The regularity of optimal irrigation patterns

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Abstract

A branched structure is observable in draining and irrigation systems, in electric power supply systems and in natural objects like the blood vessels, the river basins or the trees. Recent approaches of these networks derive their branched structure from an energy functional whose essential feature is to favor wide routes. Given a flow s in a river, a road, a tube or a wire, the transportation cost per unit length is supposed in these models to be proportional to s^α with $0 < \alpha < 1$.

The aim of this paper is to prove the regularity of paths (rivers, branches,...) when the irrigated measure is the Lebesgue density on a smooth open set and the irrigating measure is a single source. In that case we prove that all branches of optimal irrigation trees satisfy an elliptic equation and that their curvature is a bounded measure. In consequence all branching points in the network have a tangent cone made of a finite number of segments, and all other points have a tangent. An explicit counterexample disproves these regularity properties for non-Lebesgue irrigated measures.

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1 Introduction

Humans and Nature have designed many supply-demand distribution networks permitting to transport goods from an initial distribution (the supply) to another (the demand). This is obviously the case with distribution networks such as communication networks [16], electric power supply, water distribution, drainage networks [20], or gas pipelines [6]. Many observable natural flow networks connect a finite size volume to a source or an outlet. Think of trees, river basins, bronchial systems [23] or cardiovascular systems.

Probably the first mathematical transportation model was proposed by Monge, and formalized by Kantorovitch ([24], [19]). The problem was to move a pile of sand from a place to another with the least possible work. In the Monge-Kantorovitch framework, μ^+ and μ^- are measures on $\mathbb{R}^N$ which model respectively the supply and demand mass distributions. The solution to the problem is a measure π on $\mathbb{R}^N \times \mathbb{R}^N$ where $\pi(A \times B)$ represents the amount of mass going from A to B, and the marginal laws of π are μ^+ and μ^-. The measure π is called a *transport plan*. To evaluate the efficiency of a transport plan, a cost function $c : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is considered where $c(x, y)$ is the cost of transporting a unit mass from x to y. The cost associated with a transference plan is $\int_{\mathbb{R}^N \times \mathbb{R}^N} c(x, y) d\pi(x, y)$. The minimization of this functional is the Monge-Kantorovitch problem, which has yielded a rich mathematical harvest. See the recent monographs [17], [18], [11].

1.1 The Gilbert-Steiner problem modeling a discrete branched structure

In the Monge-Kantorovich framework, the cost of the structure achieving the transport is not modeled. Indeed, with this formulation, the cost behaves as if every single particle of sand went straight from its starting to its ending point. In the case of real supply-demand distribution problems, achieving this kind of single particle transport would be very costly. In most transportation networks, the aggregation of particles on common routes is preferable to individual straight ones. The local structure of human-designed distribution systems doesn't look as a set of straight wires but rather like a tree.

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The first model taking into account subadditive capacities for routes was proposed by Gilbert [16] in the case of communication networks. This author models the network as a graph such that each edge e is associated with a capacity c_e. Let $f(c)$ denote the cost per unit length of an edge with capacity c. It is assumed that $f(c)$ is subadditive and increasing, i.e., $f(a) + f(b) \geq f(a+b) \geq \max(f(a), f(b))$. Gilbert then considers the problem of minimizing the cost of networks supporting a given set of flows between terminals. The subadditivity of the cost f translates the fact that it is more advantageous to transport flows together. In the fluid mechanics context, this subadditivity follows from Poiseuille's law, according to which the resistance of a tube increases when it gets thinner (we refer to [3, 10] for a study of irrigation trees in this context). The simplest model of this kind is to take $f(c) = c^\alpha$ with $0 < \alpha < 1$.

Following [16], consider atomic sources $\mu^+ = \sum_{i=1}^k a_i \delta_{x_i}$ and sinks $\mu^- = \sum_{j=1}^l b_j \delta_{y_j}$ with $\sum_i a_i = \sum_j b_j$, $a_i, b_j \geq 0$. An irrigation graph G is a weighted directed graph with a set $E(G)$ of straight edges and a flow $w : E(G) \to (0, \infty)$ satisfying Kirchhoff's law. Observe that G can be written as a vector measure

$$G = \sum_{e \in E(G)} w(e) \mathcal{H}^1|_e \vec{e} \tag{1}$$

where $\vec{e}$ denotes the unit vector in the direction of e and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. We say that G irrigates (μ^+, μ^-) if

$$\text{div } G = \mu^- - \mu^+ \tag{2}$$

in the distributional sense. The Gilbert energy of G is defined by

$$M^\alpha(G) = \sum_{e \in E(G)} w(e)^\alpha \mathcal{H}^1(e). \tag{3}$$

We call the problem of minimizing $M^\alpha(G)$ among all finite graphs irrigating (μ^+, μ^-) the Gilbert-Steiner problem. The Monge-Kantorovich model corresponds to the limit case $\alpha = 1$ and the classical Steiner problem to $\alpha = 0$. The Gilbert model was adapted to the study of optimal pipeline or drainage networks [6, 20]. From a numerical point of view, a backtrack algorithm exploring relevant Steiner topologies is proposed in [35] to solve a problem of water treatment network. A different algorithmic approach can be found in [36].

1.2 Three continuous extensions of the Gilbert-Steiner problem

In analogy with the Monge-Kantorovich problem, the discrete Gilbert-Steiner model has been recently set in a continuous framework [31], [22] where the wells and sources are arbitrary measures, instead of a finite sum of Dirac masses. There were three approaches to this generalization, which we shall review briefly.

Xia's relaxation

Let μ^+, μ^- be two positive Radon measures in a compact convex set $X \subset \mathbb{R}^N$ with equal mass. A vector measure T on X with values in $\mathbb{R}^N$ is called by Xia [31] a *transport path* from μ^+ to μ^- if there exist two sequences μ_i^-, μ_i^+ of finite atomic measures with equal mass and a sequence of finite graphs G_i irrigating (μ_i^+, μ_i^-) such that $\mu_i^+ \to \mu^+$, $\mu_i^- \to \mu^-$ as measures and $G_i \to T$ as vector measures. The energy of T is defined by

$$M^\alpha(T) := \inf \liminf_{i \to \infty} M^\alpha(G_i)$$

where the infimum is taken over the set of all possible approximating sequences $\{\mu_i^+, \mu_i^-, G_i\}$ to T. Denote

$$M^\alpha(\mu^+, \mu^-) := \inf\{M^\alpha(T) : T \text{ is a transport path from } \mu^+ \text{ to } \mu^-\}.$$

If $\alpha \in (1 - \frac{1}{N}, 1]$, by Theorem 3.1 in [31], the above infimum is finite and attained for any pair (μ^+, μ^-). Xia showed or conjectured in a series of papers several structure and regularity properties of optimal transport paths which we shall comment later on.

Maddalena-Solimini's patterns

Maddalena and Solimini [22] gave a different (Lagrangian) formulation in the case of a single source supply $\mu^+ = \delta_S$. They model the transportation network as a set of particle trajectories, or "fibers", $\chi(\omega, \cdot)$, where $\chi(\omega, t) \in \mathbb{R}^N$ represents the location of a particle $\omega \in \Omega$ at time t and $\chi(\omega, 0) = S$. The set Ω is an abstract probability space indexing all fibers ; it is endowed with a measure $|\cdot|$ (without loss of generality one could take $\Omega = [0, 1]$ endowed with the Lebesgue measure). All the fibers are required to stop at some time $T(\omega)$ and to satisfy $\chi(\omega, 0) = S$ for all ω, i.e. all fibers start at the same root S. The set of fibers is given a structure corresponding to the intuitive notion of branches. Two fibers ω and ω' belong to the same branch at time t if $\chi(\omega, s) = \chi(\omega', s)$ for $s \leq t$. Then the partition of Ω given by the branches at time t yields a time filtration. The branch of ω at time t is denoted by $[\omega]_t$ and its measure by $|[\omega]_t|$. The energy of the set of fibers, or "irrigation pattern" is defined by

$$\tilde{E}^\alpha(\chi) = \int_\Omega \int_0^{T(\omega)} |[\omega]_t|^{\alpha-1} d\omega dt$$

It is easily checked on discrete trees that this definition extends the Gilbert energy (3). The measure μ^- irrigated by a pattern is easily defined. For every Borel set A in $\mathbb{R}^N$, $\mu^-(A)$ is the measure of the set of fibers stopping in A, $\mu^-(A) = |\{\omega,\ \chi(\omega, T(\omega)) \in A\}|$.

Traffic plans

In [2] the pattern formalism was extended to the case where the source is any Radon measure. The authors of [2] called "traffic plan" any probability measure on the set of Lipschitz paths. The equivalence of all models is proven in [21] and [4]. More precisely:

1. When the irrigated measures μ^+ and μ^- are finite atomic, the traffic plan minimizers are the same as the Gilbert finite graph minimizers.
2. For two general probability measures μ^+ and μ^-, Xia's minimizers are also optimal traffic plans and conversely.
3. When $\mu^+ = \delta_S$ is a single source, optimal patterns and optimal traffic plans are equivalent notions.

Throughout the paper we shall refer to the formalism of traffic plans which is the slight extension of the pattern formalism as explained above. The next section formalizes all definitions and recalls all properties we shall need in the sequel. They refer mainly to [31], [32], [33], [22]. The used formalism and the form given to statements follow [2], [4] and [5].

1.3 The regularity questions in a discrete and in a continuous framework

In the discrete Gilbert setting, the irrigated mass and the irrigating mass are finite atomic masses and the optimal graph has no circuits and is therefore a tree, with a finite number of vertices joined by straight edges. In addition, the following equilibrium equation is satified at all vertices:

$$\sum_{i \in I} w(e_i)^\alpha \vec{e}_i = 0 \tag{4}$$

where $w(e_i)$ and $\vec{e}_i$ are the flows and directions of all edges e_i arriving or leaving a given vertex and all $\vec{e}_i$'s are oriented inwards the corresponding edge.

One of the main challenges of the continuous model is to explore the regularity of very large and therefore virtually infinite networks. Our main goal in this paper is actually to prove that equation (4) still holds in the continuous model.

This needs some explanation. Xia defined the notion of *interior* [32] and *boundary* [33] regularity for infinite irrigation circuits. Interior regularity is the fact that, away from the supports of irrigated and irrigating measures, the network keeps locally the finite structure of the discrete case. This fact has been proven under a variety of assumptions including the case where the initial measure is atomic in [4]. In such a case the equation (4) is therefore satisfied and there is nothing to add.

Let us now consider the much more intricate case of boundary regularity, namely the *regularity of the network inside the irrigated body*. This is the case of river networks or of biological networks (blood), where the irrigated measure is a Lebesgue measure. Figure 1 shows such a river network. Models for river networks [27] could hardly raise the question of regularity in a discrete framework. The figure illustrates the relevance of this question when virtually infinitely many branches occur.

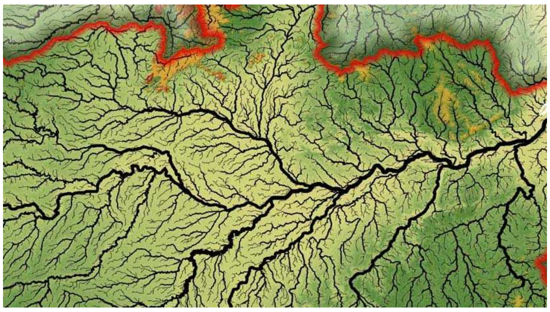

Figure 1: From the site of US Geological Survey: the branching network of the Amazon River: Is each river's direction a BV functions? It is possible to answer positively this question in the simplest available model for irrigation networks. A counterexample will also prove that the "yes' answer depends crucially on the fact that the irrigated measure is equivalent to the Lebesgue measure.

When a Lebesgue measure is irrigated the network bifurcates at an infinite, countable set of branching points. It has been proven in [33], [4] that the number of branches at each vertex is bounded by a constant depending only on α and the dimension of the ambient space. However, the generalization of the equilibrium equation (4) to the continuous framework was left open. Even worse, the existence of a tangent direction for each edge was a pending result. It is a main aim in this paper to prove this regularity result and to give an equilibrium equation, which generalizes the finite structure to infinite networks.

One of these regularity issues was raised in Xia's work [32], namely the existence of half-tangent directions at any point of the network. We shall deduce this existence from the stronger fact that each river's direction is locally BV. The BV estimate will be proven in Section 3. Section 2 is devoted to a more detailed explanation of the traffic plan model, which will be the one we will refer to, and of all the preliminary results we need for our further analysis.

For a related result concerning optimal traffic plans see Chapter 9 in [29], where the existence of tangent directions at branching points is proven when the dimension is 2 under L^p assumptions on the irrigated measure. The result is weaker (here we prove existence of tangents at any point of the network) but the hypotheses on the measures too (no lower bound on the density is required). The techniques are very similar to [30].

After proving this BV result, Section 4 formalizes by means of a differential equation the necessary optimality conditions on the tangent directions which generalizes equation (4). The ODE that we write in weak form involves the BV estimate of Section 3.

Sections 5 and 6 are aimed at the construction of a counter-example where the existence of a tangent direction fails, by properly choosing an irrigated measure. This irrigate measure won't of course satisfy the assumption of Section 3, that is, it won't be equivalent to the Lebesgue measure. The explicit counterexample is a countable atomic measure.

Section 5 is devoted to some geometric lemmas that will be preliminary to the example. They may be useful in other situations too. For instance we prove that if μ^+ and μ^- have distant and small supports, then the traffic plan consists of a single curve in a large part of the transportation. In Section 6 we give explicit choices for the irrigated measures of the counter-example and we use the lemmas of Section 5 to prove that the optimal traffic plan oscillates in a cone, thus giving raise to a non differentiable point. Providing non trivial explicit optimal traffic plans is not at all easy and we do not know of any other similar result.

2 Preliminaries on traffic plans

2.1 Basic definitions [2]

Denote by $|A| = \mathcal{L}^N(A)$ the N-dimensional measure of a subset A of $\mathbb{R}^N$. It is convenient that the supports of the irrigated measures μ^+ and μ^- are bounded in $\mathbb{R}^N$. Thus it is reasonable to consider paths contained in some compact convex N-dimensional set $X \subseteq \mathbb{R}^N$. Denote by $(\text{Lip}_1(X), d)$ the space of $1-$Lipschitz curves in X with the metric d of uniform convergence on compact sets.

Definition 2.1. *Let Ω be a measure space, its measure, being denoted by $|\cdot|$, having finite total mass. A (parametric) traffic plan is a Borel measurable map $\chi : \Omega \times \mathbb{R}^+ \to X$ such that $t \mapsto \chi_\omega(t) =: \chi(\omega, t)$ is constant for sufficiently large t and 1-Lipschitz for all $\omega \in \Omega$. Without risk of ambiguity we shall call* fiber *the path $\chi(\omega, \cdot)$, $\omega \in \Omega$ itself and the image in $\mathbb{R}^N$ of $\chi(\omega, \cdot)$. Denote by $|\chi| := |\Omega|$ the total mass transported by χ.*

In the original definition [2], traffic plans are defined as measures on the set of paths. By Skorokhod theorem such a measure also defines a parametric traffic plan in the sense of the above definition. Conversely the law of a parametric traffic plan viewed as a map $\omega \in \Omega \to \chi(\omega, .) \in \text{Lip}_1(X)$ is a traffic plan in the original sense. In this paper we will deal with parametric traffic plans but omit the mention “parametric".

Stopping time, irrigated measures, transference plan

If $\chi : \Omega \times \mathbb{R}^+ \to X$ is a traffic plan, define its stopping time by

$$T_\chi(\omega) := \inf\{t : \chi(\omega) \text{ is constant on } [t, \infty)\}.$$

Observe that $T_\chi : \Omega \to \mathbb{R}^+$ is measurable [22, 2]. The initial and final point of a fiber ω are $\chi(\omega, 0)$ and $\chi(\omega, T_\chi(\omega))$. Using these maps one can associate with any traffic plan χ its irrigating and irrigated measure defined by

$$\mu^+(\chi)(A) := |\{\omega : \chi(\omega, 0) \in A\}|,$$
$$\mu^-(\chi)(A) := |\{\omega : \chi(\omega, T_\chi(\omega)) \in A\}|$$

respectively, where A is any Borel subset of $\mathbb{R}^N$. We shall say that χ irrigates the measure $\mu(\chi) = (\mu^+(\chi), \mu^-(\chi))$ and call $TP(\mu^+, \mu^-)$ the set of traffic plans irrigating μ^+ and μ^-.

Energy of a traffic plan

Definition 2.2. *Let $\chi : \Omega \times \mathbb{R}^+ \to X$ be a traffic plan. Define the path class of $x \in \mathbb{R}^N$ in χ as the set*

$$\Omega_x^\chi := \{\omega : x \in \chi(\omega, \mathbb{R})\},$$

and the multiplicity of χ at x by $|x|_\chi = |\Omega_x^\chi|$. We shall note S_χ the support of χ, i.e. the set of points x such that $|x|_\chi > 0$.

We use the convention $0^{\alpha-1} = +\infty$ when $\alpha \in [0, 1)$.

Definition 2.3. *Let $\alpha \in [0, 1]$. We call energy of a traffic plan $\chi : \Omega \times \mathbb{R}^+ \to X$ the functional*

$$E^\alpha(\chi) = \int_\Omega \int_{\mathbb{R}^+} |\chi(\omega, t)|_\chi^{\alpha-1} |\dot{\chi}(\omega, t)| dt d\omega. \tag{5}$$

Proposition 2.4. *[2] The traffic plan energy is not changed if each fiber is re-parameterized by length. This energy decreases if all loops in the fibers are eliminated. Traffic plans normalized by length and loop-free will be called* normal.

It is proven in [2] that the traffic plan energy is equal to the Gilbert energy on a finite graph with a flow.

Definition 2.5. *In all that follows we consider traffic plans with finite energy. Without loss of generality we assume that all fibers satisfy $Z(\chi_\omega) := \int_{\mathbb{R}^+} |\chi(\omega, t)|_\chi^{\alpha-1} |\dot{\chi}(\omega, t)| dt < +\infty$.*

Definition 2.6. *A traffic plan χ is said to be* optimal for the irrigation problem *if it has minimal cost in $TP(\mu^+(\chi), \mu^-(\chi))$.*

2.2 Main properties used in the sequel [5], [4], [2]

Since most of the properties listed below are quite intuitive, if not always easy to prove, the reader is invited to read through quickly. They will be used in Sections 3-6.

Convergence, existence of minima

Definition 2.7. *We say that a sequence of traffic plans χ_n converges to a traffic plan χ if there are measure-preserving measurable maps $\varphi_n : \Omega \to \Omega$ such that $\chi_n(\varphi_n(\omega), t)$ converges to $\chi(\omega, t)$ uniformly on compact subsets of $\mathbb{R}^+$ for almost every $\omega \in \Omega$.*

The following results were proved in [22] and [2].

Theorem 2.8. *Up to a subsequence, any sequence of normal traffic plans χ_n with bounded energy converges to a traffic plan χ. In addition, $\mu^+(\chi_n) \rightharpoonup \mu^+(\chi)$, $\mu^-(\chi_n) \rightharpoonup \mu^-(\chi)$. If $\chi_n : \Omega \times \mathbb{R}^+ \to X$ is a sequence of normal traffic plans with bounded energy converging to the traffic plan χ, then*

$$E^\alpha(\chi) \leq \liminf E^\alpha(\chi_n).$$

Thus the problem of minimizing $E^\alpha(\chi)$ in $\mathrm{TP}(\mu^+, \mu^-)$ admits a solution whenever there is a feasible solution. This solution can be taken normal (see Proposition 2.4).

In the sequel we shall set

$$E^\alpha(\mu^+, \mu^-) := \min_{\mathrm{TP}(\mu^+,\mu^-)} E^\alpha(\chi).$$

Probably the most important result supporting the passage from a discrete to a continuous theory is the fact that measures are irrigable if α is large enough [31]:

Proposition 2.9. *Let μ^+ and μ^- be two positive measures supported in X, with equal mass. Then for $\alpha > 1 - \frac{1}{N}$ there is a constant depending only on α and N such that*

$$E^\alpha(\mu^+, \mu^-) \leq C \text{ diameter}(X)|\mu|^\alpha.$$

There is a thorough study of *irrigable measures* and the link between irrigability and dimension in [12], [13].

Stability of optima

The following lemma and propositions were proved in [31] and their proofs adapt immediately to traffic plans.

Lemma 2.10. *Let $\alpha \in (1 - \frac{1}{N}, 1]$. If μ_n^+ is a sequence of probability measures on X weakly converging to μ^+, then $E^\alpha(\mu_n^+, \mu^+) \to 0$ when $n \to \infty$.*

Proposition 2.11. *Let $\alpha \in (1 - \frac{1}{N}, 1]$. If χ_n is a sequence of optimal traffic plans for the irrigation problem and $\chi_n \to \chi$, then χ is optimal.*

Rectifiability and $\mathcal{H}^1$−formula

Proposition 2.12. *Let χ be traffic plan with finite energy. S_χ is countably rectifiable. More precisely there is a sequence ω_n of fibers such that $S_\chi \subset \cup_n \chi(\omega_n, \mathbb{R}^+)$.*

This result and the following representation of the energy are proven in [2] and show that the traffic plan energy indeed is a generalization of the Gilbert energy (3).

Proposition 2.13. *Let $\alpha \in [0, 1)$ and χ be a loop-free traffic plan such that $E^\alpha(\chi) < \infty$. Then,*

$$E^\alpha(\chi) = \int_{\mathbb{R}^N} |x|_\chi^\alpha d\mathcal{H}^1(x). \tag{6}$$

The following operations on traffic plans will prove useful. They are detailed in [4] and [5].

Restriction

Definition 2.14. *If $\Omega' \subset \Omega$ we call restriction of χ to $\Omega' \times \mathbb{R}^+$ the traffic plan $\chi_{|\Omega' \times \mathbb{R}^+}$ also noted $\chi_{\Omega'}$. More generally, let $\Omega' \subset \Omega$ and $D \subset \Omega' \times \mathbb{R}^+$ a subset of the form $D = \cup_{\omega \in \Omega'} \{\omega\} \times [s(\omega), t(\omega)]$. Define the restriction χ_D of χ to D as a traffic plan by $\chi_D(\omega, t) = \chi(\omega, t + s(\omega))$ if $0 \leq t \leq t(\omega) - s(\omega)$ and $\chi_D(\omega, t) = \chi(\omega, t(\omega))$ if $t \geq t(\omega) - s(\omega)$.*

Lemma 2.15. *Let $D = \cup_{\omega \in \Omega} \{\omega\} \times [s(\omega), t(\omega)]$. Then $E^\alpha(\chi_D) \leq E^\alpha(\chi)$. If Ω is a disjoint union of $\Omega_1, \dots, \Omega_l$, then $E^\alpha(\chi) \leq \sum_{i=1}^l E^\alpha(\chi_{|\Omega_i \times \mathbb{R}^+})$.*

Concatenation of a chain of traffic plans

Lemma 2.16. *Let $\chi \in TP(\mu^+, \mu^-)$ and $\xi \in TP(\nu^+, \nu^-)$ such that $\mu^- = \nu^+$. There is a traffic plan $\tilde{\chi} \in TP(\mu^+, \nu^-)$ such that each fiber of $\tilde{\chi}$ is a concatenation of a fiber of χ with a fiber of ξ. In addition, $E^\alpha(\tilde{\chi}) \leq E^\alpha(\chi) + E^\alpha(\xi)$.*

A convex hull property

We denote by $conv(E)$ the convex hull of E.

Lemma 2.17. *An optimal traffic plan χ satisfies $S_\chi \subset conv(\mathrm{supp}(\mu^-(\chi)) \cup \mathrm{supp}(\mu^+(\chi)))$. More precisely almost all fibers of the traffic plan stay in this convex hull.*

The single path property

Definition 2.18. *Let χ be a loop-free traffic plan, so that $t_x(\omega) := \chi^{-1}(\omega, \cdot)(x)$ is well defined. Let x, y in S_χ. Define*

$$\Omega_{\overrightarrow{xy}} := \{\omega \in \Omega_x^\chi \cap \Omega_y^\chi : t_x(\omega) < t_y(\omega)\},$$

the set of fibers passing through x and then through y. The restriction of χ to $\cup_{\omega \in \Omega_{\overrightarrow{xy}}} \{\omega\} \times [t_x(\omega), t_y(\omega)]$ is denoted by χ_{xy}. It is the traffic plan made of all pieces of fibers of χ joining x to y. Denote its support by $\Gamma^{xy} = S_{\chi_{xy}}$.

Definition 2.19. *A traffic plan χ has the* single path property *if for every pair (x, y) such that $|\Omega_{\overrightarrow{xy}}| > 0$, almost all fibers in $\Omega_{\overrightarrow{xy}}$ coincide between x and y with a same arc Γ^{xy} joining x to y. We say that the traffic plan has the* strict single path property *if for every x, y, either there is no fiber joining x to y or $|\Omega_{\overrightarrow{xy}}| > 0$ and* all *fibers in $\Omega_{\overrightarrow{xy}}$ coincide between x and y.*

Proposition 2.20. (Single path property) *Let $\alpha \in [0, 1)$ and χ be an optimal traffic plan. Then χ is single path and can be made strictly single path by restricting its fibers to $]0, T_\chi(\omega)[$ and removing a negligible set of fibers.*

If μ^+ is a Dirac mass, the optimal traffic plan is an optimal pattern. In that case the above proposition implies that

Proposition 2.21. *Let χ be an optimal normal pattern. Then it has the strict single path property and for almost every ω the function $t \mapsto |\chi(\omega, t)|_\chi$ is nonincreasing on $[0, T_\chi(\omega)]$.*

Interior regularity

Theorem 2.22. *Let $\alpha \in (1 - \frac{1}{N}, 1)$ and let χ be an optimal traffic plan in $\mathrm{TP}(\mu^+, \mu^-)$. Assume that the supports of μ^+ and μ^- are at positive distance. In any closed ball not meeting the supports of μ^+ and μ^-, the traffic plan has the structure of a finite graph made of segments.*

A variant for the above theorem applies when the irrigating measure is atomic.

Corollary 2.23. *Let $\alpha \in (1 - \frac{1}{N}, 1)$ and let χ be an optimal traffic plan such that $\mu^+(\chi) = \sum_{i=1}^n m_i \delta_{x_i}$ is an atomic measure. In any closed ball outside the support of $\mu^-(\chi)$, the traffic plan μ has a finite graph structure.*

Boundary regularity

The present paper is mainly concerned with boundary regularity, namely the regularity of the traffic plan inside the supports of μ^+ and μ^-. We can summarize the already known results. In [4], it was proven that:

Theorem 2.24. (bounded branching property). *Let $\alpha \in (0, 1)$. At every point x of the support of an optimal traffic plan χ in $\mathbb{R}^N$, the number of branches at x is less than a constant $\mathcal{N}(\alpha, N)$ depending only on N and α.*

This result was conjectured in [33].

Proposition 2.25. *Any optimal traffic plan χ such that $E^\alpha(\chi) < \infty$ has countably many branching points.*

We can make a synthesis of the above results in the case of an optimal pattern (see [4].)

Corollary 2.26. *Let χ be an optimal pattern. Then χ can be taken normal by restricting each fiber to $[0, T_\chi(\omega))$. All fibers have finite length and $t \to |\chi(\omega, t)|_\chi$ is positive and non increasing along the fiber. The pattern has a tree structure (fibers which separate never meet again), and has countably many branching points. If one cuts the tree at any point x, one obtains a finite number of connected components which are themselves trees with the same structure. One of them contains the source and all other ones are optimal patterns with source x.*

As a further boundary regularity property Xia [33] proves that any path in the irrigation graph with flow larger than a constant is bilipschitz, with explicit estimates on the Lipschitz constant depending on α, the dimension and the minimal value of the flow.

The landscape function

Let us also define, as in [28], the landscape function associated to a normal traffic plan χ by $z(x) = Z(\chi_\omega) = \int_0^{T(\chi_\omega)} |\chi_\omega(t)|_\chi^{\alpha-1} dt$ (χ_ω being any fiber of χ arriving at x).

Proposition 2.27. *[28] The landscape function z associated to the optimal irrigation of μ^+ from δ_0 is well defined. Moreover, if $\alpha > 1 - \frac{1}{N}$ and μ^+ has a density with respect to the Lebesgue measure which is bounded from below on X by a positive constant, then z is Hölder continuous with exponent $\beta = N(\alpha - (1 - \frac{1}{N}))$, and the Hölder constant only depends on the lower bound on the density of μ^+.*

Angle laws at bifurcations

The next elementary geometric results are proved in [16], [31], [1], [5].

Lemma 2.28. *Let χ be an optimal traffic plan and x a point of its support. Assume that the traffic plan inside $B(x, R)$ is made of k disjoint simple paths from x to $x_i \in \partial B(x, R)$, $i = 1, \dots, k$. Then these paths are straight segments. Setting m_i their flow and $\vec{e}_i = \frac{x - x_i}{|x - x_i|}$, one has*

$$\sum_{i=1}^{k} m_i^\alpha \vec{e}_i = 0. \tag{7}$$

Lemma 2.29. *Let us consider the simple irrigation case where an optimal traffic plan is made of two paths γ_1 and γ_2 with flow $\frac{1}{2}$ starting at y_0, coinciding up to their bifurcation y and ending respectively at points y_1 and y_2. In such a situation, the paths γ_1 and γ_2 are straight on the segments $[y_0, y]$ and $[y, y_1]$, $[y, y_2]$. The half line $[y_0, y)$ is the bisector of the angle made by $[y, y_1]$ and $[y, y_2]$ and the value $2\theta_\alpha$ of this angle depends only on α.*

The next lemma is an easy consequence of Lemma 2.28.

Lemma 2.30. *There is a constant $\theta_{min}(\alpha)$ depending only on α such that for every branching point x of an optimal traffic plan with locally finite branching number, the angles of pairs of vertices starting from x are all larger than $\theta_{min}(\alpha)$.*

3 Curvature Bounds

In all that follows, C denotes various constants depending only on the dimension N. Thus if C appears at different positions in the same formula it may have different values. Let μ^- be a measure on X which is equivalent to the Lebesgue measure, i.e. $c^-\mathcal{L}^N \leq \mu^- \leq c^+\mathcal{L}^N$ for some constants $0 < c^- < c^+$. In the following we consider an optimal traffic plan χ (or pattern) irrigating μ^- from a Dirac mass δ_0. By Corollary 2.26, χ has a tree structure. Consider a path $\gamma(t) = \chi(\omega, t)$ which is a fiber of χ and take $0 < a < b < T(\gamma)$ so that $|\gamma(b)|_\chi > 0$. (The notation $\gamma = \chi(\omega, .)$ is taken for brevity.) Let us denote $\Sigma = \gamma([t_0, t_1])$ and $x_i = \gamma(t_i)$ all branching points of the traffic plan belonging to Σ. By Corollary 2.25, we know that the branching points are a finite or countable set. Each x_i is the origin of a finite set of optimal subtrees χ_i of χ whose root is x_i. Each χ_i is a restriction of χ obtained by restricting all fibers to their part belonging to a connected component of χ in $\mathbb{R}^N \setminus \{x_i\}$. Without loss of generality we can assume that all of theses subtrees are indexed by I, so that we may have $x_j = x_i$ for j in a finite subset of I. Finally, let us denote by ε_i the total mass irrigated by χ_i.

The following theorem summarizes the results proven in this section.

Theorem 3.1. *Let $\gamma(s)$, $s \in [0, T(\gamma)]$ be a fiber of an optimal traffic plan (pattern) χ irrigating a measure μ^- equivalent to $\mathcal{L}^N$ from δ_0. Let $[a, b] \subset [0, T(\gamma)[$ and $\Sigma =: \gamma([a, b])$. Let ε_i, $i \in I$ the masses of all trees branching from Σ. Then $\sum_i \varepsilon_i^\alpha < \infty$ and Σ has a bounded total curvature. As a consequence it has two half-tangents at all points and a tangent at all points which are not branching points.*

The most important consequence is the following:

Corollary 3.2. *Every branching point x of an optimal traffic plan has a tangent cone made of a finite (and bounded by a constant depending on α and N) number of segments whose directions e_i and masses m_i satisfy the equation* (7).

Proof. It is sufficient to apply Theorem 3.1 to all the fibers of the traffic plan passing through x, obtaining the existence of their one-sided derivatives. This proves the existence of the full limit (and not up to subsequences) of any blow-up at x. Consequently, it is possible to study the limit configuration by means of the stability property of Corollary 2.11: the traffic plan restricted to $B(x, r)$ and rescaled by a factor r^{-1} converges to an optimal traffic plan for the irrigation of two atomic measures whose masses are the m_i masses of the branches (see [32]). The direction of the segments composing this discrete traffic plan are those of the tangents e_i. Hence, we can apply Lemma 2.28 and get the thesis. $\square$

The existence of tangent cones for particular blow-up sequences was proved by Xia in [32]. The first lemmas we will prove aims at estimating $\sum_i \varepsilon_i^\alpha$. In fact Lemma 2.28 implies in the discrete case that each time the curve γ branches and loses a mass ε_i, the change in its direction is of the order of ε_i^α. It is consequently natural to look for estimates on the sum $\sum_i \varepsilon_i^\alpha$ to get bounds on the curvature. Denote by Σ a continuum, that is, a connected set with bounded $\mathcal{H}^1$ measure and by $\Sigma^r = \{x, \text{dist}(x, \Sigma) \leq r\}$ its $r-$ neighborhood. A proof of the next lemma can be found in a paper by Tilli and Mosconi [25], but we provide a different one for the sake of completeness.

Lemma 3.3. *There is a constant C only depending on the dimension N such that*

$$|\Sigma^r| \leq C \left(\mathcal{H}^1(\Sigma) r^{N-1} + r^N \right). \tag{8}$$

for any positive radius r.

Proof. Let $r < \frac{1}{2}\text{diam}(\Sigma)$. We can cover Σ by a maximal disjoint set of balls $B_i = B(y_i, r)$, $i \in J$ with radius r and centered on Σ. Then by the assumption on the diameter and the connectedness of Σ, we get $\mathcal{H}^1(B_i \cap \Sigma) \geq r$, as a consequence of the fact that Σ cannot be contained in any of these balls. Thus

$$\text{Card}(J) r \leq \mathcal{H}^1(\Sigma). \tag{9}$$

The set J being maximal, every ball $B(x, r)$ centered at a point in Σ meets at least one B_i. Thus $\Sigma \subset \bigcup_{i \in J} B_i(y_i, 2r)$ and therefore $\Sigma^r \subset \bigcup_{i \in J} B_i(y_i, 3r)$. This implies by (9),

$$|\Sigma^r| \leq \text{Card}(J) |B(0, 1)| 3^N r^N \leq |B(0, 1)| 3^N \mathcal{H}^1(\Sigma) r^{N-1}. \tag{10}$$

Assume now $r \geq \frac{1}{2}\text{diam}(\Sigma)$. Then Σ^r is contained in a ball centered on Σ with radius $2r$ and one gets $|\Sigma^r| \leq |B(0, 2)| r^N$. $\square$

Lemma 3.4. *Let χ be a traffic plan (pattern) irrigating the measure μ^- from δ_0. Then for every subtree χ_i of χ with total mass ε_i stemming at x_i, the fibers of χ_i are contained in a ball centered at x_i and of radius $C\varepsilon_i^{\frac{1}{N}}$, where the constant C depends on the upper and lower bounds of the density of the measure μ^-.*

Proof. Take the landscape function z associated to the irrigation of μ^- from δ_0 defined in Proposition 2.27. Denote by x_i the root of χ_i and by y a point in the subtree stemming from x_i. For a point x in the image of a fiber γ we call $t(x)$ the unique value such that $\gamma(t(x)) = x$. The value $t(x)$ is also the geodesic distance in the tree from the root to x. Since the multiplicity $t \to |\gamma(t)|_\chi$ is non-increasing, one has

$$z(y) - z(x_i) = \int_{t(x_i)}^{t(y)} |\gamma(t)|_\chi^{\alpha-1} dt \geq (t(y) - t(x)) \varepsilon_i^{\alpha-1}.$$

Thus,

$$t(y) - t(x_i) \le (z(y) - z(x_i))\varepsilon_i^{1-\alpha} \le C|y - x_i|^{\beta}\varepsilon_i^{1-\alpha},$$

which yields

$$|y - x_i| \le t(y) - t(x_i) \le C|y - x_i|^{\beta}\varepsilon_i^{1-\alpha}, \text{ and therefore}$$

$$|y - x_i| \le C\varepsilon_i^{\frac{1-\alpha}{1-\beta}} = C\varepsilon_i^{\frac{1}{N}}.$$

□

Lemma 3.5. *Let Σ be a connected component of the support of an optimal traffic plan χ irrigating the measure μ^- from δ_0. Denote by ε_i, $i \in I$, the masses of all the subtrees χ_i of χ stemming from Σ, which means that they meet Σ at their root only. Then*

$$\sum_{i\in I} \varepsilon_i^{\alpha} \le C\mathcal{H}^1(\Sigma) + C.$$

Proof. By Lemma 3.4 we know that the subtree χ_i with origin x_i and mass ε_i irrigates a measure supported in the ball $B(x_i, C\varepsilon_i^{\frac{1}{N}})$. Thus, such a support is contained in Σ^r for $r \ge C\varepsilon_i^{\frac{1}{N}}$. By (8), we obtain

$$\sum_{C\varepsilon_i^{\frac{1}{N}} < r} \varepsilon_i \le C\left(\mathcal{H}^1(\Sigma)r^{N-1} + r^N\right)$$

and therefore

$$\sum_{\frac{r}{2} \le C\varepsilon_i^{\frac{1}{N}} < r} \varepsilon_i \le C\left(\mathcal{H}^1(\Sigma)r^{N-1} + r^N\right).$$

Denote $E(r) = \text{Card}(\{i,\ \frac{r}{2} \le C\varepsilon_i^{\frac{1}{N}} < r\})$. Then from the previous inequality,

$$E(r)\left(\frac{r}{2C}\right)^N \le C\left(\mathcal{H}^1(\Sigma)r^{N-1} + r^N\right)$$

which yields $E(r) \le C\left(\mathcal{H}^1(\Sigma)r^{-1} + 1\right)$. By definition of $E(r)$, the union $\cup_{n\in\mathbb{Z}} E(2^{-n}) = I$ is a disjoint union. Thus, since $\alpha > 1 - \frac{1}{N}$,

$$\sum_i \varepsilon_i^{\alpha} \le \sum_n E(2^{-n})C\left(\frac{1}{2^n}\right)^{N\alpha} \le C\mathcal{H}^1(\Sigma)\sum_n 2^{n(1-N\alpha)} + C\sum_n 2^{-nN\alpha} \le C\mathcal{H}^1(\Sigma) + C. \quad \square$$

As a following step, we will use a perturbation argument on χ to derive curvature estimates involving $\sum_i \varepsilon_i^{\alpha}$. In the perturbation we will need the following lemma.

Lemma 3.6. *Let Σ be a simple rectifiable curve from x_0 to x_1 and S the segment between the same points. Suppose $L = \mathcal{H}^1(\Sigma) < +\infty$ and set $\Lambda = \mathcal{H}^1(S)$. Take the map $p : \Sigma \to S$ defined by*

$$p(y) = ((L - d_\Sigma(y, x_0))/L)x_0 + (d_\Sigma(y, x_0)/L)x_1,$$

where d_Σ denotes the geodesic distance on Σ (we are actually creating a correspondence between Σ and S by following the segment with constant speed given by $\Lambda/L \le 1$ and we will call such an application constant speed projection). Then we have

$$|y - p(y)| \le \frac{\sqrt{L^2 - \Lambda^2}}{2}.$$

Proof. Suppose for notational simplicity that $x_0 = 0$ and $x_1 = \Lambda e_1$ (i.e., we set the segment x_0x_1 on the first coordinate axis). Take a point $y \in \Sigma$ and set $l = d_\Sigma(y, x_0)$, call a the first coordinate of y, and h the distance from y to the axis x_0x_1 (see figure 2). We have $|y - p(y)| = \sqrt{h^2 + (a - \Lambda l/L)^2}$ and we want to estimate such a quantity. Moreover, we know $\sqrt{a^2 + h^2} \le l$ et $\sqrt{(\Lambda - a)^2 + h^2} \le L - l$. Hence we have

$$h^2 \le [l^2 - a^2] \wedge [(L - l)^2 - (\Lambda - a)^2].$$

Let us suppose for simplicity that $l^2 - a^2 \le (L-l)^2 - (\Lambda - a)^2$ (the other case being symmetric), and we get

$$|y - p(y)|^2 \le l^2 - a^2 + (a - \Lambda l/L)^2 = l^2 \left(1 + \frac{\Lambda^2}{L^2}\right) - 2a\frac{l}{L}\Lambda.$$

We need to consider this quantity only under the additional condition $l^2 - a^2 \le (L-l)^2 - (\Lambda - a)^2$, i.e. $2a\Lambda \ge \Lambda^2 - L^2 + 2Ll$, and hence we have

$$|y - p(y)|^2 \le l^2 \left(1 + \frac{\Lambda^2}{L^2}\right) - \frac{l}{L}(\Lambda^2 - L^2 + 2Ll) = l\frac{L^2 - \Lambda^2}{L} - l^2\frac{L^2 - \Lambda^2}{L^2}.$$

This last expression is maximized at $l = L/2$, in which case we get exactly $(L^2 - \Lambda^2)/4$. $\square$

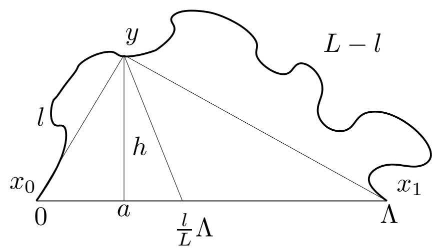

Figure 2: Constant speed projection from the curve to the segment

Lemma 3.7. *Let χ be an optimal traffic plan (pattern) irrigating the measure μ^- from δ_0. Let $\gamma(s)$, $s \in [0, T(\gamma)]$ be a fiber of χ, $[a,b] \subset [0, T(\gamma)[$ and $\Sigma =: \gamma([a,b])$. Let ε_i, $i \in I$ the masses of all trees branching from Σ. Then for every monotone polygonal line $(x_k)_{k=0,\dots n}$ approximating γ with $x_k \in \Sigma$ one has*

$$|\gamma(b)|_\chi^\alpha \sum_{k=1}^{n} \frac{(l_k - |x_k - x_{k-1}|)}{\delta_k} \le \sum_i \varepsilon_i^\alpha$$

where l_k denotes the geodesic distance in Σ of x_{k-1} to x_k and $\delta_k =: \frac{1}{2}\sqrt{l_k^2 - |x_k - x_{k-1}|^2}$.

Proof. Denote by $x_0, x_1, \dots, x_k, \dots, x_n$ an ordered sequence of points such that $x_0 = \gamma(a)$, $x_n = \gamma(b)$, $x_k = \gamma(t_k)$.

Set $\Sigma_k = \gamma([t_{k-1}, t_k])$ and call $p_k : \Sigma_k \to [x_{k-1}, x_k]$ the constant speed projection arising from the construction of Lemma 3.6. Finally call ε_i^k the masses of all subtrees χ_i^k stemming from Σ_k. We consider for each k an alternative χ_k to χ which consists roughly of replacing the path Σ_k by the segment $[x_{k-1}, x_k]$. More precisely,

- All fibers of χ passing by x_{k-1} and x_k are replaced, between x_{k-1} and x_k, with their projection p_k onto the straight line segment $[x_{k-1}, x_k]$.
- for each fiber of χ passing by x_{k-1} but not by x_k there is some y on Σ_k at which the fiber leaves Σ_k. This fiber is replaced between x_{k-1} and y by its p_k-projection onto the straight line segment from x_{k-1} to $p_k(y)$ followed by a straight path from $p_k(y)$ to y. The rest of the fiber is unchanged. (See Figure 3).
- All other fibers of χ are unchanged.

Lemma 3.6 ensures that the speed of the fibers has not increased, that all the fibers passing by x_{k-1} proceed, while on the segment $[x_{k-1}, x_k]$, in the direction from x_{k-1} to x_k, and that the distances $|y - p_k(y)|$ are estimated by δ_k.

In all that follows we shall assume that $[x_{k-1}, x_k]$ meets the support S_χ of χ on a set with zero $\mathcal{H}^1$ measure. We shall explain at the end of the proof how to get rid of this assumption, which implies $|p_k\gamma(s)|_{\chi_k} = |\gamma(s)|_\chi$ and simplifies the exposition. Using this fact, the energy of χ_k on the segment $[x_{k-1}, x_k]$ is

$$\int_{t_{k-1}}^{t_k} |p_k\gamma(s)|_{\chi_k} (p_k\gamma)'(s) ds = \int_{t_{k-1}}^{t_k} |\gamma(s)|_\chi (p_k\gamma)'(s) ds,$$

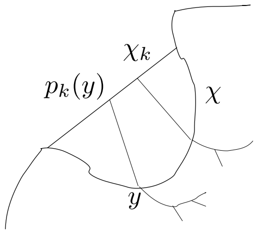

Figure 3: Replacing a piece of curve with its constant speed projection

where we identify for simplicity the vector $(p_k\gamma)'(s)$ with its scalar coordinate $(p_k\gamma)'(s).\frac{x_k-x_{k-1}}{|x_k-x_{k-1}|}$. Since χ is optimal, we must have $E^\alpha(\chi_k) \geq E^\alpha(\chi)$, which yields

$$0 \leq \int_{t_{k-1}}^{t_k} |\gamma(s)|_\chi^\alpha (p_k\gamma)'(s)ds - \int_{t_{k-1}}^{t_k} |\gamma(s)|_\chi^\alpha |\gamma'(s)|ds + \delta_k \sum_i (\varepsilon_i^k)^\alpha. \tag{11}$$

Taking in consideration that $|p_k\gamma'(s)| \leq |\gamma'(s)| = 1$ and that $|\gamma(s)|_\chi \geq |\gamma(b)|_\chi$ (because by Corollary 2.26 the multiplicity is non-increasing along fibers), Equation (11) yields

$$|\gamma(b)|_\chi^\alpha(|x_k - x_{k-1}| - l_k) + \sum_i (\varepsilon_i^k)^\alpha \delta_k \geq 0. \tag{12}$$

Summing this inequality for k ranging from 1 to n ends the proof. We are however left to explain how we can enforce that $[x_{k-1}, x_k]$ meets Σ_k on a set with zero length. To do so we can move slightly all x_k's in a ball with radius ϵ around their position on Σ. For almost all positions $\tilde{x}_k$ in these balls $B(x_k, \epsilon)$ the announced property is true. Now the polygonal line $(\tilde{x}_k)$ is no more supported by Σ. We can, however, repeat the whole above argument but we have to add to the fibers of χ_k small segments joining x_k to $\tilde{x}_k$, back and forth. Thus (12) becomes

$$|\gamma(b)|_\chi^\alpha(|x_k - x_{k-1}| - l_k) + \sum_i (\varepsilon_i^k)^\alpha \delta_k + 2k\epsilon \geq 0.$$

Since this construction can be made for every $\epsilon > 0$ we get back to (12) by letting $\epsilon \to 0$. $\square$

Remark 3.8. In the proof, we passed through the condition $\mathcal{H}^1([x_{k-1}, x_k] \cap S_\chi) = 0$ for the sake of exposition simplicity only. In fact, using the subadditivity of $s \to s^\alpha$, (12) can be obtained directly even if the multiplicity is not exactly preserved in the projection.

We shall need to define a standard approximation of a rectifiable curve by a polygonal line.

Lemma 3.9. *Let $\gamma : [0, L] \to \mathbb{R}^N$ be a rectifiable one to one curve parameterized by length. Then for every $a > 0$ one can find a polygonal curve $(x_k)_k$, $k = 0, \dots n$ such that $|x_k - x_{k-1}| = a$ for all $k \in \{1, \dots, n\}$, $x_0 = \gamma(0)$, $|x_n - \gamma(L)| \leq a$, $x_k = \gamma(t_k)$ belongs to $\gamma([0, L])$ for every $0 \leq k \leq n$, and the sequence t_k is increasing. In such a case we call the curve made by the successive segments $[x_{k-1}, x_k]$, $k = 1, \dots, n$ a* regular polygonal approximation to γ with step a*. When $a \to 0$, this polygonal curve converges uniformly to γ.*

Proof. The polygonal approximation which we shall call γ_n can be constructed for each a iteratively. One takes $x_0 = \gamma(0)$, then t_1 is defined as the smallest t such that $|\gamma(t) - x_0| = a$, t_2 as the smallest $t \geq t_1$ such that $|\gamma(t) - x_1| = a$, and so on. The process stops at $x_n = \gamma(t_n)$ such that the rest of the curve $\gamma([t_n, L])$ is contained in the open ball $B(x_n, a)$. Thus $|x_n - \gamma(L)| < a$. Let us call γ_a the curve we obtained. One has by construction $\mathcal{H}^1(\gamma_a) \leq \mathcal{H}^1(\gamma)$. By Ascoli-Arzela theorem γ_a converges to a rectifiable curve $\tilde{\gamma}$ uniformly. In addition, $\mathcal{H}^1(\tilde{\gamma}) \leq \liminf_a \mathcal{H}^1(\gamma_a) \leq \mathcal{H}^1(\gamma)$. Notice that, identifying all curves with their respective images, one has, for all $x \in \gamma_a$, $d(x, \gamma) \leq a$, which implies that $\tilde{\gamma} \subset \gamma$ (this is easily deduced from the Hausdorff convergence of a subsequence of the sequence of compact sets γ_a). Thus $\tilde{\gamma}$ is a curve whose image is contained in that of γ, which has the same starting and arrival points and a smaller length. Since γ is one to one one obtains $\gamma(s) = \tilde{\gamma}(s)$ for $s \in [0, L]$. $\square$

The statement of Theorem 3.1 will be proven by applying Lemmas 3.7, 3.5, and the next one.

Lemma 3.10. *Let Σ be a curve with finite length such that for every regular polygonal approximation $(x_k)_k$,*

$$\sum_{k=1}^{n} \frac{l_k - |x_k - x_{k-1}|}{\delta_k} \leq C,$$

where $\delta_k =: \frac{1}{2}\sqrt{l_k^2 - |x_k - x_{k-1}|^2}$ and l_k is the geodesic distance in Σ from x_{k-1} to x_k. Then Σ has a bounded total curvature.

Proof. Notice first that

$$\frac{l_k - |x_k - x_{k-1}|}{\delta_k} = 2\frac{l_k - |x_k - x_{k-1}|}{\sqrt{l_k^2 - |x_k - x_{k-1}|^2}} = 2\left(\frac{l_k - |x_k - x_{k-1}|}{l_k + |x_k - x_{k-1}|}\right)^{\frac{1}{2}}.$$

Let $\gamma_n = (x_k)_{k=0\ldots n}$ be a regular polygonal approximation of Σ. By applying the assumption to the polygonal line $(x_{2k})_k$ one has

$$\sum_k \left(\frac{\tilde{l}_k - |x_{2k} - x_{2(k-1)}|}{\tilde{l}_k + |x_{2k} - x_{2(k-1)}|}\right)^{\frac{1}{2}} \leq C,$$

where $\tilde{l}_k$ denotes the geodesic distance on Σ from $x_{2(k-1)}$ to x_{2k}. Set $a = |x_k - x_{k-1}|$ and notice that $\tilde{l}_k \geq |x_{2k} - x_{2k-1}| + |x_{2k-1} - x_{2(k-1)}| = 2a$ and that the function $l \to \frac{l-a}{l+a}$ is nondecreasing in l for $a \geq 0$. Thus,

$$\sum_k \left(\frac{2a - |x_{2k} - x_{2(k-1)}|}{2a + |x_{2k} - x_{2(k-1)}|}\right)^{\frac{1}{2}} \leq C. \tag{13}$$

Consider the isosceles triangle x_{k-1}, x_k, x_{k+1} and call $0 \leq \theta_k \leq \frac{\pi}{2}$ the absolute value of the angle of $\overrightarrow{x_{k-1}x_k}$ with $\overrightarrow{x_{k-1}x_{k+1}}$. Then $\cos\theta_{2k-1} = |x_{2k} - x_{2(k-1)}|/(2a)$ and (13) yields

$$\sum_{k \text{ odd}} \left(\frac{1 - \cos\theta_k}{1 + \cos\theta_k}\right)^{\frac{1}{2}} \leq C,$$

that is the same as

$$\sum_{k \text{ odd}} \tan\frac{\theta_k}{2} \leq C. \tag{14}$$

Analogously one can obtain $\sum_{k \text{ even}} \tan\frac{\theta_k}{2} \leq C$ and hence

$$\sum_k \tan\frac{\theta_k}{2} \leq 2C.$$

The total curvature $TC((x_k)_k)$ of a polygonal line x_k is

$$\sum_{k=1}^{n-1} |n_k - n_{k-1}|$$

where $n_k = \frac{\overrightarrow{x_k x_{k+1}}}{|x_k x_{k+1}|} - \frac{\overrightarrow{x_{k-1}x_k}}{|x_{k-1}x_k|}$. In this case, where all the segments have the same length, it is easy to see that $|n_k - n_{k-1}| = 2\sin\theta_k$. Thus

$$TC((x_k)_k) = 2\sum_k \sin\theta_k \leq 4\sum_k \tan\frac{\theta_k}{2} \leq 8C.$$

By Lemma 3.9 the polygonal curve γ_n converges uniformly to γ. Its second derivative in the distribution sense is the measure $\gamma_k'' = \mu_k = 2\sum_k(\sin\theta_k)\delta_{x_k}$ whose total mass is $TC((x_k))$. Thus $|\mu_k| \leq C$ and its weak limit γ'' is a measure with bounded total mass. We conclude that γ' has bounded variation on $[a, b]$. ◻

This also ends the proof of Theorem 3.1, which follows from Lemmas 3.5, 3.7, and 3.10.

4 An elliptic equation for traffic plans

In the previous section we have proven a regularity result on the paths γ of an optimal traffic plan χ. The the next step is to strengthen the result, by providing a differential equation which is satisfied by γ. We have just proven that γ' is BV. In addition $|\gamma|_\chi^\alpha$ is BV since it is monotone decreasing. Thus the product $|\gamma|_\chi^\alpha \gamma'$ is BV and its derivative is a measure. Our aim is to identify this measure.

Theorem 4.1. *Let γ be a fiber of an optimal traffic plan χ irrigating μ^- from δ_0. Then γ satisfies in the sense of distributions the elliptic equation*

$$-(|\gamma(t)|_\chi^\alpha \gamma'(t))' = \sum_{i\in I} \varepsilon_i^\alpha \delta_{\gamma(t_i)} \vec{\nu_i} \tag{15}$$

where $\vec{\nu_i}$ is the tangent of the branch stemming from γ at $\gamma(t_i)$ with mass ε_i. Notice that this tangent vector exists, thanks to the regularity result of Theorem 3.1, and that the right hand side is a vector measure with finite mass, thanks to Lemma 3.5.

Proof. Take an arc γ, defined on a time interval $]t_0^-, t_0^+[$, of an optimal traffic plan χ, and a function $\phi \in C_c^1(]t_0^-, t_0^+[; \mathbb{R}^N)$. We want to prove that γ satisfies (15) in weak form by testing the equation against ϕ. Let us label all the curves stemming from γ in the interval we are considering and call them γ_i, for $i \geq 1$. We assume that γ and all the curves γ_i are parameterized by arc length from the source. Let us also fix $K \in \mathbb{N}$ and let $(\gamma_i)_{i=1,\dots,K}$ be a finite set of arcs going out from γ at time t_i. Thus we have $\gamma_i(t_i) = \gamma(t_i)$ and $\lim_{t\to t_i^+}[\gamma_i(t)]_\chi = \varepsilon_i$. We denote by ε_i, for $i > K$, the total mass of the other branches stemming from γ. We also denote by $(\varepsilon_{i,j})_j$ the sequence of masses entering or leaving γ_i. Fix two small parameters h, $\delta > 0$ and a function $k \in C_c^1([0,1[)$ with $k(0) = \max k = 1$; set $R = R(\gamma) \cup \bigcup_{i=1}^K R(\gamma_i)$, the union of the ranges of γ and the K first γ_i's, and consider the mapping $S : R \to \mathbb{R}^N$ given by

$$S(x) = \begin{cases} \phi(t) & \text{if } x = \gamma(t); \\ \phi(t_i) k\left(\frac{t-t_i}{\delta}\right) & \text{if } x = \gamma_i(t). \end{cases}$$

Then we set $T_h(x) = x + hS(x)$ for $x \in R$. For any fiber ω in the traffic plan χ there is a maximal interval $I_\omega =]t^-(\omega), t^+(\omega)[$ such that we have $\chi(\omega, t) =: \chi_\omega(t) \in R$ for any $t \in I_\omega$. Let us build a new traffic plan χ' by replacing any curve χ_ω by $T_h \circ \chi_\omega$ in $]t^-(\omega), t^+(\omega)[$ and going straight between $\chi_\omega(t^\pm(\omega))$ and $T_h(\chi(t^\pm(\omega)))$ (see Figure 4). We decompose the old traffic plan χ into a traffic plan χ_0 which is the restriction of χ to the domain of γ (see definition 2.14), the traffic plans χ_i, which are the restrictions of χ to the domain of γ_i, and a remaining traffic plan $\tilde{\chi}$. By decomposition we mean that χ is the concatenation of the $\tilde{\chi}$, χ_i and χ_0. (See definition 2.16). The new traffic plan χ' is the concatenation of χ_i', $i = 0, \dots, K$ which are the images of χ_i under T_h, with the same $\tilde{\chi}$ and with an additional traffic plan $\overline{\chi}$ which corresponds to the straight line segments that we have been forced to add. Notice that the energies of the considered traffic plans add for χ. The energy of the traffic plan χ' is just sub-additive xith respect to this decomposition, since the supports of the concatenated parts are not necessarily disjoint.

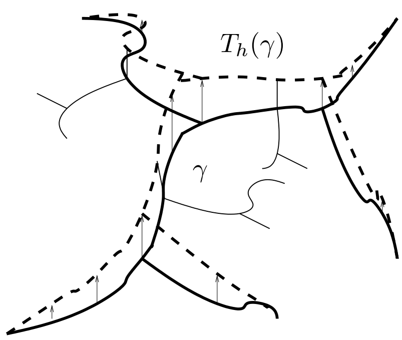

Figure 4: Perturbation of the network R

Setting $C_\phi =: ||\phi||_\infty$, it is easy to evaluate the energy of $\overline{\chi}$ by

$$E(\overline{\chi}) \leq \sum_{i,j} C_\phi h \varepsilon_{i,j}^\alpha + \sum_i C_\phi h \varepsilon_i^\alpha = hC_\phi \left(\sum_{i=1}^K Q_i(\delta) + \sum_{i=K+1}^\infty \varepsilon_i^\alpha \right),$$

where we set $Q_i(\delta) = \sum_{j\in J(\delta)_i} \varepsilon_{i,j}^\alpha$ and the set of indices $J(\delta)_i$ is the set corresponding to the j's such that the mass enters (or goes out from) the curve γ_i at a point $\gamma_i(t)$ with $|t - t_i| < \delta$. By Lemma 3.5 it follows that $E(\overline{\chi}) < \infty$. Then we evaluate the energy of χ_i' for $i = 1, \dots, K$, taking into account that T_h is one to one for h small enough and that $|\gamma_i'(t)| = 1$,

$$\begin{aligned}
E(\chi_i') &= \int_{t_i}^{t_i+\delta} |T_h(\gamma_i(t))|_{\chi_i}^\alpha |(T_h\circ\gamma_i)'(t)|dt \\
&= \int_{t_i}^{t_i+\delta} |\gamma_i(t)|_\chi^\alpha \left|\gamma_i'(t) + h\frac{d}{dt}(S\circ\gamma_i)(t)\right| dt \\
&= E(\chi_i) + h\int_{t_i}^{t_i+\delta} |\gamma_i(t)|_\chi^\alpha \gamma_i'(t)\cdot\frac{d}{dt}(S\circ\gamma_i)(t)\,dt + O(h^2) \\
&= E(\chi_i) + h\varepsilon_i^\alpha \vec{\nu_i}\left(S(\gamma_i(t_i+\delta)) - S(\gamma_i(t_i))\right) \\
&\quad + h\int_{t_i}^{t_i+\delta} \left(|\gamma_i(t)|_\chi^\alpha\gamma_i'(t) - \varepsilon_i^\alpha\vec{\nu_i}\right)\cdot\frac{d}{dt}(S\circ\gamma_i)(t)\,dt + O(h^2).
\end{aligned}$$

Thanks to Theorem 3.1, the function γ_i' has bounded variation on $[t_i, t_i+\delta]$ estimated by $CQ_i(\delta)$. Moreover, the function $t \mapsto |\gamma_i|_\chi^\alpha$ has bounded variation and its variation is bounded by $Q_i\delta$. Hence, by using also $S(\gamma_i(t_i+\delta)) = 0$ and $S(\gamma_i(t_i)) = \phi(t_i)$,

$$E(\chi_i') \le E(\chi_i) - h\varepsilon_i^\alpha\vec{\nu_i}\cdot\phi(t_i) + O(h^2) + hCQ_i(\delta)C_\phi,$$

where the vectors $\vec{\nu_i}$ are the outwards tangent vectors of the branches γ_i (i.e. $\vec{\nu_i} = \gamma_i'(t_i)$). As far as χ_0' is concerned the computations are similar and we get

$$\begin{aligned}
E(\chi_0') &= \int_{t_0^-}^{t_0^+} |\gamma(t)|_\chi^\alpha\left|\gamma'(t) + h\frac{d}{dt}(S\circ\gamma)(t)\right| dt \\
&= E(\chi_0) + h\int_{t_0^-}^{t_0^+} |\gamma_i(t)|_\chi^\alpha\gamma_i'(t)\cdot\phi'(t)\,dt + O(h^2).
\end{aligned}$$

By putting all the estimates together and using the optimality of χ we get

$$h\int_{t_0^-}^{t_0^+} |\gamma_i(t)|_\chi^\alpha\gamma_i'(t)\cdot\phi'(t)\,dt - h\sum_{i=1}^K \varepsilon_i^\alpha\vec{\nu_i}.\phi(t_i) + hCQ_i(\delta)C_\phi + O(h^2) + hC_\phi\left(\sum_{i=1}^K Q_i(\delta) + \sum_{i=K+1}^\infty \varepsilon_i^\alpha\right) \ge 0.$$

We first divide by h and let $h \to 0^+$, thus getting

$$\int_{t_0^-}^{t_0^+} |\gamma_i(t)|_\chi^\alpha\gamma_i'(t)\cdot\phi'(t)\,dt - \sum_{i=1}^K \varepsilon_i^\alpha\vec{\nu_i}.\phi(t_i) + CQ_i(\delta)C_\phi + C_\phi\left(\sum_{i=1}^K Q_i(\delta) + \sum_{i=K+1}^\infty \varepsilon_i^\alpha\right) \ge 0.$$

Now we let $\delta \to 0^+$ and we use $Q_i(\delta) \to 0$, which is a consequence of the fact that the total sum of the masses $\varepsilon_{i,j}^\alpha$ on the curves γ_i is finite. We get

$$\int_{t_0^-}^{t_0^+} |\gamma_i(t)|_\chi^\alpha\gamma_i'(t)\cdot\phi'(t)\,dt - \sum_{i=1}^K \varepsilon_i^\alpha\vec{\nu_i}.\phi(t_i) + C_\phi\sum_{i=K+1}^\infty \varepsilon_i^\alpha \ge 0.$$

Finally, we let $K \to \infty$ and we obtain, thanks to the fact that $\sum_{i=1}^\infty \varepsilon_i^\alpha < +\infty$ (Lemma 3.5),

$$\int_{t_0^-}^{t_0^+} |\gamma_i(t)|_\chi^\alpha\gamma_i'(t)\cdot\phi'(t)\,dt - \sum_{i=1}^\infty \varepsilon_i^\alpha\vec{\nu_i}\phi(t_i) \ge 0.$$

By replacing ϕ with $-\phi$ we get the equality, which is the desired weak version of the differential equation. $\square$

5 Towards a counter-example

The goal of this section is to provide a counter-example to the regularity result of Theorem 3.1, when the assumptions on μ^- are weakened. These assumptions were essentially an upper and a lower bound on the density, and in the counterexample we will get rid of both of these assumptions. We will consider a purely atomic measure μ^- (with, obviously, infinitely many atoms), thus dealing with a measure whose density is either zero or infinite. We will build such a measure in order to prove that the corresponding optimal traffic plan can have oscillating fibers. More precisely we will construct a traffic plan made of a single fiber which has no half-tangent at one of its interior points and whose tangent vector is not BV. Explicit minimizers of the branched transport energy are known only in trivial cases. Indeed, there are no easy sufficient optimality conditions, due to the lack of convexity of the problem. Hence, building our counter-example will require *ad hoc* geometric lemmas controlling the behavior of a minimizer.

Here we start with the first of these lemmas, which is actually natural and interesting in itself. This is the reason why we state it in a more powerful version with respect to what we really need in the sequel.

Lemma 5.1. *Suppose two measures μ^+, $\mu^- \in \mathcal{P}(\mathbb{R}^N)$ are concentrated on $\{0\} \times B(0,\varepsilon)$ and $\{1\} \times B(0,\varepsilon)$, respectively (we identify $\mathbb{R}^N$ as the product $\mathbb{R} \times \mathbb{R}^{N-1}$.) Let χ be an optimal traffic plan between them. Then there exist ε_0 and γ such that for every $\varepsilon \leq \varepsilon_0$ the traffic plan χ is composed by a single segment in the region $[\varepsilon^\gamma, 1-\varepsilon^\gamma] \times \mathbb{R}^{N-1}$ (γ can be chosen as any exponent smaller than $\frac{1-\alpha}{2-\alpha}$).*

Proof. First notice that a possible traffic plan between μ^+ and μ^- is the one where the measures collect their masses at $(0,0)$ and $(1,0)$, respectively, and then are linked by a straight line segment, with a total cost of $1 + C\varepsilon$ (Proposition 2.9). Thus $E^\alpha(\chi) \leq 1 + C\varepsilon$. For almost every x, the hyperplane $\{x\} \times \mathbb{R}^{N-1}$ crosses the traffic plan at a countable set of points $y_i(x)$, $i \in I(x)$. We call $m_i(x), i \in I(x)$ the flow of χ at these points $y_i(x)$ and denote by $\sum_{i\in I(x)} m_i(x)\delta_{y_i}$ the associated atomic measure for every $x \in]0,1[$. In what follows $m_i(x), i \in I(x) \subset \mathbb{N}$ are put in non-increasing order. Let us set $A(x) =: \sum_{i \in I(x)} m_i(x)^\alpha$ and notice that $A(x) \geq 1$. We have

$$\int_0^1 A(x)dx \leq E^\alpha(\chi) \leq 1 + C\varepsilon$$

and therefore, using $A(x) \geq 1$,

$$\int_0^{\varepsilon^\gamma} A(x) \leq 1 + C\varepsilon - (1-\varepsilon^\gamma) = C\varepsilon + \varepsilon^\gamma.$$

This implies that there exists $x_0 \in [0, \varepsilon^\gamma]$ such that $1 \leq A(x_0) \leq 1 + C\varepsilon^{1-\gamma}$. Thus

$$m_1(x_0)^{\alpha-1} = m_1(x_0)^{\alpha-1}\sum_i m_i(x_0) \leq \sum_i m_i(x_0)m_i(x_0)^{\alpha-1} = \sum_i m_i(x_0)^\alpha \leq 1 + C\varepsilon^{1-\gamma}.$$

It is easily deduced from this last inequality that

$$m_1(x_0) \geq \left(1 + C\varepsilon^{1-\gamma}\right)^{-\frac{1}{1-\alpha}} \geq 1 - C\varepsilon^{1-\gamma}. \tag{16}$$

By the very same argument we can find $x_1 \in [1-\varepsilon^\gamma, 1]$ such that $m_1(x_1) \geq 1 - C\varepsilon^{1-\gamma}$. By the single path property (Proposition 2.20) there is therefore a fiber γ in χ joining x_0 to x_1 whose flow exceeds $1 - 2C\varepsilon^{1-\gamma}$. In particular, for every $x \in [\varepsilon^\gamma, 1-\varepsilon^\gamma]$ one has $m_1(x) \geq 1 - 2C\varepsilon^{1-\gamma}$.

Let us now consider all fibers that do not meet the big fiber γ for $x \in [0, 2\varepsilon^\gamma]$. Let us call μ the total flow of these fibers. In order to show that such fibers cannot actually exist for ε small enough, we shall build a competitor to χ. We stop all of these thin fibers as they hit the hyperplane $\{\varepsilon^\gamma\} \times \mathbb{R}^{N-1}$. This yields an atomic measure $\mu = \mu(\varepsilon^\gamma)$. In the competitor χ_L, this measure is connected by an optimal traffic plan η_L to the point $y_1(\varepsilon^\gamma)$ on the big fiber. Then, these fibers go up to $x = 1$ and are sent to their original destination by a traffic plan η'_L contained in the hyperplane $\{1\} \times \mathbb{R}^{N-1}$. By the convex hull property (Lemma 2.17) the maximal distance of points of $\mu(\varepsilon^\gamma)$ to $y_1(\varepsilon^\gamma)$ is less than 2ε. Thus the overall energy of η_L and η'_L is less that $C\varepsilon|\mu|^\alpha$ by Proposition 2.9. Since the process also adds a flow less or equal to μ to γ, the energy increase due to this addition is less than $((m_1(x) + |\mu|)^\alpha - m_1(x)^\alpha)\,\mathrm{length}(\gamma)$.

The energy saving due to the removal of the thin fibers between ε^γ and $2\varepsilon^\gamma$ is at least $\varepsilon^\gamma|\mu|^\alpha$. In summary, the fact that $E(\chi) \geq E(\chi_L)$ implies

$$\varepsilon^\gamma |\mu|^\alpha \le C\varepsilon|\mu|^\alpha + ((m_1(x) + |\mu|)^\alpha - m_1(x)^\alpha)\,\mathrm{length}(\gamma).$$

By the concavity of $s \to s^\alpha$, we have $(m_1(x) + |\mu|)^\alpha - m_1(x)^\alpha \le m_1(x)^\alpha + \alpha|\mu|m_1(x)^{\alpha-1}$. In addition, the length of γ is clearly less than a constant, say 2, for ε small enough. Thus, since $m_1(x) \ge \frac{1}{2}$ for ε small enough, we obtain from the above inequalities

$$\varepsilon^\gamma |\mu|^\alpha \le C(\varepsilon|\mu|^\alpha + |\mu|) \tag{17}$$

for ε small enough. In this case we may also use $C\varepsilon \le \varepsilon^\gamma/2$ and get

$$\varepsilon^\gamma |\mu|^\alpha \le 2C|\mu|,$$

which implies, if $|\mu \neq 0$,

$$\varepsilon^\gamma \le C|\mu|^{1-\alpha} \le C\varepsilon^{(1-\gamma)(1-\alpha)}.$$

This is not verified, for small ε, if $\gamma < (1-\alpha)/(2-\alpha)$. □

Remark 5.2. The above value of γ is not sharp, as we expect the above result to be true for $\gamma = 1$. This stronger result is presented in the following theorem, even if we won't need it in the sequel.

Theorem 5.3. *Under the same assumptions as in Lemma 5.1, there is a constant L such that the traffic plan χ is composed by a single segment in the region $[L\varepsilon, 1 - L\varepsilon] \times \mathbb{R}^{N-1}$. (See Figure 5).*

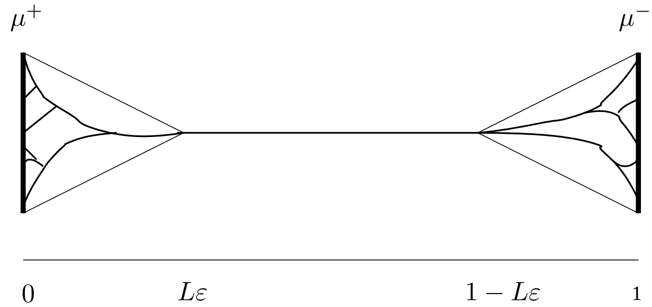

Figure 5: Illustration of the result of Theorem 5.3. The main part of the traffic plan between two measures at long distance from each other is a long segment which branches near the source and the destination, at a distance from the source and destination proportional to their diameters.

Proof. It is sufficient to prove the result for small values of ε: from Lemma 5.1 we know that in such a case we can find a long segment in the middle of the traffic plan. Following such a segment towards μ^+, call x^+ its last point, i.e. the first branching point we meet in that direction. The traffic plan between μ^+ and x^+ must satisfy the convex hull property and be contained in a cone whose base and vertex are $\{0\} \times B(0,\varepsilon)$ and x^+. Moreover, since a branching at x^+ occurs, it contains two directions at x^+ whose angle is at least a minimal angle depending on α (Lemma 2.30). This implies that x^+ belongs to the set of points such that the cone to $\{0\} \times B(0,\varepsilon)$ has a certain minimal amplitude. This set is the union of two balls and its diameter is proportional to ε. The proportionality constant L depends consequently on the value of α. □

The rest of the paper will be devoted to the following situation:

$$\mu^+ = \delta_{x_0}, \quad \mu^- = \left(\frac{1}{2} - \varepsilon\right)\delta_{x_1} + \left(\frac{1}{2} - \varepsilon\right)\delta_{x_2} + \nu, \tag{18}$$

where x_0, x_1 and x_2 are three aligned points, in that order, ν is a positive measure with total mass 2ε concentrated on the half cone $T_\theta(x_1)$ with vertex x_1, 2θ angle, and axis the half line with direction $\overrightarrow{x_1x_0}$. We shall set $e_1 = \frac{\overrightarrow{x_1x_0}}{|\overrightarrow{x_1x_0}|}$. In this situation we will always consider an optimal traffic plan χ between μ^+ and μ^-. Denote by γ_1 and γ_2 the two fibers of χ irrigating x_1 and x_2, respectively, and by γ_0 the common part of these two curves. By the strict single path property (Proposition 2.20) these curves are unique. We want to prove that if θ and ε are small enough these curves actually part at x_1, i.e. they stay together up to this point and then one of them goes on up to x_2. In the sequel, for the sake of simplicity, we will take $x_1 = 0$.

Let us consider a sequence $(\theta_h, \varepsilon_h)$ going to $(0,0)$, a sequence of measures ν_h as above, such that $|\nu_h| = 2\varepsilon_h$ and the corresponding optimal traffic plans χ_h. We fix a number $K > 1$ and, if the parting point y_h of γ_1 and γ_2 is not x_1, we set $R_h = |y_h|$. We will call y_h^0 the last point of γ^0 out of $B(y_h, KR_h)$ and y_h^2 the last point of γ_2 in $B(y_h, KR_h)$. Both of these points belong to $\partial B(y_h, KR_h)$. We will denote by r any function of one or more variables which goes to zero as its arguments go to zero.

Lemma 5.4. *For θ_h and ε_h sufficiently small, either $y_h = x_1$, or we are in the following situation (see Figure 6):*

- *the direction of $y_h^0 - y_h$ equals $e_1 + r(\theta_h, K^{-1})$;*
- *both the angles between $-e_1$ and the directions of $x_1 - y_h$ and $y_h^2 - y_h$ equal $\theta_\alpha + r(\theta_h, K^{-1}, \varepsilon_h)$ where θ_α is defined in Lemma 2.29;*
- *the four points y_h^0, y_h^2, $x_1 = 0$ and y_h are approximatively on the same plane (up to distances of the order of $R_h r(\varepsilon_h)$);*
- *in the whole ball $B(y_h, KR_h)$ the curves γ^0, γ^1 and γ^2 are very close, with respect to the Hausdorff distance, to the corresponding segments $y_h^0 y_h$, $y_h x_1$ and $y_h y_h^2$ (up to a distance of the order of $R_h r(\varepsilon_h)$).*
- *As a consequence of the above four properties the intersection point $\bar{y}_h$ of γ_h^2 with the plane Π orthogonal to e_1 and passing through x_1 satisfies $|\bar{y}_h - x_1| = 2R_h \sin(\theta_\alpha)(1 + r(\theta_h, K^{-1}, \varepsilon_h))$.*

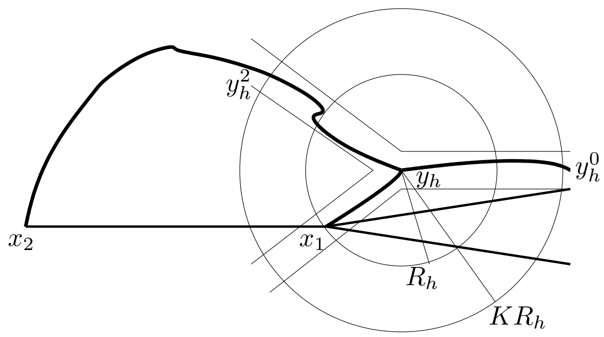

Figure 6: Illustration of Lemma 5.4: The traffic plan follows the configuration of an optimal tripode in the ball $B(y_h, KR_h)$

Proof. Let us take a subsequence such that $y_h \neq x_1$. We shall pass to the limit in an adequate blow up. Consider the restriction of the traffic plan χ_h to the ball $B(y_h, Kr_h)$ and compose it with the map $T_h : B(y_h, Kr_h) \to B(0, K)$ given by $T_h(y) = R_h^{-1}(y - y_h)$. This yields a blow up χ'_h of the traffic plan χ_h which is optimal from a measure μ_h^+ to a measure μ_h^-. The starting measure μ_h^+ contains a Dirac mass at the point $T_h(y_h^0) \in \partial B(0, K)$ with mass larger or equal to $1 - 2\varepsilon_h$, while the arrival measure μ_h^- contains two Dirac masses, one at $T_h(x_1) \in \partial B(0,1)$ and one at $T_h(y_h^2) \in \partial B(0, K)$, both with mass larger or equal than $1/2 - \varepsilon_h$. Moreover, χ'_h has the property that its two main branches part at $0 = T_h(y_h)$.

Up to subsequences, we get by Theorem 2.8 a limit traffic plan χ'_∞ which is optimal between the two measures $\mu_\infty^+ = \delta_{y^0}$ and $\mu_\infty^- = (1/2)\delta_{y^1} + (1/2)\delta_{y^2}$, with $y^0 = \lim_h T_h(y_h^0) \in \partial B(0, K)$, $y^1 = \lim_h T_h(x_1) \in \partial B(0, 1)$ and $y^2 = \lim_h T_h(y_h^2) \in \partial B(0, K)$. By Lemma 2.29 the limit configuration, being optimal is such that there is one branch arriving from y^1 to 0 which is then divided into two branches with half the mass, directed towards y^1 and y^2 respectively. The angle $y^1 0 y^2$ is equal to $2\theta_\alpha$. This fixes the relative configuration of y^0, y^1, y^2 and 0. Up to now we only used $\varepsilon_h \to 0$. By using $\theta_h \to 0$ as well, we will get information on the position of y^0 too.

The optimality of χ_h implies that the point y_h^0 must belong to the convex hull of $T_{\theta_h} \cup \{y_h\}$. This may be expressed by $y_h^0 = \lambda y_h + (1-\lambda) z_h$ with $z_h \in T_{\theta_h}$ and $\lambda \in [0,1]$. Since we have $z_h = |z_h|(e_1 + r(\theta_h))$, where e_1 is the direction of the symmetry axis of the cone (this means: the direction of a vector in the cone does not differ too much from the direction of e_1), we may rewrite $y_h^0 - y_h = (\lambda - 1)y_h + t(e_1 + r_{\theta_h})$, with $\lambda \in [0,1]$ and $t \geq 0$. Then we use $|y_h^0 - y_h| = K|y_h|$ and we divide by $|y_h^0 - y_h|$, obtaining

$$\frac{y_h^0 - y_h}{|y_h^0 - y_h|} = \frac{(\lambda - 1)y_h}{K|y_h|} + t'(e_1 + r(\theta_h)).$$

Taking the norms of the vectors, it is easy to get

$$(1 - \frac{1}{K})(1 + |r(\theta_h)|)^{-1} \leq t' \leq (1 + \frac{1}{K})(1 - |r(\theta_h)|)^{-1},$$

which implies that $t' = 1 + r(\theta_h, K^{-1})$. Thus,

$$\frac{y_h^0 - y_h}{|y_h^0 - y_h|} - e_1 = \frac{(\lambda - 1)y_h}{K|y_h|} + (t' - 1)(e_1 + r(\theta_h)) + r(\theta_h) = r(K^{-1}) + r(\theta_h, K^{-1}) + r(\theta_h).$$

This proves the first item of the Lemma. Using the uniform convergence of the blow up and the information on the limit configuration given Lemma 2.29 yields the other four statements. ☐

Theorem 5.5. *Let us consider μ^+ and μ^- as in* (18)*. If the dimension N is 2 and if θ and ε are small enough then, in any optimal traffic plan χ irrigating the measures μ and ν in* (18)*, the two main branches γ_1 and γ_2 actually part at x_1 and the fiber from x_1 to x_2 is a straight segment (see Figure 8).*

Proof. We shall use Lemma 5.4 to get information on the configuration, should the thesis be false, and then get a contradiction. Consider the curve γ_h^2 in its part between $\bar{y}_h$ and x_2. We want to find a contradiction as a consequence of the fact that the curve, as it follows quite closely the segment $y_h y_h^2$ in the whole $B(y_h, KR_h)$, gets too far from the segment x_1x_2. To do this, let us set $v_h = \bar{y}_h|\bar{y}_h|^{-1}$. Such a unit vector is orthogonal to the vector e_1 as it belongs to the plane Π.

Now consider a point of γ_h^2 which maximizes the scalar product $x \cdot v_h$ in the region we are considering. Let us notice that this maximal scalar product is neither realized by the point x_2, which gives $x_2 \cdot v_h = 0$, nor by $\bar{y}_h$, which gives $\bar{y}_h \cdot v_h = |\bar{y}_h| = R_h(2\sin(\theta_\alpha) + r(\theta_h, \varepsilon_h))$, because there is the point y_h^2 which realizes $y_h^2 \cdot v_h = R_h((1+K)\sin(\theta_\alpha) + r(\theta_h, \varepsilon_h))$. In particular, the maximum is strictly positive, and it is realized at an interior point x of the curve (or possibly a segment: in this case, just take one of the extremal points of the segment as the point x). Since the segment realizing the maximum cannot last for the whole length of γ_2, we must have a change in the direction of γ_2 at the point x, and hence a branching. Let us suppose for a while that there is only one fiber exiting x other than γ_2. Since the mass of the departing fiber is smaller than $2\varepsilon_h$ (because we are dealing with a fiber which is neither going to x_1 nor to x_2), we deduce from Lemma 5.6 that the two directions of γ_2 before and after x are very close to each other (and hence almost belong to a plane orthogonal to v_h) and that the direction w of the departing fiber is almost orthogonal to them. Since we are in dimension two (this is the key point where we use it), being orthogonal to something orthogonal to v_h means being parallel to v_h. Hence the unit vector w is either close to v_h or to $-v_h$, but we may conclude that we have $w = v_h + r(\varepsilon_h)$ because otherwise the three branches of γ_2 meeting at x would point all on the same side of the plane orthogonal to v_h passing through x (by maximality of x) and this would contradict the angle conditions of Lemma 2.29.

On the other hand, we know, by the convex hull property (Lemma 2.17) applied to the irrigation from x to T_{θ_h} (just take the resctriction of the traffic plan χ_h to this third branch at x), that w must point in the direction of the convex hull of $\{x\} \cup T_{\theta_h}$ (see Figure 7), which implies

$$w = \lambda(-x) + \mu(e_1 + r(\theta_h))$$

for some positive coefficients $\lambda, \mu \geq 0$. If we take the scalar product of this relation with the vector $v_h - e_1$ we get

$$(v_h + r(\varepsilon_h)) \cdot (v_h - e_1) = -\lambda(x \cdot v_h) + \lambda(x \cdot e_1) + \mu(e_1 + r(\theta_h)) \cdot (v_h - e_1)$$

which gives

$$0 < 1 + r(\varepsilon_h) = -\lambda(x \cdot v_h) + \lambda(x \cdot e_1) + \mu(-1 + r(\theta_h)) \leq 0,$$

where we used $(x \cdot v_h) > 0$ (because of maximality) and $(x \cdot e_1) < 0$ (because x belongs to the half space delimited by Π which includes x_2). This last fact is true because the curve γ_2 enters such an half space, and cannot come back afterwards: in this case it should in fact cross the plane Π once more, to come back in

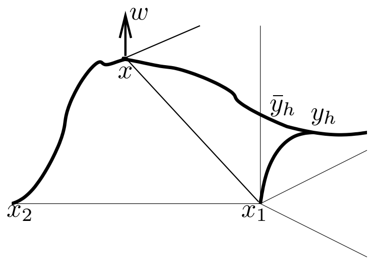

Figure 7: Illustration of the proof of Theorem 5.5. The branch stemming from the point maximizing $x \cdot v_h$ cannot be included in the convex hull of x and the cone T_{θ_h}.

the direction of x_2, at a point y; considering the irrigation from x_0 to $T_{\theta_h} \cup \{y\}$ would give a contradiction to the convex hull principle. In the end we got a contradiction, and we proved the thesis.

In the trickier case where more than one fiber exits x, we can use Lemma 5.6 and replace w by the vector we get in the statement of the Lemma. In this way we get a unit vector which shares the property of being almost orthogonal to the directions of γ_2 (and hence almost parallel to v_h, the direction being that of v_h and not of $-v_h$ as an easy consequence of the explicit formula (19) and of belonging to the convex hull of $\{x\} \cup T_{\theta_h}$ (because it is a sum of directions which only irrigate the mass in T_{θ_h}). The argument then proceeds analogously.

It remains to be proved the last part of the statement only, namely that the fiber to x_2 follows a straight line path with no branching between x_1 and x_2. To prove it the strategy is very similar to what we did before. Take a point x on such a fiber which maximizes the scalar product $x \cdot e_2$, being e_2 a unit vector orthogonal to e_1 and suppose this maximum is positive (if for both possible directions of e_2 the maximum is 0, than the curve is straight). If it is positive, it is not realized neither by x_1 nor by x_2. At the point (or at the terminal point of the segment) realizing the maximum we have a branching point. We can find a convex combination w of the directions of the fibers branching from x (thanks to Lemma 5.6) which is almost (up to $r(\varepsilon)$) in the direction of e_2. This is easily in contradiction with the fact that such a vector must be of the form $-\lambda x - \mu(e_1 + r(\theta_h))$, which follows from the convex hull principle.

This proves that the main fiber goes straight from x_1 to x_2. Analogously, if we suppose anyway the existence of a branching point in the interior of such a fiber, we get the same contradiction (an average branching direction orthogonal to e_1, which contradicts the convex hull principle). □

Lemma 5.6. *Let x be a branching point of an optimal traffic plan χ at which: a) a main fiber arrives with direction v_1 and mass $m + \varepsilon$ and leaves with direction $v_2 + r(\varepsilon)$ and mass m, b) some minor fibers with mass ε_i and directions v_i leave ($i \geq 3$ and $\sum_{i\geq 3} \varepsilon_i = \varepsilon$). Then there exists a unit vector w which is almost orthogonal to v_1 and v_2 (in the sense $w \cdot v_i = r(\varepsilon)$) and which is a linear combination with positive coefficients of the vectors $-v_1$ and $v_3, v_4 \dots$.*

Proof. The angle optimality condition of Lemma 2.28 reads

$$v_1(m+\varepsilon)^\alpha = v_2 m^\alpha + \sum_{i\geq 3} v_i \varepsilon_i^\alpha,$$

which we may rewrite as

$$v_1 - v_2 = -\frac{(m+\varepsilon)^\alpha - m^\alpha}{m^\alpha} v_1 + \sum_{i\geq 3} v_i \frac{\varepsilon_i^\alpha}{m^\alpha}. \tag{19}$$

Then take $\tilde{w}$ equal to the right hand side of (19) and $w = \tilde{w}/|\tilde{w}|$. Since $\tilde{w} = v_1 - v_2$ and $(v_1 - v_2)\cdot(v_1 + v_2) = |v_1|^2 - |v_2|^2 = 0$ we know that w is orthogonal to $v_1 + v_2$. For small ε, the right hand side of (19) is small (notice that the number of addends is uniformly bounded by a constant $N(\alpha, d)$. Thus we can estimate the whole right hand side by $\alpha m^{-1}\varepsilon + N(\alpha, d) m^{-\alpha}\varepsilon^\alpha$). This implies that the directions of v_1, v_2 and $v_1 + v_2$ are the same up to a difference of $r(\varepsilon)$. □

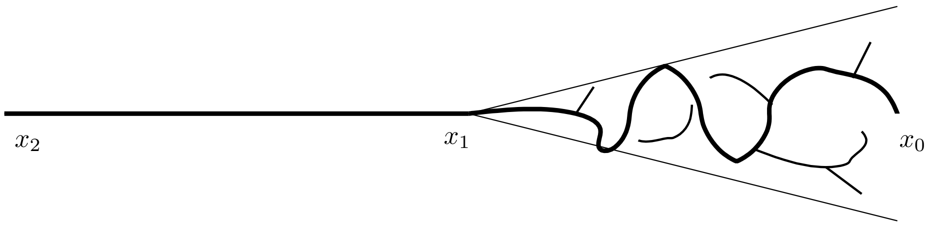

Figure 8: The final result of Theorem 5.5: the main fiber passes through x_1 and then goes straight with no branching

6 A traffic plan with an oscillatory path

In this section we shall always take ε and θ small enough to ensure that the conclusions of Theorem 5.5 holds. This implies that the arc x_2 is a straight line between x_1 and x_2.

Our counterexample will be be the following:

$$\mu^+ = \delta_{x_0},\ \mu^- = (1/2-\varepsilon)\delta_{x_1} + (1/2-\varepsilon)\delta_{x_2} + \sum_{i=1}^{\infty} \varepsilon_i \delta_{z_i},$$

where $\sum_{i=1}^{\infty} \varepsilon_i = 2\varepsilon$ and the points z_i belong to the cone $T =: T_\theta(x_1, e_1)$ and accumulate near x_1, so that the optimal traffic plan from μ to ν has a part contained in T which arrives up to x_1, and then goes straight up to x_2, according to the preceding lemmas. Moreover, we will choose the points z_i so that the traffic plan χ will be forced to follow those points (χ will be consequently composed by the straight line segments $z_i z_{i+1}$, which will converge to x_1, and by the segment $x_1 x_2$). It will be possible to choose the points satisfying the additional criterion that they oscillate from one side of T to the other, thus having as a consequence that the tangent of the traffic plan at x_1 does not exist.

For every $i \geq 2$, let us set $A_i = \{z_j\ :\ j > i\} \cup \{x_1\}$. We will call main fiber of χ the one which, by Theorem 5.5 arrives up to x_1 and then proceeds to x_2.

Lemma 6.1. *Suppose that $A_i \subset B(x_0, r)$, $|x_1 - z_i| = Ar$, $|x_1 - z_{i-1}| = A^2 r$ and that both A_i and z_i are contained in a cone with vertex z_{i-1} and angle θ_i satisfying $\theta_i \leq c\varepsilon_i^\alpha$. Suppose in addition that the main fiber of χ passes through z_{i-1}. Then, if A is sufficiently large and c and ε_i sufficiently small, the main fiber passes through z_i as well.*

Proof. In this proof, we refer to figure 9. If we cut the irrigation at a line passing through z_i and orthogonal to $z_{i-1}z_i$, thanks to the fact that the angle is small, we get that the starting measure on this line and the arrival measure on A_i have small diameter with respect to their mutual distance, and we can apply Lemma 5.1. This turns the situation into a three-point irrigation where the starting point is z_{i-1} (with mass between $1-2\varepsilon$ and 1), and the two target points are z_i (with mass ε_i) and a point $\tilde{z}_i$ with mass equal to the mass of z_{i-1} minus ε_i (i.e. almost one), which lies on a segment orthogonal to $z_{i-1}z_i$, at a small distance from x_1(small with respect to Ar). The angle between $z_{i-1}z_i$ and $z_i\tilde{z}_i$ may be estimated by $c(A)\theta_i$, since the distance $|z_i - \tilde{z}_i|$ is comparable (up to a factor depending on A) to the distance $|z_{i-1} - x_1|$ and all the points are included in the small cone of amplitude θ_i. By assumption this angle $c(A)\theta_i$ is for c small enough smaller than the minimal angle to have branching. Indeed by Lemma 5.6 this angle is $O(\varepsilon_i^a lpha)$. Thus in the three-points z_{i-}, z_i, $\tilde{z}_i$ configuration the optimal shape has no branching, which means that the main fiber of the traffic plan passes through z_i. ☐

Lemma 6.2. *Let us make the following choices, according to our previous notations (and complex notations for points in the plane): θ is an angle sufficiently small; $f : \mathbb{R} \to [-\theta/2, \theta/2]$ is a $1-$Lipschitz periodic function such as $f(t) = \theta/2 \sin t$; $x_0 = 1$, $x_1 = 0$, $x_2 = -1$; $z_n = A^{-n} e^{if(n^\gamma)}$; $\alpha, \gamma > 0$ and $\alpha + \gamma < 1$; $\varepsilon_n = cn^{(\gamma-1)/\alpha}$. Suppose moreover that A is large enough and c small enough. Under these assumptions there is only one optimal traffic plan from $\mu^+ = \delta_{x_0}$ to $\mu^- = (1/2\varepsilon)\delta_{x_1} + (1/2\varepsilon)\delta_{x_2} + \sum_{i=1}^{\infty} \varepsilon_i \delta_{z_i}$, and it is given by a single simple curve connecting x_0 to $z_1, z_2, \ldots, z_n, \ldots, x_1$ and x_2 by straight line segments. In particular, since the argument of z_n oscillates from $-\theta/2$ to $\theta/2$, there is no right hand side tangent at the point x_1.*

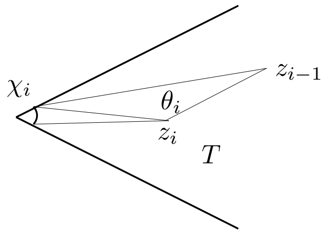

Figure 9: The angle condition for passing through z_i

Proof. Lemma 6.1 proves that the main fiber of an optimal traffic plan first passes through x_1, and then proceeds to x_2. Hence, if we can verify at each step the hypotheses of Lemma 6.1, we can get by induction that χ passes through every z_n. We only have to estimate angles. To do this it is sufficient to estimate the angles between the segments $z_{n-1}z_n$ and $z_{n-1}z_k$ for $k > n$. We will use complex notations and estimate

$$\arg\left(\frac{A^{-k}e^{if(k^\gamma)} - A^{n-1}e^{if((n-1)^\gamma)}}{A^{-n}e^{-if(n^\gamma)} - A^{n-1}e^{-if((n-1)^\gamma)}}\right).$$

We first simplify the common factor A^{n-1} and multiply by the conjugate of the denominator both the numerator and the denominator itself. We are led to consider

$$\arg\left(\left[A^{n-k-1}e^{if(k^\gamma)} - e^{if((n-1)^\gamma)}\right]\left[A^{-1}e^{if(n^\gamma)} - e^{if((n-1)^\gamma)}\right]\right).$$

Set $w = A^{n-k-2}e^{i(f(k^\gamma)-f(n^\gamma))} - A^{-1}e^{i(f((n-1)^\gamma)-f(n^\gamma))} + 1 - A^{n-k-1}e^{i(f(k^\gamma)-f((n-1)^\gamma))}$, which represents the product we have to estimate the argument of. It is easy to show $|w| \geq 1 - 3A^{-1}$, and, if A is large enough, this modulus is close to one and hence it is sufficient to estimate $\Im w$. In fact for small angles we have $\arg w \approx \Im w/|w|$. Calculating the imaginary part of w easily gives (using the fact that f is $1-$Lipschitz continuous):

$$|\Im w| \leq A^{-1}(n^\gamma - (n-1)^\gamma) + A^{n-k-1}((k^\gamma - n^\gamma)) + (k^\gamma - (n-1)^\gamma)) \leq 2A^{-1}(n^\gamma - (n-1)^\gamma) + 2A^{n-k-1}(k^\gamma - (n-1)^\gamma).$$

Lemma 6.3 gives an estimate on the last term which is independent of k and one gets using also the concavity of $n \mapsto n^\gamma$,

$$|\Im w| \leq 2A^{-1}\gamma(n-1)^{\gamma-1} + 2A^{-1}\frac{\gamma}{\log A}(n-1)^{\gamma-1}.$$

This shows that the angle θ_n which is the amplitude of the smallest cone from z_{n-1} including T_n and z_n may be estimated by $n^{\gamma-1}$. Our assumption on ε_n guarantees the inequality we need to use Lemma 6.1. $\square$

Lemma 6.3. *We have*

$$\max_{k>n} A^{n-k}(k^\gamma - n^\gamma) \leq \frac{\gamma}{\log A} n^{\gamma-1}.$$

Proof. We estimate the maximum over all $x \in [n, +\infty[$ of the function $x \mapsto A^{n-x}(x^\gamma - n^\gamma)$. This maximum exists and it is realized at an interior point because on the boundary of the domain the function tends to zero. If we call $\bar{x}$ the maximum point and we differentiate we get

$$-\log A A^{n-\bar{x}}(\bar{x}^\gamma - n^\gamma) + \gamma\bar{x}^{\gamma-1}A^{n-\bar{x}}.$$

This implies

$$\max_{k>n} A^{n-k}(k^\gamma - n^\gamma) \leq \max_{x\geq n} A^{n-x}(x^\gamma - n^\gamma) = A^{n-\bar{x}}\frac{\gamma}{\log A}\bar{x}^{\gamma-1} \leq \frac{\gamma}{\log A}n^{\gamma-1},$$

where we used the inequality $\bar{x} \geq n$ and the fact that $x \mapsto x^{\gamma-1}$ is decreasing, as $\gamma < 1$. $\square$

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