Total Variation and Cheeger sets in Gauss space

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Abstract

The aim of this paper is to study the isoperimetric problem with fixed volume inside convex sets and other related geometric variational problems in the Gauss space, in both the finite and infinite dimensional case. We first study the finite dimensional case, proving the existence of a maximal Cheeger set which is convex inside any bounded convex set. We also prove the uniqueness and convexity of solutions of the isoperimetric problem with fixed volume inside any convex set. Then we extend these results in the context of the abstract Wiener space, and for that we study the total variation denoising problem in this context.

Contents

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1 Introduction

The study of Cheeger sets has recently attracted some attention due to its relevance in describing explicit solutions of the total variation denoising problem for initial data which are characteristic functions of convex sets (or other more general cases).

Given a nonempty open bounded subset Ω of $\mathbb{R}^n$, we call Cheeger constant of Ω the quantity

$$\overline{\lambda}_\Omega := \min_{F \subseteq \Omega} \frac{P(F)}{|F|} . \tag{1}$$

Here $|F|$ denotes the n-dimensional volume of F and $P(F)$ the perimeter of F. The minimum in (1) can be taken over all nonempty sets of finite perimeter contained in Ω. A Cheeger set of Ω is any set $G \subseteq \Omega$ which minimizes (1).

Existence of Cheeger sets follows easily from the isoperimetric inequality (that guarantees that the volume of sets in minimizing sequences does not converge to 0) and the lower semicontinuity of the perimeter. The uniqueness of Cheeger sets is not true in general (a simple counterexample is given in [26] when Ω is not convex), although it is true modulo a small perturbation of Ω [17]. When Ω is convex, uniqueness is true, and when $n = 2$ an explicit construction can be given [3, 26]. The uniqueness and convexity of the Cheeger set inside bounded convex subsets of $\mathbb{R}^n$ was proved in [16] under the assumption that the set is uniformly convex and of class C^2, and extended in [1] to the general case. If the ambient set is convex, the $C^{1,1}$-regularity of Cheeger sets is a consequence of the results in [23, 24, 30]. Moreover, a Cheeger set can be characterized in terms of the mean curvature of its boundary; the sum of the principal curvatures being bounded by the Cheeger constant (see [22, 10, 26, 3] for $n = 2$ and [2, 1] for the general case).

The study of Cheeger sets is facilitated by the study of the family of geometric variational problems

$$\min\{P(F) - \mu|F| : F \subseteq \Omega\}. \tag{2}$$

Indeed, the solutions of (2) can be related to the level sets of the solution of the total variation denoising problem with Dirichlet boundary conditions

$$\min \int_\Omega |Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} + \frac{\lambda}{2} \int_\Omega (u-1)^2 \, dx. \tag{3}$$

If u is the solution of (3), then for any $t \in [0,1]$, $\{u > t\}$ is a solution of (2) $\mu = \lambda(1-t)$ and varying λ and t we can cover the whole range $\mu \in [0, \infty)$. Then, when Ω is convex, the convexity properties and uniqueness of solutions of (2) when μ is larger than the Cheeger constant can be deduced from the properties of u. Moreover the maximal Cheeger set inside Ω can be found as $\{u = \max u\}$ and it solves (2) with $\mu = \overline{\lambda}_\Omega$.

Related to Cheeger sets is the notion of calibrability (see Definition 5). We show that a set $\Omega \subseteq \mathbb{R}^n$ is calibrable if and only if Ω minimizes the problem

$$\min_{F \subseteq \Omega} P(F) - \overline{\lambda}_\Omega |F|, \tag{4}$$

or, equivalently, if Ω is a Cheeger in itself. Notice that, if G is a Cheeger set of Ω, then G is calibrable. In the convex case, calibrable sets can be characterized in terms of a bound for the mean curvature of its boundary (the sum of the principal curvatures is bounded by the Cheeger constant).

Our purpose in this paper is to extend the existence, uniqueness and convexity of Cheeger sets and to study the analog of problems (2) and (3) when E is a convex set in the Gauss space, both in the finite and the infinite dimensional (the abstract Wiener space) cases. In this context, if E is a subset of the Gauss space we consider the problem

$$(P_\mu): \qquad \min\{P_\gamma(F) - \mu\gamma(F) : F \subseteq E\}, \tag{5}$$

where γ denote the Gaussian measure in $\mathbb{R}^n$ or in the abstract Wiener space, and P_γ denotes the associated notion of perimeter. Again the study of the analog of problem (3) plays an important technical role.

In the context of the Gauss space, we say that a set $K \subseteq E$ with positive measure is a γ-*Cheeger set* of E if K is a minimum of the problem

$$\min_{F\subseteq E} \frac{P_\gamma(F)}{\gamma(F)}. \tag{6}$$

The value of (20) is the γ-*Cheeger constant* of E. Our purpose is to prove the existence of Cheeger sets inside any subset E of the Gauss space with nonempty interior. If E is convex, we also prove the existence of a maximal γ-Cheeger set of E which is convex. Moreover, it can be computed as the region where the solution u of the total γ-variation denoising problem attains its maximum.

Let us finally mention that γ-Cheeger sets in the finite dimensional Gauss space can be considered as a particular case of anisotropic Cheeger sets and we refer to [14, 19] for such approach.

Let us describe the plan of the paper. In Section 2 we define the notation to be used throughout the paper. Sections 3 to 5 are devoted to the study of calibrable and Cheeger sets in the finite dimensional Gauss space. In Section 3 we define the notion of calibrable sets and we give some characterizations in terms of the solution of the variational problem (P_{λ_E}). In Section 4 we characterize convex calibrable sets in terms of the Gaussian mean curvature of its boundary. In Section 5 we prove the existence of a maximal γ-Cheeger set inside any convex set in $\mathbb{R}^n$ with the Gauss measure. In Section 2.1 we recall the definition of abstract Wiener space and the notions of gradient and divergence in this context. In Section 6 we prove the existence of solutions of the denoising problem in the abstract Wiener space. This problem is crucial in order to study the geometric variational problems (P_μ). In Section 7 we characterize the subdifferential of the total variation in the abstract Wiener space so that we can write the Euler-Lagrange equation satisfied by solutions of the denoising problem. In Section 8 we prove the existence of solutions of problem (P_μ). In particular, we prove the existence of γ-Cheeger sets inside any subset E of the Wiener space with nonempty interior. In Section 9, assuming that E is convex and has nonempty interior, we prove uniqueness and convexity of solutions of (P_μ) for any μ larger than the γ-Cheeger constant of E. We also prove the existence of a maximal γ-Cheeger set which is convex.

2 Notation

We start with some definitions. Let us consider the Gauss space, that is, $\mathbb{R}^n$ with the Gaussian measure

$$d\gamma(x) = \gamma(x)dx = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx.$$

The divergence of a vector field $\psi \in L^p_{\rm loc}(\mathbb{R}^n, \mathbb{R}^n)$, $p \in [1, \infty]$, is defined to be the adjoint operator of (minus) the gradient, $\mathrm{div}_\gamma = -\nabla^*$, that is,

$$\int_{\mathbb{R}^n} u \mathrm{div}_\gamma \psi d\gamma = -\int_{\mathbb{R}^n} \langle \nabla u, \psi \rangle d\gamma,$$

for any $u \in C^1_c(\mathbb{R}^n)$. Then

$$\mathrm{div}_\gamma \psi(x) = \mathrm{div}\psi(x) - \langle \psi(x), x \rangle \qquad x \in \mathbb{R}^n.$$

We denote by $L^p(\mathbb{R}^n, \gamma)$ the space of all measurable functions u such that

$$\int_{\mathbb{R}^n} |u|^p d\gamma < +\infty.$$

The total variation of u is then defined as

$$|D_\gamma u|(\mathbb{R}^n) := \sup \left\{ \int_{\mathbb{R}^n} u \,\mathrm{div}_\gamma \psi \, d\gamma : \psi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), |\psi(x)| \leq 1 \right\}.$$

We say that $u \in L^1(\mathbb{R}^n, \gamma)$ has bounded total γ-variation (or simply, if clear from the context, bounded total variation) if $|D_\gamma u|(\mathbb{R}^n) < +\infty$ and we write $u \in BV(\mathbb{R}^n, \gamma)$. Given a measurable set $E \subseteq \mathbb{R}^n$, we let

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ -1 & \text{if } x \in E^c. \end{cases}$$

and

$$\mathbf{1}_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in E^c. \end{cases}$$

We say that E has finite γ-perimeter if $P_\gamma(E) := |D_\gamma \mathbf{1}_E|(\mathbb{R}^n) < +\infty$. For a set E with finite γ-perimeter we define the constant

$$\lambda_E = \frac{P_\gamma(E)}{\gamma(E)}.$$

Notice that, for a regular function u and for a smooth set E, we have the following representation formulae

$$|D_\gamma u|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\nabla u(x)| \gamma(x) dx, \qquad P_\gamma(E) = \int_{\partial E} \gamma(x) d\mathcal{H}^{n-1}(x).$$

Since the Gaussian density is bounded and locally bounded away from zero, we get $BV_{\rm loc}(\mathbb{R}^n) = BV_{\rm loc}(\mathbb{R}^n, \gamma)$ with local equivalence of the norms, and then we can use all the fine properties of the (Euclidean) functions with bounded variation and sets with finite perimeter. In particular, for $u \in BV(\mathbb{R}^n,\gamma)$ and E with finite perimeter we have

$$dD_\gamma u(x) = \gamma(x) dDu(x), \qquad dD_\gamma \mathbf{1}_E(x) = -\gamma(x)\nu_E(x) d\mathcal{H}^{n-1} \llcorner \partial^* E(x),$$

where $\partial^* E$ is the reduced boundary of E and ν_E is the outer unit normal to the boundary of E. If $E \subseteq \mathbb{R}^n$ is a set of finite perimeter with boundary of class $C^{1,1}$, we define the Gaussian mean curvature by

$$H_E^\gamma(x) = H_E(x) - \frac{1}{n-1}\langle \nu_E(x), x\rangle$$

with H_E the Euclidean mean curvature.

For a function $u \in BV(\mathbb{R}^n,\gamma)$, the following integration by parts formula holds

$$\int_{\mathbb{R}^n} u(x)\mathrm{div}_\gamma \psi(x) d\gamma(x) = -\int_{\mathbb{R}^n} \langle \psi(x), dD_\gamma u(x)\rangle, \qquad \forall \psi \in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$$

and equivalently for any set E of finite γ–perimeter

$$\int_E \mathrm{div}_\gamma \psi(x) d\gamma(x) = \int_{\partial^* E} \langle \psi(x), \nu_E(x)\rangle \gamma(x) d\mathcal{H}^{n-1}(x).$$

Thanks to a result due to Anzellotti [8], for any $\xi \in L^\infty(E,\mathbb{R}^n)$, with $\mathrm{div}_\gamma \xi \in L^2(E,\gamma)$, it is defined the normal trace of ξ on $\partial^* E$, which we denoted by $[\xi \cdot \nu_E]$, with the property $[\xi\cdot\nu_E] \in L^\infty(\partial^* E, \mathcal{H}^{n-1})$. Given ξ as above and $u \in BV(\mathbb{R}^n,\gamma)\cap L^2(\mathbb{R}^n)$, we also define the measure $(\xi \cdot D_\gamma u)$ as

$$\int_{\mathbb{R}^n} (\xi\cdot D_\gamma u)\varphi := -\int_{\mathbb{R}^n} u\,\varphi\,\mathrm{div}_\gamma(\xi)\,d\gamma - \int_{\mathbb{R}^n} u\,\xi\cdot\nabla\varphi\,d\gamma,$$

for any $\varphi \in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$. Notice that

$$(\xi\cdot D_\gamma u) = \xi\cdot\nabla u \qquad \text{for any } u \in W^{1,1}(\mathbb{R}^n).$$

2.1 Notation in the infinite dimensional case

An abstract Wiener space is defined as a triple (X,γ,H), where X is a Banach space, endowed with the norm $\|\cdot\|_X$, γ is a centered Gaussian measure, and H is the Cameron–Martin space associated to the measure γ, that is, it is a separable Hilbert space densely embedded in X, endowed with the scalar product $[\cdot,\cdot]_H$ and with the norm $|\cdot|_H$. The requirement that γ is a centered Gaussian measure means that for any $x^* \in X^*$, the measure $x^*_\# \gamma$ is a centered Gaussian measure in $\mathbb{R}$. The space $\mathcal{H} = L^2(X,\gamma)$ is called *reproducing kernel* and can be embedded in X by the map $R : \mathcal{H} \to X$ defined as

$$Rh := \int_X h(x) x d\gamma(x).$$

The space $H = R\mathcal{H}$, with the scalar product induced by $\mathcal{H}$ via R, is the Cameron-Martin space, and it is a subspace of X. A result by Fernique [13, Theorem 2.8.5] implies the existence of a positive number $\beta > 0$ such that

$$\int_X e^{\beta\|x\|} d\gamma(x) < +\infty.$$

As a consequence, the maps $x \mapsto \langle x, x^* \rangle$ belong to $L^p(X,\gamma)$ for any $x^* \in X^*$ and $p \geq 1$. In particular, any element $x^* \in X^*$ can be seen as a map $x^* \in L^2(X,\gamma)$. In this way, we obtain that $R^* : X^* \to \mathcal{H}$

$$R^* x^*(x) := \langle x, x^* \rangle$$

is the adjoint operator of R. It is possible to prove that R is a γ–Radonyfing operator (Hilbert–Schmidt in case X is Hilbert); this in particular implies that the embedding of H in X is continuous, that is there exists $c > 0$ such that

$$\|h\|_X \leq c|h|_H, \qquad \forall h \in H.$$

The covariance operator of the measure γ turns out to be $Q = RR^* \in \mathcal{L}(X^*, X)$, that is, the Fourier transform $\hat{\gamma}$ of γ is given by

$$\hat{\gamma}(x^*) = \int_X \langle x, x^* \rangle \langle x, x^* \rangle d\gamma(x) = \exp\left(-\frac{\langle Qx^*, x^* \rangle}{2}\right), \qquad \forall x^* \in X^*.$$

By considering the injective part of R, we can select (x_j^*) in X^* in such a way that $\hat{h}_j := R^* x_j^*$, or, equivalently, $h_j := R\hat{h}_j = Qx_j^*$ form an orthonormal basis of H; we then define $\lambda_j = \|x_j^*\|^{-1}$.

Given $n \in \mathbb{N}$, we also let $H_n := \langle h_1, \ldots, h_n \rangle \subseteq H$, $X_n^\perp := \overline{H_n^\perp}^X$, and $\Pi_n : X \to H_n$ be the closure of the orthogonal projection from H to X_n

$$\Pi_n(x) := \sum_{j=1}^n \langle x, x_j^* \rangle h_j \qquad x \in X.$$

As above, $\gamma(E)$ will be the Gaussian measure of a Borel set $E \subseteq X$. We denote by $C_b^1(X)$ the set of continuous and bounded functions $f : X \to \mathbb{R}$ which admit directional derivatives $\partial_h f$ which are continuous on X, for all $h \in H$. Given $f \in C_b^1(X)$ and $\phi \in C^1(X, H)$, we set

$$\begin{aligned} \nabla_\gamma f(x) &:= \sum_{j \in \mathbb{N}} \partial_j f(x) h_j, \\ \mathrm{div}_\gamma \phi(x) &:= -\sum_{j \geq 1} \partial_j^* [\phi(x), Qx_j^*], \end{aligned}$$

where $\partial_j := \partial_{h_j}$ and $\partial_j^* := \partial_j - \hat{h}_j$ is the adjoint operator of ∂_j. With this notation, there holds the integration by parts formula:

$$\int_X f \,\mathrm{div}_\gamma \phi \, d\gamma = -\int_X [\nabla_\gamma f, \phi]_H \, d\gamma. \tag{7}$$

In particular, thanks to (7), the operator ∇_γ is closable in $L^p(X,\gamma)$, and we denote by $W^{1,p}(X,\gamma)$ the domain of its closure [13, 6].

We then define the total variation of a function $u \in L^1(X,\gamma)$ as

$$|D_\gamma u|(X) := \sup\left\{ \int_X u(x) \mathrm{div}_\gamma \phi(x) d\gamma(x) : \phi \in C_b^1(X, H) : |\phi(x)|_H \leq 1 \right\}.$$

We say that u has finite γ–total variation, $u \in BV(X,\gamma)$, if $|D_\gamma u|(X) < +\infty$; in addition, a subset $E \subseteq X$ is said to have γ–finite perimeter if $P_\gamma(E) := |D_\gamma \mathbf{1}_E|(X) < +\infty$. As above, we let

$$\lambda_E := \frac{P_\gamma(E)}{\gamma(E)}.$$

Given a vector field $z \in L^p(X,\gamma)$, $p \in [1,\infty]$, we define $\operatorname{div}_\gamma z$ using test functions f in $W^{1,q}(X,\gamma)$, $\frac{1}{p} + \frac{1}{q} = 1$, by the formula

$$\int_X \operatorname{div}_\gamma z\, f\, d\gamma := -\int_X [z, \nabla_\gamma f]_H\, d\gamma, \tag{8}$$

Since the smooth functions (i.e. functions in $C_b^1(X)$) are dense in $W^{1,q}(X,\gamma)$, $\operatorname{div}_\gamma z$ is uniquely determined by its action on smooth functions. We say that $\operatorname{div}_\gamma z \in L^m(X,\gamma)$ if the previous linear functional can be extended to all test functions in $L^{m'}(X,\gamma)$ with $\frac{1}{m} + \frac{1}{m'} = 1$.

Given an open set $\Omega \subseteq X$, we consider the space of vector fields

$$\mathcal{X}_2(\Omega, H) := \left\{ z \in L^\infty(\Omega,\gamma) :\ \operatorname{div}_\gamma z \in L^2(\Omega,\gamma),\ |z|_H \le 1\ \gamma\text{-a.e. in } \Omega \right\}.$$

For each $z \in \mathcal{X}_2(X,H)$ and $u \in BV(X,\gamma) \cap L^2(X,\gamma)$ we may define

$$\int_X (z \cdot D_\gamma u)\varphi := -\int_X u\,\varphi \operatorname{div}_\gamma(z)\, d\gamma - \int_X u\,[z, \nabla_\gamma \varphi]_H\, d\gamma,$$

for any $\varphi \in C_b^1(X)$. Notice that

$$(z \cdot D_\gamma u) = [z, \nabla_\gamma u]_H \qquad \text{for any } u \in W^{1,1}(X,\gamma).$$

Finally, we let $\mathcal{U} : \mathbb{R} \to \mathbb{R}$ be the *isoperimetric function* defined as $\mathcal{U}(t) = \Phi' \circ \Phi^{-1}(t)$, $t \in \mathbb{R}$, where

$$\Phi(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{s^2}{2}}\, ds.$$

The isoperimetric function has the following asymptotic behaviour

$$\lim_{s \to 0} \frac{\mathcal{U}(s)}{s|\ln s|^{1/2}} = 1.$$

We recall the isoperimetric inequality in the Gauss space (for a proof, see for instance [28] or also [20, Proposition 3.2]).

Proposition 1. *For all Borel subset $E \subseteq X$, there holds*

$$P_\gamma(E) \ge \mathcal{U}(\gamma(E)). \tag{9}$$

We also recall the coarea formula in the space $BV(X,\gamma)$ [6, Theorem 3.5].

Proposition 2. *Let $u \in BV(X,\gamma)$. Then almost all the level sets $\{u > t\}$ have finite perimeter and the following inequality holds:*

$$|D_\gamma u|(X) = \int_{\mathbb{R}} P_\gamma(\{u > t\})\, dt. \tag{10}$$

As consequence of the isoperimetric inequality and the coarea formula, we have that for all $u \in BV(X, \gamma)$ there holds

$$|D_\gamma u|(X) \geq \int_{\mathbb{R}} \mathcal{U}\left(\gamma(\{u > s\})\right) ds. \tag{11}$$

The following result is also a consequence of the coarea formula.

Lemma 3. *There exists a sequence $r_j \to 0$ such that $P_\gamma(B_{r_j}) \to 0$.*

Proof. Let us consider the function

$$u(x) := \min\{\|x\|_X, r\}.$$

Since u 1-Lipschitz on X, we have

$$|u(x+h) - u(x)| \leq \|h\|_X \leq c|h|_H,$$

so that $|D_\gamma u|(X) \leq c\gamma(B_r)$. By the coarea formula (10), we then obtain

$$c\gamma(B_r) \geq |D_\gamma u|(X) = \int_0^r P_\gamma(B_t)dt.$$

It follows that there exists $r' \in (0, r)$ such that

$$P_\gamma(B_{r'}) \leq c\frac{\gamma(B_{r'})}{r'}.$$

The thesis now follows by observing that

$$\lim_{r \to 0} \frac{\gamma(B_r)}{r} = 0,$$

which can be easily checked by estimating $\gamma(B_r)$ with the volume of a cylinder of radius r, with finite dimensional section. □

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function, and let

$$F(t) = \gamma(\{f \leq t\}), \qquad t \in \mathbb{R}.$$

We recall the following result of [13, Corollary 4.4.2].

Theorem 4 (Bogachev)**.** *The function F is continuous on $\mathbb{R} \setminus \{t_0\}$, where*

$$t_0 = \inf\{t : F(t) > 0\}.$$

As a consequence, $\gamma(\{f = t\}) = 0$ for all $t \neq t_0$.

Additional properties of functions with bounded variation and sets with finite perimeter can be found in [6, 25].

3 Calibrability and equivalent notions

In this section we recall the notion of calibrable set and give equivalent characterizations of calibrability.

Definition 5. *We say that a set $E \subseteq \mathbb{R}^n$ of finite γ–perimeter is* calibrable *if there exists $\xi \in L^\infty(E, \mathbb{R}^n)$ such that $\|\xi\|_\infty \le 1$, $\mathrm{div}_\gamma \xi = -\lambda_E$ on E and $[\xi \cdot \nu_E] = -1$ on $\partial^* E$.*

We want to characterize the calibrability of a set in terms of minimality of some variational problems. In particular, we shall consider the following two problems:

$$(P_\mu): \qquad \min\{P_\gamma(F) - \mu\gamma(F) : F \subseteq E\},$$

$$\min\left\{\int_E (\mathrm{div}_\gamma \xi)^2 d\gamma : \xi \in L^\infty(E, \mathbb{R}^n)\ \|\xi\|_\infty \le 1, [\xi \cdot \nu_E] = -1 \text{ on } \partial^* E\right\}. \tag{12}$$

Notice that, by convexity of the last integral, a minimum always exists and two possibly different minimizers have the same divergence. We shall denote by $\xi_{\min}$ a minimizer of (12).

Remark 6. Reasoning as in [9, Lemma 5.4], one can show that, if $\xi_{\min}$ is a minimum of (12), then $\xi_{\min}$ also minimizes $\|\mathrm{div}_\gamma \xi\|_{L^p(E,\gamma)}$ for all $p \in (2, +\infty]$.

Proposition 7. *Let $E \subseteq \mathbb{R}^n$ be a finite perimeter set. Then E is calibrable if and only if $\mathrm{div}_\gamma \xi_{\min}$ is constant.*

Proof. Assume by contradiction that E calibrable, but $\mathrm{div}_\gamma \xi_{\min}$ is not constant in E, and let $E' = \{\mathrm{div}_\gamma \xi_{\min} < -\lambda_E\} \ne E$. By the results in [11, 12], E' is a set of finite γ-perimeter and $P_\gamma(E') = -\int_{E'} \mathrm{div}_\gamma \xi_{\min} d\gamma$. Then, we have

$$\begin{aligned}
P_\gamma(E) - \lambda_E \gamma(E) &= \int_E (-\mathrm{div}_\gamma \xi_{\min} - \lambda_E) d\gamma \\
&= \int_{E'} (-\mathrm{div}_\gamma \xi_{\min} - \lambda_E) d\gamma + \int_{E \setminus E'} (-\mathrm{div}_\gamma \xi_{\min} - \lambda_E) d\gamma \\
&> \int_{E'} (-\mathrm{div}_\gamma \xi_{\min} - \lambda_E) d\gamma = P_\gamma(E') - \lambda_E \gamma(E').
\end{aligned}$$

However, recalling that E is calibrable and using the vector field ξ in Definition 5, we also have

$$P_\gamma(E) - \lambda_E \gamma(E) = \int_E (-\mathrm{div}_\gamma \xi - \lambda_E) d\gamma = \int_{E'} (-\mathrm{div}_\gamma \xi - \lambda_E) d\gamma \le P_\gamma(E') - \lambda_E \gamma(E'),$$

which gives a contradiction.

Now, we assume that $\mathrm{div}_\gamma \xi_{\min}$ is constant in E, $\mathrm{div}_\gamma \xi_{\min} = c$; we have only to prove that $c = -\lambda_E$. Since

$$c\gamma(E) = \int_E \mathrm{div}_\gamma \xi_{\min} d\gamma = \int_{\partial^* E} [\xi_{\min} \cdot \nu_E] \gamma d\mathcal{H}^{n-1} = -P_\gamma(E),$$

we have $c = -\lambda_E$. □

Lemma 8. *Let E_α, E_β be the solutions of (P_μ) to the values α, β with $\alpha > \beta$; then $E_\beta \subseteq E_\alpha$. As a consequence, for almost any $\alpha > 0$ the solution of (P_α) is unique.*

The proof of this Lemma follows the usual proof in the Euclidean case can be found in [2]; it is based on the following result.

Lemma 9. *If E, F are two sets of finite perimeter in X, then*

$$P_\gamma(E \cup F) + P_\gamma(E \cap F) \leq P_\gamma(E) + P_\gamma(F). \tag{13}$$

The proof is exactly the same as in [21] for the Euclidean case, and we omit the details.
In the sequel, it will be of particular relevance the study of the following problem:

$$(Q_\lambda): \quad \min\left\{|D_\gamma u|(\mathbb{R}^n) + \frac{\lambda}{2}\int_{\mathbb{R}^n}(u - \chi_E)^2 d\gamma : u \in BV(\mathbb{R}^n, \gamma) \cap L^2(\mathbb{R}^n, \gamma)\right\}, \tag{14}$$

where $E \subseteq \mathbb{R}^n$ and $\lambda > 0$.
We collect in the next Proposition the main properties of (Q_λ).

Proposition 10. *We have the following facts:*

(i) *(Q_λ) admits a unique minimizer $u_\lambda \in BV(\mathbb{R}^n, \gamma) \cap L^2(\mathbb{R}^n, \gamma)$, for all $\lambda > 0$;*

(ii) *there exists a vector field $\xi_\lambda \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\xi_\lambda\|_\infty \leq 1$ such that*

$$u_\lambda - \frac{1}{\lambda}\mathrm{div}_\gamma \xi_\lambda = \chi_E \qquad \text{in } \mathbb{R}^n \tag{15}$$

and $(\xi_\lambda \cdot D_\gamma u_\lambda) = |D_\gamma u_\lambda|$.

(iii) *u_λ satisfies $-1 \leq u_\lambda \leq 1$;*

(iv) *if E is calibrable then*

$$u_{\lambda|E} = \left(1 - \frac{\lambda_E}{\lambda}\right) \tag{16}$$

for all $\lambda \geq \lambda_E$. If (16) holds for some $\lambda > \lambda_E$, then E is calibrable;

(v) *if $E_1 \subseteq E_2$, and then $u_{\lambda,1}, u_{\lambda,2}$ denote the corresponding solutions of (Q_λ), then $u_{\lambda,1} \leq u_{\lambda,2}$.*

Proof. Point (i) follows by the convexity of the total variation and the strict convexity of the second integral. Point (ii) is proved in [19] in a more general context (see also [15]). Point (iii) follows from a standard truncation argument. Point (iv) follows from the definition of calibrability and the Euler equation of (Q_λ). The comparison principle (v) is contained in Appendix C of [15], properly modified. □

Remark 11. If u_λ is the minimum of (Q_λ) and η is a general admissible vector field, that is $\eta \in L^\infty$ with $\mathrm{div}_\gamma \eta \in L^\infty$, $\|\eta\|_\infty \leq 1$, $[\eta \cdot \nu_E] = -1$ on ∂E, then $\chi_E + \frac{\|\mathrm{div}_\gamma \eta\|_\infty}{\lambda}$ (resp. $\chi_E - \frac{\|\mathrm{div}_\gamma \eta\|_\infty}{\lambda}$) is a supersolution (resp. subsolution), of (15). By the comparison principle [15, 19] we then have

$$\chi_E + \frac{\|\mathrm{div}_\gamma \eta\|_\infty}{\lambda} \geq u_\lambda \geq \chi_E - \frac{\|\mathrm{div}_\gamma \eta\|_\infty}{\lambda}. \tag{17}$$

Lemma 12. *If ∂E is bounded and $C^{1,1}$, then for any $\varepsilon > 0$ there exists $\lambda > 0$ such that $u_\lambda \in [1-\varepsilon, 1]$ in E and $u_\lambda \in [-1, -1+\varepsilon]$ in $\mathbb{R}^n \setminus E$. Hence $[\xi_\lambda \cdot \nu_E] = -1$ $\mathcal{H}^{n-1}$-a.e. in ∂E.*

Proof. The proof is a consequence of Remark 11. In fact, by taking η any admissible extension of $-\nu_E$ which is zero out of a tubular neighborhood of ∂E, equation (17) gives the proof of the first assertion with λ large enough. To prove the second assertion, observe that ∂E belongs to the jump set of u_λ. Then $[\xi_\lambda \cdot \nu_E] = -1$ $\mathcal{H}^{n-1}$-a.e. in ∂E follows from the identity $(\xi_\lambda \cdot D_\gamma u_\lambda) = |D_\gamma u_\lambda|$. □

Proposition 13. *Let $\lambda > 0$. For any $t \in [-1, 1]$, the sets $E_t^\lambda := \{u_\lambda > t\}$, $G_t^\lambda = \{u_\lambda \geq t\}$ are respectively the minimal and maximal solutions of $(P_{\lambda(1-t)})$.*

Proof. Since $(\xi_\lambda \cdot D_\gamma u_\lambda) = |D_\gamma u_\lambda|$, we have that $P(E_t^\lambda) = \int_{\mathbb{R}^n} (\xi_\lambda \cdot D\chi_{E_t^\lambda})$ for almost any t. For any such t and any $F \subseteq E$ we have

$$\begin{aligned} P_\gamma(E_t^\lambda) - \lambda_E \gamma(E_t^\lambda) &= \int_{E_t^\lambda} (-\operatorname{div}_\gamma \xi_\lambda - \lambda_E)\, d\gamma \leq \int_{F \cap E_t^\lambda} (-\operatorname{div}_\gamma \xi_\lambda - \lambda_E)\, d\gamma \\ &= \int_{F \cap E_t^\lambda} (-\operatorname{div}_\gamma \xi_\lambda - \lambda_E)\, d\gamma + \int_{F \setminus E_t^\lambda} (-\operatorname{div}_\gamma \xi_\lambda - \lambda_E)\, d\gamma \\ &= \int_F (-\operatorname{div}_\gamma \xi_\lambda - \lambda_E)\, d\gamma \leq P_\gamma(F) - \lambda_E \gamma(F). \end{aligned}$$

That is E_t^λ is a minimizer of $(P_{\lambda(1-t)})$. If t is any value in $[-1, 1]$, the result follows by approximating t by t_n such that $E_{t_n}^\lambda$ is a minimizer of $(P_{\lambda(1-t_n)})$. Using Lemma 8 we deduce that E_t^λ and G_t^λ are respectively the minimal and the maximal solutions of $(P_{\lambda(1-t)})$. □

Proposition 14. *Let E be a set with a $C^{1,1}$ boundary. Then E minimizes (P_{λ_E}) if and only if E is calibrable.*

Proof. Assume that E is calibrable. Let us prove that E minimizes (P_{λ_E}). In fact, if we consider a calibration ξ of E, we get

$$\lambda_E \gamma(F) = -\int_F \operatorname{div}_\gamma \xi d\gamma = -\int_{\partial^* F} [\xi \cdot \nu_F] \gamma d\mathcal{H}^{n-1} \leq P_\gamma(F),$$

whence $P_\gamma(E) - \lambda_E \gamma(E) = 0 \leq P_\gamma(F) - \lambda_E \gamma(F)$ for any $F \subseteq E$.

On the contrary, if E minimizes (P_{λ_E}), we can consider $\lambda > 0$ be large enough so that Lemma 12 holds and $\operatorname{div}_\gamma \xi_\lambda$, hence u_λ, is not constant in E. In fact, take $\lambda > 0$ such that if $\lambda_E = \lambda(1 - \bar{t})$, then $\bar{t} \in [1-\varepsilon, 1]$. Since $[\xi_\lambda \cdot \nu_E] = -1$ $\mathcal{H}^{n-1}$-a.e. in ∂E, we have

$$\frac{1}{\gamma(E)} \int_E \operatorname{div}_\gamma \xi_\lambda \, d\gamma = -\lambda_E.$$

Then $\{x \in E : \operatorname{div}_\gamma \xi_\lambda > -\lambda_E\} \neq E$. Observe that $\{x \in E : \operatorname{div}_\gamma \xi_\lambda > -\lambda_E\} = \{x \in \mathbb{R}^n : u_\lambda(x) > \bar{t}\} =: E_{\bar{t}}^\lambda$ where $\lambda(1 - \bar{t}) = \lambda_E$. Then

$$P_\gamma(E_{\bar{t}}^\lambda) - \lambda_E \gamma(E_{\bar{t}}^\lambda) = \int_{E_{\bar{t}}^\lambda} (-\operatorname{div}_\gamma \xi_\lambda - \lambda_E)\, d\gamma < 0.$$

On the other hand, by Proposition 13, $E^\lambda_{\bar t}$ is a minimizer of (P_{λ_E}). Then

$$P_\gamma(E^\lambda_{\bar t}) - \lambda_E \gamma(E^\lambda_{\bar t}) = P_\gamma(E) - \lambda_E\gamma(E) = 0.$$

This contradiction proves that $\mathrm{div}_\gamma\, \xi_\lambda$ is constant. Integrating by parts we deduce that $\mathrm{div}_\gamma\, \xi_\lambda = -\lambda_E$. Thus E is calibrable. □

4 Characterization of convex calibrable sets in the Gauss space

The following theorem, contained in [27, Section 3], extends the concavity result [27, Theorem 1.2] to the x-dependent case.

Theorem 15 (Korevaar)**.** *Let Ω be a C^1 convex and bounded domain in $\mathbb{R}^n$, and let $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\frac{\partial b}{\partial u} > 0, \qquad b \text{ jointly concave in } (x,u).$$

Assume that $u \in C(\bar\Omega) \cap C^2(\Omega)$ satisfies

$$\mathrm{div}\left(\frac{Du(x)}{\sqrt{1+|Du(x)|^2}}\right) = b(x, u(x), Du(x)),$$

coupled with the boundary conditions of vertical contact angle

$$\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu_\Omega = -1.$$

Then u is a concave function.

Theorem 16. *Let E be a bounded, convex domain in $\mathbb{R}^n$ of class $C^{1,1}$. If λ is large enough, then the solution u_λ of (Q_λ) is concave in E, with vertical contact angle at ∂E. In particular the set $E^\lambda_s = \{u_\lambda \geq s\} \cap E$ is convex for any $s \in [0,1]$ and it is the unique minimum of (P_μ) with $\mu = \lambda(1-s)$.*

Proof. The proof follows exactly as in [2, Theorem 5], using the result of Korevaar stated in Theorem 15. □

Remark 17. If the $C^{1,1}$ assumption is removed, the same result holds on $E \cap \{u_\lambda > 0\}$, where u_λ minimizes (14) with Dirichlet boundary conditions on ∂E, that is

$$\min\left\{|D_\gamma u|(\mathbb{R}^n) + \frac{\lambda}{2}\int_{\mathbb{R}^n} (u - \mathbf{1}_E)^2 d\gamma :\ u \in BV(\mathbb{R}^n,\gamma) \cap L^2(\mathbb{R}^n,\gamma),\ u \equiv 0 \text{ on } \mathbb{R}^n \setminus E\right\}, \tag{18}$$

Notice that the minimum problem (18) is equivalent to

$$\min\left\{|D_\gamma u|(E) + \int_{\partial E} |u|\, \gamma\, d\mathcal{H}^{n-1} + \frac{\lambda}{2}\int_E (u-1)^2 d\gamma : u \in BV(\mathbb{R}^n,\gamma) \cap L^2(\mathbb{R}^n,\gamma)\right\}.$$

Lemma 18. *Let $E \subseteq \mathbb{R}^n$ be a bounded convex set of class $C^{1,1}$, let $\lambda_j \to \lambda$, and let E_j be convex minimizers of (P_{λ_j}). If E_j converge to E, then $\lambda \le (n-1)\|H_E^\gamma\|_\infty$.*

Proof. Since E is of class $C^{1,1}$, the outer unit normal vector field to ∂E admits a Lipschitz extension N to a neighborhood $U = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \partial E) < \delta\}$ of ∂E, $\delta > 0$, and we have that $(n-1)H_E^\gamma = \mathrm{div}_\gamma N$ on ∂E. If $\|\mathrm{div}_\gamma N|_U\|_\infty < \lambda$, then for j large enough we have that $\partial E_j \subseteq U$ and $\|\mathrm{div}_\gamma N|_U\|_\infty < \lambda_j$. Then

$$\begin{aligned}\lambda_j \gamma(E \setminus E_j) &> \int_{E\setminus E_j} \mathrm{div}_\gamma N \, d\gamma = \int_{\partial E \setminus \partial E_j} [N \cdot \nu_E]\, \gamma \, d\mathcal{H}^{n-1} - \int_{\partial E_j \setminus \partial E} [N \cdot \nu_{E_j}]\, \gamma \, d\mathcal{H}^{n-1} \\ &\ge \int_{\partial E \setminus \partial E_j} \gamma \, d\mathcal{H}^{n-1} - \int_{\partial E_j \setminus \partial E} \gamma \, d\mathcal{H}^{n-1} = P_\gamma(E) - P_\gamma(E_j).\end{aligned}$$

Hence

$$P_\gamma(E) - \lambda_j \gamma(E) < P_\gamma(E_j) - \lambda_j \gamma(E_j).$$

This contradiction proves that $\lambda \le \|\mathrm{div}_\gamma N|_U\|_\infty$. Letting $\delta \to 0^+$ we deduce that $\lambda \le (n-1)\|H_E^\gamma\|_\infty$. □

Proposition 19. *Let E be a bounded convex subset of $\mathbb{R}^n$. Then E minimizes (P_λ) with $\lambda \ge \lambda_E$ if and only if E is of class $C^{1,1}$ and*

$$(n-1)H_E^\gamma \le \lambda. \tag{19}$$

In particular, thanks to Proposition 14, E is calibrable if and only if E satisfies (19) *with $\lambda = \lambda_E$.*

Proof. Let E be a bounded convex set of class $C^{1,1}$ satisfying (19). Reasoning as in [2, Theorem 9], from Theorem 16 and Lemma 18 it follows that E minimizes (P_λ), for all $\lambda \ge \lambda_E$.

In order to prove the reverse implication, we first assume that E is of class $C^{1,1}$. Then, since E minimizes (P_λ), by a classical first variation argument [30] we get (19). In the general case, we first assume $\lambda > \lambda_E$ and we approximate E with a sequence of $C^{1,1}$ convex sets E_j so that

$$E = \bigcap_j E_j.$$

We note that, since $\lambda_{E_j} \to \lambda_E$, we have $\lambda_{E_j} < \lambda$ for j large enough. Then we consider the problems

$$\min_{F \subseteq E_j} \{P_\gamma(F) - \lambda\gamma(F)\}.$$

By Theorem 16, there exist unique convex minima $E_{j,\lambda}$ and $E_{j,\lambda} \to E$ as $j \to +\infty$. Moreover, by [30, Theorem 3.6], $E_{j,\lambda}$ is of class $C^{1,1}$ and therefore

$$(n-1)H_{E_{j,\lambda}}^\gamma \le \lambda.$$

Since $E_{j,\lambda}$ are convex sets converging to E, we have that E satisfies (19). By letting $\lambda \to \lambda_E$, we obtain the same result for $\lambda = \lambda_E$, thus concluding the proof. □

Remark 20. As a consequence of Proposition 19, we have that every ball B_R, centered at the origin, is calibrable. Indeed, we have

$$(n-1)H^{\gamma}_{B_R} = \frac{1-R^2}{R} < \frac{e^{-\frac{R^2}{2}}R^{n-1}}{\int_0^R e^{-\frac{r^2}{2}}r^{n-1}\,dr} = \frac{P_\gamma(B_R)}{\gamma(B_R)}.$$

Similarly, every hyperplane $H = \{x \cdot \nu \leq R\}$, ν and R fixed, is also calibrable, since

$$(n-1)H^{\gamma}_{H} = -R < \frac{e^{-\frac{R^2}{2}}}{\int_{-\infty}^R e^{-\frac{r^2}{2}}\,dr} = \frac{P_\gamma(H)}{\gamma(H)}.$$

To our knowledge, in the Gauss space it is an open question whether the level sets $\{u_\lambda \geq s\}$ are convex for all $s \in [0,1]$, if E is a convex set.

Conjecture 21. *Let $\Omega_0 \supset \Omega_1$ two open, bounded and convex sets and let u be a solution of the problem*

$$\begin{cases} \Delta u = \langle x, \nabla u\rangle & on\ \Omega_0 \setminus \bar{\Omega}_1 \\ u = 0 & on\ \partial\Omega_0 \\ u = 1 & on\ \partial\Omega_1. \end{cases}$$

Then $\{u \geq t\}$ is convex for any $t \in \mathbb{R}$.

Conjecture 22. *Let u_λ be minimizer of*

$$|D_\gamma u| + \frac{\lambda}{2}\int_{\mathbb{R}^n} |u-v|^2 d\gamma,$$

with v level–set convex, i.e. $\{v > t\}$ is convex for a.e. $t \in \mathbb{R}$, then u_λ is level–set convex.

5 An isoperimetric problem inside convex sets in the Gauss space

We relate in this section the minima of (P_λ) with the minima of the constrained isoperimetric problem:

$$(I_V): \qquad \min\{P_\gamma(F) : F \subseteq E, \gamma(F) = V\},$$

with $V \in [0, \gamma(E)]$.

Given $E \subseteq \mathbb{R}^n$, we say that $K \subseteq E$ with positive measure is a γ-*Cheeger set* of E if K is a minimum of the problem

$$\min_{F \subseteq E} \frac{P_\gamma(F)}{\gamma(F)}. \tag{20}$$

We call the value of (20) the γ-*Cheeger constant* of E, and we will denote it by $\overline{\lambda}_E$. Notice that if K is of positive measure, K is a Cheeger set of E if and only if it is a minimizer of (P_{λ_K}).

From the results of Section 4, reasoning as in [2, Section 4], we obtain the following Theorem.

Theorem 23. *Let $E \subseteq \mathbb{R}^n$ be a bounded and convex set. Then, there is a convex calibrable set $K \subseteq E$ which is a maximal minimizer of (P_{λ_K}). Thus K is the maximal γ-Cheeger set of E. Moreover, for any $\lambda > \lambda_K$ there exists a unique minimizer E_λ of (P_λ), which is convex, and the map $\lambda \mapsto E_\lambda$ is increasing and continuous on $[\lambda_K, +\infty)$. In addition, for any $V \in [\gamma(K), \gamma(E)]$, there is a unique solution of problem (I_V), which is convex.*

We point out that, when $V \in (0, \gamma(K))$, the uniqueness (up to translations) of the solutions of (I_V) is an open problem, even in the Euclidean case.

Remark 24. Let E be a bounded convex set and let K be its maximal γ-Cheeger set given by Theorem 23. It follows by the previous discussion that there exists a vector field $\xi \in L^\infty(E, \mathbb{R}^n)$, with $|\xi| \le 1$ and $[\xi \cdot \nu_E] = -1$ on ∂E, such that $\mathrm{div}_\gamma \xi \in L^1(E, \gamma)$, $\mathrm{div}_\gamma \xi \equiv -\lambda_K$ on K, and $\mathrm{div}_\gamma \xi < -\lambda_K$ on $E \setminus K$.

Notice that, conversely, the existence of such vector field implies that K is a γ-Cheeger set of E.

6 The variational problem in infinite dimensions

In the next sections we work in the setting of the abstract Wiener space.

Proposition 25. *Let $f \in L^2(X, \gamma)$ and $\lambda > 0$. Then there exists a unique minimum u_λ of the problem*

$$\min \ |D_\gamma u|(X) + \frac{1}{2\lambda} \int_X |u - f|^2 d\gamma. \tag{21}$$

If $f, g \in L^2(X, \gamma)$ and $u, v \in L^2(X, \gamma)$ are the corresponding solutions, then

$$\|u - v\|_2 \le \|f - g\|_2. \tag{22}$$

Moreover $u_\lambda \to f$ in $L^2(X, \gamma)$ as $\lambda \to 0^+$.

Proof. The existence follows since the total variation is lower semicontinuous with respect to weak convergence in $L^2(X, \gamma)$, and the uniqueness follows from the strict convexity of the functional (21).

Estimate (22) follows since the subdifferential of the total variation is a monotone operator in $L^2(X, \gamma)$.

To prove the last assertion we first assume that $f \in BV(X, \gamma)$. Then, taking f as a test function, we have

$$|D_\gamma u_\lambda|(X) + \frac{1}{2\lambda} \int_X |u_\lambda - f|^2 d\gamma \le |D_\gamma f|(X).$$

Then clearly $u_\lambda \to f$ in $L^2(X, \gamma)$.

If $f \in L^2(X, \gamma)$, we approximate it in $L^2(X, \gamma)$ by functions $f_j \in BV(X, \gamma)$. We can take, for instance, the conditional expectations $f_j = \mathbb{E}_j f$. Letting $u_{\lambda,j}$ be the solutions of the corresponding problem, using (22) we have

$$\begin{aligned} \|u_\lambda - f\|_2 &\le \|u_\lambda - u_{\lambda,j}\|_2 + \|u_{\lambda,j} - f_j\|_2 + \|f_j - f\|_2 \\ &\le 2\|f_j - f\|_2 + \|u_{\lambda,j} - f_j\|_2. \end{aligned}$$

It then follows

$$\limsup_{\lambda \to 0^+} \|u_\lambda - f\|_2 \le 2\|f_j - f\|_2.$$

□

7 The characterization of the subdifferential of total variation

Let E be a normed space, and let E^* be its dual space. Let $\Psi : E \to [0, \infty]$ be any function. Let us define $\tilde{\Psi} : E^* \to [0, \infty]$ by

$$\tilde{\Psi}(x^*) := \sup \left\{ \frac{\langle x^*, y \rangle}{\Psi(y)} \ : \ y \in E \right\} \tag{23}$$

with the convention that $\frac{0}{0} = 0$, $\frac{0}{\infty} = 0$. Note that $\tilde{\Psi}(x^*) \ge 0$, for any $x^* \in E^*$. Note also that the supremum is attained on the set of $y \in E^*$ such that $\langle x^*, y \rangle \ge 0$.

Let us consider the functional $\Phi : L^2(X, \gamma) \to (-\infty, +\infty]$ defined as

$$\Phi(u) := |D_\gamma u|(X) \qquad \text{if } u \in BV(X, \gamma),$$

and $= +\infty$ if $u \in L^2(X, \gamma) \setminus BV(X, \gamma)$.

Proposition 26. *For any $\varphi \in C_b^1(X)$, one has*

$$\left| \int_X \varphi \, (z \cdot D_\gamma u) \right| \le \sup \|\varphi\|_\infty \, \|z\|_\infty \, |D_\gamma u|(X). \tag{24}$$

Proof. Take a sequence $u_n \in C_b^1(X)$ converging to u in $L^1(X, \gamma)$ and $|D_\gamma u_n|(X) \to |D_\gamma u|(X)$. Let $\varphi \in C_b^1(X)$. Then

$$\left| \int_X (z \cdot D_\gamma u_n) \varphi \right| \le \sup \|\varphi\|_\infty \, \|z\|_{L^\infty(X,\gamma)} \, |D_\gamma u_n|(X) \quad \text{ for all } \ n \in \mathbb{N}.$$

Now, taking the limit as $n \to \infty$, we get the thesis. □

The next result follows immediately from the definition of $\int_X (z \cdot D_\gamma u)$.

Lemma 27. *Let $z \in \mathcal{X}_2(X, H)$ and $u \in BV(X, \gamma) \cap L^2(X, \gamma)$. Let $u_n \in C_b^1(X)$ converging weakly to u in $L^2(X, \gamma)$. Then we have*

$$\int_X [z, \nabla_\gamma u_n]_H \, d\gamma \to \int_X (z \cdot D_\gamma u).$$

Lemma 28. *Let $z \in \mathcal{X}_2(X, H)$, $u \in BV(X, \gamma)$ be such that $\int_X (z \cdot D_\gamma u) = |D_\gamma u|(X)$. Then for almost any $t \in \mathbb{R}$ we have*

$$\int_X (z \cdot D_\gamma \mathbf{1}_{\{u > t\}}) = P_\gamma(\{u > t\}). \tag{25}$$

Moreover, $P_\gamma(\{u > t\}) < \infty$ for all $t \in \mathbb{R}$ and (25) holds for any $t \in \mathbb{R}$.

Proof. Assume that $u \geq 0$. Let $\varphi \in C_b^1(X)$.

$$\begin{aligned}\int_X (z \cdot D_\gamma u)\varphi &= -\int_X u\varphi \,\mathrm{div}_\gamma z \, d\gamma - \int_X u\,[z, \nabla_\gamma \varphi]_H \, d\gamma \\ &= -\int_0^\infty dt \int_X \mathbf{1}_{\{u>t\}} \left(\varphi \mathrm{div}_\gamma z + [z, \nabla_\gamma \varphi]_H\right) d\gamma \\ &= \int_0^\infty dt \int_X (z \cdot D_\gamma \mathbf{1}_{\{u>t\}})\varphi.\end{aligned}$$

Then

$$\begin{aligned}|D_\gamma u|(X) = \int_X (z \cdot D_\gamma u) &= \int_0^\infty dt \int_X (z \cdot D_\gamma \mathbf{1}_{\{u>t\}}) \\ &\leq \int_0^\infty |D_\gamma \mathbf{1}_{\{u>t\}}|(X)\, dt = |D_\gamma u|(X)\end{aligned}$$

and (25) follows.

Let $t \in \mathbb{R}$ be such that (25) holds. Then

$$P_\gamma(\{u > t\}) = \int_X (z \cdot D_\gamma \mathbf{1}_{\{u>t\}}) = -\int_{\{u>t\}} \mathrm{div}_\gamma z \, d\gamma \leq \|\mathrm{div}_\gamma z\|_2.$$

That is the perimeter of all level sets is equibounded. Then given $t \in \mathbb{R}$ we may approximate it by $t_n \in \mathbb{R}$ for which (25) holds. By the lower semicontinuity of the perimeter we have that

$$P_\gamma(\{u > t\}) \leq \|\mathrm{div}_\gamma\, z\|_2.$$

The last assertion follows now by approximation of $\mathbf{1}_{\{u>t\}}$ by $\mathbf{1}_{\{u>t_n\}}$. □

Theorem 29. *Let $z \in \mathcal{X}_2(X, H)$ and $u \in BV(X, \gamma) \cap L^2(X, \gamma)$, then we have*

$$\int_X u \,\mathrm{div}_\gamma z \, d\gamma + \int_X (z \cdot D_\gamma u) = 0. \tag{26}$$

Proof. Take a sequence of functions $u_n \in C_b^1(X)$ converging weakly to u in $L^2(X, \gamma)$. Then, by Lemma 27 and (8), we have

$$\int_x u \,\mathrm{div}_\gamma z \, d\gamma + \int_X (z \cdot D_\gamma u) = \lim_{n\to\infty} \left(\int_X u_n \,\mathrm{div}_\gamma z \, d\gamma + \int_X [z, \nabla_\gamma u_n]_H \, d\gamma \right) = 0.$$

□

For $v \in L^2(X, \gamma)$, we define

$$\Psi(v) := \inf\left\{ \|z\|_\infty \;:\; z \in \mathcal{X}_2(X, H),\, v = -\mathrm{div}_\gamma z \right\}. \tag{27}$$

Since H is separable, then $L^1(X, \gamma)^* = L^\infty(X, \gamma)$ and by weak* compactness of the unit ball in $L^\infty(X, \gamma)$, we know that if $\Psi(v) < \infty$, then the infimum in (27) is attained, i.e., there is some $z \in \mathcal{X}_2(X, H)$ such that $v = -\mathrm{div}_\gamma z$, and $\Psi(v) = \|z\|_\infty$.

Proposition 30. $\Psi = \tilde{\Phi}$.

Proof. Let $v \in L^2(X,\gamma)$. If $\Psi(v) = +\infty$, then we have $\tilde{\Phi}(v) \le \Psi(v)$. Thus, we may assume that $\Psi(v) < \infty$. Let $z \in \mathcal{X}_2(X,H)$ such that $v = -\mathrm{div}_\gamma z$ with test functions in $C_b^1(X)$. Then

$$\int_X vu\,d\gamma = \int_X (z\cdot D_\gamma u) \le \|z\|_\infty \Phi(u) \quad \text{for all } u \in BV(X,\gamma)\cap L^2(X,\gamma).$$

Taking the supremum in u we obtain $\tilde{\Phi}(v) \le \|z\|_\infty$, and taking the infimum in z we obtain $\tilde{\Phi}(v) \le \Psi(v)$.

In order to prove the opposite inequality, let us denote

$$D := \left\{\mathrm{div}_\gamma z : \ z \in C_b^1(X,H)\right\}.$$

Then

$$\begin{aligned}\sup_{v\in L^2} \frac{\int_X uv\,d\gamma}{\Psi(v)} \quad &\ge \sup_{v\in D,\Psi(v)<\infty} \frac{\int_X uv\,d\gamma}{\Psi(v)} \\ &\ge \sup_{z\in C_b^1(X,H)} \frac{-\int_X u\mathrm{div}_\gamma z\,d\gamma}{\|z\|_\infty} \ge \Phi(u).\end{aligned}$$

Thus, $\Phi \le \tilde{\Psi}$. This implies that $\tilde{\tilde{\Psi}} \le \tilde{\Phi}$, moreover, since $\tilde{\tilde{\Psi}} = \Psi$ [7, Proposition 1.6], we obtain that $\Psi \le \tilde{\Phi}$. □

We recall the following result which is proved in [7].

Theorem 31. *Assume that Φ is convex, lower semi-continuous and positive homogeneous of degree 1. Then $v^* \in \partial\Phi(u)$ if and only if $\tilde{\Phi}(v^*) \le 1$ and $\langle v^*, u\rangle = \Phi(u)$ (hence, $\tilde{\Phi}(v^*) = 1$ if $\Phi(u) > 0$).*

Proposition 32. *Let $u, v \in L^2(X,\gamma)$, $u \in BV(X,\gamma)$. The following assertions are equivalent:*

(a) $v \in \partial\Phi(u)$;

(b)

$$\int_X vu\,d\gamma = \Phi(u), \tag{28}$$

$$\exists z \in \mathcal{X}_2(X,H) \text{ such that } v = -\mathrm{div}_\gamma z; \tag{29}$$

(c) (29) holds and

$$\int_X (z\cdot D_\gamma u) = |D_\gamma u|(X). \tag{30}$$

Proof. By Theorem 31, we have that $v \in \partial\Phi(u)$ if and only if $\tilde{\Phi}(v) \le 1$ and $\int_\Omega vu\,dx = \Phi(u)$. Since $\tilde{\Phi} = \Psi$, the equivalence of (a) and (b) follows from the definition of Ψ. If (b) holds, integrating by parts in (28) we obtain (30). The converse implication follows in the same way. □

In a subsequent work, we shall use the results of this Section to show that all the balls of X have finite perimeter, when X is a separable Hilbert space.

8 Existence of minimizers of (P_μ)

Proposition 33. *Let $f \in L^\infty(X,\gamma)$, and let u be the (unique) minimizer of* (21)*. Then $u \in L^\infty(X,\gamma)$ and $\{u = \|u\|_\infty\}$ is of positive measure.*

Proof. The proof that $u \in L^\infty(X,\gamma)$ follows by a standard truncation argument which gives the estimate $\|u\|_\infty \le \|f\|_\infty$.

By Proposition 32, we know that there exists $z \in \mathcal{X}_2(X,H)$ with $\int_X (z \cdot D_\gamma u) = |D_\gamma u|(X)$ and such that

$$u - \operatorname{div}_\gamma z = f.$$

Multiplying the last equation by $\mathbf{1}_{\{u>t\}}$, and integrating by parts, we obtain

$$P_\gamma(\{u>t\}) = \int_X (z \cdot \mathbf{1}_{\{u>t\}})\, d\gamma = \int_X (f-u)\mathbf{1}_{\{u>t\}}\, d\gamma \le \|f-u\|_\infty \gamma(\{u>t\}).$$

Dividing both sides by $\mathcal{U}(\gamma(\{u>t\}))$, and using the isoperimetric inequality in Proposition 1 and the fact that $\mathcal{U}'(0) = +\infty$, we get a uniform lower bound on $\gamma(\{u>t\})$. □

As in the finite dimensional case, given $E \subseteq X$ we say that $K \subseteq E$ with positive measure is a γ-*Cheeger set* of E if K is a minimum of the problem

$$\min_{F \subseteq E} \frac{P_\gamma(F)}{\gamma(F)}. \tag{31}$$

We call the value of (31) the γ-*Cheeger constant* of E, and we will denote it by $\overline{\lambda}_E$. Notice that K is a γ-Cheeger set of E if and only if it is a minimizer of the problem

$$(P_\mu): \qquad \min_{F \subseteq E} \{P_\gamma(F) - \mu\gamma(F)\}, \tag{32}$$

with $\mu = \lambda_K = \overline{\lambda}_E$. If c_μ denotes the minimum of (32), we observe that $c_{\mu_1} \le c_{\mu_2}$ if $\mu_1 \ge \mu_2 \ge 0$. In particular, if $\operatorname{int}(E) \neq \emptyset$, by comparison with small balls and recalling Lemma 3 we get $c_\mu \le 0$ for all $\mu \ge 0$. In particular, since $c_{\overline{\lambda}_E} = 0$, it follows $c_\mu = 0$ for all $\mu \le \overline{\lambda}_E$, that is, $F = \emptyset$ is a solution of (P_μ) when $\mu \le \overline{\lambda}_E$.

Proposition 34. *Let $E \subseteq X$ with $\operatorname{int}(E) \neq \emptyset$. Then, there exists a solution E_μ of* (32)*. Moreover, we can choose $E_\mu \neq \emptyset$ if $\mu \ge \overline{\lambda}_E$, where $\overline{\lambda}_E$ is the γ-Cheeger constant of E. In particular, there always exists a γ-Cheeger set of E.*

Proof. We can assume $\mu \ge \overline{\lambda}_E$. Let E_j be a minimizing sequence of (32), and let $u_\mu \in BV(X,\gamma)$ be the (weak) limit of χ_{E_j}. Then

$$c_\mu \ge |D_\gamma u_\mu|(X) = \int_0^1 P_\gamma(\{u_\mu > t\})dt. \tag{33}$$

If $u_\mu \neq 0$, by the coarea formula $\{u_\mu > t\}$ is a solution of (32) for almost all $t \in (0,1)$ and the equality holds in (33). Moreover, there exists $t \in (0,1)$ such that $\{u_\mu > t\}$ is nonempty.

Let now $\mu = \overline{\lambda}_E$. In this case, we can choose the sequence E_j as a minimizing sequence of (31). Recalling that $\operatorname{int}(E) \neq \emptyset$, by the isoperimetric inequality, we then have a uniform lower bound on the volume of E_j, which in turn implies $u_{\overline{\lambda}_E} \neq 0$. In particular, there exists a nonempty γ-Cheeger set K of E.

It remains to prove that $u_\mu \neq 0$, for all $\mu > \overline{\lambda}_E$. By contradiction, if $u_\mu = 0$, we would have $c_\mu = 0$, but this is impossible since $P_\gamma(K) - \mu\gamma(K) < 0$. $\square$

Remark 35. Let us mention that the result analogous to Lemma 8 holds also in the infinite dimensional case with the same proof as in [2].

9 Uniqueness and convexity of minimizers of (P_μ)

Let C be a bounded convex subset of X and assume that C has finite perimeter. Let us consider the following problem:

$$\min\left\{|D_\gamma u|(X) + \frac{\lambda}{2}\int_X (u - \mathbf{1}_C)^2 d\gamma : u \in BV(X,\gamma) \cap L^2(X,\gamma),\ u \equiv 0 \text{ outside } C\right\}. \quad (34)$$

Proposition 36. *Let C be a bounded convex subset of X with nonempty interior. Assume that C has finite perimeter. Then problem (34) has a unique solution u_λ for all $\lambda > 0$, and we have $0 \leq u_\lambda \leq 1$. Moreover for any $\lambda > \overline{\lambda}_C$, $u_\lambda \neq 0$ is a concave function restricted to the set $\{u_\lambda > 0\}$.*

Proof. As in Propositions 25 and 33, there is a unique solution u_λ of problem (34) and it satisfies $0 \leq u_\lambda \leq 1$.

The concavity of u_λ in $\{u_\lambda > 0\}$ follows by an approximation argument. Let $C_n := \Pi_n(C) \times X_n^\perp$. Then, C_n is a cylindrical approximation of C such that $C_{n+1} \subseteq C_n$. Since C is closed we have $C = \cap_n C_n$, and $P_\gamma(C) \leq \liminf_n P_\gamma(C_n)$, by the lower semicontinuity of P_γ.

Let $\lambda > 0$, and let $u_{\lambda,n} = v_{\lambda,n} \circ \Pi_n$, where $v_{\lambda,n}$ minimizes (34) with C replaced by $\Pi_n(C)$; we point out that by Theorem 15 and Remark 17, $u_{\lambda,n}$ are concave on $\{u_{\lambda,n} > 0\}$. Then it follows that $u_{\lambda,n}$ minimizes (34) with C replaced by C_n. By Theorem 23, there exists a convex maximal γ-Cheeger set $\overline{K}_n \subseteq \Pi_n(C)$ for all $n \in \mathbb{N}$ and, thanks to the characterization given in Remark 24, the set $K_n := \overline{K}_n \times \mathbb{R}^{n\perp}$ is the maximal γ-Cheeger set of C_n. Finally, $u_{\lambda,n}$ attains its maximum on K_n. By integrating the Euler-Lagrange equation (15) on $\overline{K}_n$, we get

$$\lambda_{K_n} = \frac{P_\gamma(K_n)}{\gamma(K_n)} = \lambda\left(1 - \max_{C_n} u_{\lambda,n}\right),$$

which implies

$$1 > u_{\lambda,n}(x) = 1 - \frac{\lambda_{K_n}}{\lambda} \geq 1 - \frac{\lambda_C}{\lambda} \qquad x \in K_n\,.$$

Moreover, recalling the isoperimetric inequality (9), we also get

$$\frac{\mathcal{U}(\gamma(K_n))}{\gamma(K_n)} \leq \frac{P_\gamma(K_n)}{\gamma(K_n)} \leq \lambda_C,$$

which implies, since $\mathcal{U}(t) \sim t\sqrt{2\log 1/t}$ as $t \to 0$,

$$\gamma(K_n) \geq c > 0,$$

for some constant c independent of n. It then follows

$$\int_{C_n} u_{\lambda,n}\, d\gamma \geq \left(1 - \frac{\lambda_C}{\lambda}\right)\gamma(K_n) \geq \left(1 - \frac{\lambda_C}{\lambda}\right)c. \tag{35}$$

We now let $u_\lambda := \lim_n u_{\lambda,n} = \inf_n u_{\lambda,n}$, which is a minimizer of (34). Indeed, if $v \in BV(X,\gamma)\cap L^2(X,\gamma)$ is such that $v = 0$ out of C, then its canonical cylindrical approximation v_n is also in $BV(\mathbb{R}^n,\gamma)\cap L^2(\mathbb{R}^n,\gamma)$, $v_n = 0$ out of C_n, $v_n \to v$ in $L^2(X,\gamma)$ and $|D_\gamma v_n|(X) \to |D_\gamma v|(X)$. Then

$$\begin{aligned}
|D_\gamma u_\lambda|(X) + \frac{\lambda}{2}\int_X (u_\lambda - \mathbf{1}_C)^2 d\gamma &\leq \liminf_n |D_\gamma u_{\lambda,n}|(X) + \frac{\lambda}{2}\int_X (u_{\lambda,n} - \mathbf{1}_{C,n})^2 d\gamma \\
&\leq \lim_n |D_\gamma v_n|(X) + \frac{\lambda}{2}\int_X (v_n - \mathbf{1}_{C_n})^2 d\gamma \\
&= |D_\gamma v|(X) + \frac{\lambda}{2}\int_X (v - \mathbf{1}_C)^2 d\gamma.
\end{aligned}$$

Passing to the limit in (35) we obtain

$$\int_C u_\lambda\, d\gamma \geq \left(1 - \frac{\lambda_C}{\lambda}\right)c.$$

In particular, u_λ is not identically zero on C, and it is concave on $\{u_\lambda > 0\}$. □

Proposition 37. *For any $t \in [0,1]$, the set $E_t^\lambda = \{u_\lambda > t\}$ is a solution of*

$$(P_{\lambda(1-t)}): \qquad \min_{E\subseteq C} \{P_\gamma(E) - \lambda(1-t)\gamma(E)\}.$$

The same result holds for the set $\{u_\lambda \geq t\}$. If $\lambda > \overline{\lambda}_C$ and $t < \max u_\lambda$, the solution of $(P_{\lambda(1-t)})$ is unique (modulo γ-null sets) and convex. Moreover, there exists a maximal convex γ-Cheeger set $K \subseteq C$, which is equal to $\{u_\lambda = \|u_\lambda\|_\infty\}$ for all $\lambda > \overline{\lambda}_C = \lambda_K = \lambda(1 - \|u_\lambda\|_\infty)$, and there exists a unique convex minimizer C_μ of (32) for all $\mu > \overline{\lambda}_C$.

Proof. We observe that, as in Proposition 34, there is a solution of $(P_{\lambda(1-t)})$ for any $t \in [0,1]$. Let us denote it by F_t. By Lemma 8 and Remark 35, we have that $F_t \subseteq F_{t'}$ if $t > t'$. Let

$$w(x) := \sup\{t \in [0,1] : x \in F_t\}.$$

Then $\{w > t\} = F_t$ for a.e. $t \in (\inf w, \sup w)$, $0 \leq w \leq 1$, and $w \equiv 0$ out of C. Since

$$\int_0^1 P_\gamma(F_t)\, dt \leq \lambda \int_0^1 (1-t)\gamma(F_t)\, dt = \frac{\lambda}{2}\int_X (w - \mathbf{1}_C)^2 - \frac{\lambda}{2}\gamma(C),$$

it follows that $w \in BV(X,\gamma)$. Moreover,

$$\begin{aligned}
|D_\gamma w|(X) + \frac{\lambda}{2}\int_X (w - \mathbf{1}_C)^2\, dx &= \int_0^1 P_\gamma(F_t)\, dt - \lambda\int_0^1 (1-t)\gamma(F_t)\, dt + \frac{\lambda}{2}\gamma(C) \\
&\leq \int_0^1 P_\gamma(E_t^\lambda)\, dt - \lambda\int_0^1 (1-t)\gamma(E_t^\lambda)\, dt + \frac{\lambda}{2}\gamma(C) \\
&= |D_\gamma u_\lambda|(X) + \frac{\lambda}{2}\int_X (u_\lambda - \mathbf{1}_C)^2\, dx.
\end{aligned}$$

Since the solution of (34) is unique, we have $w = u_\lambda$. Then $\{w > t\} = E_t^\lambda$ for a.e. in $t \in [0,1]$, that is, there exists a subset $I \subseteq [0,1]$, with $|I| = 1$, such that E_t^λ is a solution of $(P_{\lambda(1-t)})$ for any $t \in I$. If $t \in [0,1]$, we may approximate it by a sequence $t_n \in I$ so that $E_{t_n}^\lambda$ is a solution of $(P_{\lambda(1-t_n)})$. Passing to the limit as $n \to +\infty$, we then obtain that E_t^λ is a solution of $(P_{\lambda(1-t)})$.

If $\lambda > \overline{\lambda}_C$ and $t < \max u_\lambda$, the convexity of E_t^λ follows from the concavity of u_λ restricted to the set $\{u_\lambda > 0\}$.

Let now E_t' be another solution of $(P_{\lambda(1-t)})$. By Lemma 8, if $t_1 < t < t_2$ we have $E_{t_2}^\lambda \subseteq E_t' \subseteq E_{t_1}^\lambda$, hence $\{u_\lambda > t\} \subseteq E_t' \subseteq \{u_\lambda \geq t\}$. By Theorem 4 we then have $E_t' = \{u_\lambda > t\} = \{u_\lambda \geq t\}$ modulo a γ-null set.

The last statements follow exactly as in [2, Section 4]. □

We point out that in the previous proof we did not use Proposition 34.

As in the finite dimensional case, Proposition 37 implies the following result.

Theorem 38. *Let C be a bounded convex subset of X with nonempty interior. Assume that C has finite perimeter. For any $V \in [\gamma(K), \gamma(C)]$, there exists a unique convex solution of the constrained isoperimetric problem*

$$\min\{P_\gamma(F) : F \subseteq C, \gamma(F) = V\}. \tag{36}$$

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