Partial regularity for quasi minimizers of perimeter

Luigi Ambrosio
Dipartimento di Matematica
via Abbiategrasso, 205
27100 Pavia, Italia

Emanuele Paolini
Scuola Normale Superiore
piazza dei Cavalieri, 7
56126 Pisa, Italia

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Abstract

Let $E \subseteq \mathbf{R}^n$ be a quasi minimizer of perimeter, that is, a set such that $P(E,\, B_\rho(x)) \leq (1+\omega(\rho))P(F, B_\rho(x))$ for all variations F with $F \triangle E \Subset B_\rho(x)$ and for a given function ω with $\lim_{\rho\to 0}\omega(\rho) = 0$. We prove that, up to a closed set with dimension at most $n-8$, for all $\alpha < 1$ the set ∂E is an $(n-1)$-dimensional $C^{0,\alpha}$ manifold. This result is obtained combining the De Giorgi and Reifenberg regularity theories for area minimizers. Moreover we prove that, in the case $n = 2$, ∂E is a bi-lipschitz curve.

Introduction

The aim of this paper is to study the regularity of quasi minimizers of perimeter. We consider the following multiplicative quasi minimality condition

$$P(E, B_\rho(x)) \leq (1+\omega(\rho))\, P(F, B_\rho(x))$$

for all variations F of the set E such that $F \triangle E \Subset B_\rho(x)$ and for a given function ω which is infinitesimal as $\rho \to 0$.

It is well known (see [24, 6]) that with the assumption $\omega(\rho) \leq c\rho^{2\alpha}$ (for some $c, \alpha > 0$) the boundary of a quasi minimizer can be split into the union of a $C^{1,\alpha}$ relatively open hypersurface and a closed singular set with Hausdorff dimension at most $n-8$ (empty if $n \leq 7$). The first partial regularity result of this kind was given by De Giorgi [10] who proved that for local minimizers (quasi minimizers with $\omega \equiv 0$) the singular set has zero $(n-1)$-dimensional Hausdorff measure. Afterwards Massari [16, 17] extended the same result to sets with generalized mean curvature in $L^p(\mathbf{R}^n)$ with $p > n$. We notice that sets with mean curvature in L^p with $p \geq n$ are quasi minimizers with $\omega(\rho) \leq o(1)\, \rho^{2\alpha}$, $\alpha = \frac{p-n}{2p}$, so in the case $p = n$ we only know that $\omega(\rho)$ is infinitesimal as $\rho \to 0$.

In 1992 De Giorgi proposed an example of a quasi minimizer in the plane having a singular point at the origin. Gonzales, Massari and Tamanini proved in [15] that the example of De Giorgi, whose boundary is the union of two bilogarithmic spirals, is indeed a set with mean curvature in $L^2(\mathbf{R}^2)$. This example shows that in general we cannot expect the boundary of a quasi minimizer to be locally a Lipschitz graph; however, De Giorgi conjectured that the boundary of a quasi minimizer is locally parameterizable (out of a singular set, if $n \geq 8$) by a bilipschitz map defined on an open ball of $\mathbf{R}^{n-1}$.

The first attempt to prove the De Giorgi conjecture has been made by the second author in [19], proving that if $n \leq 7$ the boundary of a set with mean curvature in L^n is locally parameterizable for any $\alpha < 1$ with a map τ such that both τ and τ^{-1} are $C^{0,\alpha}$; this implies that ∂E is a surface, at least in the topological sense. In this paper, adopting a new technique, we prove that the same property holds for quasi minimizers, for any number of dimensions n, out of a closed singular set Σ with dimension at most $n-8$ (Theorem 4.10). The full De Giorgi conjecture, namely the $C^{0,1}$ regularity of τ and τ^{-1}, is still an open problem; we prove it only in the case $n = 2$, by showing that any chord arc parameterization, is, thanks to the quasi minimality, a bilipschitz parameterization (Theorem 5.2).

The strategy of our proof is a combination of the De Giorgi and Reifenberg regularity theories for area minimizers: first we use a variant of De Giorgi decay theorem (for the mean flatness instead of the excess), developed in the regularity theory for varifolds, to show that if ∂E is sufficiently flat in a ball $B_\rho(x)$ then it remains flat on smaller scales (if $\omega(\rho) \leq c\rho^{2\alpha}$ there is an improvement, leading to $C^{1,\alpha}$ regularity); we also use a density lower bound for quasi minimizers to transform the mean estimate on the flatness into a pointwise one. Then, the Reifenberg topological disk condition can be applied to show that $\partial E \cap B_\eta(x)$ is parameterizable by a $C^{0,\alpha}$ map τ, with $\tau, \tau^{-1} \in C^{0,\alpha}$, for some $\eta < \rho$. Finally, using a standard dimension reduction argument we show that the set of points which are not sufficiently flat on any scale has dimension at most $n-8$.

We point out that we have confined our discussion to local minimizers in $\mathbf{R}^n$, but our arguments, of a local nature, apply to local minimizers in an open set $\Omega \subseteq \mathbf{R}^n$ as well, with only minor modifications.

Related result have been obtained by Semmes [22] for the class of chord-arc surfaces (see also [21]). Sufficient conditions for the existence of bi-lipschitz parameterizations have been found by Toro [25, 26].

1 Quasi minimizers

First, we specify our main notations. In the following n denotes a fixed integer, we assume $n \geq 2$ and set $m = n-1$. If $E \subseteq \mathbf{R}^n$, $|E|$ is the Lebesgue measure of E, $\varphi_E(x)$ is the characteristic function of E, ω_n is the Lebesgue measure of the unit ball $B_1(0)$ and

$$\partial E := \{x \in \mathbf{R}^n : \forall \rho > 0 \quad 0 < |E \cap B_\rho(x)| < \omega_n \rho^n\}.$$

Finally, $E \triangle F$ is the symmetric difference $(E \setminus F) \cup (F \setminus E)$.

We refer to [13] for the definitions concerning the notions of *Caccioppoli sets*, the perimeter $P(E, \Omega)$, the measures $D\varphi_E$ and $|D\varphi_E|$, the inner normal vector $\nu_E(x)$ and the reduced boundary $\partial^* E$; we recall that $D\varphi_E = \nu_E |D\varphi_E|$ and that

$$P(E, B) = |D\varphi_E|(B) = \mathcal{H}^m(\partial^* E \cap B)$$

for any Borel set $B \subseteq \mathbf{R}^N$ if E has finite perimeter in $\mathbf{R}^n$.

In this section we introduce the class of quasi minimizers of perimeter to which our partial regularity theorem applies; following [2] and [6], we adopt a multiplicative definition.

Definition 1.1 (Quasi minimizers) *Let $\omega : (0, +\infty) \to (0, +\infty]$ be an increasing, function with $\lim_{\rho \to 0} \omega(\rho) = 0$.*

We will denote by $\mathcal{M}_\omega$ the family of measurable sets E such that

$$P(E, B_\rho(x)) \leq (1 + \omega(\rho)) P(F, B_\rho(x))$$

whenever $x \in \partial E$, $\omega(\rho) < +\infty$ and $E \triangle F \Subset B_\rho(x)$.

The main example of quasi minimizers is given by sets with prescribed mean curvature, as we will see in Section 2. The following proposition, whose proof directly follows from the definition, will be very often used in blow-up arguments.

Proposition 1.2 (rescaling properties of quasi-minimizers) *Let $E \in \mathcal{M}_\omega$, $\lambda > 0$ and $x \in \mathbf{R}^n$. Then $\frac{1}{\lambda}(E - x) \in \mathcal{M}_{\omega'}$ with $\omega'(t) := \omega(\lambda t)$.*

In the following three propositions we estabilish upper and lower bounds for area and perimeter of quasi minimal sets.

Proposition 1.3 (density upper bound) *If $E \in \mathcal{M}_\omega$, then given $x \in \mathbf{R}^n$ and $\rho > 0$ we have*

$$P(E, B_\rho(x)) \leq n\omega_n 2^m (1 + \omega(2\rho)) \rho^m.$$

Proof:
We may suppose that $P(E, B_\rho(x)) \neq 0$ and $\omega(2\rho) < +\infty$, since otherwise the statement is trivial. Let $y \in \partial E \cap B_\rho(x)$ and $\eta < \rho$; we have

$$\begin{aligned} P(E, B_\rho(x)) &\leq P(E, B_{2\rho}(y)) \leq (1+\omega(2\rho))P(E \setminus B_{2\eta}(y), B_{2\rho}(y)) \\ &\leq (1+\omega(2\rho))(n\omega_n(2\eta)^m + P(E, B_{2\rho}(y) \setminus \overline{B_{2\eta}(y)})) \end{aligned}$$

and letting $\eta \to \rho$ the conclusion follows.

□

The proof of the following compactness theorem can be achieved by a classical comparison argument (see e.g. [15, Th. 1.1], [3, Th. 4.2.5]).

Proposition 1.4 (compactness of quasi-minimizers) *Let (E_k) be a sequence of ω_k-minimal sets and suppose that $\omega_k \to \omega$ pointwise, with $\omega(\rho) \to 0$ as $\rho \to 0$. Then there exists a subsequence (E_{k_j}) which converges in $L^1_{\rm loc}$ to a set $E \in \mathcal{M}_\omega$. Moreover, if E_k converge in $L^1_{\rm loc}$ to E then*

$$D\varphi_{E_k} \stackrel{*}{\rightharpoonup} D\varphi_E, \quad |D\varphi_{E_k}| \stackrel{*}{\rightharpoonup} |D\varphi_E|,$$

as $k \to \infty$.

Proposition 1.5 *Let $E \in \mathcal{M}_\omega$, $x \in \partial E$. Then, for all $\rho > 0$,*

$$\frac{1}{(2+\omega(\rho))^n} \leq \frac{|E \cap B_\rho(x)|}{\omega_n \rho^n} \leq 1 - \frac{1}{(2+\omega(\rho))^n}.$$

Proof:
Define $g(\rho) := |E \cap B_\rho(x)|$. This is an increasing function such that for almost all $\rho > 0$

$$\max\{P(E \cap B_\rho(x), \partial B_\rho(x)), P(E \setminus B_\rho(x), \partial B_\rho(x))\} \leq g'(\rho);$$

moreover the isoperimetric inequality gives

$$n\omega_n^{\frac{1}{n}} g(\rho)^{\frac{m}{n}} \leq P(E \cap B_\rho(x), \mathbf{R}^n) = P(E, B_\rho(x)) + P(E \cap B_\rho(x), \partial B_\rho(x)). \tag{1}$$

Comparing E with $E \setminus B_\rho(x)$ for all $\eta > \rho$ we obtain

$$P(E, B_\eta(x)) \leq (1+\omega(\eta))P(E \setminus B_\rho(x), B_\eta(x))$$

so, letting $\eta \to \rho$, if ρ is a continuity point of ω we get

$$P(E, \overline{B_\rho(x)}) \leq (1+\omega(\rho))P(E \setminus B_\rho(x), \overline{B_\rho(x)}) = (1+\omega(\rho))P(E \setminus B_\rho(x), \partial B_\rho(x)). \tag{2}$$

Putting (1) and (2) together, for almost all ρ, we obtain

$$n\omega_n^{\frac{1}{n}} g(\rho)^{\frac{m}{n}} \leq (2+\omega(\rho))g'(\rho)$$

that is

$$(g(\rho)^{\frac{1}{n}})' \geq \frac{\omega_n^{\frac{1}{n}}}{2+\omega(\rho)}$$

and integrating we get the first inequality. The second inequality is obtained if we replace E by its complementary set.

□

Proposition 1.6 (density lower bound) *There exists a constant $\rho_\omega > 0$ such that if $E \in \mathcal{M}_\omega$, $x \in \partial E$ and $\rho < \rho_\omega$ then*

$$P(E, B_\rho(x)) \geq \frac{\omega_m}{2}\rho^m.$$

Proof:
Suppose by contradiction that there exist a sequence of sets $E_k \in \mathcal{M}_\omega$, a sequence of points $x_k \in \partial E_k$ and a sequence of radii $\rho_k \to 0$ such that

$$\rho_k^{-m} P(E_k, B_{\rho_k}(x_k)) < \frac{\omega_m}{2}.$$

By the rescaling properties of quasi-minimizers the sets $F_k = \frac{1}{\rho_k}(E_k - x_k)$ belong to $\mathcal{M}_{\omega_k}$ with $\omega_k(t) := \omega(\rho_k t)$. We note that $\omega_k \to 0$ so by Lemma 1.4, up to a subsequence, we may suppose that the sequence (F_k) converges to a set $F \in \mathcal{M}_0$. From the lower semi-continuity of perimeter we obtain

$$\frac{\omega_m}{2} \geq \liminf_{k\to\infty} \rho_k^{-m} P(E_k, B_{\rho_k}(x_k)) \quad = \quad \liminf_{k\to\infty} P(F_k, B_1(0)) \geq P(F, B_1(0)). \tag{3}$$

Now, by Proposition 1.5 there exist constants $c > 0$ and $C < \omega_n$ such that $c\rho^n \leq |F_k \cap B_\rho(0)| \leq C\rho^n$ and, since $|F \cap B_\rho(0)| = \lim_{k\to\infty} |F_k \cap B_\rho(0)|$ we obtain $0 \in \partial F$. Therefore as F is a minimal set, we know that (see for example [13, (5.16)]) $P(F, B_1(0)) \geq \omega_m$ and we have a contradiction with (3).

□

Corollary 1.7 *If $E \in \mathcal{M}_\omega$ then*

$$\mathcal{H}^m(\partial E \setminus \partial^* E) = 0.$$

In particular $|D\varphi_E| = \mathcal{H}^m \llcorner \partial E$ and in Proposition 1.4 we have local convergence of ∂E_k to ∂E in the sense of Kuratowski[1].

Proof:
By Proposition 1.6 we know that the measure $\mathcal{H}^m \llcorner \partial^* E = P(E, \cdot)$ has positive m-dimensional density on the set ∂E and in particular on $\partial E \setminus \partial^* E$. Since, for all Borel sets C with $\mathcal{H}^m(C) < +\infty$, $\mathcal{H}^m \llcorner C$ has 0 m-dimensional density in $\mathcal{H}^m$ almost all points of $\mathbf{R}^n \setminus C$ (see for instance [23, Th. 3.5]), we conclude that $\mathcal{H}^m(\partial E \setminus \partial^* E) = 0$.

If E_k converge to E and satisfy the assumptions of Proposition 1.4, it is clear that any point $x \in \partial E$ can be approximated by points in ∂E_k, by the convergence of $D\varphi_{E_k}$ to $D\varphi_E$. The density lower bound and the convergence of $|D\varphi_{E_k}|$ to $|D\varphi_E|$ ensure that any limit point of ∂E_k belongs to ∂E.

□

2 Boundaries with prescribed mean curvature

In this section we present the main examples of quasi-minimal sets. Given $H \in L^1(\mathbf{R}^n)$ we say that a set $E \subseteq \mathbf{R}^n$ has (generalized) mean curvature H if E is a local minimizer of the functional

$$F_H(E) = P(E, \mathbf{R}^n) + \int_E H(x)\,\mathrm{d}x$$

that is, if for any $F \subseteq \mathbf{R}^n$ with $F \triangle E \Subset \mathbf{R}^n$ we have $F_H(E) \leq F_H(F)$. We would like to show that if $H \in L^p(\mathbf{R}^n)$ with $p \geq n$ and E is a set with mean curvature H then, for a suitable ω, $E \in \mathcal{M}_\omega$.

First of all it is not difficult to show that there exists an increasing function $\eta\colon [0, +\infty) \to [0, +\infty)$ with $\eta(t) \to 0$ as $t \to 0$ such that given any set $F \subseteq \mathbf{R}^n$ with $E \triangle F \Subset B_\rho(x)$ the following is true

$$P(E, B_\rho(x)) \leq P(F, B_\rho(x)) + \eta(\rho)|E \triangle F|^{\frac{m}{n}}. \tag{4}$$

[1] We say that a sequence of sets (X_k) converges to X in the sense of Kuratowski if

$$x_k \in X_k,\ x_k \to x \quad \Rightarrow \quad x \in X$$
$$x \in X \quad \Rightarrow \quad \exists x_k \in X_k,\ x_k \to x.$$

In fact, given such F, we know that

$$P(E, B_\rho(x)) \leq P(F, B_\rho(x)) + \int_{E \triangle F} |H(x)| \, \mathrm{d}x$$

and if we define

$$\eta(\rho) := \sup_{x \in \mathbf{R}^n} ||H||_{L^n(B_\rho(x))},$$

from Hölder inequality we get

$$\int_{E \triangle F} |H| \leq ||H||_{L^n(B_\rho(x))} |E \triangle F|^{\frac{m}{n}}$$

so that (4) holds.

It is not difficult to see (see [15]) that Proposition 1.5, Proposition 1.4 and then also Proposition 1.6 hold for the the sets satisfying (4) so that, if $E \triangle F \Subset B_\rho(x)$, $x \in \partial E$ and ρ is sufficiently small,

$$|E \triangle F|^{\frac{m}{n}} \leq \omega_n^{\frac{m}{n}} \rho^m \leq c(n) P(E, B_\rho(x))$$

for some constant $c(n) > 0$, whence

$$P(E, B_\rho(x)) \leq \frac{1}{1 - c(n)\eta(\rho)} P(F, B_\rho(x))$$

that is $E \in \mathcal{M}_\omega$ with $\omega(\rho) := \frac{c(n)\eta(\rho)}{1 - c(n)\eta(\rho)}$ for small ρ and $\omega(\rho) := +\infty$ elsewhere.

Actually it could have been possible to use inequality (4) in the definition of quasi minimizers, but the definition, as it is given, seems to be more general.

3 Flatness

In this section we introduce two quantities which measure the flatness of ∂E; the first one, analogous to the L^2 norm, is useful in the regularity theory for varifolds; the second one, analogous to the sup norm, has been used by Reifenberg in his regularity theory.

Definition 3.1 *Let E be a Caccioppoli set. Then we define the* mean flatness *and* Reifenberg flatness *of E in $B(x, \rho)$ to be respectively*

$$\begin{aligned} \mathbf{A}_E(x, \rho) &:= \rho^{-m-2} \min_{A \in G(n)} \int_{\partial E \cap \overline{B_\rho(x)}} d^2(y, A) \, \mathrm{d}\mathcal{H}^m(y), \\ \theta_E(x, \rho) &:= \rho^{-1} \min_{A \in G(n),\ A \ni x} \left(\max_{y \in \partial E \cap \overline{B_\rho(x)}} \mathrm{d}(y, A) \right) \end{aligned}$$

where $G(n)$ is the family of affine hyperplanes of $\mathbf{R}^n$ and $d(y, A)$ is the distance of the point y from A.

The following proposition comes directly from the definition.

Proposition 3.2 (rescaling properties) *If E is a Caccioppoli set and $\lambda > 0$ we have*

$$\begin{aligned} \mathbf{A}_E(x, \rho) &= \mathbf{A}_{\lambda E}(\lambda x, \lambda \rho) \\ \theta_E(x, \rho) &= \theta_{\lambda E}(\lambda x, \lambda \rho). \end{aligned}$$

If, moreover, $B_{\lambda\rho}(y) \subseteq B_\rho(x)$ we have

$$\begin{aligned} \mathbf{A}_E(y, \lambda\rho) &\leq \lambda^{-m-2} \mathbf{A}_E(x, \rho) \\ \theta_E(y, \lambda\rho) &\leq \lambda^{-1} \theta_E(x, \rho). \end{aligned}$$

Using the density lower bound we can estimate the Reifenberg flatness with the mean flatness.

Proposition 3.3 *There exists a constant $C_1 = C_1(\omega)$ such that is $E \in \mathcal{M}_\omega$, $x \in \partial E$, $\rho < \rho_\omega$ then*

$$\theta_E(x,\rho) \le C_1 \left[\mathbf{A}_E(x,2\rho)\right]^{\frac{1}{m+2}} .$$

Proof:
Let A be an hyperplane such that

$$\mathbf{A}_E(x,2\rho) = (2\rho)^{-m-2} \int_{\partial E \cap \overline{B_{2\rho}(x)}} d^2(y,A)\, \mathrm{d}\mathcal{H}^m(y)$$

and let A' be the hyperplane through x parallel to A. Given any $z \in B_\rho(x)$ define $\alpha := \frac{1}{3} d(z,A)$ and $\beta := \frac{1}{2} d(x,A)$ so that $d(z,A') \le 3\alpha + 2\beta$. Note that $\alpha \le \rho$ and $\beta \le \rho$ because $d(x,A) \le 2\rho$, so both $B_\alpha(z)$ and $B_\beta(x)$ are contained in $B_{2\rho}(x)$ and we have that $d(y,A) \ge 2\alpha$ if $y \in B_\alpha(z)$ while $d(y,A) \ge \beta$ if $y \in B_\beta(x)$. Then we have (recalling also Proposition 1.6)

$$\begin{aligned}
\mathbf{A}_E(x,2\rho) &\ge (2\rho)^{-m-2} \int_{\partial E \cap B_\alpha(z)} d^2(y,A)\, \mathrm{d}\mathcal{H}^m(y) \\
&\ge (2\rho)^{-m-2} 4\alpha^2 \frac{\omega_m}{2} \alpha^m \\
\mathbf{A}_E(x,2\rho) &\ge (2\rho)^{-m-2} \int_{\partial E \cap B_\beta(x)} d^2(y,A)\, \mathrm{d}\mathcal{H}^m(y) \\
&\ge (2\rho)^{-m-2} \beta^2 \frac{\omega_m}{2} \beta^m
\end{aligned}$$

that is

$$\begin{aligned}
\alpha &\le 2\rho \left[\frac{1}{2\omega_m} \mathbf{A}_E(x,2\rho)\right]^{\frac{1}{m+2}} \\
\beta &\le 2\rho \left[\frac{2}{\omega_m} \mathbf{A}_E(x,2\rho)\right]^{\frac{1}{m+2}} .
\end{aligned}$$

So, for every $z \in B_\rho(x)$ we have

$$d(z,A') \le 3\alpha + 2\beta \le 2\left(\frac{3}{4^{\frac{1}{m+2}}} + 2\right) \rho \left[\frac{2}{\omega_m} \mathbf{A}_E(x,2\rho)\right]^{\frac{1}{m+2}} ,$$

that is the claim with $C_1 = 2(3 \cdot 4^{\frac{-1}{m+2}} + 2)(2/\omega_m)^{\frac{1}{m+2}}$.

□

Lemma 3.4 *Let $(E_k) \subseteq \mathcal{M}_\omega$ converging in L^1_{loc} to a set E and let $x \in \mathbf{R}^n$. Then*

$$\mathbf{A}_E(x,\rho) \ge \limsup_{k\to\infty} \mathbf{A}_{E_k}(x,\rho) \qquad \forall \rho > 0.$$

Proof:
We will first prove the inequality for all $\rho > 0$ such that $\mathcal{H}^m(\partial E \cap \partial B_\rho(x)) = 0$ (this is true for almost every ρ). In view of Theorem 1.4, we know that

$$\mathcal{H}^m \llcorner \partial E_k \stackrel{*}{\rightharpoonup} \mathcal{H}^m \llcorner \partial E.$$

Let A be an hyperplane such that

$$\mathbf{A}_E(x,\rho) = \rho^{-m-2} \int_{\partial E \cap B_\rho(x)} d^2(y,A)\, \mathrm{d}\mathcal{H}^m(y),$$

define $\varphi(y) := d^2(y,A)$ and consider the measures $\mu_k := \varphi \mathcal{H}^m \llcorner \partial E_k$ and $\mu := \varphi \mathcal{H}^m \llcorner \partial E$. Since $\mu(\partial B_\rho(x)) = 0$ we have $\mu_k(B_\rho(x)) \to \mu(B_\rho(x))$ (see [13, Appendix A], [3, Th. 1.2.7]) and then

$$\begin{aligned}
\limsup_{k\to\infty} \mathbf{A}_{E_k}(x,\rho) &\le \limsup_{k\to\infty} \rho^{-m-2} \mu_k(\overline{B_\rho(x)}) \\
&= \rho^{-m-2} \mu(\overline{B_\rho(x)}) = \mathbf{A}_E(x,\rho).
\end{aligned}$$

By outer approximation the same inequality holds for any $\rho > 0$.

□

4 Partial regularity

The following decay theorem for the mean flatness plays a central role in this paper; both the statement and the proof are reminiscent of the regularity theory of varifolds developed by Allard [1] and Brakke [7] (see also [23]).

Theorem 4.1 (decay) *For all $\beta \in (0,1)$ and all $M > 0$ there exists ε_1 (depending only on β, M and ω) such that, for every $E \in \mathcal{M}_\omega$, the conditions*

$$x \in \partial E, \quad \mathbf{A}_E(x,\rho) \le \varepsilon_1, \quad M\mathbf{A}_E(x,\rho) \ge \sqrt{\omega(\rho)}$$

imply

$$\mathbf{A}_E(x,\beta\rho) \le C_3\beta^2 \mathbf{A}_E(x,\rho)$$

where C_3 is a constant depending only on n.

Proof:
If ω is a linear function a proof is given in [3]; in the general case the proof is similar, see also the decay theorem for the mean flatness of the jump set of the Mumford–Shah functional proved in [4], where no linearity assumption on ω is made. The constant C_3 is related to some pointwise estimates on harmonic functions of m variables.

□

Lemma 4.2 (iteration) *There exists a positive constant $\varepsilon_2 = \varepsilon_2(\omega)$ such that if $E \in \mathcal{M}_\omega$, $x \in \partial E$, $\mathbf{A}_E(x,r) \le \varepsilon_2$ and $\sqrt{\omega(r)} \le \varepsilon_2$ then for all $\rho \in (0,r)$*

$$\mathbf{A}_E(x,\rho) \le C_3^{m+2}\rho \max\left\{ \sup_{\eta\in[\rho,r]} \frac{\sqrt{\omega(\eta)}}{\eta}, \frac{\mathbf{A}_E(x,r)}{r} \right\}$$

and, in particular, $\lim_{\rho\to 0} \mathbf{A}_E(x,\rho) = 0$.

Proof:
Take $\beta = C_3^{-1}$, $M = \beta^{-m-3}$, $\varepsilon_2 = \varepsilon_1(\beta, M, \omega)$ (where C_3 and ε_1 are the constants given by Theorem 4.1).

We will prove, by induction, that for all integers $k \ge 0$

$$\mathbf{A}_E(x,\beta^k r) \le \beta^k \max\left\{ \sup_{\eta\in[\beta^k r,r]} \frac{r}{\eta}\sqrt{\omega(\eta)}, \mathbf{A}_E(x,r) \right\}. \tag{5}$$

The case $k = 0$ is trivial, so assuming that (5) is true for k we prove that (5) holds also for $k+1$. If $M\mathbf{A}_E(x,\beta^k r) \ge \sqrt{\omega(\beta^k r)}$ we can apply Theorem 4.1 (in fact, $\mathbf{A}_E(x,\beta^k r) \le \max\left\{\sqrt{\omega(r)}, \mathbf{A}_E(x,r)\right\} \le \varepsilon_2$) to obtain

$$\mathbf{A}_E(x,\beta^{k+1} r) \le \beta \mathbf{A}_E(x,\beta^k r) \le \beta^{k+1} \max\left\{ \sup_{\eta\in[\beta^{k+1} r,r]} \frac{r}{\eta}\sqrt{\omega(\eta)}, \mathbf{A}_E(x,r) \right\}.$$

Otherwise, by Lemma 3.2 we get

$$\begin{aligned} \mathbf{A}_E(x,\beta^{k+1} r) &\le \beta^{-m-2}\mathbf{A}_E(x,\beta^k r) \le \beta^{-m-2}\frac{\sqrt{\omega(\beta^k r)}}{M} \\ &= \beta\sqrt{\omega(\beta^k r)} \le \beta^{k+1} r \sup_{\eta\in[\beta^{k+1} r,r]} \frac{\sqrt{\omega(\eta)}}{\eta}. \end{aligned}$$

To conclude the proof, notice that given any $\rho \in (0,r)$ there exists an integer k such that $\beta^{k+1} r \le \rho \le \beta^k r$, so that

$$\begin{aligned} \mathbf{A}_E(x,\rho) &\le \beta^{-m-2}\mathbf{A}_E(x,\beta^k r) \\ &\le C_3^{m+2}\beta^k r \max\{ \sup_{\eta\in[\rho,r]} \frac{\sqrt{\omega(\eta)}}{\eta}, \mathbf{A}_E(x,r)\} \\ &\le C_3^{m+3}\rho \max\{ \sup_{\eta\in[\rho,r]} \frac{\sqrt{\omega(\eta)}}{\eta}, \mathbf{A}_E(x,r)\}. \end{aligned}$$

To prove the last statement, we claim that

$$\lim_{\rho\to 0}\rho\sup_{\eta\in[\rho,r]}\frac{\sqrt{\omega(\eta)}}{\eta}=0.$$

In fact, given any sequence $\rho_k\to 0$, let $\eta_k\in[\rho_k,r]$ be such that $\frac{\sqrt{\omega(\eta_k)}}{\eta_k}=\sup_{\eta\in[\rho,r]}\frac{\sqrt{\omega(\eta)}}{\eta}$. If $\eta_k\to 0$ then $\rho_k\frac{\sqrt{\omega(\eta_k)}}{\eta_k}\le\sqrt{\omega(\eta_k)}\to 0$, otherwise there exists $\eta>0$ such that $\eta_k\ge\eta$ for all k and we conclude noticing that $\rho_k\frac{\sqrt{\omega(\eta_k)}}{\eta_k}\le\rho_k\frac{\sqrt{\omega(r)}}{\eta}\to 0$.

□

If $X\subseteq\mathbf{R}^n$ and $\rho>0$ we will denote by $(X)_\rho$ the open ρ-neighborhood of X:

$$(X)_\rho:=\{x\in\mathbf{R}^n: d(x,X)<\rho\}.$$

Definition 4.3 *Let S be a closed set of $\mathbf{R}^n$, $x_0\in S$, $R>0$ and $0\le m\le n$ an integer. Then we say that S satisfies the (ε,R,m)-Reifenberg condition in x_0 if given any ball $B_\rho(x)\subseteq B_R(x_0)$ there exists an m-dimensional plane Σ through x such that*

$$\begin{aligned} S\cap B_\rho(x) &\subseteq (\Sigma)_{\varepsilon\rho}\\ \Sigma\cap B_\rho(x) &\subseteq (S)_{\varepsilon\rho}.\end{aligned}$$

The following theorem can be found in [20].

Theorem 4.4 (Reifenberg) *Given $0<\alpha<1$, there exists $\varepsilon_3>0$ such that if S is a closed set in $\mathbf{R}^n$ satisfying the (ε_3,R,m)-Reifenberg condition in $x_0\in S$ for some $R>0$, and some integer $0\le m\le n$, then there exists an open set U, $B_{\frac{R}{16}}(x_0)\subseteq U\subseteq B_R(x_0)$ and an homeomorphism $\tau: D^m\to S\cap\overline{U}$ (D^m is the closed m dimensional unit disk) such that both τ and τ^{-1} are α-Hölder maps.*

Lemma 4.5 *Let $E\in\mathcal{M}_\omega$, $x\in\mathbf{R}^n$, $\rho>0$, $\varepsilon\in(0,1/2)$, A an hyperplane through x and let $\pi_A:\mathbf{R}^n\to\mathbf{R}^n$ be the orthogonal projection onto A. If $\partial E\cap B_\rho(x)\subseteq(A)_{\varepsilon\rho}$ then at least one of the following statements is true:*

1. *$\pi_A(\partial E\cap B_\rho(x))\supseteq A\cap B_{(1-\varepsilon)\rho}(x)$;*
2. *$P(E\setminus B_\rho(x),\overline{B_\rho(x)})\le 4\varepsilon m\omega_m\rho^m$.*

Proof:
In this proof we assume that the set E is such that ∂E is exactly equal to the topological boundary of E. This can be done replacing E with the set $E^*:=\{x\in\mathbf{R}^n:\limsup_{\rho\to 0}\frac{|E\cap B_\rho(x)|}{\omega_n\rho^n}>0\}$ which differs from E by a zero Lebesgue measure set. Taking into account Proposition 1.5 it can be proved that the topological boundary of E^* is $\partial E^*=\partial E$.

Let $x_1,\dots,x_n$ be a system of coordinates such that $A=\{x_n=0\}$ and define $E^+=\{x_n\ge\varepsilon\rho\}\cap\overline{B_\rho(x)}$ and $E^-=\{x_n\le-\varepsilon\rho\}\cap\overline{B_\rho(x)}$. From the assumptions on E we know that ∂E intersects neither E^+ nor E^-. So, being ∂E closed and E^+, E^- compact sets, we conclude that ∂E has positive distance from E^+ and E^-. So one of the following statements must hold:

1. $E\supseteq E^-$, $E\cap E^+=\emptyset$;
2. $E\supseteq E^+$, $E\cap E^-=\emptyset$;
3. $E\cap E^+=\emptyset$, $E\cap E^-=\emptyset$;
4. $E\supseteq E^+\cup E^-$.

Suppose that 1 is true, then we claim that the first statement of the proposition holds. Let $A^+=\{x_n=\varepsilon\rho\}\cap\overline{B_\rho(x)}$ and $A^-=\{x_n=-\varepsilon\rho\}\cap\overline{B_\rho(x)}$. Given $y\in A\cap B_{(1-\varepsilon)\rho)}(x)$ denote by y^+ and y^- respectively the intersections of $\pi_A^{-1}(x)$ with A^+ and A^-. Since $y^-\in E$ while $y^+\notin E$ there must certainly be a point in the segment $[x^-,x^+]$ which belongs to ∂E (and that is contained in $B_\rho(x)$), so $y\in\pi_\Sigma(\partial E\cap B(x,r))$.

If statement 2 holds, with a symmetry with respect to A we obtain again the first statement.

Suppose that 3 is true. Let $F = E \setminus B_\rho(x)$. Then $\partial F \cap \overline{B_\rho(x)} \subseteq (A)_{\varepsilon\rho} \cap \partial B_\rho(x)$, so we obtain the second statement:

$$P(F, \overline{B_\rho(x)}) \leq \mathcal{H}^m(\partial F \cap \overline{B_\rho(x)}) \leq \mathcal{H}^m((A)_{\varepsilon\rho} \cap \partial B_\rho(x)) \leq 4\varepsilon m \omega_m \rho^m.$$

If statement 4 holds the proof is that of case 3 where instead of E we consider its complementary set.

□

Definition 4.6 (singular set) *For $E \in \mathcal{M}_\omega$ define*

$$\Sigma(E) := \{x \in \partial E \colon \limsup_{\rho \to 0} \mathbf{A}_E(x, \rho) > 0\}.$$

Note that $\Sigma(E)$ is a closed set. In fact, given any $x \in \partial E \setminus \Sigma(E)$, since $\lim_\rho \mathbf{A}_E(x,\rho) = 0$ we can find a radius ρ such that $\mathbf{A}_E(x, 2\rho) \leq 2^{-m-2}\varepsilon_2$ and for all $y \in B_\rho(x)$ we get $\mathbf{A}_E(y,\rho) \leq 2^{m+2}\mathbf{A}_E(x, 2\rho) \leq \varepsilon_2$ so that by Lemma 4.2 $\lim_\rho \mathbf{A}_E(y, \rho) = 0$; hence $B_\rho(x) \cap \Sigma(E) = \emptyset$. In view of Theorem 4.4 the following theorem gives a regularity result for the set $\partial E \setminus \Sigma(E)$.

Theorem 4.7 (partial regularity) *Let $E \in \mathcal{M}_\omega$ and $x_0 \in \partial E \setminus \Sigma(E)$. For all $0 < \varepsilon < \varepsilon_2$ (where ε_2 is the constant given in Lemma 4.2) there exists $r > 0$ such that the set ∂E satisfies the (ε, r, m)-Reifenberg condition in x_0.*

Proof:
Choose constants α_1, α_2, α_3 and r such that

$$\alpha_3 \leq \frac{\varepsilon}{2}, \quad \alpha_3 < \frac{\omega_m}{8\omega_{m-1}(1+\omega(\rho_\omega))}, \quad \alpha_3 \leq \frac{1}{2},$$

$$\alpha_2 < \left(\frac{\alpha_3}{C_1}\right)^{m+2}, \quad \alpha_1 \leq \left(\frac{2}{3C_3}\right)^{m+2}\alpha_2, \quad \alpha_1 \leq \left(\frac{2}{3}\right)^{m+2}\varepsilon_2,$$

$$r < \rho_\omega, \quad \sqrt{\omega(2r)} \leq \varepsilon_2^2, \quad \sqrt{\omega(2r)} \leq \frac{\alpha_2}{C_3^{m+2}},$$

$$\mathbf{A}_E(x, 3r) \leq \alpha_1$$

where C_1, C_3 and ε_2 are the constants given in Proposition 3.3 and Lemma 4.2.

Consider any ball $B_\rho(x) \subseteq B_r(x_0)$ with center $x \in \partial E$. By Lemma 3.2 we have $\mathbf{A}_E(x, 2r) \leq \left(\frac{3}{2}\right)^{m+2}\mathbf{A}_E(x_0, 3r) \leq \left(\frac{3}{2}\right)^{m+2}\alpha_1$. and by Lemma 4.2 we obtain

$$\begin{aligned} \mathbf{A}_E(x, 2\rho) &\leq C_3^{m+2}\max\{\sqrt{\omega(2r)}, \frac{\rho}{r}\mathbf{A}_E(x, 2r)\} \\ &\leq C_3^{m+2}\max\{\frac{\alpha_2}{C_3^{m+2}}, \left(\frac{3}{2}\right)^{m+2}\alpha_1\} \leq \alpha_2. \end{aligned}$$

Now, by Proposition 3.3 we get $\theta_E(x,\rho) \leq C_1(\mathbf{A}_E(x, 2\rho))^{\frac{1}{m+2}} < \alpha_3$ so there exists an hyperplane A through x such that

$$\rho^{-1} \sup_{y \in \partial E \cap B_\rho(x)} d(y, A) < \alpha_3$$

which means that $\partial E \cap B_\rho(x) \subseteq (A)_{\alpha_3\rho}$. To conclude the proof we only need to prove that $A \cap B_\rho(x) \subseteq (\partial E)_{\varepsilon\rho}$. By Proposition 1.6 and the definition of $\mathcal{M}_\omega$, given any $\eta \in (\rho, \rho_\omega)$

$$\frac{\omega_m}{2}\eta^m \leq P(E, B_\eta(x)) \leq P(E \setminus B_\rho(x), B_\eta(x))(1 + \omega(\eta))$$

and letting $\eta \to \rho$ we obtain

$$P(E \setminus B_\rho(x), \overline{B_\rho(x)}) \geq \frac{\omega_m}{2(1+\omega(\rho_+))}\rho^m > 4\alpha_3\omega_{m-1}\rho^m.$$

So, in view of Lemma 4.5, we obtain that the projection $\pi_A(\partial E \cap B_\rho(x))$ contains $A \cap B_{(1-\alpha_3)\rho}(x)$. Therefore given any point $y \in A \cap B_\rho(x)$ we can find a point $y' \in A \cap B_{(1-\alpha_3)\rho}(x)$ with $|y' - y| \leq \alpha_3\rho$ and a point $z \in \partial E \cap B_\rho(x)$ with $\pi_A(z) = y'$. Since $\partial E \cap B_\rho(x) \subseteq (A)_{\alpha_3\rho}$ we have $|z - y'| \leq \alpha_3\rho$ whence $|y - z| \leq 2\alpha_3\rho \leq \varepsilon\rho$.

□

Lemma 4.8 (convergence of singular points) *Let $E_k, E \in \mathcal{M}_\omega$ with $E_k \to E$. Then, for all $R > 0$,*

$$\lim_{k\to\infty} \sup_{x\in B_R(0)\cap\Sigma(E_k)} d(x, \Sigma(E)) = 0.$$

Proof:
Suppose by contradiction that there exist $\varepsilon > 0$ and a sequence (x_k) with $x_k \in \Sigma(E_k)$ which converges to a point x and such that $d(x_k, \Sigma(E)) > \varepsilon$ for all k. Since $d(x_k, x) \to 0$ then $x \notin \Sigma(E)$ so that $\lim_{\rho\to 0} \mathbf{A}_E(x, \rho) = 0$. In view of Lemma 3.4 we can find a radius $\rho > 0$ such that $\limsup_{k\to\infty} \mathbf{A}_{E_k}(x, \rho) \le \mathbf{A}_E(x, \rho)$ and $\mathbf{A}_E(x, \rho) < 2^{-m-2}\varepsilon_2$. So we can find an integer k such that $\mathbf{A}_{E_k}(x, \rho) < 2^{-m-2}\varepsilon_2$ and $|x_k - x| < \rho/2$ to obtain $\mathbf{A}_{E_k}(x_k, \rho/2) \le 2^{m+2}\mathbf{A}_{E_k}(x, \rho) \le \varepsilon_2$. By Lemma 4.2 we conclude that $x_k \notin \Sigma(E_k)$.

□

Proposition 4.9 (reduction of singularities) *Let $l \ge 0$. Given $E \in \mathcal{M}_\omega$ such that $\mathcal{H}^l(\Sigma(E)) > 0$ there exists $E_\infty \in \mathcal{M}_0$ such that $\mathcal{H}^l(\Sigma(E_\infty)) > 0$.*

Proof:
See [23, Appendix A]. First of all recall that $\mathcal{H}^l_\infty(\Sigma(E)) > 0$, where

$$\mathcal{H}^l_\infty(A) := \inf\{\sum_i \omega_l \rho_i^l \colon \bigcup_i B_{\rho_i}(x_i) \supseteq A\}.$$

By [23, 3.6(2)] we can find a point $x \in \Sigma(E)$ with positive $\mathcal{H}^l_\infty$-density, that is a point x such that there exist $\varepsilon > 0$ and a sequence $\lambda_k \to 0$ such that

$$\mathcal{H}^l_\infty(\Sigma(E) \cap B_{\lambda_k}(x)) > \varepsilon\lambda_k^l. \tag{6}$$

Let $E_k := \frac{E-x}{\lambda_k}$. Up to a subsequence we may suppose that E_k converges to a set $E_\infty \in \mathcal{M}_0$. If we suppose, by contradiction, that $\mathcal{H}^l_\infty(\Sigma(E_\infty) \cap \overline{B_2(0)}) = 0$ we can find a family of balls $B_{\rho_j}(x_j)$ such that $\bigcup_j B_{\rho_j}(x_j) \supseteq \Sigma(E_\infty) \cap \overline{B_2(0)}$ and $\sum_j \omega_l \rho_j^l < \varepsilon/2$. Since $\Sigma(E_\infty) \cap \overline{B_1(0)}$ is a compact set it has positive distance from $\mathbf{R}^n \setminus \bigcup_j B_{\rho_j}(x_j)$ and, in view of Lemma 4.8, we can find k such that $\Sigma(E_k) \cap B_1(0) \subseteq \bigcup_j B_{\rho_j}(x_j)$. This means that $\mathcal{H}^l_\infty(\Sigma(E_k) \cap B_1(0)) < \varepsilon/2$, that is

$$\mathcal{H}^l_\infty(\Sigma(E) \cap B_{\lambda_k}(x)) < \lambda_k^l \frac{\varepsilon}{2}.$$

This is in contradiction with (6).

□

Theorem 4.10 (main theorem) *Let $E \subseteq \mathbf{R}^n$, $E \in \mathcal{M}_\omega$. Then there exist a closed set $\Sigma(E)$ with Hausdorff dimension not greater than $n-8$, such that $\partial E \setminus \Sigma(E)$ is a $C^{0,\alpha}$ m-dimensional manifold for all $\alpha < 1$.*

Proof:
The regularity of the set $\partial E \setminus \Sigma(E)$ follows from Theorem 4.7 and Theorem 4.4. It is well known that if $E \in \mathcal{M}_0$ then $\mathcal{H}^l(\Sigma(E)) = 0$ for all $l > n-8$ (see for instance [23]) so, by Proposition 4.9, it is also true for any $E \in \mathcal{M}_\omega$.

□

5 The two-dimensional case

In this section we prove that quasi minimizers in $\mathbf{R}^2$ are locally bilipschitz parameterizable with an open interval of $\mathbf{R}$. The proof is achieved using as competitor in the definition of quasi minimality the set F given by the following lemma.

Lemma 5.1 *Let E be a Caccioppoli set and let $\tau\colon [t_1, t_2] \to \partial E$ be a lipschitz, injective function. Then there exists a Caccioppoli set F such that $E \triangle F$ is contained in the convex hull K of $\tau([t_1, t_2])$ and such that*

$$P(F, K) \le \mathcal{H}^1(\partial E \cap K) - \int_{t_1}^{t_2} |\tau'(t)|\, dt + |\tau(t_1) - \tau(t_2)|.$$

Proof:
First some notation. We will consider normal currents with multiplicity in $\mathbf{Z}_2$. If α is a parametric lipschitz curve, we denote by $\vec{\alpha}$ the 1-dimensional current (with multiplicity 1) induced by α. Likewise if E is a Caccioppoli set, we denote by $\vec{E}$ the 2-dimensional normal current on E with multiplicity 1. Note that every 2-dimensional normal current with multiplicity in $\mathbf{Z}_2$ can be written as $\vec{E}$ for some Caccioppoli set E and $|D\varphi_E| = \|\partial\vec{E}\|$. See [23] for basic notation and main facts about the theory of currents.

We denote by S the segment having $\tau(t_1)$ and $\tau(t_2)$ as endpoints and set $\Gamma = \tau([t_1,t_2])$. Notice that both $\vec{\tau}$ and $\partial\vec{E} \llcorner \Gamma$ are rectifiable 1-dimensional currents with multiplicity 1 supported on the rectifiable set Γ. Since there is no choice for the orientation of tangent lines (we consider orientation in $\mathbf{Z}_2$), the two currents are equal.

Consider the parameterization of S given by $\gamma\colon [0,1] \to \mathbf{R}^2$ defined by $\gamma(t) = t\tau(t_2) + (1-t)\tau(t_1)$ and let $\vec{\sigma} = \vec{\tau} - \vec{\gamma}$. Clearly $\operatorname{spt}\vec{\sigma} \subseteq \Gamma \cup S \subseteq K$, where K is the convex hull of Γ.

Since $\partial\vec{\sigma} = 0$ there exists a 2-dimensional current $\vec{R}$ (with multiplicity in $\mathbf{Z}_2$) such that $\partial\vec{R} = \vec{\sigma}$. Moreover $\sigma([0,1]) \subseteq K$ and being R constant on $\mathbf{R}^2 \setminus K$, we can also assume $\operatorname{spt}\vec{R} \subseteq K$.

Let $\vec{F} := \vec{E} - \vec{R}$. Clearly $E \triangle F \subseteq K$ (in fact $\vec{F} \llcorner (\mathbf{R}^2 \setminus K) = \vec{E} \llcorner (\mathbf{R}^2 \setminus K)$) and $\partial^* F \subseteq \partial F \subseteq \partial E \cup S$. Since

$$\partial\vec{F} \llcorner (\Gamma \setminus S) = \partial\vec{E} \llcorner (\Gamma \setminus S) - \vec{\tau} \llcorner (\Gamma \setminus S) + \vec{\gamma} \llcorner (\Gamma \setminus S) = 0 \qquad (\mathrm{mod}\ 2)$$

we obtain that $\|\partial\vec{F}\|(\Gamma \setminus S) = 0$. But $\|\partial\vec{F}\| = |D\varphi_F| = \mathcal{H}^1 \llcorner \partial^* F$ and we obtain $\mathcal{H}^1(\partial^* F \cap (\Gamma \setminus S)) = 0$. So $\partial^* F \subseteq (\partial E \setminus \Gamma) \cup S$, up to $\mathcal{H}^1$-negligible sets, and $P(F,K) = \mathcal{H}^1(\partial^* F \cap K) \leq \mathcal{H}^1(\partial E \cap K) - \mathcal{H}^1(\Gamma) + \mathcal{H}^1(S)$ whence the conclusion follows.

□

Theorem 5.2 *Let $E \subseteq \mathbf{R}^2$, $E \in \mathcal{M}_\omega$. Then ∂E is locally parameterizable over a line, with a bilipschitz map. That is, ∂E is a lipschitz 1-dimensional manifold.*

Proof:
Let z_0 be any point of ∂E. In view of Theorem 4.10 we can find an open neighbourhood B of z_0 such that there exists an homeomorphism $\tau\colon [0,L] \to \partial E \cap \overline{B}$. We may suppose that $\overline{B} \subseteq B_r(z_0)$ with $\omega(2r) < \frac{1}{16\pi}$. Suppose also that τ is a chord-arc parameterization so that τ is 1-lipschitz. Let us prove that τ^{-1} is also lipschitz.

Let $t_1, t_2 \in [0,L]$ and consider the set F given by Lemma 5.1. If K is the convex hull of $\tau([t_1,t_2])$ notice that $K \subseteq B_\rho(x)$ for some $x \in \partial E$ and some $\rho \leq |t_2 - t_1|$. Being $E \triangle F \Subset B_\rho(x)$, by the minimality of E and by Proposition 1.3 we get

$$P(E, B_\rho(x)) - P(F, B_\rho(x)) \leq \frac{\omega(\rho)}{1+\omega(\rho)} P(E, B_\rho(x)) \leq \frac{4\pi(1+\omega(2\rho))\omega(\rho)}{1+\omega(\rho)}\rho \leq \frac{1}{2}\rho$$

on the other hand Lemma 5.1 implies that

$$P(F, B_\rho(x)) - P(E, B_\rho(x)) \leq |\tau(t_2) - \tau(t_1)| - |t_2 - t_1|$$

and we conclude

$$|t_2 - t_1| \leq 2|\tau(t_2) - \tau(t_1)|.$$

which means that τ^{-1} is lipschitz.

□

References

[1] W. Allard: *On the First Variation of a Varifold,* Ann. of Math. **95** (1972) 417–491.

[2] F. J. Almgren: *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints,* Mem. Amer. Mat. Soc. 4 **165** (1976).

[3] L. Ambrosio: Corso introduttivo alla teoria geometrica della misura ed alle superfici minime, Quad. Scuola Norm. Sup. Pisa, Pantograf, Genova 1997.

[4] L. Ambrosio - D. Pallara: *Partial regularity of free discontinuity sets, I,* Ann. Scuola Norm. Sup. Pisa, **24** (1997), 1–38.

[5] E. Barozzi - E. Gonzalez - I. Tamanini: *The mean curvature of a set of finite perimeter,* Proc. AMS **99** (1987), 313–316.

[6] E. Bombieri: *Regularity theory for almost minimal currents,* Arch. Rational Mech. Anal. **78** (1982), 99–130.

[7] K. A. Brakke: The motion of a surface by its mean curvature, Princeton U.P., 1978.

[8] E. De Giorgi: *Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio a r dimensioni,* Ann. Mat. Pura Appl. Ser IV **36** (1954), 191–213.

[9] E. De Giorgi: *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio a r dimensioni,* Ric. Mat. **4** (1955), 95–113.

[10] E. De Giorgi: Frontiere orientate di misura minima, Seminario di Matematica della Scuola Normale Superiore di Pisa, Editrice Tecnico Scientifica, Pisa, 1961.

[11] E. De Giorgi - F. Colombini - L. C. Piccinini: Frontiere orientate di misura minima e questioni collegate, Quad. Scuola Norm. Sup. Pisa, Numero 1, Editrice Tecnico Scientifica Pisa, 1972.

[12] H. Federer: Geometric Measure Theory, Spinger-Verlag, Berlin, 1969.

[13] E. Giusti: Minimal Surfaces and Functions of Bounded Variation, Monographs in Mathematics, Birkhäuser, 1984.

[14] E. Gonzalez - U.Massari: *Variational mean curvatures,* Rend. Sem. Mat. Univ. Pol. Torino **52** (1994), 1–28.

[15] E. Gonzalez - U. Massari - I. Tamanini: *Boundaries of prescribed mean curvature,* Rend. Mat. Accad. Lincei, s. 9 **4** (1993), 197–206.

[16] U. Massari: *Esistenza e regolaritè delle ipersuperfici di curvatura media assegnata in $\mathbf{R}^n$,* Arch. Rat. Mech. & Anal. **55** (1974), 357–382.

[17] U. Massari: *Frontiere orientate di curvatura media assegnata in L^p.* Rend. Sem. Mat. Univ. Padova **53** (1975), 37–52.

[18] U. Massari - M.Miranda: Minimal surfaces of codimension one, North Holland, Amsterdam (1984).

[19] E. Paolini: *Regularity for minimal boundaries in $\mathbf{R}^n$ with mean curvature in L^n,* manuscripta math. (to appear).

[20] E. R. Reifenberg: *Solution of the plateau problem for m-dimensional surfaces of varying topological type,* Acta Math., **104** (1960), 1–92.

[21] S. Semmes: *Hypersurfaces in $\mathbf{R}^n$ whose unit normal has small BMO norm,* Proc. AMS, **112** (1991), 403–412.

[22] S. Semmes: *Chord-arc surfaces with small constant. II: Good parameterizations,* Adv. in Math., **85** (1991), 198–223.

[23] L. Simon: Lectures on Geometric Measure Theory, Proc. C.M.A., 1983.

[24] I. Tamanini: *Boundaries of Caccioppoli sets with Hölder continuous normal vector,* J. Reine Angew. Math. **334** (1982), 27–39.

[25] T. Toro: *Surfaces with generalized second fundamental form in L^2 are lipshitz manifolds,* J. Diff. Geom., **39** (1994), 65–101.

[26] T. Toro: *Geometric conditions and existence of bi-lipshitz parameterizations,* Duke Math. J., **77** (1995), 193–227.