Geometric properties for optimal sets and upper bound branching time under coercive L^p measures

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Abstract

In this paper we consider the quasi-static irreversible evolution of a connected network related to an average distance functional minimization problem. Our main goal is to extend some geometric properties of optimal sets to the coercive L^p measure case, and determine whether a branches is exhibited during the a minimizing movement evolution, thus changing the topology. We would give a sufficient condition for the latter. Tools belonging to minimizing movements and optimal transportation theory with free Dirichlet regions will be used extensively. Finally, we will apply our results to find an upper bound for the branching time for a particular class of configurations.

Keywords: optimal transport, Euler scheme, minimizing movements, average distance

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1 Introduction

In 1993 De Giorgi in the work [11] introduced the concept of minimizing movement to study general evolution problems endowed with some kind of variational structure. In this paper we will consider the general quasi-static, rate independent, evolution for connected networks related to an average distance functional.

Let Ω be a compact subset of $\mathbb{R}^2$, $S \subset \Omega$ a connected set with $\dim_{\mathcal{H}} S = 1$ and given Hausdorff measure, and a generic measure $f \geq 0$; we define the main functional of this paper:

$$F_f(S) := \int_\Omega \text{dist}(x, S) df.$$

Compared to the Lebesgue measure, with copes superbly in relating measure theory quantities with geometrical ones, the generic f measure losses most, if not all, of this capability. So in the paper one

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important point is to determine "how far" we can go, i.e. the minimum constraints for f required if results were to hold.

Given a domain Ω and $l \geq 0$, these sets will be used in all the paper:

$$A_l := \left\{\mathcal{X} \subseteq \Omega : \mathcal{X} \text{ compact, connected and } \mathcal{H}^1(\mathcal{X}) \leq l\right\}, \qquad A := \bigcup_{j \geq 0} A_j. \tag{1.1}$$

While a priori both A_l and A depend on the domain, to simplify notations, when the domain is clear (e.g. in a statement in which the domain is given in the hypothesis) and there is no risk of confusion we will omit this dependence.

Let us introduce a first result on the monotonicity of F_f:

Proposition 1.1. *Given a domain Ω, a measure $f \geq 0$, for any $S_1, S_2 \in A$, with $S_1 \subseteq S_2$, we have $F_f(S_1) \geq F_f(S_2)$.*

Proof. The proof is easy: as $S_1 \subseteq S_2$, for any $x \in \Omega$ we have

$$\mathrm{dist}(x, S_1) \geq \mathrm{dist}(x, S_2)$$

and integrating on Ω

$$\int_\Omega \mathrm{dist}(x, S_1) df \geq \int_\Omega \mathrm{dist}(x, S_2) df$$

which concludes the proof. □

Moreover we see from the proof of Proposition 1.1 that if there exists $\Omega' \subseteq \Omega$ with $f(\Omega') > 0$ and $\mathrm{dist}(x, S_1) > \mathrm{dist}(x, S_2)$ for any $x \in \Omega'$, then $F_f(S_1) > F_f(S_2)$.

This result says that prescribing the maximum length is the same as prescribing the length, i.e. for any measure $f \geq 0$, for any $h > 0$

$$\min_{\mathcal{X} \in A_h} F_f(\mathcal{X}) = \min_{\mathcal{H}^1(\mathcal{X}')=h} F_f(\mathcal{X}')$$

and

$$\mathrm{argmin}_{A_h} F_f = \mathrm{argmin}_{A_h \setminus \bigcup_{0 \leq h' < h} A_{h'}} F_f.$$

So in the paper, we can use these two constraints indifferently.

We aim to extend some results concerning geometric properties of optimal sets (for instance see [6], [7] and [8]) to the more general measures, and investigate the topological behavior of the minimizing movement evolution process, thus extending results in [12] to this more general case.

We will work with a particular class of measures:

Definition 1.2. *A non negative measure f is "coercive" if there exists $c_f \geq 0$ (the "coercivity constant") such that $f \geq c_f \mathcal{L}^2$.*

This paper will be structured as follows:

- in Section 2 we will present some results concerning the geometric properties of optimal sets (see [6], [7] and [8]), and extend them;

- in Section 3 we will introduce the minimizing movement theory, and analyze whether a branching behavior can happen;
- in Section 4 we will apply results from Section 3 to a particular class of configurations, thus giving the branching time estimate in this case.

Notations

The most used in this paper will be:

- Ω to denote the domain,
- f to denote a coercive measure with $f << \mathcal{L}^2$
- Σ to denote the minimizing movement function,
- ε, δ, r, ρ to denote small positive number,
- l to denote generic positive number,
- p will be used most to denote the summability class of measures, and q the conjugate exponent of p,
- S to denote generic connected compact sets in the domain,
- S_0 to denote the initial datum of an Euler scheme/minimizing movement,
- $w(k, \cdot)$, $w(k)$ ($k \in \mathbb{N}$) to denote the $(k+1)$-th set of an Euler scheme.

To avoid using excessive number of different notations, some symbols will be used in more situations: unless explicitly specified, if a notation is used in two different Definitions/ Propositions/ Lemma/ Theorems, there is no connection between them, so there is no risk of confusion. (E.g. for the measure, we are not going to use f_1 in one statement, f_2 in another, f_3 in another too ... , but we will use f in all of them and it is implicitly assumed that there is no connection between them unless explicitly specified.)

The only notable exceptions are

- A_l (with $l \geq 0$), and A: if there is a given domain Ω, they always denote the sets defined in (1.1),
- F_f which stands for the average distance functional (with dependence on the measure f),
- F will denote the average distance functional with Lebesgue measure,
- $V(\cdot)$ which stands for the Voronoi cell of the point.

We will work only domains which are closure of bounded, connected, open sets. Moreover, when we will write $F(\mathcal{X}_1 \cup \mathcal{X}_2)$ (where $\mathcal{X}_1, \mathcal{X}_2 \in A$), we will assume implicitly that $\mathcal{X}_1 \cup S_2 \in A$

2 Geometric properties

In this section we present some results concerning the geometrical properties of optimal sets, found in [6], [7] and [8], and extend them to more general measures.

In most cases we will first recall the proof in the Lebesgue measure case, then try to generalize it to more general non negative measures. The following definition may prove useful:

Definition 2.1. *Given a domain Ω, $S \in A$ a generic element, a non endpoint $P \in S$ is "smooth" if there exists $r > 0$ such that:*

(1) there exists an homeomorphism $\phi : B(P, r) \cap S \longrightarrow (0, 1)$;

(2) there exists an unique direction θ such that for any sequence $P_n \longrightarrow P$ in $B(P, r)$ the directions of the line $L(P_n, P)$ converge to θ.

A subset of S is smooth is all its non endpoints are smooth.

First we present a lower bound for the gain in energy:

Proposition 2.2. *Given a domain Ω, let be $S \subset \Omega$ be a connected set, if we add a segment λ_ε to a smooth non endpoint of S (with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$), then the "gain" $F(S) - F(S_\varepsilon)$ is comparable with $\varepsilon^{3/2}$, where $S_\varepsilon := S \cup \lambda_\varepsilon$.*

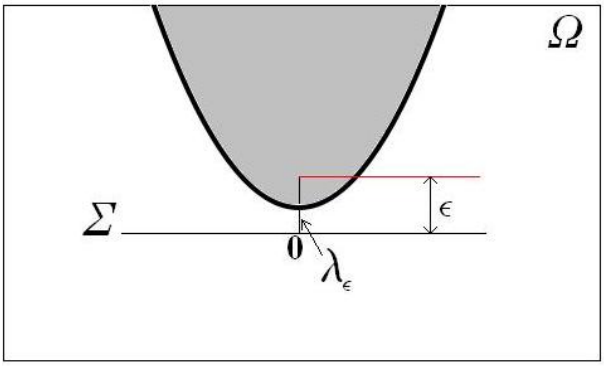

Fig. 1: All points in the shaded parabola Π, whose area is comparable with $\varepsilon^{1/2}$, gain something in path.

Proof. Upon rescaling, the configuration can be brought to the following in figure, so all the computations can be done here.

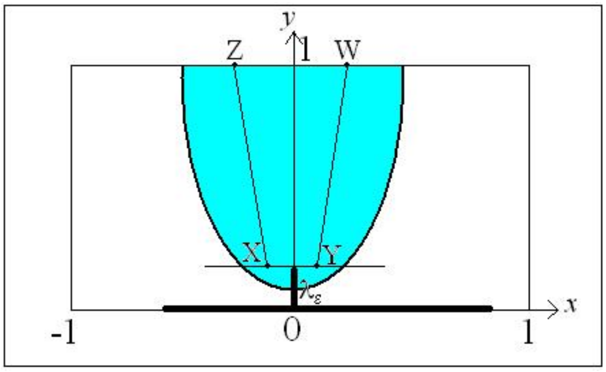

Fig. 2: For graphical purposes the borders are a bit larger, but the considered domain is $[-1,1]\times[0,1]$; notice that X, Y, W, Z are not on the border, but they are the midpoints between the y axis and the intersections of the border with $y=\varepsilon$ and $y=1$ respectively.

Adding such a segment to S, the gain is on the shaded region; if a point (x,y) can choose a shorter path, then it must satisfy

$$\text{dist}((x,y), S\cup\lambda_\varepsilon) < \text{dist}((x,y), S)$$

thus

$$(x^2+(y-\varepsilon)^2)^{1/2} < |y|,$$

which leads to

$$y > \frac{x^2}{2\varepsilon}+\frac{\varepsilon}{2}.$$

Now we have to estimate its area: as we are doing our computation in the rectangle $[-1,1]\times[0,1]\subset\mathbb{R}^2$, the parabola has boundaries defined by the last inequality and $[-1,1]\times\{1\}$. The intersections between $\{(x,y): y=\frac{x^2}{2\varepsilon}+\frac{\varepsilon}{2}\}$ and $[-1,1]\times\{1\}$ are

$$x^{\pm} := \pm\sqrt{2\varepsilon-\varepsilon^2}.$$

So the area of the shaded region is

$$2\sqrt{2\varepsilon-\varepsilon^2}-\int_{x^-}^{x^+}(\frac{t^2}{2\varepsilon}+\frac{\varepsilon}{2})dt = \frac{2}{3}\sqrt{2\varepsilon-\varepsilon^2}(2-\varepsilon).$$

The parabola contains the trapezium $XYWZ$, and $\mathcal{H}^1(XY)=\varepsilon$, $\mathcal{H}^1(WZ)=\sqrt{2\varepsilon-\varepsilon^2}$ and the height is $1-\varepsilon$. The gain in path here is at least $\varepsilon/2$ (this minimum is attained on points X and Y), so the gain for the energy functional is at least

$$\frac{\varepsilon}{2}\frac{1-\varepsilon}{2}(\varepsilon+\sqrt{2\varepsilon-\varepsilon^2}) \geq \frac{\varepsilon^{3/2}}{8},$$

so the choice $K=1/8$ is acceptable. $\square$

From the proof above we can see that given any $h > 0$, if we rescale all the configuration with the transformation

$$x \longmapsto xh, \qquad y \longmapsto yh$$

then the same argument gives that the lower bound for the gain in energy scales by the factor h^2.

All this is done under the Lebesgue measure. Now we study what happens with a more general f measure:

Proposition 2.3. *Given a domain Ω, let be $S \in A$, and a coercive measure f. Then there exists a non endpoint such that adding a segment λ_ε (with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$ small) here will cause a "gain" $F_f(S) - F_f(S_\varepsilon) = O(\varepsilon^{3/2})$, where $S_\varepsilon := S \cup \lambda_\varepsilon$.*

Proof. The proof is very similar to the previous: as the trapezium $XYWZ$ in Figure 2 has Lebesgue measure $O(\varepsilon^{1/2})$, the coercivity condition on f gives $f(XYWZ) \geq c_f |XYWZ| = O(\varepsilon^{1/2})$, with c_f the coercivity constant. Then we follow the proof of the Lebesgue measure case. □

The following results are similar to those found in [6], [7] and [8], extended to more general measures.

Proposition 2.4. *Let be Ω a given domain, $l > 0$ a fixed quantity, f a coercive measure, and $\Sigma_{opt} \in \mathrm{argmin}_{A_l} F_f$. Then Σ_{opt} cannot contain a loop (a subset homeomorphic to S^1).*

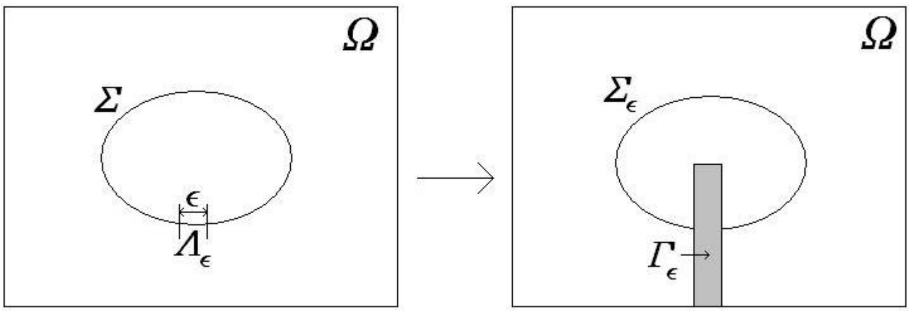

Fig. 3: This is a simple representation of what happens if we remove the portion Λ_ε.

Proof. Suppose that Σ_{opt} contains as subset E homeomorphic to S^1. If we remove the portion Λ_ε from E ($\mathcal{H}^1(\Lambda_\varepsilon) = \varepsilon > 0$), setting $E_\varepsilon := E \backslash \Lambda_\varepsilon$ we have that all the "loss" is concentrated on Γ_ε (the shaded region in Figure 1, which has Lebesgue measure no larger than $\varepsilon \mathrm{diam}(\Omega)$), as points belonging to the rest will not change their distance to Σ_ε. For the points in Γ_ε their path can be longer, but it is clear from triangle inequality

$$\mathrm{dist}(x, E_\varepsilon) \leq \mathrm{dist}(x, E) + \mathcal{H}^1(\Lambda_\varepsilon)$$

so we have

$$\int_\Omega \mathrm{dist}(x, \Sigma_\varepsilon) df \leq \int_\Omega \mathrm{dist}(x, \Sigma) df + \varepsilon f(\Gamma_\varepsilon).$$

Now we have to estimate a lower bound for $f(\Gamma_\varepsilon)$: from Hölder inequality

$$f(\Gamma_\varepsilon) \leq ||f||^{1/p}_{L^p(\Gamma_\varepsilon)} |\Gamma_\varepsilon|^{1/q}$$

and analyzing the infinitesimal orders,

$$f(\Gamma_\varepsilon) \le o(1)O(\varepsilon^{1/q}).$$

So imposing $q \le 2$, or equivalently $p \ge 2$, will conclude the proof as Proposition 2.3 states that adding such segment elsewhere the gain has order $O(\varepsilon^{3/2})$.

But we want to analyze what happens if we go beyond L^2: suppose that f is coercive. For each portion $\Lambda_\varepsilon \subset E$, the loss is concentrated on $\Gamma_\varepsilon = \Gamma_\varepsilon(\Lambda_\varepsilon)$, and if $\Lambda_\varepsilon \subset E$, $\Lambda'_\varepsilon \subset E$ are similar portions with empty intersection, their associated Γ_ε are disjoint too. So we choice points $\{X_i\}_{1=1}^N \subset E$ such that $\mathrm{dist}_E(X_k, X_{k+1}) = \varepsilon$ (dist_E is the geodesic distance on E) (it is clear that $N = \left[\frac{\mathcal{H}^1(E)}{\varepsilon}\right] + 1$ is enough), let be L_k the shortest portion of E between X_k and X_{k+1}, and $G_k := \{x \in G_k : \mathrm{dist}(x, E) < \mathrm{dist}(x, E \backslash G_k)\}$. As f does not charge Hausdorff one-dimensional sets, if all the cut loci $G_j \cap G_{j+1}$ are f-negligible for any j, we have that

$$\sum_{k=1}^N f(G_k) = f(\bigcup_{k=1}^N G_k) = f(V(E))$$

so

$$\min_{1 \le k \le N} f(G_k) \le \frac{f(V(E))}{N}.$$

But $N = \left[\frac{\mathcal{H}^1(E)}{\varepsilon}\right] + 1$, so

$$\min_{1 \le k \le N} f(G_k) = O(\varepsilon)$$

and the loss (for F_f) can be as low as comparable with ε^2. Again Proposition 2.3 concludes the proof. □

Proposition 2.5. *Let Ω be a given domain, $l > 0$ a fixed quantity, $f \in L^{\frac{4}{3}}$ a coercive measure and let be $\Sigma_{opt} \in argmin_{A_l} F_f$. Then Σ_{opt} cannot contain a cross (a subset homeomorphic to $\{x^2 + y^2 \le 1 : xy = 0\}$).*

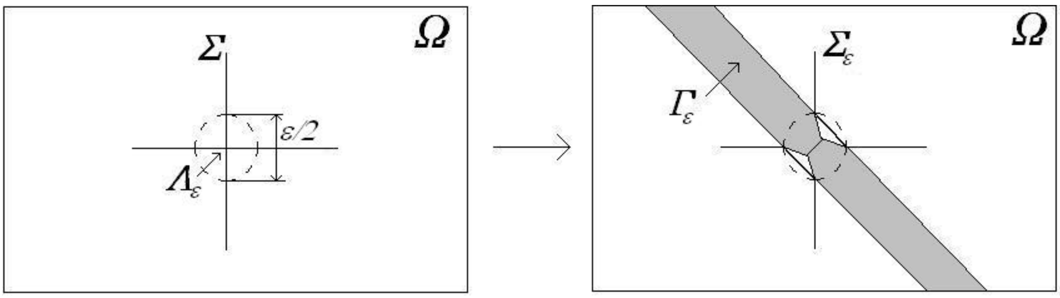

Fig. 4: Σ_ε is obtained from Σ_{opt} by replacing the infinitesimal cross Λ_ε with a slightly shorter Steiner graph.

Proof. Suppose that Σ_{opt} contains as cross Λ_ε ($\mathcal{H}^1(\Lambda_\varepsilon) = \varepsilon > 0$). If we remove the portion Λ_ε from Σ_{opt}, and replacing it with a Steiner graph Z_ε (a direct computation yields the existence of $k > 0$ such that $\mathcal{H}^1(Z_\varepsilon) < k\varepsilon$) in order to keep the connection property, setting $\Sigma_\varepsilon := \Sigma_{opt} \backslash \Lambda_\varepsilon$ we have that all the "loss" is concentrated on Γ_ε (the shaded region in Figure 2, which has Lebesgue measure

$\frac{9}{4}\pi\varepsilon^2$), as points belonging to the rest will not change their distance to Σ_ε. For the points in Γ_ε their path can be longer, but it is clear from triangle inequality

$$\text{dist}(x, \Sigma_\varepsilon) \leq \text{dist}(x, \Sigma_{opt}) + \mathcal{H}^1(\Lambda_\varepsilon)$$

so we have

$$\int_\Omega \text{dist}(x, \Sigma_\varepsilon) df \leq \int_\Omega \text{dist}(x, \Sigma_{opt}) df + \varepsilon f(\Gamma_\varepsilon)$$

and applying Hölder inequality,

$$f(\Gamma_\varepsilon) \leq ||f||^{1/p}_{L^p(\Gamma_\varepsilon)} |\Gamma_\varepsilon|^{1/q};$$

analyzing the orders we have

$$f(\Gamma_\varepsilon) \leq ||f||^{1/p}_{L^p(\Gamma_\varepsilon)} O(\varepsilon^{\frac{2}{p}}),$$

so hypothesis $f \in L^p$, $p > \frac{4}{3}$ guarantees $q < 4$ and the "loss" in energy after removing Λ_ε has order $\varepsilon^{1+2/q} = o(\varepsilon^{3/2})$. Again, Proposition 2.3, which estimates from below the "gain" in energy by adding such a portion δ_ε whose length is $\mathcal{H}^1(\Lambda_\varepsilon) - \mathcal{H}^1(Z_\varepsilon) = O(\varepsilon)$ to Σ_{opt}, will conclude the proof. $\square$

3 Evolution and branching

In this section we will present the branching problem for a quasi static irreversible minimizing movement evolution, and conditions sufficient to force it at some time. First we recall the general theory. A branching behavior is when some points (of the evolving set) increase its own multiplicity.

3.1 Minimizing movements

Let us start recalling briefly some notions about minimizing movements, in the abstract case.

Let be X a set, endowed with a convergence structure. Let be $\mathcal{F}$ a functional

$$\mathcal{F} : [0, T] \times X \times X \longrightarrow \mathbb{R} \cup \{\pm\infty\};$$

let us present the Euler scheme in the abstract case: given $\varepsilon > 0$ and $X_0 \in$ an initial datum, let be

$$\begin{cases} w(0) = X_0 \\ w(n+1) \in \text{argmin}\, \mathcal{F}((n+1)\varepsilon, \cdot, w(n)) \end{cases}$$

and let us consider the function $u_\varepsilon : [0, T] \longrightarrow X$ obtained by setting

$$u_\varepsilon(t) = w\left(\left[\frac{t}{\varepsilon}\right]\right), \tag{3.1}$$

with $[\cdot]$ denoting the integer part mapping.

In our case, we are working with the space $X = A$ endowed with the Hausdorff distance metric, and our kinetic term is (with a given measure f)

$$\mathcal{F}(t, \mathcal{X}_1, \mathcal{X}_2) := \begin{cases} F_f(\mathcal{X}_1) \; if \; \mathcal{X}_2 \subseteq \mathcal{X}_1 \; and \; \mathcal{X}_1 \in A_{t+\mathcal{H}^1(\Sigma_0)} \\ \infty \; otherwise \end{cases},$$

where $\Sigma_0 \in A$ is the initial datum.

So, given a positive time step $\eta > 0$ and an initial datum $S_0 \in A$, our Euler scheme is

$$\begin{cases} w(0) = S_0 \\ w(n+1) \in \operatorname{argmin}_{\mathcal{H}^1(\mathcal{X}) \leq \mathcal{H}^1(S_0)+(n+1)\eta, w(n) \subseteq \mathcal{X}} F_f(\mathcal{X}) \end{cases}.$$

Let us continue with the general abstract case:

Definition 3.1. *Given $T > 0$, the function $u : [0, T] \longrightarrow X$ is a minimizing movement associated with initial datum u_0 and kinetic term $\mathcal{F}$, and we will write $u \in MM(\mathcal{F}, X, u_0)$ if there exists a sequence $\varepsilon_n \downarrow 0$ for which*

$$\forall t \in [0, T] \quad u_{\varepsilon_n}(t) \to u(t).$$

In the following, when it is clear who are $\mathcal{F}, X, u_0$, we will say *"u is a MM"* instead of $u \in MM(\mathcal{F}, X, u_0)$. Expression *"rate independent"* will be used to denote this case too.

It is not difficult to generalize the above procedure by dividing $[0, T]$ in finitely many arbitrary non overlapping intervals. Namely, given a partition B of $[0, T]$ (i.e. $[0, T] = \bigcup_{i=0}^{h} [t_i, t_{i+1}]$) we consider the Euler scheme (with preassigned $\mathcal{F}, X, u_0$)

$$\begin{cases} w(0) = u_0 \\ w(t_{i+1}) \in \operatorname{argmin} \mathcal{F}(t_{i+1}, \cdot, w(t_i)) \end{cases}$$

and the function

$$u_B(t) = w(t_i) \quad \forall t \in [t_i, t_{i+1}).$$

Definition 3.2. *Given a time $T > 0$ the function $u : [0, T] \longrightarrow X$ is a general minimizing movement associated with initial datum u_0 and energy $\mathcal{F}$, and we will write $u \in GMM(\mathcal{F}, X, u_0)$ if there exists a family of finite partition $\{B_n\}_{n=0}^{\infty}$, with B_n becoming finer for $n \to \infty$ (i.e. for any $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for any $i, n > n(\varepsilon)$ $t_{i+1}^{(n)} - t_i^{(n)} < \varepsilon$, where $B_n := \{[t_i^{(n)}, t_{i+1}^{(n)}]\}_{i=0}^{h(n)}$ with $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{h(n)+1}^{(n)} = T$), such that*

$$\forall t \in [0, T] \quad u_{B_n}(t) \to u(t).$$

Back to our case, the first problem to deal with is the existence of limit functions, i.e. solutions of the rate-independent problem. Firstly, notice that given a domain Ω, for any $l > 0$ the metric space $(A_l, d_{\mathcal{H}})$ (with A_l defined in (1.1)) satisfies the following conditions:

- the convergence in $(A_l, d_{\mathcal{H}})$ is sequentially compact;
- the irreversibility condition is compatible with the convergence;

- every nondecreasing function $\psi : \mathbb{R} \longrightarrow (A_l, d_{\mathcal{H}})$ is continuous up to countably many points.

The first two conditions are easy to verify, while the third arises from the following argument: consider a generic nondecreasing function $\psi : \mathbb{R} \longrightarrow (A_l, d_{\mathcal{H}})$, and suppose that it has discontinuity points $\{x_i\}_{i\in I}$, with $x_i < x_j$ when $i < j$. As ψ is nondecreasing, we can write

$$\psi(x_1) \subset \psi(x_2) \subset \cdots \psi(x_i) \subset \psi(x_{i+1}) \subset \cdots \subset \psi(\sup_{i\in I} x_i),$$

and passing to the $\mathcal{H}^1$ measures,

$$\mathcal{H}^1(\psi(x_1)) < \mathcal{H}^1(\psi(x_2)) < \cdots \mathcal{H}^1(\psi(x_i)) < \mathcal{H}^1(\psi(x_{i+1})) < \cdots < \mathcal{H}^1(\psi(\sup_{i\in I} x_i)) < \infty,$$

which is possible only if I is finite or countable at most.

Then the following result holds (we refer to [4] for further details about the proof):

Proposition 3.3. *Under these three assumptions, every sequence $\{u_{\varepsilon_n}\}_{n=0}^{\infty}$ (as defined in* (3.1)*) of Euler schemes has an accumulation point which is a MM (or GMM, depending on the context).*

Similar to the Lebesgue measure case, for any measure $f \geq 0$, any minimizing movement $\Sigma : [0,T] \longrightarrow A$, obtained as limit of of a sequence of Euler schemes $\Sigma_{\varepsilon_n} : [0,T] \longrightarrow A$, verifies $F_f(\Sigma(t)) \geq F_f(\Sigma_{\varepsilon_n}(t)) - O(\varepsilon_n)$ for any $t \in [0,T]$.

3.2 Estimates

The next result is a lower bound estimate when adding length at an endpoint: we present first the Lebesgue measure case, then we look for conditions on the measure f:

Proposition 3.4. *Given a domain Ω, let $S \in A$ be a smooth set, and let it have an endpoint O which satisfies:*

($*'$) *there exist $\rho, \theta > 0$ and a triangle T' with a vertex in O and sides ρ, ρ, $\rho\sqrt{2(1-2\cos\theta)}$ (the order is not relevant) that does not intersect S.*

Then there exists ε_0 such for any $\varepsilon < \varepsilon_0$ adding a segment λ_ε at O, with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$ in O is more convenient that adding any connected set with same length at any non endpoint (i.e. the energy F gains more in the former case).

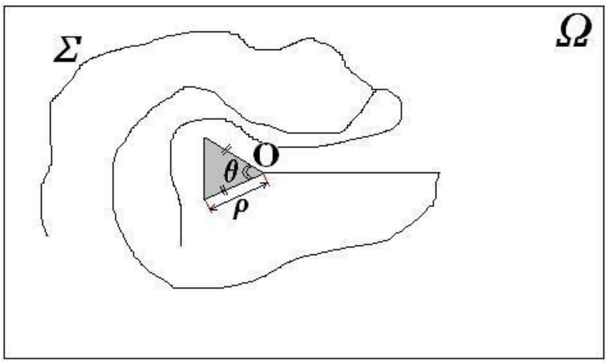

Fig. 5: the presence of the shaded triangle T' makes adding at an endpoint more convenient than at a non endpoint at least when the added portion has sufficient small length.

Proof. Adding λ_ε at a smooth non endpoint, as stated in Proposition 2.3, will decrease the energy by a quantity comparable with $\varepsilon^{3/2}$, and from the proof of Proposition 2.3, bounded by $\varepsilon^{3/2}\mathrm{diam}\,\Omega$. But adding it at O and in the shaded triangle, with ε small enough, will cause:

$$F(S_\varepsilon) - F(S) \leq \int_\Omega \mathrm{dist}(x, S_\varepsilon)dx - \int_\Omega \mathrm{dist}(x, S)dx \leq -C\varepsilon\mathcal{L}^2(T).$$

Now we estimate a lower bound value for C: if we add the segment λ_ε at O, along the bisector of the marked angle in Figure 5 (whose value is θ), then all points on $JKK'J'$ (where J', K' are midpoints of segment OJ and OK) will have a gain in path to S at least

$$\frac{\rho}{2} - \sqrt{\frac{\rho^2}{4} - \frac{\varepsilon\rho}{2}\cos\frac{\theta}{2}} \approx \varepsilon\cos\frac{\theta}{2} - O(\varepsilon^2) \tag{3.2}$$

as this is the gain of points on OJ and OK , and points inside gain even more. Notice that

$$\varepsilon\cos\frac{\theta}{2} - O(\varepsilon^2) > \frac{\varepsilon}{2}\cos\frac{\theta}{2}$$

for any $\varepsilon < \frac{\rho}{2}\cos\frac{\theta}{2}$, the total gain in energy is not less than $\frac{\varepsilon}{2}\cos\frac{\theta}{2}$ multiplied by the area of trapezium $JKK'J'$, i.e.

$$\frac{3}{8}\varepsilon\cos\frac{\theta}{2}\mathcal{L}^2(T').$$

So for ε such that $\frac{3}{8}\varepsilon\cos\frac{\theta}{2}\mathcal{L}^2(T') > \varepsilon^{3/2}\mathrm{diam}\,\Omega$, i.e.

$$\varepsilon < \left(\frac{3\cos\frac{\theta}{2}\mathcal{L}^2(T')}{8\mathrm{diam}\,\Omega}\right)^2 \bigwedge \left(\frac{\rho}{2}\cos\frac{\theta}{2}\right)$$

we have that adding λ_ε to O is more convenient than adding it at an non endpoint. □

Notice that in the entire proof few points relies on the fact that we are working with the Lebesgue measure:

- The first point is when we state that the gain by adding at a non endpoint is comparable with $\varepsilon^{3/2}$: for the general case, as we are considering measures $f \in L^p$ coercive, we have that the lower bound estimate holds without problem. For the upper bound, recalling the proof of Proposition 2.3, the gain $O(\varepsilon^{3/2})$ is due to the fact that the gain in path is $O(\varepsilon)$ and the parabola on which this gain is concentrated has Lebesgue measure $O(\varepsilon^{1/2})$. So in the general case, with coercive measure $f \in L^p$, Hölder inequality yields (for any f-measurable set Y)

  $$f(Y) \leq ||f||_{L^p(Y)}^{1/p}\mathcal{L}^2(Y)^{1/q}.$$

  So applying this result to the parabola Π (found in the proof of Proposition 2.3) on which the gain is concentrated, we have

$$f(\Pi) \leq ||f||_{L^p(\Pi)}^{1/p} \mathcal{L}^2(\Pi)^{1/q},$$

and analyzing the orders,

$$f(\Pi) \leq ||f||_{L^p(\Pi)}^{1/p} O(\varepsilon^{\frac{3}{2q}}).$$

Then the gain for F_f has order $O(\varepsilon^{1+\frac{3}{2q}})||f||_{L^p(\Pi)}^{1/p}$, so for any $q < \infty$ (thus $p > 1$) we have

$$O(\varepsilon^{1+\frac{3}{2q}})||f||_{L^p(\Pi)}^{1/p} = o(\varepsilon).$$

- The second point is when we estimate the gain by adding length at an endpoint satisfying condition $(*')$, and precisely when we compute the area of the trapezium $JKK'J'$: as the gain in path is not dependent on the measure, and we are considering coercive measures f, we have that the estimate $O(\varepsilon)$ for the gain in energy now becomes a lower bound estimate.

So combining the above two points, we have:

Proposition 3.5. *Given a domain Ω, let $S \in A$ be a smooth set, $f \in L^p$ a coercive measure with $p \in (1, \infty]$, and let it have an endpoint O which satisfies:*

$(*)$ *there exist $\rho, \theta > 0$ and a triangle T' with a vertex in O and sides ρ, $\rho, \rho\sqrt{2(1 - 2\cos\theta)}$ (the order is not relevant) that does not intersect S.*

Then there exists ε_0 such for any $\varepsilon < \varepsilon_0$ adding a segment λ_ε at O, with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$ in O is more convenient that adding any connected set with same length at any non endpoint (i.e. the energy F_f gains more in the former case).

This result provides a range of configurations in which adding at endpoints is better than adding elsewhere, thus discouraging branching behaviors. But as we will see in the following section, there are situations not satisfying these hypothesis, and definitely branching behaviors may appear.

3.3 Changing topology

Now we investigate all the situations that may appear during the evolution. Given an initial datum $S_0 \in A, \Sigma : [0, T] \longrightarrow A$ a minimizing movement function, a time $T_0 \in [0, T]$, the following behaviors are possible:

(1) Σ evolves by adding length at endpoints, i.e. there exists $\delta > 0$ such that given $t \in (T_0, T_0 + \delta)$, any simple point of $\Sigma(T_0)$ is simple in $\Sigma(t)$ too, any triple point of $\Sigma(T_0)$ is triple in $\Sigma(t)$ too, etc...;

(2) Σ evolves by adding length at a non endpoint, i.e. there exists a point of $\Sigma(T_0)$ which does not verify the condition stated in choice (1) for any $\Sigma(t)$, $t > T_0$.

In order to provide an upper bound to the branching time, we need to establish when choice (2) (which corresponds to a new branching appearing) becomes necessary preferable to choice (1).

We try to estimating the required "free space" (i.e. the minimum value for Voronoi cells of its endpoints) to evolve without changing topology.

Proposition 2.3 provides an estimate which is too weak, as the gain obtained in that way scales like $\varepsilon^{3/2}$ for ε small enough, and it is not sufficient to force a branching behavior. Something stronger is required.

The following definition is useful:

Definition 3.6. *Given a domain Ω, $S \in A$ a generic element, a non endpoint $P \in S$ is "angular" if there exists $r > 0$ such that:*

(1) there exists an homeomorphism $\varphi : B(P,r) \cap S \longrightarrow (0,1)$;

(2) there exists exactly two unitary vectors θ_1, θ_2 such that for any sequence $P_n \longrightarrow P$ in $B(P,r)$, calling $v(P_nP)$ the vector starting in P_n and pointing toward P, the accumulation points of set $\left\{\frac{v(P_nP)}{||v(P_nP)||}\right\}_{n=0}^{\infty}$ are in $\{\theta_1, \theta_2\}$, and there exist sequences $\left\{P_n^{(1)}\right\}_{n=0}^{\infty}$, $\left\{P_n^{(2)}\right\}_{n=0}^{\infty}$ such that $\left\{\frac{v(P_n^{(1)}P)}{||v(P_n^{(1)}P)||}\right\} \longrightarrow \theta_1$ and $\left\{\frac{v(P_n^{(2)}P)}{||v(P_n^{(2)}P)||}\right\} \longrightarrow \theta_2$.

Notice that geometrically, a point P is angular if the tangent vectors are not collinear here (see Figure 6).

Lemma 3.7. *Given a domain Ω, let $S \in A$ be an arbitrary element, and suppose there exists $Q \in S$ angular and let be $\delta > 0$ such that $B(Q,\delta) \cap S$ is homeomorphic to $(0,1)$. Then the Voronoi cell $V(Q)$ contains a triangle T_Q with sides $\rho_Q > 0$ and angle $\hat{Q} > 0$.*

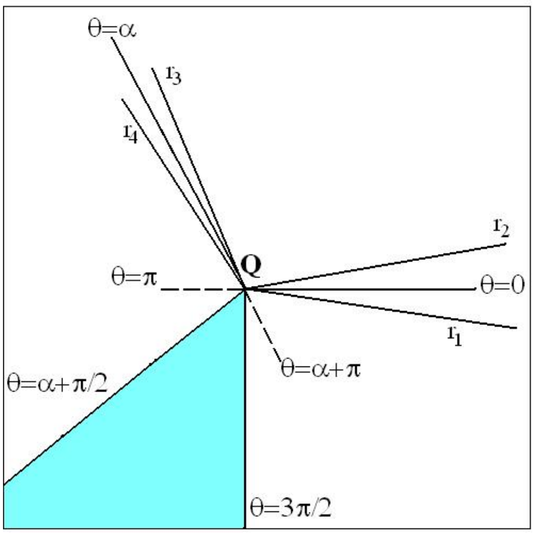

Fig. 6: All the points in the shaded area belong to $V(Q)$, and it contains a triangle.

For the proof we refer to [12].

The next result is a condition on the branching behaviors for Euler schemes.

Proposition 3.8. *Given a domain Ω, a coercive measure f, let $S_0^{(1)} \in A$ be a generic element, T a positive time and $\varepsilon > 0$ a (small) positive time step, let us consider the Euler scheme*

$$\begin{cases} w(0) := S_0^{(1)} \\ w(k) \in \mathit{argmin}_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(S_0^{(1)}) + k\varepsilon,\ w(k-1) \subseteq \mathcal{X}'} F_f(\mathcal{X}') \end{cases}$$

in the time interval $[0,T]$.

Suppose that there exist $P_0 \in S_0^{(1)}$ angular and $\eta > 0$ such that $B(P_0,\eta) \cap (w(k) \backslash w(0)) = \emptyset$ for any k. Then there is an upper bound $T^\varepsilon_{\max}$ such that $T > T^\varepsilon_{\max}$ forces a branching behavior.

Proof. As P_0 is angular, Lemma 3.7 gives the existence of a similar T_{P_0} which verifies condition $(*)$ of Proposition 3.5 for some positive ρ, θ. So from the estimate of Proposition 3.5 there is a constant $K(P_0) > 0$ (depending only on ρ, θ and the coercivity constant of f but not on ε) such that for any j

$$\min_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(w(j-1)) + v\varepsilon,\ w(j-1) \subseteq \mathcal{X}'} F_f(\mathcal{X}') \leq F_f(w(j-1)) - K(P_0)\varepsilon,$$

as this gain is achieved by simple adding a segment $Seg_\varepsilon \subset T_P$ ($\mathcal{H}^1(Seg_\varepsilon) = \varepsilon$) along the bisector of $\hat{P}_0$, which would create a branching behavior.

In order to avoid this, for any d $w(d)$ must be obtained from $w(d-1)$ by adding length at points of $ext(w(d-1))$, and the gain in energy must be more than $K(P_0)\varepsilon$, i.e.

$$F_f(w(d)) \leq F_f(w(d-1)) - K(P_0)\varepsilon \qquad \forall d = 1, \cdots, \left[\frac{T}{\varepsilon}\right]$$

which leads to

$$F_f(w(d)) \leq F_f(w(0)) - dK(P_0)\varepsilon \qquad \forall d = 1, \cdots, \left[\frac{T}{\varepsilon}\right]$$

and finally, for $d = \left[\frac{T}{\varepsilon}\right]$,

$$F_f(\left[\frac{T}{\varepsilon}\right]) \leq F_f(w(0)) - \left[\frac{T}{\varepsilon}\right] K(P_0)\varepsilon.$$

As $\frac{T}{\varepsilon} - 1 \leq \left[\frac{T}{\varepsilon}\right] \leq \frac{T}{\varepsilon}$, this leads to

$$0 \leq F_f(\left[\frac{T}{\varepsilon}\right]) \leq F_f(w(0)) - (T-\varepsilon)K(P_0),$$

which forces

$$T \leq \varepsilon + \frac{F_f(S_0^{(1)})}{K(P_0)}$$

and putting $T^\varepsilon_{\max} := \varepsilon + \frac{F_f(S_0^{(1)})}{K(P_0)}$ completes the proof. ☐

Notice that this result holds definitely for $\varepsilon > 0$ small enough, and there is no difficulty in passing to the limit $\varepsilon \to 0$, so it can be applied to the rate independent case:

Theorem 3.9. *Given a domain Ω, a coercive measure g, let $S_0^{(2)} \in A$ be a generic element, T a positive time, and $\Sigma : [0, T] \longrightarrow A$ a minimizing movement obtained as limit of the Euler schemes with time step $\{\varepsilon_n\} \downarrow 0$*

$$\begin{cases} w(0, n) = w(0) = S_0^{(2)} \\ w(k, n) \in argmin_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(S_0^{(2)}) + k\varepsilon_n,\ w(k-1,n) \subseteq \mathcal{X}'} F_g(\mathcal{X}') \end{cases} .$$

Suppose that there exist $P_1 \in S_0^{(2)}$ angular and $\eta' > 0$ such that $B(P_1, \eta) \cap (\Sigma(T) \backslash S_0^{(2)}) = \emptyset$. Then there is an upper bound $T_{\max}$ (depending only on geometrical quantities) such that $T > T_{\max}$ forces a no branching behavior.

Proof. As $(\Sigma(T) \backslash S_0^{(2)}) \cap B(P_1, \eta') = \emptyset$, $\lim_{n \to \infty} (w \left(\left[\frac{T}{\varepsilon_n} \right], n \right) \backslash S_0^{(2)}) \cap B(P_1, \eta') = \emptyset$ so there exist $\bar{n}$ such that for any $n \geq \bar{n}$

$$\lim_{n \to \infty} (w \left(\left[\frac{T}{\varepsilon_n} \right], n \right) \backslash S_0^{(2)}) \cap B(P_1, \frac{\eta'}{2}) = \emptyset.$$

By Lemma 3.7 there exists $T_{P_1} \subseteq V(P_1)$, and for any ε_k hypothesis of Proposition 3.8 are verified. Thus there exists a constant $K'(P_1)$ such that

$$F_g(w(d, n)) \leq F_g(w(d-1, n)) - K'(P_1)\varepsilon \qquad \forall d = 1, \cdots, \left[\frac{T}{\varepsilon_n} \right]$$

holds for any n, thus

$$0 \leq F_g(\left[\frac{T}{\varepsilon_n} \right]) \leq F_g(w(0)) - (T - \varepsilon_n) K'(P_1)$$

holds for any n, and the upper bound (given by Proposition 3.8)

$$T_{\max}^{\varepsilon_n} = \varepsilon_n + \frac{F_g(S_0^{(2)})}{K'(P_1)}$$

holds for any n; passing to the limit $n \longrightarrow \infty$, it reads

$$T_{\max} = \frac{F_g(S_0^{(2)})}{K'(P_1)}$$

for the rate independent case, which concludes the proof. □

4 Applications

In this section we give two examples of branching behavior, and two ways to estimate this.

4.1 Energy estimate

In Theorem 3.9 we have given an upper bound estimate for the branching time under that particular configuration: now we present an explicit example.

In order to apply this result, its hypothesis must be verified: so given a domain Ω, a coercive measure f_1, let S_0^{ini} be the initial datum, and suppose there exist $P_0 \in S_0^{ini}$ angular and $\xi > 0$ such that $B(P_0, \xi) \cap S_0^{ini}$ is homeomorphic to $(0, 1)$.

Moreover, we must ensure that this ball $B(P_0, \xi)$ is not visited, and one way to do this is imposing that any visiting here must cause a branching behavior. So we choose a particular S_0^{ini}.

Suppose that

- there exist $P_0 \in S_0^{ini}$ angular and let be $\xi > 0$ such that $B(P_0, \xi) \cap S_0^{ini}$ is homeomorphic to $(0, 1)$;
- there exist a closed injective path $\gamma : [0, 1] \longrightarrow \Omega$ such that $\gamma([0, 1]) \subseteq S_0^{ini}$: the domain Ω is now divided in two regions, Ω^+ and Ω^- with $\Omega = \Omega^+ \cup \Omega^-$ (they are the two connected components of $\Omega \backslash \gamma([0, 1])$, and they correspond to the "interior" and the "exterior" part of $\gamma([0, 1])$ – the order is not relevant – as given by the Jordan Curve Theorem);
- triangle $T_{P_0} \subset V(P_0) \cap B(P_0, \xi)$ (whose existence is given by Lemma 3.7) verifies $|T_{P_0} \cap \Omega^+| > 0$, and $ext(S_0^{ini}) \subset \Omega^-$.

The main estimate we are going to present here is Theorem 4.2, whose proof requires a series of preliminary lemma.

In the rest of this subsection we will suppose that Ω^- is large enough (both in diameter and in measure) so that all computations can be done without considering constraints imposed by $\mathrm{diam}(\Omega^-)$, $f'(\Omega^-)$ (which can only lower the branching time).

Consider now a minimizing movement $\Sigma : [0, T] \longrightarrow A$, limit of Euler schemes $\{\Sigma_{\varepsilon_n}\}_{n=0}^{\infty}$ (with time steps $\{\varepsilon_n\}_{n=0}^{\infty} \downarrow 0$):

$$\begin{cases} w(0, n) = w(0) := S_0^{ini} \\ w(k, n) \in \mathrm{argmin}_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(S_0^{ini}) + k\varepsilon_n} F_{f_1}(\mathcal{X}') \end{cases},$$

$$\Sigma_{\varepsilon_n}(t) := w\left(\left[\frac{t}{\varepsilon_n}\right], n\right), \qquad \Sigma(t) = \lim_{n\to\infty} \Sigma_{\varepsilon_n}(t) \forall t \in [0, T].$$

The notations introduced (except mute counters) will have the same meaning in all this subsection.

Lemma 4.1. *If there exist k, n such that $(w(k, n)\backslash S_0^{ini}) \cap \Omega^+ \neq \emptyset$, but $(w(k-1, n)\backslash S_0^{ini}) \cap \Omega^+ = \emptyset$, this means $w(k, n)$ is not homeomorphic to $w(k-1, n)$.*

For the proof we refer to [12].

Now we can present an upper bound estimate for the branching time.

Theorem 4.2. *Under these hypothesis, there exists a time $\bar{T}$ such that if $T > \bar{T}$, then there exists al least two sets in $\{\Sigma(t)\}_{t \in [0,T]}$ which are not homeomorphic, thus the branching time is not larger than $\bar{T}$.*

Proof. From Lemma 4.1 we see that for any k, n, $w(k,n)\backslash w(0)$ must be in $\Omega^- \cup \gamma([0,1])$, while $T_{P_0} \cap \Omega^+ \subseteq V(P_0)$ has positive measure, so for Theorem 3.9 we have that there exists a constant $K(P_0)$ such that it is not possible to evolve beyond time

$$\frac{F_{f_1}(S_0^{ini})}{K(P_0)}$$

without branching.

Now we compute $K(P_0)$ from geometric quantities: let us call P_1, P_2 the other two vertex of T_{P_0}, and c_{f_1} teh coercivity constant of f_1: by reducing the measure of the triangle we can suppose that $\mathcal{H}^1(P_0P_1) = \mathcal{H}^1(P_0P_2)$, and let be ϕ the value of $\widehat{P_1P_0P_2}$:

using the same argument found in the proof of Proposition 3.5 (the computation about the lower bound estimate) the choice

$$K(P_0) := c_{f_1}\frac{1}{8}|T_{P_0}|\cos\frac{\phi}{2}$$

is acceptable, so the upper bound estimate for the branching time is in this case

$$\bar{T} := \frac{8F(S_0^{ini})}{c_{f_1}|T_{P_0}|\cos\frac{\phi}{2}}.$$

□

The above methods relies on the fact that in this configuration there is a lower bound for the gain (for the functional F_{f_1}) at each step in each Euler scheme, and as this bound is uniform, we are able to pass to the limit $\varepsilon_n \to 0$ and obtain an estimate for the rate independent case.

4.2 Geometric-energy estimate

Now we present a sharper upper bound estimate for the branching time, based on geometrical arguments and energy considerations. The notations used in the previous subsection are null here.

Lemma 4.3. *Given a domain Ω, a measure $f_2 \geq c_{f_2}\mathcal{L}^2$ ($c_{f_2} > 0$), an element $S_1 \in A$, and suppose that there exists $Q \in \Omega$ and $R > 0$ such that the ball $B(Q,R) \cap S = \emptyset$. Then*

$$F_{f_2}(S_1) \geq \frac{4c_{f_2}\pi R^3}{27}.$$

Proof. The proof is easy: as $B(Q,R) \cap S_1 = \emptyset$, for any $r < R$ all points $x \in B(Q,r)$ verify $\text{dist}(x, S_1) \geq R - r$, so

$$F_{f_2}(S_1) = \int_\Omega \text{dist}(x,S_1)df_2 \geq \int_{B(Q,r)} \text{dist}(x,S_1)df_2 \geq c_{f_2}(R-r)\pi r^2.$$

Differentiating the expression $c_{f_2}(R-r)\pi r^2$, its maximum value is attained by $r = \dfrac{2}{3}R$, which corresponds to

$$F_{f_2}(S_1) \geq \frac{4\pi c_{f_2}}{27}R^3$$

and the proof is complete. □

Lemma 4.4. *Given a domain Ω, an element $S_2 \in A$, a point $Q' \in S_2$ and suppose that its Voronoi cell $V(Q')$ has $|V(Q')| > 0$. Then there exists $\bar{Q} \in \Omega$ and $\bar{R} > 0$ such that $B(\bar{Q}, \bar{R}) \cap S_2 = \emptyset$.*

For the proof we refer to [12].

We recall here that given $S' \in A$ (in a given domain Ω), $\varepsilon > 0$ and $H_\varepsilon \in A_\varepsilon \backslash \bigcup_{0 \leq \varepsilon' < \varepsilon} A_{\varepsilon'}$, adding H_ε to a generic point $U \in S'$ the gain in energy is (upon higher order terms)

$$F(S' \cup H_\varepsilon) \geq F(S') - \varepsilon |V(U)|. \tag{4.1}$$

Now we consider a configuration similar to the one in the previous subsection:

given a domain Ω, a coercive measure $f' \geq c' \mathcal{L}^2$ with $c' > 0$, let S_0^{dat} be the initial datum, and there exist

- $P'_0 \in S_0^{dat}$ angular and let be $\xi' > 0$ such that $B(P'_0, \xi') \cap S_0^{dat}$ is homeomorphic to $(0, 1)$;
- a closed injective path $\gamma^* : [0, 1] \longrightarrow \Omega$ such that $\gamma^*([0, 1]) \subseteq S_0^{dat}$: the domain Ω is now divided in two regions, Ω^+ and Ω^- with $\Omega = \Omega^+ \cup \Omega^-$ (they are the two connected components of $\Omega \backslash \gamma^*([0, 1])$, and they correspond to the "interior" and the "exterior" part of $\gamma^*([0, 1])$ – the order is not relevant – given by the Jordan Curve Theorem);
- triangle $T_{P'_0} \subset V(P'_0) \cap B(P'_0, \xi')$ (its existence is given by Lemma 3.7) verifies $|T_{P'_0} \cap \Omega^+| > 0$, and $ext(S_0^{dat}) \subset \Omega^-$.

Notice that S_0^{dat} is very similar to S_0^{ini}, and results as Lemma 4.1 holds.

In the rest of this subsection we will suppose that Ω^- is large enough (both in diameter and in measure) so that all computations can be done without considering constraints imposed by $\mathrm{diam}(\Omega^-), f'(\Omega^-)$.

Given a positive time T, consider a minimizing movement $\Sigma : [0, T] \longrightarrow A$ obtained as limit of Euler schemes $\{\Sigma_{\varepsilon'_n}\}_{n=0}^\infty : [0, T] \longrightarrow A$ (with time steps $\{\varepsilon'_n\}_{n=0}^\infty \downarrow 0$):

$$\begin{cases} w(0, n) = w(0) := S_0^{dat} \\ w(k, n) \in \mathrm{argmin}_{\mathcal{H}^1(S'') \leq \mathcal{H}^1(S_0^{dat}) + k\varepsilon'_n} F_{f'}(S'') \end{cases},$$

$$\Sigma_{\varepsilon'_n}(t) := w\left(\left[\frac{t}{\varepsilon'_n}\right], n\right), \qquad \Sigma(t) = \lim_{n \to \infty} \Sigma_{\varepsilon'_n}(t) \forall t \in [0, T].$$

The main estimate here is Theorem 4.5.

The notations introduced (except mute counters like k and n) will have the same meaning in the following of this subsection.

Applying the estimate of Proposition 3.5, we have a positive constant $K(P'_0)$ such that for any n, k

$$\min_{\mathcal{H}^1(\mathcal{X}'') \leq w(k-1,n) + k\varepsilon'_n,\ w(k-1,n) \subset \mathcal{X}''} F_{f'}(\mathcal{X}'') \leq F_{f'}(w(k-1, n)) - K(P'_0)\varepsilon'_n$$

thus

$$F_{f'}(w(k, n)) \leq F_{f'}(w(0)) - kK(P'_0)\varepsilon'_n \tag{4.2}$$

i.e. $\forall t \in [0,T]$

$$F_{f'}(\Sigma_{\varepsilon'_n}(t)) := F_{f'}(w(\left[\frac{t}{\varepsilon'_n}\right],n)) \le F_{f'}(S_0^{dat}) - \left[\frac{t}{\varepsilon'_n}\right] K(P'_0)\varepsilon'_n \le F_{f'}(S_0^{dat}) - (\frac{t}{\varepsilon'_n}+1)K(P'_0)\varepsilon'_n.$$

Passing to the limit $n \longrightarrow \infty$, this reads

$$\begin{array}{lll} F_{f'}(\Sigma(t)) & := F_{f'}(S_0^{dat}) - \lim_{n\to\infty} F_{f'}(w(\left[\frac{t}{\varepsilon'_n}\right],n)) & \le F_{f'}(S_0^{dat}) - \lim_{n\to\infty} \left[\frac{t}{\varepsilon'_n}\right] K(P'_0)\varepsilon'_n \\ & \le F_{f'}(S_0^{dat}) - \lim_{n\to\infty} (\frac{t}{\varepsilon'_n}+1)K(P'_0)\varepsilon'_n & = F_{f'}(S_0^{dat}) - tK(P'_0) \end{array}.$$

To avoid a branching behavior, there exists an endpoint P^* of $\Sigma(t)$ with $|V(P^*)| \ge K(P'_0)$, then for Lemma 4.4 there exists a point $X \in \Omega^-$ such that the ball

$$B(X,v) \cap \Sigma(t) = \emptyset, \qquad v = \sqrt{\frac{K(P'_0)}{\pi}},$$

and Lemma 4.3 gives

$$F_{f'}(\Sigma(t)) \ge \frac{4c'}{27}\sqrt{\frac{K(P'_0)^3}{\pi}}.$$

But we must have

$$F_{f'}(\Sigma(t)) \le F_{f'}(S_0^{dat}) - tK(P'_0)$$

and combining the above inequalities,

$$F_{f'}(S_0^{dat}) - tK(P'_0) \ge \frac{4c'}{27}\sqrt{\frac{K(P'_0)^3}{\pi}}$$

which gives $t \le \frac{F_{f'}(S_0^{dat})}{K(P'_0)} - \frac{4c'}{27}\sqrt{\frac{K(P'_0)^3}{\pi}}$. So we have proved the following result:

Theorem 4.5. *For this configuration, with the above notations, an upper bound for the branching time is given by*

$$T_{\max} := \frac{F_{f'}(S_0^{dat})}{K(P'_0)} - \frac{4c'}{27}\sqrt{\frac{K(P'_0)^3}{\pi}}.$$

Notice that the partition $\Omega^+ \cup \Omega^-$ is crucial as Lemma 4.1 makes impossible passing from one region to another without changing topology, so it prevents $\Sigma(t)$ from ever visit $T(P'_0) \cap \Omega^+$ without branching behaviors.

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