Separation of scales and almost-periodic effects in the asymptotic behaviour of perforated periodic media

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1 Introduction

We study the behaviour of solutions $u = u_{\delta,\varepsilon}$ of oscillating Dirichlet boundary problems involving two small parameters ε and δ, of the form

$$\begin{cases} -\mathrm{div}\, a\Big(\dfrac{x}{\varepsilon}, Du\Big) = h \\ \\ u = 0 \text{ on } \partial\Omega_\delta, \end{cases} \tag{1.1}$$

where Ω is a fixed bounded open subset of $\mathbb{R}^n$ (for simplicity we consider the case $n \geq 3$ only) and Ω_δ is the periodically *perforated domain* defined as follows. For all $x_i^\delta = \delta i$ ($i \in \mathbb{Z}^n$) let B_i^δ be the ball of center x_i^δ and radius $\delta^{n/(n-2)}$; the set Ω_δ is defined by

$$\Omega_\delta = \Omega \setminus \bigcup_{i\in\mathbb{Z}^n} B_i^\delta. \tag{1.2}$$

The operators we consider satisfy standard growth and monotonicity assumption, and, for the sake of simplicity, we take $h \in L^2(\Omega)$.

The case when a does not depend on x possesses a very interesting and simple limit description. In the simplest case when a is the identity then it is well known that the solutions u_δ of

$$\begin{cases} -\triangle u = h \\ u = 0 \text{ on } \partial\Omega_\delta, \end{cases} \tag{1.3}$$

converge as $\delta \to 0$ to the solution of the problem

$$\begin{cases} -\triangle u + Cu = h \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where the constant C is given by the capacitary formula

$$C = \mathrm{cap}(B_1) = \inf\Big\{\int_{\mathbb{R}^n} |Du|^2\, dx : u \in H^1(\mathbb{R}^n),\ u = 1 \text{ on } B_1(0)\Big\} \tag{1.5}$$

(see Marchenko Khrushlov [15], Cioranescu Murat [8], etc). A similar result applies in the case of a general a independent of x (see e.g. Casado Diaz and Garroni [6], where also systems are treated).

When a depends on x and we consider problems (1.1), or even problems of the general form

$$\begin{cases} -\mathrm{div}\, a_\varepsilon(x, Du) = h \\ u = 0 \text{ on } \partial\Omega_\delta, \end{cases} \tag{1.6}$$

(for the sake of clarity in the exposition we consider only the case of a_ε linear and symmetric) then a recent compactness result by Dal Maso and Murat [11] ensures that, for a fixed choice of $\delta = \delta(\varepsilon)$, upon possibly extracting a subsequence, the solutions u_ε converge to that of a limit problem of the form

$$\begin{cases} -\mathrm{div}\,\big(a_0(x, Du)\big) + \varphi u = h \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.7}$$

where the operator $A_0 = -\mathrm{div}\, a_0(x, Du)$ is the G-limit of the sequence of operators $-\mathrm{div}\, a_\varepsilon(x, Du)$ (see e.g. [18], [16], [2], [19], [7], [17]). The G-limit operator is well defined by a compactness argument; in particular, if a_ε is as in (1.1) then the G-limit (homogenized) operator $A_{\mathrm{hom}} = -\mathrm{div}\, a_{\mathrm{hom}}(Du)$ is independent from the subsequence and does not depend on x. The determination of the function $\varphi \in L^\infty$ is a subtler problem and involves a complex capacitary computation.

In this paper we address the problem of the effective computation of φ in (1.7) when $a_\varepsilon(x, z) = a(x/\varepsilon, z)$ in (1.6) with a 1-periodic, and we highlight various regimes, at which problems (1.1) behave differently (again, for the sake of simplicity here we describe the results in the case when the function a is linear, continuous and symmetric, and $n \geq 3$ only):

(i) (*separation of scales*) if $\varepsilon << \delta^{n/(n-2)}$ or $\varepsilon >> \delta$ then the whole family $u_{\varepsilon,\delta}$ converges to the solution u of a problem of the form

$$\begin{cases} -\mathrm{div}\,\big(a_{\mathrm{hom}}(Du)\big) + Cu = h \\ u = 0 \text{ on } \partial\Omega; \end{cases} \tag{1.8}$$

i.e., $\varphi = C$. In the case $\varepsilon << \delta^{n/(n-2)}$ the constant C is given by the *homogenized capacitary problem*

$$C = \mathrm{cap}_{\mathrm{hom}}(B_1) = \inf\Big\{ \int_{\mathbb{R}^n} \langle a_{\mathrm{hom}}(Du), Du\rangle\, dx : u \in H^1(\mathbb{R}^n),\ u = 1 \text{ on } B_1(0)\Big\}. \tag{1.9}$$

In a sense, we may first let $\varepsilon \to 0$ and then $\delta \to 0$. In the case $\varepsilon >> \delta$, conversely, we may let first act ε as a parameter. As a consequence the dependence on x/ε in (1.1) can be 'frozen' and we are led to consider the *parameterized capacitary problems*

$$\mathrm{cap}_y(B_1) = \inf\Big\{ \int_{\mathbb{R}^n} \langle a(y, Du), Du\rangle\, dx : u \in H^1(\mathbb{R}^n),\ u = 1 \text{ on } B_1(0)\Big\}. \tag{1.10}$$

The overall effect of letting $\varepsilon \to 0$ is then obtained by averaging, and we get

$$C = \int_{(0,1)^n} \mathrm{cap}_y(B_1)\, dy. \tag{1.11}$$

(ii) (*almost-periodic effects*) in the remaining cases, the two periods ε and δ present in (1.1) interact. As a consequence in general the family of solutions $u_{\varepsilon,\delta}$ *does not converge*. The problems satisfied by converging subsequences may be of the form (1.8) with C described by a single problem (*periodic behaviour*) of the form

$$\mathrm{cap}^0(B_1) = \inf\Big\{ \int_{\mathbb{R}^n} \langle b(x, Du), Du\rangle\, dx : u \in H^1(\mathbb{R}^n),\ u = 1 \text{ on } B_1(0)\Big\}, \tag{1.12}$$

with b a suitable scaled operator (in some cases independent of x), or by a formula of the type (1.11) (*almost-periodic behaviour*) with cap_y substituted by a suitable scaled and localized problem, but may even give rise to a problem of the form (1.7) with *non-constant* φ (*finely-tuned interplay between* δ *and* ε).

We deal with the variational case only, in which all the results above may be easily derived from the corresponding description of the Γ-limits (see [12], [10], [4], [3]) of the functionals $F_{\varepsilon,\delta}$ of the form

$$F_{\varepsilon,\delta}(u) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\, dx & \text{if } u = 0 \text{ on } \bigcup_{i\in\mathbb{Z}^n} B_i^\delta \\ +\infty & \text{otherwise} \end{cases} \tag{1.13}$$

for suitable f (see Remark 2.1(ii)). We show that the Γ-limits of converging subsequences of these functionals as $\varepsilon \to 0$ and $\delta \to 0$ are of the form

$$F(u) = \int_\Omega f_{\mathrm{hom}}(Du)\, dx + \int_\Omega \varphi |u|^2\, dx, \tag{1.14}$$

where f_{hom} is the *homogenized energy density of* f (see e.g. [4]) and φ is described above. Note that in the cases $\varepsilon << \delta^{n/(n-2)}$ and $\delta << \varepsilon$ then φ is constant and does not depend on the subsequence.

We propose a direct proof of all these results based on the use of a 'joining lemma on varying domains' (for a proof of this result in a general context see [1]) which allows to consider only sequences of functions which are constant on suitable annuli close to the points x_i^δ, so that a scaling argument immediately yields the estimates of the limit function φ by suitable capacities. This technique is explained in a general framework in Section 3. Note that we do not make use of integral representation techniques such as those in [9].

We only treat the case when f is positively homogeneous of degree 2 in the second variable and $n \geq 3$; the same method with minor changes applies for $n = 2$ or to p-homogeneous f and $1 < p \leq n$ (for changes in the statements see e.g [8], [6]). For the changes for general f and the case of vector-valued u see [1].

2 Setting of the problem

In all that follows Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 3$. If $E \subset \mathbb{R}^n$ is a Lebesgue-measurable set then $\mathcal{L}^n(E)$ is its Lebesgue measure. $B_\rho(x)$ is the open ball of centre x and radius ρ. We use standard notation for Lebesgue and Sobolev spaces. The letter c denotes a generic fixed strictly positive constant and ω a generic fixed modulus of continuity; i.e., a function $\omega : [0, +\infty) \to [0, +\infty)$ continuous in 0 and with $\omega(0) = 0$.

We recall the definition of Γ-*convergence* of a sequence of functionals F_j defined on $H^1_0(\Omega)$ (with respect to the $L^2(\Omega)$-convergence). We say that (F_j) Γ-converges to F_0 on $H^1_0(\Omega)$ if for all $u \in H^1_0(\Omega)$

(i) (Γ-*liminf inequality*) for all (u_j) sequences of functions in $H^1_0(\Omega)$ converging to u in $L^2(\Omega)$ we have

$$F_0(u) \leq \liminf_j F_j(u_j);$$

(ii) (Γ-*limsup inequality*) for all $\eta > 0$ there exists a sequence (u_j) of functions in $H^1_0(\Omega)$ converging to u in $L^2(\Omega)$ such that

$$F_0(u) \geq \limsup_j F_j(u_j) - \eta.$$

We will say that a family (F_ε) Γ-converges to F_0 if for all sequences (ε_j) of positive numbers converging to 0 (i) and (ii) above are satisfied with F_{ε_j} in place of F_j.

The functionals we consider are defined as follows. Let $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be a Borel function satisfying

(H1) (*periodicity*) $f(\cdot, z)$ is 1-periodic for all $z \in \mathbb{R}^n$;

(H2) (*positive homogeneity*) $f(x, \cdot)$ is positively homogeneous of degree 2 for all $x \in \mathbb{R}^n$;

(H3) (*growth conditions*) there exist two constants $c_1, c_2 > 0$ such that $c_1|z|^2 \leq f(x, z) \leq c_2|z|^2$ for all x, z.

It is well known (see e.g. [4] Chapter 14) that the Γ-limit G_0 of the functionals (G_ε) defined by

$$G_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\, dx \tag{2.1}$$

on $H^1_0(\Omega)$ exists and can be represented as

$$G_0(u) = \int_\Omega f_{\hom}(Du)\, dx, \tag{2.2}$$

where

$$f_{\hom}(z) = \inf\Big\{\int_{(0,1)^n} f(y, Du + z)\, dy : u \in H^1_{\rm loc}(\mathbb{R}^n) \text{ 1-periodic}\Big\} \tag{2.3}$$

for $z \in \mathbb{R}^n$ defines a convex function positively homogeneous of degree 2.

For all $\delta > 0$ we will consider the lattice $\delta\mathbb{Z}^n$ whose points will be denoted $x_i^\delta = \delta i$ ($i \in \mathbb{Z}^n$). Moreover, for all $i \in \mathbb{Z}^n$

$$B_i^\delta = B_{\delta^{n/(n-2)}}(x_i^\delta).$$

For all $\varepsilon, \delta > 0$ we consider $F_{\varepsilon,\delta} : H_0^1(\Omega) \to [0, +\infty]$ defined by

$$F_{\varepsilon,\delta}(u) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\,dx & \text{if } u = 0 \text{ on } \bigcup_{i\in\mathbb{Z}^n} B_i^\delta \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

With fixed $\delta = \delta(\varepsilon)$ we will study the Γ-limits of sequences (F_j) with

$$F_j = F_{\varepsilon_j,\delta(\varepsilon_j)}. \tag{2.5}$$

We will separately consider the following cases:

(1) (Section 4.1) $\varepsilon << \delta^{n/(n-2)}$. In this case the Γ-limit does not depend on (ε_j) and can be written in the form

$$F_0(u) = \int_\Omega f_{\rm hom}(Du)\,dx + C\int_\Omega |u|^2\,dx \tag{2.6}$$

on the whole $H_0^1(\Omega)$. The characterization of C is described in Theorem 4.2;

(2) (Section 4.2) $\varepsilon >> \delta$. The same conclusion of (1) above holds with a different characterization of C (see Theorem 4.3);

(3) (Section 5) In the remaining cases in general the Γ-limit does not exist, but we may have converging sequences (F_j) both to functionals of the form (2.6) with different C or to functionals of the form

$$F_0(u) = \int_\Omega f_{\rm hom}(Du)\,dx + \int_\Omega \varphi|u|^2\,dx \tag{2.7}$$

for some strictly positive $\varphi \in L^\infty(\Omega)$.

Remark 2.1 (i) Since the functionals we consider are weakly equi-coercive on $H_0^1(\Omega)$ (more precisely, if $\sup_j(F_j(u_j)) < +\infty$ and (u_j) is bounded in $L^2(\Omega)$ then it is weakly pre-compact in $H_0^1(\Omega)$) in the Γ-liminf inequality above we may consider only sequences (u_j) weakly converging in $H_0^1(\Omega)$;

(ii) if H is a continuous functional on $L^2(\Omega)$ then $F_j + H$ Γ-converge to $F_0 + H$. By the well-known property of convergence of minima of Γ-limits (see e.g. [4] Theorem 7.2) we deduce for instance in case (1) above that for all fixed $h \in L^2(\Omega)$ the values

$$m_\varepsilon = \inf\Big\{\int_{\Omega_{\delta(\varepsilon)}} f\Big(\frac{x}{\varepsilon}, Du\Big)\,dx - \int_{\Omega_{\delta(\varepsilon)}} hu\,dx : u = 0 \text{ on } \partial\Omega_{\delta(\varepsilon)}\Big\},$$

where Ω_δ denotes the δ-periodically perforated set

$$\Omega_\delta = \Omega \setminus (B_{\delta^{n/(n-2)}}(0) + \mathbb{Z}^n) = \Omega \setminus \bigcup_{i\in\mathbb{Z}^n} B_i^\delta, \tag{2.8}$$

converge to

$$m = \min\Big\{ \int_\Omega \Big(f_{\rm hom}(Du) + C|u|^2 - hu \Big)\, dx : u = 0 \text{ on } \partial\Omega \Big\}$$

as $\varepsilon \to 0$.

Furthermore, if f is convex in the second variable, for each ε a solution u_ε of m_ε exists, the family (u_ε) (extended to 0 on $\Omega \setminus \Omega_{\delta(\varepsilon)}$) is weakly precompact in $H^1_0(\Omega)$ and every its limit is a solution for m. If $f(x,z) = \langle a(x,z), z\rangle$ (a linear) we may then restate this Γ-convergence result in terms of convergence of solutions of elliptic PDE as in the Introduction.

3 A general Γ-convergence approach

In this section we describe a general procedure to compute the Γ-limit of functionals defined on perforated domains. In the following sections we specialize this approach to the cases (1)–(3) highlighted in the previous section.

Let $f_j : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be Borel functions satisfying the positive homogeneity condition (H2) and the growth conditions (H3) uniformly in j. We suppose that the sequence of functionals (G_j) defined on $H^1_0(\Omega)$ by

$$G_j(u) = \int_\Omega f_j(x, Du)\, dx \tag{3.1}$$

Γ-converges to a functional G_0 of the form

$$G_0(u) = \int_\Omega f_0(x, Du)\, dx. \tag{3.2}$$

In our case $f_j(x,z) = f(x/\varepsilon_j, z)$ and $f_0 = f_{\rm hom}$.

Let (δ_j) be a sequence of positive numbers converging to 0 and let (F_j) be defined on $H^1_0(\Omega)$ by

$$F_j(u) = \begin{cases} G_j(u) & \text{if } u = 0 \text{ on } \bigcup_{i\in\mathbb{Z}^n} B_i^\delta \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

Note that sometimes we use the notation $\delta = \delta_j$ not to overburden notation.

3.1 The Γ-liminf inequality

Let (u_j) converge weakly to u in $H^1_0(\Omega)$. We can suppose that $\sup_j F_j(u_j) < +\infty$. We wish to separate the contribution due to Du_j 'near the balls B_i^δ' and 'far from them'. The latter will be estimated simply by $G_0(u)$, while the former will be described by a limit capacitary formula.

The way to discriminate between 'near' and 'far' contribution is formalized by the following lemma, whose proof, together with a slightly more general statement can be found in [1].

Lemma 3.1 *Let u_j be a sequence weakly converging to u in $H^1_0(\Omega)$ as above, and let $N, k \in \mathbb{N}$. Let (δ_j) be a sequence of positive numbers converging to 0 and let*

$$Z_j = \{i \in \mathbb{Z}^n : \ \mathrm{dist}\,(x_i^\delta, \mathbb{R}^n \setminus \Omega) > \delta_j\}.$$

For all $i \in Z_j$ there exists $k_i \in \{0, \dots, k-1\}$ such that, having set

$$C_i^j = \Big\{x \in \Omega : \ 2^{-k_i-1} N \delta_j^{n/(n-2)} < |x - x_i^\delta| < 2^{-k_i} N \delta_j^{n/(n-2)}\Big\}, \tag{3.4}$$

$$u_j^i = |C_i^j|^{-1} \int_{C_i^j} u_j \, dx \quad \textit{(the mean value of } u_j \textit{ on } C_i^j\textit{)}, \tag{3.5}$$

and

$$\rho_j^i = \frac{3}{4} 2^{-k_i} N \delta_j^{n/(n-2)} \quad \textit{(the middle radius of } C_i^j\textit{)}, \tag{3.6}$$

there exists a sequence (w_j), with $w_j \rightharpoonup u$ in $H^1_0(\Omega)$ such that

$$w_j = u_j \ \textit{on } \Omega \setminus \bigcup_{i \in Z_j} C_i^j \tag{3.7}$$

$$w_j(x) = u_j^i \ \textit{if } |x - x_i^\delta| = \rho_j^i \tag{3.8}$$

and

$$\Big| \int_\Omega (f_j(x, Dw_j) - f_j(x, Du_j)) \, dx \Big| \le c \frac{1}{k}. \tag{3.9}$$

Moreover if $u_j = v_j$ with $|Dv_j|^2$ equi-integrable, setting

$$C_i^j = \Big\{x \in \Omega : \ \frac{1}{2} N \delta_j^{n/(n-2)} < |x - x_i^\delta| < \frac{3}{2} N \delta_j^{n/(n-2)}\Big\}, \tag{3.10}$$

$$v_j^i = |C_i^j|^{-1} \int_{C_i^j} v_j \, dx \quad \textit{(the mean value of } v_j \textit{ on } C_i^j\textit{)}, \tag{3.11}$$

and

$$\rho_j = N \delta_j^{n/(n-2)} \quad \textit{(the middle radius of } C_i^j\textit{)}, \tag{3.12}$$

we get the same conclusions above.

By this lemma we can use the sequence (w_j) to estimate the Γ-liminf inequality for (F_j). We first deal with the contribution of the part of Du_j 'external' to the annuli C_i^j; i.e., outside the set

$$E_j = \bigcup_{i\in Z_j} B_i^j, \qquad \text{where} \qquad B_i^j = B_{\rho_j^i}(x_i^\delta) \tag{3.13}$$

for all $i \in Z_j$.

Let k, N be fixed, let u_j^i be constructed as in (3.5). We define

$$\psi_j = \sum_{i\in Z_j} |u_j^i|^2 \chi_{Q_i^\delta}, \tag{3.14}$$

where

$$Q_i^\delta = x_i^\delta + \Big(-\frac{\delta_j}{2}, \frac{\delta_j}{2}\Big)^n .$$

The following lemma describes the asymptotic behaviour of ψ_j.

Lemma 3.2 *The sequence ψ_j converges to $|u|^2$ strongly in $L^1(\Omega)$.*

Proof. By the Poincaré inequality

$$\int_{Q_i^\delta} |u_j - u_j^i|^2\,dx \le c(k_i)\,\delta_j^2 \int_{Q_i^\delta} |Du_j|^2\,dx,$$

where $c(l)$ depends only on $l \in \{0,\dots,k-1\}$; since $k\in\mathbb{N}$ is fixed we get

$$\sum_{i\in Z_j}\int_{Q_i^\delta} |u_j - u_j^i|^2\,dx \le c\,\delta_j^2 \int_{\Omega} |Du_j|^2\,dx, \tag{3.15}$$

where $c := \max_{k_i=0,\dots,k-1} c(k_i)$. Since $\bigcup_{i\in Z_j} Q_i^\delta$ invades Ω and $u_j \to u$ in $L^2(\Omega)$ as $j\to+\infty$, by (3.15) we have that

$$\begin{aligned}\limsup_{j\to+\infty}\int_\Omega \psi_j\,dx &\le \limsup_{j\to+\infty} 2\Big(\sum_{i\in Z_j}\int_{Q_i^\delta}|u_j^i - u_j|^2 + \int_\Omega |u_j|^2\,dx\Big)\\ &= 2\int_\Omega |u|^2\,dx\,, \end{aligned}\tag{3.16}$$

and, by (3.16), (3.15) and Hölder's inequality

$$\begin{aligned}\limsup_{j\to+\infty}\int_\Omega |\psi_j - |u|^2|\,dx &\le c\limsup_{j\to+\infty}\Big(\sum_{i\in Z_j}\int_{Q_i^\delta}|u_j^i - u_j|^2\,dx\Big)^{1/2}\\ &\qquad\times\limsup_{j\to+\infty}\Big(\int_\Omega(\psi_j + |u_j|^2)\,dx\Big)^{1/2}\\ &\le c\Big(\int_\Omega |u|^2\,dx\Big)^{1/2}\lim_{j\to+\infty}\delta_j\Big(\int_\Omega |Du_j|^2\,dx\Big)^{1/2} = 0\end{aligned}$$

as desired. □

Proposition 3.3 *Let (u_j) be as above. Let $k, N \in \mathbb{N}$ and let (w_j) be given by Lemma 3.1. Then we have*

$$\liminf_{j\to+\infty}\int_\Omega f_j(x,Du_j)\,dx \geq \int_\Omega f_0(x,Du)\,dx + \liminf_{j\to+\infty}\int_{E_j} f_j(x,Dw_j)\,dx - \frac{c}{k}. \quad (3.17)$$

Proof. We define

$$v_j^{k,N} = \begin{cases} u_j^i & \text{on } B_i^j,\ i\in Z_j \\ w_j & \text{otherwise}\,. \end{cases}$$

The sequence $(v_j^{k,N})_j$ is bounded in $H_0^1(\Omega)$; hence, it is pre-compact in $L^2(\Omega)$. Since $\mathcal{L}^n(\{v_j^{k,N}-w_j\})\to 0$ and $w_j\to u$ in $L^2(\Omega)$ as $j\to+\infty$, $v_j^{k,N}$ converges strongly to u in $L^2(\Omega)$.

By Lemma 3.1 and condition (H2)

$$\begin{aligned} F_j(u_j)+c\frac{1}{k} &\geq F_j(w_j) = \int_{\Omega\setminus E_j} f_j(x,Dw_j)\,dx + \int_{E_j} f_j(x,Dw_j)\,dx \\ &= \int_\Omega f_j(x,Dv_j^{k,N})\,dx + \int_{E_j} f_j(x,Dw_j)\,dx \\ &= G_j(v_j^{k,N}) + \int_{E_j} f_j(x,Dw_j)\,dx\,. \end{aligned} \quad (3.18)$$

By the Γ-liminf inequality of the functionals G_j (3.1)

$$\liminf_{j\to+\infty} G_j(v_j^{k,N}) \geq \int_\Omega f_0(x,Du)\,dx \quad (3.19)$$

and (3.17) follows immediately. □

We now turn to the estimate of the contribution due to Du_j on E_j. From now on, we suppose that $N>2^k$ so that the construction of w_j in Lemma 3.1 keeps $w_j=u_j$ on B_i^δ. With fixed $j\in\mathbb{N}$ and $i\in Z_j$ such that $u_j^i\neq 0$ let $\zeta: B_N(0)\to\mathbb{R}$ be defined by

$$\zeta(y) = \begin{cases} \dfrac{1}{u_j^i}\Big(u_j^i - w_j\Big(x_i^\delta - \delta_j^{n/(n-2)}y\Big)\Big) & y\in B_{\frac{3}{4}2^{-k_i}N}(0) \\ 0 & \text{otherwise}\,. \end{cases}$$

If $u_j^i=0$ we simply set $\zeta=0$. Note that

$$\zeta\in H_0^1(B_N(0)) \qquad \text{and} \qquad \zeta=1 \text{ on } B_1(0). \quad (3.20)$$

By a change of variables we obtain

$$\int_{B_i^j} f_j(x,Dw_j)\,dx = \delta_j^n |u_j^i|^2 \int_{B_N(0)} f_j(x_i^\delta - \delta_j^{n/(n-2)}x, D\zeta)\,dx; \quad (3.21)$$

hence, if we set

$$\varphi_{N,j}(x) = \inf\Big\{\int_{B_N(0)} f_j(x - \delta_j^{n/(n-2)}y, D\zeta)\, dy :\ \zeta \in H^1_0(B_N(0)),\ \zeta = 1 \text{ on } B_1(0)\Big\} \tag{3.22}$$

the computation of the liminf on the right hand side of (3.17) is translated into computing the limit

$$\liminf_{j\to+\infty} \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \varphi_{N,j}(x_i^\delta). \tag{3.23}$$

By considering the functions ψ_j and φ_j^N defined by (3.14) and by

$$\varphi_j^N = \sum_{i\in Z_j} \varphi_{N,j}(x_i^\delta)\chi_{Q_i^\delta}, \tag{3.24}$$

respectively, the limit (3.23) is translated into

$$\liminf_{j\to+\infty} \int_\Omega \varphi_j^N \psi_j\, dx. \tag{3.25}$$

By Lemma 3.2 it is sufficient to compute the weak* limit φ^N in $L^\infty(\Omega)$ of the functions φ_j^N as $j \to +\infty$. For our problem this will be done differently in the cases (1)–(3) described in Section 2. We then have

$$\liminf_{j\to+\infty} \int_{E_j} f_j(x, Dw_j)\, dx \geq \int_\Omega \varphi^N |u|^2\, dx, \tag{3.26}$$

and a Γ-liminf inequality is achieved by taking the supremum in N.

3.2 The Γ-limsup inequality

The Γ-limsup inequality is obtained by suitably modifying a recovery sequence for the Γ-limit of G_j. Let $u \in H^1_0(\Omega)$ and let (v_j) be a sequence converging to u weakly in $H^1_0(\Omega)$ such that $\lim_j G_j(v_j) = G_0(u)$. Let

$$\Omega(\delta_j) = \{x \in \Omega :\ \mathrm{dist}\,(x, \partial\Omega) > \delta_j\};$$

we may assume that $\mathrm{spt}\, v_j \subset \Omega(\delta_j)$ (see e.g. [4] Proposition 11.7) and that $|Dv_j|^2$ is equi-integrable (see e.g. [4] Appendix D, [13]).

By Lemma 3.1, taking the equi-integrability of $|Dv_j|^2$ into account, we may also suppose that v_j equals a constant v_j^i on $\partial B_{\rho_j}(x_i^\delta)$, where

$$\rho_j = N\delta_j^{n/(n-2)}.$$

The construction of a recovery sequence will be then obtained easily if, fixed η, we construct functions ζ_j^i in $H^1_0(B_N(0))$ with $\zeta_j^i = 1$ on $B_1(0)$ such that, setting

$$u_j(x) = \begin{cases} v_j(x) & \text{on } \Omega \setminus \bigcup_{i\in Z_j} B_{\rho_j}(x_i^\delta) \\ v_j^i\Big(1 - \zeta_j^i\Big(\frac{x - x_i^\delta}{\delta_j^{n/(n-2)}}\Big)\Big) & \text{on } B_{\rho_j}(x_i^\delta), \end{cases} \tag{3.27}$$

we have

$$\limsup_j \int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f_j(x, Du_j)\, dx \le \int_\Omega \varphi |u|^2\, dx + \eta,$$

where $\varphi = \sup_N \varphi^N$ is suggested by the liminf inequality. Indeed, with this choice of (u_j), we obtain

$$\begin{aligned}
\limsup_{j\to+\infty} \int_\Omega f_j(x, Du_j)\, dx &\le \int_\Omega f_0(x, Du)\, dx + \limsup_{j\to+\infty} \int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f_j(x, Du_j)\, dx \\
&\le \int_\Omega f_0(x, Du)\, dx + \int_\Omega \varphi |u|^2\, dx + \eta\,, \qquad (3.28)
\end{aligned}$$

and the Γ-limsup inequality is verified.

4 Separation of Scales

In this section we study the extreme cases $\varepsilon << \delta^{n/(n-2)}$ and $\varepsilon >> \delta$. In both cases the Γ-limit of the whole family $(F_{\varepsilon,\delta})$ exists and it is described by an extra term of the form $C \int_\Omega |u|^2\, dx$, whose computation highlights a separation of scales effect.

4.1 Highly-oscillating energies in perforated domains

We treat the case $\varepsilon << \delta^{n/(n-2)}$ first. In this case the limit is computed as if by first letting $\varepsilon \to 0$, thus obtaining a homogenized functional, and then applying the theory of perforated domains for a fixed functional.

Remark 4.1 We define

$$\mathrm{cap}_{\mathrm{hom}}(B_1) = \inf\Big\{\int_{\mathbb{R}^n} f_{\mathrm{hom}}(D\zeta)\, dz \ : \ \zeta \in H^1(\mathbb{R}^n), \quad \zeta = 1 \text{ on } B_1(0)\Big\}.$$

It can be easily checked that

$$\begin{aligned}
\mathrm{cap}_{\mathrm{hom}}(B_1) &= \lim_{N\to+\infty} \min\Big\{\int_{B_{N+\frac{1}{N}}(0)} f_{\mathrm{hom}}(D\zeta)\, dz \ : \ \zeta \in H^1(B_{N+\frac{1}{N}}(0)) \\
&\qquad \zeta = 1 \text{ on } \partial B_{N+\frac{1}{N}}(0) \quad \zeta = 0 \text{ on } B_{1-\frac{1}{N}}(0)\Big\} \\
&= \lim_{N\to+\infty} \min\Big\{\int_{B_{N-\frac{1}{N}}(0)} f_{\mathrm{hom}}(D\zeta)\, dz \ : \ \zeta \in H^1(B_{N-\frac{1}{N}}(0)) \\
&\qquad \zeta = 1 \text{ on } \partial B_{N-\frac{1}{N}}(0) \quad \zeta = 0 \text{ on } B_{1+\frac{1}{N}}(0)\Big\}.
\end{aligned}$$

Theorem 4.2 *Let f satisfy* (H1)–(H3) *and let $F_{\varepsilon,\delta}$ be given by* (2.4). *Let $\delta : (0,+\infty) \to (0,+\infty)$ be such that*

$$\lim_{\varepsilon\to 0}\delta(\varepsilon)=0 \qquad \lim_{\varepsilon\to 0}\frac{\delta^{n/n-2}(\varepsilon)}{\varepsilon}=+\infty;$$

then there exists the Γ-limit

$$\Gamma\text{-}\lim_{\varepsilon\to 0}F_{\varepsilon,\delta(\varepsilon)}(u)=\int_\Omega f_{\mathrm{hom}}(Du)\,dx+\mathrm{cap}_{\mathrm{hom}}(B_1)\int_\Omega |u|^2\,dx$$

for all $u\in H^1_0(\Omega)$.

Proof. We fix a sequence (ε_j) of positive numbers converging to 0 and let $\delta_j=\delta(\varepsilon_j)$. Let $F_j=F_{\varepsilon_j,\delta_j}$. Note that we sometime simply write δ in place of δ_j.

We first deal with the Γ-liminf inequality. Let u_j be weakly converging to u in $H^1_0(\Omega)$, such that $\sup_j F_j(u_j)<\infty$. Let $k\in\mathbb{N}$ and $N>2^k$, and let w_j be as in Lemma 3.1; by Proposition 3.3 to compute the Γ-liminf inequality we have to study the contribution on the set E_j given by (3.13).

For all $i\in\mathbb{Z}^n$ let $y_i^\varepsilon=\varepsilon_j[\frac{x_i^\delta}{\varepsilon_j}]$, so that $x_i^\delta\in y_i^\varepsilon+[0,\varepsilon_j)^n$. Taking into account that $\varepsilon_j << \delta_j^{n/(n-2)}$, we deduce the following inclusions

$$B_{(1-\frac{1}{N})\delta_j^{\frac{n}{n-2}}}(y_i^\varepsilon)\subset B_i^\delta \tag{4.1}$$

and

$$B_i^j\subset B_{\rho_j}(x_i^\delta)\subset B_{(N+\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \tag{4.2}$$

for j large enough. There follows that w_j can be extended outside B_i^j as

$$w_{j,i}=\begin{cases} w_j & \text{on } B_i^j\\ u_j^i & \text{on } B_{(N+\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon)\setminus B_i^j\,. \end{cases} \tag{4.3}$$

Let $u_j^i\neq 0$. By (4.3) and conditions (H1) and (H2), by a change of variables, we get

$$\begin{aligned}\int_{B_i^j} f\Big(\frac{x}{\varepsilon_j},Dw_j\Big)\,dx &= \int_{B_{(N+\frac{1}{N})\delta_j^{n/n-2}(y_i^\varepsilon)}} f\Big(\frac{x}{\varepsilon_j},Dw_{j,i}\Big)\,dx\\ &= \delta_j^n|u_j^i|^2\int_{B_{(N+\frac{1}{N})}(0)} f\Big(z\frac{\delta_j^{n/n-2}}{\varepsilon_j},D\zeta_j^i\Big)\,dz, \end{aligned}\tag{4.4}$$

where

$$\zeta_j^i(z)=w_{j,i}(z\delta_j^{n/n-2}+y_i^\varepsilon)/u_j^i\,.$$

Note that by (4.1) and (4.2) $\zeta_j^i(z)=1$ on $\partial B_{(N+\frac{1}{N})}(0)$ and $\zeta_j^i=0$ on $B_1(0)$.

If we denote $\eta_j = \varepsilon_j/\delta_j^{n/n-2}$, by (4.4) we have

$$\begin{aligned}
&\int_{\bigcup_i B_i^j} f\Big(\frac{x}{\varepsilon_j}, Dw_j\Big)\,dx \\
\geq \quad & \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \min\Big\{\int_{B_{(N+\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta(z)\Big)\,dz \;:\; \zeta\in H^1(B_{N+\frac{1}{N}}(0)) \\
& \qquad\qquad \zeta = 1 \text{ on } \partial B_{(N+\frac{1}{N})}(0)\ \zeta = 0 \text{ on } B_{(1-\frac{1}{N})}(0)\Big\}; \qquad (4.5)
\end{aligned}$$

hence, by (4.5), Lemma 3.2 and the Γ-convergence of the functionals (2.1) to that in (2.2), we have

$$\begin{aligned}
&\liminf_{j\to+\infty} \int_{\bigcup_i B_i^j} f\Big(\frac{x}{\varepsilon_j}, Dw_j\Big)\,dx \\
\geq \quad & \min\Big\{\int_{B_{(N+\frac{1}{N})}(0)} f_{\rm hom}(D\zeta(z))\,dz \;:\; \zeta\in H^1(B_{N+\frac{1}{N}}(0)) \\
& \qquad\qquad \zeta = 1 \text{ on } \partial B_{(N+\frac{1}{N})}(0)\ \zeta = 0 \text{ on } B_{(1-\frac{1}{N})}(0)\Big\} \int_\Omega |u|^2\,dx\,.
\end{aligned}$$

Passing to the limit in the inequality given by Proposition 3.3 first as N and then as k tend to $+\infty$, by Remark 4.1 we have that

$$\liminf_{j\to+\infty} F_j(u_j) \geq \int_\Omega f_{\rm hom}(Du)\,dx + {\rm cap}_{\rm hom}(B_1)\int_\Omega |u|^2\,dx$$

as desired. By the arbitrariness of u_j the Γ-liminf inequality is proved.

Now we pass to compute the Γ-limsup inequality. Given $u\in H^1_0(\Omega)$ we want to construct a recovery sequence (u_j) for the Γ-limit of F_j. Following the approach of Section 3.2, it remains to define u_j on $B_{\rho_j}(x_i^\delta)$.

We denote

$$\begin{aligned}
m_\eta^N \quad = \quad & \min\Big\{\int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta}, D\zeta(z)\Big)\,dz \;:\; \zeta\in H^1(B_{N-\frac{1}{N}}(0)) \\
& \qquad\qquad \zeta = 1 \text{ on } \partial B_{(N-\frac{1}{N})}(0)\ \zeta = 0 \text{ on } B_{(1+\frac{1}{N})}(0)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
m^N \quad = \quad & \min\Big\{\int_{B_{(N-\frac{1}{N})}(0)} f_{\rm hom}(D\zeta(z))\,dz \;:\; \zeta\in H^1(B_{N-\frac{1}{N}}(0)) \\
& \qquad\qquad \zeta = 1 \text{ on } \partial B_{(N-\frac{1}{N})}(0)\ \zeta = 0 \text{ on } B_{(1+\frac{1}{N})}(0)\Big\},
\end{aligned}$$

and fix $M\in\mathbb{N}$; by Remark 4.1 and for N large enough

$$m^N \leq {\rm cap}_{\rm hom}(B_1) + \frac{1}{M}\,.$$

By the Γ-convergence of the functionals (2.1) to that in (2.2), we have that m_η^N converges to m^N as η tends to 0 (see [4] Theorem 7.2). Considering $\eta_j = \varepsilon_j/\delta_j^{n/n-2}$, from the convergence of minima we deduce that there exists a sequence $\zeta_j \in H^1(B_{N-\frac{1}{N}}(0))$ with $\zeta_j = 1$ on $\partial B_{(N-\frac{1}{N})}(0)$ and $\zeta_j = 0$ on $B_{(1+\frac{1}{N})}(0)$ such that

$$\lim_{j\to+\infty} \int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta_j(z)\Big)\, dz \le \mathrm{cap}_{\mathrm{hom}}(B_1) + \frac{1}{M}\,. \tag{4.6}$$

By a change of variables we get

$$\int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta_j(z)\Big)\, dz = \frac{1}{\delta_j^n} \int_{B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon)} f\Big(\frac{x}{\varepsilon_j}, D\tilde\zeta_j^i(x)\Big)\, dx, \tag{4.7}$$

where

$$\tilde\zeta_j^i(x) = \zeta_j\Big(\frac{x-y_i^\varepsilon}{\delta_j^{n/n-2}}\Big)\,.$$

Reasoning as for the Γ-liminf inequality we may suppose that

$$B_i^\delta \subset B_{(1+\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \qquad \text{and} \qquad B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \subset B_{\rho_j}(x_i^\delta)\,. \tag{4.8}$$

Since $\tilde\zeta_j^i(x) = 1$ on $\partial B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon)$ and $\tilde\zeta_j^i(x) = 0$ on $B_{(1+\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon)$, by (4.8) we can define

$$\zeta_j^i(x) = \begin{cases} \tilde\zeta_j^i & \text{on } B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \\ 1 & \text{on } B_{\rho_j}(x_i^\delta) \setminus B_{(N-\frac{1}{N})\delta_j^{n/n-2}}(y_i^\varepsilon) \end{cases}$$

so that $\zeta_j^i = 1$ on $\partial B_{\rho_j}(x_i^\delta)$ and $\zeta_j^i = 0$ on B_i^δ. By (4.7) and condition (H2), we get

$$\int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta_j(z)\Big)\, dz = \frac{1}{\delta_j^n} \int_{B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, D\zeta_j^i(x)\Big)\, dx\,. \tag{4.9}$$

Now we can construct the recovery sequence u_j by setting

$$u_j = \begin{cases} v_j & \text{on } \Omega \setminus \bigcup_i B_{\rho_j}(x_i^\delta) \\ v_j^i\, \zeta_j^i(x) & \text{on } B_{\rho_j}(x_i^\delta), \end{cases} \tag{4.10}$$

and prove that it converges weakly to u in $H^1(\Omega)$. In fact (u_j) is bounded in $H^1(\Omega)$ and $v_j - u_j$ tends to 0 in measure. Since $v_j \to u$ in $L^2(\Omega)$, then also $u_j \to u$ in $L^2(\Omega)$ and hence weakly in $H^1(\Omega)$.

By (3.28), (4.10), (4.9), Lemma 3.2 and (4.6) we have

$$\begin{aligned}\limsup_{j\to+\infty} F_j(u_j) &\le \int_\Omega f_{\rm hom}(Du)\,dx \\ &\quad + \limsup_{j\to+\infty} \sum_{i\in Z_j} \delta_j^n |v_j^i|^2 \int_{B_{(N-\frac{1}{N})}(0)} f\Big(\frac{z}{\eta_j}, D\zeta_j(z)\Big)\,dz \\ &\le \int_\Omega f_{\rm hom}(Du)\,dx + \Big(\mathrm{cap}_{\rm hom}(B_1) + \frac{1}{M}\Big)\int_\Omega |u|^2\,dx\,.\end{aligned}$$

By the arbitrariness of M we conclude the Γ-limsup inequality; hence, the Γ-convergence of the functionals $F_{\varepsilon,\delta(\varepsilon)}$ as $\varepsilon\to 0$. □

4.2 Slowly-oscillating energies in perforated domains

Now we treat the case $\varepsilon >> \delta$. In this case the limit is computed as if first applying the limit process to functionals in which x/ε acts as a parameter, and then averaging the outcome.

We consider for the sake of simplicity the case of continuous f:

(H4) (*continuity*) $f(\cdot, z)$ is continuous for all $z\in\mathbb{R}^n$.

This condition can be easily dropped, at the expense of a much heavier notation.

Theorem 4.3 *Let f satisfy* (H1)–(H4) *and let $F_{\varepsilon,\delta}$ be given by* (2.4). *Let $\delta : (0,+\infty)\to(0,+\infty)$ be such that*

$$\lim_{\varepsilon\to 0}\frac{\delta(\varepsilon)}{\varepsilon} = 0\,.$$

There exists the Γ-limit

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)}(u) = \int_\Omega f_{\rm hom}(Du)\,dx + \int_{(0,1)^n} a(x)\,dx \int_\Omega |u|^2\,dx$$

for all $u\in H^1_0(\Omega)$, where

$$a(x) = \inf\Big\{\int_{\mathbb{R}^n} f(x, D\zeta)\,dy : \ \zeta\in H^1(\mathbb{R}^n),\ \zeta = 1 \ \textit{on}\ B_1(0)\Big\}\,. \tag{4.11}$$

Before proving Theorem 4.3 we make some general observations from which the Γ-limsup inequality will easily follow, and that will be used also in the next sections.

Remark 4.4 In this Section and in the next one, we will consider several cases for which the Γ-limsup inequality will be obtained by considering the recovery sequence (3.27) introduced in Section 3.2, but the functions ζ_j^i will be constructed in a different way with respect to the previous section. In this case the function $\varphi_{N,j}$ defined as in (3.22) takes the form

$$\begin{aligned}\varphi_{N,j}(x) &= \inf\Big\{\int_{B_N(0)} f\Big(x-\frac{\delta_j^{n/(n-2)}}{\varepsilon_j}y, D\zeta\Big)\,dy : \\ &\qquad\qquad \zeta\in H_0^1(B_N(0)),\ \zeta=1 \text{ on } B_1(0)\Big\}\,. \qquad (4.12)\end{aligned}$$

With fixed $j\in\mathbb{N}$ and $i\in Z_j$ we take ζ_j^i in $H_0^1(B_N(0))$ with $\zeta_j^i=1$ on $B_1(0)$ such that

$$\int_{B_N(0)} f\Big(\frac{x_i^\delta}{\varepsilon_j}-\frac{\delta_j^{n/(n-2)}}{\varepsilon_j}y, D\zeta_j^i\Big)\,dy \le \varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)+\frac{1}{j}\,. \qquad (4.13)$$

By a change of variables we obtain

$$\frac{1}{\delta_j^n}\int_{B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, D\zeta_j^i\Big(\frac{x-x_i^\delta}{\delta_j^{n/(n-2)}}\Big)\Big)\,dx \le \varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)+\frac{1}{j}$$

and

$$\int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, D\Big(v_j^i\Big(1-\zeta_j^i\Big(\frac{x-x_i^\delta}{\delta_j^{n/(n-2)}}\Big)\Big)\Big)\Big)\,dx \le \sum_{i\in Z_j}\delta_j^n|v_j^i|^2\varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)+\frac{1}{j}\,.$$

Hence, if we define

$$\varphi_j^N=\sum_{i\in Z_j}\varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)\chi_{Q_i^\delta}, \qquad (4.14)$$

where $\varphi_{N,j}$ is given by (4.12), and

$$\psi_j=\sum_{i\in Z_j}v_j^i\chi_{Q_i^\delta} \qquad (4.15)$$

with v_j^i given by (3.11), we have

$$\begin{aligned}&\limsup_{j\to+\infty}\int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, Du_j\Big)\,dx \\ =\ &\limsup_{j\to+\infty}\int_{\bigcup_i B_{\rho_j}(x_i^\delta)} f\Big(\frac{x}{\varepsilon_j}, D\Big(v_j^i\Big(1-\zeta_j^i\Big(\frac{x-x_i^\delta}{\delta_j^{n/(n-2)}}\Big)\Big)\Big)\Big)\,dx \\ \le\ &\limsup_{j\to+\infty}\sum_{i\in Z_j}\delta_j^n|v_j^i|^2\varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)=\limsup_{j\to+\infty}\int_\Omega \psi_j\varphi_j^N\,dx\,. \qquad (4.16)\end{aligned}$$

Proof of Theorem 4.3. We fix a sequence (ε_j) of positive numbers converging to 0 and let $\delta_j=\delta(\varepsilon_j)$. We have already shown in Section 3.2 that to get the Γ-liminf inequality we have to study (in the notation of that section) the weak* convergence in $L^\infty(\Omega)$ of the functions φ_j^N to φ^N, as $j\to+\infty$. In our case φ_j^N is given by (4.14).

If we define

$$a_N(x)=\inf\Big\{\int_{B_N(0)} f(x,D\zeta)\,dy:\ \zeta\in H_0^1(B_N(0)),\ \zeta=1\text{ on }B_1(0)\Big\} \tag{4.17}$$

by hypothesis (H4), we have that

$$\|\varphi_{N,j}-a_N\|_\infty\le\omega\Big(N\frac{\delta_j^{n/(n-2)}}{\varepsilon_j}\Big) \tag{4.18}$$

and

$$\Big|a_N\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)-a_N\Big(\frac{y}{\varepsilon_j}\Big)\Big|\le\omega\Big(\frac{\delta_j}{\varepsilon_j}\Big) \tag{4.19}$$

for all $y\in Q_i^\delta$. Hence, if we define

$$a_j^N=\sum_{i\in Z_j}a_N\Big(\frac{x_i^\delta}{\varepsilon_j}\Big)\chi_{Q_i^\delta}\qquad\text{and}\qquad\bar a_N=\int_{(0,1)^n}a_N(y)\,dy$$

since a_N is 1-periodic, we have $a_N(\frac{\cdot}{\varepsilon_j})\rightharpoonup^*\bar a_N$ in L^∞ and by (4.19) also $a_j^N\rightharpoonup^*\bar a_N$ as $j\to+\infty$. By (4.18) $\varphi_j^N\rightharpoonup^*\varphi^N=\bar a_N$ and hence

$$\lim_{N\to+\infty}\varphi^N=\int_{(0,1)^n}a(x)\,dx\,. \tag{4.20}$$

By Proposition 3.3, (3.26), Lemma 3.2 and (4.20) we get the Γ-liminf inequality.

The Γ-limsup inequality is obtained by considering the recovery sequence (3.27) with ζ_j^i constructed by (4.13), and recalling (4.16) and Lemma 3.2. □

5 Interaction between homogenization processes

In this section we treat the remaining cases when ε is between the scales $\delta^{n/n-2}$ and δ. We will suppose that (δ_j) and (ε_j) are such that

$$\lim_{j\to\infty}\frac{\delta_j^{n/(n-2)}}{\varepsilon_j}=q\in[0,+\infty)\qquad\lim_{j\to\infty}\frac{\varepsilon_j}{\delta_j}<+\infty \tag{5.1}$$

hold. We define the *localized capacitary formula*

$$a^q(x)=\inf\Big\{\int_{\mathbb{R}^n}f(x-qy,D\zeta)\,dy:\ \zeta\in H^1(\mathbb{R}^n),\ \zeta=1\text{ on }B_1(0)\Big\}\,. \tag{5.2}$$

Note that when $q=0$, a^0 coincides with the function a defined in (4.11).

Theorem 5.1 (Periodic interaction of scales) *Let f satisfy* (H1)–(H4) *and let $F_j = F_{\varepsilon_j,\delta_j}$ with $F_{\varepsilon,\delta}$ as in* (2.4)*. Let $\varepsilon_j \to 0$ and let $\delta_j \to 0$ be such that* (5.1) *holds. Suppose that $\delta_j = \frac{k_j}{M}\varepsilon_j$ with $k_j \in \mathbb{N}$ prime with $M \in \mathbb{N}$. Then there exists the Γ-limit*

$$\Gamma\text{-}\lim_{j\to+\infty} F_j(u) = \int_\Omega f_{\rm hom}(Du)\,dx + C\int_\Omega |u|^2\,dx,$$

on $H^1_0(\Omega)$, where

$$C = \frac{1}{M^n}\sum_{h\in\{0,\dots,M-1\}^n} a^q\Big(\frac{h}{M}\Big). \tag{5.3}$$

Proof. Let $\varphi_{N,j}$ be the 1-periodic function defined as in (4.12), and let

$$a^q_N(x) = \inf\Big\{\int_{B_N(0)} f(x-qy, D\zeta)\,dy:\ \zeta\in H^1_0(B_N(0)),\ \zeta = 1 \text{ on } B_1(0)\Big\}. \tag{5.4}$$

As $\delta_j = \frac{k_j}{M}\varepsilon_j$ then

$$\frac{x_i^\delta}{\varepsilon_j} = i\frac{k_j}{M} = k + \frac{h}{M} \qquad k\in\mathbb{Z}^n,\quad h\in\{0,...,M-1\}^n. \tag{5.5}$$

By (5.5) and the periodicity of $\varphi_{N,j}$

$$\sum_{i\in Z_j}\delta_j^n|u_j^i|^2\varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) = \sum_{h\in\{0,\dots,M-1\}^n}\Big(\sum_{i\in I_h}\delta_j^n|u_j^i|^2\Big)\varphi_{N,j}\Big(\frac{h}{M}\Big) = \int_\Omega \psi_j\varphi_j^N\,dx, \tag{5.6}$$

where

$$I_h = \frac{h}{M} + \mathbb{Z}^n \cap Z_j$$

and ψ_j, φ_j^N are defined in (3.14) (4.14), respectively. Note that

$$\varphi_j^N(x) = \sum_{h\in\{0,\dots,M-1\}^n}\sum_{i\in I_h}\varphi_{N,j}\Big(\frac{h}{M}\Big)\chi_{Q_i^\delta}(x)$$

and $\|\varphi_{N,j} - a_N^q\|_\infty \to 0$ as $j\to+\infty$; hence,

$$\varphi_j^N \rightharpoonup^* \varphi^N = \sum_{h\in\{0,\dots,M-1\}^n}\frac{1}{M^n}a_N^q\Big(\frac{h}{M}\Big)$$

and

$$\lim_{N\to+\infty}\varphi^N = \sum_{h\in\{0,\dots,M-1\}^n}\frac{1}{M^n}a^q\Big(\frac{h}{M}\Big).$$

Recalling Lemma 3.2 we obtain the Γ-liminf inequality.

In order to obtain the Γ-limsup inequality, by (4.16) it is sufficient to use the scheme of Section 3.2 with ζ_j^i as in (4.13). □

Remark 5.2 In the particular case when $\delta_j/\varepsilon_j \in \mathbb{N}$ (i.e., $M = 1$) the constant C is given by the single problem defining $a^q(0)$.

Theorem 5.3 (Almost-periodic interaction of scales) *Let f satisfy* (H1)–(H4) *and let $F_j = F_{\varepsilon_j,\delta_j}$ with $F_{\varepsilon,\delta}$ as in* (2.4)*. Let $\varepsilon_j \to 0$ and let $\delta_j \to 0$ be such that* (5.1) *holds. Suppose that $\delta_j = (k_j + r)\varepsilon_j$ with $k_j \in \mathbb{N}$ and $r \notin \mathbb{Q}$. Then there exists the Γ-limit*

$$\Gamma\text{-}\lim_{j\to+\infty} F_j(u_j) = \int_\Omega f_{\rm hom}(Du)\,dx + C\int_\Omega |u|^2\,dx$$

on $H^1_0(\Omega)$, where

$$C = \int_{(0,1)^n} a^q(x)\,dx\,. \tag{5.7}$$

Proof. The sequence φ^N_j defined in (4.14) is bounded in $L^\infty(\Omega)$; hence, up to subsequences, there exists $\varphi^N \in L^\infty(\Omega)$ such that $\varphi^N_j \rightharpoonup^* \varphi^N$ in $L^\infty(\Omega)$ as $j \to +\infty$. In order to identify the limit φ^N, it suffices to test it against characteristic functions of n-cubes. Hence, if we prove that

$$\int_A \varphi^N_j\,dx \to \mathcal{L}^n(A)C \tag{5.8}$$

for every n-cube A, we have $\varphi^N = C$.

We define

$$\tilde\varphi^N_j = \sum_{i\in Z_j} \delta^n_j \frac{M^n}{\varepsilon^n_j}\varphi_{N,j}(\frac{x^\delta_i}{\varepsilon_j})\chi_{Q_{\frac{\varepsilon}{M}}(x^\delta_i)+\mathbb{Z}^n},$$

where

$$Q_{\frac{\varepsilon}{M}}(x^\delta_i) = x^\delta_i + \Big(-\frac{\varepsilon_j}{M}, \frac{\varepsilon_j}{M}\Big)^n.$$

Note that also $\tilde\varphi^N_j \rightharpoonup^* \varphi^N$ in $L^\infty(\Omega)$. By the continuity of $\varphi_{N,j}$

$$\Big|\varphi_{N,j}\Big(\frac{x^\delta_i}{\varepsilon_j}\Big) - \varphi_{N,j}\Big(\frac{x}{\varepsilon_j}\Big)\Big| \le \omega\Big(\frac{1}{M}\Big)$$

if $x \in Q_{\frac{\varepsilon}{M}}(x^\delta_i) + \mathbb{Z}^n$; hence, we study the weak* convergence of

$$x \mapsto \sum_{i\in Z_j} \delta^n_j \frac{M^n}{\varepsilon^n_j}\varphi_{N,j}\Big(\frac{x}{\varepsilon_j}\Big)\chi_{Q_{\frac{\varepsilon}{M}}(x^\delta_i)+\mathbb{Z}^n}(x)\,.$$

Let A be an n-cube with edges parallel to the coordinate axes and of side length l, we compute

$$\int_A \sum_{i\in Z_j} \delta^n_j \frac{M^n}{\varepsilon^n_j}\varphi_{N,j}\Big(\frac{x}{\varepsilon_j}\Big)\chi_{Q_{\frac{1}{M}}(\frac{x^\delta_i}{\varepsilon_j})+\mathbb{Z}^n}\Big(\frac{x}{\varepsilon_j}\Big)\,dx$$

$$
\begin{aligned}
&= \varepsilon_j^n \int_{\frac{1}{\varepsilon_j}A} \sum_{i\in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n} \varphi_{N,j}(z) \chi_{Q_{\frac{1}{M}}(\frac{x}{\varepsilon_j})+\mathbb{Z}^n}(z)\, dz \\
&= [(l/\varepsilon_j)-1]^n \varepsilon_j^n \int_{(0,1)^n} \varphi_{N,j}(z) \sum_{i\in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n} \chi_{Q_{\frac{1}{M}}(\frac{x_i^\delta}{\varepsilon_j})+\mathbb{Z}^n}(z)\, dz \\
&\quad + \varepsilon_j^n \int_{R_j} \varphi_{N,j}(z) \sum_{i\in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n} \chi_{Q_{\frac{1}{M}}(\frac{x_i^\delta}{\varepsilon_j})+\mathbb{Z}^n}(z)\, dz,
\end{aligned} \tag{5.9}
$$

where we have decomposed $(1/\varepsilon_j)A$ as the union of $[(l/\varepsilon_j)-1]^n$ unit cubes and of a set R_j, with $\mathcal{L}^n(R_j) \le 2n(l/\varepsilon_j)^{n-1}$.

By an application of Birkhoff's Theorem (see e.g. [14]) as in [5] Appendix A and (5.9) we deduce

$$
\begin{aligned}
& \lim_{j\to+\infty} \int_A \sum_{i\in Z_j} \delta_j^n \frac{M^n}{\varepsilon_j^n} \varphi_{N,j}\Big(\frac{x}{\varepsilon_j}\Big) \chi_{Q_{\frac{\varepsilon_j}{M}}(x_i^\delta)+\mathbb{Z}^n}(x)\, dx \\
= \; & \mathcal{L}^n(A) \int_{(0,1)^n} a_N^q(z)\, dz \int_{(0,1)^n} \chi_{Q_{\frac{1}{M}}}(z) M^n\, dz = \mathcal{L}^n(A) \int_{(0,1)^n} a_N^q(z)\, dz,
\end{aligned}
$$

where a_N^q is defined by (5.4). By (5.8) we have

$$
\varphi^N = \int_{(0,1)^n} a_N^q(z)\, dz;
$$

hence, by Lemma 3.2

$$
\lim_j \int_\Omega \varphi_j^N \psi_j\, dx = \varphi^N \int_\Omega |u|^2\, dx, \tag{5.10}
$$

where ψ_j is defined as in (3.14), and

$$
\lim_{N\to+\infty} \varphi^N = \int_{(0,1)^n} a^q(x)\, dx\,. \tag{5.11}
$$

By (3.26)

$$
\liminf_{j\to+\infty} \int_{E_j} f\Big(\frac{x}{\varepsilon_j}, Dw_j\Big)\, dx \ge \int_{(0,1)^n} a^q(x)\, dx \int_\Omega |u|^2\, dx \tag{5.12}
$$

and we obtain the Γ-liminf inequality.

Recalling (4.16), we choose ζ_j^i as in (4.13), and by (5.10), (5.11) we get the Γ-limsup inequality. □

Corollary 5.4 (Non-existence) *If $\delta : (0,+\infty) \to (0,+\infty)$ is a continuous function such that*

$$
\lim_{\varepsilon\to 0} \delta(\varepsilon) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)} = 0, \qquad \textit{and} \qquad \lim_{\varepsilon\to 0} \frac{\delta^{n/(n-2)}(\varepsilon)}{\varepsilon} = q \in [0,+\infty),
$$

then the Γ-limit of the functionals $F_{\varepsilon,\delta(\varepsilon)}$ as $\varepsilon \to 0$ does not exist.

Remark 5.5 The case $\delta_j = \varepsilon_j$ (more generally, $\delta_j = s\varepsilon_j$ with fixed $s > 0$) is covered by Theorem 5.1 and Theorem 5.3. Note that the condition $\lim_{j\to+\infty} \delta_j/\varepsilon_j = 1$ does not allow to conclude the existence of the Γ-limit of F_j as shown by Example 5.6 below.

Example 5.6 *(Finely-tuned interplay between scales)* We finally give an example when the extra term in the limit is not described by a constant: if $\delta_j = \varepsilon_j + \varepsilon_j^2$ then

$$\Gamma\text{-}\lim_{j\to+\infty} F_j(u) = \int_\Omega f_{\rm hom}(Du)\,dx + \int_\Omega a(x)|u(x)|^2\,dx\,.$$

In fact, by the periodicity of $\varphi_{N,j}$ defined as in (4.12)

$$\sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) = \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \varphi_{N,j}(i\varepsilon_j)$$

If we consider the function a_N defined by (4.17), by condition (H4)

$$\Big|a_N\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) - a_N(x_i^\delta)\Big| \le \omega(\varepsilon_j^2); \tag{5.13}$$

hence, by (4.18) and (5.13), we have that

$$\begin{aligned}
\lim_{j\to+\infty} \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 \varphi_{N,j}\Big(\frac{x_i^\delta}{\varepsilon_j}\Big) &= \lim_{j\to+\infty} \sum_{i\in Z_j} \delta_j^n |u_j^i|^2 a_N(x_i^\delta) \\
&= \int_\Omega a_N(x)|u(x)|^2\,dx
\end{aligned} \tag{5.14}$$

and

$$\lim_{N\to+\infty} \int_\Omega a_N(x)|u(x)|^2\,dx = \int_\Omega a(x)|u(x)|^2\,dx\,. \tag{5.15}$$

Reasoning as in the proof of Theorems 5.1 and 5.3 we get the Γ-liminf and the Γ-limsup inequalities.

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