence for Problem (4.7). By the assumptions, there exists a constant $M > 0$ such that $||u_h||_\infty \le M$ and

$$|E^j u_h|(\Omega) \le 2||u_h||_\infty \mathcal{H}^{n-1}(J_{u_h}) \le 2M\mathcal{H}^{n-1}(K) < \infty. \tag{4.9}$$

Hence, by the growth assumptions (i) on f, (4.2) is satisfied with $\phi(t) = t^p$. Therefore, by Theorem 4.1, the sequence (u_h) converges (up to a subsequence) strongly in $L^1(\Omega,\mathbb{R}^n)$ to some function $u \in SBD(\Omega)$. Moreover, we have also $u(x) \in F(x)$ a.e. $x \in \Omega$.

On the other hand $|E^j u_h|$ converge (up to a subsequence) weakly $\star$ to some positive measure ν. It easily follows from (4.9) that the measure ν is concentrated on the set K. Hence ν is singular with respect to the Lebesgue measure. Therefore, by Theorem 1.2, we have that

$$\int_\Omega f(x,\mathcal{E}u)\,dx \le \liminf_{h\to\infty} \int_\Omega f(x,\mathcal{E}u_h)\,dx.$$

Now let us prove that u verifies the constraint $J_u \subset K$ up to a $\mathcal{H}^{n-1}$-negligible set. This is obtained by slicing method.

To this aim, we recall the notations for one-dimensional sections of BD functions.
Given $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, we set

$$\pi^\xi := \{y \in \mathbb{R}^n\colon\ (y,\xi) = 0\}$$

and for every $y \in \pi^\xi$ and for every $B \in \mathcal{B}(\Omega)$,

$$B^\xi_y := \{t \in \mathbb{R}\colon\ y + t\xi \in B\} \quad \text{and} \quad B^\xi := \{y \in \pi^\xi\colon\ B^\xi_y \neq \emptyset\}.$$

For every $u \in L^1(\Omega, \mathbb{R}^n)$ we set

$$u_y^\xi(t) := (u(y+t\xi), \xi).$$

It has been proved in [4] that, if $u \in SBD(\Omega)$ then for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$, $u_y^\xi \in \mathrm{SBV}(\Omega_y^\xi)$. Viceversa, assume that

$$u_y^\xi \in \mathrm{SBV}(\Omega_y^\xi) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Omega^\xi \quad \text{and} \quad \int_{\Omega^\xi} |Du_y^\xi|(\Omega_y^\xi) d\mathcal{H}^{n-1}(y) < \infty$$

for every $\xi = \xi_i + \xi_j$, $i, j = 1, \cdots, n$ with $(\xi_i)_{i=1}^n$ being an orthonormal basis in $\mathbb{R}^n$. Then $u \in SBD(\Omega)$.
Setting $J_u^\xi := \{x \in J_u \colon (u^+(x) - u^-(x), \xi) \neq 0\}$, it follows from Fubini's theorem that

$$\mathcal{H}^{n-1}\big(J_u \setminus J_u^\xi\big) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathcal{S}^{n-1}. \tag{4.10}$$

From the structure theorem for BD functions (see [4, theorem 5.1]) we have also

$$J_{u_y^\xi} = \big(J_u^\xi\big)_y^\xi \quad \text{for a.e. } y \in \Omega^\xi.$$

Now we can prove that the limit u of the minimizing sequence (u_h) for Problem (4.7) satifies the constraint $J_u \subset K$ up to a $\mathcal{H}^{n-1}$-negligible set. Let $\xi \in \mathbf{S}^{n-1}$ be such that (4.10) holds. Following the proof of Theorem 4.1, we get that the sequence of one-dimensional section $(u_{h,y}^\xi)$ of the minimizing sequence (u_h) satisfies the assumptions of the SBV compactness theorem in [3, Theorems 4.7 and 4.8] and from $J_{u_h} \subset K$ we have also

$$J_{u_{h,y}^\xi} = (J_u^\xi)_y^\xi \subset K_y^\xi \text{ with } \mathcal{H}^0(K_y^\xi) < \infty \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Omega^\xi.$$

Therefore the limit function u_y^ξ has also its jump set contained in the finite set K_y^ξ. In fact, it is easy to see that the jump set $J_{u_y^\xi}$ is contained in the set of limits of the jump points of $u_{h,y}^\xi$. Now from $(J_u^\xi)_y^\xi = J_{u_y^\xi} \subset K_y^\xi$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$, we get $J_u^\xi \subset K$ up to a $\mathcal{H}^{n-1}$-negligible set and hence also $J_u \subset K$ up to a $\mathcal{H}^{n-1}$-negligible set. □

Remark 4.3. Note that the set K has been taken non closed in order to avoid the easy case where the minimizing sequences (u_h) and their limit u belong to the space

$$LD(\Omega \setminus K) := \{u \in L^1(\Omega \setminus K, \mathbb{R}^n) \colon Eu \in L^1(\Omega \setminus K, \mathrm{M}_{\mathrm{sym}}^{n\times n})\}$$

for which the lower semicontinuity of the functional

$$\int_\Omega f(x, \mathcal{E}u)\, dx = \int_{\Omega\setminus K} f(x, \mathcal{E}u)\, dx$$

in the strong topology of $L^1(\Omega \setminus K, \mathbb{R}^n)$ follows from [12, Theorem 3.1].

As we have seen in the previous example, the minimizing sequences for problem (4.7) satisfy the assumptions of both Theorems 1.2 and 4.1. However, unlike the assumptions (1.1) in Theorem 1.1 and (1.2) in [19], which are consistent with the compactness criterion in SBV, the assumption of Theorem 1.2 on the measures $|E^j u_h|$ is not always compatible with the compactness criterion in Theorem 4.1.

In the following example, we construct a sequence $(u_h) \subset SBD(\Omega)$ that satisfies the compactness criterion in SBD while $|E^j u_h|$ converges to a measure proportional to the Lebesgue measure.

Example 4.4. We consider in $\mathbb{R}^2$ the open squares

$$\Omega := \big(0,2\big) \times \big(0,2\big) \quad \text{and} \quad \Omega_h := \Big(\frac{1}{h} - \frac{1}{h^2}, \frac{1}{h} + \frac{1}{h^2}\Big) \times \Big(\frac{1}{h} - \frac{1}{h^2}, \frac{1}{h} + \frac{1}{h^2}\Big).$$

We set

$$E_h := \bigcup_{(i,j)\in I_h\times I_h} \big(\Omega_h + (i,j)\big) \quad \text{with} \quad I_h := \{0, 2/h, 4/h, \cdots, 2-2/h\}.$$

Let (u_h) be the sequence defined by $u_h := (\chi_{E_h}, 0)$ and let $E_{h,i,j} := \Omega_h + (i,j)$. By easy computations we get

$$E^j u_h = E u_h = \sum_{(i,j)\in I_h\times I_h} (1,0) \odot \nu_{E_{h,i,j}} \mathcal{H}^1 \llcorner \partial E_{h,i,j}$$

where $\nu_{E_{h,i,j}}$ is the unit normal vector to $\partial E_{h,i,j}$. Hence, we have

$$|E^j u_h|(\Omega) = 2\sqrt{2} + 4 \quad \text{and} \quad \mathcal{H}^1(J_{u_h} \cap \Omega) = 8.$$

Thus, the sequence (u_h) satisfies the assumptions of Theorem 4.1. However, the sequence $|E^j u_h|$ converges weakly $*$ to the measure $(\sqrt{2}+2)\mathcal{L}^2 \llcorner \Omega$.

Indeed, let $M_h^{i,j}$, $N_h^{i,j}$ be the two vertical sides of the square $E_{h,i,j}$ and $L_h^{i,j}$, $K_h^{i,j}$ be its horizontal sides. It is easy to see that

$$|E^j u_h| = \sum_{(i,j)\in I_h\times I_h} \Big(\mathcal{H}^1 \llcorner M_h^{i,j} + \mathcal{H}^1 \llcorner N_h^{i,j} + \frac{\sqrt{2}}{2}\mathcal{H}^1 \llcorner L_h^{i,j} + \frac{\sqrt{2}}{2}\mathcal{H}^1 \llcorner K_h^{i,j}\Big). \tag{4.11}$$

Now let $\varphi \in C_c(\Omega)$. It is easy to see

$$\lim_{h\to\infty} \sum_{(i,j)\in I_h\times I_h} \int_{S_h^{i,j}} \varphi d\mathcal{H}^1 = \int_\Omega \varphi\, dx \quad \text{for } S_h^{i,j} = M_h^{i,j}, N_h^{i,j}, K_h^{i,j}, L_h^{i,j}.$$

Therefore we get from (4.11) that

$$\lim_{h\to\infty} \int_\Omega \varphi\, d|E^j u_h| = (\sqrt{2}+2) \int_\Omega \varphi\, dx.$$

□

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Department of Maths and Applied Maths, University of Cape Town, Rondebosch, 7701 South Africa

E-mail address: ebobisse@maths.uct.ac.za