SOME REMARKS ON THE VISCOUS APPROXIMATION OF CRACK GROWTH

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Abstract. We describe an existence result for quasistatic evolutions of cracks in antiplane elasticity obtained in [16] by a vanishing viscosity approach, with free (but regular enough) crack path. We underline in particular the motivations for the choice of the class of admissible cracks and of the dissipation potential. Moreover, we extend the result to a model with applied forces depending on time.

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1. Introduction

The problem of determining the quasistatic growth of cracks in a brittle body subject to applied loads depending on time has been extensively studied in recent years (see [2] for an account on the results). Lately much attention has been focused on models that allow following the evolution of local minimizers: some results in this direction have been obtained in [23, 20, 12] under the assumption of a prescribed crack path, while in [8, 14] the crack path is determined by energy criteria.

In this work we describe a model, presented in [16], where the crack path is not a priori known. To solve the problem we use the well known method of vanishing

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viscosity, which enforces local minimization and provides also an energy equality (see e.g. [17, 18] and references therein).

We are interested in an irreversible quasistatic evolution, that is, we want to find an increasing family of cracks such that at each instant Griffith's criterion holds. Under suitable regularity assumptions on the crack, this criterion can be expressed in terms of the energy release rate, i.e., the opposite of derivative of the elastic energy with respect to the crack's elongation. Griffith's criterion states that

- the crack must increase in time,
- the energy release rate cannot exceed a threshold depending on the material's toughness,
- the crack can grow only if the energy release rate equals that threshold.

In order to establish the existence of such an evolution, we first construct a family of sets satisfying a viscously regularized version of Griffith's criterion, then we pass to the limit as the viscous parameter tends to zero. This can be done provided the class of admissible cracks is sequentially compact (with respect to a convergence of sets) and the energy release rate is (lower semi-) continuous with respect to the same convergence.

These two properties are fulfilled by a suitable class of regular cracks, with a uniform bound on the curvature; it is possible to treat the case of multiple cracks if their number is a priori bounded and they are well separated (see Definition 5.1). In this paper we underline the points where the compactness and the continuity are needed and motivate our choice of the admissible cracks.

The viscous regularization depends on the definition of a dissipation potential: we discuss some possible choices for the case of many cracks and make some comments about their mechanical interpretation in Remark 2. Moreover, in the case of many cracks it is useful to introduce some reparametrized evolutions in order to write the energy balance and to study the behaviour of the system when the evolution presents a discontinuity in time; for this point we follow an abstract construction of [18].

The existence of quasistatic evolutions is proven in [16], where we employed the method already used in [12] in the case of prescribed path. In this paper we consider the case when the evolution is driven by applied forces and imposed boundary displacements, both depending on time.

Notation. Throughout the paper, the symbol $\cdot$ denotes the scalar product in $\mathbb{R}^n$, $|\cdot|_2$ the corresponding Euclidean norm defined by $|v|_2 := \left(\sum_{h=1}^n v_h^2\right)^{\frac{1}{2}}$, and $\mathrm{dist}\,(\cdot,\cdot)$ the induced distance. We will consider also the norm $|\cdot|_1$ defined by $|v|_1 := \sum_{h=1}^n |v_h|$.

The symbol $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Fixed an open bounded subset $\Omega \subset \mathbb{R}^2$, the symbols $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$ stand respectively for the norm and the duality in $L^2(\Omega)$, $L^2(\Omega;\mathbb{R}^2)$, or $L^2(\partial\Omega;\mathcal{H}^1)$ (it will be clear from the context which of the cases we refer to).

Given a function $u \in H^1(\Omega\backslash\Gamma)$, where Γ is a closed subset of Ω, we will regard its gradient ∇u as an element of $L^2(\Omega;\mathbb{R}^2)$, by extending it to 0 on Γ (of course, this extension is not the distributional gradient of any extension of u).

Given a function $h \in BV([0,T])$, its time derivative $\mathrm{D}h$ is decomposed as

$$\mathrm{D}h = \widetilde{\mathrm{D}}h + \mathrm{D}^j h = \dot{h}\,\mathrm{d}\mathcal{L}^1 + \mathrm{D}^c h + \mathrm{D}^j h\,,$$

where $\dot{h}$ is the density of the absolutely continuous part of $\mathrm{D}h$ with respect to the Lebesgue measure and $\mathrm{D}^j h$ is concentrated in the jump set $J(h)$, which is at most countable.

2. Setting of the problem

We consider a brittle body subject to time-dependent boundary displacements and external forces, in the setting of antiplane elasticity. The horizontal section of the body is represented by a bounded connected open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary. The displacements are functions $u\colon \Omega \to \mathbb{R}$ and the cylinder $\Omega \times \mathbb{R}$ is subject to deformations of the type

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + u(x_1, x_2)) . \tag{2.1}$$

We study the evolution of a fixed number M of non-interacting cracks in Ω. Since in this work we do not address the problem of crack initiation, we assume that Ω contains an initial crack Γ_0, composed of M closed nondegenerate arcs of curve $\Gamma_0^1, \dots, \Gamma_0^M$. In order to treat the problem from the mathematical point of view, some regularity of the cracks is needed, as we will see in the sequel: we assume that each curve Γ_0^m $(m = 1, \dots, M)$ is of class $C^{1,1}$ and has no self-intersections. We suppose that $\Omega \backslash \Gamma_0$ is the finite union of Lipschitz domains (possibly overlapping): this allows us to employ the Poincaré inequality.

We do not consider cracks growing along the boundary of Ω (i.e., the imposed boundary condition will always be satisfied); so, each curve is assumed to be contained in Ω, except for at most one of the endpoints, which may belong to $\partial\Omega$. More precisely, we are interested in studying the evolution of cracks either starting from $\partial\Omega$, or well contained in Ω. The cracks considered in the latter case have two mobile tips; to simplify the notation, such a crack is represented by two curves Γ_0^m and Γ_0^h, starting from the same point and such that their union is of class $C^{1,1}$. Any two of the curves $\Gamma_0^1, \dots, \Gamma_0^M$ are either disjoint, or their intersection consists only of the initial point.

Each curve Γ_0^m is parametrized starting from the initial point, that is, the point contained in $\partial\Omega$ or in an other curve Γ_0^h $(h \neq m)$. This convention will be used for the whole paper.

Given the initial condition Γ_0, the admissible cracks are sets of the type $\Gamma = \Gamma^1 \cup \dots \cup \Gamma^M$, such that, for every m, Γ^m is a closed arc of curve of class $C^{1,1}$, without self-intersections, that extends Γ_0^m starting from the final point of the parametrization. More precisely, we have $\Gamma^m \supset \Gamma_0^m$ and $\Gamma^m \backslash \Gamma_0^m \subset\subset \Omega \backslash \bigcup_{h \neq m} \Gamma^h$. In the sequel we will highlight the properties needed for the class of admissible cracks, henceforth denoted by $\mathcal{R}^M$, and later we will give its complete definition (see Definition 5.1).

We suppose that the body has a perfectly elastic behaviour outside the cracked region and also that no force is transmitted across the cracks. We consider a linearized model, so the bulk energy associated to a displacement u is $\frac{1}{2} \|\nabla u\|_2^2$.

We fix a relatively open subset $\partial_N\Omega$ of $\partial\Omega$ with a finite number of connected components, called the Neumann part of the boundary. We define the Dirichlet part of the boundary as $\partial_D\Omega = \partial\Omega \setminus \overline{\partial_N\Omega}$, which turns out to be also a relatively open subset of $\partial\Omega$ with a finite number of connected components. We suppose that $\partial_D\Omega \neq \emptyset$.

The body is subject to prescribed boundary displacements on $\partial_D\Omega$ and to applied loads given by a system of time dependent body and surface forces. We assume that the imposed boundary displacements are given by the trace on $\partial_D\Omega$ of a function $\psi \in AC([0,T];H^1(\Omega\backslash\Gamma_0))$.

To avoid interactions between cracks and applied surface forces, we impose that all cracks remain at a positive distance from the part $\partial_S\Omega \subset\subset \partial_N\Omega$ which contains the support of the surface forces. We assume that the density of the surface forces is a function $g \in AC[0,T];L^2(\partial_S\Omega;\mathcal{H}^1))$ and that the density of the volume forces is a function $f \in AC([0,T];L^2(\Omega))$. This corresponds to the case of dead loads, in which the density of the body force per unit volume (and of the surface force per unit area) in the reference configuration does not depend on the deformation. Notice that in the setting considered here the cracks do not disconnect the domain.

Given an instant $t \in [0,T]$ and a crack $\Gamma \in \mathcal{R}^M$, the associated displacement $u(t;\Gamma)$ is the solution to the minimum problem

$$\min\left\{\tfrac{1}{2}\int_{\Omega\backslash\Gamma}|\nabla v(x)|^2\,\mathrm{d}x-\int_{\Omega\backslash\Gamma}f(t)(x)\,v(x)\,\mathrm{d}x-\int_{\partial_S\Omega}g(t)(x)\,v(x)\,\mathrm{d}\mathcal{H}^1(x)\colon\right.$$
$$\left. v\in H^1(\Omega\backslash\Gamma),\ v=\psi(t)\text{ on }\partial_D\Omega\right\}, \tag{2.2}$$

where the equality on the boundary is intended in the sense of traces. Therefore, once Γ is known, u is uniquely determined, so that the main unknown of the problem is Γ. The minimum elastic energy associated to the crack Γ, to the boundary displacement $\psi(t)$, and to the loads $f(t)$ and $g(t)$ is denoted by

$$\mathcal{E}(t;\Gamma):=\tfrac{1}{2}\int_{\Omega\backslash\Gamma}|\nabla u(t;\Gamma)|^2\,\mathrm{d}x-\int_{\Omega\backslash\Gamma}f(t)\,u(t;\Gamma)\,\mathrm{d}x-\int_{\partial_S\Omega}g(t)\,u(t;\Gamma)\,\mathrm{d}\mathcal{H}^1\,. \tag{2.3}$$

According to Griffith's theory [10], the energy spent to produce a crack is proportional to its length; we assume that the proportionality constant is 1. Hence, the total energy associated to a crack Γ at time t is

$$\mathcal{E}(t;\Gamma)+\mathcal{H}^1(\Gamma)\,.$$

In order to study this energy as the crack increases, it is useful to introduce a sort of derivative with respect to the curve Γ: this is the subject of the next section.

3. Energy release rate

The energy release rate is a fundamental quantity in Griffith's theory in order to measure the amount of energy spent during the crack's growth. We start by defining it in the case of a single curve, so for the moment $M=1$.

Given $t \in [0,T]$ and a curve $\Gamma \in \mathcal{R}^1$, we consider a curve Γ' of class $C^{1,1}$ without self-intersections, such that $\Gamma' \supsetneq \Gamma$ and $\Gamma'\backslash\Gamma \subset\subset \Omega$, and we denote by $\gamma\colon [0,l_{\Gamma'}] \to \overline{\Omega}$ its arc-length parametrization (chosen with the conventions stated above). It is possible to prove that the elastic energy $\mathcal{E}(t;\gamma([0,l]))$ is a C^1-function of the crack length l. The opposite of its derivative is the energy release rate of the system in the configuration corresponding to the crack Γ and to the time t:

$$G(t;\Gamma):=-\partial_l\mathcal{E}(t;\gamma([0,l]))\big|_{l=\mathcal{H}^1(\Gamma)}\,. \tag{3.1}$$

We recall that $u(t;\Gamma)$ is the unique weak solution of the equation

$$\begin{cases} -\Delta u(t;\Gamma) = f(t) & \text{in } \Omega\backslash\Gamma\,, \\ u(t;\Gamma) = \psi(t) & \text{on } \partial_D\Omega\,, \\ \partial_\nu u(t;\Gamma) = g(t) & \text{on } \partial_S\Omega\,, \\ \partial_\nu u(t;\Gamma) = 0 & \text{on } (\partial_N\Omega\setminus\partial_S\Omega)\cup\Gamma\,, \end{cases}$$

where ν is the outer (unit) normal vector to $\Omega\setminus\Gamma$, that is, $u(t;\Gamma)\in H^1(\Omega\setminus\Gamma)$, $u(t;\Gamma)=\psi(t)$ on $\partial_D\Omega$ in the sense of traces, and

$$\int_\Omega \nabla u(t;\Gamma)\cdot\nabla w\,\mathrm{d}x = \int_\Omega f(t)\,w\,\mathrm{d}x + \int_{\partial_S\Omega} g(t)\,w\,\mathrm{d}\mathcal{H}^1\,,$$

for every $w\in H^1(\Omega\setminus\Gamma)$ with $w=0$ on $\partial_D\Omega$. It is well known that the solution is regular (H^2) in the interior of $\Omega\setminus\Gamma$ and up to the two sides of the crack, far from the crack tip. Since $f(t)\in L^2(\Omega)$, it is possible to characterize the singularity of $u(t;\Gamma)$: there exists a unique constant $K(t;\Gamma)$ such that

$$u(t;\Gamma) - K(t;\Gamma)\,S \in H^2(\Omega'\backslash\Gamma) \quad \text{for every } \Omega'\subset\subset\Omega\,,$$

where $S=\rho^{\frac12}\sin\frac{\theta}{2}$ near the crack tip; here (ρ,θ) are polar coordinates around the crack tip such that the axis $\{\theta=0\}$ is oriented as the tangent vector to the tip and the discontinuity set of θ coincides with Γ.

The energy release rate and the stress intensity factor are related by the following equality:

$$G(t;\Gamma) = \frac{\pi}{4}\,K(t;\Gamma)^2\,.$$

Therefore, $G(t;\Gamma)$ turns out to be independent of the choice of the continuation Γ' of Γ, which justifies the notation.

We will exploit a further characterization of the energy release rate in terms a volume integral depending on the deformation gradient. Since Γ is of class $C^{1,1}$, we can take a vector field V of class $C^{0,1}$ with compact support in Ω, such that on Γ we have $V(\gamma(s))=\zeta(\gamma(s))\,\dot\gamma(s)$, where $\gamma\colon[0,l_\Gamma]\to\overline{\Omega}$ is the arc-length parametrization of Γ and ζ is a cut-off function, equal to one in a neighbourhood of $\gamma(l_\Gamma)\in\Omega$. Then it is possible to prove that

$$\begin{aligned} G(t;\Gamma) = &\int_{\Omega\backslash\Gamma}\left[\frac{(\mathrm{D}_1u)^2-(\mathrm{D}_2u)^2}{2}(\mathrm{D}_1V^1-\mathrm{D}_2V^2)+\mathrm{D}_1u\,\mathrm{D}_2u\,(\mathrm{D}_2V^1+\mathrm{D}_1V^2)\right]\mathrm{d}x \\ &-\int_{\Omega\backslash\Gamma}\nabla u\cdot V\,f\,\mathrm{d}x\,, \end{aligned} \tag{3.2}$$

where $u:=u(t;\Gamma)$. The first integrand appearing in the above equation corresponds to the Eshelby or Hamilton tensor.

In the case of many cracks, for every $m=1,\dots,M$ we choose some $C^{1,1}$ extensions $\Gamma^{m\prime}$ of Γ^m as before and denote by γ^m their arc-length parametrizations (where $\gamma^m(0)$ is the initial point of the curve). Then the m-th component of the energy release rate is the partial derivative

$$G^m(t;\Gamma) := -\partial_{l^m}\mathcal{E}(t;\gamma^1([0,l^1])\cup\cdots\cup\gamma^M([0,l^M]))\big|_{l^1=\mathcal{H}^1(\Gamma^1),\dots,l^M=\mathcal{H}^1(\Gamma^M)}\,. \tag{3.3}$$

We set $G(t;\Gamma):=(G^1(t;\Gamma),\dots,G^M(t;\Gamma))$. It is straightforward to see that the above results hold also in the context of many cracks for every fixed m, up to choosing the neighbourhood Ω' and the support of V small enough aroung the tip of Γ^m.

The facts exposed in the present section are well known in the case of smooth cracks: see, e.g., [9, 11] for a straight crack and [12, 13] under strong regularity hypotheses on the crack path. The results for $C^{1,1}$ curves, necessary in the present approach, were proven in [15]: see Theorem 1.9 for the existence of the stress intensity factor, Theorem 2.1 for the derivability and the computation of the energy release rate in terms of the stress intensity factor, and Proposition 2.4 for the integral characterization.

4. Quasistatic evolution

In this section we present the strategy to prove the existence of evolutions satisfying Griffith's criterion using a viscous regularization. This technique, which enforces the local minimality, has been used in many different contexts: we refer in particular to the approaches of [18] in an abstract finite-dimensional setting and [12] for the evolution of a crack with prescribed path of class C^2. Our result allows the treatment of the problem of crack growth without prescribing a priori the path, which belongs to the class $\mathcal{R}^M$. During the exposition we shall see the properties that $\mathcal{R}^M$ should satisfy, postponing the precise definition to the beginning of the next section.

Remark 1. Let Γ be a simple curve of class $C^{1,1}$ and let γ be its arc-length parametrization, with $\gamma(0) \in \partial\Omega$ and $\gamma(s) \in \Omega$ for $s > 0$. Setting

$$\mathcal{R}^M := \{\gamma([0,s])\colon s_0 \le s \le \mathcal{H}^1(\Gamma)\}$$

in the following results, the reader can recover a theorem with prescribed crack path.

Here we only give a sketch of the proof of the existence of quasistatic evolutions, referring to [16] for the complete one. The main difference with respect to [16] is that in this work we consider a model where volume and surface forces are applied on the body; this requires in particular some modifications to a technical point, which will be discussed in the next section.

The evolution is approximated by time discretization: we fix $\{t_{n,i}\}_{0\le i\le n}$ such that

$$0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = T \quad \text{and} \quad \lim_{n\to\infty} \max_{1\le i\le n} (t_{n,i} - t_{n,i-1}) = 0\,. \tag{4.1}$$

Then we introduce a viscous parameter $\varepsilon > 0$ and define by induction a sequence of solutions to incremental minimum problems. Let $\Gamma_{\varepsilon,n,0} := \Gamma_0$; for $i \ge 1$ we set $\Gamma_{\varepsilon,n,i} = \Gamma^1_{\varepsilon,n,i} \cup \cdots \cup \Gamma^M_{\varepsilon,n,i}$ to be a solution to

$$\min_{\Gamma\in\mathcal{R}^M} \left\{ \mathcal{E}(t_{n,i};\Gamma) + \mathcal{H}^1(\Gamma\backslash\Gamma_{\varepsilon,n,i-1}) + \frac{\varepsilon}{2}\,\frac{d(\Gamma;\Gamma_{\varepsilon,n,i-1})^2}{t_{n,i}-t_{n,i-1}} \right\}, \tag{4.2}$$

where $d(\cdot,\cdot)$ is the dissipation distance defined by

$$d(\Gamma_1;\Gamma_2) := \begin{cases} \left(\sum_{m=1}^M \mathcal{H}^1(\Gamma_1^m\backslash\Gamma_2^m)^2\right)^{\frac12} & \text{if } \Gamma_1 \supset \Gamma_2\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.3}$$

for every $\Gamma_1, \Gamma_2 \in \mathcal{R}^M$ with $\Gamma_1 = \Gamma_1^1 \cup \cdots \cup \Gamma_1^M$ and $\Gamma_2 = \Gamma_2^1 \cup \cdots \cup \Gamma_2^M$. We follow [12, 18] for the choice of the exponent in (4.3); in Remark 2 below we provide some comments on other possible definitions of the dissipation distance.

In the sequel we will study the properties of the solutions to the approximate problems (4.2); then we will let n tend to $+\infty$ and ε tend to zero, finding in the limit a continuous-time quasistatic evolution. From a discrete version of Griffith's criterion we will be able to pass to a viscous criterion and later to the desired rate-independent one. The notion of convergence for this passage to the limit is the one in the sense of Hausdorff, whose definition is given below.

Definition 4.1. The Hausdorff distance between two compact subsets $\Gamma, \Gamma' \subset \overline{\Omega}$ is given by

$$d_H(\Gamma'; \Gamma) := \max\left\{\sup_{x\in\Gamma'} \operatorname{dist}(x, \Gamma), \sup_{x\in\Gamma} \operatorname{dist}(x, \Gamma')\right\}.$$

with the conventions $d_H(x; \emptyset) = \operatorname{diam}(\Omega)$ and $\sup\emptyset = 0$. A sequence Γ_n of compact subsets of $\overline{\Omega}$ converges to Γ in the Hausdorff metric if $d_H(\Gamma_n; \Gamma) \to 0$.

The existence of solutions to (4.2) is a consequence of the direct method of the calculus of variations. Of course, given a minimizing sequence $\{\Gamma^{(k)}_{\varepsilon,n,i}\}_{k\in\mathbb{N}}$ for problem (4.2), we extract a subsequence converging in $\mathcal{R}^M$. In order to find a limit in $\mathcal{R}^M$, this class will have to satisfy the following

Property 1. $\mathcal{R}^M$ is sequentially compact with respect to the Hausdorff convergence.

Hence we get a family $\{\Gamma_{\varepsilon,n,i}\}_{1\le i\le n}$, which is nondecreasing in i because of the definition of the dissipation distance. We have to prove that for every i, $\Gamma_{\varepsilon,n,i}$ is a solution to (4.2); this is guaranteed by standard semicontinuity hypotheses. Actually in our setting the following stronger properties hold, as we shall see later:

Property 2. The Hausdorff measure is continuous in $\mathcal{R}^M$ with respect to the Hausdorff convergence.

Property 3. The minimum elastic energy (2.3) is continuous in $\mathcal{R}^M$ with respect to the Hausdorff convergence.

This proves that $\Gamma_{\varepsilon,n,i}$ is a solution to (4.2), being the Hausdorff limit of a minimizing sequence. We define the piecewise constant interpolation $\Gamma_{\varepsilon,n}(t) = \Gamma^1_{\varepsilon,n}(t) \cup \cdots \cup \Gamma^M_{\varepsilon,n}(t)$, where

$$\Gamma^m_{\varepsilon,n}(0) := \Gamma^m_0, \quad \Gamma^m_{\varepsilon,n}(t) := \Gamma^m_{\varepsilon,n,i} \quad \text{for } t \in (t_{n,i-1}, t_{n,i}].$$

We introduce also some notation for the vector of the lenghts

$$l^m_{\varepsilon,n}(t) := \mathcal{H}^1(\Gamma^m_{\varepsilon,n,i}), \quad l_{\varepsilon,n}(t) := (l^1_{\varepsilon,n}(t), \ldots, l^M_{\varepsilon,n}(t)),$$

for the one of the energy release rates

$$G^m_{\varepsilon,n}(t) := G^m(t_{n,i}; \Gamma_{\varepsilon,n,i}), \quad G_{\varepsilon,n}(t) = (G^1_{\varepsilon,n}(t), \ldots, G^M_{\varepsilon,n}(t)),$$

and for the displacements

$$u_{\varepsilon,n}(t) := u(t_{n,i}; \Gamma_{\varepsilon,n,i}),$$

where $t \in (t_{n,i-1}, t_{n,i}]$. Following standard methods for the viscous approximation, besides the piecewise constant interpolation we consider an affine one: for $t \in (t_{n,i-1}, t_{n,i}]$ we set

$$\hat{l}^m_{\varepsilon,n}(t) := p^m_{\varepsilon,n}(t)(t - t_{n,i-1}) + \mathcal{H}^1(\Gamma^m_{\varepsilon,n,i-1}), \quad \hat{l}_{\varepsilon,n}(t) := (\hat{l}^1_{\varepsilon,n}(t), \ldots, \hat{l}^M_{\varepsilon,n}(t)),$$

where the slopes are defined by

$$p^m_{\varepsilon,n}(t) := \frac{\mathcal{H}^1(\Gamma^m_{\varepsilon,n,i}) - \mathcal{H}^1(\Gamma^m_{\varepsilon,n,i-1})}{t_{n,i}-t_{n,i-1}}\,, \quad p_{\varepsilon,n}(t) := (p^1_{\varepsilon,n}(t),\dots,p^M_{\varepsilon,n}(t))\,.$$

Notice that

$$|p_{\varepsilon,n}(t)|_2 = \frac{d(\Gamma_{\varepsilon,n,i};\Gamma_{\varepsilon,n,i-1})}{t_{n,i}-t_{n,i-1}}\,.$$

It is easy to prove (see [16, Proposition 4.2] for the details) that there exists a constant $C>0$ such that for every $\varepsilon>0$ and $n\in\mathbb{N}$ we have

$$\varepsilon\int_0^T |p_{\varepsilon,n}(t)|_2^2\,\mathrm{d}t \le C\,. \tag{4.4}$$

It is possible to provide a version of Griffith's criterion in the discrete setting, for each of the M components of the crack. Fix $m\le M$ and $t\in(t_{n,i-1},t_{n,i}]$. Obviously we have

$$p^m_{\varepsilon,n}(t)\ge 0\,. \tag{4.5}$$

Let $\Gamma^{m,s}_{\varepsilon,n}$ be the curve of length s contained in $\Gamma^m_{\varepsilon,n}(T)$; in (4.2) we compare $\Gamma_{\varepsilon,n,i}$ with

$$\Gamma^{(s)}_{\varepsilon,n} := \Gamma^1_{\varepsilon,n,i}\cup\dots\cup\Gamma^{m-1}_{\varepsilon,n,i}\cup\Gamma^{m,s}_{\varepsilon,n}\cup\Gamma^{m+1}_{\varepsilon,n,i}\cup\dots\cup\Gamma^M_{\varepsilon,n,i}\,.$$

We obtain

$$\mathcal{E}(t_{n,i};\Gamma_{\varepsilon,n,i}) + l^m_{\varepsilon,n,i} + \frac{\varepsilon}{2}\,\frac{(l^m_{\varepsilon,n,i}-l^m_{\varepsilon,n,i-1})^2}{t_{n,i}-t_{n,i-1}} \le \mathcal{E}(t_{n,i};\Gamma^{(s)}_{\varepsilon,n}) + s + \frac{\varepsilon}{2}\,\frac{(s-l^m_{\varepsilon,n,i-1})^2}{t_{n,i}-t_{n,i-1}}\,.$$

If $s>l^m_{\varepsilon,n,i}$, we divide by $s-l^m_{\varepsilon,n,i}$ and get

$$1+\frac{\mathcal{E}(t_{n,i};\Gamma^{(s)}_{\varepsilon,n})-\mathcal{E}(t_{n,i};\Gamma_{\varepsilon,n,i})}{s-l^m_{\varepsilon,n,i}} + \frac{\varepsilon}{2}\,\frac{s+l^m_{\varepsilon,n,i}-2\,l^m_{\varepsilon,n,i-1}}{t_{n,i}-t_{n,i-1}} \ge 0\,,$$

so, passing to the limit as $s\to l^m_{\varepsilon,n,i}$ we find

$$1-G^m_{\varepsilon,n}(t)+\varepsilon p^m_{\varepsilon,n}(t)\ge 0\,. \tag{4.6}$$

Moreover, if $p^m_{\varepsilon,n}(t)\ne 0$ we can take $s\in(l^m_{\varepsilon,n,i-1},l^m_{\varepsilon,n,i})$ and as $s\to l^m_{\varepsilon,n,i}$ we get

$$p^m_{\varepsilon,n}(t)\,[1-G^m_{\varepsilon,n}(t)+\varepsilon p^m_{\varepsilon,n}(t)] = 0\,. \tag{4.7}$$

Equations (4.5)–(4.7) are a discrete version of Griffith's criterion.

We can now pass to the limit as $n\to\infty$ for $\varepsilon>0$ fixed; for this, we need a uniform bound on the length in the class $\mathcal{R}^M$.

Property 4. The length of the curves in $\mathcal{R}^M$ is uniformly bounded.

Then, thanks to the compactness of $\mathcal{R}^M$ (Property 1) and to the monotonicity of the discrete evolutions (Helly Theorem), there is a subsequence $n_k\to\infty$, independent of t, such that for every $t\in[0,T]$ $\Gamma_{\varepsilon,n_k}(t)$ converges in the Hausdorff metric to a set $\Gamma_\varepsilon(t)=\Gamma^1_\varepsilon(t)\cup\dots\cup\Gamma^M_\varepsilon(t)$ in $\mathcal{R}^M$, nondecreasing in t. The map $t\mapsto\Gamma_\varepsilon(t)$ is called a viscous solution.

As for the length, by Property 2 we have that $l^m_{\varepsilon,n}(t)$ converge to $l^m_\varepsilon(t) := \mathcal{H}^1(\Gamma^m_\varepsilon(t))$ for every t. It is possible to prove that also $\hat{l}^m_{\varepsilon,n}$ converges to the same pointwise limit [16, Proposition 5.1]; as a consequence of (4.4), this convergence is also weak in $H^1([0,T])$. We set

$$l_\varepsilon(t) := (l^1_\varepsilon(t),\dots,l^M_\varepsilon(t)) \quad\text{and}\quad \dot{l}_\varepsilon(t) := \big(\dot{l}^1_\varepsilon(t),\dots,\dot{l}^M_\varepsilon(t)\big)\,.$$

By (4.4) we obtain

$$\varepsilon \int_0^T \left|\dot{l}_\varepsilon(t)\right|_2^2 \,\mathrm{d}t \le C \,. \tag{4.8}$$

In order to pass to the limit in the discrete Griffith's criterion (4.5)–(4.7), we need the following

Property 5. In $\mathcal{R}^M$, the energy release rates are bounded uniformly and continuous with respect to the convergence of times and the Hausdorff convergence of curves.

Hence for every t the vector of energy release rates $G_{\varepsilon,n}(t)$ converges to

$$G_\varepsilon(t) = \big(G_\varepsilon^1(t), \dots, G_\varepsilon^M(t)\big) := \big(G^1(t;\Gamma_\varepsilon(t)), \dots, G^M(t;\Gamma_\varepsilon(t))\big) \,;$$

by the uniform bound, the convergence is also strong in $L^2([0,T])$. This is sufficient to pass to the limit and find the viscous version of Griffith's criterion [16, Proposition 5.2]: for every ε, m, and t

$$\dot{l}_\varepsilon^m(t) \ge 0 \,, \tag{4.9}$$

$$1 - G_\varepsilon^m(t) + \varepsilon\, \dot{l}_\varepsilon^m(t) \ge 0 \,, \tag{4.10}$$

$$\dot{l}_\varepsilon^m(t)\, [1 - G_\varepsilon^m(t) + \varepsilon\, \dot{l}_\varepsilon^m(t)] = 0 \,. \tag{4.11}$$

Remark 2. One could consider, for any $q \ge 1$, the dissipation distance

$$d_q(\Gamma_1;\Gamma_2) := \begin{cases} \Big(\sum_{m=1}^M \mathcal{H}^1(\Gamma_1^m \setminus \Gamma_2^m)^q\Big)^{\frac{1}{q}} & \text{if } \Gamma_1 \supset \Gamma_2 \,, \\ +\infty & \text{otherwise.} \end{cases}$$

Then one could define a viscous solution using d_q in the incremental problem (4.2) (with no changes in the other exponents).

Let us see in detail the case $q = 1$ and $M = 2$. Using the same notation as in the proof of (4.6), we have for an elongation of the first component

$$\begin{aligned} &\mathcal{E}(t_{n,i};\Gamma_{\varepsilon,n,i}) + l_{\varepsilon,n,i}^1 + \frac{\varepsilon}{2}\, \frac{[(l_{\varepsilon,n,i}^1 - l_{\varepsilon,n,i-1}^1) + (l_{\varepsilon,n,i}^2 - l_{\varepsilon,n,i-1}^2)]^2}{t_{n,i} - t_{n,i-1}} \\ &\le \mathcal{E}(t_{n,i};\Gamma_{\varepsilon,n}^{(s)}) + s + \frac{\varepsilon}{2}\, \frac{[(s - l_{\varepsilon,n,i-1}^1) + (l_{\varepsilon,n,i}^2 - l_{\varepsilon,n,i-1}^2)]^2}{t_{n,i} - t_{n,i-1}} \,. \end{aligned}$$

If $s > l_{\varepsilon,n,i}^1$, dividing by $s - l_{\varepsilon,n,i}^1$ we obtain

$$1 + \frac{\mathcal{E}(t_{n,i};\Gamma_{\varepsilon,n}^{(s)}) - \mathcal{E}(t_{n,i};\Gamma_{\varepsilon,n,i})}{s - l_{\varepsilon,n,i}^1} + \frac{\varepsilon}{2}\left(\frac{s + l_{\varepsilon,n,i}^1 - 2\, l_{\varepsilon,n,i-1}^1}{t_{n,i} - t_{n,i-1}} + 2\, \frac{l_{\varepsilon,n,i}^2 - l_{\varepsilon,n,i-1}^2}{t_{n,i} - t_{n,i-1}}\right) \ge 0 \,,$$

so, passing to the limit as $s \to l_{\varepsilon,n,i}^1$ we find

$$1 - G_{\varepsilon,n}^1(t) + \varepsilon\, |p_{\varepsilon,n}(t)|_1 \ge 0 \,.$$

Arguing as in the proof of (4.7) we get also

$$p_{\varepsilon,n}^1(t)\, [1 - G_{\varepsilon,n}^1(t) + \varepsilon\, |p_{\varepsilon,n}(t)|_1] = 0 \,.$$

In the same way, we get for the second crack

$$1 - G_{\varepsilon,n}^2(t) + \varepsilon\, |p_{\varepsilon,n}(t)|_1 \ge 0 \,, \quad p_{\varepsilon,n}^2(t)\, [1 - G_{\varepsilon,n}^2(t) + \varepsilon\, |p_{\varepsilon,n}(t)|_1] = 0 \,.$$

As $n \to +\infty$ we find the viscous laws

$$1 - G_\varepsilon^1(t) + \varepsilon\big|\dot{l}_\varepsilon(t)\big|_1 \ge 0 \,, \quad \dot{l}_\varepsilon^1(t)\, [1 - G_\varepsilon^1(t) + \varepsilon\big|\dot{l}_\varepsilon(t)\big|_1] = 0$$

and

$$1-G^2_\varepsilon(t)+\varepsilon\big|\dot l_\varepsilon(t)\big|_1\geq 0\,,\quad \dot l^2_\varepsilon(t)\,[1-G^2_\varepsilon(t)+\varepsilon\big|\dot l_\varepsilon(t)\big|_1]=0\,.$$

In particular, for every fixed t, only two cases are possible: at most one between $\dot l^1_\varepsilon(t)$ and $\dot l^2_\varepsilon(t)$ is different from zero, i.e., only one of the cracks can move; or, if they both evolve, $G^1_\varepsilon(t)$ has to be equal to $G^2_\varepsilon(t)$.

Hence, the choice $q=1$ in the dissipation distance leads to a strong interdependence of the different crack tips (even when they are very far from each other and for any choice of the loads); this does not seem natural from the mechanical point of view. It is possible to see that for $q\neq 2$ the term $d_q(\Gamma^{(s)}_{\varepsilon,n};\Gamma_{\varepsilon,n,i-1})^2 - d_q(\Gamma_{\varepsilon,n,i};\Gamma_{\varepsilon,n,i-1})^2$ depends on all the components of the curves, even if only one of them is elongated. Therefore, $q=2$ is the only case where the viscous term in Griffith's equation for one component is not influenced by the other components.

In the limit case $q=+\infty$ one sets

$$d_\infty(\Gamma_1;\Gamma_2):=\begin{cases}\max\limits_{m=1,\dots,M}\mathcal{H}^1(\Gamma_1^m\backslash\Gamma_2^m) & \text{if } \Gamma_1\supset\Gamma_2\,,\\ +\infty & \text{otherwise.}\end{cases}$$

We can repeat the arguments above in the cases $l^1_{\varepsilon,n,i}-l^1_{\varepsilon,n,i-1}<l^2_{\varepsilon,n,i}-l^2_{\varepsilon,n,i-1}$ and $l^1_{\varepsilon,n,i}-l^1_{\varepsilon,n,i-1}>l^2_{\varepsilon,n,i}-l^2_{\varepsilon,n,i-1}$. In the former we find

$$1-G^1_{\varepsilon,n}(t)\geq 0\,,\quad p^1_{\varepsilon,n}(t)\,[1-G^1_{\varepsilon,n}(t)]=0\,.$$

in the latter we get

$$1-G^1_{\varepsilon,n}(t)+\varepsilon p^1_{\varepsilon,n}(t)\geq 0\,,\quad p^1_{\varepsilon,n}(t)\,[1-G^1_{\varepsilon,n}(t)+\varepsilon p^1_{\varepsilon,n}(t)]=0\,.$$

The case $l^1_{\varepsilon,n,i}-l^1_{\varepsilon,n,i-1}=l^2_{\varepsilon,n,i}-l^2_{\varepsilon,n,i-1}$ leads just to inequalities:

$$1-G^1_{\varepsilon,n}(t)+\varepsilon p^1_{\varepsilon,n}(t)\geq 0\,,\quad p^1_{\varepsilon,n}(t)\,[1-G^1_{\varepsilon,n}(t)]\leq 0\,.$$

Hence, when one of the cracks has speed strictly higher than the others, its evolution is governed by Griffith's usual viscous equation; on the contrary, all the other components evolve with the viscous energy release rate equal to one. This kind of interplay of different cracks does not seem natural, either.

These considerations motivate the choice of the exponent $q=2$ in the definition of the dissipation distance. For other considerations about the effects of the choice of the dissipation potential in viscous regularizations, we refer to [21, Section 2.2].

We now consider the limit as $\varepsilon\to 0$, passing from a rate-dependent context to a rate-independent one. As in the previous passage, by Properties 1 and 4 we find, up to subsequences, a limit crack $\Gamma(t)=\Gamma^1(t)\cup\dots\cup\Gamma^M(t)$ in $\mathcal{R}^M$, nondecreasing in t: the function $t\mapsto\Gamma(t)$ is said to be an approximable quasistatic evolution. The limit length vector

$$l(t)=(l^1(t),\dots,l^M(t)):=(\mathcal{H}^1(\Gamma^1(t)),\dots,\mathcal{H}^1(\Gamma^M(t)))$$

is a nondecreasing function of t and may present some jumps. Finally, the vector of energy release rates $G_\varepsilon(t)$ converges (Property 5) to

$$G(t)=(G^1(t),\dots,G^M(t)):=(G^1(t;\Gamma(t)),\dots,G^M(t;\Gamma(t)))\,.$$

Also in this case, passing to the limit in Griffith's viscous criterion, we find its usual rate-independent version in the continuity points of $l(t)$; for the details, see [16, Theorem 6.1].

Theorem 4.2 (Griffith's criterion)**.** *Let $t \mapsto \Gamma(t) = \Gamma^1(t) \cup \cdots \cup \Gamma^M(t)$ be an approximable quasistatic evolution. Then the BV function $t \mapsto l(t)$ satisfies:*

(a) $\dot{l}^m(t) \geq 0$ *for every $m = 1, \ldots, M$ and for a.e. $t \in [0, T]$;*
(b) $G^m(t) \leq 1$ *for every $m = 1, \ldots, M$ and for every $t \notin J(l^m)$;*
(c) *if $G^m(\bar{t}) < 1$ for some m and $\bar{t} \notin J(l^m)$, then l^m is locally constant around $\bar{t}$.*

Notice that (c) implies the usual form for the activation criterion

$$[G(\cdot) - \mathbf{1}] \cdot \widetilde{\mathrm{D}}l = 0\,,$$

where $\widetilde{\mathrm{D}}l := (\widetilde{\mathrm{D}}l^1, \ldots, \widetilde{\mathrm{D}}l^M)$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^M$.

5. The class of admissible cracks

In the previous section, we described the strategy to pass to the limit in the viscous incremental problems under some hypotheses on the class of admissible cracks. If $\mathcal{R}^M$ is given by Remark 1, one recovers a result of existence with prescribed path as in [12]. To obtain instead a result with free crack path, one has to construct a class $\mathcal{R}^M$ such that Properties 1–5 hold.

In particular, in order to satisfy Property 5 we will exploit the integral characterization of the energy release rate given in (3.2). That integral makes sense if the tangent to each component of the crack is Lipschitz continuous; therefore, we have to require that the curves in $\mathcal{R}^M$ are of class $C^{1,1}$.

On the other hand, the class $C^{1,1}$ is compact with respect to the Hausdorff convergence, under some constraints on the curvature; this allows us to fulfill Property 1. As under these assumptions the class C^2 is not compact, it is necessary to extend some results about the energy release rate, known in the case of C^2 cracks, to cracks of class $C^{1,1}$.

Such a generalization is not trivial and is one of the major difficulties in our result. We solve the problem thanks to a transformation of the domain that, around the tip of each curve, maps the crack into a segment and modifies the coefficients of the Laplacian in a suitable way. The details of the construction of such a diffeomorphism are contained in [15] together with the proofs of the results stated in Section 3.

The previous discussion motivates the following definition of the class $\mathcal{R}^M$.

Definition 5.1. Given $\eta > 0$, $\mathcal{R}^M$ is the class of sets $\Gamma = \Gamma^1 \cup \cdots \cup \Gamma^M$, union of closed arcs of curve of class $C^{1,1}$, such that for every m and h with $m \neq h$

(a) $\Gamma^m \supset \Gamma_0^m$, $\Gamma^m \backslash \Gamma_0^m \subset\subset \Omega$, and $\Gamma^m \cap \Gamma^h = \Gamma_0^m \cap \Gamma_0^h$;
(b) for every point $x \in \Gamma^m \backslash \Gamma_0^m$ there exist two open balls $C_1, C_2 \subset \Omega$ of radius η, such that $(C_1 \cup C_2) \cap (\Gamma^m \cup \partial\Omega) = \emptyset$ and $\overline{C}_1 \cap \overline{C}_2 = \{x\}$;
(c) for every point $x \in \Gamma^m \backslash \Gamma_0^m$ the open ball C_3 of radius 2η centred at x satisfies $C_3 \cap \Gamma^h = \emptyset$.

As for the parameter η, we take η so small that for every $m = 1, \ldots, M$ the curvature of Γ_0^m is controlled from above by $\frac{1}{\eta}$ at a.e. point and the class $\mathcal{R}^M$ is not empty. The parameter η is fixed and will never pass to the limit in this work. Under these hypotheses, it is easy to see that, for every $\Gamma \in \mathcal{R}^M$, $\mathcal{H}^1(\Gamma) \leq L$ and $\mathrm{dist}\,(\Gamma \backslash \Gamma_0, \partial\Omega) \geq D$, where $L, D > 0$ depend only on η, Ω, and Γ_0. In particular, Property 4 holds.

Property 1 (the sequential compactness of $\mathcal{R}^M$) is then satisfied: indeed, given a sequence in $\mathcal{R}^M$, the arc-length parametrizations of each component converge to the parametrization of a $C^{1,1}$ curve, thanks to the uniform bound on the curvatures. By properties (b) and (c) above, the M limit curves have no self-intersections and none of them intersects the other ones; therefore, the limit crack belongs to $\mathcal{R}^M$ (see [15, Proposition 2.9] for the details of the proof).

The convergence of the lengths (Property 2) is a consequence of the regularity of the curves. The convergence of the energies and of the energy release rates (Properties 3 and 5) is instead consequence of a delicate result about the convergence in the sense of Mosco [19] of Sobolev spaces on $\Omega\backslash\Gamma_n$ to the corresponding Sobolev space on $\Omega\backslash\Gamma$; see, e.g., [22, 4, 3, 7, 5, 6], where, instead of regular curves, compact sets with an a priori bound on the number of connected components are considered. Since in our case the domain has a much simpler geometry, the proof of the convergence of the energies can be significantly simplified; therefore, for the reader's convenience we sketch it in the next proposition, following the lines of [3, 7, 5, 6].

Remark 3. In the sequel we shall make use of the Poincaré inequality. To employ it, we use the fact that for every $\Gamma \in \mathcal{R}^M$, $\Omega\backslash\Gamma$ can be seen as a finite union of Lipschitz domains (possibly overlapping). Notice moreover that, due to the definition of $\mathcal{R}^M$, the choice of these domains can be made in such a way that there is a Poincaré constant depending only on Ω, Γ_0, and η, and independent of Γ. For the existence of such a constant, we refer to [1, Proposition 3.2].

In the next result we show that the energy is continuous with respect to the Hausdorff convergence of the cracks in $\mathcal{R}^M$ under some natural convergence assumptions on the data. Notice that, as mentioned above, the gradients are extended to zero on the cracks in order to regard them as vector fields in $L^2(\Omega;\mathbb{R}^2)$.

Proposition 1. *Let $\Gamma_n \in \mathcal{R}^M$ be a sequence converging to Γ in the Hausdorff metric and let $\psi_n, \psi \in H^1(\Omega\backslash\Gamma_0)$, $f_n, f \in L^2(\Omega)$, and $g_n, g \in L^2(\partial_S\Omega;\mathcal{H}^1)$. Assume that $\psi_n \to \psi$ strongly in $H^1(\Omega\backslash\Gamma_0)$, $f_n \to f$ strongly in $L^2(\Omega)$, and $g_n \to g$ strongly in $L^2(\partial_S\Omega;\mathcal{H}^1)$. Let u_n, u be the solutions to the following problems*

$$\begin{cases} -\Delta u_n = f_n & \text{in } \Omega\setminus\Gamma_n\,, \\ u_n = \psi_n & \text{on } \partial_D\Omega\,, \\ \partial_\nu u_n = g_n & \text{on } \partial_S\Omega\,, \\ \partial_\nu u_n = 0 & \text{on } (\partial_N\Omega\setminus\partial_S\Omega)\cup\Gamma_n\,, \end{cases} \qquad \begin{cases} -\Delta u = f & \text{in } \Omega\setminus\Gamma\,, \\ u = \psi & \text{on } \partial_D\Omega\,, \\ \partial_\nu u = g & \text{on } \partial_S\Omega\,, \\ \partial_\nu u = 0 & \text{on } (\partial_N\Omega\setminus\partial_S\Omega)\cup\Gamma\,, \end{cases}$$

respectively. Then ∇u_n converges to ∇u strongly in $L^2(\Omega;\mathbb{R}^2)$, and hence the elastic energy $E_n(\Gamma_n) := \frac{1}{2}\|\nabla u_n\|_2^2 - \langle f_n, u_n\rangle - \langle g_n, u_n\rangle$ converges to $E(\Gamma) := \frac{1}{2}\|\nabla u\|_2^2 - \langle f, u\rangle - \langle g, u\rangle$.

Proof. We begin by proving the weak convergence of the gradients in $L^2(\Omega;\mathbb{R}^2)$. Taking $u_n - \psi_n$ as test function in the equation satisfied by u_n we deduce that $\|u_n\|_{H^1(\Omega\backslash\Gamma_n)}$ is bounded uniformly. Here we use the fact that we can apply the Poincaré inequality on the sets $\Omega\backslash\Gamma_n$ with a constant independent of n (Remark 3). Therefore, up to a subsequence, we have that there exist $u^* \in L^2(\Omega)$ and $\Phi \in L^2(\Omega;\mathbb{R}^2)$ such that $u_n \rightharpoonup u^*$ weakly in $L^2(\Omega)$ and $\nabla u_n \rightharpoonup \Phi$ weakly in $L^2(\Omega;\mathbb{R}^2)$.

Taking $\varphi \in C_0^\infty(\Omega\setminus\Gamma)$, as Γ_n converges to Γ in the Hausdorff distance, we have that for n sufficiently large $\varphi \in C_0^\infty(\Omega\setminus\Gamma_n)$; using the definition of distributional derivative we easily check that Φ coincides with the distributional gradient of u^* on $\Omega\setminus\Gamma$, hence $u^* \in H^1(\Omega\setminus\Gamma)$. We can then deduce that $u^* = \psi$ on $\partial_D\Omega$, since, due to the definition of $\mathcal{R}^M$, in a (fixed) neighbourhood of the boundary of Ω the sets Γ_n, Γ coincide with Γ_0, so that $u_n \rightharpoonup u^*$ weakly in $H^1(\omega \setminus \Gamma_0)$ for some $\omega \subset \Omega$ with $\partial\omega \cap \partial\Omega = \partial\Omega$.

In order to prove the strong convergence of ∇u_n we first show that u^* solves the same equation as u, and therefore, by the uniqueness of the solution, $u^* = u$, so that the limit does not depend on the subsequence. Then using the equations again we conclude that the convergence is in the strong sense.

The standard way to show that u^* solves the problem on $\Omega \setminus \Gamma$ is to use the weak form of the equations. To this aim, we will prove that for each test function $w \in H^1(\Omega \setminus \Gamma)$ with $w = 0$ on $\partial_D\Omega$ there exists a sequence w_n such that $w_n \in H^1(\Omega \setminus \Gamma_n)$, $w_n = 0$ on $\partial_D\Omega$, and $\nabla w_n \to \nabla w$ strongly in $L^2(\Omega;\mathbb{R}^2)$. Therefore we can take w_n as test function in the equation satisfied by u_n and then passing to the limit, we conclude that u^* is the solution.

In order to prove this approximation result, let B be an open ball in $\mathbb{R}^2$ containing $\overline{\Omega}$ and let $w \in H^1(\Omega \setminus \Gamma)$ with $w = 0$ on $\partial_D\Omega$. Define $\tilde{w} = w\,1_{\Omega\setminus\Gamma}$ in Ω and $\tilde{w} = 0$ in $B \setminus \Omega$. As $w = 0$ on $\partial_D\Omega$ we have that $\tilde{w} \in H^1(B \setminus (\partial_N\Omega \cup \Gamma))$.

Let z_n be the weak solution of

$$\begin{cases} -\Delta z_n = -\mathrm{div}\nabla\tilde{w} & \text{in } B \setminus (\partial_N\Omega \cup \Gamma_n)\,, \\ z_n = 0 & \text{on } \partial B\,, \\ \partial_\nu z_n = \nabla\tilde{w}\cdot\nu & \text{on } \partial_N\Omega \cup \Gamma_n\,, \end{cases} \tag{5.1}$$

i.e., $z_n \in H^1(B \setminus (\partial_N\Omega \cup \Gamma_n))$ and $\int_{B\setminus(\partial_N\Omega\cup\Gamma_n)}(\nabla z_n - \nabla\tilde{w})\nabla\varphi_n = 0$ for every $\varphi_n \in H^1(B \setminus (\partial_N\Omega \cup \Gamma_n))$ with $\varphi_n = 0$ on ∂B. Let us recall that $\nabla\tilde{w}$ coincides with the distributional gradient of $\tilde{w}$ only on $B \setminus (\partial_N\Omega \cup \Gamma)$ and is extended by 0 on $\partial_N\Omega \cup \Gamma$.

Taking z_n as test function in the above equation we deduce that $\|\nabla z_n\|$ is bounded uniformly. The same arguments used for the sequence u_n in the first part of the proof, show that there exists $z \in H^1(B \setminus (\partial_N\Omega \cup \Gamma))$ with $z = 0$ on ∂B such that, up to a subsequence,

$$\nabla z_n \rightharpoonup \nabla z \quad \text{weakly in } L^2(B;\mathbb{R}^2)\,.$$

We want to prove that $z = \tilde{w}$; due to the uniqueness of the solutions, this is equivalent to see that z solves

$$\begin{cases} -\Delta z = -\mathrm{div}\nabla\tilde{w} & \text{in } B \setminus (\partial_N\Omega \cup \Gamma)\,, \\ z = 0 & \text{on } \partial B\,, \\ \partial_\nu z = \nabla\tilde{w}\cdot\nu & \text{on } \partial_N\Omega \cup \Gamma\,. \end{cases}$$

To check that $\int_{B\setminus(\partial_N\Omega\cup\Gamma)}(\nabla z - \nabla\tilde{w})\nabla\varphi = 0$ for every $\varphi \in H^1(B \setminus (\partial_N\Omega \cup \Gamma))$ with $\varphi = 0$ on ∂B, we use the "harmonic conjugates".

Let $Z_n = (\nabla z_n - \nabla\tilde{w})\,1_{B\setminus(\partial_N\Omega\cup\Gamma_n)}$; then div $Z_n = 0$ in $\mathcal{D}'(B)$. Therefore there exists $v_n \in H^1(B)$ such that $\nabla v_n = R\,Z_n$ a.e. in B, where R is the rotation defined by $R(x_1, x_2) = (-x_2, x_1)$. As $\partial_\tau v_n = \partial_\nu(z_n - \tilde{w}) = 0$ on $\partial_N\Omega \cup \Gamma_n$, we deduce that v_n is constant on each connected component of $\partial_N\Omega \cup \Gamma_n$, respectively. The sequence ∇v_n is bounded in $L^2(B;\mathbb{R}^2)$; moreover, up to an additive constant we may assume that $\int_B v_n = 0$ for every n. Hence, applying the Poincaré inequality we have that, up to a subsequence, v_n converges to a function v weakly in $H^1(B)$. Then $\nabla v = R(\nabla z - \nabla\tilde{w})$.

We now want to show that v is constant on each connected component of $\partial_N\Omega$ and of Γ. The fact that v is constant on each connected component of $\partial_N\Omega$ and of Γ_0^m follows from the weak convergence of v_n to v in $H^1(B)$ (notice that in our case each connected component Γ_n^m and Γ^m contains Γ_0^m). It remains to show that v remains constant on each connected component of Γ: for this part we refer for instance to the proof in [6, Lemma 3.5].

As $v \in H^1(B)$ is constant on each connected component of $\partial_N\Omega$ and of Γ, we can approximate it in the strong convergence of $H^1(B)$ by a sequence of functions $\zeta_n \in C^1(\overline{B})$ such that each ζ_n is constant in a neighbourhood of each connected component of $\partial_N\Omega \cup \Gamma$. Let ρ_n be a sequence of functions in $C^1_0(B)$ such that $\rho_n = 1$ on supp $\nabla\zeta_n$ and $\rho_n = 0$ in a neighbourhood of $\partial_N\Omega \cup \Gamma$. Given $\varphi \in H^1(B \setminus (\partial_N\Omega \cup \Gamma))$ with $\varphi = 0$ on ∂B we have

$$\int_B R\,\nabla\zeta_n\nabla\varphi = \int_B R\,\nabla\zeta_n\nabla(\varphi\,\rho_n) = 0\,,$$

since div $R\,\nabla\zeta_n = 0$ and $\varphi\,\rho_n \in H^1_0(B)$. Passing to the limit we get $\int_B R\,\nabla v\nabla\varphi = 0$, and hence $z = \tilde{w}$.

Using again z_n as test function in the equation it satisfies, we deduce the strong convergence of the gradients (extended by 0):

$$\nabla z_n \to \nabla\tilde{w} \qquad \text{strongly in } L^2(B;\mathbb{R}^2)\,.$$

Notice that $z_n 1_{B\setminus\overline{\Omega}} \to 0$ strongly in $H^1(B \setminus \overline{\Omega})$. Since $\partial\Omega$ is Lipschitz there exists a sequence of functions $\hat{w}_n \in H^1(B)$ such that $\hat{w}_n \to 0$ strongly in $H^1(B)$ and $\hat{w}_n = z_n$ in $B \setminus \overline{\Omega}$. At this point, setting

$$w_n = (z_n - \hat{w}_n)1_{\Omega\setminus\Gamma_n}\,,$$

we have that $w_n \in H^1(\Omega\setminus\Gamma_n)$, $w_n = 0$ on $\partial_D\Omega$, $\nabla w_n \to \nabla w$ strongly in $L^2(\Omega;\mathbb{R}^2)$, as desired.

To conclude the proof we take w_n as test function in the equation satisfied by u_n and obtain that $\langle\nabla u_n, \nabla w_n\rangle = \langle f_n, w_n\rangle + \langle g_n, w_n\rangle$. Passing to the limit we get $\langle\nabla u^*, \nabla w\rangle = \langle f, w\rangle + \langle g, w\rangle$.

Therefore $u^* = u$ and the whole sequence converges. To prove the strong convergence of the gradients and the convergence of the energies it is now enough to take $u_n - \psi_n$ as test function in the equation satisfied by u_n. □

The previous proposition gives the convergence of the energies (Property 3). The convergence of the energy release rates (Property 5) is now a consequence of the convergence of the gradients proven above and of the integral formula (3.2). Indeed, due to our regularity assumptions on the curves belonging to $\mathcal{R}^M$, it is possible to construct suitable vector fields V_n on Ω, each of them tangent to the crack Γ_n, which converge to a vector field V tangent to the crack Γ in such a way that we can pass to the limit in the integral formula (3.2). For the details of the proof we refer to [15, Theorem 2.12].

In this section we showed how to define the class $\mathcal{R}^M$ in such a way that Properties 1–5 are satisfied. This completes the proof of the existence of quasistatic evolution sketched above. In the final part of the work we will mention the results concerning the energy balance.

6. Parametrized solutions and energy balance

In the previous sections we explained how to find a quasistatic evolution of cracks satisfying Griffith's criterion through the method of vanishing viscosity. In the resulting evolution, the length is a BV function of time and may present jumps in time. However, in the usual time scale it is difficult to state the properties of the jump points: so, following the method of [18], we reparametrize the time interval in such a way that the reparametrized evolutions are absolutely continuous in time. This allows us to write the energy balance with the correct dissipative term during the jumps. In this section we state the results that are proven in [16, Section 8].

Following the lines of [18], it is possible to construct a reparametrized time $\tilde t\colon [0,S] \to [0,T]$ and a reparametrized length $\tilde l\colon [0,S] \to [0,L]^M$. Both these functions are absolutely continuous; moreover, both $\tilde t$ and the components $\tilde l^m$ are nondecreasing. For $s \in [0,S]$ we define $\widetilde\Gamma(s) = \widetilde\Gamma^1(s) \cup \dots \cup \widetilde\Gamma^M(s)$ as the set in $\mathcal{R}^M$, contained in $\Gamma(T)$, such that, for every m, $\widetilde\Gamma^m(s) \supset \Gamma_0^m$ and $\mathcal{H}^1(\widetilde\Gamma^m(s)) = \tilde l^m(s)$. As for the energy release rates, we set $\widetilde G(s) = \big(\widetilde G^1(s), \dots, \widetilde G^M(s)\big) := G(\tilde t(s); \widetilde\Gamma(s))$. By Property 5 and by the continuity of $\tilde t$ and $\tilde l$, $s \mapsto \widetilde G(s)$ is continuous.

By construction, the functions $\tilde t$ and $\tilde l$ enjoy the normalization property

$$\tilde t'(s) + \Big|\tilde l'(s)\Big|_1 = 1$$

for a.e. s. Therefore, three cases are possible:

(a) $\tilde t'(s) > 0$ and $\tilde l'(s) = (0, \dots, 0)$; in this case $\widetilde G^m(s) \le 1$ for every m;
(b) $\tilde t'(s) > 0$ and $\tilde l'(s) \ne (0, \dots, 0)$; in this case $\widetilde G^m(s) = 1$ for every m such that $(\tilde l^m)'(s) \ne 0$, while $\widetilde G^m(s) \le 1$ for every other m;
(c) $\tilde t'(s) = 0$ and $\tilde l'(s) \ne (0, \dots, 0)$; in this case $\widetilde G^m(s) \ge 1$ for every m such that $(\tilde l^m)'(s) \ne 0$, while $\widetilde G^m(s) \le 1$ for every other m.

In case (a), there is no motion (sticking regime); case (b) corresponds instead to a rate-independent growth in the quasistatic time scale (sliding regime). These two situations reflect the usual criterion of Griffith described above. Case (c), instead, was not characterized in Theorem 4.2 and corresponds to the jumps of the quasistatic evolution: the length is discontinuous in the original time scale and the path of the crack is parametrized by $\tilde l$ in the rescaled setting as the time $\tilde t$ is frozen.

Using the reparametrized functions $\tilde t$ and $\tilde l$, it is possible also to prove the following energy balance: for every $s \in [0,S]$

$$\begin{aligned}
&\mathcal{E}(\tilde t(s); \widetilde\Gamma(s)) + \Big|\tilde l(s)\Big|_1 + \int_0^s \tilde l'(\sigma) \cdot (\widetilde G(\sigma) - \mathbf{1})^+ \, \mathrm{d}\sigma \\
&= \mathcal{E}(0; \Gamma_0) + \Big|\tilde l(0)\Big|_1 + \int_0^s \Big\langle \nabla \tilde u(\sigma), \nabla \tilde\psi'(\sigma) \Big\rangle \, \mathrm{d}\sigma \\
&\quad - \int_0^s \Big\langle \tilde f(\sigma), \tilde\psi'(\sigma) \Big\rangle \, \mathrm{d}\sigma - \int_0^s \Big\langle \tilde f'(\sigma), \tilde u(\sigma) \Big\rangle \, \mathrm{d}\sigma \\
&\quad - \int_0^s \Big\langle \tilde g(\sigma), \tilde\psi'(\sigma) \Big\rangle \, \mathrm{d}\sigma - \int_0^s \langle \tilde g'(\sigma), \tilde u(\sigma) \rangle \, \mathrm{d}\sigma \,.
\end{aligned}$$

Here $\tilde f(\sigma) = f(\tilde t(\sigma))$, $\tilde g(\sigma) = g(\tilde t(\sigma))$, $\tilde\psi(\sigma) = \psi(\tilde t(\sigma))$, and $\tilde u(\sigma) = u(\tilde t(\sigma); \widetilde\Gamma(\sigma))$, while the symbol $(v)^+$ stands for the vector whose components are the positive parts of the components of the vector v. This means that the energy spent to pass from the configuration at time 0 to the configuration at time s equals the integral of the power of external forces (including those acting on $\partial_D\Omega$) up to an additive term that is positive only when the energy release rate overcomes the toughness. This happens when $\tilde t' = 0$, indeed this term describes the energy dissipated during the jumps. Coming back to the original time scale t we obtain

$$\begin{aligned}
&\mathcal{E}(t;\Gamma(t)) + l(t) - \mathcal{E}(0;\Gamma_0) - l(0) = \int_0^t \Big\langle \nabla u(\tau;\Gamma(\tau)), \nabla\dot{\psi}(\tau)\Big\rangle \,\mathrm{d}\tau \\
&- \int_0^t \Big\langle f(\tau), \dot{\psi}(\tau)\Big\rangle \,\mathrm{d}\tau - \int_0^t \Big\langle \dot{f}(\tau), u(\tau;\Gamma(\tau))\Big\rangle \,\mathrm{d}\tau \\
&- \int_0^t \Big\langle g(\tau), \dot{\psi}(\tau)\Big\rangle \,\mathrm{d}\tau - \int_0^t \langle \dot{g}(\tau), u(\tau;\Gamma(\tau))\rangle \,\mathrm{d}\tau \\
&- \int_{\tilde{s}(0)}^{\tilde{s}(0^+)} \tilde{l}'(\tau)\cdot(\widetilde{G}(\tau) - \mathbf{1})^+ \,\mathrm{d}\tau - \sum_{\bar{t}\in J(l)\cap(0,t)} \int_{\tilde{s}(\bar{t}^-)}^{\tilde{s}(\bar{t}^+)} \tilde{l}'(\tau)\cdot(\widetilde{G}(\tau) - \mathbf{1})^+ \,\mathrm{d}\tau \\
&- \int_{\tilde{s}(t^-)}^{\tilde{s}(t)} \tilde{l}'(\tau)\cdot(\widetilde{G}(\tau) - \mathbf{1})^+ \,\mathrm{d}\tau\,,
\end{aligned}$$

where $\tilde{s}$ is the inverse of $\tilde{t}$, well defined where $\tilde{t}$ is not constant.

In models of quasistatic evolution based on local minimization, the jumps in the slow time scale represent the limit of brutal propagations, i.e., fast dynamic motions through unstable states. This regime can be studied thanks to a suitable reparametrization of the time and of the length. The reparametrized variables allow one to express, in the energy balance, the terms representing the energy dissipated "instantaneously" during each jump.

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